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S. M. SALAMON

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DIFFERENTIAL GEOMETRY OF QUATERNIONIC MANIFOLDS

By S. M. SALAMON (*)

Introduction

This paper consists of an exposition of a theory of quaternionic manifolds which has been advertised in [S_1]; [S_3] and developed independently by L. Bérard Bergery ([BO]; [Bes]). A quaternionic manifold is defined by a G-structure admitting a torsion-free connection, where G denotes the maximal subgroup $GL(n, \mathbb{H})\,GL(1, \mathbb{H})$ of $GL(4n, \mathbb{R})$. This encompasses virtually all definitions that have been given in the past by various authors. The torsion condition is a natural generalization of the holonomy restriction that is used to define quaternionic Kähler manifolds, and was considered by Bonan in [Bo_1]. Here, in contrast to [S_2], we study almost exclusively those properties of a quaternionic manifold that can be expressed without reference to a particular Riemannian metric.

Unlike complex manifolds, quaternionic manifolds can be distinguished locally by a curvature tensor whose vanishing is equivalent to integrability of the G-structure. Integrable G-structures were investigated in [M]; [Ku], and shown to be locally equivalent to quaternionic projective space $\mathbb{H}P^n$; they therefore constitute a very restrictive class. Even more special are the locally affine manifolds with an integrable $GL(n, \mathbb{H})$-structure discussed in [Eh]; [So]. Our definition, on the other hand, does not require the existence of any obvious quaternionic coordinates.

In the first three sections we develop the algebra which is needed to understand the properties of a quaternionic manifold. This consists of some elementary, if somewhat involved, representation theory, and culminates in an explicit description of curvature (theorem 3.4). The central results of the paper are theorems 4.1 and 5.5, which characterize the quaternionic structure by the existence of certain complexes of differential operators. Our methods involve first of all a decomposition of the exterior algebra of the de Rham complex, and then the adoption of techniques from conformal geometry.

We single out one special operator, already familiar in Penrose's twistor theory [P], that can be used to reduce the structure group of a quaternionic manifold to $GL(n, \mathbb{H})$. These ideas lead to a new collection of examples (theorem 7.2) that do not fit into any category previously studied. This is based on the notion of self-dual connections on bundles over 4-manifolds, and so provides a link with Yang-Mills theory. For instance, we show that the tangent bundle of both the sphere S^4 and complex projective plane $\mathbb{C}P^2$ is a quaternionic manifold.

As a final consequence of the above approach, we see that associated to any quaternionic manifold there corresponds a complex manifold, the so-called twistor space (corollary 7.4). This correspondence accounts for many of the formal similarities between quaternionic and complex geometry, and its implications will be pursued in a subsequent paper.

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1. The structure group $GL(n, \mathbb{H})\,GL(1, \mathbb{H})$

Pointwise, the definition of a quaternionic manifold will be modelled on the quaternionic projective space $\mathbb{H}P^n$. Take homogeneous coordinates on $\mathbb{H}P^n$ with $(q_0, \ldots, q_n)$ and $(q_0 u, \ldots, q_n u)$, $u \in \mathbb{H}^*$, defining the same point. On the open set U_r defined by $q_r \neq 0$, there are inhomogeneous coordinates $q_{ir} = q_i q_r^{-1}$, $i \neq r$. The n-tuple

$$dq_{ir} = (dq_i - q_{ir}\, dq_r)\, q_r^{-1}, \qquad i \neq r,$$

of quaternion valued differential forms defines for each $x \in U_r$ an isomorphism $T_x \mathbb{H}P^n \xrightarrow{\cong} \mathbb{H}^n$, and so gives rise to a section η_r over U_r of the principal frame bundle P of $\mathbb{H}P^n$. The section $\eta_s \in \Gamma(U_s, P)$ corresponding to a different coordinate patch is defined by the n-tuple

$$dq_{is} = (dq_{ir} - q_{is}\, dq_{sr})\, q_{sr}^{-1}, \qquad i \neq s.$$

Thus at each $x \in U_r \cap U_s$,

$$\eta_s = \eta_r\, A\, q,$$

where q is an element of the subgroup $GL(1, \mathbb{H})$ of $GL(4n, \mathbb{R})$ of nonzero quaternions acting by right multiplication, and A is an element of the centralizer $GL(n, \mathbb{H})$ of $GL(1, \mathbb{H})$ in $GL(4n, \mathbb{R})$. Consequently the transition function $A\,q$ takes values in the product

$$G = GL(n, \mathbb{H})\, GL(1, \mathbb{H}) \subset GL(4n, \mathbb{R}),$$

and the sections η_r generate a principal G-subbundle Q of P. The bundle Q constitutes a reduction of the structure group of P from $GL(4n, \mathbb{R})$ to G, and Q itself is called a G-structure.

Since $A\,q=(A\,|q|)(|q|^{-1}q)$, we also have

$$G=GL(n, \mathbb{H})\,Sp(1)\cong GL(n, \mathbb{H})\times_{Z_2} Sp(1),$$

where $Sp(1)$ denotes the group of unit quaternions in $GL(1, \mathbb{H})$. Thus G has as $2:1$ universal covering the group $\tilde{G}=GL(n, \mathbb{H})\times Sp(1)$. By considering also the subgroup $Sp(n)$ of $GL(n, \mathbb{H})$ consisting of unitary transformations, we obtain the maximal compact subgroup $Sp(n)\,Sp(1)$ of G. For $n\geqq 2$, this is a maximal Lie subgroup of $SO(4n)$ [G], although $Sp(1)\,Sp(1)=SO(4)$. As a rank one symmetric space, $\mathbb{H}P^n$ has linear isotropy group $Sp(n)\,Sp(1)$, and $Sp(n)\,Sp(1)$ is one of the possible holonomy groups of a Riemannian manifold which is not locally symmetric [Be]. A Riemannian manifold whose linear holonomy lies in $Sp(n)\,Sp(1)$ is called *quaternionic Kähler* in analogy with the complex case ([G]; [I]; [S_2]). On such a manifold, the Levi-Civita connection on the holonomy bundle extends to a torsion-free connection on the underlying G-structure Q. This leads to:

DEFINITION 1.1. — A quaternionic manifold is a real $4n$-dimensional manifold M, $n\geqq 2$, together with a G-structure Q admitting a torsion-free connection.

In other words Q is a G-subbundle of the frame bundle of M, and there exist local coordinates $x_1, \ldots, x_{4n}$ on M for which the frame $(\partial/\partial x_1, \ldots, \partial/\partial x_{4n})$ is tangent to Q at a given point. Equivalently Q is "1-flat", i. e. to the first order it is locally isomorphic to the standard G-structure $\mathbb{R}^{4n}\times G$ on $\mathbb{R}^{4n}$. We shall investigate the obstruction to the existence of a torsion-free G-connection in the next section.

On the coordinate patch U_r of $\mathbb{H}P^n$, the moving frame $\eta_r\in\Gamma(U_r, Q)$ converts the unit imaginary quaternions i, j, k into almost complex structures I, J, K on U_r. Using η_s instead of η_r produces different I, J, K, but preserves the family $\{aI+bJ+cK : a^2+b^2+c^2=1\}$. The same holds for any quaternionic manifold M: regarding an element $\eta\in Q$ as an isomorphism $T_x M \xrightarrow{\cong} \mathbb{H}^n$, set

$$Z=\{\eta^{-1}\circ R_u\circ\eta : u\in \operatorname{Im}\mathbb{H}),\ |u|=1,\ \eta\in Q\},$$

where R_u denotes right multiplication by u on $\mathbb{H}^n$. Then Z is a well defined bundle with fibre S^2 over M. For if η, η' belong to the same fibre of Q, $\eta'=\eta\,A\,q$ for some $A\,q\in G$, $q\in Sp(1)$, and

$$\eta'^{-1}\circ R_u\circ\eta=R_{q^{-1}uq}.$$

The last equation shows that Z is none other than the bundle $Q\times_G S^2$ associated to the adjoint action of $Sp(1)$ on the sphere $S^2\subset \operatorname{Im}\mathbb{H}\cong\mathfrak{sp}(1)$. Any local section of Q converts the basis i, j, k of $\operatorname{Im}\mathbb{H}$ into local almost complex structures I, J, K on M satisfying $IJ=-JI=K$.

In the sequel we shall frequently consider vector bundles associated to the G-structure Q. Actually it is more convenient to work with the double cover $\tilde{G}$ in order to handle the factors $GL(n, \mathbb{H})$ and $Sp(1)$ independently. Regard Q as an element of the Čech cohomology group $H^1(M, G)$ with coefficients in the sheaf of germs of smooth G-valued functions. In the exact sequence

$$H^1(M, \mathbb{Z}_2) \to H^1(M, \tilde{G}) \to H^1(M, G) \xrightarrow{\delta} H^2(M, \mathbb{Z}_2),$$

$\varepsilon = \delta(Q)$ defines a canonical cohomology class, which is the obstruction to Q lifting to a principal $\tilde{G}$-bundle $\tilde{Q}$. Reducing to maximal compact subgroups relates ε to the second Steifel-Whitney class w_2 of M ([S_3]; [MR]).

By restricting to an open subset if necessary, we assume that $\varepsilon = 0$, and choose a lifting $\tilde{Q}$ of Q. Given a representation $\rho : \tilde{G} \to \operatorname{Aut} V$, let $\mathbf{V}$ denote the associated vector bundle $\tilde{Q} \times_G V$; it is defined independently of the choice of lifting if ρ factors through G. For example, copying notation of [S_2], the complexified cotangent bundle of M has the form

$$(1.1) \qquad (T^* M)^c \cong \mathbf{E} \otimes_{\mathbb{C}} \mathbf{H},$$

where $E \cong \mathbb{C}^{2n}$, $H \cong \mathbb{C}^2$ are basic complex representations of $GL(n, \mathbb{H})$, $Sp(1)$ respectively. However it may not be possible to define vector bundles **E**, **H** globally. Complex conjugation in (1.1) corresponds to the antilinear involution $j_E \otimes j_H$, where j_E, j_H denote quaternion multiplication on E, H.

2. Torsion and the first prolongation

To understand the torsion of a G-connection, one must consider the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ of G. Let T denote the representation of G corresponding to the tangent bundle, so that

$$\mathfrak{g} \subset \operatorname{End} T = T \otimes T^*.$$

Then $\mathfrak{g}^{(1)}$ is defined to be the kernel of the natural skewing mapping

$$(2.1) \qquad \partial : \quad \mathfrak{g} \otimes T^* \to T \otimes \Lambda^2 T^*.$$

The difference $\nabla - \nabla'$ of any two G-connections is essentially a tensor ξ belonging to $\mathfrak{g} \otimes T^*$ (more precisely ξ is a section of the associated vector bundle with fibre $\mathfrak{g} \otimes T^*$). The element $\partial\xi$ then represents the difference $\tau(\nabla) - \tau(\nabla')$ of the torsions, so if ∇ and ∇' are both without torsion, ξ belongs to $\ker \partial = \mathfrak{g}^{(1)}$. On the other hand, given an arbitrary G-connection ∇ which always exists, the projection $[\tau(\nabla)]$ of its torsion in the quotient

$$(2.2) \qquad \operatorname{coker} \partial = T \otimes \Lambda^2 T^* / \partial(\mathfrak{g} \otimes T^*)$$

is a tensor C_0 depending only on the G-structure. It is this tensor that measures the obstruction to the existence of a torsion-free G-connection.

The calculation of $\ker \partial$, $\operatorname{coker} \partial$ for $G = GL(n, \mathbb{H})\,Sp(1)$ is contained implicitly in the work of Ochiai [Oc], along with results for other groups. Because of its importance in the sequel, we shall carry out this calculation directly using some elementary representation theory. From (1.1), the cotangent representation is $T^* = E \otimes H$ (for notational convenience we work exclusively over $\mathbb{C}$, so that real representations are automatically complexified). Now $Sp(1) \cong SU(2)$ leaves invariant a skew form $\omega \in \Lambda^2 H^*$ which induces an isomorphism $H \cong H^*$. We shall let $\{h, \tilde{h}\}$ denote any basis of the complex vector space H compatible with the SU(2)-structure; hence $\tilde{h} = jh$ and $\omega(h, \tilde{h}) = 1$.

The irreducible complex representations of Sp(1) are precisely the symmetric powers $S^k H$, $k \geqq 0$ (homogeneous polynomials of degree k in 2 variables). Their tensor products behave according to the Clebsch-Gordan formula

$$S^j H \otimes S^k H \cong \bigotimes_{r=0}^{\min(j,k)} S^{j+k-2r} H, \tag{2.3}$$

which follows by taking traces with ω. For example

$$\operatorname{End} H = H \otimes H^* \cong H \otimes H \cong S^2 H \oplus \mathbb{C},$$

exhibits $S^2 H$ as the adjoint representation of Sp(1).

Modulo real 1-dimensional representations of the respective centres, the algebra of finite-dimensional representations of $GL(n, \mathbb{H})$ is isomorphic to that of the maximal compact subgroup $U(2n)$ of $GL(2n, \mathbb{C})$. Irreducible complex representations of $GL(n, \mathbb{H})$ can then be identified with weights

$$(a_1, a_2, \ldots, a_{2n}), \qquad a_i \in \mathbb{Z}, \qquad a_1 \geqq a_2 \ldots \geqq a_{2n},$$

and standard methods are available to decompose the tensor product of two irreducible representations [Z]. However for the most part we shall proceed from first principles, leaving the reader to verify that certain modules are irreducible and to determine their weights. The basic modules are

$$E = (1, 0, \ldots, 0), \qquad E^* = (0, \ldots, 0, -1),$$

and (omitting complexification signs),

$$\mathfrak{gl}(n, \mathbb{H}) \cong E^* \otimes E \cong \mathbb{C} \oplus \mathfrak{sl}(n, \mathbb{H}).$$

Any complex irreducible representation of $\tilde{G} = GL(n, \mathbb{H}) \times Sp(1)$ has the form $A \otimes S^k H$ for some irreducible $GL(n, \mathbb{H})$-module A. If A is contained in $(\otimes^p E) \otimes (\otimes^q E^*)$ with $p+q+k$ even, $A \otimes S^k H$ is (the complexification of) a real G-module. We are now in a position to determine the homomorphism (2.1), for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{H}) \oplus \mathfrak{sp}(1) \cong E^* E \oplus S^2 H$. We have

$$\mathfrak{g} \otimes T^* \cong (E^* E \oplus S^2 H) \otimes EH,$$

and since $\Lambda^2 H$ is trivial,

$$T \otimes \Lambda^2 T^* \cong E^* H \otimes \Lambda^2 (EH) \cong E^* H \otimes (S^2 E \oplus \Lambda^2 E S^2 H). \tag{2.4}$$

(Tensor products are indicated either in the usual way or simply by juxtaposition.) There is a contraction $\varphi : E^*\otimes S^2 E \to E$, so by Schur's lemma E must appear as a summand in $E^*\otimes S^2 E$, and

$$E^*\otimes S^2 E \cong E\oplus C, \tag{2.5}$$

where $C=\ker\varphi$. Similarly,

$$E^*\otimes \Lambda^2 E \cong E\oplus D, \tag{2.6}$$

and C and D are both irreducible. Combining the above gives

LEMMA 2.1:

$$\mathfrak{g}\otimes T^* \cong 3\,EH\oplus CH\oplus DH\oplus ES^3 H,$$
$$T\otimes \Lambda^2 T^* \cong 2\,EH\oplus CH\oplus DH\oplus ES^3 H\oplus DS^3 H.$$

Above $n\,EH$ denotes an isotypic component isomorphic to the direct sum of n copies of EH. It is now a straightforward, if somewhat tiresome, matter to verify that ∂ has "full rank", that is to say rank ∂ is as large as is permitted by Schur's lemma. This will then imply

PROPOSITION 2.2:

$$\mathfrak{g}^{(1)}=\ker\partial\cong EH,$$
$$\operatorname{coker}\partial\cong DS^3 H.$$

Proof. — By Schur's lemma it suffices to produce elements of the image $\partial(\mathfrak{g}\otimes T^*)$ which have nonzero components in the summands of $T\otimes\Lambda^2 T^*$ except $DS^3 H$. For the multiplicity one summands CH, DH, $ES^3 H$, this is easy, so we shall check only the restriction of ∂ to the submodule $3\,EH$.

There is one copy of EH in each of the three terms on the right-hand side of

$$\mathfrak{g}\otimes T^* \cong (\mathbb{C}\oplus\mathfrak{sl}(n,\mathbb{H})\oplus\mathfrak{sp}(1))\otimes EH.$$

Take any basis $\{e_i\}_{i=1}^{2n}$ of E and an SU(2)-basis $\{h, \tilde{h}\}$ of H, and let $\{e^i\}$ denote the dual basis of E^* so that $e^i he_i\tilde{h}-e^i\tilde{h}e_i h$ (summation) is an invariant in $E^*HEH=T\otimes T^*$. Then

$$\begin{aligned}
&\alpha_1=(e^i he_i\tilde{h}-e^i\tilde{h}e_i h)\,e_1 h\in\mathbb{C}\otimes EH,\\
&\alpha_2=(e^i he_1\tilde{h}-e^i\tilde{h}e_1 h)\,e_i h-\frac{1}{2n}\alpha_1\in\mathfrak{sl}(n,\mathbb{H})\otimes EH,\\
&\alpha_3=2\,e^i he_i he_1\tilde{h}-(e^i\tilde{h}e_i h+e^i he_i\tilde{h})\,e_1 h\in\mathfrak{sp}(1)\otimes EH,
\end{aligned}\tag{2.7}$$

are representatives of the element $e_1 h$ in each of the three copies of EH.

On the other hand, by (2.4) there are contractions

$$\psi_1 :\quad T\otimes\Lambda^2 T^*\to EH\subset E^*H\otimes S^2 E,$$
$$\psi_2 :\quad T\otimes\Lambda^2 T^*\to EH\subset E^*H\otimes\Lambda^2 ES^2 H$$

(each defined up to a constant). Calculation gives

$$\psi_1\,\partial\alpha_1 = e_1\,h,$$

$$\psi_1\,\partial\alpha_2 = e_1\,h - \frac{1}{2n}\psi_1\,\partial\alpha_1 = \left(1-\frac{1}{2n}\right)e_1\,h,$$

$$\psi_1\,\partial\alpha_3 = -3\,e_1\,h,$$

and

$$\psi_2\,\partial\alpha_1 = -e_1\,h,$$

$$\psi_2\,\partial\alpha_2 = e_1\,h - \frac{1}{2n}\psi_2\,\partial\alpha_1 = \left(1+\frac{1}{2n}\right)e_1\,h,$$

$$\psi_2\,\partial\alpha_3 = -e_1\,h.$$

Hence the homomorphism $\partial : 3\,\mathrm{EH} \to 2\,\mathrm{EH}$ is represented by the matrix

$$\begin{pmatrix} 1 & 1-\dfrac{1}{2n} & -3 \\ -1 & 1+\dfrac{1}{2n} & -1 \end{pmatrix},$$

∂ maps onto $2\,\mathrm{EH}$, and $\ker\partial$ is spanned by the element

$$\alpha = \frac{n+1}{n}\alpha_1 + 2\,\alpha_2 + \alpha_3. \quad \blacksquare \tag{2.8}$$

The above calculations remain valid when $n=1$, although in this case $\Lambda^2\,\mathrm{E}^* \cong \mathrm{C}$ and $\mathrm{D}=0$. Consequently $\operatorname{coker}\partial = 0$, and there is no obstruction to the existence of a torsion-free G-connection. Indeed $\mathrm{G} = \mathrm{GL}(1, \mathbb{H})\,\mathrm{Sp}(1) \cong \mathbb{R}^+ \times \mathrm{SO}(4)$, so a G-structure is equivalent to an orientation and a conformal class, and the Levi-Civita connection of any compatible Riemannian metric preserves the G-structure and has zero torsion.

3. Curvature and integrability

Following Guillemin [Gu], we first review some facts about G-structures for an arbitrary subgroup G of $\mathrm{GL}(\mathrm{N}, \mathbb{R})$ with Lie algebra $\mathfrak{g} \subset \mathrm{T} \otimes \mathrm{T}^*$, $\mathrm{T} = \mathbb{R}^{\mathrm{N}}$. The homomorphism (2.1) is a special case of the Spencer complex

$$\ldots \to \mathfrak{g}^{(r)} \otimes \Lambda^{s-1}\,\mathrm{T}^* \to \mathfrak{g}^{(r-1)} \otimes \Lambda^s\,\mathrm{T}^* \to \mathfrak{g}^{(r-2)} \otimes \Lambda^{s+1}\,\mathrm{T}^* \to \ldots \tag{3.1}$$

In this the higher prolongations $\mathfrak{g}^{(r)}$ are defined inductively by

$$\mathfrak{g}^{(r)} = \ker\partial : \quad \mathfrak{g}^{(r-1)} \otimes \mathrm{T}^* \to \mathfrak{g}^{(r-2)} \otimes \Lambda^2\,\mathrm{T}^*,$$

where $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(-1)} = \mathrm{T}$, and ∂ denotes antisymmetrization. Equivalently,

$$\mathfrak{g}^{(r)} = (\mathfrak{g} \otimes S^r T^*) \cap (T \otimes S^{r+1} T^*),$$

and the fact that $\partial^2 = 0$ in (3.1) follows by construction. The cohomology at the point $\mathfrak{g}^{(r-1)} \otimes \Lambda^s T^*$ is denoted by $H^{r,s}(\mathfrak{g})$; note that $H^{r,0}(\mathfrak{g}) = 0 = H^{r,1}(\mathfrak{g})$ for all r.

A G-structure Q is said to be *integrable* or *flat* if it is locally isomorphic to the standard G-structure $\mathbb{R}^N \times G$. This means that in a neighbourhood of each point of the manifold, there exist coordinates $x_1, \ldots, x_N$ such that the moving frame $(\partial/\partial x_1, \ldots, \partial/\partial x_N)$ is a section of Q. On the overlap of two such charts, the derivative of the transition mapping will lie in the subgroup G of $GL(N, \mathbb{R})$.

One can construct a series of tensors C_r, $r \geqq 0$, whose vanishing is a necessary condition for integrability. More precisely, if $C_0, \ldots, C_{r-1}$ vanish, then C_r is a well-defined tensor in $H^{r,2}(\mathfrak{g})$ [i. e. C_r is a section of $Q \times_G H^{r,2}(\mathfrak{g})$]. The tensor C_0 has already been defined as the projection in $H^{0,2}(\mathfrak{g})$ of the torsion of any G-connection [*see* (2.2)]. If $C_0 = 0$, there exists a torsion-free G-connection ∇ whose curvature R lies in $\mathfrak{g} \otimes \Lambda^2 T^*$. The first Bianchi identity implies that $\partial R = 0$, and C_1 is the corresponding class [R] in $H^{1,2}(\mathfrak{g})$ which is independent of the choice of ∇.

The space

$$\sum_{r \geqq -1} \mathfrak{g}^{(r)},$$

is isomorphic to the graded Lie algebra of vector fields preserving infinitesimally the standard G-structure $\mathbb{R}^N \times G$. The Lie algebra $\mathfrak{g}$ is said to be of *finite type* k if $\mathfrak{g}^{(r)} = 0$ exactly when $r \geqq k$; in this case the group of local automorphisms of the standard G-structure is finite-dimensional. If $\mathfrak{g}^{(k)} = 0$, then $H^{r,2}(\mathfrak{g}) = 0$ for all $r \geqq k+1$, and it follows essentially from the Frobenius theorem that the vanishing of $C_0, C_1, \ldots, C_k$ is actually a sufficient condition for integrability. For more details, we refer the reader to [Gu]; [SS].

The following theorem was proved explicitly by Kulkarni [Ku], and is a special case of quite general results of [KN]; [Oc]. Here it is given an elementary proof using only notation of the last section.

THEOREM 3.1. — *The Lie algebra of* $GL(n, \mathbb{H})\,Sp(1)$ *has finite type* 2.

Proof. — We must show that the kernel $\mathfrak{g}^{(2)}$ of

$$\partial : \quad \mathfrak{g}^{(1)} \otimes T^* \to \mathfrak{g} \otimes \Lambda^2 T^*,$$

is zero. From proposition 2.2,

$$\mathfrak{g}^{(1)} \otimes T^* \cong EH \otimes EH \cong S^2 E \oplus \Lambda^2 E \oplus (S^2 E \oplus \Lambda^2 E)\, S^2 H. \tag{3.2}$$

By Schur's lemma, all we have to do is find elements in each of these 4 irreducible components which have nonzero images under ∂. In the notation of (2.7), (2.8), put

$$\beta_i = \alpha_i \otimes e_2 \tilde{h} \in (\mathfrak{g} \otimes T^*) \otimes T^*,$$
$$\beta = \alpha \otimes e_2 \tilde{h} \in \mathfrak{g}^{(1)} \otimes T^*.$$

Then β represents the element $e_1 h \otimes e_2 \tilde{h}$ in $EH \otimes EH$, so has nonzero components in all 4 submodules. The proof is completed by the following facts which can all be verified readily. The element $\partial\beta_1$ has nonzero components in both the submodules $\Lambda^2 ES^2 H$, $S^2 E$ of $\mathbb{C} \otimes \Lambda^2 T^*$, and $\partial\beta_3$ has nonzero components in both S^2ES^2H, $\Lambda^2 E$ in $\mathfrak{sp}(1) \otimes \Lambda^2 T^*$. ∎

Let M be a quaternionic manifold with G-structure Q, where from now on G again denotes the group $GL(n, \mathbb{H})\,Sp(1)$, $n \geqq 2$. In the Spencer cohomology language, proposition 2.2 says that $H^{0,2}(\mathfrak{g}) = DS^3 H$, and by definition $C_0 = 0$. The tensor C_1 belongs to the cohomology $H^{1,2}(\mathfrak{g})$ of the sequence

$$\mathfrak{g}^{(1)} \otimes T^* \to \mathfrak{g} \otimes \Lambda^2 T^* \to T \otimes \Lambda^3 T^*.$$

In order to decompose these spaces, we first introduce some additional representations of $GL(n, \mathbb{H})$.

There must exist modules L, L′ such that

$$E \otimes S^2 E \cong S^3 E \oplus L,$$
$$E \otimes \Lambda^2 E \cong \Lambda^3 E \oplus L',$$

and since no contractions are possible on the left-hand side, L and L′ are irreducible. The existence of a nonzero homomorphism

$$E \otimes \Lambda^2 E \hookrightarrow E \otimes E \otimes E \to S^2 E \otimes E.$$

then implies via Schur's lemma that $L \cong L'$. Next

$$\text{(3.3)} \qquad \begin{cases} E^* \otimes S^3 E \cong S^2 E \oplus U, \\ E^* \otimes \Lambda^3 E \cong \Lambda^2 E \oplus V, \end{cases}$$

with U, V irreducible, and so

$$\text{(3.4)} \qquad \begin{cases} E^* \otimes E \otimes S^2 E \cong S^2 E \oplus U \oplus E^* L, \\ E^* \otimes E \otimes \Lambda^2 E \cong \Lambda^2 E \oplus V \oplus E^* L. \end{cases}$$

From (2.5), (2.6), both left-hand members in (3.4) contain $E \otimes E$, so it must be the case that

$$\text{(3.5)} \qquad E^* L \cong S^2 E \oplus \Lambda^2 E \oplus W,$$

for some W. In fact W is irreducible; in terms of weights

$$L = (2, 1, 0, \ldots, 0), \qquad U = (3, 0, \ldots, 0, -1),$$
$$W = (2, 1, 0, \ldots, 0, -1), \qquad V = (1, 1, 1, 0, \ldots, 0, -1).$$

Finally we compute $\Lambda^3(EH)$ which is contained in

(3.6) $\mathrm{EH}\otimes\Lambda^2(\mathrm{EH})\cong\mathrm{EH}\otimes(\mathrm{S}^2\,\mathrm{E}\oplus\Lambda^2\,\mathrm{ES}^2\,\mathrm{H})\cong(\mathrm{S}^3\,\mathrm{E}\oplus 2\,\mathrm{L}\oplus\Lambda^3\,\mathrm{E})\,\mathrm{H}\oplus(\Lambda^3\,\mathrm{E}\oplus\mathrm{L})\,\mathrm{S}^3\,\mathrm{H}.$

Now $\Lambda^3(\mathrm{EH})$ certainly contains $\Lambda^3\,\mathrm{ES}^3\,\mathrm{H}$, and a dimension count reveals that the only other component is LH. Thus

$$\Lambda^3(\mathrm{EH})\cong\Lambda^3\,\mathrm{ES}^3\,\mathrm{H}\oplus\mathrm{LH}. \tag{3.7}$$

Combining (2.3), (3.3), (3.4), (3.5), (3.7) yields

LEMMA 3.2:

$$\mathfrak{g}\otimes\Lambda^2\,\mathrm{T}^*\cong 2\,\mathrm{S}^2\,\mathrm{E}\oplus 2\,\Lambda^2\,\mathrm{E}\oplus\mathrm{U}\oplus\mathrm{W}\oplus(2\,\mathrm{S}^2\,\mathrm{E}\oplus 3\,\Lambda^2\,\mathrm{E}\oplus\mathrm{V}\oplus\mathrm{W})\,\mathrm{S}^2\,\mathrm{H}\oplus\Lambda^2\,\mathrm{ES}^4\,\mathrm{H},$$

$$\mathrm{T}\otimes\Lambda^3\,\mathrm{T}^*\cong\mathrm{S}^2\,\mathrm{E}\oplus\Lambda^2\,\mathrm{E}\oplus\mathrm{W}\oplus(\mathrm{S}^2\,\mathrm{E}\oplus 2\,\Lambda^2\,\mathrm{E}\oplus\mathrm{V}\oplus\mathrm{W})\,\mathrm{S}^2\,\mathrm{H}\oplus(\Lambda^2\,\mathrm{E}\oplus\mathrm{V})\,\mathrm{S}^4\,\mathrm{H}.$$

The components of $\mathfrak{g}\otimes\Lambda^2\,\mathrm{T}^*$ minus those of $\partial(\mathfrak{g}^{(1)}\otimes\mathrm{T}^*)$ all occur in $\mathrm{T}\otimes\Lambda^3\,\mathrm{T}^*$ with the exception of U [*see* (3.2)]. Picking elements of $\mathfrak{g}\otimes\Lambda^2\,\mathrm{T}^*$ and using Schur's lemma, it is once again straightforward to check that $\partial:\mathfrak{g}\otimes\Lambda^2\,\mathrm{T}^*\to\mathrm{T}\otimes\Lambda^3\,\mathrm{T}^*$ has full rank. Hence

PROPOSITION 3.3. — $\mathrm{H}^{1,2}(\mathfrak{g})\cong\mathrm{U}$.

The curvature R of any torsion-free G-connection now has the form

$$\mathrm{R}=\partial(v_i\otimes t^i)+\mathrm{R}_\mathrm{U}, \tag{3.8}$$

where $v_i\in\mathfrak{g}^{(1)}$, $t^i\in\mathrm{T}^*$, and $\mathrm{R}_\mathrm{U}\in\mathrm{U}$ (with respect to any frame in Q). In analogy with the conformal case, R_U is called the *Weyl tensor* of M (*see* [Oc]); it can be identified with C_1. The above does not apply when $n=1$ and M is a 4-manifold with conformal structure; for in that case $\mathrm{H}^{1,2}(\mathfrak{g})=\mathrm{U}\oplus\mathrm{S}^4\,\mathrm{H}$, corresponding to the fact that in the presence of an orientation, the Weyl conformal tensor has two irreducible components [AHS]. One can see that M behaves like a quaternionic manifold provided that part of R lying in $\mathrm{S}^4\,\mathrm{H}$ vanishes; in this case M is said to be *self-dual.* It is well known that the 4-manifold M is conformally flat iff Weyl vanishes identically. However for $n\geqq 2$, it is necessary to check that there is no further integrability obstruction C_2 in $\mathrm{H}^{2,2}(\mathfrak{g})$.

THEOREM 3.4. — $\mathrm{H}^{2,2}(\mathfrak{g})=0$, *so the* G-*structure of a quaternionic manifold is integrable iff* $\mathrm{R}_\mathrm{U}=0$.

Proof. — This is almost identical to that of theorem 3.1. One must show that

$$\partial:\quad\mathfrak{g}^{(1)}\otimes\Lambda^2\,\mathrm{T}^*\to\mathfrak{g}\otimes\Lambda^3\,\mathrm{T}^*,$$

is injective, where the module $\mathfrak{g}^{(1)}\otimes\Lambda^2\,\mathrm{T}^*$ is decomposed in (3.6).

Setting

$$\gamma_i=\alpha_i\otimes(e_2\,\tilde{h}\wedge e_3\,h)\in(\mathfrak{g}\otimes\mathrm{T}^*)\otimes\Lambda^2\,\mathrm{T}^*,$$

the proof follows from verification of the following statements. The element $\partial\gamma_1$ has nonzero components in both submodules $\Lambda^3\mathrm{E}\,\mathrm{S}^3\,\mathrm{H}$, LH of $\mathbb{C}\otimes\Lambda^3\,\mathrm{T}^*$; $\partial\gamma_2$ has nonzero component in $\mathrm{S}^3\mathrm{E}\,\mathrm{H}$ in $\mathfrak{sl}(n,\mathbb{H})\otimes\Lambda^3\,\mathrm{T}^*$; and finally $\partial\gamma_3$ has nonzero components in LH, $\Lambda^3\mathrm{E}\,\mathrm{H}$, $\mathrm{LS}^3\,\mathrm{H}$ in $\mathfrak{sp}(1)\otimes\Lambda^3\,\mathrm{T}^*$. ■

In section 1, we saw that the quaternionic manifold $\mathbb{HP}^n$ possesses a natural integrable G-structure Q. Elements of the group $\mathrm{PGL}(n+1, \mathbb{H})=\mathrm{SL}(n+1, \mathbb{H})/\mathbb{Z}_2$ of projective transformations act as automorphisms of Q. Denote the isotropy subgroup of $\mathrm{PGL}(n+1, \mathbb{H})$ fixing a given point of $\mathbb{HP}^n$ by G_1, so that there is an exact sequence

$$1 \to \mathrm{N} \to \mathrm{G}_1 \to \mathrm{G} \to 1, \tag{3.9}$$

N being the kernel of the linear isotropy representation. Counting dimensions, the Lie algebra of G_1 is isomorphic as a vector space to $\mathfrak{g}\oplus\mathfrak{g}^{(1)}$. Indeed N is isomorphic to the abelian group $(\mathfrak{g}^{(1)}, +)$, and G_1 is a semidirect product $\mathrm{G}\ltimes \mathrm{N}$. Kulkarni proves in [Ku] that any local automorphism of Q is the restriction of an element of $\mathrm{PGL}(n+1, \mathbb{H})$ (the essential point is that $\mathrm{PGL}(n+1, \mathbb{H})$ is not a connected component of a larger group of automorphisms).

Now suppose that M is some other quaternionic manifold with an integrable $\mathrm{GL}(n, \mathbb{H})\,\mathrm{Sp}(1)$-structure. From the definition of integrability, there must be an open covering $\{\mathrm{U}_\alpha\}$ of M and diffeomorphisms φ_α of U into $\mathbb{HP}^n$ such that every $\varphi_\beta\circ\varphi_\alpha^{-1}$ is a projective transformation. In the language of Kobayashi [Ko], M is then *projectively flat*. Using a development argument, as for a conformally flat manifold, yields

COROLLARY 3.5. — *A compact simply-connected quaternionic manifold with* $\mathrm{R_U}=0$ *is diffeomorphic to* $\mathbb{HP}^n$.

4. Exterior forms

In this section we shall study properties of the exterior forms on a quaternionic manifold M. The space of r-forms

$$\Lambda^r \mathrm{T}^*=\Lambda^r(\mathrm{E}\otimes\mathrm{H}),$$

may be regarded as a subspace of $(\otimes^r \mathrm{E})\otimes(\otimes^r \mathrm{H})$, and using (2.3), there must exist representations L_k^r of $\mathrm{GL}(n, \mathbb{H})$ such that

$$\Lambda^r \mathrm{T}^*\cong \bigoplus_{k=0}^{[r/2]} \mathrm{L}_k^r\otimes \mathrm{S}^{r-2k}\mathrm{H}. \tag{4.1}$$

This is essentially the Lepage decomposition of [Bo_2], theorem 4. If one symmetrizes completely on H, one must antisymmetrize completely on E, so $\mathrm{L}_0^r=\Lambda^r \mathrm{E}$ for all r. Consequently for $0\leqq r\leqq 2n$, $\Lambda^r\mathrm{T}^*$ contains an irreducible subspace

$$\mathrm{A}^r\cong\Lambda^r\mathrm{E}\otimes\mathrm{S}^r\mathrm{H}.$$

The subalgebras $\mathfrak{gl}(n, \mathbb{H})$, $\mathfrak{gl}(1, \mathbb{H})$ are each other's centralizer in $\mathfrak{gl}(4n, \mathbb{R})$, and are said to constitute a *reductive dual pair*. It follows from the general theory of such objects that L_k^r is always irreducible (*see* for example [H], theorem 8), although we shall not need to know this. Apart from L_0^r, we have already met $\mathrm{L}_1^2=\mathrm{S}^2\mathrm{E}$ and $\mathrm{L}_1^3=\mathrm{L}$; in general in terms of weights,

$$\mathrm{L}_k^r = (\underbrace{2, \ldots, 2}_{k}, \underbrace{1, \ldots, 1}_{r-2k}, 0, \ldots, 0).$$

For $0 \leqq r \leqq 2n$, let B^r denote the complementary subspace to A^r in $\Lambda^r \mathrm{T}^*$; this is well defined by the action of $\mathrm{Sp}(1)$ and (4.1). Thus

$$\Lambda^r \mathrm{T}^* = \mathrm{A}^r \oplus \mathrm{B}^r, \qquad 0 \leqq r \leqq 2n, \tag{4.2}$$

where as a representation of $\mathrm{Sp}(1)$, B^r is a sum of spaces $\mathrm{S}^{r-2k}\mathrm{H}$, $k \geqq 1$. Note that $\mathrm{B}^0 = 0 = \mathrm{B}^1$. Let $p : \Lambda^r \mathrm{T}^* \to \mathrm{A}^r$ denote the projection.

Setting $\mathrm{D} = p \circ d$ defines differential operators

$$0 \to \mathbf{A}^0 \xrightarrow{\mathrm{D}=d} \mathbf{A}^1 \xrightarrow{\mathrm{D}} \mathbf{A}^2 \xrightarrow{\mathrm{D}} \ldots \to \mathbf{A}^{2n} \to 0, \tag{4.3}$$

where $\mathbf{A}^r = \mathrm{Q} \times_{\mathrm{G}} \mathrm{A}^r$ is the associated vector bundle.

So far we have only used the fact that M has a G-structure, where G is the group $\mathrm{GL}(n, \mathbb{H})\,\mathrm{Sp}(1)$, $n \geqq 2$. Using the expression *almost quaternionic* for such a manifold (*cf.* [Bo_1, Ma]), we have:

THEOREM 4.1. — *An almost quaternionic manifold* M *is quaternionic iff* (4.3) *is a complex.*

Proof. — Since $\mathrm{D}^2 : \mathbf{A}^0 \to \mathbf{A}^2$ is simply $p \circ d^2$, this always vanishes.

Accordingly consider $\mathrm{D}^2 : \mathbf{A}^1 \to \mathbf{A}^3$. If $\alpha \in \Gamma(\mathrm{M}, \mathbf{A}^1)$ with

$$d\alpha = \mathrm{D}\alpha + \beta, \qquad \beta \in \Gamma(\mathrm{M}, \mathbf{B}^2),$$

then

$$0 = p\, d^2 \alpha = \mathrm{D}^2 \alpha + p\, d\beta.$$

Now if f is a function,

$$p\, d(f\beta) = f\, p\, d\beta + p(df \otimes \beta) = f\, p\, d\beta,$$

since $df \otimes \beta$ belongs to $\mathrm{EH} \otimes \mathrm{S}^2\mathrm{E}$ which has no component in A^3. Hence $\Phi = p \circ d : \mathbf{B}^2 \to \mathbf{A}^3$ is a homomorphism which vanishes iff $\mathrm{D}^2 : \mathbf{A}^1 \to \mathbf{A}^3$ does.

If ∇ is any G-connection, thought of as a differential operator $\mathbf{T}^* \to \mathbf{T}^* \otimes \mathbf{T}^*$ acting on the cotangent bundle $\mathbf{T}^* = \mathrm{T}^*\mathrm{M}$, its torsion may be defined by

$$\tau = d + \partial\nabla : \mathbf{T}^* \to \Lambda^2 \mathbf{T}^*,$$

where ∂ is the antisymmetrization. Hence

$$\Phi(\beta) = p(\tau - \partial\nabla)(\beta) = p\,\tau(\beta),$$

where τ acts on β as an antiderivation. This realizes Φ as a component of τ, but it must of course be a component independent of the choice of ∇. Indeed relative to any frame in the principal G-bundle Q,

$$\Phi \in (B^2)^* \otimes A^3 \cong S^2 E^* \otimes \Lambda^3 ES^3 H,$$

and because the right-hand side cannot contain $ES^3 H$, Φ must belong to $DS^3 H$. It follows that Φ can be identified with the tensor C_0, which therefore vanishes if (4.3) is a complex.

Conversely if $C_0 = 0$, ∇ can be chosen torsion-free. In this case $p \circ d : \mathbf{B}^r \to \mathbf{A}^{r+1}$ must vanish for all $r \leqq 2n-1$, and (4.2) is a complex. ■

Theorem 4.1 is completely analogous to the corresponding statement involving a complex manifold and the Dolbeault complex $(\Lambda^{0,*}, \bar\partial)$. The obstruction to $\bar\partial^2 = 0$ is the Nijenhuis tensor

$$N(X, Y) = [IX, IY] - I[IX, Y] - I[X, IY] - [X, Y],$$

where I is the almost complex structure. The vanishing of N means that the exterior derivative of a (1, 1)-form has no (0, 3)-component. We can obtain a similar interpretation of C_0 as follows. Let M be almost quaternionic, and consider the bundle Z of almost complex structures defined in section 1. Fixing $x \in M$, each point I of the fibre Z_x gives rise to a decomposition

$$(T_x^* M)^c = \Lambda_I^{1,0} \oplus \Lambda_I^{0,1},$$

so one can speak of the space $\Lambda_I^{r,0}$ of $(r, 0)$-forms relative to I.

PROPOSITION 4.2:

$$\mathbf{A}^r = \sum_{I \in Z} \Lambda_I^{0,r},$$

$$\mathbf{B}^r = \bigcap_{I \in Z} (\Lambda_I^{r-1,1} \oplus \Lambda_I^{r-2,2} \oplus \ldots \oplus \Lambda_I^{1,r-1}).$$

Proof. — The sum (which denotes finite linear combinations) and the intersection are to be interpreted fibrewise. Fix a frame in the fibre Q_x to identify $(T_x^* M)^c$ with the vector space $E \otimes H$. Choosing $I \in Z_x$ reduces the structure group G of Q_x to $G \cap GL(2n, \mathbb{C}) = GL(n, \mathbb{H})\, U(1)$.

The representation H of Sp(1) then decomposes as $H = \mathbb{C}h \oplus \mathbb{C}jh$ for some basis $\{h, jh\}$ to give

(4.4) $$\Lambda_I^{1,0} = E \otimes \mathbb{C}h, \qquad \Lambda_I^{0,1} = E \otimes \mathbb{C}jh.$$

Then

$$\Lambda_I^{r,0} = \Lambda^r (E \otimes \mathbb{C}h) = \Lambda^r E \otimes \mathbb{C}h^r,$$

and summing over $I \in Z_x$ gives the required expression for $\mathbf{A}^r$.

Next choose a reduction of the structure group G of $T_x M$ to $Sp(n)\, Sp(1)$, which amounts to introducing a metric which is Hermitian relative to every $I \in Z_x$. This makes

$$(\Lambda^r T_x^* M)^c = (\Lambda_I^{r,0} \oplus \Lambda_I^{0,r}) \oplus (\Lambda_I^{r-1,1} \oplus \ldots \oplus \Lambda_I^{1,r-1}),$$

an orthogonal sum for all I. Therefore an element is orthogonal to A^r iff it belongs to

$$\bigcap_{I \in Z_x} (\Lambda_I^{r-1,1} \oplus \ldots \oplus \Lambda_I^{1,r-1}),$$

and the proposition follows. ■

Consequently, the vanishing of

$$C_0 = \Phi = p \circ d : \mathbf{B}^2 \to \mathbf{A}^3,$$

is equivalent to the next assertion. The exterior derivative of a 2-form of type (1, 1) relative to *all* $I \in Z$ has no (0, 3) component relative to *any* $I \in Z$. Observe that if $I \in Z_x$, then $-I \in Z_x$, and $\Lambda_{-I}^{r,0} = \Lambda_I^{0,r}$; this explains why A^r is (the complexification of) a real space.

Let P(**H**) denote the projective bundle with fibre $P(\mathbf{H}_x) = \mathbb{C}P^1$; it is defined globally even if **H** is not. As a corollary of the correspondence $I \leftrightarrow \mathbb{C}h$ arising from (4.4), we have

$$(4.5) \qquad Z \cong P(\mathbf{H}).$$

One of the justifications for our definition of a quaternionic manifold is that if $C_0 = 0$, the manifold Z has a natural complex structure (*see* corollary 7.4 or [S_3], theorem 2). This fact interprets the assertion in the previous paragraph.

5. Invariant differential operators

Let M be any quaternionic manifold, and as usual let Q denote the principal G-bundle of distinguished frames, $G = GL(n, \mathbb{H})\,Sp(1)$. At each point q of some fibre Q_x, consider the set $\mathscr{H}_q$ of horizontal subspaces in $T_q Q$ arising from torsion-free connections on Q. The set of such connections is parametrized at $x \in M$ by $\mathfrak{g}^{(1)}$, so $\mathscr{H}_q \cong \mathfrak{g}^{(1)}$. Each element $H \in \mathscr{H}_q$ can also be regarded as a 2-frame at x, i. e. the 2-jet of a local diffeomorphism from $\mathbb{R}^{4n}$ into M. The bundle Q_1 over M with fibre $\bigcup \{\mathscr{H}_q : q \in Q_x\}$ is then a G_1-subbundle of the second order frame bundle of M (*see* [Ko]), with G_1 as in (3.9). Indeed Q_1 consists of those frames which provide first order contact between the standard G-structure $\mathbb{R}^{4n} \times G$ and Q. The existence of Q_1 enables us to introduce first order invariant differential operators on a quaternionic manifold. Like G, the conformal group $CO(n) = \mathbb{R}^+ \times SO(n)$ has Lie algebra of type 2 [$n \geqq 3$; in fact $CO(4) = GL(1, \mathbb{H})\,Sp(1)$], and many of the techniques used by Fegan [F] in dealing with conformally invariant operators carry over to the quaternionic case. We outline the relevant facts.

Let **V**, **W** be vector bundles on M associated to Q and G-modules V, W. A first order differential operator from **V** to **W** is a homomorphism $D : J_1(\mathbf{V}) \to \mathbf{W}$, where $J_1(\mathbf{V})$ is the bundle of 1-jets of sections of **V**. Thus D defines a mapping of sections

$\Gamma(M, \mathbf{V}) \to \Gamma(M, \mathbf{W})$, though by abuse of notation one often writes $D : \mathbf{V} \to \mathbf{W}$. The vector bundle $J_1(\mathbf{V})$ fits into a short exact sequence

$$0 \to \mathbf{V} \otimes \mathbf{T}^* \xrightarrow{i} J_1(\mathbf{V}) \to \mathbf{V} \to 0, \tag{5.1}$$

and the symbol of D is the composition

$$\sigma = D \circ i : \mathbf{V} \otimes \mathbf{T}^* \to \mathbf{W}.$$

Now $J_1(\mathbf{V})$ is associated to the principal G_1-bundle Q_1, and $\mathbf{W} = Q_1 \times_G W$ is associated to Q_1 by means of the isotropy representation $G_1 \to G$. It therefore makes sense to demand that a homomorphism $D : J_1(\mathbf{V}) \to \mathbf{W}$ be G_1-equivariant; in this case we say that D is *quaternionic invariant*.

The symbol σ of a quaternionic invariant operator must arise from a G-homomorphism $V \otimes T^* \to W$ which we also denote by σ. Conversely, suppose that ∇ is a torsion-free connection on the principal bundle Q; it defines a homomorphism $J_1(\mathbf{V}) \to \mathbf{V} \otimes \mathbf{T}^*$ that splits (5.1). Given any G-homomorphism $\sigma : V \otimes T^* \to W$, one can define a differential operator $D = \sigma \circ \nabla$ with symbol σ. The problem is that if ∇' is a different torsion-free G-connection, $D' = \sigma \circ \nabla'$ will not in general equal D. However $\xi = \nabla - \nabla'$ is a section of the vector bundle with fibre $\mathfrak{g}^{(1)}$ (*see* section 2). If $\rho : G \to \operatorname{Aut} V$ is the homomorphism defining V, let $d\rho : \mathfrak{g} \to \operatorname{End} V$ denote the Lie algebra representation. Composing the inclusion $\mathfrak{g}^{(1)} \subset \mathfrak{g} \otimes T^*$ with $d\rho$ gives a G-homomorphism

$$\beta : \quad \mathfrak{g}^{(1)} \otimes V \to V \otimes T^*, \tag{5.2}$$

and for $v \in J_1(\mathbf{V})$, $\xi(v)$ belongs to $\operatorname{im} \beta$ (as a subbundle of $\mathbf{V} \otimes \mathbf{T}^*$). Therefore D will be independent of the choice of ∇ iff $\operatorname{im} \beta \subset \ker \sigma$.

This explains [F], theorem 2.1, proposition 2.2:

PROPOSITION 5.1. — *A surjective homomorphism* $\sigma : V \otimes T^* \to W$ *of* G-*modules is the symbol of a quaternionic invariant operator iff* $\operatorname{im} \beta \subset \ker \sigma$.

As a corollary, all invariant operators on a quaternionic manifold M may be constructed from a fixed torsion-free G-connection ∇. For given $D : \mathbf{V} \to \mathbf{W}$ invariant with surjective symbol σ, $D - \sigma \circ \nabla$ is invariant of degree 0, and so arises from a G-homomorphism $\varphi : V \to W$. However by (2.3), the powers of H occurring in V and W have different parity, so $\varphi = 0$ and $D = \sigma \circ \nabla$. It is now possible to extend the definition of invariance to apply to operators $D : \mathbf{V} \to \mathbf{W}$ when $\mathbf{V}$, $\mathbf{W}$ are vector bundles associated to representations V, W of $\tilde{G}$ and some lifting $\tilde{Q}$:

DEFINITION 5.2. — A quaternionic invariant first order operator on some open set of M has the form $D = \sigma \circ \nabla$, where $\sigma : V \otimes T^* \to W$ is a surjective $\tilde{G}$-homomorphism with $\operatorname{im} \beta \subset \ker \sigma$.

For the group $CO(n)$, invariant operators can be constructed using conformal weights. This amounts to tensoring with a suitable real line bundle associated to a representation of the centre of $CO(n)$, and is a trick which also works in our case. The centre of G is isomorphic to $\mathbb{R}^*$, being generated by scalar multiples $\lambda 1$ of the identity in $GL(n, \mathbb{H}) \subset G$. We say that an irreducible G-space V has weight m if $\lambda 1$ acts as multiplication by λ^m. With this convention, any Sp(1)-module $S^r H$ has weight 0, and both E and the cotangent space $T^* = E \otimes H$ have weight -1.

Let d denote the representation of G defined by

$$d^{2n} = \otimes^{2n} d \cong \Lambda^{2n} E^*; \tag{5.3}$$

hence for any $m \in \mathbb{R}$, d^m is (the complexification of) a real 1-dimensional space of weight m. The next result should be compared with [F], theorem 1.1.

THEOREM 5.3. — *Let* V, W *be irreducible* G-*modules with* $W \subset V \otimes T^*$. *Then for a unique* $m \in \mathbb{R}$, *there is a quaternionic invariant differential operator* $D : \mathbf{d}^m \mathbf{V} \to \mathbf{d}^m \mathbf{W}$.

Proof. — A key ingredient is the fact that each irreducible component of $V \otimes T^* \cong V \otimes E \otimes H$ occurs with multiplicity one. In view of (2.3), it suffices to show that the same is true of the components of $B \otimes E$, where B is any irreducible $GL(n, \mathbb{H})$-space. However this follows from a multiplicity formula of Kostant [Kos], theorem 4.8. For each $m \in \mathbb{R}$, there is thus (up to a constant) a unique homomorphism

$$\sigma : \quad d^m \otimes V \otimes T^* \to d^m \otimes W.$$

Labelling the homomorphism β in (5.2) with a subscript to indicate the representation involved, we have

$$\beta_V : \quad \mathfrak{g}^{(1)} \otimes V \to V \otimes T^*,$$
$$\beta_d : \quad \mathfrak{g}^{(1)} \otimes d \to d \otimes T^*.$$

The nonvanishing of the coefficient of α_1 in (2.8) implies that β_d is an isomorphism. Extend both β_V, β_d to homomorphisms

$$\mathfrak{g}^{(1)} \otimes d^m \otimes V \to d^m \otimes V \otimes T^*,$$

so that they act as the identity on the extra factor d^m, $d^{m-1} \otimes V$ respectively. In the definition of $\beta_{d^m \otimes V}$, $\mathfrak{g}$ acts on $d^m \otimes V$ as a derivation, so

$$\beta_{d^m \otimes V} = m\, \beta_d + \beta_V.$$

By Schur's lemma and the above remarks, $\sigma \circ \beta_d$ and $\sigma \circ \beta_V$ are proportional, so for some m, $\sigma \circ (\beta_{d^m \otimes V}) = 0$. Now apply proposition 5.1. ■

COROLLARY 5.4. — *There is an invariant operator*

$$D : \quad \Lambda^p \mathbf{E}' S^q \mathbf{H}' \to \Lambda^{p+1} \mathbf{E}' S^{q+1} \mathbf{H}', \qquad p, q \geqq 0, \tag{5.4}$$

where $E' = d^m E$, $H' = d^{-m} H$ *for some* m *independent of* p *and* q.

Proof. — The symbol of D is the natural homomorphism

$$\sigma : \quad d^{m(p-q)} \Lambda^p E\, S^q H \otimes EH \to d^{m(p-q)} \Lambda^{p+1} E\, S^{q+1} H.$$

Let m, n be the weights for which

$$(5.5)\qquad \begin{cases} \mathrm{D}: & \mathbf{d}^m\mathbf{E}\to\mathbf{d}^m\Lambda^2\mathbf{E}\mathbf{H},\\ \mathrm{D}: & \mathbf{d}^n\mathbf{H}\to\mathbf{d}^n\mathbf{E}\mathrm{S}^2\mathbf{H},\end{cases}$$

are invariant. Since (5.4) can be built up from (5.5) and the derivation law, the corollary is valid for $\mathrm{E}'=d^m\mathrm{E}$, $\mathrm{H}'=d^n\mathrm{H}$. But the existence of invariant operators $\mathrm{D}:\mathbf{A}^r\to\mathbf{A}^{r+1}$ in (4.2) shows that $m+n=0$. ■

Generalizing the notation of section 4, we set $\mathrm{A}^{p,q}=\Lambda^p\mathrm{E}'\otimes\mathrm{S}^q\mathrm{H}'$, so that $\mathrm{A}^{r,r}=\mathrm{A}^r$. This is more consistent with writing the group G as $\mathrm{GL}(n,\mathbb{H})\,\mathrm{GL}(1,\mathbb{H})$ rather than $\mathrm{GL}(n,\mathbb{H})\,\mathrm{Sp}(1)$. Setting $\mathrm{A}^{0,-1}$ in addition allows us to state

THEOREM 5.5. — *On a quaternionic manifold there is for each* $r\geqq -1$ *an elliptic complex*

$$0\to\mathbf{A}^{0,r}\xrightarrow{\mathrm{D}}\mathbf{A}^{1,r+1}\xrightarrow{\mathrm{D}}\mathbf{A}^{2,r+2}\to\ldots\to\mathbf{A}^{2n,r+2n}\to 0$$

(*defined only locally if* $\varepsilon\neq 0$ *and* r *is odd*).

Proof. — The operators D are defined by the corollary and have the form $\sigma\circ\nabla$, where ∇ is a torsion-free G-connection. Thus

$$\mathrm{D}^2=\sigma\circ\nabla\circ\sigma\circ\nabla=\sigma^2\circ\nabla,$$

where

$$\sigma^2:\quad \mathrm{A}^{p,q}\otimes\mathrm{T}^*\otimes\mathrm{T}^*\to\mathrm{A}^{p+2,q+2}.$$

Now if

$$\partial:\quad \mathrm{T}^*\otimes\mathrm{T}^*\to\Lambda^2\mathrm{T}^*\subsetneq\mathrm{T}^*\otimes\mathrm{T}^*,$$

denotes antisymmetrization, $\sigma^2=\sigma^2\circ\partial$, and $\mathrm{R}=\partial\circ\nabla^2$ is the curvature of ∇. If $a\in\Gamma(\mathrm{M},\mathrm{A}^{p,q})$, using (3.8),

$$\mathrm{D}^2(a)=\sigma^2\,\mathrm{R}(a)=\sigma^2\,\partial(\beta(v_i\otimes a)\otimes t^i)+\sigma^2\,\mathrm{R}_{\mathrm{U}}(a)=\sigma(\sigma\beta(v_i\otimes a)\otimes t^i)=0.$$

Here $\sigma^2\circ\mathrm{R}_{\mathrm{U}}=0$ because U does not involve $\mathrm{S}^2\mathrm{H}$, and $\sigma\beta(v_i\otimes a)=0$ by invariance.

We leave the reader to verify that for any nonzero real tangent vector $\mathrm{X}\in\mathrm{TM}$, the symbol sequences

$$\to\mathbf{A}^{p-1,q-1}\xrightarrow{\sigma_{\mathrm{X}}}\mathbf{A}^{p,q}\xrightarrow{\sigma_{\mathrm{X}}}\mathbf{A}^{p+1,q+1}\to$$

are exact. The restriction $r\geqq -1$ is necessary since $\sigma_{\mathrm{X}}:\mathbf{A}^{p,0}\to\mathbf{A}^{p+1,1}$ is not injective for $p\geqq 2$. ■

The complexes of theorem 5.5 are related, in the context of K-theory, to the virtual exterior powers $\Lambda^{2n+r}(\mathbf{E}^*-\mathbf{H})$. They may also be built up from the Dirac operator defined by any reduction of the structure group to $\mathrm{Sp}(n)\,\mathrm{Sp}(1)$; this is explained in [BS], section 4; [S_2], section 7.

6. The structure group $\mathrm{GL}(n, \mathbb{H})$

Several authors, including more recently Sommese [So], have studied the particular case of $\mathrm{GL}(n, \mathbb{H})$-structures, so we consider briefly how these fit into our scheme. A $\mathrm{GL}(n, \mathbb{H})$-structure on a manifold M is defined by two anti-commuting almost complex structures I, J; this defines a whole family

$$\{ a\mathrm{I}+b\mathrm{J}+c\mathrm{K} : a^2+b^2+c^2=1,\ \mathrm{K}=\mathrm{IJ} \},$$

of almost complex structures on M parametrized by the imaginary quaternions. This is the same thing as an almost quaternionic manifold for which the $\mathrm{Sp}(1)$ vector bundle **H**, and so $\mathrm{Z}\cong\mathrm{P}(\mathbf{H})$, has been trivialized.

Obata has proved that a $\mathrm{GL}(n, \mathbb{H})$-structure admits a torsion-free $\mathrm{GL}(n, \mathbb{H})$-connection iff both I and J are integrable [O], section 11, corollary 2. This result may be deduced by applying the techniques of sections 2 and 3 with the Lie algebra $\mathfrak{g}=\mathfrak{gl}(n, \mathbb{H})\oplus\mathfrak{sp}(1)$ replaced by $\mathfrak{gl}(n, \mathbb{H})$. With little extra work one obtains

THEOREM 6.1:

$$\mathfrak{gl}(n, \mathbb{H})^{(1)}=0,$$

$$\mathrm{H}^{0,2}(\mathfrak{gl}(n, \mathbb{H}))\cong\mathrm{DS}^3\mathrm{H}\oplus\mathrm{ES}^3\mathrm{H},$$

$$\mathrm{H}^{1,2}(\mathfrak{gl}(n, \mathbb{H}))\cong\mathrm{U}\oplus\mathrm{S}^2\mathrm{E}.$$

The first equation tells us that $\mathfrak{gl}(n, \mathbb{H})$ has type 1, so a torsion-free $\mathrm{GL}(n, \mathbb{H})$-connection is unique if it exists; Bonan [Bo_1] calls it the *Obata connection.* The obstruction to existence consists essentially of two tensors, which in terms of the underlying almost quaternionic structure are the components $\Phi\in\mathbf{D}\,\mathrm{S}^3\,\mathbf{H}$ (as in section 4) and $\Phi'\in\mathbf{E}\,\mathrm{S}^3\,\mathbf{H}$ of the torsion of any $\mathrm{GL}(n, \mathbb{H})$-connection. Let h be a constant section of the vector bundle **H**, so that the projective class defines an almost complex structure $a\mathrm{I}+b\mathrm{J}+c\mathrm{K}$ with a, b, c constants, in accordance with (4.4). Obata's theorem now follows from the fact that the Nijenhuis tensor of this almost complex structure is represented by the component of $\Phi+\Phi'$ proportional to $h^3=h\otimes h\otimes h$ in $(\mathbf{D}\oplus\mathbf{E})\,\mathrm{S}^3\,\mathbf{H}$.

The curvature of the Obata connection also has two components, namely $\mathrm{R}_{\mathrm{U}}\in\mathbf{U}$ (as in section 3) and $\mathrm{R}'\in\mathrm{S}^2\,\mathbf{E}$. The latter is also the curvature of the "canonical" line bundle $\mathbf{d}^{2n}=\Lambda^{2n}\,\mathbf{E}^*$ (5.3), and measures the obstruction to an $\mathrm{SL}(n, \mathbb{H})$-reduction. The $\mathrm{GL}(n, \mathbb{H})$-structure is integrable iff $\mathrm{R}_{\mathrm{U}}=0=\mathrm{R}'$; so in this special case M is locally affine ([Eh]; [L]; [So]). Perhaps the most important class of manifolds with a torsion-free $\mathrm{GL}(n, \mathbb{H})$-connection are Riemannian ones with holonomy lying in $\mathrm{Sp}(n)$, the so-called *hyperkähler manifolds.* These may equivalently be defined as quaternionic Kähler manifolds with zero Ricci tensor [C]; the Obata connection coincides with the Levi-Civita connection, $\mathrm{R}'=0$, and R_{U} equals the Riemann curvature tensor. Following work of Beauville [B], many examples are now known, and are being studied from the viewpoint of algebraic and symplectic geometry. A quotient construction for hyperkähler manifolds involving moment maps has been discovered by Hitchin and Roček ([HKLR]; [K]).

Over a quaternionic manifold M, we have seen that it is more natural to work with the weighted bundle $\mathbf{H}'=\mathbf{A}^{0,1}=\mathbf{d}^{-m}\mathbf{H}$. Indeed, corollary 5.4 provides us with an invariant differential operator

$$\mathrm{D}: \quad \mathbf{A}^{0,1} \to \mathbf{A}^{1,2}, \tag{6.1}$$

well known in four dimensions. Recall from section 3 that a quaternionic 4-manifold is really one with a self-dual conformal structure. The vector bundle $\mathbf{A}^{0,1}$ can then be interpreted as the negative spin bundle, and (6.1) is called the *twistor* (or *anti-Dirac*) *operator* [P]. For $n \geqq 1$, it is overdetermined, since local solutions of $\mathrm{D}s=0$ define covariant constant sections of a certain second order vector bundle [AHS], section 4, and form a finite-dimensional complex vector space whose dimension attains its maximum value of $2n+2$ when $\mathrm{M}=\mathbb{H}\,\mathrm{P}^n$.

Provided $\mathbf{A}^{0,1}$ is globally defined, any nowhere-zero section s of it defines an almost complex structure $\mathrm{I}=\mathbb{C}\,s$, namely the projectivization in $\mathrm{Z}=\mathrm{P}(\mathbf{A}^{0,1})$. However, as $\mathbf{A}^{0,1}$ is a quaternionic line bundle, it is trivialized by the linearly independent sections s, js, so $\mathrm{J}=\mathbb{C}\,(s+js)$ and $\mathrm{K}=\mathbb{C}\,(is+js)$ determine, together with I, a $\mathrm{GL}(n, \mathbb{H})$-structure on M. This structure is unchanged when s is multiplied by any nowhere-zero *real* scalar, reductions from $\mathrm{G}=\mathrm{GL}(n, \mathbb{H})\,\mathrm{Sp}(1)$ to $\mathrm{GL}(n, \mathbb{H})$ being parametrized by $\mathrm{G}/\mathrm{GL}(n, \mathbb{H}) \cong \mathbb{R}\,\mathrm{P}^3$.

THEOREM 6.2. — *Let* M *be a quaternionic manifold with* $\varepsilon=0$, *and let s be a nowhere-vanishing section of* $\mathbf{A}^{0,1}$. *If* $\mathrm{D}s=0$, *the* $\mathrm{GL}(n, \mathbb{H})$-*structure defined by s admits a torsion-free connection with* $\mathrm{R}'=0$.

Proof. — The section s can be written in the form $l \otimes h$, where l belongs to the real line bundle $\mathbf{d}^{-m}$, and h is a section of $\mathbf{H}$ of constant norm. Suppose that $\mathrm{D}s=0$, and fix a torsion-free G-connection ∇ on M. Then there is an induced connection ∇' on $\mathbf{E}$, which we extend to a $\mathrm{GL}(n, \mathbb{H})$-connection on $\mathbf{T}^*=\mathbf{E}\otimes\mathbf{H}$ by setting $\nabla' h=0=\nabla'(jh)$. From

$$(\nabla'-\nabla)\,s=s\otimes l^{-1}\,\nabla' l-\nabla s,$$

it follows that the tensor $\xi=\nabla'-\nabla$ has no component in the subbundle $\mathbf{E}\,\mathrm{S}^3\,\mathbf{H}$ of $\mathrm{End}\,\mathbf{T}\otimes\mathbf{T}^*$. The same is true of the torsion $\tau(\nabla')=\partial\xi$, so using theorem 6.1, the Obata connection ∇'' exists. The equation $0=\mathrm{D}s=\sigma(s\otimes l^{-1}\,\nabla'' l)$ forces $\nabla'' l=0$, so the line bundle $\mathbf{d}$ is flat. ∎

7. New examples

Let M be a quaternionic manifold, and consider a complex vector bundle F over M together with a connection

$$\nabla_{\mathrm{F}}: \quad \mathrm{F} \to \mathrm{F}\otimes\mathbf{T}^*.$$

The latter may be used to extend the operators of corollary 5.4 into

$$D_F: \quad F \otimes \mathbf{A}^{p,q} \to F \otimes \mathbf{A}^{p+1,q+1},$$

by setting $D_F(f \otimes a) = p(\nabla_F f \otimes a) + f \otimes D\,a$, where p denotes the obvious projection. The proof of theorem 5.5 implies that D_F^2 is proportional to $p(\Omega_F)$, where $\Omega_F \in \mathrm{End}\,F \otimes \Lambda^2 \mathbf{T}^*$ is the curvature of ∇_F, and here p is the projection to $\mathrm{End}\,F \otimes \mathbf{A}^2$ arising from the decomposition (4.2).

Definition 7.1. — A quaternionic connection on a vector bundle F over a quaternionic manifold M is one whose curvature satisfies $p(\Omega_F) = 0$.

Accordingly the curvature 2-forms of a quaternionic connection lie in the subbundle $\mathbf{B}^2$. In the special case in which M is a conformal 4-manifold, we may identify $A^2 = \Lambda_-^2$, $B^2 = \Lambda_+^2$ with the eigenspaces of the * operator, and (4.2) becomes the celebrated splitting

$$\Lambda^2 T^* M = \Lambda_-^2 \oplus \Lambda_+^2.$$

In this case, definition 7.1 reduces to that of a *self-dual connection* or *instanton* [AHS], section 2. For another example, consider quaternionic projective space with the opening notation of section 1. On the open set U_r, $\mathbf{B}^2$ is spanned by the real components of the quaternion valued 2-forms $dq_{ir} \wedge d\bar{q}_{jr}$. Using this fact, the construction of [ADHM] may be generalized to give quaternionic connections on $\mathrm{Sp}(n)$-vector bundles on $\mathbb{H}\,\mathbf{P}^n$ [S_3].

Now assume that M is a quaternionic manifold with $\varepsilon = 0$. Suppose that F has a quaternionic connection preserving a $\mathrm{GL}(m, \mathbb{H})$-structure; by this we mean that F has fibre $\mathbb{H}^m$ (so the complex rank $2m$ of F is even) and that ∇_F commutes with fibre-wise multiplication by j. A real rank $4m$ vector bundle $\hat{F}$ can now be defined over M by

$$\hat{F}^c = F \otimes_{\mathbb{C}} \mathbf{A}^{0,1}.$$

Complex conjugation on the right-hand side equals $j \otimes j$; for example if F is $\mathbf{A}^{1,0}$, then

$$\hat{F} = \mathbf{A}^{1,0} \otimes \mathbf{A}^{0,1} = \mathbf{T}^*, \tag{7.1}$$

is none other than the cotangent bundle (1.1) of M.

Theorem 7.2. — *If* M *and* F *are as above, the total space of* $\hat{F}$ *is a quaternionic manifold.*

Proof. — Choose a torsion-free G-connection ∇ on M, and consider the connection $\nabla_{\hat{F}}$ induced by ∇ and ∇_F on $\hat{F}$ by means of the derivation law. Let $\{f_i\}$, $1 \leqq i \leqq 4m$, be a real basis of the vector bundle $\hat{F}$ consisting of sections over some open set U of M, with corresponding coordinate functions λ^i, and connection forms ω_i^j. Thus if $\pi : \hat{F} \to M$ denotes the projection, a typical element of $\pi^{-1}(U)$ has the form $\lambda^i f_i$ (summation), and $\nabla_{\hat{F}} f_i = f_j \otimes \omega_i^j$.

The tangent bundle of the total space of $\hat{F}$ admits a splitting

$$T\hat{F} = \mathscr{H} \oplus \mathscr{V}, \tag{7.2}$$

where $\mathscr{H}$ is the horizontal subbundle relative to F, and $\mathscr{V}$ is tangent to the fibres. The former is annihilated by the forms

$$\omega^j = d\lambda^j + \lambda^i \omega_i^j,$$

which may be regarded as a dual basis of $\{f_j\}$. Since

$$(7.3)\qquad \left\{\begin{aligned} \mathscr{H} &\cong \pi^*(\mathbf{E}^* \otimes \mathbf{H}),\\ \mathscr{V} &\cong \pi^*(\mathrm{F} \otimes \mathbf{A}^{0,1}) \cong \pi^*(\mathrm{F} \otimes \mathbf{d}^{-m}\mathbf{H}),\end{aligned}\right.$$

we have

$$\mathrm{T}\hat{\mathrm{F}} \cong \pi^*(\mathbf{E}^* \oplus \mathbf{d}^{-m}\mathrm{F}) \otimes \pi^*\mathbf{H}.$$

Comparison with (1.1) shows that the manifold $\hat{\mathrm{F}}$ now has an almost quaternionic structure, i.e. a reduction of structure group to $\mathrm{GL}(m+n, \mathbb{H})\,\mathrm{Sp}(1)$. The $\mathrm{GL}(n, \mathbb{H})\,\mathrm{Sp}(1)$ and $\mathrm{GL}(m, \mathbb{H})\,\mathrm{Sp}(1)$ structures of $\mathscr{H}$ and $\mathscr{V}$ respectively can be "added" because they share a common $\mathrm{Sp}(1)$ action.

Equip the bundles $\mathscr{H}$, $\mathscr{V}$ with the natural connections arising from (7.3), and give the manifold F a $\mathrm{GL}(m+n, \mathbb{H})\,\mathrm{Sp}(1)$-connection $\hat{\nabla}$ by taking the direct sum in (7.2). In order to compute the torsion $\hat{\tau}$ of $\hat{\nabla}$ at a point of $\pi^{-1}(x)$ for $x \in \mathrm{M}$ arbitrary, we may suppose that $\omega_j^i|_x = 0$, so that $\hat{\nabla}\omega^i = 0$ at all points of $\pi^{-1}(x)$. First observe that for any 1-form α on M, $\hat{\tau}(\pi^*\alpha) = \pi^*(\tau\alpha) = 0$, since the torsion τ of ∇ is zero.

Evaluating on $\pi^{-1}(x)$, we have

$$\hat{\tau}(\omega^j) = d\omega^j = \lambda^i\, d\omega_i^j|_x = \lambda^i (\Omega_{\hat{\mathrm{F}}})_i^j|_x,$$

where $(\Omega_{\hat{\mathrm{F}}})_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$ are the curvature 2-forms of $\Omega_{\hat{\mathrm{F}}}$. We can now write

$$(7.4)\qquad \hat{\tau} = \lambda^i\, \Omega_{\hat{\mathrm{F}}}(f_i).$$

By hypothesis, the composition

$$\mathrm{F} \otimes \mathbf{A}^{0,1} \xrightarrow{\mathrm{D_F}} \mathrm{F} \otimes \mathbf{A}^{1,2} \xrightarrow{\mathrm{D_F}} \mathrm{F} \otimes \mathbf{A}^{2,3},$$

is zero, but this implies that $\Omega_{\hat{\mathrm{F}}}$, and so $\hat{\tau}$, has no components involving $\mathrm{S}^3\mathrm{H}$. The result follows from proposition 2.2. ■

One can verify that the quaternionic structure of $\hat{\mathrm{F}}$ does not depend upon the choice of connection ∇ on M made in the above proof. Indeed, a section s of $\hat{\mathrm{F}}$ defines a quaternionic submanifold of $\hat{\mathrm{F}}$ (one whose tangent spaces are invariant by the locally defined I, J, K) iff $\mathrm{D_F}\, s = 0$.

Unfortunately, the only quaternionic Kähler manifold with positive scalar curvature that has $\varepsilon = 0$ is $\mathbb{H}\mathrm{P}^n$ [S2], theorem 6.2. However the problem that ε is often non-zero can be evaded by insisting that the manifold $\hat{\mathrm{F}}$ of theorem 7.2, rather than F, be globally defined. For example, the locally defined vector bundle $\mathbf{A}^{1,0} = \mathbf{E}'$ over a quaternionic Kähler manifold has a natural $\mathrm{Sp}(n)$-connection which is quaternionic [S2], theorem 3.1. In this case $\hat{\mathrm{F}} = \mathrm{T}^*\mathrm{M} \cong \mathrm{TM}$ [*see* (7.1)]; whence:

COROLLARY 7.3. — *The tangent bundle of a quaternionic Kähler manifold is itself a quaternionic manifold.*

It is curious to see how this result forces one into the domain of Riemannian geometry, even though definition 1.1 makes no reference to a metric. In addition, one can show that unless M is flat, the above structure on TM does not underlie any quaternionic Kähler metric.

As explained, theorem 7.2 is valid when M is a self-dual 4-manifold, and F an $Sp(m)$-bundle with a self-dual connection. In particular, if M is also Einstein, the positive spin bundle $\mathbf{A}^{1,0}$ is itself self-dual [AHS], proposition 2.2; thus the tangent bundle of both S^4 and $\mathbb{C}P^2$ is a quaternionic manifold.

The set of irreducible self-dual connections on $Sp(m)$-bundles over S^4 have been classified by Atiyah, Drinfeld, Hitchin and Manin [ADHM]. For $m=1$ and $c_2(F)=-k$, this set modulo gauge-equivalence forms a smooth manifold $\mathscr{M}_k$ of dimension $8k-3$. Since gauge-equivalent connections on F give isomorphic quaternionic structures on $\hat{F}$, $\mathscr{M}_k$ may be regarded as a moduli space for the latter. Choosing an identification $S^4=\mathbb{H}\cup\{\infty\}$ and factoring out by only those gauge equivalences fixing the fibre F_∞ gives an enlarged moduli space $\tilde{\mathscr{M}}_k$ of dimension $8k$. It is probably no coincidence that $\tilde{\mathscr{M}}_k$ is itself a hyperkähler manifold ([A]; [HKLR]).

Explicit constructions of self-dual connections on bundles over $\mathbb{C}P^2$ have been given by Donaldson in [D]. For our purposes one may consider U(2) vector bundles F′ with $c_1(F')=1$ and $c_2(F')=-k$. If L denotes the hyperplane section bundle over $\mathbb{C}P^2$, then $F=F'\otimes L^{-1/2}$ has an SU(2)-structure, and the 8-dimensional quaternionic manifold $\hat{F}$ is well defined.

Finally we apply theorem 7.2 to an arbitrary quaternionic manifold by taking F to be trivial with fibre $\mathbb{H}$. This allows us to construct a $GL(1+n, \mathbb{H})$-structure and prove:

COROLLARY 7.4. — *For any quaternionic manifold* M, *the total space of the associated bundle* Z *is a complex manifold.*

Proof. — Working locally, we may assume that M is a quaternionic manifold with $\varepsilon=0$. Let $Y=\hat{F}\setminus M$ denote the quaternionic manifold consisting of the total space of $\hat{F}=\mathbb{H}\otimes\mathbf{A}^{0,1}$ minus its zero section. Identify $\hat{F}$ with the real vector bundle underlying $\mathbf{A}^{0,1}$ by associating the real element $1\otimes a+j\otimes ja\in\hat{F}$ to $a\in\mathbf{A}^{0,1}$. This is one of a possible $\mathbb{C}P^1$ worth of identifications, and its choice defines both a tautologous section $s\in\Gamma(Y, \pi^*\mathbf{A}^{0,1})$ and a projection

$$\nu:\quad Y\to Z=P(\mathbf{A}^{0,1}).$$

By definition, the twistor operator $\hat{D}$ on Y acts on sections of the vector bundle $\pi^*\mathbf{A}^{0,1}$. The torsion formula (7.4) implies that $\hat{\nabla}$ induces the same operator on $\pi^*\mathbf{A}^{0,1}$ as would a torsion-free $GL(1+n, \mathbb{H})Sp(1)$-connection; in other words, $\hat{D}$ factors through $\hat{\nabla}$. In the notation of the previous proof, with a real basis $\{f_1, f_2, f_3, f_4\}$ *of* $\hat{F}$,

$$\hat{D}s=\hat{\sigma}\,\hat{\nabla}(s)=\hat{\sigma}\,\hat{\nabla}(\lambda^i\pi^* f_i)=\hat{\sigma}(\pi^* f_i\otimes\omega^i)=0.$$

By theorem 6.2, the manifold Y admits a torsion-free $GL(1+n, \mathbb{H})$-connection, and in particular s determines a complex structure on Y. Relative to the latter, there is a holomorphic action by the group $\mathbb{C}^*$ of non-zero scalars defining ν, and the quotient Z acquires a natural complex structure. ■

When M is quaternionic Kähler with positive scalar curvature, techniques of [S_2], section 6 show that Y is hyperhähler and Z is Kähler. For $M=\mathbb{H}P^n$, Y can be identified with $\mathbb{H}^{n+1}$ minus the origin, and $Z \cong \mathbb{C}P^{2n+1}$. In general Z is known as the *twistor space* of M in analogy with [AHS], theorem 4.1; indeed corollary 7.4 may be proved directly by applying [AHS], proposition 3.1 to the twistor operator on M. Points of M correspond to projective lines in Z invariant under the real structure induced by multiplication by j in $A^{0,1}$, and each line has normal bundle the sum of an appropriate number of copies of the (positive) Hopf bundle. To summarize, a quaternionic manifold can be thought of as a quotient of a complex manifold which is foliated in a special way by $\mathbb{C}P^1$'s.

Many properties of M discussed in this paper may be converted into holomorphic data on Z. For example there is a "twistor transform" relating the elliptic complex of theorem 5.5 to the Dolbeault complex on Z tensored by the r-th power of a holomorphic line bundle. This approach can be used to define additional elliptic complexes for $r<-1$ involving second order operators (the basic ideas are contained in [S_4]; [E]). Finally, to return full circle to definition 7.1, we remark that any vector bundle F with a quaternionic connection defined over a quaternionic manifold M pulls back to a holomorphic bundle over the complex manifold Z in accordance with [AHS], theorem 5.2 and [BO], theorem 7.2.

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S. M. SALAMON
Institute for Advanced Study,
Trinity College, Oxford,
Oxford OX13LB.
Grande-Bretagne

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