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BASIC COVARIANT DIFFERENTIAL OPERATORS ON HERMITIAN SYMMETRIC SPACES (1)

By Hans Plesner JAKOBSEN (2)

Dedicated to the memory of Stephen M. Paneitz.

Introduction

Let $\mathscr{D} = G/K$ be a Hermitian symmetric space of the non-compact type. By a covariant differential operator (CDO) we mean a matrix-valued differential operator D which intertwines two holomorphically induced representations of G. Specifically, there must be, for $i=1, 2$, a finite-dimensional vector space V_i and a representation U_i of G on the space of V_i-valued holomorphic functions on $\mathscr{D}$ such that D is $\mathrm{Hom}(V_1, V_2)$- valued and such that, furthermore,

$$D(U_1(g) f) = U_2(g)(Df),$$

for all holomorphic V_1-valued functions f, and for all $g \in G$. We may, and will often, assume that G is simply connected. In the bounded realization of $\mathscr{D}$, D is forced to be of constant coefficients.

We present here a complete classification of those operators for which either $\dim V_1 = 1$ or $\dim V_2 = 1$. Along with a class of elementary operators —containing all first order operators— which we also describe, these are basic in the sense that most, if not all, other operators D can be determined from the knowledge of these.

Dual to the notion of a CDO is that of a homomorphism between generalized Verma modules. Any CDO gives rise, by duality, to a homomorphism, and conversely. Explicit results concerning this duality have been obtained in joint work with Michael Harris ([5], [6]). The results of [7] and [9] thus have got dual analogues and, working in the opposite direction, Proposition 7.3 of [10] is seen to determine the full set of first order operators.

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In a series of articles in the mid-seventies Lepowsky made many important contributions to the theory of homomorphisms between generalized Verma modules. Particular attention was paid to "scalar" modules ([13] to [17]). Recently there has been significant contributions to the general situation by Boe [2] and by Boe and Collingwood [3].

The article is organized as follows: In chapter 1 we recall some of the results obtained in collaboration with Harris. Further, based upon a refinement due to Boe of a result of Lepowsky's, we prove a result which has the classification of first order operators as a corollary. In chapter 2 we classify the set of homomorphisms into scalar modules and in chapter 3 address the situations in which the homomorphism originates in a scalar module. Finally, in chapter 4, we use these results to fill in some of the finer details in the description of the set of homomorphisms into scalar modules. We conclude with an example from outside the realm of Hermitian symmetric spaces.

1. Covariant differential operators

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. We assume that $\mathfrak{k}$ has a non-empty center η; in this case $\eta=\mathbb{R}\cdot h_0$ for an $h_0\in\eta$ whose eigenvalues under the adjoint action on $\mathfrak{p}^{\mathbb{C}}$ are $\pm i$. Let

$$\mathfrak{p}^+=\{z\in\mathfrak{p}^{\mathbb{C}}\,|\,[h_0, z]=iz\},$$

and

$$\mathfrak{p}^-=\{z\in\mathfrak{p}^{\mathbb{C}}\,|\,[h_0, z]=-iz\}.$$

Let $\mathfrak{k}_1=[\mathfrak{k}, \mathfrak{k}]$ and let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{k}=\mathfrak{k}_1\oplus\mathbb{R}\,h_0$, $\mathfrak{h}=(\mathfrak{h}\cap\mathfrak{k}_1)\oplus\mathbb{R}\cdot h_0$, $(\mathfrak{h}\cap\mathfrak{k}_1)^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_1^{\mathbb{C}}$, and $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We let σ denote the conjugation in $\mathfrak{g}^{\mathbb{C}}$ relative to the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$. The sets of compact and non-compact roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}$ are denoted Δ_c and Δ_n, respectively. $\Delta=\Delta_c\cup\Delta_n$. We choose an ordering of Δ such that

$$\mathfrak{p}^+=\sum_{\alpha\in\Delta_n^+}\mathfrak{g}^\alpha,$$

and set

$$\mathfrak{g}^+=\sum_{\alpha\in\Delta^+}\mathfrak{g}^\alpha,\qquad \mathfrak{g}^-=\sum_{\alpha\in\Delta^-}\mathfrak{g}^\alpha,$$

and

$$\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha.$$

Throughout, β denotes the unique simple non-compact root. For $\gamma \in \Delta$ let H_γ be the unique element of $i\mathfrak{h} \cap [(\mathfrak{g}^{\mathbb{C}})^\gamma, (\mathfrak{g}^{\mathbb{C}})^{-\gamma}]$ for which $\gamma(H_\gamma)=2$. Then for all γ_1 in Δ

$$(1.1) \qquad \langle \gamma_1, \gamma \rangle = \frac{2(\gamma_1, \gamma)}{(\gamma, \gamma)} = \gamma_1(H_\gamma),$$

where $(.,.)$ is the bilinear form on $(\mathfrak{h}^{\mathbb{C}})^*$ obtained from the Killing form of $\mathfrak{g}^{\mathbb{C}}$. The reflexion corresponding to $\gamma \in \Delta$ is denoted by σ_γ;

$$(1.2) \qquad \sigma_\gamma(\gamma_1) = \gamma_1 - \langle \gamma_1, \gamma \rangle \gamma.$$

For $\alpha \in \Delta_n^+$ choose $z_\alpha \in (\mathfrak{g}^{\mathbb{C}})^\alpha$ such that

$$(1.3) \qquad [z_\alpha, z_\alpha^\sigma] = H_\alpha,$$

and let $z_{-\alpha} = z_\alpha^\sigma$. Following the notation of [18] we let γ_r denote the highest root. Then $\gamma_r \in \Delta_n^+$, and $H_{\gamma_r} \notin [\mathfrak{h} \cap \mathfrak{k}_1]^{\mathbb{C}}$.

If Λ_0 is a dominant integral weight of $\mathfrak{k}_1$ and if $\lambda \in \mathbb{R}$ we denote by $\Lambda = (\Lambda_0, \lambda)$ the linear functional on $\mathfrak{h}^{\mathbb{C}}$ given by

$$(1.4) \qquad \Lambda|_{(\mathfrak{h} \cap \mathfrak{k}_1)^{\mathbb{C}}} = \Lambda_0, \qquad \Lambda(H_{\gamma_r}) = \lambda.$$

Such a Λ determines an irreducible finite-dimensional $\mathscr{U}(\mathfrak{k}^{\mathbb{C}})$-module which we, for convenience, denote by V_τ, where $\tau = \tau_\Lambda$ is the corresponding representation of the connected, simply connected Lie group $\tilde{K}$ with Lie algebra $\mathfrak{k}$. Further, let

$$(1.5) \qquad M(V_\tau) = \mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \underset{\mathscr{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+)}{\otimes} V_\tau$$

denote the generalized Verma module of highest weight Λ, and let M_Λ denote the Verma module of which $M(V_\tau)$ is a quotient.

In what follows, we choose to represent our Hermitian symmetric space $\mathscr{D}$ as a bounded domain in $\mathfrak{p}^-$. This is different from the situations in the articles to which we appeal for proofs of the following claims. However, the discrepancy can be removed by interchanging $\mathfrak{p}^+$ and $\mathfrak{p}^-$. This we may do since the element h_0 is only determined up to a sign.

Consider an (irreducible) finite-dimensional $\mathscr{U}(\mathfrak{k}^c)$-module V_τ. Through the process of holomorphic induction, the space $\mathscr{P}(V_\tau)$ of V_τ-valued polynomials on $\mathfrak{p}^-$ becomes a $\mathscr{U}(\mathfrak{g}^{\mathbb{C}})$-module consisting of $\mathfrak{k}$- (or $\tilde{K}$-) finite vectors. We maintain the notation $\mathscr{P}(V_\tau)$ for this module and let dU_τ denote the corresponding representation of $\mathfrak{g}^{\mathbb{C}}$. Explicitly, let

$$(\delta(z_0)f)(z) = \frac{d}{dt}\bigg|_{t=0} f(z+tz_0),$$

for z_0, $z \in \mathfrak{p}^-$, and $f \in C^\infty(\mathfrak{p}^-)$. Then, for $p \in \mathcal{P}(V_\tau)$ we have [8]:

$$(1.6)\quad \begin{aligned} &(dU_\tau(x) f)(z) = -(\delta(x) f)(z) \quad \text{for} \quad x \in \mathfrak{p}^-,\\ &(dU_\tau(x) f)(z) = d\tau(x) f(z) - (\delta([x, z]) f)(z) \quad \text{for} \quad x \in \mathfrak{k}^{\mathbb{C}},\\ &\text{and}\\ &(dU_\tau(x) f)(z) = d\tau([x, z]) f(z) - \frac{1}{2}(\delta([[x, z], z]) f)(z) \quad \text{for} \quad x \in \mathfrak{p}^+. \end{aligned}$$

It follows from these formulas (especially the first) that the space

$$(1.7)\quad W(\tau) = \text{Span}\{ dU_\tau(u) \cdot v \mid v \in V_\tau,\ u \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \},$$

is contained in any invariant subspace. In particular, $W(\tau)$ is irreducible.

Let V_τ and V_{τ_1} be finite-dimensional (irreducible) $\mathcal{U}(\mathfrak{k}^{\mathbb{C}})$-modules, and let D be a constant coefficient holomorphic differential operator on $\mathfrak{p}^-$ with values in $\text{Hom}(V_\tau, V_{\tau_1})$.

Definition 1.1. — $D : \mathcal{P}(V_\tau) \to \mathcal{P}(V_{\tau_1})$ *is covariant iff*

$$\forall x \in \mathfrak{g}^{\mathbb{C}}; \qquad D\,dU_\tau(x) = dU_{\tau_1}(x)\,D.$$

Let $\tilde{G}$ denote the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. We remark here that dU_τ is always the differential of a representation U_τ of $\tilde{G}$ on the space of holomorphic V_τ-valued functions on $\mathcal{D}$. By holomorphy and analyticity, Definition 1.1 is then equivalent to demanding that D should intertwine U_τ and U_{τ_1}.

Along with $\mathcal{P}(V_\tau)$ we consider the space $\mathcal{E}(V_{\tau'})$ of holomorphic constant coefficient differential operators on $\mathfrak{p}^-$ with values in the contragredient module, $V'_\tau = V_{\tau'}$, to V_τ. For $p \in \mathcal{P}(V_\tau)$ and $q \in \mathcal{E}(V_{\tau'})$ let

$$(1.8)\quad (q, p) = \left(q\left(\frac{\partial}{\partial z}\right), p(\cdot)\right)(0).$$

This bilinear pairing clearly places $\mathcal{P}(V_\tau)$ and $\mathcal{E}(V_{\tau'})$ in duality and as a result, $\mathcal{E}(V_{\tau'})$ becomes a $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$-module. The following result was stated in [5]. The proof is straightforward (*cf.* the appendix to [6]).

Proposition 1.2. — *As* $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$*-modules,*

$$\mathcal{P}(V_\tau)' = \mathcal{E}(V_{\tau'}) = M(V_{\tau'}).$$

The following is essentially contained in [5] and [6].

Proposition 1.3. — *A homomorphism* $\varphi : M(V_{\tau'_1}) \to M(V_{\tau'})$ *gives rise, by duality, to a covariant differential operator* $D_\varphi : \mathcal{P}(V_\tau) \to \mathcal{P}(V_{\tau_1})$*, and conversely.*

Proof. — By Proposition 1.2, we may view φ as a homomorphism from $\mathscr{E}(V_{\tau'_1})$ to $\mathscr{E}(V_{\tau'})$. $V_{\tau'_1} \subset \mathscr{E}(V_{\tau'_1})$ and thus there exists an element T_φ in $\mathscr{E}(\mathrm{Hom}(V_{\tau'_1}, V_{\tau'}))$ such that $\varphi(v) = T_\varphi(v)$ for $v \in V_{\tau'_1}$. *Since* φ is a module map it then follows that

$$\forall q \in \mathscr{E}(V_{\tau'_1}), \quad \varphi(q) = T_\varphi(q) \qquad \text{(pointwise)}.$$

D_φ is then the transpose of T_φ. The converse is equally obvious.

We now turn our attention to homomorphisms between $M(V_\tau)$'s. Naturally, our generalized Verma modules only form a small subclass of the class of all such. We shall comment further upon this in chapter 3.

It is part of the results of Bernstein-Gel'fand-Gel'fand (B-G-G) in [1] that the existence of a non-zero homomorphism $M(V_{\tau_1}) \to M(V_{\tau_2})$ implies the existence of a non-zero homomorphism $M_{\Lambda_1} \to M_{\Lambda_2}$ where $\tau_i \equiv \tau_{\Lambda_i}$; $i = 1, 2$. Conversely, a map $M_{\Lambda_1} \to M_{\Lambda_2}$ yields a quotient map—the so-called standard map. This, however, is often zero, but there may be "non-standard" maps. Almost all of the homomorphisms of the next chapter are non-standard.

Lepowsky [13] and Boe [2], based on [13], have given explicit criteria for the vanishing of the standard map. We present here Boe's criterion for our situation. "Condition (A)" refers to the well-known ingredient in B-G-G [1]. Let τ_i, $i = 1, 2$, be as above and assume the existence of a non-zero homomorphism $M_{\Lambda_1} \to M_{\Lambda_2}$.

PROPOSITION 1.4 [2]. — *The standard map is zero if and only if there is a sequence* $\gamma_1, \ldots, \gamma_r$ *of positive roots satisfying condition* (A) *for the pair* $(\Lambda_1 + \rho, \Lambda_2 + \rho)$ *such that* $(\sigma_{\gamma_1}(\Lambda_2 + \rho))(\mu) \notin \mathbb{N}$ *for some* $\mu \in \Delta_c^+$.

One particular feature of the present framework is that only certain subsets of the Weyl group are relevant. One such subset, specialized to our situation is

$$W^c = \{ w \in W \mid w^{-1} \Delta_c^+ \subset \Delta^+ \}.$$

The general analogues of this have been studied by Deodhar [4] and Boe [2].

We present here our approach [11]. Though there are certain analogies to the above, we feel that it is quite different in spirit. The following is essentially Proposition 3.6 of that article.

PROPOSITION 1.5. — *Let* $\tau_i = \tau_{\Lambda_i}$ *for* $i = 1, 2$. *Let* $\varphi \neq 0$ *be a homomorphism from* $M(V_{\tau_1})$ *to* $M(V_{\tau_2})$. *Then there exists a sequence* $\gamma_1, \ldots, \gamma_s$ *of elements of* Δ_n^+ *which satisfies condition* (A) *for the pair* $(\Lambda_1 + \rho, \Lambda_2 + \rho)$.

Thus, instead of having to work with the full set Δ as in the Theorem of (B-G-G), we can restrict our attention to Δ_n^+. In [11] we have described how Δ_n^+ can be represented in a 2-dimensional diagram. These diagrams will be used in the next chapter.

We also mention Proposition 7.3 from [10] which, in the dual picture, states that whenever the necessary (B-G-G) condition for a first order covariant differential operator to exist is satisfied, it does. It is in fact standard, and hence it is natural to examine whether this result can be obtained from Proposition 1.4. This is indeed so, as the proof of the following stronger result shows.

PROPOSITION 1.6. — *Let $\tau=\tau_\Lambda$ and let $\gamma\in\Delta_n^+$. Suppose that $\Lambda_1=\Lambda-m\gamma$ is the highest weight of an irreducible $\mathfrak{k}^{\mathbb{C}}$-submodule V_{τ_1} of $(\overset{m}{\otimes}\mathfrak{p}^-)\otimes V_\tau$ for some $m\in\mathbb{N}$ and that $(\Lambda+\rho)(H_\gamma)=m$. If $m\geqq 2$ assume further that γ is long. Then the standard map $M_{\tau_1}\to M_\tau$ is non-zero.*

Proof. — We have that $\{\gamma\}$ satisfies condition (A) for the pair $(\Lambda_1+\rho, \Lambda_2+\rho)$. If γ is long it is conjugate to β by reflections by compact roots. Hence there can be no other sequence $\gamma_1, \ldots, \gamma_s$ of positive non-compact roots satisfying condition (A) for this pair. Further, this implies that there can be no other sequence at all. Namely, by the procedure of removing reflections by compact roots described in the proof of Proposition 3.6 in [11] (Proposition 1.5), if there is another sequence, there must also be a two element sequence μ, γ_1 with $\mu\in\Delta_c^+$ and $\gamma_1\in\Delta_n^+$. However, using that γ is long, the equations

$$\Lambda+\rho-m\gamma=\sigma_{\gamma_1}\sigma_\mu(\Lambda+\rho),$$

$$\langle\Lambda+\rho, \mu\rangle=q>0 \quad \text{and} \quad \langle\Lambda+\rho-q\mu, \gamma_1\rangle=m,$$

where $q=m$ or $2m$, depending on whether $\gamma=\gamma_1+\mu$ or $\gamma=\gamma_1+2\mu$, are easily seen to imply that $\langle\Lambda-m\gamma, \mu\rangle\leqq-\langle\rho, \mu\rangle$.

Using the assumptions on Λ_1, if $m=1$ a similar argument holds for short roots ($\mathfrak{g}=\mathfrak{sp}(n, \mathbb{R})$ or $\mathfrak{so}(2n-1, 2)$). Finally, when $m\geqq 2$ and γ is short it is again easy to apply Proposition 1.4 to determine whether or not the standard map is zero. Both situations occur.

2. Homomorphisms into scalar modules

By a scalar module we mean an $M(V_\tau)$ where $\dim V_\tau=1$. Equivalently; $\tau=\tau_{(0,\lambda)}$. In this chapter we determine for which λ's and for which τ_1's there can be a non-trivial homomorphism $M(V_{\tau_1})\to M(V_{(0,\lambda)})$.

We begin by quoting some key facts from the existing literature:

Let $\gamma_1=\beta, \gamma_2, \ldots, \gamma_r$ be a maximal set of orthogonal roots in Δ_n^+, constructed so that γ_i is the element in $\Delta_n^+\cap\{\gamma_1, \ldots, \gamma_{i-1}\}^\perp$ with the smallest height; $i=2, \ldots, r$. Let $\delta_i=\gamma_1+\ldots+\gamma_i$; $i=1, \ldots, r$.

PROPOSITION 2.1 ([19]). — *The set of highest weights of the irreducible submodules of the $\mathfrak{k}^{\mathbb{C}}$-module $\mathcal{U}(\mathfrak{p}^-)$ are*

$$\{-i_1\delta_1-\ldots-i_r\delta_r\,|\,(i_1, \ldots, i_r)\in(\mathbb{Z}_+)^r\}.$$

There are no multiplicities.

Secondly we observe that the results of [21], chapter 5, or [18], easily are seen to imply the following fact which also can be proved by simple case-by-case computations.

Let p denote the dimension of an "off-diagonal" root space in $\mathfrak{g}$ for a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ (*cf.* [18]; (2.2.2)).

PROPOSITION 2.2. — *There exists a non-zero homomorphism*

$$M(V_{(0,\lambda_i)-\delta_i}) \to M(V_{(0,\lambda_i)}).$$

when $\lambda_i = -(i-1)\cdot p/2$; $i=1, \ldots, r$.

Let $p_{-\delta_i}$ denote the element of $\mathcal{U}(\mathfrak{p}^-)$ of highest weight $-\delta_i$. Consider $z_\beta p_{-\delta_i}$. Clearly there are elements $\mu(i, 1), \ldots, \mu(i, s)$ of Δ_c^- and elements $\hat{p}_i, p_1, \ldots, p_s$ of $\mathcal{U}(\mathfrak{p}^-)$ such that

$$z_\beta p_{-\delta_i} = \sum_{j=1}^{s} p_j k_{\mu(i, j)} + \hat{p}_i (H_\beta - \lambda_i) + p_{-\delta_i} z_\beta, \tag{2.1}$$

where $k_{\mu(i, j)}$ is non-zero and belongs to $\mathfrak{g}^{\mu(i, j)}$ and λ_i is as in Proposition 2.2. Evidently $\hat{p}_i$ has weight $\beta - \delta_i$.

LEMMA 2.3. — *Let* $\mu(i, j)$ *be as above. Then for all* $i' \geqq i$,

$$[k_{\mu(i, j)}, p_{-\delta_{i'}}] = 0.$$

Proof. — Consider the diagram of Δ_n^+ ([11]). $p_{-\delta_i}$ is a sum of monomials corresponding to the various ways one can write δ_i as a sum of elements of Δ_n^+. The signs are of no importance. It is then easy to determine which $\mu(i, j)$'s may occur in (2.1) and that these all have zero inner product with δ_i, when $i' \geqq i$. Since $p_{-\delta_{i'}}$ is a highest weight vector the claim follows.

Before stating the main result of this section we mention that we in [9] studied homomorphisms between scalar modules (i. e. also originating in a such) for $SU(n, n)$ and $Mp(n, \mathbb{R})$.

B. Ørsted studied composition series for, in particular, unitary scalar module for $SU(n, n)$ in [22], and Boe treated the general case of homomorphisms between scalar modules on Hermitian symmetric spaces in [2].

We finally remark that if there is a non-zero homomorphism of an $M(V_{\tau_1})$ into a scalar module $M(V_{(0,\lambda)})$ then clearly τ_1 has got to occur in $\mathcal{U}(\mathfrak{p}^-) \otimes V_{(0,\lambda)}$.

PROPOSITION 2.4. — (*a*) *If there is a non-trivial homomorphism*

$$M(V_{(0,\lambda)-\sum_{s=1}^{r} i_s \delta_s}) \to M(V_{(0,\lambda)}),$$

then at most one i_s *is different from* 0.

(*b*) *There is a non-zero homomorphism*

$$M(V_{(0,\lambda)-n\delta_s}) \to M(V_{(0,\lambda)}),$$

exactly when $\lambda=\lambda_s+(n-1)$ *where* λ_s *is given by Proposition* 2.2 *and* $n\in\mathbb{N}$.

Proof. — The existence of a homomorphism into $M(V_{(0,\lambda)})$ is equivalent to the existence of an element p of $\mathcal{U}(\mathfrak{p}^-)$ which, when viewed as an element of $M(V_{(0,\lambda)})$, is annihilated by $\mathfrak{g}^+$. By Proposition 2.1 we may assume that

$$(2.2)\qquad p=\prod_{s=1}^{r} p_{-\delta_s}^{i_s}$$

for some r-touple $(i_1,\ldots,i_r)\in(\mathbb{Z}_+)^r$, and it is then necessary and sufficient that $z_\beta p=0$ in $M(V_{(0,\lambda)})$. Though $\mathfrak{p}^-$ is commutative we insist on writing

$$p=p_{-\delta_1}^{i_1}p_{-\delta_2}^{i_2}\ldots p_{-\delta_r}^{i_r}.$$

This, namely, by Lemma 2.3, has the effect that we, when computing inside $M(V_{(0,\lambda)})$, may ignore altogether the terms of the form $p_s k_{\mu(i,s)}$ in (2.1) as these annihilate $V_{(0,\lambda)}$. Now observe that it follows by induction from (2.1) and Lemma 2.3 that

$$(2.3)\qquad z_\beta p_{-\delta_s}^{i_s}=p_{-\delta_s}^{i_s}z_\beta+p_{-\delta_s}^{i_s-1}\hat{p}_s(i_s H_\beta-i_s\lambda_s-i_s(i_s-1))\,(\mathrm{mod}\,\mathcal{U}(\mathfrak{g}^{\mathbb{C}})\,\mathfrak{k}_1^{\mathbb{C}}).$$

From this, and the above remark, it then follows that inside $M(V_{(0,\lambda)})$,

$$\begin{aligned}(2.4)\quad z_\beta p_{-\delta_1}^{i_1}\ldots p_{-\delta_r}^{i_r}&=p_{-\delta_1}^{i_1-1}\hat{p}_1 p_{-\delta_2}^{i_2}\ldots p_{-\delta_r}^{i_r}(i_1\lambda-i_1\lambda_1-i_1(i_1-1)-2i_1(i_2+\ldots+i_r))\\&+p_{-\delta_1}^{i_1}p_{-\delta_2}^{i_2-1}\hat{p}_2\ldots p_{-\delta_r}^{i_r}(i_2\lambda-i_2\lambda_2-i_2(i_2-1)-2i_2(i_3+\ldots+i_r))\\&+\ldots+p_{-\delta_1}^{i_1}\ldots p_{-\delta_{r-1}}^{i_{r-1}}p_{-\delta_r}^{i_r-1}\hat{p}_r(i_r\lambda-i_r\lambda_r-i_r(i_r-1)).\end{aligned}$$

We want this expression to vanish. Clearly, when exactly one of the exponents $i_1,\ldots,i_r$ is non-zero, this is possible for a unique λ. Assume then that at least two exponents are non-zero. With no loss of generality we may assume that one of these is i_r. Now observe that the polynomials in (2.4) are linearly independent [elements of $\mathcal{U}(\mathfrak{p}^-)$ are identified with polynomials on $\mathfrak{p}^+$ *via* the Killing form, *cf.* Proposition 1.2]. Indeed, it suffices to consider the cases in which the non-zero exponents all are equal to 1. In the cases one can easily find points at which all but e. g. the polynomial containing $\hat{p}_r$, vanish (*cf.* the diagrams of Δ_n^+ in [11]). This implies that

$$(2.5)\qquad \lambda-\lambda_r-(i_r-1)=0\quad\text{and}\quad(\lambda-\lambda_k-(i_k-1)-2i_r)=0,$$

where k is the biggest integrer below r at which the exponent is non-zero. The equations (2.5) clearly imply that

$$\lambda_r-\lambda_k=i_r+i_k,$$

and since $\lambda_r-\lambda_k=(k-r)p/2$ by Proposition 2.2, this is a contradiction.

3. Homomorphisms from scalar modules

In this chapter we analyze for which values of λ and for which irreducible representations τ one can have a non-trivial homomorphism

$$M(V_{(0,\lambda)}) \to M(V_\tau), \tag{3.1}$$

or, equivalently, for which τ's there exists a $p \in \mathscr{U}(\mathfrak{p}^-) \otimes V_\tau$ of weight $(0, \lambda)$ such that

$$\forall x \in \mathfrak{g}^+, \quad xp = 0. \tag{3.2}$$

We now begin to examine what one can deduce about λ and τ from the existence of such a p:

First observe that since p is inside a tensor product and since the tensor product of two finite-dimensional $\mathfrak{k}_1^{\mathbb{C}}$-modules contains the trivial module if and only if the modules are the contragredients of each other, it follows that there are non-negative integers $n_1, \ldots, n_r$ such that

$$\tau = \tau_\Lambda; \qquad \Lambda = (0, \lambda) + \omega_1 \left(\sum_{i=1}^{r} n_i \delta_i \right). \tag{3.3}$$

In this formula, ω_1 is the Weyl group element $\omega_1(\beta) = \gamma_r$; $\omega_1(\Delta_c^+) = \Delta_c^-$. It is convenient to introduce the following notation:

$$\hat{\gamma}_i = \omega_1(\gamma_i), \qquad \hat{\delta}_i = \omega_1(\delta_i); \qquad i = 1, \ldots, r. \tag{3.4}$$

Observe that the lowest weight vector in the polynomial representation of highest weight $-\delta_i$ has weight $-\hat{\delta}_i$. Let $q_i = q_{-\hat{\delta}_i}$ denote this vector (only given, of course, up to multiplication by a non-zero constant). Further we let v_0 denote the highest weight vector of V_τ and assume that $\{v_0, v_1, \ldots, v_N\}$ is a basis of V_τ. Then there are elements $p_0, p_1, \ldots, p_N$ of $\mathscr{U}(\mathfrak{p}^-)$ such that

$$p = p_0 v_0 + \sum_{i=1}^{N} p_i v_i. \tag{3.5}$$

Since p is annihilated in particular by $(\mathfrak{k}_1^{\mathbb{C}})^-$ and since $v_i \in \mathscr{U}((\mathfrak{k}_1^{\mathbb{C}})^-) v_0$ for all $i = 1, \ldots, N$, it follows that, up to a constant multiple,

$$p_0 = \prod_{i=1}^{r} q_i^{n_i}. \tag{3.6}$$

If $\mu \in \Delta_c^+$ and if $k_{-\mu} \in (\mathfrak{k}_1^{\mathbb{C}})^{-\mu}$ does not annihilate v_0, we assume that $k_{-\mu} v_0$ is proportional to one of the basis vectors which we then denote by $v_{-\mu}$. It follows that there is a unique $k_\mu \in (\mathfrak{k}_1^{\mathbb{C}})^\mu$ satisfying $k_\mu v_{-\mu} = v_0$. If p_μ denotes the coordinate function corresponding to $v_{-\mu}$, then

$$p_\mu = -[k_\mu, p_0] \tag{3.7}$$

since $k_\mu p=0$ and hence, in particular, the coordinate function of $k_\mu p$ corresponding to v_0 must vanish.

We assume that at least one n_i is different from zero. The case where they all vanish is contained in chapter 2. Let $i_0=\max\{i=1, 2, \ldots, n \mid n_i\neq 0\}$ and consider

$$z_{\hat{\gamma}_{i_0}}\left(p_0 v_0+\sum_{i=1}^{N} p_i v_i\right). \tag{3.8}$$

If this is zero then so are the coordinate functions of this expression and, specifically, so is the leading term in $z_{-\hat{\gamma}_{i_0}}$ in the coordinate function $\hat{p}_0$ corresponding to v_0. Let q_{i_0} be written as

$$q_{i_0}=z_{-\hat{\gamma}_{i_0}}\hat{q}_{i_0}+r_{i_0}, \tag{3.9}$$

With $\hat{q}_{i_0}$ and r_{i_0} elements of $\mathscr{U}(\mathfrak{p}^-)$ that do not depend on $z_{-\hat{\gamma}_{i_0}}$. The contribution c_0 to $\hat{p}_0$ from p_0 is easily computed from Proposition 2.4 [*cf.* (2.3)] along with the observation that for all $i<i_0$, $z_{\hat{\gamma}_{i_0}} q_i=q_i z_{\hat{\gamma}_{i_0}}$;

$$q_{i_0} c_0=n_{i_0}((\lambda+2 n_{i_0})-\lambda_{i_0}-(n_{i_0}-1))\hat{q}_{i_0} p_0, \tag{3.10}$$

where the $2 n_{i_0}$ contribution stems from the $n_{i_0}\hat{\hat{\delta}}_{i_0}$ in (3.3).

The only other terms that can contribute to $\hat{p}_0$ are those $p_\mu v_{-\mu}$ for which $\hat{\gamma}_{i_0}-\mu\in\Delta_n^+$, since we must be able to pick out a term proportional to k_μ from $[z_{\hat{\gamma}_{i_0}}, p_\mu]$. $\hat{\gamma}_{i_0}$ is long and therefore we must have $(\hat{\gamma}_0, \mu)>0$; in fact, for $\gamma=\hat{\gamma}_{i_0}$, $2(\mu, \gamma)/(\gamma, \gamma)=1$ (*cf.* the proof of Proposition 6.2 in [10]). This, on the other hand, is a sufficient condition for $v_{-\mu}$ to be non-zero. Also note that it follows that $[z_{\hat{\gamma}_{i_0}}, k_\mu]=0$.

LEMMA 3.1. — *Let m be a monomial in $\mathscr{U}(\mathfrak{p}^-)$ of weight $-\sum n_i\hat{\gamma}_i$ and assume that $n_{i_0}=0$. Then $z_{-\hat{\gamma}_{i_0}+\mu}$ does not occur in m.*

Proof. — If it did, we could write

$$-\sum n_i\hat{\gamma}_i=-(\hat{\gamma}_{i_0}-\mu)-\alpha,$$

where α is a sum of elements of Δ_n^+. Since innder products between elements of Δ_n^+ are non-negative and since $\hat{\gamma}_{i_0}$ has a non-zero inner product with $-\hat{\gamma}_{i_0}+\mu$ this is impossible since $\hat{\gamma}_{i_0}$ is perpendicular to the left hand side.

Applying this Lemma to $\hat{q}_{i_0}$, and $q_i (i<i_0)$, and observing that

$$[z_{\hat{\gamma}_{i_0}}, -[k_\mu, z_{-\hat{\gamma}_{i_0}}]]=[H_{\hat{\gamma}_{i_0}}, k_\mu]=k_\mu,$$

we see that if we let d_i denote the number of elements μ in Δ_c^+ for which $(\hat{\gamma}_i, \mu)>0$ $(i=1, \ldots, r)$, it follows that modulo lower order terms in $z_{-\hat{\gamma}_{i_0}}$,

$$\hat{p}_0\sim n_{i_0}((\lambda+2 n_{i_0})-\lambda_{i_0}-(n_{i_0}-1)+d_{i_0}) z_{-\hat{\gamma}_{i_0}}^{n_{i_0}-1}\hat{q}_{i_0}^{n_{i_0}}\prod_{\substack{i=1\\ i\neq i_0}}^{r} q_i^{n_i},$$

hence

$$(3.11)\qquad \lambda+n_{i_0}+1-\lambda_{i_0}+d_{i_0}=0.$$

So, λ is determined by the biggest $\hat{\delta}_i$ in (3.3). We will now show that there is only one non-zero n_i in (3.3). Since we are assuming $n_{i_0}\neq 0$ let us then further assume that n_i is non-zero for some $i<i_0$, and let i_1 denote the largest such i. We begin our analysis by insisting that p_0 is written as

$$(3.12)\qquad p_0=q_1^{n_1}\cdot q_2^{n_2}\cdot\ldots\cdot q_r^{n_r}.$$

We again look at the coordinate function corresponding to v_0, but this time we consider

$$z_{\hat{\gamma}_{i_1}}\left(p_0 v_0+\sum_{i=1}^{r} p_i v_i\right),$$

and we look at the leading term in $z_{-\hat{\gamma}_{i_1}}$. The effect of our way of writting p_0 is, of course, that in the computation of the contribution to this term from $p_0 v_0$ we may work modulo $\mathcal{U}(\mathfrak{g})\mathfrak{k}_1^{\mathbb{C}}$ just as in the proof of Proposition 2.4 (but the reason here being that $\{k\cdot v_0 \mid k\in\mathfrak{k}_1^{\mathbb{C}}\}\cap\mathbb{C}\cdot v_0=\{0\}$) The remaining part of the computation is carried out as above and we obtain the following equation for λ:

$$(3.13)\qquad n_{i_1}((\lambda+2n_{i_1})-\lambda_{i_1}-(n_{i_1}-1)+d_{i_1})+n_{i_0}((\lambda+2n_{i_0}+2n_{i_1})-\lambda_{i_0}-(n_{i_0}-1)+d_{i_1})=0.$$

We insert the value of λ from (4.12) and obtain

$$(3.14)\qquad n_{i_1}(n_{i_0}+n_{i_1})+(n_{i_1}+n_{i_0})(d_{i_1}-d_{i_0})+n_{i_1}(\lambda_{i_0}-\lambda_{i_1})=0.$$

To keep notation at a minimum we just refer to the diagrams of Δ_n^+ in [11] for a proof of the fact that

$$(d_{i_1}-d_{i_0})=(i_0-i_1)p,$$

which, by Proposition 2.2, can be formulated as

$$(d_{i_1}-d_{i_0})=2(\lambda_{i_1}-\lambda_{i_0}).$$

Thus, (3.14) is an absurdity and hence there can be at most one $n_i\neq 0$.

Another structural equation which can be read off the diagrams of Δ_n^+ is

$$(3.15)\qquad \rho(\hat{\gamma}_i)=d_i+1.$$

PROPOSITION 3.2. — *There exists a non-zero homomorphism* $\mathrm{M}(\mathrm{V}_{(0,\lambda)})\to\mathrm{M}(\mathrm{V}_\tau)$ *exactly when* $\tau\equiv(0,\lambda)+n\omega_1(\delta_i)$ *for some* $n\in\mathbb{N}$, $i\in\{1,\ldots,r\}$, *and* $\lambda=\lambda_i-n-\rho(\hat{\gamma}_i)$ $(=\lambda_i-n-1-d_i)$. *The homomorphism is unique.*

Proof. — By the preceding analysis we know that the conditions on τ and λ are necessary. Further, for a τ of this form there is exactly one $\mathfrak{k}_1^{\mathbb{C}}$-fixed vector in $\mathcal{U}(\mathfrak{p}^-)\otimes\mathrm{V}_\tau$

of weight $(0, \lambda)$ so the uniqueness is clear. Let now n and i be fixed. According to Proposition 1.6 there is a non-zero homomorphism

$$\varphi_c: \quad M(V_{(0, \lambda_c)+n\omega_1(\delta_i)-\hat{\gamma}_i}) \to M(V_{(0, \lambda_c)+n\omega_1(\delta_i)}),$$

when $\lambda_c + 2n + \rho(\hat{\gamma}_i) = 1$ [*cf.* the remark following Proposition 8.1 in [10]; λ_c is in fact "the last possible place of unitarity" for the one-parameter family of (irreducible quotients of) modules $M(V_\tau)$].

Let $\alpha_1, \ldots, \alpha_t$ denote those elements of Δ_n^+ for which there is a highest weight vector in $\mathfrak{p}^- \otimes V_\tau$ of weight $(0, \lambda_c) + n\omega_1(\delta_i) - \alpha_j$ and let $V_{\alpha_j} \subset \mathfrak{p}^- \otimes V_\tau$ denote the corresponding $\mathfrak{k}$-module ($j = 1, \ldots, t$; $\alpha_1 = \hat{\gamma}_i$). Of course, $\mathscr{U}(\mathfrak{p}^-) \otimes V_\tau \subset \bigoplus_{j=1}^{t} \mathscr{U}(\mathfrak{p}^-) \otimes V_{\alpha_j}$ and by expanding the highest weights of each of the spaces $\mathscr{U}(\mathfrak{p}^-) \otimes V_{\alpha_j}$ on the basis for Δ and by paying attention to the coefficients to the simple compact roots, it is easy to see that the image of φ_c contains all the $\mathfrak{k}$-modules of highest weights

$$-\left(\sum_{j=1}^{i} m_j \hat{\gamma}_j\right) + n\omega_1(\delta_i) + (0, \lambda_c), \tag{3.16}$$

where $n \geqq m_i \geqq \ldots \geqq m_1 \geqq 0$ and $m_i \geqq 1$. This implies that there is an invariant subspace $I_c \subset \mathscr{P}(V_{\tau'_c})$ such that none of the contragredients to the representations of $\mathfrak{k}$ as given by (3.16) are contained in I_c. (τ_c is τ with $\lambda = \lambda_c$.) Next recall from Proposition 2.4 that there is a non-trivial homomorphism

$$\varphi_b: \quad M(V_{(0, \lambda_b) - n\delta_i}) \to M(V_{(0, \lambda_b)})$$

when $\lambda_b = \lambda_i + n - 1$. This means that there is an invariant subspace $I_b \subset \mathscr{P}(V_{(0, -\lambda_b)})$ consisting exactly of those $\mathfrak{k}$-types whose contragredients are of the form $(0, \lambda_b) - \sum_{j=1}^{r} n_j \delta_j$ with $\sum_{j=1}^{r} n_i < n$. Now we form the tensor product $\mathscr{P}(V_{\tau'_c}) \otimes \mathscr{P}(V_{(0, -\lambda_b)})$ which we view as a subspace of the space of $V_{\tau'_c} \otimes V_{(0, -\lambda_b)}$-valued polynomials on $\mathscr{D}$. By restricting $I_c \otimes I_b$ to the diagonal in $\mathscr{D} \times \mathscr{D}$ along the lines of [8] we obtain an invariant subspace I_{b+c} of $\mathscr{P}(V_{\tau'})$ ($\lambda = \lambda_b + \lambda_c$). It is clear that the contragredients to all representations $-\sum_{j=1}^{i} n_j \hat{\gamma}_j + n\omega_1(\delta_i) + (0, \lambda)$ with $n \geqq n_i \geqq \ldots \geqq n_1 \geqq 0$, but the one in which $n_1 = n$, are contained in I_{b+c} and that the $\mathfrak{k}_1^{\mathbb{C}}$-fixed vector q_0 of weight $(0, -\lambda)$ does not belong to I_{b+c}. It follows that the $\mathfrak{k}_1^{\mathbb{C}}$-fixed vector $\tilde{q}_0$ of $M(V_\tau)$ of weight $(0, \lambda)$ belongs to

$$I_{a+b}^0 = \{ q \in M(V_\tau) \mid \forall p \in I_{b+c} : (q, p) = 0 \}$$

(*cf.* Proposition 1.2) and it is easy to see that this element is of lowest order in I_{b+c}^0. Thus, by the invariance of I_{b+c}^0, $\mathfrak{p}^+ \tilde{q}_0 = 0$ when computed inside $M(V_\tau)$.

4. Further applications

We present here two examples of applications of the results of chapter 3. Further applications, in the spirit of the Jantzen-Zuckerman translation functor and along the lines of ([19], Lemma 4.5.9), will be presented elsewhere.

Example 1. — With this we return to the description of the set of homomorphisms into a scalar module. What needs to be analyzed further are the situations in which there are several homomorphisms (of course corresponding to different δ_i's) into the same scalar module. Specifically, consider

$$M(V_{(0,\ \lambda_i+n_i-1)-n_i\delta_i}) \xrightarrow{\varphi_i} M(V_{(0,\ \lambda_i+n_i-1)}).$$

Assume that $j>i$ and that $n_j=n_i+(j-i)p/2$ is an integer. (With the exception of $sp(n, \mathbb{R})$, n_j is always an integer. For $sp(n, \mathbb{R})$ we must have that $j=i+2m$ for some $m\in\mathbb{N}$.) We then get the following picture

$$\text{(4.1)}\qquad \begin{array}{ccc} M(V_{(0,\ \lambda_i+n_i-1)-n_i\delta_i}) & \xrightarrow{\varphi_j} & M(V_{(0,\ \lambda_i+n_i-1)}). \\ \Big\downarrow \ \varphi_{ji}=0 & \nearrow \varphi_i & \\ M(V_{(0,\ \lambda_i+n_i-1)-n_i\delta_i}) & & \end{array}$$

The somewhat surprising fact that φ_{ji} must be zero is a direct consequence of Proposition 3.2. Suppose namely that φ_{ji} is non-zero and let $\mathfrak{g}_j^{\mathbb{C}}$ be the subalgebra of $\mathfrak{g}^{\mathbb{C}}$ corresponding to $\gamma_i, \ldots, \gamma_j$. The highest weight vector in $M(V_{(0,\ \lambda_j+n_j-1)-n_j\delta_j})$ is mapped by φ_{ji} into an element $p\in\mathcal{U}(\mathfrak{p}^-)\otimes V_{(0,\ \lambda_i+n_i-1)-n_i\delta_i}$ which, in fact, belongs to $\mathcal{U}(\mathfrak{p}_j^-)\otimes V_{(0,\ \lambda_i+n_i-1)-n_i\delta_i}$, where $\mathfrak{p}_j^-$ denotes the "$\mathfrak{p}^-$" of $\mathfrak{g}_j^{\mathbb{C}}$. This fact follows easily by looking at the coordinate functions of p in some basis of $V_{(0,\ \lambda_i+n_i-1)-n_i\delta_i}$. It follows from this that there is a non-zero homomorphism $\tilde{\varphi}_{ji}$ for the analogous modules for $\mathfrak{g}_j^{\mathbb{C}}$, where now $M(V_{(0,\ \lambda_j+n_j-1)-n_j\delta_j})$ is a scalar module. However, if p_0 denotes the coordinate function of p with respect to the highest weight vector in $V_{(0,\ \lambda_i+n_i-1)-n_i\delta_i}$, then p_0 has weight $(0, -2n_i+2n_j)-n_i(\delta_j-\delta_i)$ as element of $\mathcal{U}(\mathfrak{p}^-)$ and this is not, since $n_j>n_i$, of the form given by Proposition 3.2 (*cf.* (3.12)).

Our second example illustrates how results about generalized Verma modules on Hermitian symmetric spaces yield results about homomorphisms between modules outside this realm.

Example 2. — Let $\mathfrak{g}=gl(2n_i+n_j, \mathbb{C})$ and consider the following subalgebras:

$$\mathfrak{n}^-=\left\{\begin{pmatrix}0&0&0\\ z_1^-&0&0\\ z_2^-&z_3^-&0\end{pmatrix}\ \middle|\ \begin{array}{c} z_1^-\in\operatorname{Hom}(\mathbb{C}^{n_j}, \mathbb{C}^{n_i}),\ z_2^-\in M(n_i, \mathbb{C}) \\ \text{and } z_3^-\in\operatorname{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})\end{array}\right\},$$

$$\mathfrak{n}^+=\left\{\begin{pmatrix}0&z_1^+&z_2^+\\ 0&0&z_3^+\\ 0&0&0\end{pmatrix}\ \middle|\ \begin{array}{c} z_1^+\in\operatorname{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}),\ z_2^+\in M(n_i, \mathbb{C}) \\ \text{and } z_3^+\in\operatorname{Hom}(\mathbb{C}^{n_j}, \mathbb{C}^{n_i})\end{array}\right\},$$

and

$$\mathfrak{l}=\left\{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \middle| a_1, a_3 \in \mathrm{M}(n_i, \mathbb{C}) \text{ and } a_2 \in \mathrm{M}(n_j, \mathbb{C})\right\}.$$

The subalgebras of $\mathfrak{l}$ corresponding to the entries a_1, a_2, and a_3 are denoted $\mathfrak{l}_1$, $\mathfrak{l}_2$, and $\mathfrak{l}_3$, respectively; $\mathfrak{l}=\mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{l}_3$.

By "the variable $z_i^{\pm}$" ($i=1, 2, 3$) we mean the corresponding matrix as above. Let $\lambda=(\lambda_1, \lambda_2, \lambda_3)$ denote a triple of reals and let

$$\lambda(a)=\lambda_1 \operatorname{tr}(a_1)+\lambda_2 \operatorname{tr}(a_2)+\lambda_3 \operatorname{tr}(a_3)$$

$$\text{for} \quad a=\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in \mathfrak{l}.$$

Let $\mathscr{I}$ denote the left ideal in $\mathscr{U}(\mathfrak{g})$ generated by $\mathfrak{n}^+$ and the elements $a-\lambda(a)$ for $a \in \mathfrak{l}$. By the scalar module $\mathrm{S}(\lambda_1, \lambda_2, \lambda_3)$ of highest weight $(\lambda_1, \lambda_2, \lambda_3)$ we mean

$$\mathrm{S}(\lambda_1, \lambda_2, \lambda_3)=\mathscr{U}(\mathfrak{g})/\mathscr{I}.$$

There is a bijective correspondence between homomorphisms

$$\mathrm{S}(\lambda_1', \lambda_2', \lambda_3') \to \mathrm{S}(\lambda_1, \lambda_2, \lambda_3)$$

and polynomials p in the non-commuting variables z_1^-, z_2^-, and z_3^- satisfying

$$(4.2) \qquad z_1^+ p = z_3^+ p = 0 \quad \text{in } \mathrm{S}(\lambda_1, \lambda_2, \lambda_3)$$

$$\text{and} \qquad [a, p]=\left(\sum_{i=1}^{3} (\lambda_i'-\lambda_i) \operatorname{tr} a_i\right) p \qquad \text{for} \quad a \in \mathfrak{l}.$$

Let us agree to write our polynomials p as sums of polynomials of the form $p_1(z_1^-) p_2(z_2^-) p(z_3^-)$.

In case $n_i=n_j$, it follows from Proposition 2.4 that for $\lambda_2-\lambda_3=-n+b$, $b \in \mathbb{N}$, there is a homomorphism

$$(4.3) \qquad \mathrm{S}(\lambda_1, \lambda_2-b, \lambda_3+b) \to \mathrm{S}(\lambda_1, \lambda_2, \lambda_3),$$

defined by the polynomial $p_{3,b}=(\det z_3^-)^b$. Likewise, then, for $\lambda_1-\lambda_2=-n+c$, $c \in \mathbb{N}$, there is a homomorphism

$$(4.4) \qquad \mathrm{S}(\lambda_1-c, \lambda_2+c, \lambda_3) \to \mathrm{S}(\lambda_1, \lambda_2, \lambda_3),$$

defined by $p_{1,c}=(\det z_1^-)^c$.

To avoid having to deal with some special cases which are of no interest in relation to the features we wish to reveal, we assume from now on that $n_i \neq n_j$.

We suppose that p satisfies (4.2). Let $\bar{p}=\sum_{\alpha}\bar{p}_{1,\alpha}(z_1^-)\bar{p}_{2,\alpha}(z_2^-)\bar{p}_{3,\alpha}(z_3^-)$ denote the leading term in z_2^- in p. Either $\bar{p}=(\det z_2^-)^a$ for some $a\in\mathbb{N}$, or not. In the first case (which is the generic) it follows easily from the structure of the root system together with Proposition 2.4, that

$$\lambda_1-\lambda_3=a-i-j, \tag{4.5}$$

and that the hypothetical homomorphism originates in the module $S(\lambda_1-a, \lambda_2, \lambda_3+a)$.

In the second case it follows also from Proposition 2.4 that

$$\lambda_2-\lambda_3=-s_1+u_1, \tag{4.6}$$

for some $s_1=1, \ldots, \min\{i, j\}$, and $u_1\in\mathbb{N}$, and that the polynomials $\bar{p}_{3,\alpha}$ must be in the sum of the $\mathfrak{k}$-modules ($\mathfrak{k}=\mathfrak{l}_2\oplus\mathfrak{l}_3$) generated by the polynomials $p^{u_x}_{-\delta_{s_x}}(z_3^-)$ for which $\lambda_2-\lambda_3=-s_x+u_x$.

We now use the fact that $\bar{p}$ is $\mathfrak{l}_3$-invariant and that the $\bar{p}_{2,\alpha}$'s are polynomials. Further, as we may now clearly do with no loss of generality, we assume that the $\bar{p}_{3,\alpha}$'s belong to just one $\mathfrak{k}$-module. It follows that the $\bar{p}_{2,\alpha}$'s are uniquely determined up to a multiple of $(\det z_2^-)^b$ for some $b=0, 1, \ldots$ At this point we invoke the assumption that $z_1^+ p=0$. Due to the $(\mathfrak{l}_1\oplus\mathfrak{l}_2)$-invariance of p it follows from Proposition 3.2 and the structure of the root system that

$$\lambda_1-\lambda_2=b+s_1-i-j, \tag{4.7}$$

where the b comes from a possible factor of $(\det z_2^-)^b$, as explained above. It thus follows that

$$\lambda_1-\lambda_3=b+u_1-i-j,$$

and it is straightforward to see that the homomorphism into $S(\lambda_1, \lambda_2, \lambda_3)$ (if it exists) originates in the module $S(\lambda_1-b-u_1, \lambda_2, \lambda_3+b+u_1)$.

Thus we have proved that in all cases, when $n_i\neq n_j$, the homomorphisms must originate in modules of the form $S(\lambda_1-a, \lambda_2, \lambda_3+a)$ with $a\in\mathbb{N}$, and this is considerably simpler than what can be deduced directly from Bernstein-Gelfand-Gelfand. That there can be at most one homomorphism between scalar modules is a result of Lepowsky's [17].

We finally mention that some preliminary computations indicate that there does exist homomorphisms in the above çases.

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