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MCKENZIE Y. WANG
WOLFGANG ZILLER

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ON NORMAL HOMOGENEOUS EINSTEIN MANIFOLDS

BY MCKENZIE Y. WANG (1) AND WOLFGANG ZILLER (2)

A Riemannian metric g is called Einstein if its Ricci tensor satisfies $\mathrm{Ric}(g)=cg$ for some constant c. For $c>0$, most known examples of Einstein manifolds are compact homogeneous spaces; *see*, for example, [25], [20], [12], [7], [21], [27], [28], [2]. Not every simply connected compact homogeneous space admits a homogeneous Einstein metric [24], but a general classification of homogeneous Einstein metrics seems to be difficult. In this paper we study the Einstein condition for a "natural" metric that exists on every simply connected compact homogeneous space.

Let G/H be compact and simply connected. Then G is compact, and the semisimple part of G acts transitively on G/H. Hence we will assume that G is a compact, connected, semisimple Lie group, and H is a closed subgroup. We let $\mathfrak{g}$, $\mathfrak{h}$ denote the respective Lie algebras. Any bi-invariant metric on $\mathfrak{g}$ induces an orthogonal splitting $\mathfrak{g}=\mathfrak{h}\perp\mathfrak{m}$, and if we identify $\mathfrak{m}$ with $T_{eH}(G/H)$, the restriction of the bi-invariant metric to $\mathfrak{m}$ induces a G-invariant metric on G/H by left translation. Such a metric is called a *normal homogeneous metric*. A canonical choice for a bi-invariant metric on g is the negative of the Killing form, denoted by B. The induced metric on G/H, denoted by g_B, will be called the *standard homogeneous metric* on G/H.

The Einstein condition for g_B can be described as follows. Let χ be the isotropy representation of H^0, the identity component of H, on $T_{eH}(G/H)=\mathfrak{m}$. We also denote by χ the corresponding representation of $\mathfrak{h}$ on $\mathfrak{m}$. For any (orthogonal) representation π of $\mathfrak{h}$ and any bi-invariant metric Q on $\mathfrak{h}$ we let $C_{\pi,Q}$ be the Casimir operator defined by $-\sum_i \mathrm{tr}(\pi(X_i)\pi(X_i))$, where $\{X_i\}$ is a Q-orthonormal basis of $\mathfrak{h}$. Then we have (*see* (1.7), (1.12)).

THEOREM 1. — *The standard homogeneous metric* g_B *on* G/H *is Einstein iff* $C_{\chi,B|\mathfrak{h}}=a\,\mathrm{Id}$ *for some constant a.*

Equivalently, if $\mathfrak{m}=\mathfrak{m}_0\oplus\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_k$ is the decomposition of $\mathfrak{m}$ into non-trivial $\mathbb{R}$-irreducible summands $\mathfrak{m}_1,\ldots,\mathfrak{m}_k$ and a space $\mathfrak{m}_0$ on which χ is trivial, then g_B is

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Einstein iff $m_0 = 0$ and $B^*(\lambda_i, \lambda_i + 2\delta) = B^*(\lambda_j, \lambda_j + 2\delta)$ for all i, j. Here λ_i is the dominant weight of χ on m_i, 2δ is the sum of the positive roots of $\mathfrak{h}$, and B^* is the metric on $\mathfrak{h}^*$ induced by $B|\mathfrak{h}$.

If the isotropy representation of H on m is irreducible over $\mathbb{R}$, then g_B is obviously Einstein. Such spaces are called *isotropy irreducible*. If the representation of H^0 on m is also $\mathbb{R}$-irreducible, the spaces are called *strongly isotropy irreducible*. The irreducible compact symmetric spaces are of course strongly isotropy irreducible. The non-symmetric strongly isotropy irreducible spaces were classified by J. Wolf [25]. This classification is not quite complete, see the correction to [25] and the paper [23].

If the isotropy representation is reducible, Theorem 1 becomes rather restrictive, and it enables us to classify all the standard homogeneous metrics which are Einstein in the case when G is simple. Note that in such a case the only normal homogeneous metric on G/H, up to scaling, is the standard homogeneous metric. It is natural to assume that G/H is simply connected (hence H is connected) since if g_B is Einstein on G/H, then g_B on its universal cover is also Einstein.

Our main result is

THEOREM 2. — *Let* G *be a compact, connected, simple Lie group and* H *a closed, connected subgroup such that* G *acts almost effectively on* G/H *and* G/H *is simply connected. If* g_B *is Einstein and* G/H *is not strongly isotropy irreducible, then the Lie algebras* $(\mathfrak{g}, \mathfrak{h})$ are given in Table I of Chapter 1.

We will see in Chapter 5 that there are de Rham irreducible spaces with G semi-simple but not simple whose standard homogeneous metric is Einstein. However, it would be more natural in such a case to classify all normal homogeneous Einstein metrics.

Chapter 1 contains a general discussion of the Einstein condition for g_B. More generally, in (1.9) we study the Ricci tensor of any naturally reductive metric on G/H in terms of the Casimir operator of its isotropy representation. We then describe some of the more interesting examples in our classification. A table of our full classification follows.

In Chapter 2 we develop the necessary tools for computing Einstein constants and describe some facts we need from representation theory and from [23]. The details of our classification are given in Chapter 3 (for the quotients of the classical groups) and in Chapter 4 (for the quotients of the exceptional groups).

Applications of Theorems 1 and 2 are given in Chapter 5. We first determine the connected isometry groups of the manifolds in Theorem 2 and show that none of the manifolds are isometric. Second, we use Theorem 2 to classify all the left invariant Einstein metrics on compact simple Lie groups that are obtained from the bi-invariant metric by scaling in the direction of a subgroup. Third, we examine fibrations of the Einstein manifolds in Theorem 2 where the fibres and base are again normal homogeneous Einstein. For such a fibration we can scale the metric on the total space in the direction of the fibres, and in most cases we obtain another Einstein metric which is not normal homogeneous.

Chapters 1 and 5 can be read independently of the rest of the paper by any reader who is more interested in the classification results and their applications. However, the classification for quotients of the classical Lie group in Chapter 3 is conceptual, and contains results of independent interest. In particular, we mention.

THEOREM 3. — *Let* π *be an n-dimensional almost faithful orthogonal representation of a compact connected Lie group* H, *and let* χ *be the isotropy representation of* $SO(n)/\pi(H)$, *i.e.,* $\Lambda^2\pi = ad_H \oplus \chi$. *Then* $C_{\chi,Q} = a\,\mathrm{Id}$ *for some constant a and some bi-invariant metric* Q *on* $\mathfrak{h}$ *iff* π *is the isotropy representation of a symmetric space of compact type, or* (*a*) $H = G_2$, $\pi = o \overset{1}{\equiv} \bullet$ *or* $\mathrm{id} \oplus o \overset{1}{\equiv} \bullet$ (*b*) $H = \mathrm{Spin}(7)$, $\pi = o—o \overset{1}{=} \bullet$, (*c*) $H = \mathrm{Spin}(7) \cdot SO(m)$, $m \geqq 3$, $\pi = [o—o \overset{1}{=} \bullet \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_m]$.

The classification for the quotients of $SO(n)$ follows easily from this result. Similar theorems are proved in Chapter 3 for quotients of the unitary (resp. symplectic) groups and compact hermitian (resp. quaternionic) symmetric spaces. These results are in the same spirit as results in [23].

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CHAPTER ONE

The Einstein condition and description of results

1. PRELIMINARIES AND THE RICCI TENSOR. — Let G be a compact, connected, semisimple Lie group and H a closed subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras and by π the embedding of H in G. The homogeneous space $G/\pi(H)$ will be denoted by M. We assume that G acts almost effectively on M, i.e., $\mathfrak{g}$ and $\mathfrak{h}$ have no non-trivial ideal in common.

For X, Y in $\mathfrak{g}$, set $B(X, Y) = -\mathrm{tr}((\mathrm{ad}\ X) \circ (\mathrm{ad}\ Y))$. B is the negative of the Killing form of $\mathfrak{g}$; it is positive definite, and gives an (Ad H)-invariant orthogonal splitting $\mathfrak{g} = \mathfrak{h} \perp \mathfrak{m}$, with respect to which $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair. We may identify $\mathfrak{m}$ with the tangent space of M at the identity coset: for $X \in \mathfrak{m} \subset \mathfrak{g}$, let X^* be the vector field generated by the action of the one-parameter subgroup $\exp(tX)$ of G on M and associate X with $X^*(eH)$. Then $[X, Y]_{\mathfrak{m}} = -[X^*, Y^*]_{eH}$.

We recall next the isotropy representation χ of H on $T_{eH}(M)$. An element h in H acts on M by left translation and fixes the identity coset eH. dh is an automorphism of $T_{eH}(M)$ and the isotropy representation is given by $h \mapsto dh$. χ induces in turn a representation of $\mathfrak{h}$ on $T_{eH}(M)$, which will again be denoted by χ. Using the identification of $\mathfrak{m}$ with $T_{eH}(M)$ these representations get identified with the adjoint representation on $\mathfrak{m}$, i.e., for $h \in H$, $\chi(h) = \mathrm{Ad}_{\mathfrak{m}}(h)$ and for $X \in \mathfrak{h}$, $Y \in \mathfrak{m}$, $\chi(X)Y = [X, Y]$. Since we assume that G acts almost effectively on M, the isotropy representations of M are almost faithful and faithful respectively.

From $\mathfrak{m} \approx T_{eH}(M)$, we also see immediately that $B \mid \mathfrak{m}$ induces an invariant Riemannian metric g_B on M which will be called the *standard homogeneous metric*. Notice that every homogeneous space G/H with G compact, semisimple has such a metric. We are interested in characterizing when g_B is Einstein, i.e., has constant Ricci curvature.

A preliminary simplification results from observing that the Einstein condition is a local one, and so we can assume that M is simply connected, which in turn implies that H is connected. Then the embedding π of H in G is uniquely determined by $\mathfrak{h} \subset \mathfrak{g}$. In the remainder of this paper we shall therefore mainly work with the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. We shall say that the pair $(\mathfrak{g}, \mathfrak{h})$ is Einstein or that $\mathfrak{h}$ is Einstein in $\mathfrak{g}$, meaning that g_B is Einstein for M.

For the convenience of the reader, we derive below an expression for the Ricci tensor of the standard homogeneous metric g_B, which is implied by (12 *a*) and (18) on pp. 608-9 of [12].

Let $X, Y \in \mathfrak{m}$. Define $A(X, Y) = -\mathrm{tr}_{\mathfrak{h}}(\mathrm{pr}_{\mathfrak{h}} \circ \mathrm{ad}\, X \circ \mathrm{ad}\, Y)$, where $\mathrm{pr}_{\mathfrak{h}}$ is the projection of $\mathfrak{g}$ onto $\mathfrak{h}$ with respect to the orthogonal splitting $\mathfrak{g} = \mathfrak{h} \perp \mathfrak{m}$ and $\mathrm{tr}_{\mathfrak{h}}$ is the trace of linear operators on $\mathfrak{g}$ restricted to $\mathfrak{h}$. If $\{Z_i\}$ is an orthonormal basis of $\mathfrak{h}$ with respect to B, then

$$A(X, Y) = -\sum_i B([X, [Y, Z_i]], Z_i) = -\sum_i B([Z_i, [Z_i, X]], Y) \tag{1.1}$$

since B is ad g-invariant. Moreover, A is ad $\mathfrak{h}$-invariant, thus defining a positive semi-definite invariant tensor on M.

(1.2) PROPOSITION. — *Ric* $(g_B) = (1/4)\,B + (1/2)\,A$.

Proof. — Let $X \in \mathfrak{m}$ be a unit vector. By Theorem X.3.5(3) of [15] we obtain

$$B(R(X, X_i)X_i, X) = \frac{1}{4}B([X, X_i]_{\mathfrak{m}}, [X, X_i]_{\mathfrak{m}}) + B([X, X_i]_{\mathfrak{h}}, [X, X_i]_{\mathfrak{h}})$$

$$= -\frac{3}{4}B([X, X_i]_{\mathfrak{m}}, [X, X_i]_{\mathfrak{m}}) + B([X, X_i], [X, X_i]),$$

where $\{X_i\}$ is an orthonormal basis for $\mathfrak{m}$ with respect to B such that $X = X_1$. It follows that

$$\mathrm{Ric}\, g_B(X, X) = \frac{3}{4}\mathrm{tr}_{\mathfrak{m}}(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}\, X)^2 + B(X, X) - A(X, X).$$

Since $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and B is ad g-invariant, the matrix of ad X with respect to $\{Z_i, X_j\}$ has the form

$$\begin{pmatrix} 0 & a(X) \\ -a(X)^t & b(X) \end{pmatrix}$$

It follows that

$$\mathrm{tr}_{\mathfrak{m}}(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{ad}\, X)^2 = \mathrm{tr}(b(X)^2) = -B(X, X) + 2\,\mathrm{tr}(a(X)\,a(X)^t) = -B(X, X) + 2A(X, X).$$

Hence $\mathrm{Ric}\, g_B(X, X) = (1/4)\,B(X, X) + (1/2)\,A(X, X)$, as asserted. ■

2. THE EINSTEIN CONDITION AND CASIMIR OPERATORS. — We first deduce some immediate consequences of Proposition 1.2 and then go on to relate the tensor A to a Casimir operator of the isotropy representation of M.

Since $\mathfrak{m}$ is an orthogonal representation of H, let us write it as a sum of a trivial representation $\mathfrak{m}_0$ (of possibly zero dimension) and irreducible orthogonal representations $\mathfrak{m}_i$, $i > 0$, with dominant weights λ_i.

(1.3) COROLLARY. — *If* g_B *is Einstein, then either* H *is trivial or* $\mathfrak{m}_0 = 0$. *In the first case* g_B *is a bi-invariant metric of* G.

Proof. — Let $\mathrm{Ric}(g_B) = C\,g_B$. Since $A \,|\, \mathfrak{m}_0 = 0$, $\mathfrak{m}_0 \neq 0$ implies that $C = 1/4$. But then $A \equiv 0$, and the definition of A implies that $\mathfrak{m} = \mathfrak{m}_0$. This contradicts the assumption that G acts almost effectively on M unless H is trivial. ■

(1.4) *Remark.* — Since $\mathfrak{m}_0$ is a subalgebra of g by the Jacobi identity, the condition $\mathfrak{m}_0 \neq 0$ is equivalent to the existence of a subalgebra $\mathfrak{k}$ such that $\mathfrak{h} \oplus \mathfrak{k} \subset \mathfrak{g}$. Hence if g_B is Einstein, no such subalgebra $\mathfrak{k}$ can exist unless $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. This already restricts the possibilities for H.

(1.5) COROLLARY. — *If* H *is a torus in* G, *then* g_B *is Einstein iff the torus is maximal and all roots of* G *have the same length with respect to* B. *Hence* G *is locally a product of* $SU(n)$, $SO(2n)$, E_6, E_7, *or* E_8.

Proof. — That the torus must have maximal rank follows from (1.3). If H is a maximal torus, $\mathfrak{m}=\bigoplus_i \mathfrak{m}_i$ is just the root space decomposition of $\mathfrak{g}$. Therefore, by the definition of A, $A\,|\,\mathfrak{m}_i=-B^*(\alpha_i,\alpha_i)\,B$, where $\pm\alpha_i$ is the root corresponding to $\mathfrak{m}_i$, and B^* is the inner product induced by B on $\mathfrak{g}^*$. The result follows immediately. ■

Remark. — A theorem of Matsushima ([17], Theorem 3) implies that up to a holomorphic transformation there is a unique Kähler-Einstein metric on G/T. But examining the Kähler condition for an invariant metric on G/T ([1], p. 1149) one sees that g_B for G/T is never Kähler.

(1.6) COROLLARY. — *Suppose* $\mathrm{Ric}(g_B)=C g_B$ *on* M, *then* $1/4\leqq C\leqq 1/2$. M *is locally symmetric iff* $C=1/2$. $C=1/4$ *iff* $H=\{e\}$, *i.e.,* g_B *is a bi-invariant metric of* G.

Proof. — In the proof of Proposition (1.2), we established that

$$A(X,X)=\frac{1}{2}B(X,X)+\frac{1}{2}\mathrm{tr}_{\mathfrak{m}}(\mathrm{pr}_{\mathfrak{m}}\circ \mathrm{ad}\,X)^2.$$

Clearly, $\mathrm{tr}_{\mathfrak{m}}(\mathrm{pr}_{\mathfrak{m}}\circ \mathrm{ad}\,X)^2\leqq 0$ and so $1/4\leqq C\leqq 1/2$. Now $C=1/4$ iff $A\equiv 0$ iff $H=\{e\}$ since the isotropy representation is almost faithful. Lastly, $C=1/2$ iff $\mathrm{tr}_{\mathfrak{m}}(\mathrm{pr}_{\mathfrak{m}}\circ \mathrm{ad}\,X)^2=0$ for all $X\in\mathfrak{m}$ iff $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$. ■

The Einstein constant C can be calculated by taking the trace of (1.2) and using (1.1). We get $C=1/4+1/2\sum_i (\dim H_i)\,(1-\alpha_i)/\dim(G/H)$, where H_i are the simple factors of H and $B_{H_i}=\alpha_i B_G\,|\,\mathfrak{h}_i$.

To obtain a necessary and sufficient condition for g_B to be Einstein we need to examine the tensor A more closely. The main observation is that A is the Casimir operator of the isotropy representation with respect to $B\,|\,\mathfrak{h}$. We explain this connection below.

Let $\mathfrak{h}$ be a compact Lie algebra, (i.e., $\mathfrak{h}=\mathfrak{z}\oplus[\mathfrak{h},\mathfrak{h}]$ where $\mathfrak{z}$ is the center of $\mathfrak{h}$ and $[\mathfrak{h},\mathfrak{h}]$ is semisimple,) and φ be a faithful representation of $\mathfrak{h}$. Suppose that $\langle\,,\rangle$ is an ad $\mathfrak{h}$-invariant non-degenerate symmetric bilinear form on $\mathfrak{h}$. Then the Casimir operator of φ with respect to $\langle\,,\rangle$ is defined by

$$C_{\varphi,\langle,\rangle}=-\sum_i \varphi(X_i)\circ\varphi(Y_i),$$

where $\{X_i\}$, $\{Y_i\}$ are bases of $\mathfrak{h}$ dual with respect to $\langle\,,\rangle$, i.e., $\langle X_i, Y_j\rangle=\delta_{ij}$. $C_{\varphi,\langle,\rangle}$ is independent of the choice of $\{X_i\}$ and $\{Y_i\}$, and commutes with every $\varphi(X)$. Hence if φ is an irreducible complex representation then $C_{\varphi,\langle,\rangle}$ is a scalar operator. If in addition φ is orthogonal, i.e. $\varphi(X)$ is skew symmetric for every X, and if $\langle\,,\rangle$ is positive definite, then this scalar is nonnegative.

Now if we let $\mathfrak{h}$ be the Lie algebra of H, φ be the isotropy representation χ of $\mathfrak{h}$ (the differential of the isotropy representation of H), and $\langle , \rangle = B \mid \mathfrak{h}$, then (1.1) implies that

$$A(X, Y) = B(C_{\chi, B \mid \mathfrak{h}} X, Y).$$

Combining this with (1.2) we get

(1.7) COROLLARY. — *If we regard the Ricci tensor as a symmetric endomorphism of* $\mathfrak{m}$, *then*

$$\mathrm{Ric}(g_B) = \frac{1}{4}\mathrm{Id} + \frac{1}{2} C_{\chi, B \mid \mathfrak{h}}.$$

Hence g_B *is Einstein iff* $C_{\chi, B \mid \mathfrak{h}}$ *is a multiple of the identity.* ■

Note that the condition $C_{\chi, B \mid \mathfrak{h}} = a\,\mathrm{Id}$ is only a condition on the isotropy representation χ, despite the presence of the restriction of the Killing form B of $\mathfrak{g}$ to $\mathfrak{h}$. We only need to observe that

$$B(X, Y) = B_{\mathfrak{h}}(X, Y) - \mathrm{tr}(\chi(X)\chi(Y)) \tag{1.8}$$

for $X, Y \in \mathfrak{h}$, where $B_{\mathfrak{h}}$ is the negative of the Killing form of $\mathfrak{h}$.

A formula analogous to (1.7) also holds for the Ricci tensor of naturally reductive metrics. This is not needed for this paper, but since it is of independent interest, we include the derivation below. We will assume that G is a connected but not necessarily compact Lie group. Let g be an invariant Riemannian metric on G/H. Then $\mathfrak{h}$ is a compact Lie algebra (although H need not be compact) and there exists an $\mathrm{ad}(\mathfrak{h})$-invariant subspace $\mathfrak{m}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. The metric g is *naturally reductive* with respect to the transitive group G and the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ if for all X, Y, Z in $\mathfrak{m}$ we have $g(X, [Z, Y]_{\mathfrak{m}}) + g([Z, X]_{\mathfrak{m}}, Y) = 0$.

A theorem of Kostant (*see* [13], p. 355 Theorem 4 or [7] p. 5) says that given a naturally reductive metric with respect to a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ there exists a unique $\mathrm{ad}(\bar{\mathfrak{g}})$-invariant non-degenerate symmetric bilinear form Q on the ideal $\bar{\mathfrak{g}} = \mathfrak{m}\mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ such that $Q(\mathfrak{m}, \bar{\mathfrak{g}} \cap \mathfrak{h}) = 0$ and $Q \mid \mathfrak{m} = g$. Conversely, if Q is an $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form such that $Q \mid \mathfrak{h}$ is non-degenerate and $Q \mid \mathfrak{h}^{\perp}$ is positive definite, then with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$, $Q \mid \mathfrak{h}^{\perp}$ is a naturally reductive metric. Since $\bar{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$ that acts transitively on M, we will henceforth assume that $\bar{\mathfrak{g}} = \mathfrak{g}$. Notice though that Q and $Q \mid \mathfrak{h}$ are in general not positive definite.

(1.9) PROPOSITION. — *Let g be a naturally reductive metric on* M *which is the restriction to* $\mathfrak{h}^{\perp}$ *of an* $\mathrm{ad}(\mathfrak{g})$*-invariant non-degenerate symmetric bilinear form* Q *on* $\mathfrak{g}$. If we define S *by* $B(X, Y) = Q(SX, Y)$, *then*

$$\mathrm{Ric}(g) = \frac{1}{4} S + \frac{1}{2} C_{\chi, Q \mid \mathfrak{h}}.$$

Proof. — We define as before

$$A(X, Y) = -\mathrm{tr}_{\mathfrak{h}}(\mathrm{pr}_{\mathfrak{h}} \circ \mathrm{ad}\, X \circ \mathrm{ad}\, Y).$$

Using the bi-invariance of Q and $Q(\mathfrak{h},\mathfrak{m})=0$, we get

$$A(X,Y)=-\sum_i Q([Y_i,[Z_i X]],Y),$$

where Y_i, Z_i are dual bases of $\mathfrak{h}$: i.e., $Q(Y_i,Z_j)=\delta_{ij}$. The proof of (1.2) carries over if we replace B by Q at appropriate places and shows that

$$\mathrm{Ric}(g)(X,Y)=\frac{1}{4}B(X,X)+\frac{1}{2}A(X,Y).$$

The definition of C_χ then implies that

$$A(X,Y)=Q(C_{\chi,Q\mid\mathfrak{h}}X,Y).$$

(Notice that in this formula we have to use Q instead of the metric B since we do not necessarily have $B(\mathfrak{h},\mathfrak{m})=0$.) ■

Unlike the case of $C_{\chi,B\mid\mathfrak{h}}$, $C_{\chi,Q\mid\mathfrak{h}}$ can have eigenvalues of either sign since $Q\mid\mathfrak{h}$ need not be positive definite. Hence $\mathrm{Ric}(g)$ can also have eigenvalues of either sign. Notice also that the Einstein condition is not equivalent to $C_{\chi,Q\mid\mathfrak{h}}=a\,\mathrm{Id}$ anymore. This concludes our detour to consider the Ricci tensor of naturally reductive metrics.

For $\mathfrak{h}$ semisimple and $\langle\,,\rangle$ the negative of the Killing form of $\mathfrak{h}$, the calculation of $C_{\varphi,\langle\,,\rangle}$ for an irreducible complex representation φ is classical. Exactly the same calculation holds when $\mathfrak{h}$ is compact and $\langle\,,\rangle$ is any $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form. For the convenience of the reader, we include the calculation below.

We pause first to review some basic facts about the structure and representation theory of compact Lie algebras. Let $\langle\,,\rangle$ be an $\mathrm{ad}(\mathfrak{h})$-invariant non-degenerate symmetric bilinear form on $\mathfrak{h}$ and $t\subset\mathfrak{h}$ be a maximal abelian subalgebra. We may extend $\langle\,,\rangle$ to a non-degenerate symmetric form on $\mathfrak{h}\otimes\mathbb{C}$ which will also be denoted by $\langle\,,\rangle$. Using $\langle\,,\rangle$, we may write $\mathfrak{h}=\mathfrak{z}\perp\mathfrak{h}'$ and $t=\mathfrak{z}\perp t'$, where $\mathfrak{z}=$center of $\mathfrak{h}$ and $\mathfrak{h}'$ is semisimple. $\mathfrak{h}'_0=t'\otimes\mathbb{C}$ is then a Cartan subalgebra for $\mathfrak{h}'\otimes\mathbb{C}$. Now $\mathfrak{h}\otimes\mathbb{C}=\mathfrak{z}\otimes\mathbb{C}\perp\mathfrak{h}'\otimes\mathbb{C}$. Let

$$\mathfrak{h}'_0\oplus\sum_\alpha\mathfrak{h}'_\alpha$$

be the root space decomposition of $\mathfrak{h}'\otimes\mathbb{C}$ with respect to $\mathfrak{h}'_0$. Note that $\langle\mathfrak{h}'_\alpha,\mathfrak{h}'_\beta\rangle=0$ whenever $\alpha+\beta\neq 0$.

For every positive root α we can find vectors $E_\alpha\in\mathfrak{h}'_\alpha$, $E_{-\alpha}\in\mathfrak{h}'_{-\alpha}$ such that $\langle E_\alpha,E_\alpha\rangle=\langle E_{-\alpha},E_{-\alpha}\rangle=0$, $\langle E_\alpha,E_{-\alpha}\rangle=1$. Then $[E_\alpha,E_{-\alpha}]=H_\alpha$, the element dual to α with respect to $\langle\,,\rangle$. As is customary, we let δ denote one half the sum of the positive roots of $\mathfrak{h}\otimes\mathbb{C}$.

Every irreducible complex representation φ of $\mathfrak{h}$ has a cyclic vector v corresponding to a dominant integral form λ, which determines the representation up to equivalence. v is unique up to a scalar multiple, and is characterized by $\varphi(E_\alpha)v=0$ for all positive roots α.

(1.10) LEMMA. — *Let* $\mathfrak{h}$ *be a compact Lie algebra with an ad* $(\mathfrak{h})$*-invariant non-degenerate symmetric bilinear form* $\langle\,,\rangle$ *and* φ *an irreducible complex representation of* $\mathfrak{h}$ *with dominant weight* λ. *Then* $C_{\varphi,\langle\,,\rangle}=-\langle\lambda,\lambda+2\delta\rangle^{*}\mathrm{Id}$, *where* $\langle\,,\rangle^{*}$ *denotes the bilinear form on* $\mathfrak{h}^{*}\otimes\mathbb{C}$ *induced by* $\langle\,,\rangle$.

Proof. — We already noted that $C_{\varphi,\langle\,,\rangle}$ is scalar. In the notation of the preceding paragraphs, we choose $\{h_1,\ldots,h_k\}\subset\mathfrak{h}_0'$, $\{h_{k+1},\ldots,h_r\}\subset\mathfrak{z}\otimes\mathbb{C}$ and $\{h_1^{*},\ldots,h_k^{*}\}\subset\mathfrak{h}_0'$, $\{h_{k+1}^{*},\ldots,h_r^{*}\}\subset\mathfrak{z}\otimes\mathbb{C}$ such that $\langle h_i,h_j^{*}\rangle=\delta_{ij}$, $1\leqq i,\ j\leqq r$. Then $h_1,\ldots,h_r$, E_α, $E_{-\alpha}$, $(\alpha>0)$ and $h_1^{*},\ldots,h_r^{*}$, $E_{-\alpha}$, E_α, $(\alpha>0)$ are dual bases of $\mathfrak{h}\otimes\mathbb{C}$ with respect to $\langle\,,\rangle$. Hence

$$-C_{\varphi,\langle\,,\rangle}=\sum_{i=1}^{r}\varphi(h_i)\varphi(h_i^{*})+\sum_{\alpha>0}\varphi(E_\alpha)\circ\varphi(E_{-\alpha})+\sum_{\alpha>0}\varphi(E_{-\alpha})\circ\varphi(E_\alpha)$$

$$=\sum_{i=1}^{r}\varphi(h_i)\varphi(h_i^{*})+\sum_{\alpha>0}\varphi(H_\alpha)+2\sum_{\alpha>0}\varphi(E_{-\alpha})\circ\varphi(E_\alpha)$$

Let v be a dominant weight vector of φ. Then

$$-C_{\varphi,\langle\,,\rangle}v=\left(\sum_{i=1}^{r}\lambda(h_i)\lambda(h_i^{*})+\sum_{\alpha>0}\lambda(H_\alpha)\right)v=(\langle\lambda,\lambda\rangle^{*}+\langle\lambda,2\delta\rangle^{*})v.\quad\blacksquare$$

Combining (1.7) with (1.10) we obtain

(1.11) THEOREM. — *Let* $\mathfrak{m}=\mathfrak{m}_0\oplus\mathfrak{m}_1\oplus\ldots\oplus\mathfrak{m}_k$ *be the decomposition of the isotropy representation into a trivial representation* $\mathfrak{m}_0$ *and irreducible real representations* $\mathfrak{m}_i$, $1\leqq i\leqq k$, *with dominant weights* λ_i. *Then* g_B *is Einstein iff* $\mathfrak{m}_0=0$ *and for every* $i\neq j$, $B^{*}(\lambda_i,\lambda_i+2\delta)=B^{*}(\lambda_j,\lambda_j+2\delta)$. ($B^{*}$ *is the inner product on* $\mathfrak{g}^{*}$ *induced by* B.)

Proof. — The definition of $C_{\chi,B}$ implies that $C_{\chi,B}\mathfrak{m}_i\subset\mathfrak{m}_i$. $\mathfrak{m}_i\otimes\mathbb{C}$ is either V_{λ_i} or $V_{\lambda_i}\oplus V_{\lambda_i^{*}}$ where V_{λ_i} is the complex irreducible representation with dominant weight λ_i and $*$ denotes the contragredient representation. If $\mathfrak{m}_i\otimes\mathbb{C}=V_{\lambda_i}$, then $C_{\chi,B}\,|\,\mathfrak{m}_i=-B^{*}(\lambda_i,\lambda_i+2\delta)\,\mathrm{Id}$ by (1.9). If $\mathfrak{m}_i\otimes\mathbb{C}=V_{\lambda_i}\oplus V_{\lambda_i^{*}}$ we observe that the map which takes λ to λ^{*} is an isometry with respect to B^{*} and that $\delta^{*}=\delta$. Hence $B^{*}(\lambda_i,\lambda_i+2\delta)=B^{*}(\lambda_i^{*},\lambda_i^{*}+2\delta)$ and again we have $C_{\chi,B}\,|\,\mathfrak{m}_i=-B^{*}(\lambda_i,\lambda_i+2\delta)\,\mathrm{Id}$. ■

The above proof yields immediately.

(1.12) COROLLARY. — (M,g_B) *is Einstein iff* $B^{*}(\lambda_i,\lambda_i+2\delta)=B^{*}(\lambda_j,\lambda_j+2\delta)$ *for all* $i\neq j$ *and* $m_0=0$, *where* $\{\lambda_i\}$ *are the dominant weights of the irreducible* complex *representations of* $\mathfrak{m}\otimes\mathbb{C}$. ■

An immediate consequence of Corollary (1.7) and Theorem (1.11) is the following corollary, which will be used as an inductive method for classification in Chapter 4.

(1.13) COROLLARY. — *Let* G *be a compact, connected, semisimple group.*

(*a*) *If* $H\subset K\subset G$ *are closed connected subgroups such that* $B_{\mathfrak{g}}\,|\,\mathfrak{k}=C.B_{\mathfrak{k}}$ *for some constant* C *and* $(G/H,g_{B_{\mathfrak{g}}})$ *is Einstein, then* $(K/H,g_{B_{\mathfrak{k}}})$ *is Einstein.*

(*b*) *If* $H_i \subset K_i \subset G$ *are closed connected subgroups with* K_i *simple,* $\Pi H_i \subset \Pi K_i \subset G$, *and* $(G/(\Pi H_i), g_{B_8})$ *is Einstein, then for each i,* $(K_i/H_i, g_{B_{\mathfrak{k}_i}})$ *is Einstein.* ■

Many of the examples we will obtain are easily seen to be Einstein by the following.

(1.14) COROLLARY. — *Let* $\chi = \chi_1 \oplus \ldots \oplus \chi_k$ *where* χ_i *is an irreducible real representation with dominant weight* λ_i. *If for each* $i \neq j$ *there exists an automorphism of* $\mathfrak{h}$ *which takes* λ_i *to* λ_j *and permutes the* $\{\lambda_k\}$, *then* g_B *is Einstein.*

Proof. — If A is such an automorphism, then $\lambda_i \circ A = \lambda_j$, $\chi_i \circ A = \chi_j$, and $\delta \circ A = \delta$. We only have to show that A is an isometry of $B|\mathfrak{h}$ since then $B^*(\lambda_i, \lambda_i + 2\delta) = B^*(\lambda_j, \lambda_j + 2\delta)$. But this follows from (1.8) since any automorphism is an isometry of $B_{\mathfrak{h}}$ and $\operatorname{tr}(\chi(X) \circ \chi(Y)) = \sum_i \operatorname{tr}(\chi_i(X)\chi_i(Y))$ is invariant under A since A permutes the χ_i's. ■

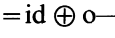
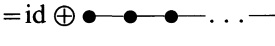
3. SOME EXAMPLES. — In this section we describe some pairs (G, H) for which g_B is easily seen to be Einstein using (1.11) or (1.14).

We begin by establishing some notation and conventions in representation theory. Let μ_m, ν_{2m}, and ρ_m denote respectively the standard complex representations of $SU(m)$ (or $U(m)$), $Sp(m)$, and $SO(m)$ (or $Spin(m)$) of dimensions m, $2m$, and m. If λ is the dominant weight of an irreducible complex representation π_λ of a compact simple Lie algebra $\mathfrak{h}$, we often describe π_λ by giving the *diagram* of π_λ. Suppose α is a simple root, let $\lambda^\alpha = [2B^*(\lambda, \alpha)]/[B^*(\alpha, \alpha)]$. λ^α is a non-negative integer and is independent of the choice of the bi-invariant metric on $\mathfrak{h}$. The diagram of π_λ consists of the Dynkin diagram of $\mathfrak{h}$ with λ^α placed above the vertex corresponding to α.

$\Lambda^2\pi$ and $S^2\pi$ denote respectively the second exterior and symmetric power of π, and we have

$$\Lambda^2(\pi \hat{\otimes} \pi') = [\Lambda^2\pi \hat{\otimes} S^2\pi'] \oplus [S^2\pi \hat{\otimes} \Lambda^2\pi'],$$
$$S^2(\pi \hat{\otimes} \pi') = [S^2\pi \hat{\otimes} S^2\pi'] \oplus [\Lambda^2\pi \hat{\otimes} \Lambda^2\pi'].$$

($\hat{\otimes}$ denotes the external tensor product while $\otimes$ is used to denote the internal tensor product.) If π is a non-self-contragredient representation, then $\pi \oplus \pi^*$ has an orthogonal and a symplectic structure. The corresponding real/quaternionic representation is denoted by $[\pi]_{\mathbb{R}}/[\pi]_{\mathbb{H}}$.

If $SO(n)/H$ is a homogeneous space with isotropy representation χ and the inclusion $H \subset SO(n)$ is given by the orthogonal representation π, then $\Lambda^2\pi = \mathrm{Ad}_H \oplus \chi$ since $\Lambda^2\rho_n = \mathrm{Ad}_{SO(n)}$ and $\mathrm{Ad}_{SO(n)}|H = \mathrm{Ad}_H \oplus \chi$. This can be used to compute the isotropy representation χ. Furthermore, if $H \subset K \subset G$ and if χ_1 is the isotropy representation of H in K and χ_2 that of K in G, then the isotropy representation of H in G is $\chi_1 \oplus \chi_2|H$. We also observe that $S^2\rho_n = \mathrm{id} \oplus$  , ..., $S^2\nu_{2n} = \mathrm{Ad}_{Sp(n)}$, and $\Lambda^2\nu_{2n} = \mathrm{id} \oplus$  .

Examples of homogeneous manifolds whose standard metric is Einstein include group manifolds G with G compact, connected, and semisimple, and symmetric spaces of compact type. They include also the reductive strongly isotropy irreducible spaces G/H with H compact, connected and G acting effectively on G/H. In [25] such spaces are completely classified (*see* also [23]). It turns out that G must be compact and simple if (G, H) is not a Riemannian symmetric pair.

As we saw in (1.5), $(G/T, g_B)$ is Einstein iff T is a maximal torus and all roots of G have the same length with respect to B. We now describe some non-trivial examples.

Example 1. — (*a*) $G=SU(nk)$, $H=S(U(k)\times\ldots\times U(k))$ (n times), $k\geqq 2$, $n\geqq 3$. $\pi: H\to G$ is given by $\oplus[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\mu_k\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}]$. The isotropy representation is easily seen to be $\oplus[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\mu_k\hat{\otimes}\ldots\hat{\otimes}\mu_k^*\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}]_{\mathbb{R}}$. By (1.14) $(G/H, g_B)$ is Einstein.

(*b*) $G=Sp(kn)$, $H=Sp(k)\times\ldots\times Sp(k)$ (*n* times), $k\geqq 1$, $n\geqq 3$. $\pi: H\to G$ is given by $\oplus[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\nu_{2k}\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}]$. The isotropy representation is

$$\oplus[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\nu_{2k}\hat{\otimes}\ldots\hat{\otimes}\nu_{2k}\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}].$$

Again, by (1.14) $(G/H, g_B)$ is Einstein.

(*c*) $G=SO(nk)$, $H=SO(k)\times\ldots\times SO(k)$ (*n* times), $k\geqq 3$, $n\geqq 3$. $\pi: H\to G$ is given by $\oplus[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\rho_k\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}]$ and the isotropy representation is

$$\oplus[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\rho_k\hat{\otimes}\ldots\hat{\otimes}\rho_k\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}].$$

By (1.14) $(G/H, g_B)$ is Einstein. (This example was observed previously in [7], p. 59.)

Example 2. — (*a*) $G=SO(n^2)$, $H=SO(n).SO(n)$, $n\geqq 3$, and $\pi: H\to G$ is given by $\rho_n\hat{\otimes}\rho_n$. By computing $\Lambda^2(\rho_n\hat{\otimes}\rho_n)$ we see easily that the isotropy representation of G/H is $[\Lambda^2\rho_n\hat{\otimes}(S^2\rho_n-\mathrm{id})]\oplus[(S^2\rho_n-\mathrm{id})\hat{\otimes}\Lambda^2\rho_n]$. $(G/H, g_B)$ is normal homogeneous Einstein by (1.14).

(*b*) $G=SO(4n^2)$, $H=Sp(n)\cdot Sp(n)$, $n\geqq 2$, $\pi: H\to G$ is given by $\nu_{2n}\hat{\otimes}\nu_{2n}$. By computing $\Lambda^2(\nu_{2n}\hat{\otimes}\nu_{2n})$, we see that the isotropy representation of G/H is

$$[S^2\nu_{2n}\hat{\otimes}(\Lambda^2\nu_{2n}-\mathrm{id})]\oplus[(\Lambda^2\nu_{2n}-\mathrm{id})\hat{\otimes}S^2\nu_{2n}].$$

By (1.14) $(G/H, g_B)$ is Einstein.

Remark. — The normal homogeneous Einstein spaces in Example 2 can be obtained from symmetric spaces, just as the non-symmetric strongly isotropy irreducible quotients of $SO(n)$ by connected subgroups can be obtained from Riemannian symmetric spaces of compact type. (*See* [25] pp. 147,8, and [23].)

Let us consider the symmetric spaces $G/K=SO(2n)/(SO(n)\cdot SO(n))$ and $Sp(2n)/(Sp(n)\cdot Sp(n))$. The isotropy representations χ are respectively

$$\rho_n\hat{\otimes}\rho_n: SO(n)\times SO(n)\to SO(n^2) \qquad \text{and} \qquad \nu_{2n}\hat{\otimes}\nu_{2n}: Sp(n)\times Sp(n)\to SO(4n^2).$$

The spaces in Example 2 are just $SO(\dim G/K)/\chi(K)$.

The following interesting family of examples also arises from symmetric spaces.

Example 3. – Let G/H be a compact irreducible simply connected symmetric space with H simple. Let π be the isotropy representation of G/H and $n=\dim G/H$. Then $SO(n)/\pi(H)$ is isotropy irreducible (*see* [23]) with isotropy representation χ determined by $\Lambda^2\pi=ad_H\oplus\chi$. In the above we include the spheres $G/H=SO(n+1)/SO(n)$, in which case $\pi(H)=SO(n)$ and χ is 0-dimensional. In Chapter 2 section 3 we will see that

$$C_{\chi,\,B^*_{SO(n)}}=\left(\frac{2\dim H}{n(n-2)}\right)Id \qquad \text{and} \qquad C_{\pi,\,B^*_{SO(n)}}=\left(\frac{\dim H}{n(n-2)}\right)Id.$$

For irreducible symmetric spaces G/H for which H is not simple the above equalities are satisfied only for $SO(2k)/(SO(k).SO(k))$ and $Sp(2k)/(Sp(k).Sp(k))$. In these cases, example 2 shows that the standard metrics of $SO(n)/\pi(H)$ are Einstein.

Now let G_i/H_i, $i=1,\ldots,k$ be a family of irreducible symmetric spaces of the above types of dimension n_i with corresponding representations π_i and χ_i. Then $G/H=G_1/H_1\times\ldots\times G_k/H_k$ is a new symmetric space. We examine $SO(n)/\pi(H)$, where $n=\dim G/H$ and π is the isotropy representation of G/H. The isotropy representation of $SO(n)/\pi(H)$ is

$$\chi=\bigoplus_{i=1}^{k}[id\hat{\otimes}\ldots\hat{\otimes}\chi_i\hat{\otimes}\ldots\hat{\otimes}id]\oplus\bigoplus_{i<j}[id\hat{\otimes}\ldots\hat{\otimes}\pi_i\hat{\otimes}\ldots\hat{\otimes}\pi_j\hat{\otimes}\ldots\hat{\otimes}id],$$

as can be seen from the inclusions

$$H=H_1\times\ldots\times H_k\xrightarrow{(\pi_1,\ldots,\pi_k)}SO(n_1)\times\ldots\times SO(n_k)\to SO(n),$$

and the fact that the isotropy representation of $SO(n)/(SO(n_1)\times\ldots\times SO(n_k))$ is $\bigoplus_{i<j}[id\hat{\otimes}\ldots\hat{\otimes}\rho_{n_i}\hat{\otimes}\ldots\hat{\otimes}\rho_{n_j}\hat{\otimes}\ldots\hat{\otimes}id]$. One easily sees that for the standard inclusion $SO(n_i)\subset SO(n)$ we have $B_{SO(n)}=((n-2)/(n_i-2))\,B_{SO(n_i)}$. Hence

$$C_{\chi_i,\,B^*_{SO(n)}\mid\mathfrak{h}_i}=\left(\frac{n_i-2}{n-2}\right)C_{\chi_i,\,B^*_{SO(n_i)}\mid\mathfrak{h}_i}=\left(\frac{2\dim H_i}{n_i(n-2)}\right)Id \quad \text{and} \quad C_{\pi_i,\,B^*_{SO(n)}\mid\mathfrak{h}_i}=\left(\frac{\dim H_i}{n_i(n-2)}\right)Id.$$

Therefore $C_{\chi,\,B^*_{SO(n)}\mid\mathfrak{h}}=a\,Id$ iff $(\dim H_i/n_i)$ is independent of i, in which case we obtain a large family of Einstein standard homogeneous metrics.

An interesting special case is if $G_i/H_i=(H_i\times H_i)/\Delta H_i$ with $\pi_i=Ad_{H_i}$, where H_i is any compact simple Lie group. Then the above shows that the standard metric of $SO(\dim H)/Ad(H)$ is always Einstein provided that H is compact and semisimple. Another special case is if we let $G_i/H_i=G/H$ for all i, where G/H is an arbitrary irreducible compact symmetric space as above. (We obtain example 1 (*c*) from $G/H=S^n$.)

By examining a list of irreducible symmetric spaces one easily shows that the only other possibilities for which $(\dim H_i)/n_i = (\dim H_j)/n_j$ (or equivalently $\dim G_i/\dim H_i = \dim G_j/\dim H_j$) are given by

G_i/H_i	G_j/H_j
$SO(2n)/SO(n)\cdot SO(n)$	$SU(2n-2)/SO(2n-2)$
$Sp(2n)/Sp(n)\cdot Sp(n)$	$SU(4n+2)/Sp(2n+1)$
$SO(32)/SO(16)\cdot SO(16)$	$E_8/Spin(16)$
$SO(20)/SO(10)\cdot SO(10)$	$E_7/SU(8)$
$SO(14)/SO(7)\cdot SO(7)$	$E_6/Sp(4)$
$SO(6)/SO(5)$	E_6/F_4
$SO(5)/SO(4)$	$SU(6)/Sp(3)$

Most of the other Einstein standard metrics on $SO(n)/\pi(H)$ also come from compact symmetric spaces G/H where $\pi: H \to SO(n)$ is the isotropy representation of G/H: the symmetric spaces $M = S^n \times P^{n+1}\mathbb{C}$ and $M = S^6 \times P^5\mathbb{C}$ give rise to the normal homogeneous Einstein spaces $SO(3n+2)/(SO(n)\cdot U(n+1))$ and $SO(26)/(SO(6)\cdot Sp(5)\cdot Sp(1))$. The only new normal homogeneous Einstein space $SO(n)/H$ which does not come from symmetric spaces is described in the next example.

Example 4. — Let $G = Spin(8)$ and $H = G_2$. Suppose $\pi: H \to G$ is given by $[o \overset{1}{\equiv} \bullet] \oplus [o \equiv \bullet]$. Then the inclusions $G_2 \subset so(7) \subset so(8)$ show that the isotropy representation is $[o \overset{1}{\equiv} \bullet] \oplus [o \overset{1}{\equiv} \bullet]$. Obviously, (1.14) implies that g_B is Einstein. Now G/H is diffeomorphic to $S^7 \times S^7$ because G/H is a 7-sphere bundle over S^7; however, by Corollary 5.4, p. 215, of [15], g_B is not a product metric.

Notice that this example is also the only homogeneous space G/H with G simple, H connected, and isotropy representation a sum of equivalent irreducible real representation. This follows from our classification, since for any such space the standard metric is Einstein.

Example 5. — $G = E_8$, $H = SU(5)\cdot SU(5)$, where $\pi: H \to G$ is determined by the fact that H is a maximal subgroup of maximal rank in G. (*See* [5], p. 219.) Wolf calculated the complexified isotropy representation of G/H (*see* pp. 282 *a*, *b* in [26]) to be

$$[\overset{1}{o}\!-\!o\!-\!o\!-\!o \hat{\otimes} o\!-\!o\!-\!\overset{1}{o}\!-\!o]_{\mathbb{R}} \oplus [\overset{1}{o}\!-\!o\!-\!o\!-\!o \hat{\otimes} \overset{1}{o}\!-\!o\!-\!o\!-\!o]_{\mathbb{R}}.$$

By (1.14), $(E_8/(SU(5)\cdot SU(5)), g_B)$ is Einstein. In fact, this is the only isotropy reducible space of the form G/H where H is a maximal subgroup of maximal rank in G and G is compact, connected, and simple. Hence we have shown that the standard metric of every homogeneous space G/H where H is a maximal connected subgroup of maximal rank in a compact connected simple Lie group G is Einstein.

Example 6. — $G = F_4$, $H = Spin(8)$ and $\pi: H \to G$ is given by the inclusions $Spin(8) \subset Spin(9) \subset F_4$. The complexified isotropy representation is $\rho_8 \oplus \Delta_8^+ \oplus \Delta_8^-$. Hence by (1.14), $(F_4/Spin(8), g_B)$ is Einstein.

A glance at Table IB shows that many examples with G an exceptional Lie group satisfy the conditions of (1.14) and hence are Einstein without any computation. On the other hand we have:

Example 7. — $G=E_8$, $H=SO(9)$, respectively Spin (9), with inclusions given by $SO(9) \subset SU(9) \subset E_8$ and Spin (9) $\subset$ Spin (16) $\subset E_8$ (where Spin (9) $\subset$ Spin (16) is given by the spin representation). One can show that the isotropy representations are given by $\overset{2}{o}\!\!-\!\!o\!\!-\!\!o=\bullet \oplus 2[o\!\!-\!\!o\!\!-\overset{1}{o}=\bullet]$ in the first case and by $o\!\!-\!\!o\!\!-\overset{1}{o}=\bullet \oplus o\!\!-\overset{1}{o}\!\!-\overset{1}{o}=\bullet$ in the second case. A computation shows, surprisingly, that the Casimir contants are all equal.

We conclude this section with an example involving a number theoretical condition.

Example 8. — Let $G=SU(pq+l)$, $H=S(U(p)\times U(q)\times U(l))$ with $p, q, l \geqq 2$ and $\pi: H \to G$ given by $[\mu_p \hat{\otimes} \mu_q \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \mu_l]$. Actually, π is not effective, since the kernel of $\mu_p \hat{\otimes} \mu_q$ is a one-dimensional central subgroup, so that $\pi(H)=H/U(1)$. One can show that the isotropy representation is $[\mu_p \hat{\otimes} \mu_q \hat{\otimes} \mu_l^*]_{\mathbb{R}} \oplus [\mathrm{id} \hat{\otimes} \mathrm{ad} \hat{\otimes} \mathrm{ad}]$ and a computation shows that g_B is Einstein iff $p^2+q^2-lpq=-1$ (and hence $l \geqq 3$). There are infinitely many positive integral solutions of this equation, e. g., if $l=3$, $p_0=q_0=1$, then $p_n=q_{n-1}$, $q_n=3q_{n-1}-p_{n-1}$ give recursively solutions of $p^2+q^2-lpq=-1$ with $l=3$. To describe all solutions of this equation, let $d=l^2-4$ and consider $\mathbb{Q}(\sqrt{d})$. Thisquadratic extension is the same as $\mathbb{Q}(\xi)$, where $\xi=(l+\sqrt{l^2-4})/2$. ξ has minimal polynomial x^2-lx+1. Let σ be the generator of the Galois group of $\mathbb{Q}(\xi)/\mathbb{Q}$. The Galois conjugate ξ^σ is $(l-\sqrt{l^2-4})/2$. Consider the lattice $\mathbb{Z}[\xi] \subset \mathbb{Q}(\xi)$. A typical element in $\mathbb{Z}[\xi]$ can be written as $m_1-m_2\xi$. The norm

$$N(m_1-m_2\xi)=(m_1-m_2\xi)(m_1-m_2\xi^\sigma)=m_1^2+m_2^2-lm_1m_2.$$

Hence finding all integer solutions for a given l is equivalent to the determination of elements of norm -1 in $\mathbb{Z}[\xi]$. It is a well-known result in number theory (Theorem 1, p. 118 of [6]) that all solutions of $N(m_1-m_2\xi)=-1$ have the form $\pm\xi^s n_i$ where $s \in \mathbb{Z}$ and $\{n_i\}$ is a set of pairwise non-associate elements of norm -1 in $\mathbb{Z}[\xi]$. $\{n_i\}$ may be empty (for certain values of l), but if it is non-empty, it gives rise to infinitely many solutions. One can show that for infinitely many values of l $\{n_i\}$ is non-empty, and for $l=3$ it easily follows that the above solutions are the only ones.

We finally remark that the quotients of $SU(n)$ and $Sp(n)$ in Table IA are related to symmetric spaces in a similar fashion as the strongly isotropy irreducible quotients of $SU(n)$ and $Sp(n)$ are. (*See* Chapter 3.)

4. TABLE OF RESULTS. — We now summarize in table form our classification of normal homogeneous Einstein manifolds $M=G/H$ where G is compact, connected, *simple*, H is a compact connected subgroup, and where G/H is not strongly isotropy irreducible.

In view of remarks made in section 1 we shall only list the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The embedding $d\pi: \mathfrak{h} \to \mathfrak{g}$ is specified by $\varphi \circ \pi$, where φ is the lowest dimensional basic representation of $\mathfrak{g}$, or by the embedding of $\mathfrak{h}$ into one of the maximal subalgebras of maximal rank of $\mathfrak{g}$.

A *regular subalgebra* in these tables means a subalgebra of maximal rank, an R-*subalgebra* is a subalgebra which is contained in a regular subalgebra, and an S-*subalgebra*

is one which is not contained in any regular subalgebra. In the case of an R− or S− subalgebra, the index of the subalgebra is also given. Recall that for simple compact connected groups $H \subset G$, $\pi_3(H) = \mathbb{Z} \to \pi_3(G) = \mathbb{Z}$ is multiplication by an integer, which is called the *index* of $\mathfrak{h}$ in $\mathfrak{g}$. *See* section 2.2 for more details.

Note that the items in Tables IA are not necessarily mutually exclusive.

TABLE I A

Normal homogeneous Einstein metrics – G classical, simple

No.	$\mathfrak{g}$	$\mathfrak{h}$	Embedding of $\mathfrak{h}$ in $\mathfrak{g}$	χ	Remarks
1.	$su(n)$, $so(2n)$	any maximal abelian subalgebra	all embeddings are conjugate	root space decomposition	$n \geqq 3$
2a. . . .	$su(nk)$	$s(u(k)\oplus \dots \oplus u(k))$ n copies	$\sum_1^n [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\mu_k\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]$	$\Sigma [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\mu_k\hat{\otimes} \dots \hat{\otimes}\mu_k^*\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]_{\mathbb{R}}$	$k \geqq 2$, $n \geqq 3$ regular subalgebra
2b. . . .	$sp(nk)$	$sp(k)\oplus \dots \oplus sp(k)$ n copies	$\sum_1^n [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\nu_{2k}\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]$	$\Sigma [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\nu_{2k}\hat{\otimes} \dots \hat{\otimes}\nu_{2k}\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]$	$k \geqq 1$, $n \geqq 3$ regular subalgebra
2c. . . .	$so(nk)$	$so(k)\oplus \dots \oplus so(k)$ n copies	$\sum_1^n [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\rho_k\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]$	$\Sigma [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\rho_k\hat{\otimes} \dots \hat{\otimes}\rho_k\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]$	$k \geqq 3$, $n \geqq 3$ regular subalgebra
3a. . . .	$so(n^2)$	$so(n)\oplus so(n)$	$\rho_n\hat{\otimes}\rho_n$	$[\Lambda^2\rho_n\hat{\otimes}(S^2\rho_n - \mathrm{id})]\oplus[(S^2\rho_n - \mathrm{id})\hat{\otimes}\Lambda^2\rho_n]$	iff k is even $n \geqq 3$
3b. . . .	$so(4n^2)$	$sp(n)\oplus sp(n)$	$\nu_{2n}\hat{\otimes}\nu_{2n}$	$[S^2\nu_{2n}\hat{\otimes}(\Lambda^2\nu_{2n} - \mathrm{id})]\oplus [(\Lambda^2\nu_{2n} - \mathrm{id})\hat{\otimes}S^2\nu_{2n}]$	$n \geqq 2$
4.	$so(n)$, $n = \dim \mathfrak{h}$	$\mathfrak{h}$ semi-simple, non-simple $\mathfrak{h} \neq so(4)$	ad		
5.	$so\left(\sum_1^l \dim \pi_i\right)$	$\mathfrak{h}_1\oplus \dots \oplus \mathfrak{h}_l$ $(\mathfrak{h}_i, \pi_i)$ as in example 3 and $\dfrac{\dim \pi_i}{\dim H_i}$ independent of i	$\sum_{i=1}^l [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\pi_i\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]$	$\sum_{i<j} [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\pi_i\hat{\otimes} \dots \hat{\otimes}\pi_j\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]\oplus \sum_1^l [\mathrm{id}\hat{\otimes} \dots \hat{\otimes}\chi_i\hat{\otimes} \dots \hat{\otimes}\mathrm{id}]$ where $\Lambda^2\pi_i = \mathrm{ad}\, h_i \oplus \chi_i$	$l > 1$
6.	$su(pq+l)$	$\dfrac{u(l)\oplus u(p)\oplus u(q)}{u(1)\oplus u(1)}$ $p^2+q^2-lpq = -1$	$[\mu_l\hat{\otimes}\mathrm{id}\hat{\otimes}\mathrm{id}]\oplus [\mathrm{id}\hat{\otimes}\mu_p\hat{\otimes}\mu_q]$	$[\mu_l\hat{\otimes}\mu_p^*\hat{\otimes}\mu_q^*]_{\mathbb{R}}\oplus [\mathrm{id}\hat{\otimes}\mathrm{ad}\hat{\otimes}\mathrm{ad}]$	$p \geqq 2$ $q \geqq 2$ $l \geqq 3$
7a. . . .	$sp(3n-1)$	$sp(n)\oplus u(2n-1)$	$[\nu_{2n}\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\mu_{2n-1}]_{\mathbb{H}}$	$[\nu_{2n}\hat{\otimes}\mu_{2n-1}]_{\mathbb{R}}\oplus[\mathrm{id}\hat{\otimes}S^2\mu_{2n-1}]_{\mathbb{R}}$	$n \geqq 1$ regular subalgebra
7b. . . .	$so(3n+2)$	$so(n)\oplus u(n+1)$	$[\rho_n\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\mu_{n+1}]_{\mathbb{R}}$	$[\rho_n\hat{\otimes}\mu_{n+1}]_{\mathbb{R}}\oplus[\mathrm{id}\hat{\otimes}\Lambda^2\mu_{n+1}]_{\mathbb{R}}$	$n \geqq 3$ regular subalgebra
8.	$so(26)$	$sp(1)\oplus sp(5)\oplus so(6)$	$[\nu_2\hat{\otimes}\nu_{10}\hat{\otimes}\mathrm{id}]\oplus [\mathrm{id}\hat{\otimes}\mathrm{id}\hat{\otimes}\rho_6]$	$[\nu_2\hat{\otimes}\nu_{10}\hat{\otimes}\rho_6]\oplus[\overset{2}{\circ}\hat{\otimes}\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\overset{2}{=}\circ\hat{\otimes}\circ\!\!-\!\!\circ\!\!-\!\!\circ]$	
9.	$so(8)$	G_2	$[\circ\overset{1}{\equiv}\bullet]\oplus[\circ\equiv\bullet]$	$2[\circ\overset{1}{\equiv}\bullet]$	

TABLE I B

Normal homogeneous Einstein metrics — G exceptional

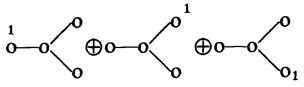
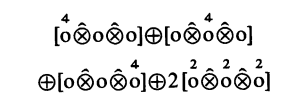
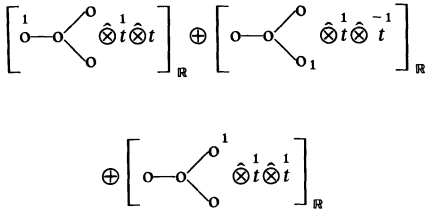
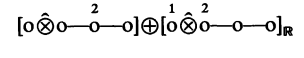
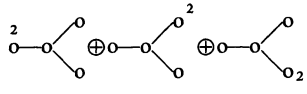
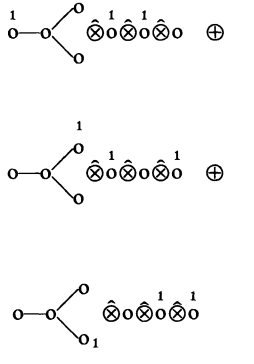
No.	$\mathfrak{g}$	$\mathfrak{h}$	Embedding of $\mathfrak{h}$ in $\mathfrak{g}$	χ	Remarks
1	F_4	$spin(8)$	$spin(8)\subset spin(9)\subset F_4$		regular subalgebra
2	E_6	$so(3)\oplus so(3)\oplus so(3)$	$3[so(3)]\subset 3[su(3)]\subset E_6$ $so(3)\subset su(3)$ by $\overset{2}{o}$		R-subalgebra index of $so(3)=4$
3*.	E_6	$spin(8)\oplus\mathbb{R}^2$	$spin(8)\oplus\mathbb{R}\oplus\mathbb{R}$ $\subset spin(10)\oplus\mathbb{R}\subset E_6$		regular subalgebra
4	E_6	$su(2)\oplus so(6)$	$su(2)\oplus so(6)$ $\subset su(2)\oplus su(6)\subset E_6$		R-subalgebra index of $so(6)=2$
5	E_7	$so(8)$	$so(8)\subset su(8)\subset E_7$		R-subalgebra index $=2$
6	E_7	$spin(8)\oplus 3[su(2)]$	$spin(8)\oplus 3[su(2)]$ $=spin(8)\oplus spin(4)\oplus su(2)$ $\subset spin(12)\oplus su(2)\subset E_7$		regular subalgebra

TABLE I B

Normal homogeneous Einstein metrics – G exceptional

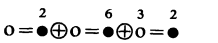
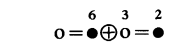
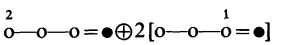
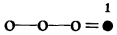
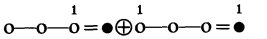
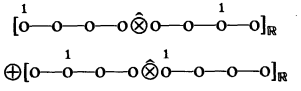
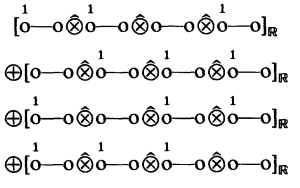
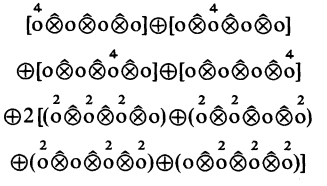
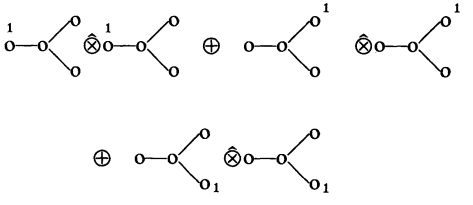
No.	$\mathfrak{g}$	$\mathfrak{h}$	Embedding of $\mathfrak{h}$ in $\mathfrak{g}$	χ	Remarks
7*	E_7	$7[su(2)]$	$7[su(2)]=3[spin(4)]\oplus su(2)$ $\subset spin(12)\oplus su(2)\subset E_7$	$(1234)\oplus(1256)\oplus(3456)\oplus(1357)$ $\oplus(2457)\oplus(1467)\oplus(2367)$	regular subalgebra
8	E_8	$so(5)$			maximal S-subalgebra index = 12
9	E_8	$so(9)$	$so(9)\subset su(9)\subset E_8$		R-subalgebra index = 2
10	E_8	$spin(9)$	$spin(9)\subset spin(16)\subset E_8$ 		R-subalgebra index = 2
11	E_8	$su(5)\oplus su(5)$	$\mathrm{ad}\,h\oplus\chi$		maximal subalgebra of maximal rank
12	E_8	$4[su(3)]$	$4[su(3)]\subset su(3)\oplus E_6\subset E_8$		regular subalgebra
13	E_8	$4[so(3)]$	$4[so(3)]\subset 4[su(3)]$ $\subset su(3)\oplus E_6\subset E_8$		R-subalgebra index of $so(3)=4$
14	E_8	$spin(8)\oplus spin(8)$	$spin(8)\oplus spin(8)$ $\subset spin(16)\subset E_8$		regular subalgebra

TABLE I B

Normal homogeneous Einstein metrics — G exceptional

No.	g	h	Embedding of h in g	χ	Remarks
15*	E_8	$8[su(2)]$	$8[su(2)]=4[spin(4)]\subset spin(16)\subset E_8$	$(1234)\oplus(5678)\oplus(1356)\oplus(2478)\oplus(1378)\oplus(2456)\oplus(1458)\oplus(2367)\oplus(1467)\oplus(2358)\oplus(1257)\oplus(3468)\oplus(1268)\oplus(3457)$	regular subalgebra
16.	E_8	$so(5)\oplus so(5)$	$so(5)\oplus so(5)\subset spin(16)\subset E_8$ $o=\overset{1}{\bullet}\hat{\otimes}o=\overset{1}{\bullet}$	$2[o=\overset{2}{\bullet}\hat{\otimes}\overset{1}{o}=\bullet]\oplus 2[\overset{1}{o}=\bullet\hat{\otimes}o=\overset{2}{\bullet}]$ $\oplus[\overset{2}{o}=\bullet\hat{\otimes}o=\bullet]\oplus[o=\bullet\hat{\otimes}\overset{2}{o}=\bullet]$	R-subalgebra index of $so(5)=2$
17.	E_8	$su(3)\oplus su(3)$	$su(3)\oplus su(3)\subset su(9)\subset E_8$ $\overset{1}{o}—o\hat{\otimes}\overset{1}{o}—o$	$3[\overset{1}{o}—o\hat{\otimes}\overset{1}{o}—\overset{1}{o}]\oplus[\overset{3}{o}—o\hat{\otimes}o—o]_{\mathbb{R}}$ $\oplus[o—o\hat{\otimes}\overset{3}{o}—o]_{\mathbb{R}}$	R-subalgebra index of $su(3)=3$
18.	E_6, E_7, E_8	any maximal abelian subalgebra	all embeddings are conjugate	root space decomposition	

* The notation (1, 2, 3, 4) stands for the represention $\overset{1}{o}\hat{\otimes}\overset{1}{o}\hat{\otimes}\overset{1}{o}\hat{\otimes}\overset{1}{o}\hat{\otimes}o\hat{\otimes}o\hat{\otimes}o$, etc., and $\overset{1}{t}$ stands for the effective irreducible representation of a one-dimensional torus.

CHAPTER TWO

Computation of the Einstein constants

In this chapter we discuss the practical aspects of computing Einstein constants, and collect various useful facts from representation theory.

1. FACTS FROM REPRESENTATION THEORY. — First we consider how the isotropy representations of quotients of the classical groups can be determined. Let μ_n, ν_{2n}, ρ_n denote respectively the usual complex representations of $SU(n)$, $Sp(n)$, and $SO(n)$ (or of the corresponding Lie algebras) on $\mathbb{C}^n$, $\mathbb{C}^{2n}$, and $\mathbb{C}^n$. Then $\mu_n \otimes \mu_n^* = \mathrm{id} \oplus \mathrm{ad}_{su(n)}$ (id is the trivial 1-dimensional representation), $S^2 \nu_{2n} = \mathrm{ad}_{sp(n)}$, and $\Lambda^2 \rho_n = \mathrm{ad}_{so(n)}$. Now let G be $SU(n)$, $Sp(n)$, or $SO(n)$, and $\pi: H \to G$ be an almost faithful representation of a compact, connected group H. Since $\mathrm{ad}_{\mathfrak{g}} = \mathrm{ad}_{\mathfrak{h}} \oplus \chi$, the isotropy representation χ of $G/\pi(H)$ is determined by $\pi \otimes \pi^* = \mathrm{id} \oplus \mathrm{ad}_{\mathfrak{h}} \oplus \chi$ in the unitary case, $S^2 \pi = \mathrm{ad}_{\mathfrak{h}} \oplus \chi$ in the symplectic case, and $\Lambda^2 \pi = \mathrm{ad}_{\mathfrak{h}} \oplus \chi$ in the orthogonal case. Note that the above relationships still hold if the representations are replaced by their complexifications. Moreover, the Einstein condition can be expressed in terms of the complexified isotropy representation (*see* (1.12)).

When H is not semi-simple it is more convenient to allow $G = U(n)$ and consider quotients of $U(n)$ rather than of $SU(n)$. Let μ_n denote also the n-dimensional complex representation of $U(n)$. Since $\mu_n \otimes \mu_n^* = \mathrm{ad}_{u(n)}$, if $\pi: H \to U(n)$ is almost faithful, we have upon restriction $\pi \otimes \pi^* = \mathrm{ad}_{\mathfrak{h}} \oplus \chi$, where χ is the isotropy representation of $U(n)/\pi(H)$. However, in this case $U(n)/\pi(H)$ need not be almost effective.

Let $\mathfrak{h}$ be compact. Then $\mathrm{id} \subset \pi \otimes \pi^*$, $\mathrm{id} \subset \Lambda^2 \pi$ if π is symplectic, and $\mathrm{id} \subset S^2 \pi$ if π is orthogonal. Furthermore, in each case the multiplicity of id is one if π is irreducible. The condition $\Lambda^2 \pi = \mathrm{id}$ holds only for the 2-dimensional representation of $su(2)$ and $S^2 \pi \neq \mathrm{id}$ if $\pi \neq \mathrm{id}$. If $\Lambda^2 \pi = \mathrm{ad}_{\mathfrak{h}}$, then $\mathfrak{h} = so(n)$ and $\pi = \rho_n$. If $S^2 \pi = \mathrm{ad}_{\mathfrak{h}}$, then $\mathfrak{h} = sp(n)$ and $\pi = \nu_{2n}$. If $\pi \otimes \pi^* = \mathrm{id} \oplus \mathrm{ad}_{\mathfrak{h}}$, then $\mathfrak{h} = su(n)$ and $\pi = \mu_n$.

The following well-known isomorphisms will be used frequently:

$$\Lambda^2(\pi \oplus \sigma) = \Lambda^2 \pi \oplus \Lambda^2 \sigma \oplus [\pi \otimes \sigma],$$

$$S^2(\pi \oplus \sigma) = S^2 \pi \oplus S^2 \sigma \oplus [\pi \otimes \sigma],$$

$$\Lambda^2(\pi \hat{\otimes} \sigma) = [\Lambda^2 \pi \hat{\otimes} S^2 \sigma] \oplus [S^2 \pi \hat{\otimes} \Lambda^2 \sigma],$$

$$S^2(\pi \hat{\otimes} \sigma) = [S^2 \pi \hat{\otimes} S^2 \sigma] \oplus [\Lambda^2 \pi \hat{\otimes} \Lambda^2 \sigma].$$

Certain irreducible summands of χ are found by

(2.1) LEMMA. — *Let π_λ be an irreducible complex representation of a compact Lie algebra $\mathfrak{h}$ with dominant weight λ.*

(*a*) $\pi_{2\lambda-\alpha} \subset \Lambda^2 \pi_\lambda$ *with multiplicity* 1 *if* α *is a simple root with* $(\lambda, \alpha) \neq 0$. *If* π_λ *is in addition orthogonal, then* $\pi_{2\lambda-\alpha} \not\subset \mathrm{ad}_h \subset \Lambda^2 \pi_\lambda$ *unless* $(\mathfrak{h}, \pi_\lambda)$ *is one of the following:*

$(so(n), \rho_n)$, $\left(so(8), \begin{array}{c} \circ\!-\!\!-\!\circ \begin{array}{l} \diagup \circ^{1} \\ \diagdown \circ \end{array} \end{array} \right)$, $(so(7), \circ\!-\!\!-\!\circ = \overset{1}{\bullet})$, $(G_2, \circ \equiv \overset{1}{\bullet})$, *or* $(sp(n) \oplus sp(1), \nu_{2n} \hat{\otimes} \nu_2$

and α *is the simple root of* $sp(1)$).

(*b*) $\pi_{2\lambda} \subset S^2 \pi_\lambda$ *with multiplicity* 1. *If* π_λ *is symplectic, then* $\pi_{2\lambda} \not\subset \mathrm{ad}_{\mathfrak{h}} \subset S^2 \pi_\lambda$ *unless* $(\mathfrak{h}, \pi_\lambda) = (sp(n), \nu_{2n})$.

(*c*) $\pi_{\lambda+\lambda^*} \subset \pi_\lambda \otimes \pi_\lambda^*$ *with multiplicity* 1. *If* $\pi_\lambda \neq \pi_\lambda^*$, $\pi_{\lambda+\lambda^*} \not\subset \mathrm{ad}_{\mathfrak{h}}$ *unless*
$(\mathfrak{h}, \pi) = (su(n), \mu_n)$ *or* $(u(n), \mu_n)$.

For a proof, see, for example, [23]. It is also implicit in the proofs of Propositions in [25]. Here we only note that if v_λ is a dominant weight vector of π_λ and $v_{\lambda-\alpha} = \pi_\lambda(E_{-\alpha}) v_\lambda$, then $v_\lambda \wedge v_{\lambda-\alpha}$, $v_\lambda \circ v_\lambda$ (symmetric product), and $v_\lambda \otimes v_\lambda^*$ are clearly weight vectors in $\Lambda^2 \pi_\lambda$, $S^2 \pi_\lambda$, $\pi_\lambda \otimes \pi_\lambda^*$ respectively of multiplicity 1 and are moreover dominant. This proves the first part of each statement in (2.1).

The dominant weight of any irreducible summand in $\pi_\lambda \otimes \pi_\lambda^*$, $\Lambda^2 \pi_\lambda$, or $S^2 \pi_\lambda$ is respectively of the form $\lambda + \lambda^* - \Sigma n_i \alpha_i$, $2\lambda - \alpha - \Sigma n_i \alpha_i$, or $2\lambda - \Sigma n_i \alpha_i$, where α_i are the simple roots of $\mathfrak{h}$, n_i are non-negative integers, and α is a simple root of $\mathfrak{h}$ such that $(\lambda, \alpha) \neq 0$.

The following Lemma is useful for comparing Casimir constants. It is well-known, but we include a proof for the convenience of the reader.

(2.2) LEMMA. — *Let* π_{λ_1} *and* π_{λ_2} *be irreducible complex representations of* $\mathfrak{h}$ *and* Q *a bi-invariant metric on* $\mathfrak{h}$. (*We extend* Q *to a non-degenerate symmetric form on* $\mathfrak{h}^* \otimes \mathbb{C}$, *denoted by* Q.)

(*a*) *If* $\mathfrak{h}$ *is compact and* $\lambda_2 = \lambda_1 - \Sigma m_j \alpha_j$, *where* α_j *are the simple roots of* $\mathfrak{h}$ *and* m_j *are non-negative integers, then* $-Q(\lambda_1, \lambda_1 + 2\delta) \geqq -Q(\lambda_2, \lambda_2 + 2\delta)$ *with equality iff* $\lambda_1 = \lambda_2$.

(*b*) *If* $\mathfrak{h}$ *is semi-simple and* $\lambda_1^\alpha \geqq \lambda_2^\alpha$ *for every simple root* α, *then* $-Q(\lambda_1, \lambda_1 + 2\delta) \geqq -Q(\lambda_2, \lambda_2 + 2\delta)$ *with equality iff* $\lambda_1 = \lambda_2$.

Proof. — (*a*)

$$\begin{aligned} -Q(\lambda_1, \lambda_1 + 2\delta) + Q(\lambda_2, \lambda_2 + 2\delta) \\ &= -2\Sigma m_j Q(\lambda_1, \alpha_j) - 2\Sigma m_j Q(\alpha_j, \delta) + \Sigma m_i m_j Q(\alpha_i, \alpha_j) \\ &= -\Sigma m_j Q(\alpha_j, \alpha_j)(\lambda_1^{\alpha_j} + 1) + \Sigma m_i m_j Q(\alpha_i, \alpha_j) \end{aligned}$$

since $Q(2\delta, \alpha_j) = Q(\alpha_j, \alpha_j)$. Because λ_2 is dominant, we have

$$\lambda_2^{\alpha_j} = \lambda_1^{\alpha_j} - \sum_i m_i \frac{2Q(\alpha_i, \alpha_j)}{Q(\alpha_j, \alpha_j)} \geqq 0.$$

Hence $-(1/2)\lambda_1^{\alpha j}(\alpha_j, \alpha_j) \geqq -\sum_i m_i Q(\alpha_i, \alpha_j)$. Therefore,

$$-Q(\lambda_1, \lambda_1 + 2\delta) + Q(\lambda_2, \lambda_2 + 2\delta) \geqq -\Sigma m_j Q(\alpha_j, \alpha_j)(1 + (1/2)\lambda_1^{\alpha j}) \geqq 0$$

with equality iff $m_j = 0$ for each j.

(*b*) The proof of (*a*) remains valid if the m_j's are arbitrary non-negative real numbers. But if $\lambda_1^{\alpha} \geqq \lambda_2^{\alpha}$, then $\lambda_1 - \lambda_2$ is dominant with $\lambda_1 - \lambda_2 = \Sigma\, m_j \alpha_j$ for some non-negative rational numbers m_j (since the entries of the inverse of the Cartan matrix $2Q(\alpha_i, \alpha_j)/Q(\alpha_j, \alpha_j)$ consist of positive rational numbers). ■

2. COMPUTING CASIMIR CONSTANTS. — It will be convenient to use a particular normalization of the Killing form. For a simple Lie group G, let B_G be the negative of the Killing form and B_G^* be the induced metric on $\mathfrak{g}^*$. Let μ be the maximal root of G and B'_G be the multiple of B_G defined by $B'^*_G(\mu, \mu) = -2$. We write

$$B_G = \alpha_G B'_G \qquad \text{and hence} \quad B'^*_G = \alpha_G B^*_G.$$

Then $\alpha_G = -(1/2) B_G(\bar{\mu}, \bar{\mu}) = -2/B^*_G(\mu, \mu)$, where $\bar{\mu}$ is dual to μ with respect to B'_G.

Below we list the values of α_G for the simple Lie groups. (*See*, for example, [7] p. 40 for their calculation.)

TABLE II

$\alpha_G = -(1/2) B_G(\bar{\mu}, \bar{\mu})$

G	α_G
SU(*n*)	$2n$
Sp(*n*)	$2(n+1)$
SO(*n*), $n \geqq 5$	$2(n-2)$
SO(3)	4
G_2	8
F_4	18
E_6	24
E_7	36
E_8	60

Since

$$B_{so(n)}(A, B) = -(n-2)\,\mathrm{tr}(AB), \qquad B_{sp(n)}(A, B) = -2(n+1)\,\mathrm{tr}(AB),$$

and

$$B_{su(n)}(A, B) = -2n\,\mathrm{tr}(AB),$$

we have

$$B'_{so(n)}(A, B) = -\frac{1}{2}\mathrm{tr}(AB), \qquad n \geqq 5,$$

$$B'_{so(3)}(A, B) = -\frac{1}{4}\mathrm{tr}(AB), \qquad B'_{sp(n)}(A, B) = -\mathrm{tr}(AB),$$

and

$$B'_{su(n)}(A, B) = -\mathrm{tr}(AB).$$

In this paper we consider the Einstein condition for standard homogeneous metrics on G/H with G simple. Therefore, by (1.12) it is equivalent to compare the Casimir constants of irreducible summands of χ defined using B'_G instead of B_G. For any irreducible representation π_λ of $H(H \subset G)$, we introduce the notation $E(\pi_\lambda)$ for the Casimir constant $-B'^*_G(\lambda, \lambda+2\delta)$. We sometimes use the same notation for Casimir constants with respect to other bi-invariant metrics on H. When we do so the bi-invariant metric used will be clearly stated.

If $\mathfrak{g}$, $\mathfrak{h}$ are both simple Lie algebras and $\mathfrak{h} \subset \mathfrak{g}$, then the *index of* $\mathfrak{h}$ *in* $\mathfrak{g}$ is the constant $[\mathfrak{g}:\mathfrak{h}]$ so that $B'_{\mathfrak{g}} = [\mathfrak{g}:\mathfrak{h}]\, B'_{\mathfrak{h}}$. Dynkin [8] showed that this constant is an integer, and Oniščik [18] showed that his integer is equal to the index of the homomorphism $\pi_3(H) \to \pi_3(G)$ where $H \subset G$ are Lie groups whose Lie algebras are $\mathfrak{h}$ and $\mathfrak{g}$ respectively. In Chapter 4, to indicate the index of $\mathfrak{h}$ in an exceptional Lie algebras $\mathfrak{g}$, we shall place it at the upper right hand corner of the symbol of $\mathfrak{h}$. For example, the principal 3-dimensional subalgebra of G_2 is written as A_1^{28}.

To compute the indices of simple subalgebras of the classical groups we also introduce the *index of a representation.* If π is a complex representation of a simple Lie algebra $\mathfrak{h}$, the index of π is the constant $i(\pi)$ so that

$$-\operatorname{tr}(\pi(X)\,\pi(Y)) = i(\pi)\, B'_{\mathfrak{h}}(X, Y) \quad \text{for all } X, Y \in \mathfrak{h}.$$

$i(\pi)$ clearly satisfies

$$i(\pi \oplus \sigma) = i(\pi) + i(\sigma), \quad i(\mathrm{id}) = 0, \qquad \text{and} \qquad i(\pi \otimes \sigma) = i(\pi) \dim \sigma + i(\sigma) \dim \pi.$$

If $\pi = \pi_\lambda$ is irreducible and $\{X_i\}$, $\{Y_i\}$ are dual bases of $\mathfrak{h}$ with respect to $B'_{\mathfrak{h}}$, we get

$$\begin{aligned} -i(\pi) \dim \mathfrak{h} &= -i(\pi) \sum_i B'_{\mathfrak{h}}(X_i, Y_i) = \sum_{i,j} B'_{\mathfrak{h}}(\pi(X_i)\,\pi(Y_i)\,X_j, Y_j) \\ &= \sum_j B'_{\mathfrak{h}}(-C_{\pi, B'_{\mathfrak{h}}}(X_j), Y_j) = (\dim \pi)\, B'^*_{\mathfrak{h}}(\lambda, \lambda+2\delta). \end{aligned}$$

Thus

$$i(\pi_\lambda) = -\left(\frac{\dim \pi}{\dim \mathfrak{h}}\right) B'^*_{\mathfrak{h}}(\lambda, \lambda+2\delta). \tag{2.3}$$

If $\mathfrak{h} \subset \mathfrak{g}$ with $\mathfrak{h}$, $\mathfrak{g}$ simple and φ is a representation of $\mathfrak{g}$, then $[\mathfrak{g}:\mathfrak{h}] = i(\varphi\,|\,\mathfrak{h})/i(\varphi)$, where $\varphi\,|\,\mathfrak{h}$ is the restriction of φ to $\mathfrak{h}$. Note that $i(\mu_n) = i(\nu_{2n}) = 1$, $i(\rho_n) = 2$ if $n > 3$, and $i(\rho_3) = 4$. Hence if π is a unitary n-dimensional representation of $\mathfrak{h}$ with $\mathfrak{h}$ simple, we have $[su(n):\pi(\mathfrak{h})] = i(\pi)$, so that in particular $i(\pi)$ is an integer. If π is symplectic then $[sp(n):\pi(\mathfrak{h})] = i(\pi)$, and if π is orthogonal $[so(n):\pi(\mathfrak{h})] = i(\pi)/2$, $n \geqq 5$.

For the standard inclusions $so(m) \subset so(n)\,(3 < m \leqq n)$, $sp(m) \subset sp(n)$, $su(m) \subset su(n)$, we have $[\mathfrak{g}:\mathfrak{h}] = 1$. Also, $[so(2n):su(n)] = [su(2n):sp(n)] = [so(4n):sp(n)] = 1$, while $[so(n):so(3)] = 2$ if $n \geqq 5$, and $[su(n):so(n)] = [sp(n):su(n)] = 2$ except that $[su(3):so(3)] = 4$. Table V of [8] contains the indices of the basic representations of the simple Lie algebras.

Let χ be the isotropy representation of G/H with G compact, connected, simple, and $\mathfrak{h}=\mathfrak{h}_1\oplus\ldots\oplus\mathfrak{h}_l\oplus\mathfrak{t}$ with $\mathfrak{h}_i$ simple and $\mathfrak{t}$ abelian. If $\pi_\lambda\subset\chi\otimes\mathbb{C}$, then $\pi_\lambda=\pi_{\lambda_1}\hat{\otimes}\ldots\hat{\otimes}\pi_{\lambda_l}\hat{\otimes}\pi_{\lambda_0}$, and from the above it follows that

$$(2.4)\qquad E(\pi_\lambda)=-B'^*_G(\lambda,\lambda+2\delta)=-B'^*_G(\lambda_0,\lambda_0)-\sum_i\left(\frac{1}{[\mathfrak{g}:\mathfrak{h}_i]}\right)B'^*_{\mathfrak{h}_i}(\lambda_i,\lambda_i+2\delta_i),$$

where δ_i is one half the sum of the positive roots of $\mathfrak{h}_i$. We can compute $[\mathfrak{g}:\mathfrak{h}_i]$ as indicated before. So we are left with computing $-B'^*_{\mathfrak{h}}(\lambda,\lambda+2\delta)$ for a simple Lie algebra $\mathfrak{h}$, which we from now on abbreviate by $(\lambda,\lambda+2\delta)$.

Let $\mathfrak{h}$ be simple, $\{\alpha_i\}$ be the simple roots of $\mathfrak{h}$, and τ_i be the (dominant weight of the) basic representation corresponding to α_i defined by $2(\tau_i,\alpha_j)=\delta_{ij}(\alpha_j,\alpha_j)$. Since $\{\tau_i\}$ is a basis of the Cartan subalgebra of $\mathfrak{h}$ dual to $\{2\alpha_i/(\alpha_i,\alpha_i)\}$ with respect to $B'^*_{\mathfrak{h}}$, the inverse matrix of $g^{ij}=4(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)$ is $g_{ij}=(\tau_i,\tau_j)$. Since $\lambda=\Sigma\lambda^{\alpha_i}\tau_i$ and $\delta=\Sigma\tau_i$ we have

$$(2.5)\qquad (\lambda,\lambda+2\delta)=\sum_{i,j}\lambda^{\alpha_i}\lambda^{\alpha_j}g_{ij}+2\sum_i\lambda^{\alpha_i}\Big(\sum_j g_{ij}\Big).$$

The matrix (g_{ij}) for each simple Lie algebra is given in Table II, pp. 117-8 of [8]. From this one easily obtains Table III of Casimir constants of all the basic representations and Table IV for a few other representations that will occur frequenctly in the later chapters.

3. SYMMETRIC SPACES AND ISOTROPY IRREDUCIBLE SPACES. — We next collect some results in [23] which will enable us to compute in a uniform fashion the Einstein constants of symmetric spaces and strongly isotropy irreducible spaces.

Let G/K be an n-dimensional irreducible symmetric space of compact type with (orthogonal) isotropy representation π. Since by (1.6, 1.7) $C_{\pi,B_G|\mathfrak{k}}=(1/2)\,\mathrm{Id}$, we have

$$(2.6)\qquad E(\pi)=\frac{1}{2}\alpha_G$$

which depends only G and not on the subgroup K. If χ is the isotropy representation of $SO(n)/\pi(K)$, i.e., $\Lambda^2\pi=\mathrm{ad}_{\mathfrak{k}}\oplus\chi$, then we have

$$(2.7)\qquad C_{\chi,B_G|\mathfrak{k}}=2\,C_{\pi,B_G|\mathfrak{k}}=\mathrm{Id},$$

even if $SO(n)/\pi(K)$ is not strongly isotropy irreducible.

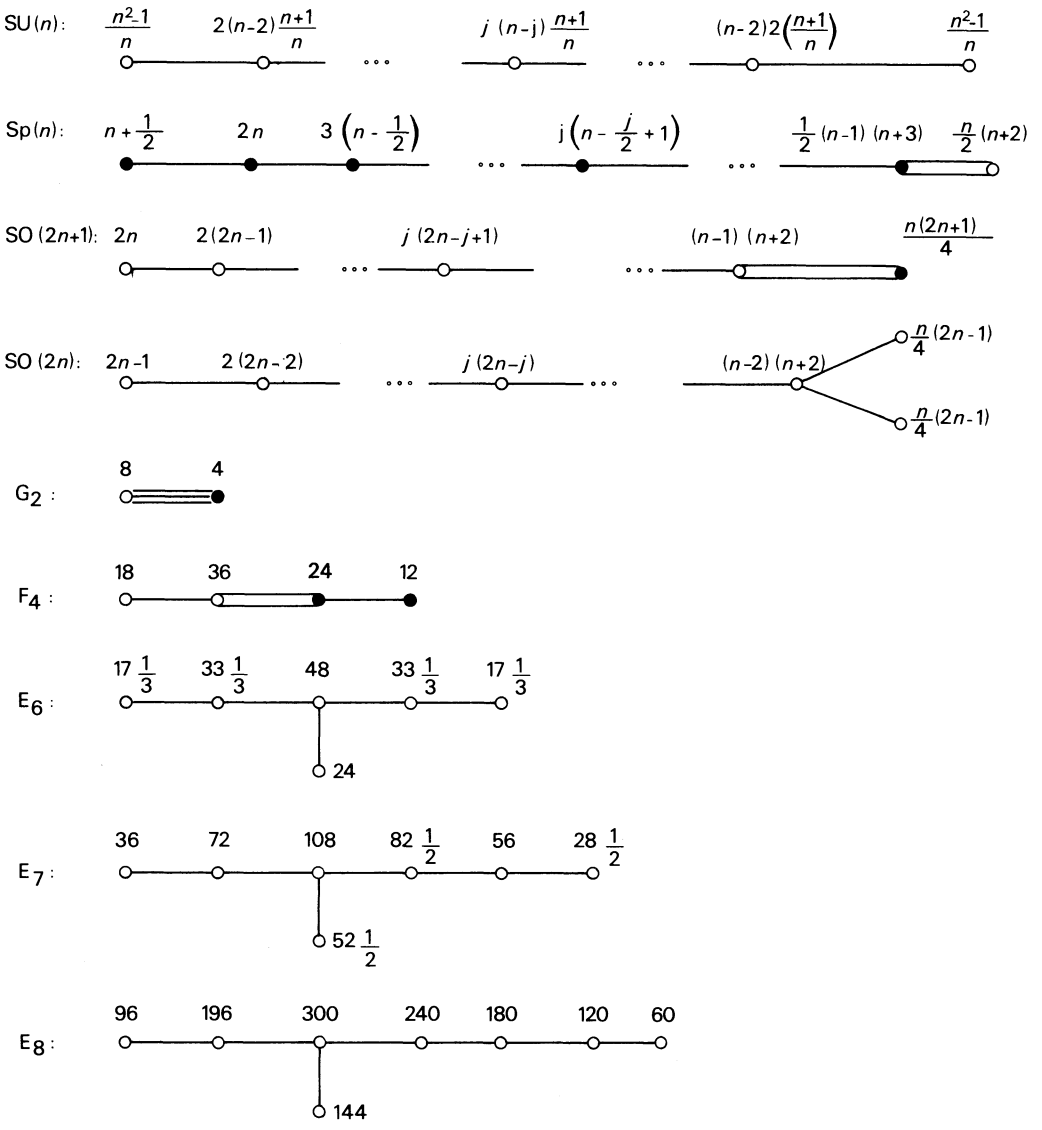
We distinguish four types of symmetric spaces: the hermitian symmetric spaces $G/H\cdot S^1$, the quaternionic symmetric spaces $G/H\cdot Sp(1)$, the real symmetric spaces G/H with H simple, and the Grassmannians over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

If L/H is strongly isotropy irreducible but not symmetric, then L is simple (Theorem 1.1 in [25], p. 62). If L is in addition a classical group, then L/H is related to a symmetric space G/K as follows: (for details, see [23])

(A) $L=SU(n)$. — Let $G/H\cdot S^1$ be an irreducible compact hermitian symmetric space of (real) dimension $2n$. Its isotropy representation $\pi=[\pi_\lambda\hat{\otimes}\pi_{\lambda_0}]_{\mathbb{R}}$ and $SU(n)/\pi_\lambda(H)$ is strongly isotropy irreducible with isotropy representation $\chi=\pi_{\lambda+\lambda^*}$. Notice that if

TABLE III

Casimir constants of basic representations



$G/H \cdot S^1$ is a complex Grassmannian the above construction yields $SU(pq)/SU(p) \cdot SU(q)$, while for the real Grassmannian $SO(n+2)/SO(n) \cdot SO(2)$, which is hermitian symmetric, the construction yields the symmetric space $SU(n)/SO(n)$.

Since $\pi_{\lambda+\lambda^*} \hat{\otimes} \mathrm{id} \subset (\pi_\lambda \hat{\otimes} \pi_{\lambda_0}) \otimes (\pi_\lambda^* \hat{\otimes} \pi_{\lambda_0}^*) \subset \Lambda^2 \pi$ we have $C_{\chi, B_G} = \mathrm{Id}$ by (2.7). Let X_0 be in the complexified Lie algebra of S^1 with $\lambda_0(X_0) = 1$. Then by (1.8) we have $B_G(X_0, X_0) = -2n$ and $B_G^*(\lambda_0, \lambda_0) = -1/2n$. Since $C_{\pi_\lambda \hat{\otimes} \pi_{\lambda_0}, B_G} = (1/2)\,\mathrm{Id}$, we have $C_{\pi_\lambda, B_G} = (1/2 - 1/2n)\,\mathrm{Id}$. We next relate B_G to $B'_{SU(n)}$. Since $B'_{SU(n)}(A, B) = -\mathrm{tr}(AB)$,

(1.8) implies that $B'_{SU(n)}|\mathfrak{h} = 1/2(B_G - B_H)$. In [23], we show that $B_H = (1-(n-1)/\dim H)\, B_G$ if $\mathfrak{h}$ is simple, and hence $B'_{SU(n)}|\mathfrak{h} = ((n-1)/2\dim H)\, B_G$. Therefore, with respect to $\pi_\lambda : \mathfrak{h} \subset su(n)$, for simple $\mathfrak{h}$, $E(\chi) = 2\dim H/(n-1)$ and $E(\pi_\lambda) = \dim H/n$.

Conversely, every non-symmetric strongly isotropy irreducible quotient of $SU(n)$ arises in this fashion from a compact hermitian symmetric space.

(B) $L = Sp(n)$. — Let $G/H \cdot Sp(1)$ be an irreducible compact quaternionic symmetric space of (real) dimension $4n$. The isotropy representation $\pi = \pi' \hat{\otimes} \overset{1}{o}$ and $Sp(n)/\pi'(H)$ is strongly isotropy irreducible. Notice that if $G/H \cdot Sp(1)$ is $SU(n+2)/S(U(n)\cdot U(2))$, the above construction yields the symmetric space $Sp(n)/U(n)$, and for $SO(n+4)/SO(n)\cdot SO(4)$ it yields the strongly isotropy irreducible space $Sp(n)/SO(n)\cdot Sp(1)$, which was not included in [25]. If $G/H \cdot Sp(1)$ is not also hermitian symmetric, then $\pi' = \pi_\lambda$ and $\chi = \pi_{2\lambda}$.

Since $\pi_{2\lambda} \hat{\otimes} \mathrm{id} \subset S^2\pi_\lambda \hat{\otimes} \Lambda^2(\overset{1}{o}) \subset \Lambda^2\pi$, (2.7) implies that $C_{\chi, B_G} = \mathrm{Id}$. If $\overset{1}{o}$ has dominant weight λ_0 and $X_0 \in sp(1) \otimes \mathbb{C}$ with $\lambda_0(X_0) = 1$, then (1.8) implies that

$$B_G(X_0, X_0) = B_{Sp(1)}(X_0, X_0) - 4n = -4n - 8 = -4(n+2).$$

Since $\delta_0 = \lambda_0$, we have

$$B^*_G(\lambda_0, \lambda_0 + 2\delta_0) = 3\, B^*_G(\lambda_0, \lambda_0) = \frac{-3}{4(n+2)}$$

and hence

$$C_{\pi_\lambda, B_G} = \left(\frac{1}{2} - \frac{3}{4(n+2)}\right)\mathrm{Id} = \frac{2n+1}{4(n+2)}\,\mathrm{Id}.$$

Together with $B'_{Sp(n)}(A, B) = -\mathrm{tr}(AB)$, (1.8) implies that $B'_{Sp(n)}|\mathfrak{h} = 1/2(B_G - B_H)$. If $\mathfrak{h}$ is simple, we show in [23] that $B_H = (1 - n(2n+1)/\dim H\,(n+2))\, B_G$. Therefore,

$$B'_{Sp(n)}|\mathfrak{h} = \left(\frac{n(2n+1)}{2(n+2)\dim H}\right) B_G,$$

so that $E(\chi) = (2(n+2)\dim H)/n(2n+1)$ and $E(\pi_\lambda) = \dim H/2n$.

Conversely every non-symmetric strongly isotropy irreducible quotient of $Sp(n)$ arises in this fashion from compact quaternionic symmetric spaces.

(C) $L = SO(n)$. — Let G/H be an n-dimensional real symmetric space with $\mathfrak{h}$ simple. Its isotropy representation is of the form π_λ. Then $SO(n)/\pi_\lambda(H)$ is strongly isotropy irreducible with isotropy representation χ. Either there exists only one simple root α with $(\lambda, \alpha) \neq 0$, in which case $\chi = \pi_{2\lambda - \alpha}$, or there exist two such simple roots, in which case $(\mathfrak{h}, \pi_\lambda) = (su(k), \mathrm{ad})$, $n = k^2 - 1$, and $\chi \otimes \mathbb{C} = \pi_{2\lambda-\alpha} \oplus \pi^*_{2\lambda-\alpha}$. In either case we have $C_{\chi, B_G} = \mathrm{Id}$ and $C_{\pi_\lambda, B_G} = (1/2)\,\mathrm{Id}$. From [23], we have $B_H = (1 - n/2\dim H)\, B_G$ and since $B'_{SO(n)}(A, B) = -1/2\,\mathrm{tr}(AB)$ we have $B'_{SO(n)}|\mathfrak{h} = 1/2(B_G - B_H) = (n/4\dim H)\, B_G$.

TABLE IV

G	π_λ	$(\lambda, \lambda+2\delta)$
$SU(n)$	$\overset{2}{o}—o—\ldots—o$	$2(n-1)(n+2)/n$
$SU(n)$	$Ad=\overset{1}{o}—o—\ldots—o—\overset{1}{o}$	$2n$
$SU(2)$	$\overset{k}{o}$	$(1/2)k^2+k$
$SU(3)$	$\overset{k}{o}—\overset{l}{o}$	$(2/3)(k^2+kl+l^2)+2(k+l)$
$Sp(n)$	$Ad=\overset{2}{\bullet}—\bullet—\ldots—\bullet=o$	$2(n+1)$
$SO(n)$	$S^2\rho_n-id=\overset{2}{o}—o—o—\ldots$	$2n$

TABLE V

Non-symmetric strongly isotropy irreducible quotients of the classical groups

L/H	π	χ
$SU(n)/\pi_\lambda(H)$, H simple	$\pi=\pi_\lambda$ $E(\pi)=\dim H/n$	$\chi=\pi_{\lambda+\lambda^*}$ $E(\chi)=2\dim H/(n-1)$
$SU(pq)/\pi(SU(p)\cdot SU(q))$ $1<p\leqq q$, $(p,q)\neq(2,2)$	$\pi=\mu_p\hat{\otimes}\mu_q$ $E(\pi)=(1/q)((p^2-1)/p)+(1/p)((q^2-1)/q)$	$\chi=ad_{su(p)}\hat{\otimes}ad_{su(q)}$ $E(\chi)=(1/q)(2p)+(1/p)(2q)$
$Sp(n)/\pi(H)$, H simple	$\pi=\pi_\lambda$ $E(\pi)=\dim H/2n$	$\chi=\pi_{2\lambda}$ $E(\chi)=2(n+2)\dim H/(2n^2+n)$
$Sp(n)/\pi(SO(n)\cdot Sp(1))$, $n\geqq 3$	$\pi=\rho_n\hat{\otimes}\nu_2$ $E(\pi)=(1/4)(n-1)+(1/n)(3/2)$	$\chi=(S^2\rho_n-id)\hat{\otimes}ad_{sp(1)}$ $E(\chi)=(1/4)(2n)+(1/n)(4)$
$SO(n)/\pi(H)$, H simple	$\pi=\pi_\lambda$ $E(\pi)=2\dim H/n$	$\chi=\pi_{2\lambda-\alpha}$ or $\pi_{2\lambda-\alpha}\oplus\pi_{2\lambda-\alpha^*}$ $E(\chi)=4\dim H/n$
$SO(4n)/\pi(Sp(n)Sp(1))$, $n\geqq 2$	$\pi=\nu_{2n}\hat{\otimes}\nu_2$ $E(\pi)=(n+1/2)+(1/n)(3/2)$	$\chi=(\Lambda^2\nu_{2n}-id)\hat{\otimes}ad_{sp(1)}$ $E(\chi)=(2n)+(1/n)(4)$
$SO(7)/\pi(G_2)$	$\pi=\overset{1}{o}\equiv\bullet$, $E(\pi)=4$	$\chi=\overset{1}{o}\equiv\bullet$, $E(\chi)=4$

Table VI

G/K	π	$\Lambda^2\pi=ad_\mathfrak{h}\oplus\chi$
$SO(p+q)/SO(p)\cdot SO(q)$ $p\geqq q>1$	$\pi=\rho_p\hat{\otimes}\rho_q$ $E(\pi)=(1/q)(p-1)+(1/p)(q-1)$	$\chi=[ad_{so(p)}\hat{\otimes}(S^2\rho_q-id)]$ $\oplus[(S^2\rho_p-id)\hat{\otimes}ad_{so(q)}]$ $E(\chi)=(1/q)(2(p-2))+(1/p)(2q)$, $(1/q)(2p)+(1/p)(2(q-2))$
$Sp(p+q)/Sp(p).Sp(q)$ $p\geqq q>1$	$\pi=\nu_{2p}\hat{\otimes}\nu_{2q}$ $E(\pi)=(1/2q)(p+1/2)+(1/2p)(q+1/2)$	$\chi=[(\Lambda^2\nu_{2p}-id)\hat{\otimes}ad_{sp(q)}]$ $\oplus[ad_{sp(p)}\hat{\otimes}(\Lambda^2\nu_{2q}-id)]$ $E(\chi)=(1/2q)(2p)+(1/2p)(2(q+1))$, $(1/2q)(2(p+1))+(1/2p)(2q)$

Thus $E(\chi)=(4\dim H)/n$ and $E(\pi_\lambda)=(2\dim H)/n$. The only non-symmetric strongly isotropy irreducible spaces of the form $SO(n)/H$ with H non-simple are $SO(4k)/Sp(k)\cdot Sp(1)$, $k\geqq 2$. Conversely, every non-symmetric strongly isotropy irreducible quotient of $SO(n)$ arises this way except for $(\mathfrak{h}, \pi_\lambda)=(G_2, o\overset{1}{\equiv}\bullet)$, where $\chi=\ o\overset{1}{\equiv}\bullet$.

In Table V we list π_λ and χ for the non-symmetric strongly isotropy irreducible quotients of the classical groups. If the subgroup is non-simple, the index and normalized Casimir constants for each simple factor can easily be read off from the table, and hence also the Casimir constants with respect to any other bi-invariant metric on $\mathfrak{h}$. (This will be useful in Chapter 3.) Table VI supplies the same information for the real and quaternionic Grassmannians, for which the above construction does not yield strongly isotropy irreducible spaces.

4. IRREDUCIBLE SUMMANDS IN $\Lambda^2\pi_\lambda$ AND $S^2\pi_\lambda$. — In the next chapter we need to know irreducible summands in $\Lambda^2\pi_\lambda$ other than those given by (2.1). Here we describe them. Let $\mathfrak{h}$ be a compact Lie algebra, and α, β be simple roots of $\mathfrak{h}$. Then we call $\alpha_1, \ldots, \alpha_k$ a *chain of simple roots connecting* α *and* β if $\alpha_1=\alpha$, $\alpha_k=\beta$, $(\alpha_i, \alpha_{i+1})\neq 0$, and $(\alpha_i, \alpha_j)=0$ whenever $j\geqq i+2$. Such chains were first considered by Dynkin ([9], p. 266).

(2.8) PROPOSITION. — *Let π_λ be an effective irreducible representation of $\mathfrak{h}$ with $(\lambda, \alpha)\neq 0$, $(\lambda, \beta)\neq 0$ for two distinct simple roots α and β of $\mathfrak{h}$, and let $\alpha_1, \ldots, \alpha_k$ be a chain of simple roots connecting α and β with the additional property that $(\lambda, \alpha_i)=0$ for $2\leqq i\leqq k-1$. Then $\pi_{2\lambda-\alpha_1-\ldots-\alpha_k}\subset\Lambda^2\pi_\lambda$ with multiplicity 1. Furthermore, if π is orthogonal, then $\pi_{2\lambda-\alpha_1-\ldots-\alpha_k}\subset \mathrm{ad}_{\mathfrak{h}}$ unless $\pi=\mathrm{ad}_{su(k+1)}$.*

Proof. — Let v be a dominant weight vector of π_λ. Then it follows from the hypotheses and any one of the standard formulas for the multiplicities of weights of π_λ that $\lambda-\alpha$, $\lambda-\alpha_1-\alpha_2, \ldots, \lambda-\alpha_1-\alpha_2-\ldots-\alpha_{k-1}$, $\lambda-\alpha_k$, $\lambda-\alpha_{k-1}-\alpha_k, \ldots, \lambda-\alpha_2-\alpha_3-\ldots-\alpha_k$, and $\lambda-\alpha_1-\alpha_2-\ldots-\check{\alpha}_i-\ldots-\alpha_k$ (α_i deleted) have multiplicitly one and that $\lambda-\alpha_1-\alpha_2-\ldots-\alpha_{k-1}-\alpha_k$ has multiplicity k. The corresponding weight vectors are then of the form $v_i=X_{-\alpha_i}X_{-\alpha_{i-1}}\ldots X_{-\alpha_1}v$, $w_i=X_{-\alpha_{k+i-1}}\ldots X_{-\alpha_k}v$, $(1\leqq i\leqq k-1)$, and

$$y_i=X_{-\alpha_k}\ldots X_{-\alpha_{i+1}}X_{-\alpha_{i-1}}\ldots X_{-\alpha_2}X_{-\alpha_1}v \quad (2\leqq i\leqq k-1).$$

A basis for the weight vectors with weight $\lambda-\alpha_1-\ldots-\alpha_k$ are then $z_i=X_{-\alpha_i}y_i$.

Let $A=\{z\in\pi_\lambda\otimes\pi_\lambda \mid z \text{ has weight } 2\lambda-\alpha_1-\ldots-\alpha_k\}$. A basis for $A\cap\Lambda^2(\pi_\lambda)$ is given by $v\wedge z_1$, $v\wedge z_2, \ldots, v\wedge z_k$, $v_1\wedge w_{k-1}$, $v_2\wedge w_{k-2}, \ldots, v_{k-1}\wedge w_1$, and hence $A\cap\Lambda^2(\pi_\lambda)$ has dimension $2k-1$. On the other hand, the only possible representation in $\Lambda^2(\pi_\lambda)$ in which $2\lambda-\alpha_1-\ldots-\alpha_k$ can be a non-dominant weight vector are $\pi_{2\lambda-\alpha_1}$ and $\pi_{2\lambda-\alpha_k}$. But as before we see that $2\lambda-\alpha_1-\ldots-\alpha_k$ has multiplicity $k-1$ in $\pi_{2\lambda-\alpha_1}$ and $\pi_{2\lambda-\alpha_2}$. Hence there is one linearly independent weight vector in $A\cap\Lambda^2(\pi_\lambda)$ not accounted for, and this must be a dominant weight vector for an irreducible summand $\pi_{2\lambda-\alpha_1-\ldots-\alpha_k}$ in $\Lambda^2(\pi_\lambda)$.

If π_λ is orthogonal and $\pi_{2\lambda-\alpha_1-\ldots-\alpha_k} \subset \mathrm{ad}_\mathfrak{h}$, then $\mu = 2\lambda - \alpha_1 - \ldots - \alpha_k$ is the maximal root of a simple ideal of $\mathfrak{h}$ and $(\mu, \alpha_1) > 0$, $(\mu, \alpha_k) > 0$. By the effectiveness of π_λ, $\mathfrak{h}$ must be simple and $\mathfrak{h} = su(k+1)$ follows immediately. ■

Remarks. — (*a*) By a similar argument one can show that under the same conditions $\pi_{2\lambda-\alpha_1-\ldots-\alpha_k} \subset S^2(\pi_\lambda)$ with multiplicity 1.

(*b*) One can obtain further irreducible summands in $\Lambda^2\pi_\lambda$ and $S^2\pi_\lambda$ under other hypotheses using similar methods. For example, if $\lambda^\alpha \geqq 3$, then besides $\pi_{2\lambda-\alpha} \subset \Lambda^2(\pi_\lambda)$ we also have $\pi_{2\lambda-3\alpha} \subset \Lambda^2(\pi_\lambda)$. Likewise, if $\lambda^\alpha \geqq 2$ besides $\pi_{2\lambda} \subset S^2(\pi_\lambda)$ we also have $\pi_{2\lambda-2\alpha} \subset S^2(\pi_\lambda)$. Such information considerably simplifies the classification of strongly isotropy irreducible quotients of $SO(n)$ and $Sp(n)$. (Compare Lemma 7.4 and Theorem 8.1 in [25].)

(*c*) Irreducible summands other than $\pi_{\lambda+\rho}$ in $\pi_\lambda \otimes \pi_\rho$ were studied by Dynkin (Theorem 3.1 in [9]). In the special case where $\rho = \lambda^*$, he showed that if $\alpha_1, \ldots, \alpha_k$ is a chain of simple roots joining α_1, α_k such that in addition $(\lambda, \alpha_1) \neq 0$, $(\lambda, \alpha_2) = \ldots = (\lambda, \alpha_k) = 0$, $(\lambda^*, \alpha_1) = \ldots = (\lambda^*, \alpha_{k-1}) = 0$, $(\lambda^*, \alpha_k) \neq 0$, then

$$\pi_{\lambda+\lambda^*-\alpha_1-\ldots-\alpha_k} \subset \pi_\lambda \otimes \pi_\lambda^*$$

with multiplicity 1.

Hence if $SU(n)/\pi_\lambda(H)$ is isotropy irreducible (and is positive dimensional), which by (2.1) is equivalent to $\pi_\lambda \otimes \pi_\lambda^* = \mathrm{id} \oplus \mathrm{ad}_\mathfrak{h} \oplus \pi_{\lambda+\lambda^*}$, the above implies that $\mathrm{ad}_\mathfrak{h} = \pi_{\lambda+\lambda^*-\alpha_1-\ldots-\alpha_k}$. By looking at each simple Lie algebra one can immediately enumerate the possibilities for $(\mathfrak{h}, \pi_\lambda)$. This gives a short proof of Theorem 6.1 in [25].

CHAPTER THREE

Quotients of the classical Lie groups

We will now classify the Einstein metrics among $(G/H, g_B)$, where G is a classical, compact, connected, simple Lie group. H will be described by an almost faithful representation $\pi: H \to G$. If χ is the isotropy representation of G/H, the Einstein condition is equivalent to $C_{\chi, B|\mathfrak{h}} = a \cdot \mathrm{Id}$ by (1.7). In each of the following sections we begin by classifying those homogeneous spaces $G/\pi(H)$ for which $C_{\chi, Q} = a\,\mathrm{Id}$ for *some* (positive definite) bi-invariant metric Q on $\mathfrak{h}$ ((3.1), (3.4), (3.6), (3.8)). These results are of independent interest; in particular (3.8) gives a new characterization of symmetric spaces (not necessarily irreducible) in terms of their isotropy representations. Then we specialize to the case where $Q = B|\mathfrak{h}$ and use results in Chapter 2 to give the classification of normal homogeneous Einstein metrics on quotients of the classical groups. All dimensions will be taken over the complex numbers unless otherwise stated.

1. THE UNITARY CASE

(3.1) THEOREM. — *Let π be an almost faithful n-dimensional unitary representation of* H, *and let χ be the isotropy representation of* $U(n)/\pi(H)$, *i.e.*, $\pi \otimes \pi^* = \mathrm{ad}_{\mathfrak{h}} \oplus \chi$. *If* $C_{\chi, Q} = a \cdot \mathrm{Id}$ *for some constant a and some bi-invariant metric* Q *on* $\mathfrak{h}$, *then one of the following holds:*

(*a*) *there exists a hermitian symmetric space* K/H *whose isotropy representation is* $[\pi]_{\mathbb{R}}$;

(*b*) $H = \mathrm{Sp}(m) \cdot S^1$ *and* $\pi = \nu_{2m} \hat{\otimes} \varphi$, *so that* $U(2m)/\pi(H) = SU(2m)/\mathrm{Sp}(m)$, $m > 1$.

(*c*) $H = (\mathrm{Sp}(m) \cdot S^1) \cdot H_2$ *with* $\pi = [(\nu_{2m} \hat{\otimes} \varphi) \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \pi_2]$, $m > 1$, *and* $[\pi_2]_{\mathbb{R}}$ *is the isotropy representation of an irreducible hermitian symmetric space.*

Proof. — Note first that H cannot be semi-simple because then $\pi(H) \subset SU(n) \subset U(n)$ and so χ contains a trivial representation, contradicting $C_{\chi, Q} = a\,\mathrm{Id}$.

We can therefore assume that $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r \oplus \mathfrak{t}$, where $\mathfrak{h}_i$ is simple, and $\mathfrak{t} = \mathbb{R}^k$ with $k \geqq 1$. Let $\pi_1 \hat{\otimes} \ldots \hat{\otimes} \pi_r \hat{\otimes} \varphi$ be an irreducible summand of π. Then φ is 1-dimensional, say with dominant weight λ_φ. Let $\{\lambda_\varphi\}$ be the set of dominant weights of t appearing in the irreducible summands of π. $\{\lambda_\varphi\}$ must contain a basis for $\mathfrak{t}^* \otimes \mathbb{C}$ since π is faithful. Hence the number of irreducible summands of π is at least k. On the other hand, each irreducible summand of π contributes to $\pi \otimes \pi^*$ the summand

$$(\pi_1 \hat{\otimes} \ldots \hat{\otimes} \pi_r \hat{\otimes} \varphi) \otimes (\pi_1^* \hat{\otimes} \ldots \hat{\otimes} \pi_r^* \hat{\otimes} \varphi^*) = (\pi_1 \otimes \pi_1^*) \hat{\otimes} \ldots \hat{\otimes} (\pi_r \otimes \pi_r^*) \hat{\otimes} \mathrm{id},$$

which contains a trivial representation of $\mathfrak{h}$. Since this cannot lie in χ by hypothesis, and since $\mathrm{ad}_{\mathfrak{h}}$ contains exactly k trivial representations, π must contain exactly k irreducible summands of the form $\pi_1 \hat{\otimes} \ldots \hat{\otimes} \pi_r \hat{\otimes} \varphi$ with $\{\varphi\}$ linearly independent.

Next we show that for each simple factor $\mathfrak{h}_i$ in $\mathfrak{h}$ there exists a unique irreducible summand of π which remains non-trivial when restricted to $\mathfrak{h}_i$. If $\pi_1 \hat{\otimes} \ldots \pi_i \ldots \hat{\otimes} \varphi$ and $\tilde{\pi}_1 \hat{\otimes} \ldots \tilde{\pi}_i \ldots \hat{\otimes} \tilde{\pi}_r \hat{\otimes} \tilde{\varphi}$ in π are such that $\pi_i \neq \mathrm{id}$, $\tilde{\pi}_i \neq \mathrm{id}$, then

$$(\pi_1 \otimes \tilde{\pi}_1^*) \hat{\otimes} \ldots (\pi_i \otimes \tilde{\pi}_i^*) \ldots \hat{\otimes} (\varphi \otimes \tilde{\varphi})$$

lies in χ since it cannot be contained in $\mathrm{ad}_{\mathfrak{h}}$ in view of $\varphi \otimes \tilde{\varphi} \neq \mathrm{id}$. Now $\pi_i \otimes \tilde{\pi}_i^*$ is reducible since $\mathfrak{h}_i$ is simple. Let $\pi_i = \pi_\lambda$ and $\tilde{\pi}_i = \pi_\mu$. Then $\pi_i \otimes \tilde{\pi}_i^*$ contains $\pi_{\lambda + \mu^*}$ and another irreducible representation whose dominant weight is of the form $\lambda + \mu^* - \Sigma n_i \alpha_i$. By (2.2) it has a smaller Casimir constant than $\pi_{\lambda + \mu^*}$. This contradicts $C_{\chi, Q} = a \cdot \mathrm{Id}$.

The above shows that after re-grouping and re-numbering we can write

$$\mathfrak{h} = (\mathfrak{h}_1 \oplus \mathfrak{t}_1) \oplus (\mathfrak{h}_2 \oplus \mathfrak{t}_2) \oplus \ldots \oplus (\mathfrak{h}_k \oplus \mathfrak{t}_k)$$

(where $\mathfrak{h}_i$ is semi-simple but not necessarily simple, and $\mathfrak{t}_i$ are 1-dimensional with $\mathfrak{t} = \mathfrak{t}_1 \oplus \ldots \oplus \mathfrak{t}_k$) and

$$\pi = \bigoplus_{i=1}^{k} [\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \pi_i \hat{\otimes} \ldots \hat{\otimes} \mathrm{id}],$$

where π_i is an n_i-dimensional faithful, irreducible, non-self-contragredient representation of $\mathfrak{h}_i \oplus \mathfrak{t}_i$. Then

$$\pi \otimes \pi^* = \bigoplus_{i=1}^{k} [\mathrm{id} \hat{\otimes} \dots \hat{\otimes} (\mathrm{id} \oplus \mathrm{ad}_{\mathfrak{h}_i} \oplus \chi_i) \hat{\otimes} \dots \hat{\otimes} \mathrm{id}]$$

$$\oplus \bigoplus_{i<j} [\mathrm{id} \hat{\otimes} \dots \hat{\otimes} \pi_i \hat{\otimes} \dots \hat{\otimes} \pi_j^* \hat{\otimes} \dots \hat{\otimes} \mathrm{id}]$$

where $\pi_i \otimes \pi_i^* = \mathrm{id} \oplus \mathrm{ad}_{\mathfrak{h}_i} \oplus \chi_i$. Hence

$$\chi = \bigoplus_i [\mathrm{id} \hat{\otimes} \dots \hat{\otimes} \chi_i \hat{\otimes} \dots \hat{\otimes} \mathrm{id}] \oplus \bigoplus_{i<j} [\mathrm{id} \hat{\otimes} \dots \hat{\otimes} \pi_i \hat{\otimes} \dots \hat{\otimes} \pi_j^* \hat{\otimes} \dots \hat{\otimes} \mathrm{id}]_{\mathbb{R}}.$$

If λ is the dominant weight of π_i, then $\pi_{\lambda+\lambda^*} \subset \pi_i \otimes \pi_i^*$, and by (2.1 *c*), $\pi_{\lambda+\lambda^*} \subset \chi_i$ unless $(\mathfrak{h}_i \oplus \mathfrak{t}_i, \pi_i) = (u(m), \mu_m)$. In the latter case $[\mu_m]_{\mathbb{R}}$ is the isotropy representation of the irreducible hermitian symmetric space $\mathrm{SU}(m+1)/\mathrm{S}(\mathrm{U}(m)\cdot\mathrm{U}(1))$. If $\pi_{\lambda+\lambda^*} \subset \chi_i$, then any other irreducible summand of χ_i has dominant weight $\lambda+\lambda^*-\Sigma n_i \alpha_i$, and hence has a smaller Casimir constant by (2.2). Thus $\chi_i = \pi_{\lambda+\lambda^*}$, i.e., $(u(n_i), \mathfrak{h}_i \oplus \mathfrak{t}_i)$ is strongly isotropy irreducible. By section 2.3 (A), if this isotropy irreducible space is non-symmetric, then $[\pi_i]_{\mathbb{R}}$ is the isotropy representation of an irreducible hermitian symmetric space. If it is symmetric, then since H_i is semi-simple, either $\mathrm{SU}(n_i)/\mathrm{H}_i = \mathrm{SU}(m)/\mathrm{SO}(m)$ or $\mathrm{SU}(2m)/\mathrm{Sp}(m)$. If instead we consider the ineffective quotients $\mathrm{U}(n_i)/\mathrm{H}_i \cdot \mathrm{S}^1$, then in the first case $[\pi_i]_{\mathbb{R}}$ is the isotropy representation of the hermitian symmetric space $\mathrm{SO}(m+2)/\mathrm{SO}(m)\cdot\mathrm{SO}(2)$. Thus we have shown that for the π_i's occurring in π, $[\pi_i]_{\mathbb{R}}$ is the isotropy representation of some irreducible hermitian symmetric space unless

$$(\mathfrak{h}_i \oplus \mathfrak{t}_i, \pi_i) = (sp(m) \oplus \mathbb{R}, \nu_{2m} \hat{\otimes} \varphi), \qquad m > 1.$$

If for all i, $[\pi_i]_{\mathbb{R}}$ is the isotropy representation of an irreducible hermitian symmetric space $\mathrm{K}_i/\mathrm{H}_i \cdot \mathrm{S}^1$, where K_i is the connected isometry group, then we are in case (*a*) with $\mathrm{K} = \mathrm{K}_1 \times \dots \times \mathrm{K}_k$ and $\mathrm{H} = (\mathrm{H}_1 \times \mathrm{S}^1) \times \dots \times (\mathrm{H}_k \times \mathrm{S}^1)$ (at least locally). Notice that for K and H just described, on $\mathrm{H}_i \times \mathrm{S}^1$ we may take the bi-invariant metric $\mathrm{B}_{\mathfrak{t}_i} \mid \mathfrak{h}_i \oplus \mathfrak{t}_i$ and let Q be the orthogonal sum of these metrics. By (2.6) and (2.7), $\mathrm{C}_{\pi_i, \mathrm{Q}} = 1/2\,\mathrm{Id}$ and $\mathrm{C}_{\chi_i, \mathrm{Q}} = \mathrm{Id}$, so that $\mathrm{C}_{\chi, \mathrm{Q}} = \mathrm{Id}$. This gives the converse of 3.1 (*a*).

Next if for some i, $\pi_i = \nu_{2m} \hat{\otimes} \varphi$ with $m > 1$, then $\chi_i = (\Lambda^2 \nu_{2m} - \mathrm{id}) \hat{\otimes} \mathrm{id}$. With respect to the normalized metric $(\ ,\ )$ on $sp(m)$, the Casimir constants are $m+(1/2)$ (for ν_{2m}) and $2m$ (for $\Lambda^2 \nu_{2m} - \mathrm{id}$). Thus, with respect to any bi-invariant metric on $\mathfrak{h}_i \oplus \mathfrak{t}_i$, $2\,\mathrm{E}(\pi_i) > \mathrm{E}(\chi_i)$. This in particular shows that $[\pi_i]_{\mathbb{R}}$ cannot be the isotropy representation of a hermitian symmetric space. Notice also that $\mathrm{E}(\pi_i)/\mathrm{E}(\chi_i)$ can be any number in $(1/2+1/(4m), \infty)$ for an appropriate choice of metric on $\mathfrak{t}_i$. This shows that such a π_i cannot occur if $k \geqq 3$ since (with respect to Q) we must then have $\mathrm{E}(\chi_i) = 2\,\mathrm{E}(\pi_i)$. If $k=1$ we obtain (*b*). If $k=2$ and $\chi_1 \neq 0$, $\chi_2 \neq 0$, we must have $\mathrm{E}(\chi_1) = \mathrm{E}(\chi_2) = \mathrm{E}(\pi_1) + \mathrm{E}(\pi_2)$, or equivalently, $\mathrm{E}(\pi_1)/\mathrm{E}(\chi_1) + \mathrm{E}(\pi_2)/\mathrm{E}(\chi_2) = 1$. Hence at most one of the π_i, say π_1, can be $\nu_{2m} \hat{\otimes} \varphi$, so then $[\pi_2]_{\mathbb{R}}$ must be the isotropy representation of an irreducible hermitian symmetric space, which is case (*c*). ■

(3.2) *Remark.* — We now examine the possibilities for $(\mathfrak{h}_2, \pi_2)$ in (3.1 *c*). Clearly, $(u(k), \mu_k)$ is possible because $\chi_2 = 0$, and so an appropriate choice of Q gives $E(\chi_1) = E(\pi_1) + E(\pi_2)$.

If $\chi_2 \neq 0$, then $E(\pi_1)/E(\chi_1) + E(\pi_2)/E(\chi_2) = 1$ can only be satisfied if $E(\pi_2)/E(\chi_2) < 1/2 - 1/(4m)$ for some bi-invariant metric on $\mathfrak{h}_2 \oplus \mathfrak{t}_2$. If $\mathfrak{h}_2$ is simple, it follows from Table V that $E(\pi_2)/E(\chi_2) = 1/2 - 1/(2\dim\pi_2) + \varepsilon$ for any $\varepsilon > 0$ by suitably scaling the metric on $\mathfrak{t}_2$. Hence $(\mathfrak{h}_2, \pi_2)$ is admissible iff $\dim\pi_2 < 2m$. If $\mathfrak{h}_2$ is not simple, then $\mathfrak{h}_2 = su(p) \oplus su(q)$, $p \leqq q$. From Table V, we see that for an appropriate choice of bi-invariant metric, $E(\pi_2)/E(\chi_2) \leqq 1/2 - 1/(2p^2) + \varepsilon$, for any $\varepsilon > 0$. Hence $(\mathfrak{h}_2, \pi_2)$ is admissible iff $p^2 < 2m$.

(3.3) *Remark.* — The converse of Theorem (3.1) is true provided that in case (*c*), $(\mathfrak{h}_2, \pi_2)$ is one of the admissible pairs in (3.2). The construction of Q is obvious by the discussion in (3.2) and the proof of (3.1).

(3.4) COROLLARY. — *Let π be n-dimensional almost faithful unitary representation of* H, *and let χ be the isotropy representation of* $SU(n)/\pi(H)$, *i. e.*, $\pi \otimes \pi^* = \mathrm{id} \oplus \mathrm{ad}_H \oplus \chi$. *If* $C_{\chi, Q} = a\,\mathrm{Id}$ *for some constant a and some bi-invariant metric Q on* $\mathfrak{h}$, then one of the following holds:

(a) there exists a hermitian symmetric space $K/H \cdot S^1$ *whose isotropy representation is* $[\pi \hat{\otimes} \varphi]_{\mathbb{R}}$ (φ *is given by the inclusion of* S^1 *into the center of* $U(n)$);

(*b*) $H = Sp(m)$ *and* $\pi = \nu_{2m}$;

(*c*) $H = Sp(m) \cdot H_2$ *with* $\pi = [\nu_{2m} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \pi_2]$ *and* $[\pi_2]_{\mathbb{R}}$ *is the isotropy representation of an irreducible hermitian symmetric space.*

Proof. — Set $\tilde{H} = H \cdot S^1$ and $\tilde{\pi} = \pi \hat{\otimes} \varphi$, and apply the previous theorem. ■

Next we determine those homogeneous spaces $U(n)/\pi(H)$ for which $C_{\chi, B'_{u(n)}} = a \cdot \mathrm{Id}$.

For the rest of this section $E(\pi_\lambda)$ will stand for $-B'^{*}_{u(n)}(\lambda, \lambda + 2\delta)$. The corresponding determination for $SU(n)/\pi(H)$ follows as before.

(3.5) THEOREM. — *If π is an n-dimensional, almost faithful unitary representation of* H, *and if* $(U(n)/\pi(H), g_B)$ *is Einstein but not strongly isotropy irreducible, then either:*

(*a*) $\mathfrak{h} = u(m) \oplus \ldots \oplus u(m)$ (*k times*), $n = km$, $m \geqq 1$, $k > 2$, *and*

$$\pi = \oplus\,(\mathrm{id} \hat{\otimes} \ldots \hat{\otimes}\, \mu_m \hat{\otimes} \ldots \hat{\otimes}\, \mathrm{id}),$$

or;

(*b*) $\mathfrak{h} = u(l) \oplus s(u(p) \oplus u(q))$, $n = pq + l$, $p \geqq 2$, $q \geqq 2$, $l \geqq 3$,

$$\pi = [\mu_l \hat{\otimes} \mathrm{id} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mu_p \hat{\otimes} \mu_q], \qquad \textit{and} \qquad p^2 + q^2 + 1 = lpq.$$

Proof. — We will apply (3.1) with $Q = B'_{u(n)}$ and use the same notation as that in the proof of (3.1). Since $B'_{u(n)} \mid u(n_i) = B'_{u(n_i)}$, we can compute $E(\pi_i)$ using $B'^{*}_{u(n_i)}$. If $\pi_i = \pi_\lambda \hat{\otimes} \varphi$ with $\chi_i \neq 0$, then $\chi_i = \pi_{\lambda + \lambda^*} \hat{\otimes} \mathrm{id}$. Since $B'_{u(n_i)} \mid \mathfrak{h}_i \oplus \mathfrak{t}_i = -\mathrm{tr}(\pi_i \circ \pi_i)$, we have $E(\varphi) = 1/n_i$, where $n_i = \dim \pi_i$.

We first consider (3.1 *a*), where π_i is the isotropy representation of an irreducible symmetric space. We claim that in this case if $\chi_i \neq 0$ then $E(\chi_i) > 2E(\pi_i)$. Indeed, by Table V and the above remarks, $E(\pi_i) = (\dim \mathfrak{h}_i + 1)/n_i$, $E(\chi_i) = 2\dim \mathfrak{h}_i/(n_i - 1)$ if $\mathfrak{h}_i$ is simple and $E(\pi_i) = (p^2+q^2-1)/pq$, $E(\chi_i) = 2(p^2+q^2)/pq$ for the Grassmannian $SU(p+q)/S(U(p)\cdot U(q))$. The claim follows now by comparing $E(\chi_i)$ and $2E(\pi_i)$ case by case.

If the number k of irreducible summands in π is $\geqq 3$, then we must have $E(\pi_i) = E(\pi_j)$ and whenever $\chi_i \neq 0$ also $E(\chi_i) = 2E(\pi_i)$. Hence all $\chi_i = 0$, i. e.,

$$(\mathfrak{h}_i \oplus \mathfrak{t}_i,\ \pi_i) = (u(m_i),\ \mu_{m_i}).$$

For this case $E(\pi_i) = m_i$, so necessarily $m_i = m_j$, which is case (*a*).

If $k = 2$, then necessarily $\chi_1 = 0$ or $\chi_2 = 0$. Say $\chi_1 = 0$, $\chi_2 \neq 0$, then $\pi_1 = \mu_l$, and $E(\chi_2) = E(\pi_1) + E(\pi_2)$. Since $E(\pi_1) = l$, $E(\chi_2) - E(\pi_2)$ must be an integer, and using the above values, one easily checks that this is only possible if

$$(\mathfrak{h}_2 \oplus \mathfrak{t}_2,\ \pi_2) = (u(p) \oplus u(q),\ \mu_p \hat{\otimes} \mu_q).$$

Now $E(\chi_2) - E(\pi_2) = (p^2+q^2+1)/pq$ and we get case (*b*).

If we are in case (3.1 *c*), then using Chapter 2, $E(\pi_1) = m + 1/2 + 1/(2m)$ and $E(\chi_1) = 2m$. Hence we must have $E(\pi_2)/E(\chi_2) = 1/2 - 1/(4m) - 1/(4m^2)$ for some integer m, and one easily checks that this is impossible. ■

2. The symplectic case

(3.6) Theorem. — *Let π be an almost faithful, symplectic representation of H of complex dimension $2n$, and let χ be the isotropy representation of $Sp(n)/\pi(H)$, i. e., $S^2\pi = ad_{\mathfrak{h}} \oplus \chi$. Then $C_{\chi, Q} = a \cdot Id$ for some constant a and some bi-invariant metric Q on $\mathfrak{h}$ iff one of the following holds:*

(*a*) *$Sp(n)/\pi(H)$ is strongly isotropy irreducible (i. e., the representation $\pi \hat{\otimes}_1 o$ of $H \cdot Sp(1)$ is the isotropy representation of a quaternionic symmetric space or*

$$Sp(n)/\pi(H) = Sp(p+q)/Sp(p)\,Sp(q));$$

(*b*) $H = Sp(m_1) \ldots Sp(m_k)$ *and* $\pi = \bigoplus_{i=1}^{k} [id \hat{\otimes} \ldots \hat{\otimes} \nu_{2m_i} \hat{\otimes} \ldots \hat{\otimes} id]$;

(*c*) $H = Sp(m) \cdot H_2$, *and* $\pi = [\nu_{2m} \hat{\otimes} id] \oplus [id \hat{\otimes} \pi_2]$ *with* $\pi_2 \hat{\otimes}_1 o$ *the isotropy representation of a quaternionic symmetric space.*

Proof. — The irreducible summands in π are either symplectic or occur with their contragredients. As in the unitary case, for each simple factor H_i of H, there exists only one irreducible summand of π excluding its contragredient whose restriction to H_i is non-trivial. Again, using analogous arguments we can write $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$

and $\pi = \bigoplus_{i=1}^{k} [\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \pi_i \hat{\otimes} \ldots \hat{\otimes} id]$, where π_i is a faithful representation of $\mathfrak{h}_i$. If π_i is irreducible, then π_i is symplectic and $\mathfrak{h}_i$ is semi-simple. Otherwise, $\pi_i = \sigma_i \oplus \sigma_i^*$, and $\mathfrak{h}_i$ has a 1-dimensional center since $\mathrm{id} \subset \sigma_i \otimes \sigma_i^* \subset S^2 \pi_i$. It follows that

$$\chi = \oplus [\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \chi_i \hat{\otimes} \ldots \hat{\otimes} \mathrm{id}] \oplus \bigoplus_{i<j} [\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \pi_i \hat{\otimes} \ldots \hat{\otimes} \pi_j \hat{\otimes} \ldots \hat{\otimes} \mathrm{id}],$$

where $S^2 \pi_i = \mathrm{ad}_{\mathfrak{h}_i} \oplus \chi_i$.

First we consider the case where π_i is symplectic with dominant weight λ. Then $\pi_{2\lambda} \subset S^2 \pi_\lambda$, and by (2.1) either $\pi_{2\lambda} \subset \chi_i$ or $\pi_i = \nu_{2m}$. In the latter case $\pi_i \hat{\otimes}^1 \mathrm{o}$ is the isotropy representation of the quaternionic symmetric space $\mathrm{Sp}(m+1)/\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$. If $\pi_{2\lambda} \subset \chi_i$, then since any other irreducible summand in χ_i has the form $\pi_{2\lambda - \sum n_i \alpha_i}$, by (2.2) we must have $\chi_i = \pi_{2\lambda}$, i.e., $\mathrm{Sp}((1/2)\dim \pi_i)/\pi_i(\mathrm{H}_i)$ is strongly isotropy irreducible. If $\mathrm{Sp}((1/2)\dim \pi_i)/\pi_i(\mathrm{H}_i)$ is non-symmetric, then by Chapter 2, $\pi_i \hat{\otimes}^1 \mathrm{o}$ is the isotropy representation of a quaternionic symmetric space. It cannot be symmetric since H_i is semi-simple and π_i is irreducible.

Next we consider the case where $\pi_i = \sigma_i \oplus \sigma_i^*$. Then $S^2 \sigma_i \subset \chi_i$. Again by (2.2) $S^2 \sigma_i$ must be irreducible, which is possible only if $(\mathfrak{h}_i, \pi_i) = (u(m), \mu_m \oplus \mu_m^*)$ (*see* [23]). So $\pi_i \hat{\otimes}^1 \mathrm{o}$ is the isotropy representation of the quaternionic symmetric space $\mathrm{SU}(m+2)/\mathrm{S}(\mathrm{U}(m) \cdot \mathrm{U}(2))$. Hence in all cases $\pi_i \hat{\otimes}^1 \mathrm{o}$ is the isotropy representation of a quaternionic symmetric space.

We now examine which combinations of $(\mathfrak{h}_i, \pi_i)$ can occur. First, observe that for any bi-invariant metric on $\mathfrak{h}_i$, we have $\mathrm{E}(\chi_i) > 2\mathrm{E}(\pi_i)$ if $\chi_i \neq 0$. If $\mathfrak{h}_i$ is semi-simple, this follows from Table V, and for the symmetric space $\mathrm{Sp}(m)/\mathrm{U}(m)$ this follows easily from Tables III and IV.

If all $\chi_i = 0$, i.e., $(\mathfrak{h}_i, \pi_i) = (sp(m_i), \nu_{2m_i})$, we have case (*b*). Conversely, for case (*b*), clearly we can find a bi-invariant metric Q on $\mathfrak{h}$ such that $\mathrm{E}(\pi_i) = \mathrm{E}(\pi_j)$ and so $C_{\chi, Q} = a\,\mathrm{Id}$. If for $i \neq j$, $\chi_i \neq 0$, $\chi_j \neq 0$, we get a contradiction since then $\mathrm{E}(\chi_i) = \mathrm{E}(\chi_j) = \mathrm{E}(\pi_i) + \mathrm{E}(\pi_j)$. Hence at most one $\chi_i \neq 0$, say $\chi_1 \neq 0$, while $\chi_2 = \ldots = \chi_k = 0$. If $k \geq 3$, we must have $\mathrm{E}(\chi_1) = \mathrm{E}(\pi_1) + \mathrm{E}(\pi_i)$ and $\mathrm{E}(\pi_1) = \mathrm{E}(\pi_i)$, which is impossible. If $k = 2$ we must have $\mathrm{E}(\chi_1) = \mathrm{E}(\pi_1) + \mathrm{E}(\pi_2)$, which can clearly be achieved with an appropriate choice of Q since $\mathrm{E}(\chi_1) - \mathrm{E}(\pi_1) > 0$. This is case (*c*). Case (*a*) corresponds to $k = 1$ and the case $\mathrm{Sp}(p+q)/\mathrm{Sp}(p) \cdot \mathrm{Sp}(q)$. ■

Remarks. — Unlike hermitian symmetric spaces, quaternionic symmetric spaces are automatically irreducible. (*a*), (*b*), (*c*) are not mutually exclusive. The possibilities for H_2 in (*c*) do not include all of (*a*) (the missing possibility is $\mathrm{Sp}(p+q)/\mathrm{Sp}(p) \cdot \mathrm{Sp}(q)$).

(3.7) THEOREM. — *If π is a complex $2n$-dimensional almost faithful symplectic representation of* H, *and* $(\mathrm{Sp}(n)/\pi(\mathrm{H}), g_B)$ *is Einstein but not strongly isotropy irreducible, then either*

(*a*) $\mathfrak{h} = sp(m) \oplus \ldots \oplus sp(m)$ (*k times*), $n = km$, $k \geqq 3$, $m \geqq 1$, *and*

$$\pi = \oplus\,[\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \nu_{2m} \hat{\otimes} \ldots \hat{\otimes} \mathrm{id}],$$

or;

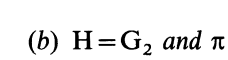
(*b*) $\mathfrak{h} = sp(m) \oplus u(2m-1)$, $n = 3m-1$, $m \geqq 2$, *and* $\pi = [\nu_{2m} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mu_{2m-1}]_{\mathbb{H}}$.

Proof. — We apply (3.6). For (3.6*b*) we observe that $B'_{sp(m_i)} = B'_{sp(n)}$, and hence $E(\pi_i) = m_i + 1/2$. Hence we must have $m_i = m_j$, which yields (*a*). For spaces in (3.6*c*), we need (since $\chi_2 \neq 0$) $E(\chi_2) = E(\pi_1) + E(\pi_2)$. But then $m + 1/2 = E(\pi_1) = E(\chi_2) - E(\pi_2)$, and hence $E(\chi_2) - E(\pi_2) - 1/2$ must be a positive integer. A case by case analysis using Table V shows that this is not satisfied if $\mathfrak{h}_2$ is simple or if $\pi_2 = \rho_n \hat{\otimes} \nu_2$. If $(\mathfrak{h}_2, \pi_2) = (u(k), [\mu_k]_{\mathbb{H}})$, then $E(\pi_2) = (1/2)k$ and $E(\chi_2) = k+1$ since $B'_{sp(k)} \mid u(k) = 2B'_{u(k)}$, $\chi_2 = [S^2 \mu_k]_{\mathbb{R}}$, and $B^*_{u(k)}(\varphi, \varphi) = 1/k$. Thus k must be $2m-1$, which yields (*b*). ■

3. Orthogonal case

(3.8) Theorem. — *Let π be an n-dimensional almost faithful orthogonal representation of* H, *and let χ be the isotropy representation of* $SO(n)/\pi(H)$, *i.e.*, $\Lambda^2 \pi = \mathrm{ad}_{\mathfrak{h}} \oplus \chi$. *Then* $C_{\chi, Q} = a \cdot \mathrm{Id}$ *for some constant a and some bi-invariant metric* Q *on* $\mathfrak{h}$ *iff one of the following holds:*

(*a*) π *is the isotropy representation of a symmetric space;*

(*b*) $H = G_2$ *and* $\pi =$ $(n=7)$ *or* $\pi =$  $(n=8)$;

(*c*) $H = \mathrm{Spin}(7)$, $\pi =$, $n=8$, *or*

$$H = \mathrm{Spin}(7) \cdot SO(m),\ (m \geqq 3),\ \pi = [\text{o—o=}\overset{1}{\bullet} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_m],\ n = m+8.$$

Proof. — $\pi = \pi_1 \oplus \ldots \oplus \pi_k \oplus [\sigma_1 \oplus \sigma_1^*] \oplus \ldots \oplus [\sigma_l \oplus \sigma_l^*]$, where π_i, σ_i are irreducible, $\sigma_i \neq \sigma_i^*$, and π_i are orthogonal. Notice that *one* π_i can be id. This case will be dealt with last.

Case 1: $\mathfrak{h}$ *semi-simple, no* id *in* π. — In this case there are no σ_i's in π since $\Lambda^2(\sigma_i \oplus \sigma_i^*) \supset \sigma_i \otimes \sigma_i^*$, which contains id, so that $\mathrm{id} \subset \chi$ by semi-simplicity. As in the unitary and symplectic cases, for each simple factor of H, there is exactly one irreducible summand of π whose restriction to it is non-trivial. So we may write $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$, $\mathfrak{h}_i$ not necessarily simple, and $\pi = \overset{k}{\underset{i=1}{\oplus}}\,[\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \pi_i \hat{\otimes} \ldots \hat{\otimes} \mathrm{id}]$, where π_i is an irreducible, orthogonal faithful representation of $\mathfrak{h}_i$. Then

$$\chi = \underset{i}{\oplus}\,[\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \chi_i \hat{\otimes} \ldots \mathrm{id}] \oplus \underset{i<j}{\oplus}\,[\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \pi_i \hat{\otimes} \ldots \hat{\otimes} \pi_j \hat{\otimes} \ldots \hat{\otimes} \mathrm{id}]$$

where $\Lambda^2 \pi_i = \mathrm{ad}_{\mathfrak{h}_i} \oplus \chi_i$.

If $\mathfrak{h}_i$ is simple and π_i has dominant weight λ, and if α is a simple root of $\mathfrak{h}_i$ with $(\lambda, \alpha) \neq 0$, then $\pi_{2\lambda-\alpha} \subset \Lambda^2 \pi_i$. By (2.1) either $(\mathfrak{h}_i, \pi_i) = (so(m), \rho_m)$, $\left(so(8), \text{o—o}\begin{smallmatrix}\text{o}\\ \diagup\\ \diagdown\\ \text{o}\end{smallmatrix}\right)$, $(so(7), \text{o—o}\overset{1}{=}\bullet)$, $(G_2, \text{o}\overset{1}{\equiv}\bullet)$, or $\pi_{2\lambda-\alpha} \subset \chi_i$. In the first two cases, π_i is respectively the isotropy representation of the irreducible symmetric space $SO(m+1)/SO(m)$ and $SO(9)/SO(8)$ (after renumbering the roots of $so(8)$). If $\pi_{2\lambda-\alpha} \subset \chi_i$, there are two further cases. If there are two simple roots α, β of $\mathfrak{h}_i$ with $(\lambda, \alpha) \neq 0$, $(\lambda, \beta) \neq 0$, then since $\mathfrak{h}_i$ is simple, there is a chain of simple roots connecting α and β. By (2.8) either $\pi_{2\lambda-\alpha-\alpha_2-\ldots-\alpha_{k-1}-\beta} \subset \chi_i$ in addition to $\pi_{2\lambda-\alpha} \subset \chi_i$, which contradicts our hypothesis by (2.2), or else $\pi_i = \mathrm{ad}(su(m))$, which is the isotropy representation of $SU(m) \times SU(m)/\Delta SU(m)$. If there is only one simple root α of $\mathfrak{h}_i$ with $(\lambda, \alpha) \neq 0$, then any irreducible summand in χ_i has the form $\pi_{2\lambda-\alpha-\Sigma n_i \alpha_i}$, which has a smaller Casimir constant by (2.2). Thus $\chi_i = \pi_{2\lambda-\alpha}$, i.e., $SO(\dim \pi_i)/\pi_i(H_i)$ is strongly isotropy irreducible. Since π_i is assumed to be irreducible, and H_i is simple, $SO(\dim \pi_i)/\pi_i(H_i)$ is non-symmetric. Hence π_i is the isotropy representation of an irreducible symmetric space except when $(\mathfrak{h}_i, \pi_i) = (G_2, \text{o}\overset{1}{\equiv}\bullet)$.

Next let us consider π_i corresponding to semi-simple, non-simple $\mathfrak{h}_i$. By faithfulness, we may assume that $\mathfrak{h}_i = \mathfrak{h}_i' \oplus \mathfrak{h}_i''$ ($\mathfrak{h}_i'$ $\mathfrak{h}_i''$ not necessarily simple), and $\pi_i = \pi_i' \hat{\otimes} \pi_i''$ with π_i' and π_i'' faithful, and both orthogonal or both symplectic.

If π_i' and π_i'' are orthogonal, then

$$\begin{aligned}\Lambda^2(\pi_i' \hat{\otimes} \pi_i'') &= [\Lambda^2 \pi_i' \hat{\otimes} S^2 \pi_i''] \oplus [S^2 \pi_i' \hat{\otimes} \Lambda^2 \pi_i''] \\ &= [(\mathrm{ad}\, \mathfrak{h}_i' \oplus \chi' \hat{\otimes} (i\mathfrak{d} \oplus \Psi'')] \oplus [(i\mathfrak{d} \oplus \Psi') \hat{\otimes} (\mathfrak{ad}\, \mathfrak{h}_i'' \oplus \chi'']\end{aligned}$$

Since Ψ', $\Psi'' \neq 0$, we must have $\chi' = \chi'' = 0$ since $C_{\chi, Q} = a\,\mathrm{Id}$. But this implies that $(\mathfrak{h}_i', \pi_i') = (so(m'), \rho_{m'})$ and $(\mathfrak{h}_i'', \pi_i'') = (so(m''), \rho_{m''})$. Thus $\pi_i = \rho_{m'} \hat{\otimes} \rho_{m''}$ is the isotropy representation of the irreducible symmetric space $SO(m'+m'')/SO(m') \cdot SO(m'')$.

If π_i' and π_i'' are symplectic, then

$$\Lambda^2(\pi_i' \hat{\otimes} \pi_i'') = [(\mathrm{id} \oplus \Psi') \hat{\otimes} (\mathrm{ad}_{\mathfrak{h}_i''} \oplus \chi'')] \oplus [(\mathrm{ad}_{\mathfrak{h}_i'} \oplus \chi') \hat{\otimes} (\mathrm{id} \oplus \Psi'')].$$

If Ψ' and Ψ'' are non-zero, then $\chi' = \chi'' = 0$, and hence $\mathfrak{h}_i' = sp(m')$, $\pi_i' = \nu_{2m'}$ $\mathfrak{h}_i'' = sp(m'')$, $\pi_i'' = \nu_{2m''}$. Thus, $\pi_i = \pi_i' \hat{\otimes} \pi_i''$ is the isotropy representation of the irreducible symmetric space $Sp(m'+m'')/Sp(m') \cdot Sp(m'')$. If Ψ' or Ψ'' is 0, say $\Psi' = 0$, then $\mathfrak{h}_i' = sp(1)$, $\pi_i' = \nu_2$.

In this case

$$\Lambda^2(\pi_i' \hat{\otimes} \pi_i'') = [\mathrm{id} \hat{\otimes} (\mathrm{ad}_{\mathfrak{h}_i''} \oplus \chi'')] \oplus [\mathrm{ad}_{\mathfrak{h}_i'} \hat{\otimes} (\mathrm{id} \oplus \Psi'')].$$

Hence $\mathrm{id} \hat{\otimes} \chi'' \subset \chi_i$. If λ'' is the dominant weight of π_i'', then by (2.1) either $(\mathfrak{h}_i'', \pi_i'') = (sp(m), \nu_{2m})$ or $\pi_{2\lambda''} \subset \chi''$. In the first case, $\pi_i = \pi_i' \hat{\otimes} \pi_i'' = \nu_2 \hat{\otimes} \nu_{2m}$ is the isotropy representation of an irreducible symmetric space since $\mathfrak{h}_i''$ is semi-simple. In

the second case, any other irreducible summand of χ'' is of the form $\pi_{2\lambda''-\Sigma n_i\alpha_i}$, which by (2.2) has a smaller Casimir constant. Thus, $C_{\chi,Q}=a\,\mathrm{Id}$ implies that $\mathrm{Sp}((1/2))\dim\pi_i'')/\pi_i''(\mathfrak{h}_i'')$ is strongly isotropy irreducible and non-symmetric since $\mathfrak{h}_i''$ is semi-simple. By Chapter 2, $\pi_i=\pi_i'\hat{\otimes}\pi_i''=\nu_2\hat{\otimes}\pi_i''$ is the isotropy representation of a quaternionic symmetric space.

Therefore, we have proved that for all i, π_i is the isotropy representation of an irreducible symmetric space unless $(\mathfrak{h}_i, \pi_i)=(G_2, \overset{1}{\circ}\!\!\equiv\!\!\bullet)$ or $(so(7), \circ\!\!-\!\!\circ\!\!=\overset{1}{\bullet})$.

If all π_i come from irreducible symmetric spaces K_i/H_i, by taking their product we obtain case (*a*). Conversely, choose as the bi-invariant metric on $\mathfrak{h}_i$ the restriction to $\mathfrak{h}_i$ of the negative of the Killing form of the connected isometry group of the symmetric space. Since by (2.7), (2.8) $C_{\pi_i,Q}=1/2\,\mathrm{Id}$, $C_{\chi_i,Q}=\mathrm{Id}$, we have $C_{\chi,Q}=\mathrm{Id}$.

For $(G_2, \overset{1}{\circ}\!\!\equiv\!\!\bullet)$ we observe that $\pi_i=\chi_i=\overset{1}{\circ}\!\!\equiv\!\!\bullet$, and hence $E(\pi_i)=E(\chi_i)$ for any bi-invariant metric on $\mathfrak{h}$. For $(so(7), \circ\!\!-\!\!\circ\!\!=\overset{1}{\bullet})$ we have $\chi_i=\overset{1}{\circ}\!\!-\!\!\circ\!\!=\bullet$ and with respect to $-B'_{so(7)}$, $E(\pi_i)=21/4$, $E(\chi_i)=6$, and hence $E(\pi_i)/E(\chi_i)=7/8$ with respect to any bi-invariant metric on $\mathfrak{h}$. If $k\geqq 3$ we must have $E(\chi_i)=2E(\pi_i)$ and hence neither case can occur. If $k=2$, and $\chi_1\neq 0$, $\chi_2\neq 0$, we must have $E(\chi_1)=E(\chi_2)=E(\pi_1)+E(\pi_2)$, i.e., $E(\pi_1)/E(\chi_1)+E(\pi_2)/E(\chi_2)=1$. If one of χ_1 or $\chi_2=0$, say $\chi_2=0$, we must have $E(\chi_1)=E(\pi_1)+E(\pi_2)$. This shows that $(G_2, \overset{1}{\circ}\!\!\equiv\!\!\bullet)$ cannot occur and only one of π_1 or π_2, say π_1, could be $(so(7), \circ\!\!-\!\!\circ\!\!=\overset{1}{\bullet})$. π_2 will then have to be ρ_m for some m or the isotropy representation of an irreducible symmetric space with $E(\chi_2)=8E(\pi_2)$ for some bi-invariant metric on $\mathfrak{h}_2$. We will now show that this second case is impossible. If $\mathfrak{h}_2$ is simple, then with respect to any bi-invariant metric on $\mathfrak{h}_2$ we have $E(\chi_2)=2E(\pi_2)$ by Table V. If π_2 is the isotropy representation of a real Grassmannian or a quaternionic Grassmanian it follows from Table VI that $E(\chi_2)<3E(\pi_2)$ for any bi-invariant metric on $\mathfrak{h}_2$. The only remaining $(\mathfrak{h}_2, \pi_2)$ to consider are of the form $\mathfrak{h}_2=sp(1)\oplus\mathfrak{h}''$ with $\mathfrak{h}''$ simple, $\pi_2=\nu_2\hat{\otimes}\pi''$ with π'' symplectic, and $\mathrm{Sp}((1/2)\dim\pi'')/\pi''(H'')$ strongly isotropy irreducible, non-symmetric. It follows from [23] that $\chi_2=[\mathrm{id}\hat{\otimes}\chi'']\oplus[\mathrm{ad}_{sp(1)}\hat{\otimes}\pi_{2\lambda''-\alpha}]$, where λ'' is the dominant weight of π'' and α is the unique simple root of $\mathfrak{h}''$ with $(\lambda'', \alpha)\neq 0$. By taking the trace of the Casimir operator of $\Lambda^2\pi''=\mathrm{id}\oplus\pi_{2\lambda''-\alpha}$ (*see* [23]), it follows that $E(\pi_{2\lambda''-\alpha})=(2m''/(m''+1))E(\pi'')$, where $m''=\dim_{\mathbb{C}}\pi''$. Since $E(\mathrm{ad}_{sp(1)})=4$, $E(\nu_2)=3/2$, we get $E(\chi_2)<3E(\pi_2)$ for any bi-invariant metric on $\mathfrak{h}_2$. For future reference we note that also $E(\pi_2)<E(\chi_2)$ for any bi-invariant metric on $\mathfrak{h}_2$.

Hence $(G_2, \overset{1}{\circ}\!\!\equiv\!\!\bullet)$ can only occur by itself $(k=1)$, while $(so(7), \circ\!\!-\!\!\circ\!\!=\overset{1}{\bullet})$ can occur by itself or with $(so(m), \rho_m)$ $(k=2)$.

Case 2: $\mathfrak{h}$ non semi-simple, no id *in* π. — We write $\mathfrak{h}=\mathfrak{h}_1\oplus\ldots\oplus\mathfrak{h}_r\oplus\mathfrak{t}$, where $\dim_{\mathbb{R}}\mathfrak{t}=l$. As in the symplectic case, there must be exactly l summands of the form $[\pi_1\hat{\otimes}\ldots\hat{\otimes}\pi_r\hat{\otimes}\varphi]\oplus[\pi_1^*\hat{\otimes}\ldots\hat{\otimes}\pi_r^*\hat{\otimes}\varphi^*]$ with $\varphi\neq\mathrm{id}$. Any other summand in π is orthogonal and of the form $\pi_1\hat{\otimes}\ldots\hat{\otimes}\pi_r\hat{\otimes}\mathrm{id}$. For each simple factor $\mathfrak{h}_i$, there is exactly one summand in π whose restriction to $\mathfrak{h}_i$ is non-trivial. Hence we may re-write

$\mathfrak{h}=\mathfrak{h}'\oplus\mathfrak{h}''$, where $\mathfrak{h}'=\mathfrak{h}_1\oplus\ldots\oplus\mathfrak{h}_k$ ($\mathfrak{h}_i$ semi-simple), $\mathfrak{h}''=\mathfrak{h}_{k+1}\oplus\ldots\oplus\mathfrak{h}_{k+l}$ ($\mathfrak{h}_i$ non-semi-simple with 1-dimensional center), and $\pi=[\pi'\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}(\pi''\oplus\pi''^*)]$, where $\pi'=\bigoplus_{i=1}^{k}[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\pi_i\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}]$ with π_i an orthogonal, faithful representation of $\mathfrak{h}_i$, and $\pi''=\bigoplus_{j=1}^{l}[\mathrm{id}\hat{\otimes}\ldots\hat{\otimes}\sigma_j\hat{\otimes}\ldots\hat{\otimes}\mathrm{id}]$ with $\sigma_j\neq\sigma_j^*$ a faithful representation of $\mathfrak{h}_{j+k}$. Then $\chi=[\chi'\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\chi'']\oplus[\pi'\hat{\otimes}(\pi''\oplus\pi''^*)]$, where χ' and χ'' are respectively the isotropy representations of $\mathrm{SO}(\dim\pi')/\pi'(\mathrm{H}')$ and $\mathrm{SO}(2\dim\pi'')/(\pi''\oplus\pi''^*)(\mathrm{H}'')$.

Since there exists a bi-invariant metric Q on $\mathfrak{h}$ with $C_{\chi,Q}=a.\,\mathrm{Id}$ it follows that $C_{\chi',Q}=a.\,\mathrm{Id}$ and $C_{\chi'',Q}=a.\,\mathrm{Id}$. Since $\mathfrak{h}'$ is semi-simple, by Case 1, either π' is the isotropy representation of a symmetric space or $(\mathfrak{h}',\pi')=$ one of $(\mathrm{G}_2,\ \mathrm{o}\overset{1}{\equiv}\bullet)$, $(so(7),\ \mathrm{o}\text{—}\mathrm{o}\overset{1}{=}\bullet)$, or $(so(7)\oplus so(m),\ [\mathrm{o}\text{—}\mathrm{o}\overset{1}{=}\bullet\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\rho_m])$. Moreover, since

$$\Lambda^2(\pi''\oplus\pi''^*)=\Lambda^2\pi''\oplus\Lambda^2\pi''^*\oplus[\pi''\otimes\pi''^*],\qquad \chi''=\Lambda^2\pi''\oplus\Lambda^2\pi''^*\oplus\tilde{\chi},$$

where $\tilde{\chi}$ is the isotropy representation of $\mathrm{U}(\dim\pi'')/\pi''(\mathrm{H}'')$. Hence we also have $C_{\tilde{\chi},Q}=a\,\mathrm{Id}$. By (3.1), $[\pi']_{\mathbb{R}}$ is the isotropy representation of a hermitian symmetric space, or $(\mathfrak{h}'',\pi'')=$ one of

$$(sp(m)\oplus\mathbb{R},\ \nu_{2m}\hat{\otimes}\varphi)\qquad\text{or}\quad(sp(m)\oplus\mathbb{R}\oplus\mathfrak{k},[\nu_{2m}\hat{\otimes}\varphi\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\mathrm{id}\hat{\otimes}\sigma])$$

where $[\sigma]_{\mathbb{R}}$ is the isotropy representation of an irreducible hermitian symmetric space. To exclude the last two cases, we first observe that $C_{\chi'',Q}=a.\,\mathrm{Id}$ also implies that $C_{\Lambda^2\pi'',Q}=a.\,\mathrm{Id}$. But if π'' contains $\nu_{2m}\hat{\otimes}\varphi$, then $\Lambda^2\pi''$ contains

$$\Lambda^2\nu_{2m}\hat{\otimes}\mathrm{S}^2\varphi=[\mathrm{id}\hat{\otimes}\mathrm{S}^2\varphi]\oplus[(\Lambda^2\nu_{2m}-\mathrm{id})\hat{\otimes}\mathrm{S}^2\varphi]$$

which have different Casimir constants.

If both π' and $[\pi'']_{\mathbb{R}}$ are isotropy representations of a symmetric space, we get 3.8 (*a*). So we next let $(\mathfrak{h}',\pi')=(\mathrm{G}_2,\ \mathrm{o}\overset{1}{\equiv}\bullet)$ or $(so(7),\ \mathrm{o}\text{—}\mathrm{o}\overset{1}{=}\bullet)$, and $[\pi'']_{\mathbb{R}}$ to be the isotropy representation of a hermitian symmetric space. It suffices to consider the case of an irreducible hermitian symmetric space. Let $\mathfrak{h}''=\tilde{\mathfrak{h}}\oplus\mathbb{R}$, $\pi''=\tilde{\pi}\hat{\otimes}\varphi$. If $\tilde{\mathfrak{h}}$ is simple and $\tilde{\pi}=\pi_\lambda$, then we show in [23] that $\Lambda^2\pi_\lambda=\pi_{2\lambda-\alpha}$, where α is the unique simple root with $(\lambda,\alpha)\neq 0$. Taking the trace of Casimir operators we get

$$\mathrm{E}(\pi_{2\lambda-\alpha})=\frac{2(m-2)}{(m-1)}\mathrm{E}(\pi_\lambda),$$

where $m=\dim\pi_\lambda$, for any bi-invariant metric on $\tilde{\mathfrak{h}}$. Since $\Lambda^2\pi''=\Lambda^2\pi_\lambda\hat{\otimes}\mathrm{S}^2\varphi$ and $\mathrm{E}(\mathrm{S}^2\varphi)=4\,\mathrm{E}(\varphi)$ we have $\mathrm{E}(\chi'')<4\,\mathrm{E}(\pi'')$ (and $\mathrm{E}(\pi')<\mathrm{E}(\chi'')$) for any bi-invariant metric. This contradicts

$$\frac{\mathrm{E}(\pi')}{\mathrm{E}(\chi')}+\frac{\mathrm{E}(\pi'')}{\mathrm{E}(\chi'')}=1.$$

If $\mathfrak{h}''$ is non-simple, then we are in the case of complex Grassmannians, and

$$\pi'' = \mu_p \hat{\otimes} \mu_q \hat{\otimes} \varphi.$$

Then

$$\Lambda^2 \pi'' = ([\Lambda^2 \mu_p \hat{\otimes} S^2 \mu_q] \oplus [S^2 \mu_p \hat{\otimes} \Lambda^2 \mu_q]) \hat{\otimes} S^2 \varphi.$$

From Tables III and IV, again it follows that $E(\chi'') < 4E(\pi'')$ and $E(\pi'') < E(\chi'')$. The same argument shows that

$$(\mathfrak{h}', \pi') = (so(7) \oplus so(m), [\mathrm{o—o} \overset{1}{=\!\!\bullet} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_m])$$

is not possible. ■

Case 3: $\mathrm{id} \subset \pi$. — Let $\pi = \pi' \oplus \mathrm{id}$, where π' contains no id. Then

$$\Lambda^2 \pi = \Lambda^2 \pi' \oplus \pi' = \mathrm{ad}_{\mathfrak{h}} \oplus \chi' \oplus \pi'$$

and hence $\chi = \chi' \oplus \pi'$ Therefore, by hypothesis $C_{\chi', Q} = a\,\mathrm{Id}$ and $C_{\pi', Q} = a\,\mathrm{Id}$. In cases 1 and 2, we already studied $C_{\chi', Q} = a\,\mathrm{Id}$. If in addition $C_{\pi', Q} = a\,\mathrm{Id}$, then π' cannot be the isotropy representation of a symmetric space since it follows from Tables V, VI and earlier remarks that $E(\chi') > E(\pi')$ with respect to any bi-invariant metric on $\mathfrak{h}'$. If $(\mathfrak{h}', \pi') = (G_2, \mathrm{o} \overset{1}{\equiv\!\!\bullet})$, then $\chi' = \pi'$, which yields precisely the second case in (3.6(*b*)). If

$$(\mathfrak{h}', \pi') = (so(7), \mathrm{o—o} \overset{1}{=\!\!\bullet}), \qquad \text{or} \quad (so(7) \oplus so(m), [\mathrm{o—o} \overset{1}{=\!\!\bullet} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_m]),$$

we saw already that $E(\chi') \neq E(\pi')$, contradicting $C_{\chi, Q} = a\,\mathrm{Id}$.

This completes the proof of Theorem (3.8). ■

(3.9) THEOREM. — *If π is an n-dimensional almost faithful orthogonal representation of* H, *and* $(\mathrm{SO}(n)/\pi(\mathrm{H}), g_{\mathrm{B}})$ *is Einstein but not strongly isotropy irreducible, then one of the following holds:*

(*a*) $\mathfrak{h} = so(m) \oplus so(m)$, $\pi = \rho_m$, $n = m^2$, $m \geqq 3$; $h = sp(m) \oplus sp(m)$, $\pi = \nu_{2m} \hat{\otimes} \nu_{2m}$, $n = 4m^2$, $m \geqq 2$;

(*b*) $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_l$, $\pi = \bigoplus_i [\mathrm{id} \hat{\otimes} \ldots \hat{\otimes} \pi_i \hat{\otimes} \ldots \hat{\otimes} \mathrm{id}]$, $n = \sum_i \dim \pi_i$, $l \geqq 2$, *where* $(\mathfrak{h}_i, \pi_i)$ *is the isotropy representation of an irreducible symmetric space with* $\mathfrak{h}_i$ *simple or* $(\mathfrak{h}_i, \pi_i)$ *is as in* (*a*). *Furthermore, we require that* $\dim \pi_i / \dim \mathrm{H}_i$ *is independent of* i, $1 \leqq i \leqq l$;

(*c*) $\mathfrak{h} = so(k) \oplus u(k+1)$, $\pi = [\rho_k \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mu_{k+1}]_{\mathbb{R}}$, $n = 3k+2$, $k \geqq 3$;

(*d*) $\mathfrak{h} = sp(1) \oplus sp(5) \oplus so(6)$, $\pi = [\nu_2 \hat{\otimes} \nu_{10} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \rho_6]$, $n = 26$;

(*e*) $\mathfrak{h} = G_2$, $\pi = \mathrm{o} \overset{1}{\equiv\!\!\bullet} \oplus \mathrm{id}$, $n = 8$.

Proof. — We apply (3.8). Since (3.8 *b*) gives rise to case (*e*), while a computation shows that (3.8 *c*) does not yield any new Einstein metrics, we shall assume from now on that $(\mathfrak{h}_i, \pi_i)$ is the isotropy representation of an irreducible symmetric space.

Now $B'_{so(n)} \mid so(n_i) = B'_{so(n_i)}$ $(n_i = \dim \pi_i)$ and so $C_{\chi_i, B'_{so(n_i)}} = a\,\mathrm{Id}$, which already excludes many cases. If π_i comes from a real or quaternionic Grassmannian, then by Table VI,

$$(\mathfrak{h}_i, \pi_i) = (so(m) \oplus so(m), \rho_m \hat{\otimes} \rho_m), \qquad (sp(m) \oplus sp(m), \nu_{2m} \hat{\otimes} \nu_{2m}),$$
$$(so(m), \rho_m), \qquad \text{or} \quad (sp(m) \oplus sp(1), \nu_{2m} \hat{\otimes} \nu_2).$$

If $(\mathfrak{h}_i, \pi_i) = (\mathfrak{k} \oplus sp(1), \pi_\lambda \hat{\otimes} \overset{1}{\mathrm{o}})$ is the isotropy representation of a quaternionic symmetric space $\neq \mathrm{Sp}(m+1)/\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ with $\mathfrak{k}$ simple, then as before

$$\chi_i = [\pi_{2\lambda - \alpha} \hat{\otimes} \overset{2}{\mathrm{o}}] \oplus [\pi_{2\lambda} \hat{\otimes} \mathrm{o}].$$

Let $\dim \pi_\lambda = 2n$, then $B'_{so(4n)} = B'_{so(n)}$ and $[so(4n) : sp(1)] = i(n\nu_2) = n$, so Table V implies that

$$E(\pi_{2\lambda} \hat{\otimes} \mathrm{o}) = 2(n+2)\dim \mathrm{H}/n(2n+1) \qquad \text{and} \qquad E(\pi_{2\lambda - \alpha} \hat{\otimes} \overset{2}{\mathrm{o}}) = 2\dim \mathrm{H}/(2n+1) + 4/n,$$

where we used the fact (*see* the proof of (3.8), Case 1) that $E(\pi_{2\lambda - \alpha}) = (4n/(2n+1))\,E(\pi_\lambda)$. But then $E(\pi_{2\lambda} \hat{\otimes} \mathrm{o}) = E(\pi_{2\lambda - \alpha} \hat{\otimes} \overset{2}{\mathrm{o}})$ implies that $\dim \mathrm{H} = 2n+1$, and one easily checks that this is never satisfied.

If $(\mathfrak{h}_i, \pi_i) = (\mathfrak{k} \oplus \mathfrak{t}, [\pi_\lambda \hat{\otimes} \varphi]_{\mathbb{R}})$ is the isotropy representation of an irreducible hermitian symmetric space with $\mathfrak{k}$ simple then $\chi_i = [\Lambda^2 \pi_\lambda \hat{\otimes} S^2 \varphi]_{\mathbb{R}} \oplus [\pi_{\lambda + \lambda^*} \hat{\otimes} \mathrm{id}]$ unless $\pi_\lambda = \mu_m$. Since

$$B'_{so(2n)} \mid \mathfrak{k} \oplus \mathfrak{t} = -\frac{1}{2}\mathrm{tr}(\pi_i \circ \pi_i) = -\mathrm{tr}(\pi_\lambda \hat{\otimes} \varphi) \circ (\pi_\lambda \hat{\otimes} \varphi)$$

we have $E(\varphi) = 1/n$. If $\mathfrak{k}$ is simple we have $B'_{so(2n)} \mid su(n) = B'_{su(n)}$. So

$$E(\Lambda^2 \pi_\lambda) = \frac{2(n-2)}{(n-1)} E(\pi_\lambda)$$

(*see* the proof of (3.8)), and hence by Table V

$$E(\Lambda^2 \pi_\lambda \hat{\otimes} S^2 \varphi) = 2(n-2)\dim \mathrm{H}/n(n-1) + 4/n,$$

while $E(\pi_{\lambda + \lambda^*} \hat{\otimes} \mathrm{id}) = 2\dim \mathrm{H}/(n-1)$. Therefore, we have equality iff $\dim \mathrm{H} = n-1$, and one easily checks that this is never satisfied. Similarly, if

$$(\mathfrak{h}_i, \pi_i) = (s(u(p) \oplus u(q)), [\mu_p \hat{\otimes} \mu_q \hat{\otimes} \varphi]_{\mathbb{R}}), \quad p \geqq q > 1,$$

then $E(\pi_{\lambda + \lambda^*} \hat{\otimes} \mathrm{id}) = 2(p^2 + q^2)/pq$ and

$$E(\Lambda^2 \mu_p \hat{\otimes} S^2 \mu_q \hat{\otimes} S^2 \varphi) = 2(p^2 + q^2 - p + q - 4)/pq,$$
$$E(S^2 \mu_p \hat{\otimes} \Lambda^2 \mu_q \hat{\otimes} S^2 \varphi) = 2(p^2 + q^2 + p - q - 4)/pq,$$

which shows that this case cannot occur either. Hence we are left with the following possibilities:

	$(\mathfrak{h}_i, \pi_i)$	$E(\pi_i)$	$E(\chi_i)$
(1)	$\mathfrak{h}_i$ simple	$2\dim H_i/n_i$	$4\dim H_i/n_i$
(2)	$(so(m)\oplus so(m), \rho_m\hat{\otimes}\rho_m)$,	$2\dim H_i/n_i$	$4\dim H_i/n_i$
(3)	$(sp(m)\oplus sp(m), \nu_{2m}\hat{\otimes}\nu_{2m})$, $m\geqq 2$	$2\dim H_i/n_i$	$4\dim H_i/n_i$
(4)	$(sp(m)\oplus sp(1), \nu_{2m}\hat{\otimes}\nu_2)$, $m\geqq 2$	$m+1/2+3/2m$	$2m+4/m$
(5)	$(u(m), [\mu_m]_{\mathbb{R}})$, $m\geqq 2$	m	$2(m-1)$
(6)	$(so(m), \rho_m)$, $m\geqq 3$	$m-1$	$\chi_i=0$

Since $E(\chi_i)=2E(\pi_i)$ in cases (1)-(3) and $E(\chi_i)<2E(\pi_i)$ in cases (4)-(5) it follows that (4) or (5) cannot be combined with (1)-(5). If we combine (4) or (5) with (6) we obtain (*c*) and (*d*) in (3.9). If (1)-(3) or (6) are combined with each other we obtain (*a*) and (*b*). ■

CHAPTER FOUR

Subgroups of the exceptional Lie groups

1. GENERAL REMARKS. — In [8], Dynkin classified the semi-simple subalgebras of the exceptional simple Lie algebras up to "L-equivalence". Two homomorphisms $\pi_1: \mathfrak{h}\to\mathfrak{g}$, $\pi_2: \mathfrak{h}\to\mathfrak{g}$ are L-equivalent if for every linear representation φ of g, the representations $\varphi\circ\pi_1$ and $\varphi\circ\pi_2$ are equivalent.

In classifying normal homogeneous Einstein metrics of quotients of the exceptional groups, we shall consider in turn regular subalgebras, R-subalgebras, and S-subalgebras, as did Dynkin. However, we shall use his classification crucially only in the case of S-subalgebras.

Recall that a subalgebra $\pi: \mathfrak{h}\to\mathfrak{g}$ is *regular* if there is a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ with associated root space decomposition $\mathfrak{g}=\mathfrak{t}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ such that there exists a subset R of roots of $\mathfrak{g}$ and $\mathfrak{t}'\subset\mathfrak{t}$ with $\mathfrak{h}=\mathfrak{t}'\oplus\sum_{\alpha\in R}\mathfrak{g}_\alpha$. For example, if $\mathfrak{h}\subset\mathfrak{g}$ and $\operatorname{rank}\mathfrak{h}=\operatorname{rank}\mathfrak{g}$, then $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{g}$. In section 2, we shall classify all Einstein regular subalgebras of the exceptional Lie algebras using the classification of Borel-de Siebenthal [5].

An R-*subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is one which is contained in but not equal to a regular subalgebra. Obviously, an R-subalgebra cannot be strongly isotropy irreducible. Each R-subalgebra $\mathfrak{h}$ is contained in a maximal regular subalgebra of maximal rank $\mathfrak{k}$ of $\mathfrak{g}$. ($\mathfrak{k}$ is not necessarily unique.) Using the inclusions $\mathfrak{h}\subset\mathfrak{k}\subset\mathfrak{g}$ and results in chapters 1 and 2 enable us to classify Einstein R-subalgebras of $\mathfrak{g}$ without reliance on Dynkin's classification of R-subalgebras, which is up to L-equivalence only.

A subalgebra of $\mathfrak{g}$ that is not regular and not an R-subalgebra is an S-*subalgebra*. In classifying Einstein S-subalgebras, we rely on Dynkin's classification (p. 233, [8]). Notice

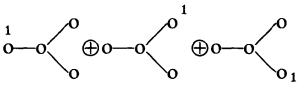
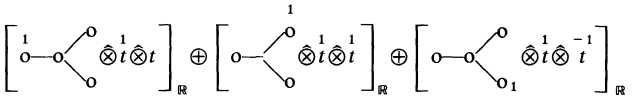
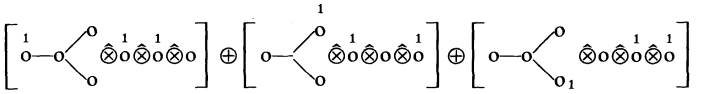
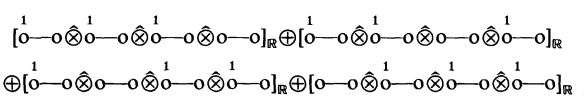
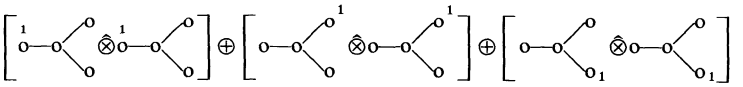
that in this case Dynkin showed that if two S-subalgebras are L-equivalent but not conjugate in $\mathfrak{g}$, then there is an outer automorphism of $\mathfrak{g}$ that takes one S-subalgebra into the other (p. 128 [8]). Consequently, the corresponding homogeneous spaces with the normal metrics g_B are actually isometric.

In this chapter, $(\ ,\ )$ will denote $B'^{*}_{G}(\ ,\ )$.

2. Regular subalgebras

(4.1) Proposition. — *Let* $\mathfrak{h}$ *be an Einstein regular subalgebra of an exceptional simple Lie algebra* $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{h})$ *is not strongly isotropy irreducible. Then* $(\mathfrak{g}, \mathfrak{h})$ *must be one of the following:*

Table VII

$\mathfrak{g}$	$\mathfrak{h}$	χ
F_4	D_4	
E_6	$D_4 \oplus \mathbb{R}^2$	
E_7	$D_4 \oplus 3A_1$	
E_7	$7A_1$	$[1\,2\,3\,4] \oplus [1\,2\,5\,6] \oplus [3\,4\,5\,6] \oplus [1\,3\,5\,7]$ $\oplus [2\,4\,5\,7] \oplus [1\,4\,6\,7] \oplus [2\,3\,6\,7]$
E_8	$A_4 \oplus A_4$	
E_8	$4A_2$	
E_8	$D_4 \oplus D_4$	
E_8	$8A_1$	$[1\,2\,3\,4] \oplus [5\,6\,7\,8] \oplus [1\,3\,5\,6] \oplus [2\,4\,7\,8] \oplus [1\,3\,7\,8]$ $\oplus [2\,4\,5\,6] \oplus [1\,4\,5\,8] \oplus [2\,3\,6\,7] \oplus [1\,4\,6\,7] \oplus [2\,3\,5\,8]$ $\oplus [1\,2\,5\,6] \oplus [3\,4\,6\,8] \oplus [1\,3\,6\,8] \oplus [3\,4\,5\,7]$
E_6, E_7, E_8	maximal abelian subalgebras	root space decomposition

Remark. — That $(\mathfrak{g}, \mathfrak{h})$ is Einstein follows immediately from (1.14). It is, however, not true that if $\mathfrak{h}$ is an Einstein regular subalgebra then all dominant weight of the irreducible summands of $\chi \otimes \mathbb{C}$ are permuted by outer automorphisms of $\mathfrak{h}$. $so(n) \oplus u(n+1) \subset so(3n+2)$ and $sp(n) \oplus u(2n-1) \subset sp(3n-1)$ furnish counter-examples.

Our first observation is an immediate consequence of (1.4):

(4.2) LEMMA. — *An Einstein regular subalgebra must be of maximal rank in* $\mathfrak{g}$. ■

We shall henceforth assume that $\mathfrak{h} = \mathfrak{t} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$, where $R \subset \Delta(\mathfrak{g}) =$ root system of $\mathfrak{g}$. It follows that the weights of the isotropy representation of G/H consist of the complementary roots $\Delta(\mathfrak{g}) - R$. Since the dominant weights in the isotropy representation are roots, all irreducible summands of $\chi \otimes \mathbb{C}$ are inequivalent representations. Furthermore, the coefficients $\Lambda^\alpha = 2(\lambda, \alpha)/(\alpha, \alpha)$ are 0, 1, 2 or 3.

Before giving the detailed proof of (4.1) we first present an outline of the proof, followed by a description of some facts and methods used constantly in the proof.

In [5], A. Borel and J. de Siebenthal classified all maximal subalgebras of maximal rank in the simple Lie algebras. Let $\mathfrak{g}$ be a simple exceptional Lie algebra, and $\mathfrak{k}$ be any maximal subalgebra of maximal rank. It turns out that all such $\mathfrak{k}$'s are Einstein in $\mathfrak{g}$. (*See* (4.6).) If $\mathfrak{k}$ is simple, then we examine the subalgebras of maximal rank in $\mathfrak{k}$ to see which are Einstein in $\mathfrak{g}$. A useful necessary condition is (1.13 *a*).

If $\mathfrak{k}$ is not simple, say, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, then note that any regular subalgebra of $\mathfrak{k}$ is of the form $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_i$ is a regular subalgebra of $\mathfrak{k}_i$. In view of (4.2) and (1.13 *b*) we may restrict our attention to regular subalgebras of the form $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r$, where $\mathfrak{k} = \mathfrak{a} \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_r$ with $\mathfrak{a}$ abelian, $\mathfrak{k}_i$ simple, and $\mathfrak{h}_i$ Einstein, regular and of maximal rank in $\mathfrak{k}_i$.

In any event, if $\mathfrak{k}_i$ is a classical Lie algebra (we allow $\mathfrak{k} = \mathfrak{k}_i$) then by comparing Table I A and the list of isotropy irreducible spaces we see that $\mathfrak{h}_i$, being of maximal rank and Einstein in $\mathfrak{k}_i$, must either be symmetric or one of cases 1, 2, or 7 in Table I A (with $\mathfrak{k}$ even if we are in case 2 *c*). If $\mathfrak{k}_i$ is an exceptional Lie algebra, then the admissible $\mathfrak{h}_i$'s are obtained by induction.

At this point we note that many non-semi-simple regular subalgebra are not Einstein as a result of

(4.3) LEMMA. — *Let* $\mathfrak{k}_1 \oplus \mathfrak{t} \subset \mathfrak{k}_1 \oplus \mathfrak{k}_2 \subset \mathfrak{g}$, where $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$ *are semi-simple regular subalgebras in* $\mathfrak{g}$ *and* $\mathfrak{t}$ *is abelian. Let* $B_\mathfrak{g} | \mathfrak{k}_2 = c\, B_{\mathfrak{k}^2}$ *and* $\mathfrak{k}_1 \oplus \mathfrak{t}$ *be Einstein in* $\mathfrak{g}$. *Then*

(*a*) *all roots of* $\mathfrak{k}_2$ *have the same length;*

(*b*) *if* $\pi_{\lambda_1} \hat{\otimes} \pi_{\lambda_2}$ *is an irreducible summand of the complexified isotropy representation of* $G/(K_1 \times K_2)$, *then all weights of* π_{λ_2} *have the same length;*

(*c*) *the roots of* $\mathfrak{g}$ *do not all have the same length.*

Proof. — (*a*) is an immediate consequence of (1.13 *a*) and (1.5). To see (*b*), we restrict $\pi_{\lambda_1} \hat{\otimes} \pi_{\lambda_2}$ to $\mathfrak{k}_1 \oplus \mathfrak{t}$, getting $\sum_w \pi_{\lambda_1} \hat{\otimes} \varphi_w$, where w runs through all weights of π_{λ_2} and φ_w is the 1-dimensional complex representation of $\mathfrak{t}$ with weight w. Hence (*b*) follows.

Lastly, if all roots of $\mathfrak{g}$ have the same length $\sqrt{2}$ with respect to the normalized Killing form, then Einstein constants coming from K_2/T are all equal to 2 since roots of $\mathfrak{k}_2$ are roots of $\mathfrak{g}$. On the other hand, since $G/(K_1 \times T)$ is assumed to be almost effective, there exists an irreducible summand of $\chi \otimes \mathbb{C}$, say π_λ, with $\pi_\lambda \mid \mathfrak{k}_1 \neq \mathrm{id}$. Now $B'^*_G(\lambda, \lambda+2\delta) = B'^*_G(\lambda, \lambda) + B'^*_G(\lambda, 2\delta) > 2$ since λ is a root of $\mathfrak{g}$, giving a contradiction. ■

To proceed further in the classification we need to compare Einstein constants. We shall establish the following convention.

(4.4) CONVENTION. — Let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, where $\mathfrak{g}$ is a simple exceptional Lie algebra. Denote by χ_1 and χ_2 the isotropy representations of K/H and G/K respectively. Let π_{λ_1} be an irreducible summand of $\chi_1 \otimes \mathbb{C}$ with dominant weight λ_1 and π_{λ_2} be an irreducible summand of $\chi_2 \otimes \mathbb{C} \mid \mathfrak{h}$.

For most cases, to show that $(\mathfrak{g}, \mathfrak{h})$ is not Einstein it suffices to compare the constants of λ_1 and λ_2 for suitably chosen irreducible summands.

To obtain the irreducible summands π_{λ_1} and π_{λ_2} we use the Borel-de Siebenthal classification theory [5]. For the convenience of the reader, we recall this theory, following Wolf. (*See* section 8.10 of [26]. This material is not in the earlier editions of the book.)

(4.5) *Borel-de Siebenthal theory.* — For each regular subalgebra $\mathfrak{h}$ of maximal rank in $\mathfrak{g}$, there exists a sequence of subalgebras $\mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \ldots \subset \mathfrak{k}_r$ such that $\mathfrak{k}_0 = \mathfrak{h}$, $\mathfrak{k}_r = \mathfrak{g}$, and $\mathfrak{k}_i$ is a maximal subalgebra of maximal rank in $\mathfrak{k}_{i+1}$. The maximal subalgebras of maximal rank in a simple Lie algebra are obtained as follows.

Let $\{\alpha_1, \ldots, \alpha_l\}$ be a fundamental system of roots of $\mathfrak{g}$ and $\mu = \sum_{i=1}^{l} m_i \alpha_i$ be the maximal root. Choose i so that $m_i = 1$ and let $\mathfrak{k}'$ be the simple Lie algebra whose fundamental system is $\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_l\}$. Then $\mathfrak{k} = \mathfrak{k}' \oplus \mathbb{R}$ is a maximal subalgebra of maximal rank in $\mathfrak{g}$. Moreover, $(\mathfrak{g}, \mathfrak{k})$ is hermitian symmetric with isotropy representation $[\pi_{-\alpha_i}]_{\mathbb{R}}$.

To obtain semi-simple maximal subalgebras of maximal rank, choose i so that m_i is a prime. Then let $\mathfrak{k}$ be the Lie algebra whose fundamental system is $\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_l, -\mu\}$. The possibilities for m_i are 2, 3, and 5. In the first case, $(\mathfrak{g}, \mathfrak{k})$ is symmetric but non-hermitian with isotropy representation $\pi_{-\alpha_i}$. In the second case $(\mathfrak{g}, \mathfrak{k})$ is non-symmetric with isotropy representation $[\pi_{-\alpha_i}]_{\mathbb{R}}$. There is only one subalgebra with $m_i = 5$: $A_4 \oplus A_4 \subset E_8$, whose isotropy representation is

$$[\overset{1}{\mathrm{o}}\text{—}\mathrm{o}\text{—}\mathrm{o}\text{—}\mathrm{o} \hat{\otimes} \mathrm{o}\text{—}\mathrm{o}\text{—}\overset{1}{\mathrm{o}}\text{—}\mathrm{o}]_{\mathbb{R}} \oplus [\mathrm{o}\text{—}\mathrm{o}\text{—}\overset{1}{\mathrm{o}}\text{—}\mathrm{o} \hat{\otimes} \overset{1}{\mathrm{o}}\text{—}\mathrm{o}\text{—}\mathrm{o}\text{—}\mathrm{o}]_{\mathbb{R}}.$$

Using Corollary (1.14) we obtain immediately

(4.6) LEMMA. — *The maximal subalgebras of maximal rank in the simple compact Lie algebras are all Einstein.* ■

The above description allows us to select π_{λ_1} and π_{λ_2} immediately. Another useful observation is that if $\mathfrak{h}$ is contained in $\mathfrak{k}_1$ and $\mathfrak{k}_2$, then we immediately obtain two

irreducible summands π_{λ_1} and π_{λ_2} (from the isotropy representations of K_1/H and K_2/H), whose Einstein constants we can compare.

In the present case of regular subalgebras, the computation of Einstein constants is facilitated by the following observations:

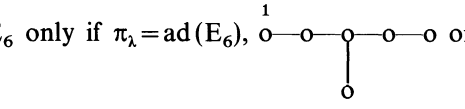
(*a*) λ_i are roots of $\mathfrak{g}$, and so $B'^*_G(\lambda_i, \lambda_i)$ are easily known just from Borel-de Siebenthal. Hence the computation of $B'^*_G(\lambda_i, \lambda_i + 2\delta)$ is reduced to that of $B'^*_G(\lambda_i, 2\delta)$. Note that only the coefficients of λ_i on the semi-simple part of $\mathfrak{h}$ are required. When $\mathfrak{h}$ is non- semi-simple, the coefficients of λ_2 can be obtained from the extended Dynkin diagram of $\mathfrak{k}$ since the ordering of the roots of $\mathfrak{h}$ agrees with that of $\mathfrak{k}$. If $\mathfrak{h}$ is semi-simple, this is no longer true and we have to perform separate calculations to determine these coefficients.

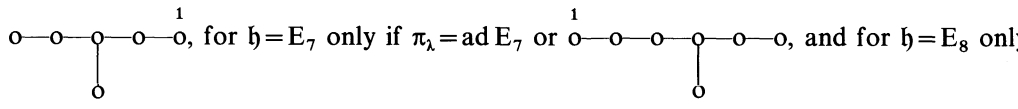
(*b*) In computing $B'^*_G(\lambda_i, \lambda_i + 2\delta)$ we sometimes need to compute the indices of the simple factors of $\mathfrak{h}$ in $\mathfrak{g}$. Since $\mathfrak{h}$ is a regular subalgebra, we see that $[\mathfrak{g}: \mathfrak{h}_i] = [B'^*_G(\mu_{\mathfrak{h}_i}, \mu_{\mathfrak{h}_i})]/[B'^*_G(\mu_{\mathfrak{g}}, \mu_{\mathfrak{g}})]$, where $\mu_{\mathfrak{g}}$ is the maximal root of $\mathfrak{g}$ and $\mu_{\mathfrak{h}_i}$ is the maximal root of $\mathfrak{h}_i$. Notice that $[\mathfrak{g}: \mathfrak{h}_i] = 1$, 2, or 3.

For the rest of the observations we shall assume that all roots of $\mathfrak{g}$ are of the same length. Then since each λ_i is a root, we are interested only in $B'^*_G(\lambda_i, 2\delta)$. Let λ be a dominant weight of $\chi \otimes \mathbb{C}$.

(*c*) For *any root* α of $\mathfrak{h}$, $\lambda^\alpha = B'^*_G(\lambda, \alpha) = 0$ or 1. This follows from the Schwartz inequality and the fact that $\lambda \neq \pm \alpha$.

(*d*) On each simple factor of $\mathfrak{h}$, only one coefficient λ^α may be non-zero. Assume otherwise that λ^α, $\lambda^\beta \neq 0$ for simple roots α, β of $\mathfrak{h}_i$. Then there is a chain of simple roots of $\mathfrak{h}_i$, $\alpha_1, \ldots, \alpha_s$, with $\alpha_1 = \alpha$ and $\alpha_s = \beta$. But then $\gamma = \alpha_1 + \ldots + \alpha_s$ would have $\lambda^\gamma \geqq 2$, contradicting (*c*).

(*e*) All roots of $\mathfrak{h}$ also have the same length. So the simple factors of $\mathfrak{h}$ occur among A_n, D_n, E_6, E_7, or E_8, and have index 1. Since $B'^*_G(\lambda, \lambda) = 2$ this places severe restrictions on λ. By Table 2, p. 117, of [8], a dominant weight λ satisfies $(\lambda, \lambda) \leqq 2$ for $\mathfrak{h} = D_n$ only if $\pi_\lambda = \rho_n$, $\mathrm{ad}(D_n)$, or $\Delta^{\pm}_{2n}$ $(4 \leqq n \leqq 8)$, for $\mathfrak{h} = E_6$ only if $\pi_\lambda = \mathrm{ad}(E_6)$,  or

 , for $\mathfrak{h} = E_7$ only if $\pi_\lambda = \mathrm{ad}\, E_7$ or , and for $\mathfrak{h} = E_8$ only

if $\pi_\lambda = \mathrm{ad}(E_8)$. But if $\pi_\lambda = \mathrm{ad}\,\mathfrak{h}$, then 0 is a weight, giving a contradiction since $rk\ \mathfrak{h} = rk\ \mathfrak{g}$ and all weights of π_λ are roots of $\mathfrak{g}$.

(*f*) The following two tables of values are very useful for analyzing $\chi \otimes \mathbb{C}$, in view of the restriction $B'^*_G(\lambda, \lambda) = 2$.

Finally, in order to avoid repetitions, we observe that except for a few regular subalgebras of E_7 and E_8 listed on p. 139 of [8], any two isomorphic regular subalgebras of the simple exceptional Lie algebras are conjugate in $\mathfrak{g}$.

TABLE VIII A

$B'^{*}_{G}(\lambda, \lambda)$ *for some basic representations*

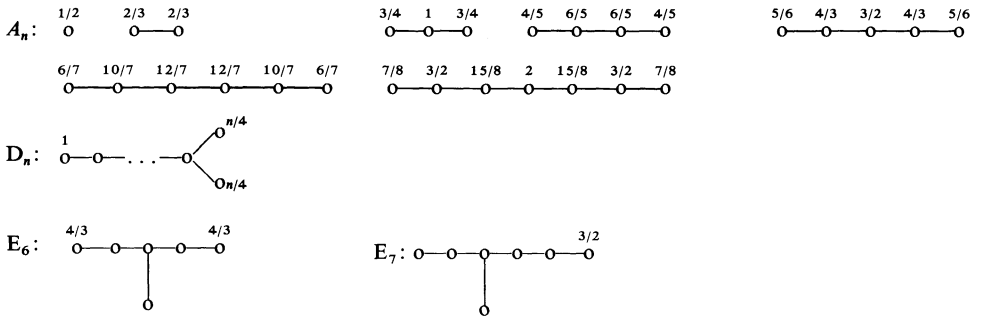
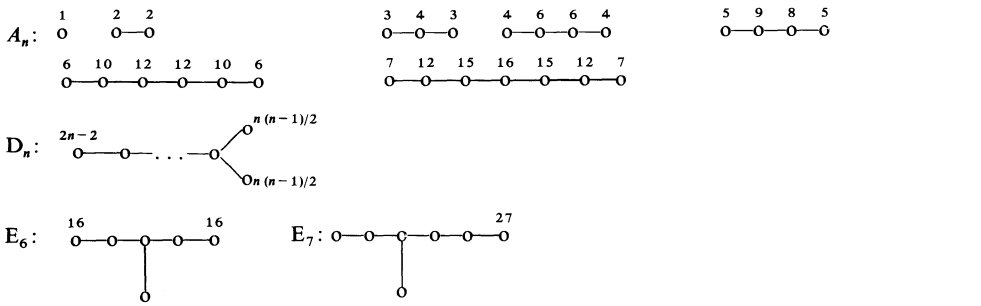


TABLE VIII B

$B'^{*}_{G}(\lambda, 2\delta)$ *for some basic representations*



Using all of the above, we can prove (4.1). The details are described below.

Proof of Proposition (4.1). — We shall consider each exceptional Lie algebra $\mathfrak{g}$ in turn. The extended Dynkin diagram will be listed. Subalgebras of $\mathfrak{g}$ whose roots are the short roots of $\mathfrak{g}$ will be denoted by $\tilde{A}_1$, $\tilde{A}_2$, etc.

Case I : $\mathfrak{g}=G_2$. — $\begin{pmatrix} 1 & 2 & 3 \\ o & -o & \equiv\bullet \\ -\mu & \alpha_1 & \alpha_2 \end{pmatrix}$.

The maximal subalgebras of maximal rank are A_2 and $A_1 \oplus \tilde{A}_1$. There are only two subalgebras of maximal rank: $A_1 \oplus \mathbb{R}$ and $\mathbb{R} \oplus \tilde{A}_1$, both contained in $A_1 \oplus \tilde{A}_1$. The isotropy representation of $A_1 \oplus \tilde{A}_1$ in G_2 is $\overset{1}{o} \hat{\otimes} \overset{3}{\bullet}$. So by (4.3 *b*) $A_1 \oplus \mathbb{R}$ is not Einstein. For $\mathbb{R} \oplus \tilde{A}_1$, let $\lambda_1 = -\mu$ and $\lambda_2 = -\alpha_1$. Clearly,

$$(\lambda_1, \lambda_1 + 2\delta) = (\lambda_1, \lambda_1) = (\lambda_2, \lambda_2) < (\lambda_2, \lambda_2 + 2\delta).$$

So $\mathbb{R} \oplus \tilde{A}_1$ is not Einstein.

Case II : $\mathfrak{g}=F_4$. — $\begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ o & -o & -o & =\bullet & -\bullet \\ -\mu & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}$.

Maximal subalgebras of maximal rank	Isotropy representation
B_4	$\circ\!-\!\circ\!-\!\circ\!=\!\overset{1}{\bullet}\ =\pi_{-\alpha_4}$
$A_1 \oplus C_3$	$[\overset{1}{\circ}\hat{\otimes}\overset{1}{\circ}\!=\!\bullet\!-\!\bullet]\ =\pi_{-\alpha_1}$
$A_2 \oplus \tilde{A}_2$	$[\circ\!-\!\overset{1}{\circ}\hat{\otimes}\overset{2}{\bullet}\!-\!\bullet]_{\mathbb{R}}\ =[\pi_{-\alpha_2}]_{\mathbb{R}}$

We need to consider the following subalgebras:

1. $B_3 \oplus \mathbb{R} \subset B_4^1 \subset F_4$. We can take $\lambda_1 = -\mu$, $\lambda_2 = -\alpha_4$. Since $B_3 \oplus \mathbb{R} \subset B_4^1$ is symmetric, $(\lambda_1, \lambda_1 + 2\delta) = 7$. $\pi_{\lambda_2} \mid B_3 = \circ\!-\!\circ\!=\!\overset{1}{\bullet}$ and $(\lambda_2, \lambda_2 + 2\delta) = 11/2$, so $B_3 \oplus \mathbb{R}$ is not Einstein.

2. $D_3 \oplus \tilde{A}_1 \subset B_4^1 \subset F_4$. We take π_{λ_1} to be $\circ\!-\!\overset{1}{\circ}\!-\!\circ\hat{\otimes}\overset{2}{\bullet}$ and π_{λ_2} to be $\overset{1}{\circ}\!-\!\circ\!-\!\circ\hat{\otimes}\overset{1}{\bullet}$. Since $D_3 \oplus \tilde{A}_1 \subset B_4$ is symmetric, $(\lambda_1, \lambda_1 + 2\delta) = 7$ and $(\lambda_2, \lambda_2 + 2\delta) = 9/2$. Hence $D_3 \oplus \tilde{A}_1$ is not Einstein.

3. $B_2 \oplus 2A_1 \subset B_4 \subset F_4$ and $B_2 \oplus 2A_1 \subset C_3 \oplus A_1$. $\pi_{\lambda_1} = \overset{1}{\circ}\!=\!\bullet\hat{\otimes}\overset{1}{\circ}\hat{\otimes}\overset{1}{\circ}$ and $\pi_{\lambda_2} = \circ\!=\!\overset{1}{\bullet}\hat{\otimes}\overset{1}{\circ}\hat{\otimes}\circ$. Since $B_2 \oplus 2A_1 \subset B_4$ and $B_2 \oplus A_1 \subset C_3$ are symmetric, we have $(\lambda_1, \lambda_1 + 2\delta) = 7$ and $(\lambda_2, \lambda_2 + 2\delta) = 4$. Hence $B_2 \oplus 2A_1$ is not Einstein.

4. $C_3 \oplus \mathbb{R} \subset C_3 \oplus A_1 \subset F_4$. This is clearly not Einstein because we can let λ_1 be the root of A_1 so that $(\lambda_1, \lambda_1 + 2\delta) = 2$ while $\lambda_2 = -\alpha_1$ so that $(\lambda_2, \lambda_2 + 2\delta) > (\lambda_2, \lambda_2) = 2$.

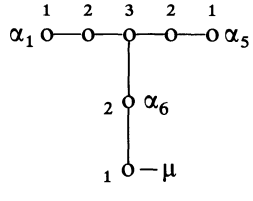
5. $\tilde{A}_2 \oplus (A_1 \oplus \mathbb{R}) \subset \tilde{A}_2 \oplus A_2 \subset F_4$ and $(\tilde{A}_2 \oplus \mathbb{R}) \oplus A_1 \subset C_3 \oplus A_1 \subset F_4$. We let π_{λ_1} be the isotropy representation of $(A_1 \oplus \mathbb{R}) \subset A_2$. Since $A_1 \oplus \mathbb{R} \subset A_2$ is symmetric, $(\lambda_1, \lambda_1 + 2\delta) = 3$. Let π_{λ_2} be the isotropy representation of $\tilde{A}_2 \oplus \mathbb{R} \subset C_3$. This is the symmetric space $Sp(3)/U(3)$ hence $(\lambda_2, \lambda_2 + 2\delta) = 4$ since C_3 has index 1 in F_4. So $\tilde{A}_2 \oplus A_1 \oplus \mathbb{R}$ is not Einstein.

6. $\tilde{A}_2 \oplus \mathbb{R}^2 \subset \tilde{A}_2 \oplus A_1 \oplus \mathbb{R} \subset F_4$. This is not Einstein since $(\lambda_1, \lambda_1 + 2\delta) = 2$, while λ_2 can be taken to be $-\alpha_2$ since $\tilde{A}_2 \oplus \mathbb{R}^2 \subset \tilde{A}_2 \oplus A_2 \subset F_4$. Certainly, $(\lambda_2, \lambda_2 + 2\delta) > (\lambda_2, \lambda_2) = 2$.

7. $D_4 \subset B_4 \subset F_4$. This is Einstein (*see* Example 6 in Section 1-3).

All other regular subalgebras of maximal rank are not Einstein either because 1.13 is contradicted or because they are ruled out by (4.3).

Case III: $\mathfrak{g} = E_6$. —



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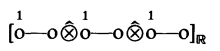
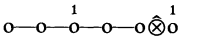
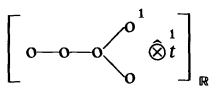
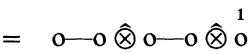
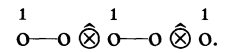
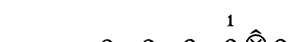
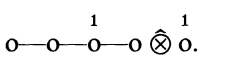
Maximal subalgebras of maximal rank	Isotropy representation
$3A_2$	
$A_5 \oplus A_1$	
$D_5 \oplus \mathbb{R}$	

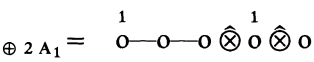
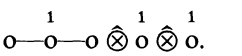
1. $A_2 \oplus A_2 \oplus A_1 \oplus \mathbb{R} \subset 3A_2 \subset E_6$. This is obviously not Einstein since $\pi_{\lambda_1}|_{2A_2 \oplus A_1} =$  while $\pi_{\lambda_2}|_{2A_2 \oplus A_1} =$  .

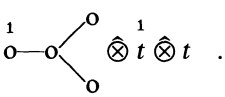
2. $A_4 \oplus A_1 \oplus \mathbb{R} \subset A_5 \oplus A_1 \subset E_6$. This is not Einstein; just take

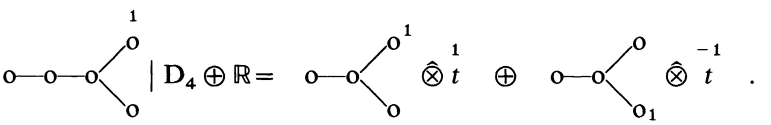
$\pi_{\lambda_1}|_{A_4 \oplus A_1} =$  and $\pi_{\lambda_2}|_{A_4 \oplus A_1} =$  .

3. $A_3 \oplus 2A_1 \oplus \mathbb{R} \subset A_5 \oplus A_1 \subset E_6$ and $A_3 \oplus 2A_1 \oplus \mathbb{R} = D_3 \oplus D_2 \oplus \mathbb{R} \subset D_5 \oplus \mathbb{R} \subset E_6$.

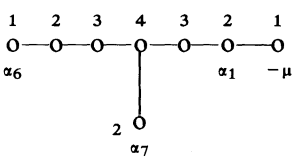
Hence to see this is not Einstein, let

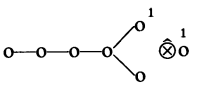
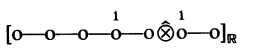
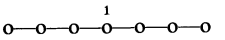
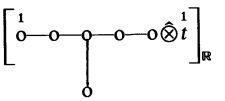
$\pi_{\lambda_1}|_{A_3 \oplus 2A_1} =$  and let $\pi_{\lambda_2}|_{A_3 \oplus 2A_1} =$  .

4. $D_4 \oplus \mathbb{R}^2 \subset D_5 \oplus \mathbb{R} \subset E_6$. This is Einstein by (1.14) once we compute the isotropy representation. From the first inclusion we get  . Next notice that

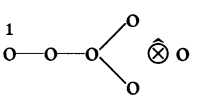
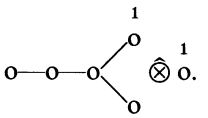


All other regular subalgebras of maximal rank fail to be Einstein in E_6 because they are not Einstein in one of the maximal subalgebras of maximal rank or because of (4.3).

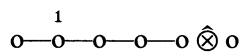
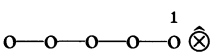
Case IV: $\mathfrak{g} = E_7$. –  .

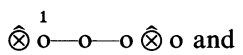
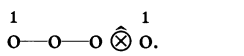
Maximal subalgebra of maximal rank	Isotropy representation
$D_6 \oplus A_1$	
$A_5 \oplus A_2$	
A_7	
$E_6 \oplus \mathbb{R}$	

1. $D_5 \oplus \mathbb{R} \oplus A_1 \subset D_6 \oplus A_1 \subset E_7$. This is not Einstein: let

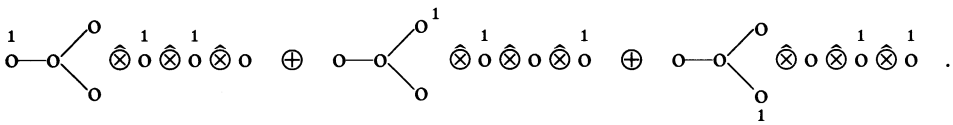
$\pi_{\lambda_1} \mid D_5 \oplus A_1 =$  and $\pi_{\lambda_2} \mid D_5 \oplus A_1 =$  .

2. $(A_5 \oplus \mathbb{R}) \oplus A_1 \subset D_6 \oplus A_1 \subset E_7$. This is not Einstein:

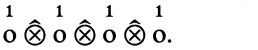
$\pi_{\lambda_1} \mid A_5 \oplus A_1 =$  and $\pi_{\lambda_2} \mid A_2 \oplus A_1 =$  .

3. $D_3 \oplus D_3 \oplus A_1 \subset D_6 \oplus A_1 \subset E_7$. This is not Einstein since we easily see that $\chi \otimes \mathbb{C}$ contains  and  .

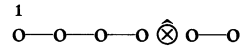
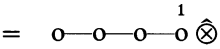
4. $D_4 \oplus D_2 \oplus A_1 \subset D_6 \oplus A_1 \subset E_7$. The isotropy representation has to be



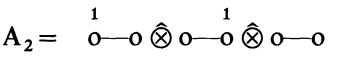
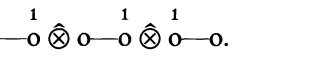
Then by (1.14) this is Einstein.

5. $7A_1 \subset D_4 \oplus 3A_1 \subset D_6 \oplus A_1 \subset E_7$. This is clearly Einstein. To compute χ explicitly, one restricts the isotropy representation in (4) to $7A_1$; the isotropy representation of $4A_1 \subset D_4$ is  .

6. $(A_4 \oplus \mathbb{R}) \oplus A_2 \subset A_5 \oplus A_2 \subset E_7$. This is not Einstein: let

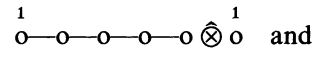
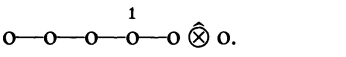
$\pi_{\lambda_1} \mid A_4 \oplus A_2 =$  and $\pi_{\lambda_2} \mid A_4 \oplus A_2 =$  .

7. $2A_2 \oplus \mathbb{R} \oplus A_2 \subset A_5 \oplus A_2 \subset E_7$. This is not Einstein: let

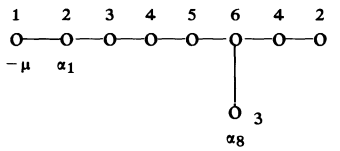
$\pi_{\lambda_1} \mid 3A_2 =$  and $\pi_{\lambda_2} \mid 3A_2 =$  .

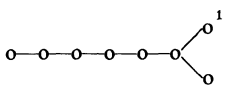
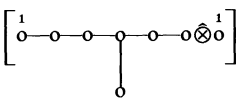
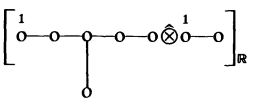
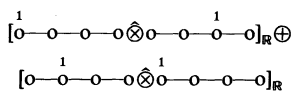
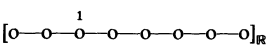
8. $A_6 \oplus \mathbb{R} \subset A_7 \subset E_7$. This is clearly not Einstein by Table VIII.

9. $A_5 \oplus A_1 \oplus \mathbb{R} \subset A_7 \subset E_7$. This is not conjugate to the subalgebra in (2). To see that it also is not Einstein, let

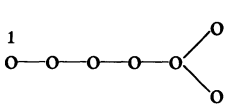
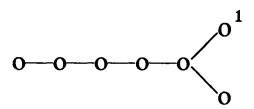
$\pi_{\lambda_1} \mid A_5 \oplus A_1 =$  and $\pi_{\lambda_2} \mid A_5 \oplus A_1 =$  .

All other subalgebras are ruled out by (4.3) or are contained in $D_6 \oplus A_1$ but are not Einstein in $D_6 \oplus A_1$.

Case V: $\mathfrak{g} = E_8$. —  .

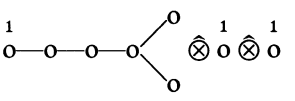
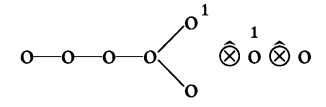
Maximal subalgebras of maximal rank	Isotropy representation
D_8	
$E_7 \oplus A_1$	
$E_6 \oplus A_2$	
$A_4 \oplus A_4$	
A_8	

1. $D_7 \oplus \mathbb{R} \subset D_8 \subset E_8$. This is not Einstein: let

$\pi_{\lambda_1} \mid D_7 =$  and $\pi_{\lambda_2} \mid D_7 =$ 

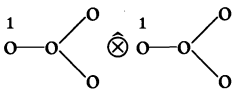
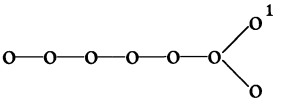
2. $A_7 \oplus \mathbb{R} \subset D_8 \subset E_8$. This is not Einstein by Table VIII and a dimension count.

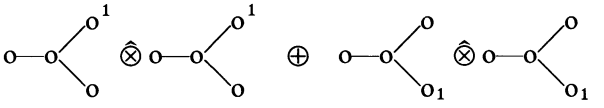
3. $D_6 \oplus D_2 \subset D_8 \subset E_8$. This is not Einstein: let

$\pi_{\lambda_1} =$  and $\pi_{\lambda_2} =$ 

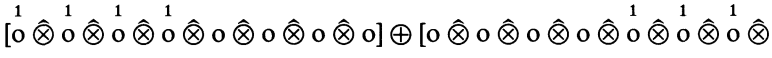
by Table VIII A.

4. $D_4 \oplus D_4 \subset D_8 \subset E_8$. This is obviously Einstein by Tables VIII A and VIII B.

$\pi_{\lambda_1} =$  and 

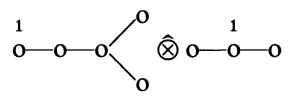
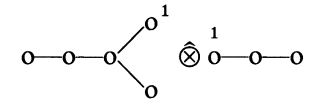
restricts to  .

5. $4D_2 \subset D_4 \oplus D_4 \subset E_8$. Again, this is obviously Einstein. The isotropy representation is



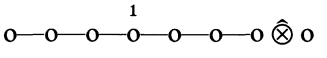
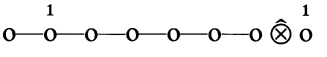
plus the restriction of the isotropy representation in (4) to $4D_2$.

6. $D_5 \oplus D_3 \subset D_8 \subset E_8$. This is not Einstein:

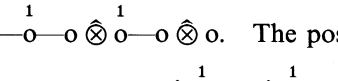
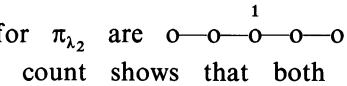
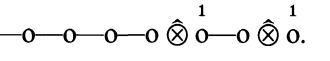
$\pi_{\lambda_1} =$  and $\pi_{\lambda_2} =$ 

by Table VIII A.

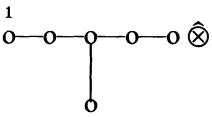
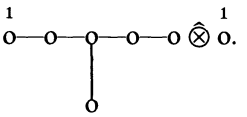
7. $A_7 \oplus A_1 \subset E_7 \oplus A_1 \subset E_8$. This is not Einstein:

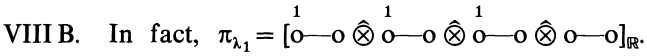
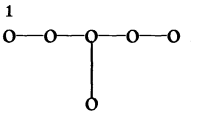
$\pi_{\lambda_1} =$  and $\pi_{\lambda_2} =$ 

by Table VIII A.

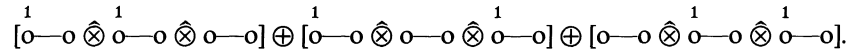
8. $A_5 \oplus A_2 \oplus A_1 \subset E_7 \oplus A_1 \subset E_8$. Now $\pi_{\lambda_1} =$  . The possibilities for π_{λ_2} are  or  . A dimension count shows that both possibilities must occur. By Table VIII B, $A_5 \oplus A_2 \oplus A_1$ cannot be Einstein.

9. $(E_6 \oplus \mathbb{R}) \oplus A_1 \subset E_7 \oplus A_1 \subset E_8$. This is not Einstein:

$\pi_{\lambda_1} \mid E_6 \oplus A_1 =$  and $\pi_{\lambda_2} \mid E_6 \oplus A_1 =$  .

10. $4A_2 \subset E_6 \oplus A_2 \subset E_8$. This is obviously Einstein by Tables VIII A and VIII B. In fact, $\pi_{\lambda_1} =$  . Furthermore, 

restricts to



These facts determine the isotropy representation of $4A_2$ in E_8.

All other subalgebras are seen to be not Einstein as in the other cases. ■

3. R-SUBALGEBRAS

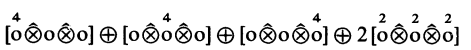
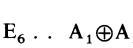
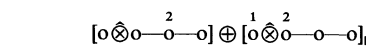
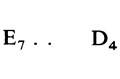
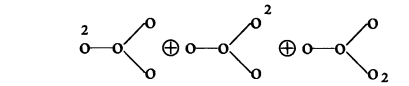
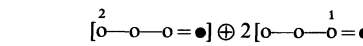
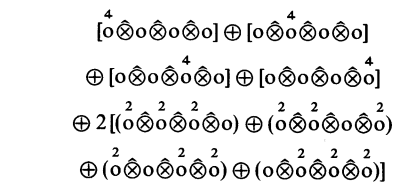
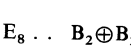
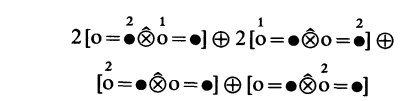
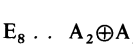
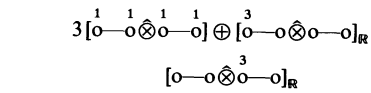
(4.7) PROPOSITION. — *Let* $\mathfrak{g}$ *be an exceptional simple Lie algebra and* $\mathfrak{h}$ *an Einstein* R-*subalgebra. Then* $\mathfrak{h}$ *occurs in Table IX. Every subalgebra* $\mathfrak{h}$ *in the table is Einstein. Each* $\mathfrak{h}$ *is listed with some maximal regular subalgebra* $\mathfrak{k} \supset \mathfrak{h}$, *where the containment is described by giving the induced representation.*

Let $\mathfrak{h}$ be an R-subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is contained in some maximal subalgebra of maximal rank $\mathfrak{k}$. Of course, rank $\mathfrak{h} <$ rank $\mathfrak{g}$. As in the case of regular subalgebras, $\mathfrak{k}$ may be simple. Then $\mathfrak{h} \subset \mathfrak{k}$ must be Einstein and we can use either induction or the classification results in Chapter 3 to determine the Einstein subalgebras $\mathfrak{h}$ of $\mathfrak{k}$ of strictly smaller rank. In listing these Einstein subalgebras of $\mathfrak{k}$ we must remember to consider the symmetric and isotropy irreducible subalgebras. When $\mathfrak{k}$ is non-simple, then $\mathfrak{k} = \mathbb{R}^d \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_r$, where $d = 0$ or 1 and the $\mathfrak{k}_i$'s are simple. We have to consider subalgebras $\mathfrak{h}$ of the form $\mathfrak{a} \oplus \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r$, where $\mathfrak{a} \subset \mathbb{R}^d$, $\mathfrak{h}_i$ is Einstein in $\mathfrak{k}_i$ and at least one $\mathfrak{h}_i$ has rank strictly less than that of $\mathfrak{k}_i$. That this is enough is shown by

(4.8) LEMMA. — *Let* $\mathfrak{g}$ *be an exceptional simple Lie algebra and* $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ *be a maximal subalgebra of maximal rank. If* $\mathfrak{h} \subset \mathfrak{k}_1 \oplus \mathfrak{k}_2$ *is not of the form* $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ *with* $\mathfrak{h}_i \subset \mathfrak{k}_i$, *then* $\mathfrak{h}$ *cannot be Einstein in* $\mathfrak{g}$.

Proof. — $\mathfrak{h}$ must be of the form $\mathfrak{h}_1 \oplus \Delta\mathfrak{h}_0 \oplus \mathfrak{h}_2 \subset (\mathfrak{h}_1 \oplus \mathfrak{h}_0) \oplus (\mathfrak{h}_0 \oplus \mathfrak{h}_2) \subset \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\mathfrak{h}_0 \neq 0$. We note from paragraph (4.2) that the complexified isotropy representation of $G/(K_1 \times K_2)$ always contains an irreducible factor of the form $\pi_{\lambda_1} \hat{\otimes} \pi_{\lambda_2}$ with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.

TABLE IX

$\mathfrak{g}$	$\mathfrak{h}$	Inclusion	$\mathfrak{k}$	χ
E_6 . .	$3A_1$		$3A_2$	
E_6 . .	$A_1 \oplus A_3$		$A_1 \oplus A_5$	
E_7 . .	D_4		A_7	
E_8 . .	B_4		A_8	
E_8 . .	$4A_1$	$3A_1 \subset E_6$ $A_1 \subset A_2 : \overset{2}{o}$	$E_6 \oplus A_2$	
E_8 . .	$B_2 \oplus B_2$		$A_4 \oplus A_4$ or D_8	
E_8 . .	$A_2 \oplus A_2$		A_8 or D_8	

If $\mathfrak{k}$ is not semi-simple, then we may assume that $\mathfrak{k}_1 = \mathbb{R}$. Then $\mathfrak{h}_0 = \mathbb{R}$, $\mathfrak{h}_1 = 0$ and we obtain a contradiction immediately to $m_0 = 0$ in (1.3). So we may assume that $\mathfrak{k}$ is semi-simple with $\mathfrak{k}_1$ simple.

Indeed, $\mathfrak{h}_0$ must be semi-simple, because the same argument in the previous paragraph can be used. So let $\mathfrak{h}_0'$ be a simple non-trivial ideal in $\mathfrak{h}_0$. We write $\mathfrak{h}_0 = \mathfrak{h}_0' \oplus \mathfrak{h}_0''$. Now $\pi_{\lambda_1} \mid \mathfrak{h}_1 \oplus \mathfrak{h}_0$ contains an irreducible summand of the form $\sigma_1 \hat{\otimes} \sigma_0$ with $\sigma_0 \neq \mathrm{id}$: if not, then $\mathfrak{h}_0 \subset \operatorname{Ker} \pi_{\lambda_1}$, contradicting the fact that all π_{λ_1}'s have finite kernels. The same argument shows that we can even assume that $\sigma_0 \mid \mathfrak{h}_0' \neq \mathrm{id}$. Likewise, $\pi_{\lambda_2} \mid \mathfrak{h}_0 \oplus \mathfrak{h}_2$ contains an irreducible summand of the form $\tau_0 \hat{\otimes} \tau_2$ with $\tau_0 \mid \mathfrak{h}_0' \neq \mathrm{id}$.

Thus $\sigma_1 \hat{\otimes} \sigma_0 \hat{\otimes} \tau_0 \hat{\otimes} \tau_2 \mid \mathfrak{h}_1 \oplus \Delta\mathfrak{h}_0 \oplus \mathfrak{h}_2$ breaks up into irreducible summands with different Einstein constants because $\sigma_0 \hat{\otimes} \tau_0 \mid \Delta\mathfrak{h}_0$ behaves that way. (Let M_1, M_2 be respectively the dominants weights of σ_0 and τ_0. Then $\sigma_0 \hat{\otimes} \tau_0 \mid \Delta\mathfrak{h}_0$ contains $\pi_{M_1+M_2}$ and at least another summand with dominant weight $M_1 + M_2 - \Sigma n_i \alpha_i$, which has strictly smaller Einstein constant.) ■

Below we compile a table of Einstein subalgebras of low dimensional simple Lie algebras, the induced representations specifying their embedding, and the Einstein constants of their isotropy representations. This table will be used throughout the proof of (4.7).

TABLE X

Einstein subalgebras of low dimensional Lie algebras.

$\mathfrak{h}$	Einstein subalgebra (induced representation on $\mathfrak{h}$; Einstein constant)
B_4^*	$A_1^6 \oplus A_1^6(\rho_3 \hat{\otimes} \rho_3;\ 8/3)$, $A_1^2 \oplus A_1^2 \oplus A_1^2([\rho_3 \hat{\otimes} \mathrm{id} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_3 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \rho_3];\ 4)$
D_5^*	$B_2^1 \oplus B_2^1([\rho_5 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_5];\ 8)$, $B_2^3(\mathrm{ad};\ 4)$, $B_4^1(\overset{1}{\mathrm{o}}\!-\!\mathrm{o}\!-\!\mathrm{o}\!=\!\bullet \oplus \mathrm{id},\ 8)$, $A_1^{10} \oplus A_1^{10}([\mathrm{o} \hat{\otimes} \overset{4}{\mathrm{o}}] \oplus [\mathrm{o} \hat{\otimes} \overset{4}{\mathrm{o}}];\ 12/5)$, $A_1^2 \oplus B_3^1([\mathrm{o} \hat{\otimes} \overset{2}{\mathrm{o}}\!-\!\mathrm{o}\!=\!\bullet] \oplus [\mathrm{o} \hat{\otimes} \overset{1}{\mathrm{o}}\!-\!\mathrm{o}\!=\!\bullet];\ 8)$
D_8^*	$B_7^1(\overset{1}{\mathrm{o}}\!-\!\mathrm{o}\!-\!\mathrm{o}\!-\!\mathrm{o}\!-\!\mathrm{o}\!-\!\mathrm{o}\!=\!\bullet \oplus \mathrm{id};\ 14)$, $A_1^2 \oplus B_6^1([\mathrm{o} \hat{\otimes} \overset{2}{\mathrm{id}}] \oplus [\mathrm{o} \hat{\otimes} \overset{1}{\mathrm{o}}\!-\!\mathrm{o}\!-\!\mathrm{o}\!-\!\mathrm{o}\!-\!\mathrm{o}\!=\!\bullet];\ 14)$, $B_2^1 \oplus B_5^1([\rho_5 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_{11}];\ 14)$, $B_3^1 \oplus B_4^1([\rho_7 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_9];\ 14)$, $C_1^4 \oplus C_4^1(v_2 \hat{\otimes} v_8;\ 9)$, $C_2^2 \oplus C_2^2(v_4 \hat{\otimes} v_4;\ 5)$, $B_4^2(\Delta_9;\ 9)$, $D_2^4 \oplus D_2^4(\rho_4 \hat{\otimes} \rho_4;\ 3/4)$, $A_2^3 \oplus A_2^3([\mathrm{ad} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mathrm{ad}];\ 4)$
A_8^*	$B_4^2(\overset{2}{\mathrm{o}}\!-\!\mathrm{o}\!-\!\mathrm{o}\!=\!\bullet;\ 9)$, $A_2^3 \oplus A_2^3(\overset{1}{\mathrm{o}}\!-\!\mathrm{o} \hat{\otimes} \overset{1}{\mathrm{o}}\!-\!\mathrm{o};\ 4)$
A_2	$A_1^4(\overset{2}{\mathrm{o}};\ 3)$, $A_1^1 \oplus \mathbb{R}(\overset{1}{\mathrm{o}};\ 3)$, $\mathbb{R}^2(2)$
A_4	$\mathbb{R}^4(2)$, $B_2^2(\overset{1}{\mathrm{o}}\!=\!\bullet;\ 5)$, $A_3^1 \oplus \mathbb{R}(\overset{1}{\mathrm{o}}\!-\!\mathrm{o}\!-\!\mathrm{o};\ 5)$, $A_2^1 \oplus A_1^1 \oplus \mathbb{R}(5)$
A_5	$\mathbb{R}^5(2)$, $A_3^2(\mathrm{o}\!-\!\overset{1}{\mathrm{o}}\!-\!\mathrm{o};\ 6)$, $C_3^1(\overset{1}{\bullet}\!-\!\bullet\!=\!\mathrm{o};\ 6)$, $A_4^1 \oplus \mathbb{R}(5)$, $A_3^1 \oplus A_1^1 \oplus \mathbb{R}(5)$, $A_2^1 \oplus A_2^1 \oplus \mathbb{R}(5)$, $A_2^5(\overset{2}{\mathrm{o}}\!-\!\mathrm{o};\ 16/5)$, $A_1^3 \oplus A_2^2(\overset{1}{\mathrm{o}} \hat{\otimes} \overset{1}{\mathrm{o}}\!-\!\mathrm{o};\ 13/3)$, $\mathbb{R}^2 \oplus A_1^1 \oplus A_1^1 \oplus A_1^1(\overset{1}{\mathrm{o}} \hat{\otimes} \mathrm{o} \hat{\otimes} \mathrm{o}] \oplus [\mathrm{o} \hat{\otimes} \overset{1}{\mathrm{o}} \hat{\otimes} \mathrm{o}] \oplus [\mathrm{o} \hat{\otimes} \mathrm{o} \hat{\otimes} \overset{1}{\mathrm{o}}];\ 2)$
A_7^*	$\mathbb{R}^7(2)$, $D_4^2\left(\begin{array}{c}\overset{1}{\mathrm{o}}\!-\!\mathrm{o}\\ \qquad\ \mathrm{o}\\ \qquad\ \mathrm{o}\end{array};\ 8\right)$, $C_4^1(\overset{1}{\bullet}\!-\!\bullet\!-\!\bullet\!=\!\mathrm{o};\ 8)$, $A_1^4 \oplus A_3^2(\overset{1}{\mathrm{o}} \hat{\otimes} \overset{1}{\mathrm{o}}\!-\!\mathrm{o}\!-\!\mathrm{o};\ 5)$
C_3	$A_2 \oplus \mathbb{R}(\mathrm{o}\!-\!\overset{1}{\mathrm{o}};\ 4)$, $C_1^1 \oplus C_2^1([\overset{1}{\bullet} \hat{\otimes} \bullet\!=\!\mathrm{o}] \oplus [\bullet \hat{\otimes} \overset{1}{\bullet}\!=\!\mathrm{o}];\ 4)$, $C_1^1 \oplus C_1^1 \oplus C_1^1([\overset{1}{\bullet} \hat{\otimes} \bullet \hat{\otimes} \bullet] \oplus [\bullet \hat{\otimes} \overset{1}{\bullet} \hat{\otimes} \bullet] \oplus [\bullet \hat{\otimes} \bullet \hat{\otimes} \overset{1}{\bullet}];\ 3)$, $C_1^3 \oplus A_1^8(\overset{1}{\bullet} \hat{\otimes} \overset{2}{\mathrm{o}};\ 17/6)$
D_6	$\mathbb{R}^6(2)$, $A_5 \oplus \mathbb{R}(\overset{1}{\mathrm{o}}\!-\!\mathrm{o}\!-\!\mathrm{o}\!-\!\mathrm{o}\!-\!\mathrm{o};\ 10)$, $D_5^1 \oplus \mathbb{R}(10)$, $B_4^1 \oplus A_1^1(10)$, $D_4^1 \oplus D_2^1(10)$, $B_3^1 \oplus B_2^1(10)$, $D_3^1 \oplus D_3^1(10)$, $C_1^3 \oplus C_3^1(\overset{1}{\bullet} \hat{\otimes} \overset{1}{\bullet}\!-\!\bullet\!=\!\mathrm{o};\ 22/3)$, $A_1^2 \oplus A_1^2 \oplus A_1^2 \oplus A_1^2([\rho_3 \hat{\otimes} \mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_3 \hat{\otimes} \mathrm{id} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \rho_3 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \rho_3];\ 4)$, $D_2^1 \oplus D_2^1 \oplus D_2^1([\rho_4 \hat{\otimes} \mathrm{id} \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_4 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \mathrm{id} \hat{\otimes} \rho_4];\ 6)$

* For these subalgebras only the Einstein subalgebras of rank strictly less than rank $\mathfrak{h}$ are listed.

Remarks. — In using Table X, one must bear in mind that if $\mathfrak{h}$ does not have index 1 in $\mathfrak{g}$, then we must divide the constant $(\lambda_1, \lambda_1 + 2\delta)$ by the index to get the correct constant. Also, we need a list of Einstein subalgebras for E_6 and E_7, but this has to be compiled during the proof of (4.7).

Proof of (4.7). — Recall that we are still using the convention in (4.4). We shall use the maximal subalgebras of maximal rank $\mathfrak{k}$ and their isotropy representations as listed in section 2. We shall also use freely computations of indices of simple subalgebras in $\mathfrak{g}$ developed in chapter 2.

I. $\mathfrak{g}=G_2$. – The maximal subalgebras of maximal rank are $A_1^1\oplus\tilde{A}_1^3$ and A_2^1. The diagonally embedded A_1 in $A_1^1\oplus\tilde{A}_1^3$ coincides with $A_1^4\subset A_2^1$ and so is eliminated by (4.8). By (4.8) and (1.4), as well as the condtiion rank $\mathfrak{h}<$ rank $\mathfrak{g}$, we are done. ($A_1^1\subset A_1^1\oplus\tilde{A}_1^3$ is not Einstein because the isotropy representation contains three trivial copies.)

II. $\mathfrak{g}=F_4$. – After ruling out R-subalgebras using (4.8), (1.4), etc., we are left to consider only the following ones:

1. $\mathfrak{k}=B_4^1$.

(*a*) $A_1^6\oplus A_1^6$. To compute $\circ\!-\!\circ\!-\!\circ=\overset{1}{\bullet}\mid\mathfrak{h}$, observe that if $0,\ \pm y_1,\ \pm y_2,\ \pm y_3,\ \pm y_4$ are the weights of ρ_9, and if $0,\ \pm 2x_i$ $(i=1,2)$ are the weights of ρ_3 on each simple factor, then $1/2(y_1+y_2+y_3+y_4)\mid\mathfrak{h}=3x_1+x_2$. Hence $\circ\!-\!\circ\!-\!\circ=\overset{1}{\bullet}\mid\mathfrak{h}\supset\overset{3}{\circ}\hat{\otimes}\overset{1}{\circ}$. Since χ_1 contains $\overset{4}{\circ}\hat{\otimes}\overset{2}{\circ}$, $\mathfrak{h}$ is obviously not Einstein.

(*b*) $A_1^2\oplus A_1^2\oplus A_1^2$. By the same method in (*a*), $\circ\!-\!\circ\!-\!\circ=\overset{1}{\bullet}\mid 3A_1^2\supset\overset{1}{\circ}\hat{\otimes}\overset{1}{\circ}\hat{\otimes}\overset{1}{\circ}$ with constant 9/4.

2. $\mathfrak{k}=A_1^1\oplus C_3^1$. For this we only need to consider $\mathfrak{h}$ of the form $\mathfrak{h}_1\oplus\mathfrak{h}_2$, where $\mathfrak{h}_1\subset A_1^1$ is non-trivial and $\mathfrak{h}_2$ is an Einstein subalgebra of C_3 of rank less than 3. So essentially we need to consider $A_1^1\oplus(A_1^3\oplus A_1^8)$. To compute $\bullet\!-\!\bullet=\overset{1}{\circ}\mid A_1^3\oplus A_1^8$, we let $\pm z_1,\ \pm z_2,\ \pm z_3$ be the weights of ν_6. Then the dominant weight is just $z_1+z_2+z_3$. $\nu_6\mid A_1^3\oplus A_1^8=\overset{1}{\circ}\hat{\otimes}\overset{2}{\circ}$, so it has weights $\pm x_1,\ \pm(x_1+2x_2),\ \pm(x_1-2x_2)$. Hence $z_1+z_2+z_3\mid A_1^3\oplus A_1^8=3x_1$. This shows that

$$\overset{1}{\circ}\hat{\otimes}\bullet\!-\!\bullet=\overset{1}{\circ}\mid A_1^1\oplus(A_1^3\oplus A_1^8)\supset\overset{1}{\circ}\hat{\otimes}\overset{3}{\circ}\hat{\otimes}\circ.$$

This irreducible factor has constant $4\neq 17/6$.

3. $\mathfrak{k}=A_2^1\oplus\tilde{A}_2^2$. We need only consider

(*a*) $A_1^4\oplus\tilde{A}_2^2$. Since $\overset{1}{\circ}\!-\!\circ\mid A_1^4=\overset{2}{\circ}$, $\overset{2}{\circ}\hat{\otimes}\bullet\!-\!\overset{2}{\bullet}\subset\chi\otimes\mathbb{C}$ with constant 13/3.

(*b*) $A_2^1\oplus\tilde{A}_1^8$. Since $\overset{2}{\bullet}\!-\!\bullet\mid\tilde{A}_1^8=\overset{4}{\bullet}\oplus\bullet$, $\overset{1}{\circ}\!-\!\circ\hat{\otimes}\overset{4}{\bullet}\subset\chi\otimes\mathbb{C}$. Since the isotropy representation of $A_2^1\oplus\tilde{A}_1^8\subset A_2^1\oplus\tilde{A}_2^2$ is $\circ\!-\!\circ\hat{\otimes}\overset{4}{\bullet}$, the subalgebra is clearly not Einstein.

III. $\mathfrak{g}=E_6$.

1. $\mathfrak{k}=A_2\oplus A_2\oplus A_2$.

(*a*) $3A_1^4$. $\overset{1}{\circ}\!-\!\circ\hat{\otimes}\overset{1}{\circ}\!-\!\circ\hat{\otimes}\overset{1}{\circ}\!-\!\circ\mid 3A_1^4=\overset{2}{\circ}\hat{\otimes}\overset{2}{\circ}\hat{\otimes}\overset{2}{\circ}$ and has constant 3. The isotropy representation of $3A_1^4$ in $3A_2^1$ is $[\overset{4}{\circ}\hat{\otimes}\circ\hat{\otimes}\circ]\oplus[\circ\hat{\otimes}\overset{4}{\circ}\hat{\otimes}\circ]\oplus[\circ\hat{\otimes}\circ\hat{\otimes}\overset{4}{\circ}]$, which also has constant 3. Hence $3A_1^4$ is Einstein.

(*b*) $A_1^4 \oplus A_2^1 \oplus (A_1^1 \oplus \mathbb{R})$ or similar R-subalgebras. $\chi \otimes \mathbb{C}$ contains the isotropy representation of $A_1^1 \oplus \mathbb{R} \subset A_2^1$, which has constant 3 since we are dealing with a symmetric space.

$$\overset{1}{\circ}\!-\!\circ \,\hat{\otimes}\, \overset{1}{\circ}\!-\!\circ \,\hat{\otimes}\, \overset{1}{\circ}\!-\!\circ \mid \mathfrak{h} \supset \overset{2}{\circ} \,\hat{\otimes}\, \overset{1}{\circ}\!-\!\circ \,\hat{\otimes}\, \overset{1}{\circ} \,\hat{\otimes}\, \overset{1}{t},$$

which has constant $> 1+8/3+3/2 \neq 3$. Thus $\mathfrak{h}$ is not Einstein.

2. $\mathfrak{k} = D_5 \oplus \mathbb{R}$. For R-subalgebras contained in $\mathfrak{k}$, we need only the Einstein subalgebras of D_5^1 of rank at most 4. Let λ_2 be the dominnant weight of $\circ\!-\!\circ\!-\!\circ\!\left\langle\begin{smallmatrix}\overset{1}{\circ}\\ \circ\end{smallmatrix}\right. \hat{\otimes}\, \overset{1}{t}$ and write $\lambda_2 = \lambda_2' + \lambda_2''$ where $\lambda_2' = \lambda_2 \mid D_5$. Since $E_6/(SO(10) \times SO(2))$ is symmetric, $(\lambda_2, \lambda_2 + 2\delta) = 12$. $(\lambda_2', \lambda_2' + 2\delta) = 45/4$ and $(\lambda_2'', \lambda_2'') = 3/4$.

(*a*) $B_2^1 \oplus B_2^1 \oplus \mathbb{R}$. By using the weights of $[\rho_5 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_5]$, we see that $\circ = \overset{1}{\bullet} \,\hat{\otimes}\, \circ = \overset{1}{\bullet} \,\hat{\otimes}\, \overset{1}{t}$ occurs in $\circ\!-\!\circ\!-\!\circ\!\left\langle\begin{smallmatrix}\overset{1}{\circ}\\ \circ\end{smallmatrix}\right. \hat{\otimes}\, \overset{1}{t} \mid \mathfrak{h}$. This summand has constant $5+3/4 \neq 8$. Hence $\mathfrak{h}$ is not Einstein.

(*b*) $A_1^2 \oplus B_3^1 \oplus \mathbb{R}$. $\circ\!-\!\circ\!-\!\circ\!\left\langle\begin{smallmatrix}\overset{1}{\circ}\\ \circ\end{smallmatrix}\right. \hat{\otimes}\, \overset{1}{t} \mid A_1^2 \oplus B_3^1 \oplus \mathbb{R} = \overset{1}{\circ} \,\hat{\otimes}\, \circ\!-\!\circ = \overset{1}{\bullet} \,\hat{\otimes}\, \overset{1}{t}$, which has constant $6+3/4 \neq 8$. So $\mathfrak{h}$ is not Einstein.

(*c*) $B_2^3 \oplus \mathbb{R}$. Let 0, $\pm x_1 \pm x_2$ be the weights of ρ_5, then $\circ = \overset{2}{\bullet}$ is the adjoint representation. Let $\pm z_i$, $i = 1, \ldots, 5$ be the weights of ρ_{10}. Then $1/2(z_1 + \ldots + z_5) = (3/2)\, x_1 + (1/2)\, x_2$. Thus $\circ\!-\!\circ\!-\!\circ\!\left\langle\begin{smallmatrix}\overset{1}{\circ}\\ \circ\end{smallmatrix}\right. \hat{\otimes}\, \overset{1}{t} \mid \mathfrak{h} = \overset{1}{\circ} = \overset{1}{\bullet} \,\hat{\otimes}\, \overset{1}{t}$, which has constant $15/6+3/4 = 13/4 \neq 4$. So $\mathfrak{h}$ is not Einstein.

(*d*) $A_1^{10} \oplus A_1^{10} \oplus \mathbb{R} \subset B_2^1 \oplus B_2^1 \oplus \mathbb{R} \subset D_5 \oplus \mathbb{R}$. Since $\circ = \overset{1}{\bullet} \mid A_1^{10} = \overset{3}{\circ}$, using (*a*), we find that $\overset{3}{\circ} \,\hat{\otimes}\, \overset{3}{\circ} \,\hat{\otimes}\, \overset{1}{t}$ occurs in $\chi \otimes \mathbb{C}$ and its constant is $9/4 \neq 12/5$. So $\mathfrak{h}$ is not Einstein.

3. $\mathfrak{k}=A_1\oplus A_5$. To consider R-subalgebras $\mathfrak{h}\subset\mathfrak{k}$, we need only the Einstein subalgebras of A_5 of rank <5.

(*a*) $A_1^1\oplus A_3^2$. Since

$$\Lambda^3(\overset{1}{o}\!-\!o\!-\!o)=\ \overset{2}{o}\!-\!o\!-\!o\oplus o\!-\!o\!-\!\overset{2}{o},$$

$$\overset{1}{o}\hat{\otimes}o\!-\!o\!-\!\overset{1}{o}\!-\!o\!-\!o\ \big|\ \mathfrak{h}=\ [\overset{1}{o}\hat{\otimes}\overset{2}{o}\!-\!o\!-\!o]\oplus[\overset{1}{o}\hat{\otimes}o\!-\!o\!-\!\overset{2}{o}]$$

which have constant 6. Thus $\mathfrak{h}$ is Einstein.

(*b*) $A_1^1\oplus C_3^1$. $\Lambda^3(\overset{1}{\bullet}\!-\!\bullet=o)=\ \bullet\!-\!\overset{1}{\bullet}=o\oplus\overset{1}{\bullet}\!-\!\bullet=o$, $\overset{1}{o}\hat{\otimes}\overset{1}{\bullet}\!-\!\bullet=o\subset\chi\otimes\mathbb{C}$ with constant $5\neq 6$. Thus $\mathfrak{h}$ is not Einstein.

(*c*) $A_1^1\oplus A_2^5$. $\Lambda^3(\overset{2}{o}\!-\!o)=\ \overset{3}{o}\!-\!o\oplus o\!-\!\overset{3}{o}$, so $\overset{1}{o}\hat{\otimes}\overset{3}{o}\!-\!o\subset\chi\otimes\mathbb{C}$ with constant $39/10\neq 16/5$.

(*d*) $A_1^1\oplus A_1^3\oplus A_2^2$. $\Lambda^3(\overset{1}{o}\hat{\otimes}\overset{1}{o}\!-\!o)\supset\overset{3}{o}\hat{\otimes}o\!-\!o$, so $\overset{1}{o}\hat{\otimes}\overset{3}{o}\hat{\otimes}o\!-\!o\subset\chi\otimes\mathbb{C}$ with constant $4\neq 13/3$.

IV. $\mathfrak{g}=E_7$.

1. $\mathfrak{k}=A_2\oplus A_5$. To consider R-subalgebras $\mathfrak{h}\subset\mathfrak{k}$, we need this time all Einstein subalgebras of A_5 from Table X.

(*a*) $A_1^4\oplus A_1^1$. Since $\overset{1}{o}\!-\!o\ \big|\ A_1^4=\ \overset{2}{o}$, $\overset{2}{o}\hat{\otimes}o\!-\!o\!-\!\overset{1}{o}\!-\!o\!-\!o\subset\chi\otimes\mathbb{C}$ with constant $31/3\neq 3$. Thus $\mathfrak{h}$ is not Einstein.

(*b*) $A_2^1\oplus A_3^2$. Since $\Lambda^2(o\!-\!\overset{1}{o}\!-\!o)$ contains $\overset{1}{o}\!-\!o\!-\!\overset{1}{o}$, $\overset{1}{o}\!-\!o\hat{\otimes}\overset{1}{o}\!-\!o\!-\!\overset{1}{o}$ is a summand of $\chi\otimes\mathbb{C}$ and has constant $20/3\neq 6$. So $\mathfrak{h}$ is not Einstein.

(*c*) $A_2^1\oplus C_3^1$. Since $\Lambda^2(\overset{1}{\bullet}\!-\!\bullet=o)=\ \bullet\!-\!\overset{1}{\bullet}=o\oplus\bullet\!-\!\bullet=o$, $\overset{1}{o}\!-\!o\hat{\otimes}\bullet\!-\!\bullet=o$ is a summand in $\chi\otimes\mathbb{C}$ with constant $8/3\neq 6$. So $\mathfrak{h}$ is not Einstein.

(*d*) $A_2^1\oplus(A_1^3\oplus A_2^2)$. Since $\Lambda^2(\overset{1}{o}\hat{\otimes}\overset{1}{o}\!-\!o)$ contains $o\hat{\otimes}\overset{2}{o}\!-\!o$, $\overset{1}{o}\!-\!o\hat{\otimes}o\hat{\otimes}\overset{2}{o}\!-\!o$ is a summand in $\chi\otimes\mathbb{C}$ with constant $6\neq 13/3$. So $\mathfrak{h}$ is not Einstein.

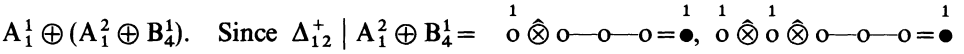
(*e*) $A_2^1\oplus A_2^5$. Since $\Lambda^2(\overset{2}{o}\!-\!o)=\ \overset{2}{o}\!-\!\overset{1}{o}$, $\overset{1}{o}\!-\!o\hat{\otimes}\overset{2}{o}\!-\!\overset{1}{o}$ is a summand of $\chi\otimes\mathbb{C}$ with constant $24/5\neq 16/5$. So $\mathfrak{h}$ is not Einstein.

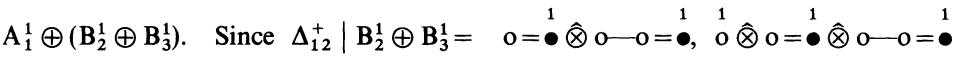
2. $\mathfrak{k}=A_7$.

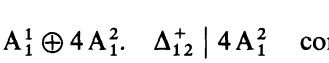
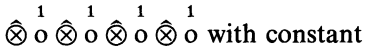
(*a*) $A_1^4\oplus A_3^2$. Since $\Lambda^4(\overset{1}{o}\hat{\otimes}\overset{1}{o}\!-\!o\!-\!o)$ contains $o\hat{\otimes}o\!-\!\overset{2}{o}\!-\!o$ with constant $6\neq 5$, $\mathfrak{h}$ is not Einstein.

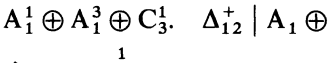
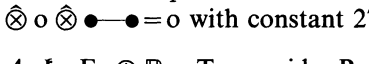
(*b*) C_4^1. Since $\Lambda^4(\overset{1}{\bullet}\!-\!\bullet\!-\!\bullet=o)=\ \bullet\!-\!\bullet\!-\!\overset{1}{\bullet}=o\oplus\bullet\!-\!\bullet\!-\!\bullet=o\oplus\mathrm{id}$, C_4 is clearly not Einstein.

3. $\mathfrak{k}=A_1\oplus D_6$.

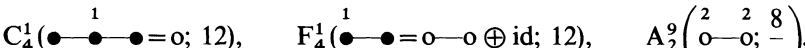
(*a*) $A_1^1 \oplus (A_1^2 \oplus B_4^1)$. Since $\Delta_{12}^+ \mid A_1^2 \oplus B_4^1 =$  is a summand in $\chi \otimes \mathbb{C}$ with constant $45/4 \neq 10$. So $\mathfrak{h}$ is not Einstein.

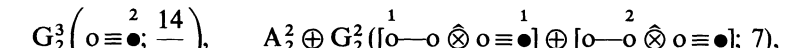
(*b*) $A_1^1 \oplus (B_2^1 \oplus B_3^1)$. Since $\Delta_{12}^+ \mid B_2^1 \oplus B_3^1 =$  is a summand of $\chi \otimes \mathbb{C}$ with constant $31/4 \neq 10$. So $\mathfrak{h}$ is not Einstein.

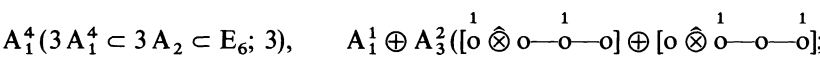
(*c*) $A_1^1 \oplus 4A_1^2$. $\Delta_{12}^+ \mid 4A_1^2$ contains  and so $\chi \otimes \mathbb{C}$ contains  with constant $9/2 \neq 4$. So $\mathfrak{h}$ is not Einstein.

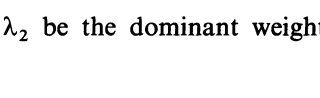
(*d*) $A_1^1 \oplus A_1^3 \oplus C_3^1$. $\Delta_{12}^+ \mid A_1 \oplus C_3$ contains  and so $\chi \otimes \mathbb{C}$ contains  with constant $27/4 \neq 22/3$. So $\mathfrak{h}$ is not Einstein.

4. $\mathfrak{k} = E_6 \oplus \mathbb{R}$. To consider R-subalgebras $\mathfrak{h} \subset \mathfrak{k}$, we need a list of the Einstein subalgebras of E_6 of rank $\leqq 5$. The following list gives also the induced representation corresponding to the inclusion $\mathfrak{h} \subset \mathfrak{k}$ and the constants:

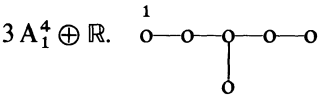






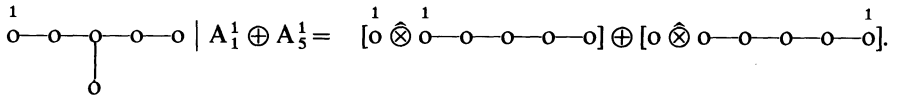
Let λ_2 be the dominant weight of  and write $\lambda_2 = \lambda_2' + \lambda_2''$ as in

III(2). Then $(\lambda_2'', \lambda_2'') = 2/3$ by an analogous calculation.

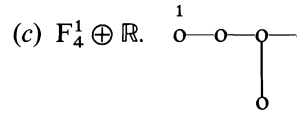
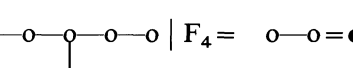
(*a*) $3A_1^4 \oplus \mathbb{R}$.  $\mid \mathfrak{h}$ contains , which has constant

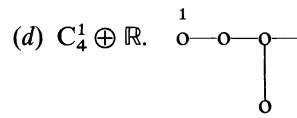
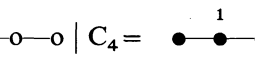
$8/3 \neq 3$.

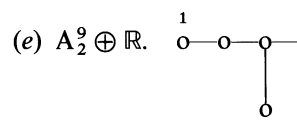
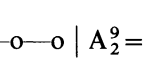
(*b*) $A_1^1 \oplus A_3^2 \oplus \mathbb{R}$.

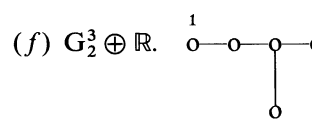
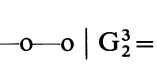


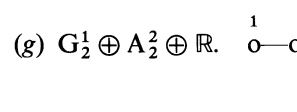
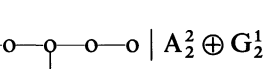
So if we further restrict to $A_1^1 \oplus A_3^2$, we get $[o \hat{\otimes} o - \overset{1}{o} - o] \oplus [o \hat{\otimes} \overset{1}{o} - o - \overset{1}{o}]$. We therefore get a constant $14/3 \neq 6$.

(*c*) $F_4^1 \oplus \mathbb{R}$.  $F_4 =$  $\oplus$ id, which has constant 12. So there is a summand in $\chi \otimes \mathbb{C}$ with constant $38/3 \neq 12$.

(*d*) $C_4^1 \oplus \mathbb{R}$.  $C_4 =$  has constant $26/3 \neq 12$.

(*e*) $A_2^9 \oplus \mathbb{R}$.  $A_2^9 =$  and the corresponding summand has constant $22/9 \neq 8/3$.

(*f*) $G_2^3 \oplus \mathbb{R}$.  $G_2^3 =$  and the corresponding summand has constant $34/9 \neq 14/3$.

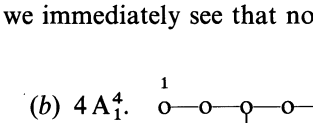
(*g*) $G_2^1 \oplus A_2^2 \oplus \mathbb{R}$.  $A_2^2 \oplus G_2^1 =$  so the corresponding summand has constant $6 \neq 7$.

So none of the above subalgebras are Einstein.

V. $\mathfrak{g} = E_8$.

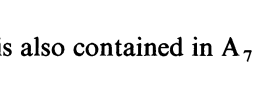
1. $\mathfrak{k} = E_6 \oplus A_2$.

(*a*) $\mathfrak{h} \oplus A_2^1$, where rank $\mathfrak{h} < 6$. By the results of IV(4), using 8/3 in place of the 2/3, we immediately see that none of these subalgebras are Einstein.

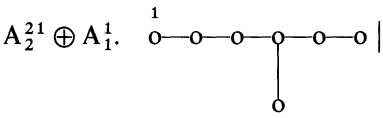
(*b*) $4A_1^4$.  $3A_2 = [\rho_4 \hat{\otimes} \rho_4 \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_4 \hat{\otimes} \rho_4] \oplus [\rho_4 \hat{\otimes} \mathrm{id} \hat{\otimes} \rho_4]$,

and from this it is clear that $4A_1^4$ is Einstein in E_8.

2. $\mathfrak{k} = E_7 \oplus A_1$. For this we need a list of Einstein subalgebras of rank < 7 with the corresponding Einstein constants: $D_4^2(8)$, $A_2^{21}(16/7)$, $G_2^1 \oplus C_3^1(10)$, $A_1^3 \oplus F_4^1(76/3)$.

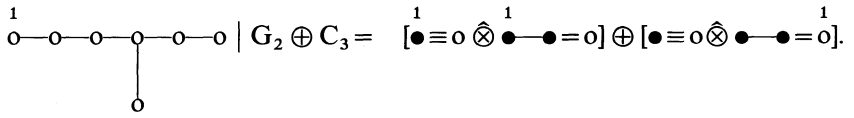
(*a*) $D_4^2 \oplus A_1^1$. $D_4^2 \oplus A_1^1$ is also contained in $A_7 \oplus A_1$. So  is a summand

of $\chi \otimes \mathbb{C}$ with constant $15/2 \neq 8$.

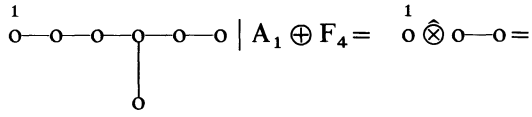
(*b*) $A_2^{21} \oplus A_1^1$.  with corresponding constant

$45/7 \neq 16/7$.

(*c*) $G_2^1 \oplus C_3^1 \oplus A_1^1$.

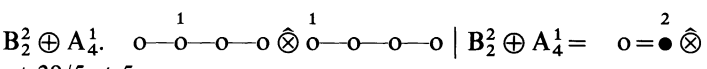


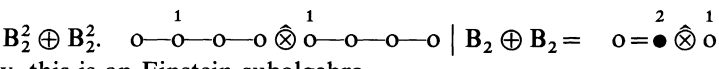
The corresponding constant for the first summand is $9 \neq 10$.

(*d*) $A_1^3 \oplus F_4^1 \oplus A_1^1$.  and so the corresponding constant is $14 \neq 76/3$.

So none of the above subalgebras are Einstein.

3. $2A_4$.

(*a*) $B_2^2 \oplus A_4^1$.  with constant $39/5 \neq 5$.

(*b*) $B_2^2 \oplus B_2^2$.  with constant 5. Clearly, this is an Einstein subalgebra.

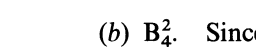
(*c*) $B_2^2 \oplus A_3^1 \oplus \mathbb{R}$ and $B_2^2 \oplus A_2^1 \oplus A_1^1 \oplus \mathbb{R}$ are now easily seen not to be Einstein.

4. $\mathfrak{k} = A_8$.

(*a*) $A_2^3 \oplus A_2^3$. Since

$$\Lambda^3(\mu_9) \mid 2A_2 = \Lambda^3(\rho_3 \hat{\otimes} \rho_3) = [\overset{3}{o} - o \hat{\otimes} o - o] \oplus [o - o \hat{\otimes} \overset{3}{o} - o] \oplus [\overset{1}{o} - \overset{1}{o} \hat{\otimes} \overset{1}{o} - \overset{1}{o}],$$

this is clearly an Einstein R-subalgebra.

(*b*) B_4^2. Since $\Lambda^3 \rho_9 =$  with constant 9, this is again an Einstein R-subalgebra.

5. $k = D_8$.

(*a*) $2A_2^3 \subset 2D_4 \subset D_8$ and $2C_2^2 \subset D_8$. One easily verifies that the isotropy representations are the same as those in 4(*a*) and 3(*b*) respectively. Hence these subalgebras are also Einstein.

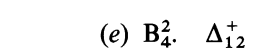
(*b*) $D_2^4 \oplus D_2^4 \subset C_2^2 \oplus C_2^2$. Since

$$\overset{2}{\bullet} = o \,\hat{\otimes}\, \bullet = \overset{1}{o} \mid 2D_2 = ([o \,\hat{\otimes}\, o] \oplus [o \,\hat{\otimes}\, \overset{2}{o}] \oplus [\overset{2}{o} \,\hat{\otimes}\, \overset{1}{o}]) \,\hat{\otimes}\, ([\overset{1}{o} \,\hat{\otimes}\, \overset{1}{o}] \oplus [o \,\hat{\otimes}\, o]),$$

$2D_2^4$ cannot be Einstein.

(*c*) $A_1^4 \oplus C_4^1$. Since $\Delta_{12}^+ \mid A_1 \oplus C_4$ contains $o \,\hat{\otimes}\, \bullet - \bullet - \bullet = \overset{1}{o}$ with constant $12 \neq 9$, $\mathfrak{h}$ is not Einstein.

(*d*) $\Delta_{12}^+ \mid A_1^2 \oplus B_6^1$ contains $\overset{1}{o} \,\hat{\otimes}\, \Delta_{13}$, $\Delta_{12}^+ \mid B_2^1 \oplus B_5^1$ contains $\Delta_5 \,\hat{\otimes}\, \Delta_{11}$ and $\Delta_{12}^+ \mid B_3^1 \oplus B_4^1$ contains $\Delta_7 \,\hat{\otimes}\, \Delta_9$ with respective constants 81/4, 65/4, and 57/4, all of which are not equal to 14. So these subalgebras are not Einstein.

(*e*) B_4^2. $\Delta_{12}^+ \mid B_4 =$  with constant 9. So this is an Einstein subalgebra.

This completes the proof of (4.7). ■

4. S-SUBALGEBRAS

(4.8) PROPOSITION. — *Let* $\mathfrak{g}$ *be an exceptional simple Lie algebra and* $\mathfrak{h}$ *an Einstein* S-*subalgebra which is not strongly isotropy irreducible. Then* $\mathfrak{h}$ *is the maximal subalgebra* $B_2^{12} \subset E_8$ *with isotropy representation* $o = \overset{6}{\bullet} \oplus \overset{3}{o} = \overset{2}{\bullet}$.

Proof. — A table of all S-subalgebras of the exceptional simple Lie algebras together with their inclusion relations can be found on p. 233 in [8]. For the non-simple maximal S-subalgebras, the corresponding isotropy representations are given in Table 35 in [8]. From the Table one checks easily that $\mathfrak{h}$ is Einstein iff it is strongly isotropy irreducible.

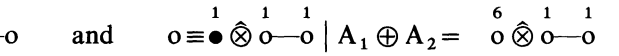
If $\mathfrak{h}$ is a 3-dimensional subalgebra, Theorem 5.2, Corollary 5, and Corollary 8.7 in [14] imply that $\mathfrak{h}$ is Einstein in any simple Lie algebra $\mathfrak{g}$ (not necessarily exceptional) iff $\operatorname{ad} \mathfrak{g} \mid \mathfrak{h} = \overset{2}{o} \oplus \overset{4}{o}$ or $\operatorname{ad} \mathfrak{g} \mid \mathfrak{h} = \overset{2}{o} \oplus \overset{6}{o}$. Hence $(\mathfrak{g}, \mathfrak{h})$ must be isotropy irreducible and there are no such pairs with $\mathfrak{g}$ exceptional.

If $\mathfrak{h}$ is a simple S-subalgebra, then the isotropy representations are listed in Table 24 in [8]. All subalgebras there are isotropy irreducible except for $B_2^{12} \subset E_8$, and one checks that it is Einstein.

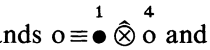
Below we analyse as in sections 2 and 3 the remaining S-subalgebras in Table 39. We retain convention (4.4).

1. $A_1^{28} \oplus A_1^8 \subset G_2^1 \oplus A_1^8 \subset F_4$. $\chi \otimes \mathbb{C} = [\overset{10}{o} \,\hat{\otimes}\, o] \oplus [\overset{6}{o} \,\hat{\otimes}\, \overset{4}{o}]$ with constants 15/7 and $6/7 + 3/2$.

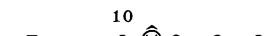
2. $A_1^{28} \oplus A_2^2 \subset G_2^1 \oplus A_2^2 \subset E_6$.

$\pi_{\lambda_1} =$ and 

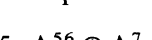
with respective constants 15/7 and 6/7+3.

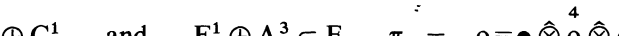
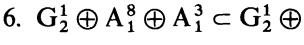
3. $G_2^1 \oplus A_1^8 \subset F_4 \subset E_6$ and $G_2^1 \oplus A_1^8 \subset G_2^1 \oplus A_2^2 \subset E_6$. We immediately obtain summands and  in $\chi \otimes \mathbb{C}$, which obviously have different constants.

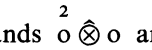
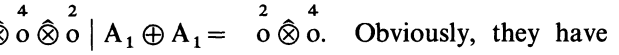
4. $A_1^{28} \oplus C_3^1 \subset G_2^1 \oplus C_3^1 \subset E_7$.

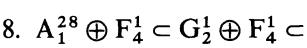
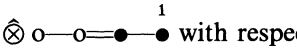
$\pi_{\lambda_1} =$  and

with respective constants $15/7 \neq 6/7+6$.

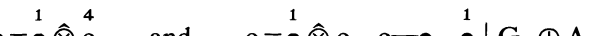
5. $A_1^{56} \oplus A_1^7 \subset G_2^2 \oplus A_1^7 \subset E_7$. $\pi_{\lambda_1} =$  and
with respective constants 15/14 and 15/7.

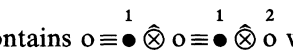
6. $G_2^1 \oplus A_1^8 \oplus A_1^3 \subset G_2^1 \oplus C_3^1$ and $F_4^1 \oplus A_1^3 \subset E_7$. $\pi_{\lambda_1} =$  and
$\pi_{\lambda_2} =$  with respective constants 3/2+4/3 and 11/2.

7. $A_1^{31} \oplus A_1^8 \subset A_1^{28} \oplus A_1^8 \oplus A_1^3 \subset G_2^1 \oplus A_1^8 \oplus A_1^3 \subset E_7$. Immediately we obtain summands  and  Obviously, they have different constants.

8. $A_1^{28} \oplus F_4^1 \subset G_2^1 \oplus F_4^1 \subset E_8$. We have summands  and
 with respective constants 15/7 and 90/7.

9. $G_2^1 \oplus G_2^1 \oplus A_1^8 \subset G_2^1 \oplus F_4^1 \subset E_8$.

$\pi_{\lambda_1} =$ and 

contains  with constants 11/2 and 17/2 respectively.

Hence none of the above S-subalgebras are Einstein. All other S-subalgebras are not Einstein because they fail to be Einstein in some maximal S-subalgebra.

This completes the proof of (4.8). ■

CHAPTER FIVE

Geometrical properties and applications

1. ISOMETRIES AND CURVATURE. — In this section we determine the connected isometry groups of our Einstein manifolds, and show that no two of them are isometric.

Let G/H be a simply connected normal homogeneous Einstein manifold with G compact, connected, and simple. In this section we assume that G acts *effectively* on

G/H. Since G and H are connected, we may pass to their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ whenever convenient. If $(G/H, g_B)$ is (strongly) isotropy irreducible, then J. Wolf showed ([25], Theorem 17.1, p. 141) that $I_0(M, g_B)=G$ unless $G/H=\mathrm{Spin}(7)/G_2$ with $I_0(M, g_B)=SO(8)$, or $G/H=G_2/SU(3)$ with $I_0(M, g_B)=SO(7)$.

(5.1) THEOREM. — *If* $(G/H, g_B)$ *is an effective, simply connected, normal homogeneous Einstein manifold with* G *compact, connected, and simple and such that* G/H *is not (strongly) isotropy irreducible, then* $I_0(M, g_B)=G$.

The proof of this theorem relies on a result of Oniščik, which we will describe shortly.

Let $(\tilde{\mathfrak{g}}, \mathfrak{g}, \mathfrak{k})$ be a triple of compact Lie algebras, where $\mathfrak{g}$ and $\mathfrak{k}$ are subalgebras of $\tilde{\mathfrak{g}}$. Then $(\tilde{\mathfrak{g}}, \mathfrak{g}, \mathfrak{k})$ is a *decomposition* if $\tilde{\mathfrak{g}}=\mathfrak{g}+\mathfrak{k}$. If $\tilde{G}$, G, and K are the corresponding connected Lie groups, then $(\tilde{\mathfrak{g}}, \mathfrak{g}, \mathfrak{k})$ is a decomposition iff G acts transitively on $\tilde{G}/K$. $(\tilde{\mathfrak{g}}, \mathfrak{k})$ is called an *extension* of $(\mathfrak{g}, \mathfrak{h})$ if $(\tilde{\mathfrak{g}}, \mathfrak{g}, \mathfrak{k})$ is a decomposition and $\mathfrak{g}\cap\mathfrak{k}=\mathfrak{h}$. The extension is *effective* if $\tilde{\mathfrak{g}}$ and $\mathfrak{k}$ have no non-trivial ideal in common. The decompositions $(\tilde{\mathfrak{g}}, \mathfrak{g}, \mathfrak{k})$ with $\tilde{\mathfrak{g}}$ simple are listed in Table VII of [18].

$(\tilde{\mathfrak{g}}, \mathfrak{k})$ is called a *type I extension* of $(\mathfrak{g}, \mathfrak{h})$ if there exists a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{h}\oplus\mathfrak{a}\subset\mathfrak{g}$, $\tilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{a}$, $\mathfrak{k}=\mathfrak{h}\oplus\mathfrak{a}$, and the inclusion $\mathfrak{k}\subset\tilde{\mathfrak{g}}$ restricted to $\mathfrak{a}$ is given by the diagonal embedding. Obviously, $(\mathfrak{g}, \mathfrak{h})$ has a type I extension iff the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is non-empty.

$(\tilde{\mathfrak{g}}, \mathfrak{k})$ is called a *type II extension* of $(\mathfrak{g}, \mathfrak{h})$ if $\tilde{\mathfrak{g}}$ is simple. All such extensions are listed in Table VII of [18]. The correspondence between our and his notation is given by $(\tilde{\mathfrak{g}}\ \mathfrak{g}, \mathfrak{k}, \mathfrak{h})=(G, G', G'', U)$.

Type III extensions are defined next. Let $(\mathfrak{m}, \mathfrak{m}', \mathfrak{m}'')$ be a decomposition with $\mathfrak{m}$ simple, and $\mathfrak{a}$ be a simple subalgebra with $\mathfrak{m}''\nsubseteq\mathfrak{a}\subset\mathfrak{m}$. Let $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{a}$, $\mathfrak{k}=\mathfrak{m}'\oplus\mathfrak{m}''$ $\mathfrak{g}=\Delta\mathfrak{a}$, and $\mathfrak{h}=k\cap\Delta\mathfrak{a}$, where $\Delta\mathfrak{a}$ is the image of $\mathfrak{a}$ under the diagonal embedding $\mathfrak{a}\to\mathfrak{m}\oplus\mathfrak{a}=\tilde{\mathfrak{g}}$. Then $(\tilde{\mathfrak{g}}, \mathfrak{k})$ is called a type III extension of $(\mathfrak{g}, \mathfrak{h})$. Notice that in this case $\tilde{G}/K$ is differentially a product manifold $M/M'\times A/M''$. Moreover, by Table VII in [18], in most cases the only possibility for $\mathfrak{a}$ is $\mathfrak{m}$ itself. The exceptions are given by

$(\mathfrak{m}, \mathfrak{m}', \mathfrak{m}'')=$ (*a*) $(so(7), G_2, so(5))$, $so(5)\subset so(6)=\mathfrak{a}\subset so(7)$

(*b*) $(so(4n), so(4n-1), sp(n))$, $sp(n)\subset su(2n)=\mathfrak{a}\subset so(4n)$

(*c*) $(so(8), spin(7), so(5))$, $so(5)\subset so(6)\subset so(7)\subset so(8)$

(*d*) $(so(8), spin(7), so(5)\oplus so(2))$, $so(5)\oplus so(2)\subset so(7)=\mathfrak{a}\subset so(8)$

(*e*) $(so(8), spin(7), so(6))$, $so(6)\subset so(7)=\mathfrak{a}\subset so(8)$.

(In [7], p. 17, it was incorrectly claimed that $\mathfrak{a}=\mathfrak{m}$ in all cases, but this does not affect the proofs there.)

We can now state Oniščik's theorem (Theorem 6.2 in [18]).

(5.2) (Oniščik). Let $(\mathfrak{g}, \mathfrak{h})$ be an effective pair of compact Lie algebras with $\mathfrak{g}$ simple. Then any effective compact extension of $(\mathfrak{g}, \mathfrak{h})$ is either a type I extension or a type I extension of an extension of type II or III.

Proof of (5.1). — Let $\tilde{G}=I_0(M, g_B)$ and K be the isotropy group of $\tilde{G}$ at eH. Then $H=K\cap G$ and $(\tilde{\mathfrak{g}}\ \mathfrak{k})$ is a non-trivial effective compact extension of $(\mathfrak{g}, \mathfrak{h})$. In view of

(5.2) we shall examine extensions of $(\mathfrak{g}, \mathfrak{h})$ of types I, II, and III. Since g_B is Einstein, (1.3) implies that $(\mathfrak{g}, \mathfrak{h})$ has no type I extensions.

For type II extensions we use Table VII in [18]. It follows from this table that $\mathfrak{g}$ is either classical or G_2, and that $\mathfrak{h}$ is either simple or $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$ with $\mathfrak{h}_1$ simple and $\mathfrak{h}_2$ of rank 1. Looking through Table I (Chapter 1), we find that the only $(\mathfrak{g}, \mathfrak{h})$ with a type II extension $(\tilde{\mathfrak{g}}, \mathfrak{k})$ is $(sp(2), sp(1)\oplus u(1))$ with $(\tilde{\mathfrak{g}}, \mathfrak{k})=(su(4), s(u(3)\oplus u(1)))$. Now $\tilde{G}/K=P^3\mathbb{C}$. Homogeneous Einstein metrics on $P^n\mathbb{C}$ were completely determined in [27]. Using the notation of pp. 6-7 of [27], and writting $P^{2n+1}\mathbb{C}$ as $Sp(n+1)/Sp(n)\cdot U(1)$, one easily shows that $B'_{Sp(n+1)}$ induces the metric $\langle\ ,\ \rangle$ on $P^{2n+1}\mathbb{C}$ with $t=1/2$, while the symmetric metric corresponds to $t=1$. In [27] it is shown that the only homogeneous Einstein metrics on $P^{2n+1}\mathbb{C}$ are given by $t=1/(n+1)$ and $t=1$. Hence (as we already know) $B'_{Sp(n+1)}$ is Einstein iff $n=1$. Furthermore, it follows from [27] that the sectional curvature of $B'_{Sp(n+1)}$ satisfies $1/16\leqq K\leqq 1$ with both limits assumed. Hence $B'_{Sp(n+1)}$ can never be isometric to the symmetric metric on $P^{2n+1}\mathbb{C}$. In particular, the connected isometry group of $B'_{Sp(n+1)}$ must be $Sp(n+1)$, which shows that for $(Sp(2)/Sp(1)\cdot U(1), g_B)$, $I_0(G/H, g_B)=G$.

Now let $(\tilde{\mathfrak{g}}, \mathfrak{k})$ be a type III extension of $(\mathfrak{g}, \mathfrak{h})$ constructed from a decomposition $(\mathfrak{m}, \mathfrak{m}', \mathfrak{m}'')$. Such extensions are easily enumerated with the help of Table 7 in [27]. In most cases, $\mathfrak{g}=\Delta\mathfrak{a}\approx\mathfrak{m}$, and the exceptions are formed from the decompositions described just before (5.2). By going through Table I, we see that the only $(\mathfrak{g}, \mathfrak{h})$ with a type III extension is $(spin(8), G_2)$, where $\tilde{\mathfrak{g}}=spin(8)\oplus so(8)$, $\mathfrak{k}=spin(7)\oplus so(7)$, $spin(7)\subset spin(8)$ by $\overset{1}{\circ}\!\!-\!\!\circ\!\!-\!\!\circ\!\!-\!\!\bullet$, $so(7)\subset so(8)$ by $\rho_7\oplus\mathrm{id}$, and $\mathfrak{g}\subset\tilde{\mathfrak{g}}$ by the diagonal embedding of $spin(8)$ into $spin(8)\oplus so(8)$. Notice that $M=G/H$ is differentially the product manifold $S^7\times S^7$. Every $\tilde{G}$-invariant metric on M is a product metric since the isotropy representation of $\tilde{G}/K$ is $[\rho_7\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\rho_7]$, which consists of two inequivalent K-representations. On the other hand, the metric g_B on M is irreducible by Corollary X.5.4 in [15]. Hence the connected isometry group of g_B is *Spin* (8).

By (5.2), the proof of (5.1) is complete. ■

Remark. — We would like to take this opportunity to correct some misleading statements in [27]. Among the homogeneous metrics on $P^{2n+1}\mathbb{C}$ there are two normal metrics, corresponding to $t=1$ and $t=1/2$. All other metrics are not even naturally reductive. Similarly, on $S^{15}=Spin(9)/Spin(7)$, among all the $Spin(9)$-invariant metrics, there are exactly two normal metrics on S^{15}-the symmetric metric and $B'_{Spin(9)}$. All other metrics are again not naturally reductive. In particular, among distance spheres on $P^2\mathrm{Ca}$, one (whose radius is 2/3 the distance to the cut locus) is normal homogeneous and all others are not naturally reductive. But, as follows from the present paper, $B'_{Spin(9)}$ is not Einstein. Hence the $Spin(9)$ invariant Einstein metric on S^{15} is not naturally reductive. The non-symmetric Einstein metric on $P^{2n+1}\mathbb{C}$ is not naturally reductive unless $n=1$, in which case it is normal homogeneous.

(5.3) COROLLARY. — *Let* G/H *be an effective, simply connected, normal homogeneous Einstein manifold with* G *compact, connected, and simple and such that* G/H *is not (strongly)*

isotropy irreductible. Then

(i) *If* G^*/H^* *is another such homogeneous space, then* $(G/H, g_B)$ *is isometric to* $(G^*/H^*, g_B)$ *iff there exists an isomorphism* $\varphi: G \to G^*$ *such that* $\varphi(H) = H^*$. *Hence no two spaces in Table I are isometric.*

(ii) $(G/H, g_B)$ *is not isometric to any strongly isotropy irreducible homogeneous space.*

Proof. — (5.1) implies that $I_0(G/H, g_B) = G \approx G^* = I_0(G^*/H^*, g_B)$ which in turn implies (i). Let G'/H' be strongly isotropy irreducible, and assume that $(G'/H', g_B)$ is isometric to $(G/H, g_B)$. Then $G \approx I_0(G'/H', g_B)$. If $I_0(G'/H', g_B) = G'$, then G/H would be strongly isotropy irreducible, a contradiction. If $I_0(G'/H', g_B) \not\cong G'$, then since $G/H = I_0(G'/H', g_B)/K$ for some K, and $K \supset H'$, $I_0(G'/H', g_B)/K$ is strongly isotropy irreducible. Hence G/H is strongly isotropy irreducible, a contradiction. ■

(5.4) PROPOSITION. — *Let* $(G/H, g_B)$ *be a normal homogeneous Einstein manifold with* G *compact, connected, and simple, Then*

(i) g_B *has non-negative sectional curvature.*

(ii) g_B *has positive sectional curvature iff* G/H *is isometric to a rank* 1 *symmetric space, or* $G/H = Sp(2)/Sp(1) \cdot U(1) = P^3\mathbb{C}$ *with* $1/16 \leqq K \leqq 1$, *or* $G/H = Sp(2)/SU(2)$ *which is an isotropy irreducible rational* 7*-sphere with* $1/37 \leqq K \leqq 1$.

(iii) $(G/H, g_B)$ *is irreducible as a Riemannian manifold.*

Proof. — (i) and (iii) do not require the Einstein condition, and follow immediately from X.3.6 and X.5.4 in [15]. To see (ii), first by Berger's classification [3], if G/H is not diffeomorphic to a rank 1 symmetric space, then it is either $Sp(2)/SU(2)$ or $SU(5)/Sp(2) \cdot U(1)$. The first case is strongly isotropy irreducible, and the second case is not Einstein. If G/H is diffeomorphic to a rank 1 symmetric space, using the classification of compact homogeneous spaces diffeomorphic to rank 1 symmetric spaces (*see*, e. g., [27]) and looking through Table I, we see that G/H must be $Sp(2)/Sp(1) \cdot U(1)$. The pinching estimates follow from [27] and [10]. ■

2. NORMAL HOMOGENEOUS EINSTEIN MANIFOLDS WITH G NON-SIMPLE. — We now give some de Rham irreducible examples to show the necessity of the assumption that G is simple in our classification theorem.

(5.5) PROPOSITION. — (i) *Let* $G = K \times K \times \ldots \times K$ (l *times*, $l \geqq 3$) *and* $H = K$ *with* K *simple and* $H \to G$ *given by* $k \mapsto (k, k, \ldots, k)$. *Then the standard metric on* G/H *is Einstein and* G/H *is not strongly isotropy irreducible.*

(ii) *Let*

$$H = SO(n) \times SO(m) \subset [SO(n) \times SO(n)] \times SO(m) \subset SO(n) \times SO(n+m) = G,$$

where the first embedding is $\Delta \times \mathrm{id}$ *and the second is* $id \times ([\rho_n \hat{\otimes} \mathrm{id}] \oplus [\mathrm{id} \hat{\otimes} \rho_m])$. *Then the standard metric on* G/H *is Einstein iff* $(n-1) \cdot (n-2) = m(m+n-2)$.

(iii) *Let* $H = Sp(n) \times Sp(m) \subset [Sp(n) \times Sp(n)] \times Sp(m) \subset Sp(n) \times Sp(n+m) = G$, *with embedding as in* (ii). *Then the standard metric on* G/H *is Einstein iff* $(2n+1)(n+1) = 2m(m+n+1)$.

Remark. — Examples for solutions in (ii) are $(n, m)=(11,6)$ *and* (66,40), and in (iii) $(n, m)=(1,1)$ and (289, 357). Notice that G/H in (ii) resp. (iii) is diffeomorphic to the Stiefel manifold $SO(n+m)/SO(m)$ resp. $Sp(m+n)/Sp(m)$ and so by [20] and [12] carries a normal homogeneous Einstein metric for any value of m and n although g_B is seldom Einstein.

Proof. — For (i), $\chi=\bigoplus_{i=1}^{l-1} \mathrm{ad}_{\mathfrak{k}}$, so $C_{\chi,\,Q}=a\,\mathrm{Id}$ for any bi-invariant metric Q on $\mathfrak{k}$. For (ii)

$$\chi=[\mathrm{ad}_{so(n)}\hat{\otimes}\mathrm{id}]\oplus[\rho_n\hat{\otimes}\rho_m] \qquad \text{and} \qquad B_G\,|\,so(n)=2(2n+m-4)\,B'_{SO(n)},$$
$$B_G\,|\,so(m)=2(n+m-2)\,B'_{SO(m)}.$$

The Einstein constants are now easily computed, giving the condition in (ii). For (iii), similarily,

$$\chi=[\mathrm{ad}_{sp(n)}\hat{\otimes}\mathrm{id}]\oplus[\nu_{2\,n}\hat{\otimes}\nu_{2\,m}]$$

and

$$B_G\,|\,sp_{(n)}=2(2n+m+2)\,B'_{Sp(n)}, \qquad B_G\,|\,sp(m)=2(n+m+1)\,B'_{Sp(m)}. \quad \blacksquare$$

If G is not simple, it would be more appropriate to classify all normal homogeneous Einstein metrics (or more generally all naturally reductive Einstein metrics) than to classify only the Einstein standard metrics. In principle, such a classification is possible using the methods developed in this paper, although in practice it seems rather cumbersome. By [11] (which uses (1.9)), if G/H carries a naturally reductive Einstein metric g which is not locally symmetric, then the scalar curvature is positive, and hence $\mathfrak{g}$ is a compact Lie algebra. It is natural also to assume that (G/H, g) is an irreducible Riemannian manifold. The classification would then go inductively as follows.

Let $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ with $\mathfrak{g}_1$ simple. Then $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2\oplus\mathfrak{h}_3$ with $\mathfrak{h}_1\oplus\mathfrak{h}_2\subset\mathfrak{g}_1$, $\mathfrak{h}_2\oplus\mathfrak{h}_3\subset\mathfrak{g}_2$, and the inclusion $\mathfrak{h}\subset\mathfrak{g}$ is given by $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)\to(\mathfrak{h}_1\oplus\mathfrak{h}_2, \mathfrak{h}_2\oplus\mathfrak{h}_3)$. We can assume that $\mathfrak{h}_2\neq 0$ since otherwise the metric would be a product metric. For simplicity we assume also that $\mathfrak{h}_2$ is simple.

Let $B_1=B_{G_1}$, $B_2=B_{G_2}$, then

$$Q=\beta_1 B_1\perp Q_2\,(Q_2=Q\,|\,g_2), \qquad \text{and} \qquad B_G=B_{G_1}\perp B_{G_2}=\alpha_{G_1}B'_1\perp B_2.$$

Using B_1 and Q_2, we have orthogonal decompositions

$$\mathfrak{g}_1=\mathfrak{h}_1\oplus\mathfrak{h}_2\oplus\mathfrak{m}_1, \qquad \mathfrak{g}_2=\mathfrak{h}_2\oplus\mathfrak{h}_3\oplus\mathfrak{m}_2.$$

The isotropy representation

$$\chi=[\chi_1\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\mathrm{ad}_{\mathfrak{h}_2}\hat{\otimes}\mathrm{id}]\oplus[\mathrm{id}\hat{\otimes}\chi_2].$$

By (1.9), if the metric $Q\,|\,\mathfrak{h}^{\perp}$ is Einstein with Einstein constant E, then

$$C_{\chi_1,\,Q\,|\,\mathfrak{h}_1\oplus\mathfrak{h}_2}=(2E-1/(2\beta_1))\,\mathrm{Id}.$$

The possibilities for pairs $(\mathfrak{g}_1, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ satisfying this condition can be classified by the methods of this paper. If $\mathfrak{g}_1$ is a classical Lie algebra and $Q|\mathfrak{h}_1 \oplus \mathfrak{h}_2$ is positive definitive, the classification was done in Chapter 3. Similarily, one can classify the admissible pairs where the metric on $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ is non-degenerate but not positive definite or where $\mathfrak{g}_1$ is an exceptional Lie algebra.

The simplest case for which the Einstein condition is not over-determined is when $(\mathfrak{g}_1, \mathfrak{h}_1 \oplus \mathfrak{h}_2)$ is strongly isotropy irreducible and $\mathfrak{g}_2 = \mathfrak{h}_2$ is also simple. (5.5 (ii), (iii)) are special instances of this case. We may assume that $Q_2 = B_2$ after re-normalization. The Einstein condition becomes a quadratic equation in β_1, and one can show that it always has two positive real solutions. One of these corresponds to a normal homogeneous Einstein metric, the other corresponds to a naturally reductive one. G/H is diffeomorphic to G_1/H_1. The corresponding metric on G_1/H_1 can be described as the metric obtained by scaling in the direction of H_2 in the fibration $G_1/H_1 \to G_1/(H_1 \times H_2)$. This situation was examined by G. Jensen [12] when $G_1/(H_1 \times H_2)$ is irreducible symmetric and by [7] when $G_1/(H_1 \times H_2)$ is strongly isotropy irreducible but non-symmetric.

3. LEFT INVARIANT EINSTEIN METRICS. — Let G be a compact semisimple Lie group. Then any metric on $\mathfrak{g}$ determines a unique left invariant metric on G. If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, one can consider for $t > 0$ the metric $g_t = t(B_G|\mathfrak{h}) \perp (B_G|\mathfrak{h}^\perp)$ as a left-invariant metric on G. If $t = 1$, g_t is Einstein since $\mathrm{Ric}(g_B) = (1/4)\,B$. G. Jensen [12] first considered the question when g_t is Einstein for $t \neq 1$. Subsequently D'Atri and Ziller obtained the following

(5.7) THEOREM ([7] Corollary 2, p. 44). — *If $\mathfrak{h}$ is not an ideal in $\mathfrak{g}$, then there exists a unique $t \neq 1$ with g_t Einstein iff the standard metric on G/H is Einstein and $B_H = c\,B_G|\mathfrak{h}$ for some $c > 0$. Furthermore, $t < 1$ and g_t is normal homogeneous with respect to $G \times H$.*

In [7] these metrics were examined when G/H is strongly isotropy irreducible. If H is simple, $B_H = c\,B_G|\mathfrak{h}$ is automatically satisfied. If H is not simple, there are only six cases for which $B_H = c\,B_G|\mathfrak{h}$. Five of these are listed in [7], p. 46. The sixth case is $sp(1) \oplus so(4) \subset sp(4)$, which comes from one of the two families of isotropy irreducible spaces omitted in [25].

Next we use the results of this paper to obtain a complete classification of the Einstein metrics g_t, $t \neq 1$, in (5.7) if G is simple.

(5.8) THEOREM. — *Let G be a compact, connected, simple Lie group and H a* semi-simple *subgroup such that G/H is normal homogeneous Einstein but not strongly isotropy irreducible. Then $B_H = c\,B_G|\mathfrak{h}$, $c > 0$, (and hence g_t is a left invariant Einstein metric for some $t < 1$) unless G/H is given by No. 8 in Table I A or No. 6 in Table I B.*

Remark. — In particular, each member of the infinite family of normal homogeneous Einstein manifolds given by Nos. 4 and 5 of Table I A gives rise to a left invariant Einstein metric on $SO(n)$. By Theorem 5, p. 24, of [7], two such g_t are isometric iff the corresponding standard metrics g_B on G/H are isometric.

Proof. — If $\mathfrak{h} \subset \mathfrak{g}$ is a simple subalgebra and c is defined by $B_H = c\,B_G|\mathfrak{h}$, then it follows from our definitions in Chapter 2 that $c = \alpha_H/(\alpha_G[\mathfrak{g}:\mathfrak{h}])$. Hence if $\mathfrak{h}_i$ are the simple

factors of $\mathfrak{h}$, $B_H = c\,B_G|\mathfrak{h}$ implies that $\alpha_{H_i}/[g:\mathfrak{h}_i]$ is independent of i. One can now examine each case in Tables I A and B. If $\mathfrak{h}$ is of maximal rank in $\mathfrak{g}$ and all roots of $\mathfrak{g}$ have the same length, then $[\mathfrak{g}:\mathfrak{h}_i]=1$, and hence we only need to check that α_{H_i} is independent of i. Notice that Table I B, No. 4 is an example where α_{H_i} and the indices are different, but c is still the same for all i. The only non-trivial case is Table I A, No. 5 (No. 4 being a special case of No. 5). But in section 2-3 C we showed that in this case

$$B'_{SO(n_i)}|\mathfrak{h}_i = \left(\frac{n_i}{4\dim\mathfrak{h}_i}\right)B_{G_i}|\mathfrak{h}_i = \left(\frac{n_i}{4\dim\mathfrak{h}_i - 2n_i}\right)B_{H_i},$$

where $n_i = \dim\pi_i$, and since $B'_{SO(n)}|so(n_i) = B'_{SO(n_i)}$, we have

$$B_{SO(n)}|\mathfrak{h}_i = \left(\frac{n_i(n-2)}{2\dim\mathfrak{h}_i - n_i}\right)B_{H_i},$$

so

$$B_{H_i} = \left(\frac{2\dim\mathfrak{h}_i}{n_i} - 1\right)\left(\frac{1}{n-2}\right)B_{SO(n)}|\mathfrak{h}_i.$$

But since $\dim\mathfrak{h}_i/n_i$ is independent of i, we have $B_H = c\,B_G$. ■

Remark. — It was shown in [7] that a left invariant metric $\langle\ ,\ \rangle$ on a simple Lie group G is naturally reductive with respect to some transitive group of isometries iff there exists a subalgebra $\mathfrak{h}\subset\mathfrak{g}$ (written as $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1\oplus\ldots\oplus\mathfrak{h}_r$ where $\mathfrak{h}_i$ is simple and $\mathfrak{h}_0$ is abelian) such that $\langle\ ,\ \rangle$ is given by $(\alpha B_G|\mathfrak{h}^{\perp})\perp(g|\mathfrak{h}_0)\perp(\alpha_1 B_G|\mathfrak{h}_1)\perp\ldots\perp(\alpha_r B_G|\mathfrak{h}_r)$, where $\alpha,\ \alpha_i>0$ and g is an arbitrary metric on $\mathfrak{h}_0$. If $\langle\ ,\ \rangle$ is Einstein, then $g=\alpha_0 B_G|\mathfrak{h}_0$, and if we normalize $\langle\ ,\ \rangle$ so that $\alpha=1$, then the Einstein condition in [7], p. 33, in our present notation becomes

$$\alpha_0 = 4E, \qquad (1-\alpha_i^2)c_i + \alpha_i^2 = 4E\alpha_i$$

$$C_{\chi,\ \Sigma(\alpha_i-1)B|\mathfrak{h}_i} = \frac{1}{2}(1-4E)\,\mathrm{Id},$$

where χ is the isotropy representation of G/H, $B_{H_i}=c_i B_G$, and E is the Einstein constant. Hence we again need $C_{\chi,\ Q}=a\,\mathrm{Id}$ with respect to some bi-invariant metric Q. Notice that these equations will be over-determined unless G/H is strongly isotropy irreducible, a case which was examined completely in [7].

4. FIBRATIONS OF EINSTEIN MANIFOLDS. — For this section we need the following result, which was obtained independently by L. Bérard-Bergery (see the forthcoming book by A. Besse on Einstein manifolds) and T. Matsuzawa [16]:

(5.9) THEOREM (Bérard-Bergery, Matsuzawa). — *Let* $F\to M\to B$ *be a Riemannian submersion with totally geodesic fibres. Assume that the metrics on* B, M, *and* F *are Einstein with Einstein constants* E_B, E_M, E_F *respectively and* $E_F>0$. *Furthermore, if* M *is not locally a Riemannian product of* F *and* B, *then the metric* g_t *obtained by scaling the*

metric on M *in the direction of* F *by a factor* $t>0$ *is Einstein iff* $t=1$ *or* $t=E_F/(E_B-E_F)$. *In particular,* g_t *gives rise to a different Einstein metric on* M *iff* $E_F \neq 1/2\, E_B$.

The only previously known examples which satisfy the assumptions in (5.9) are (*a*) the Hopf fibrations

$$S^3 \to S^{4n+3} \to P^n \mathbb{H}$$
$$S^7 \to S^{15} \to S^8$$
$$S^2 \to P^{2n+1} \mathbb{C} \to P^n \mathbb{H},$$

where (5.9) gives rise to the non-symmetric Einstein metrics found by G. Jensen [12], Bourguignon-Karcher [4], and W. Ziller [27], and (*b*) fibrations of the form $K_2 \to G/K_1 \to G/K_1 \cdot K_2$, where $G/K_1 K_2$ is isotropy irreducible and $B_{K_2} = c\, B_G | \mathfrak{k}_2$. If K_2 is non-abelian, it was shown in [12] and [7] that there are two distinct homogeneous Einstein metrics on G/K_1. For each of these, we may apply (5.9) again. In all the above examples, $E_F \neq 1/2\, E_B$, but in this section we will obtain several fibrations with $E_F = 1/2\, E_B$.

Let $H \subset K \subset G$ where G, K are compact semisimple and $B_K = c\, B_G | \mathfrak{k}$ for some constant $c>0$. We consider the Riemannian submersion with totally geodesic fibres

$$K/H \to G/H \to G/K$$

where the metrics are given by B_G and $B_G | \mathfrak{k}$. Note that even if G is simple, K/H need not be effective, so we denote by $\bar{K}/\bar{H}$ the corresponding (almost) effective quotient. In such a case we shall only assume that $B_{\bar{K}} = c\, B_G | \bar{\mathfrak{k}}$.

If the standard metrics on G/H and G/K are Einstein, then, since the isotropy representation of $\bar{K}/\bar{H}$ is contained in that of G/H, the standard metric on $\bar{K}/\bar{H}$ is automatically Einstein. If in addition G is simple and $\bar{K}/\bar{H}$ is not flat, then (5.9) yields a new Einstein metric on G/H iff $E_F \neq 1/2\, E_B$. Of course, if $\bar{H}$ is trivial we are back in the situation considered in the previous section, where $E_F < 1/2\, E_B$. Hence we shall assume that $\dim \bar{H} > 0$. From Table I one can easily compile a complete list of fibrations of the above type where $\dim \bar{H} > 0$ and $\bar{K}/\bar{H}$ is not flat. This list is given in Table XI, where we follow the same order as that in Table I. When G is exceptional one sometimes has to refer to Chapter 4 to obtain all possibilities for K. Otherwise, the inclusions $H \subset K \subset G$ are easily deduced from Table I.

(5.10) THEOREM. — *Let* $\bar{K}/\bar{H} \to G/H \to G/K$ *be one of the fibrations in Table XI. Then besides the standard homogeneous metric there is always another Einstein metric* g_t *except in the following cases for which* $E_F = 1/2\, E_B$:

(*a*) *No.* 1

$$SO(8) \supset U(4) \supset T^4$$
$$SU(4) \supset S(U(2) \times U(2)) \supset T^3$$
$$SO(2n+2) \supset SO(2n)\, SO(2) \supset T^{n+1};$$

TABLE XI

Fibrations of Einstein metrics $K/H \to G/H \to G/K$

No.	g	h	k	Remarks
1	one of $su(n)$, $so(2n)$, E_6, E_7, E_8	$\mathfrak{t}$ = Lie algebra of maximal torus	maximal rank with $(G/K, g_B)$ Einstein, $B_{\mathfrak{t}} = c \cdot B_{\mathfrak{g}} \mid \mathfrak{t}$	
2a	$so(n_1 n_2 k)$	$n_1 n_2 so(k)$	$n_1 so(n_2 k)$	$k \geqq 3$, $n_i \geqq 2$
b	$so(nk)$	$n\,so(k)$	$so((n-1)k) \oplus so(k)$	$k \geqq 3$, $n \geqq 3$
3*a*	$sp(n_1 n_2 k)$	$n_1 n_2 sp(k)$	$n_1 sp(n_2 k)$	$k \geqq 1$, $n_i \geqq 2$
b	$sp(nk)$	$n\,sp(k)$	$sp((n-1)k) \oplus sp(k)$	$k \geqq 1$, $n \geqq 3$
4*a*	$su(n_1 n_2 k)$	$s(n_1 n_2 u(k))$	$s(n_1 u(n_2 k))$	$k \geqq 2$, $n_i \geqq 2$
b	$su(nk)$	$s(\underset{l}{nu}(k))$	$s(u((n-1)k) \oplus u(k))$	$k \geqq 2$, $n \geqq 3$
5	$so(nk)$	$\bigoplus_{i=1} \mathfrak{h}_i$	$\bigoplus_{i=i_s+1} nso(k)$	see Table I A No. 5
	$\dim \pi_i / \dim h_i$ independent of i		$\bigoplus_{i=i_s+1} \mathfrak{h}_i \subset so(k)$	$0 = i_1 < i_2 < \ldots < i_{n+1} = l$ $n \geqq 2$
6	$su(pq+l)$	$[u(l) \oplus u(p) \oplus u(q)]/u(1) \oplus u(1)$	$s(u(l) \oplus u(pq))$	$p^2+q^2+1 = lpq$ $p, q \geqq 2$, $l \geqq 3$
7	$sp(3n-1)$	$sp(n) \oplus u(2n-1)$	$sp(n) \oplus sp(2n-1)$	$n \geqq 1$
8	$so(3n+2)$	$so(n) \oplus u(n+1)$	$so(n) \oplus so(2n+2)$	$n \geqq 3$
9	$so(26)$	$sp(1) \oplus sp(5) \oplus so(6)$	$so(20) \oplus so(6)$	$\nu_2 \hat{\otimes} \nu_{10}$
10.	$spin(8)$	G_2	$spin(7)$	
11.	F_4	$spin(8)$	$spin(9)$	
12.	E_6	$3\,so(3)$	$3\,su(3)$	
13.	E_6	$spin(8) \oplus 2\,so(2)$	$spin(10) \oplus so(2)$	
14.	E_6	$su(2) \oplus so(6)$	$su(2) \oplus su(6)$	
15.	E_7	$so(8)$	$su(8)$	
16.	E_7	$spin(8) \oplus 3\,su(2)$	$so(12) \oplus su(2)$	
17*a*. . . .	E_7	$7\,su(2)$	$spin(12) \oplus su(2)$	
b			$spin(8) \oplus spin(4) \oplus su(2)$	
18.	E_8	$so(9)$	$su(9)$	ρ_9
19.	E_8	$spin(9)$	$spin(16)$	Spin representation
20.	E_8	$4\,su(3)$	$E_6 \oplus su(3)$	
21.	E_8	$4\,so(3)$	$4\,su(3)$	ρ_3
22.	E_8	$spin(8) \oplus spin(8)$	$spin(16)$	
23*a*. . . .	E_8	$8\,su(2)$	$spin(16)$	
b			$spin(8) \oplus spin(8)$	
c			$E_7 \oplus su(2)$	
24*a*. . . .	E_8	$so(5) \oplus so(5)$	$spin(16)$	$o \overset{1}{=} \bullet \hat{\otimes} o \overset{1}{=} \bullet$
b			$su(5) \oplus su(5)$	
25*a*. . . .	E_8	$su(3) \oplus su(3)$	$su(9)$	$\mu_3 \hat{\otimes} \mu_3$
b			$so(8) \oplus so(8)$	$su(3) \subset so(8)$ by ad
c			$spin(16)$	

(*b*) *No.* 3*a*

$$\mathrm{Sp}(6) \supset 3\,\mathrm{Sp}(2) \supset 6\,\mathrm{Sp}(1)$$

$$\mathrm{Sp}(6) \supset 2\,\mathrm{Sp}(3) \supset 6\,\mathrm{Sp}(1);$$

(*c*) *No.* $4a$, $n_1 n_2 = 4$, $k \geqq 2$,

(*d*) *No.* 14.

Proof. – We have $E_B = (1/4) - (1/2)\, B^*_G(\lambda, \lambda + 2\delta)$ and since $B_{\bar{K}} = c\, B_G \big| \bar{\mathfrak{k}}$, $E_F = c[(1/4) - (1/2)\, B^*_{\bar{K}}(\mu, \mu + 2\delta)]$, where λ and μ are dominant weights of the isotropy representations of G/K and $\bar{K}/\bar{H}$. The following observations simplify the calculations considerably:

(*a*) If G/K and $\bar{K}/\bar{H}$ are both symmetric, then $E_F = (1/2)\, E_B$ iff $c = 1/2$. This excludes No. 7, 8, 11, 13, 15, 16, and 22 and for No. 14 it shows that $E_F = (1/2)\, E_B$.

(*b*) If G/K is symmetric and $c < 1/2$, then $E_F < (1/2)\, E_B$. This excludes No. 19, $23a$, $24a$, and $25c$.

(*c*) If $c \leqq 1/4$, then $E_F < (1/2)\, E_B$. This excludes No. 12, 21,$23b$, $24b$, and $25b$.

(*d*) If $E_B > c$ then $E_F < (1/2)\, E_B$. This excludes No. $2a$, $3a$, $4a$, 5, 18 and $25a$.

Here we have used $0 < -B^*_G(\lambda, \lambda + 2\delta)$, $-B^*_{\bar{K}}(\mu, \mu + 2\delta) \leqq 1/2$, which follows from (1.6) and (1.7). The remainder of the cases are settled by a direct calculation. ■

Remarks. – (*a*) It is not always the case that $E_F < (1/2)\, E_B$. Hence in contrast to the situation when $\bar{H}$ is trivial, sometimes $t > 1$ and sometimes $t < 1$ for the new Einstein metric.

(*b*) One can easily show that the new Einstein metric on G/H is not naturally reductive except in the following two cases:

In No. 10 the new Einstein metric is the product metric on $\mathrm{Spin}(8)/G_2 = S^7 \times S^7$, and in No. 7, $n = 1$, the new Einstein metric is the symmetric metric on $\mathrm{Sp}(2)/\mathrm{Sp}(1)\, U(1) = P^3\, \mathbb{C}$. This follows since one shows, using (5.2), that except in the above two cases G is the full isometry group of g_t and no subgroup of G acts transitively on G/H.

Note added in proof. – To be precise, one should include in Table 1 A and Table 1 B the case of a biinvariant metric on a compact, simple, simply connected Lie group G, i.e. $H = \{e\}$, which was mentioned in (1.3) and (1.6). In (5.1) and (5.3) (ii) this case should then be excluded, since the metric is isometric to the strongly isotropy irreducible symmetric space $G \times G/\Delta G$. The proof of (5.1) and (5.3) are easily modified.

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McKENZIE Y. WANG,
Department of Mathematics & Statistics,
McMaster University,
Hamilton, Ontario,
Canada, L8S 4K1;

Wolfgang ZILLER,
Department of Mathematics,
University of Pennsylvania,
Philadelphia, PA 19104,
U.S.A.