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HOMOGENEOUS KÄHLER MANIFOLDS ADMITTING A TRANSITIVE SOLVABLE GROUP OF AUTOMORPHISMS

BY JOSEF DORFMEISTER

ABSTRACT. — In this paper we prove the "fundamental conjecture" due to Gindikin and Vinberg for homogeneous Kähler manifolds admitting a solvable transitive group of holomorphic isometries. We generalize a proof of Gindikin, Piatetskii-Shapiro and Vinberg (which worked with split solvable groups) using "modifications" of solvable Kähler algebras.

RÉSUMÉ. — Nous éprouvons la « conjecture fondamentale » de Gindikin et Vinberg pour les variétés kählériennes homogènes admettant un groupe soluble, transitive des isométries holomorphes. Nous généralisons une épreuve de Gindikin, Piatetskii-Shapiro et Vinberg en introduisant des « modifications » des algèbres kählériennes solubles.

In this paper we consider the manifolds described in the title. We prove that the "fundamental conjecture" of Gindikin and Vinberg [8] holds in this case.

Our main tool is the "modification" of a solvable Kähler algebra (*see* 3.1 for a definition). We use it to remove obstacles which arise when going through [7], part III, § 3. Thus we prove that a solvable Kähler algebra can be modified to yield a sum of an abelian Kähler ideal and a normal j-algebra. As an application we describe Kählerian NC algebras and prove that they are modifications of the product of an abelian Kähler algebra and a normal j-algebra. We would like to point out that our results improve over the recent results of [10]. We are also able to recapture the results of [14].

In paragraphs 1, 2 we investigate certain abelian ideals of a solvable Kähler algebra. In paragraph 3 we introduce the notion of a modification and prove various results about modifications. Sections 4, 5 and 6 are devoted to the proof of the algebraic main theorem 3.7. In paragraph 7 we collect geometric consequences of this theorem. In particular we prove the fundamental conjecture for Kähler manifolds admitting a solvable transitive group of automorphisms.

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1.1. A *solvable Kähler algebra* is a solvable Lie algebra $\mathfrak{s}$, together with an inner product $\langle .,. \rangle$ and an orthogonal map $j : \mathfrak{s} \to \mathfrak{s}$ satisfying

$$(1.1) \qquad j^2 = -\mathrm{id},$$

$$(1.2) \qquad [jx, jy] = j[jx, y] + j[x, jy] + [x, y] \quad \text{for} \quad x, y \in \mathfrak{s},$$

$$(1.3) \qquad \langle [x, y], jz \rangle + \langle [y, z], jx \rangle + \langle [z, x], jy \rangle = 0 \quad \text{for} \quad x, y, z \in \mathfrak{s}.$$

From [7], Part II, 1, it follows that solvable Kähler algebras correspond to simply connected homogeneous Kähler manifolds admitting a simply transitive solvable group of holomorphic isometries. Kähler algebras in the sense of [7], Part II, 1, will be called "general Kähler algebras".

1.2. The following lemma is an easy consequence of (1.2).

LEMMA. — *Let $\mathfrak{r}$ be an abelian ideal of $\mathfrak{s}$, then $j\mathfrak{r}$ and $\mathfrak{r}+j\mathfrak{r}$ are subalgebras of $\mathfrak{s}$ and $\mathfrak{r} \cap j\mathfrak{r}$ is an abelian ideal of $\mathfrak{r}+j\mathfrak{r}$.*

We apply [7], Part II, §6, and see that the orthogonal complement of $\mathfrak{r} \cap j\mathfrak{r}$ in $\mathfrak{r}+j\mathfrak{r}$ is a solvable Kähler algebra. Let $\tilde{\mathfrak{r}}$ denote the orthogonal complement of $\mathfrak{r} \cap j\mathfrak{r}$ in $\mathfrak{r}$. Then $\mathfrak{r}+j\mathfrak{r} = (\mathfrak{r} \cap j\mathfrak{r}) + (\tilde{\mathfrak{r}}+j\tilde{\mathfrak{r}})$ where $\tilde{\mathfrak{r}}$ is an abelian ideal in $\tilde{\mathfrak{r}}+j\tilde{\mathfrak{r}}$ satisfying $\tilde{\mathfrak{r}} \cap j\tilde{\mathfrak{r}} = 0$.

1.3. Let now $\mathfrak{s}$ be a solvable Kähler algebra and $\mathfrak{r}$ an abelian ideal satisfying $\mathfrak{s} = \mathfrak{r}+j\mathfrak{r}$, $\mathfrak{r} \cap j\mathfrak{r} = 0$. We follow the approach of the proof of [12]; Appendix, Theorem 1, and represent the manifold corresponding to $\mathfrak{s}$ as a "tube domain": Fix $y_0 \in \mathfrak{r}$ and define, for $h \in j\mathfrak{r}$, a map $C_h : \mathfrak{r} \to \mathfrak{r}$ by $C_h(y) := [h, y-y_0] - jh$. Then the map $h \mapsto C_h$ is an injective homomorphism for the Lie algebra $j\mathfrak{r}$ into the Lie algebra $\mathrm{aff}(\mathfrak{r})$ of affine transformations of $\mathfrak{r}$. Put $\mathfrak{h} := \{C_h;\ h \in j\mathfrak{r}\}$. Then $C_h y_0 = -jh$ implies that the map $C_h \mapsto C_h y_0$ is an isomorphism of vector spaces. Therefore, the orbit U of the Lie group H generated by $\mathfrak{h}$ is an open subset of $\mathfrak{r}$. We put

$$D(U) := \{x+iy;\ x, y \in \mathfrak{r},\ y \in U\} \subset \mathfrak{r}^{\mathbb{C}}$$

and get that

$$\varphi : \quad \mathfrak{r}+j\mathfrak{r} \to \mathrm{Lie\,Aut}\,D(u), \qquad \varphi(a+jb)(x+iy) = a + C_{jb}\,x + i\,C_{jb}\,y$$

is an isomorphism from $\mathfrak{r}+j\mathfrak{r}$ onto a solvable Lie algebra generating a transitive Lie group on D(U). Moreover, φ is $\mathbb{C}$-linear relative to the natural complex structures. Hence we may consider the tube domain $D(U) = \mathfrak{r}+iU$ as the realization of the complex manifold corresponding to $\mathfrak{r}+j\mathfrak{r}$. The Lie algebra $\mathfrak{s} = \mathfrak{r}+j\mathfrak{r}$ under consideration is then of type $\mathfrak{g}_{-1}+\mathfrak{g}_0$ where $\mathfrak{g}_{-1}$ consists of all translations with elements from $\mathfrak{r}$ and $\mathfrak{g}_0$ is an affine Lie algebra of infinitesimal automorphisms of U.

We denote by $(.,.)$ a scalar product on $\mathfrak{r}$ and extend it as a hermitian form to $\mathfrak{r}+j\mathfrak{r}$. We denote this hermitian form again by $(.,.)$. We represent the Kähler metric on D(U) in the form $(H_0(z)u, v)$. By assumption, j acts as i on the tangent spaces and is orthogonal relative to the Kähler metric as well as to $(.,.)$. Therefore $H_0(z)$ is a $\mathbb{C}$-linear endomorphism of $\mathfrak{r}^{\mathbb{C}}$. Moreover, the Kähler metric is invariant under the group generated by $\mathfrak{g}_{-1}+\mathfrak{g}_0$. As a consequence we have

$$H_0(x+iy)=H(y) \qquad \text{and} \qquad H(Wy)={}^t(dW)^{-1}H(y)(dW)^{-1}$$

for all $y\in U$, $W\in\exp\mathfrak{g}_0$ where $\exp\mathfrak{g}_0$ denotes the Lie group generated by $\mathfrak{g}_0$. Finally, we evaluate the Kähler condition $\nabla_x jY=j\nabla_x Y$. Here we use vector fields on D(U), the defining equation for $\nabla_A B$ and the Kähler condition and get

$$H(y;a)b=H(y;b)a \quad \text{for all } a, b\in\mathfrak{r}, \quad y\in U.$$

Where

$$H(y;a):=\frac{d}{dt}\Big|_0 H(y+ta).$$

As $H(y)$ is $\mathbb{C}$-linear we may write

$$H(y)=A(y)+iB(y) \qquad \text{with} \quad A(y), B(y)\in\operatorname{End}_{\mathbb{R}}\mathfrak{r}.$$

Moreover, $H(y)$ is hermitian whence ${}^tA(y)=A(y)$ and ${}^tB(y)=-B(y)$. Obviously, the Kähler condition gives

$$A(y;a)b=A(y;b)a \qquad \text{and} \qquad B(y;a)b=B(y;b)a.$$

Clearly

$$0=(B(y;u)x, x)=(B(y;x)u, x)=-(u, B(y;x)x) \qquad \text{for} \quad u, x\in\mathfrak{r}.$$

Therefore $B(y;x)x=0$. This implies $B(y;a)b=-B(y;b)a$; together with the Kähler condition we get $B(y;a)=0$ for all $y\in U$, $a\in\mathfrak{r}$. But then $B(y)=B$ is constant. From above we know that B is skew-adjoint and invariant under the action of $\exp\mathfrak{g}_0$.

1.4. We keep the notation of 1.3 and introduce the invariant Kähler metric

$$\tilde{g}_z(u,v)=(A(\operatorname{Im}z)u, v) \quad \text{on } D(U).$$

From

$${}^tA(y)=A(y) \qquad \text{and} \qquad A(y;a)b=A(y;b)a$$

we get (locally) a function $\tilde{\eta}: U\to\mathbb{R}^+$ satisfying

$$\tilde{g}_{iy}(u,v)=d_y^2\log\tilde{\eta}(u,v) \quad \text{for all } u, v\in\mathfrak{r}, \quad y\in U.$$

We evaluate the invariance of $\tilde{g}$ under $\exp\mathfrak{g}_0$. It is easy to see that the directional derivatives of $d_{Wx}\log\tilde{\eta}(dWu)$ and $d_x\log\tilde{\eta}(u)$ coincide for all $x\in U$. Hence, there exists a linear map $\lambda(W; -): \mathfrak{r} \to \mathbb{R}$ satisfying

$$d_{Wx} \log \tilde{\eta}(dW\,u) = d_x \log \tilde{\eta}(u) + \lambda(W; u).$$

Now we see that the directional derivatives of $\log \tilde{\eta}(Wx)$ and $\log[\tilde{\eta}(x) e^{\lambda(W;x)}]$ coincide for all $x \in U$. This implies $\tilde{\eta}(Wx) = \tilde{\eta}(x) e^{\lambda(W;x)} e^{c(W)}$ where $c(W) \in \mathbb{R}$.

THEOREM. — U *is convex.*

Proof. — The proof consists of several steps. We first replace W by a one-parameter group $W_t = \exp tT$ and write $\lambda(W; u) = \langle l(W), u \rangle$:

$$(1) \qquad d_x \log \tilde{\eta}(Tx) = \langle d_{Id}\, l(T), u \rangle + d_{Id}\, c(T).$$

We recall that $\mathfrak{g}_0$ consists of elements of type C_{jh}, $h \in \mathfrak{r}$. We claim that $\varphi_x: \mathfrak{r} \to \mathfrak{r}$, $h \mapsto C_{jh} x$ is bijective for all x in a neighborhood of y_0. For a proof we note that φ_x is linear and that $\varphi_{y_0} = \mathrm{id}$ holds. Hence $\det \varphi_x \neq 0$ in a neighborhood of y_0. Moreover, φ_x is affine in x whence φ_x^{-1} is rational in x and also $C_{j\varphi_x^{-1}(u)}$ is rational in x. Finally, $C_{j\varphi_x^{-1}(u)} x = \varphi_x \varphi_x^{-1}(u) = u$. Hence, from (1) we get with $T = C_{j\varphi_x^{-1}(u)}$

$$(2) \qquad d_x \log \tilde{\eta}(u) = \langle d_{Id}\, l(C_{j\varphi_x^{-1}(u)}), x \rangle + d_{Id}\, c(C_{j\varphi_x^{-1}(u)}).$$

As a consequence of this we see that $d_x \log \tilde{\eta}(u)$ is rational in x. We put $\tilde{\eta}_x'' = d_x^2 \log \tilde{\eta}$ and $\tilde{\eta}_x' = -d_x \log \tilde{\eta}$. Then $(U, \tilde{\eta}_x'')$ is a complete, connected riemannian manifold (as a connected group acts transitively on it). Let $a \in U$ and $u \in \mathfrak{r}$ and assume $a + \tau u \in U$ for all $0 \leqq \tau \leqq \tau_0$ and $a + \tau_0 u \notin U$. Because η_x' is rational in x we may apply part (*a*) of the proof of [6], Lemma 8.2, and get $\tilde{\eta}(a + \tau u) \to \infty$ if $\tau \to \tau_0$. Now (*c*) of the proof of [6], Theorem 3.9, shows that U is convex.

2.1. Let $\mathfrak{s}$ be an arbitrary solvable Kähler algebra and $\mathfrak{r}$ a commutative ideal in $\mathfrak{s}$. Then $\mathrm{ad}\,\mathfrak{s}$ is a solvable Lie algebra of endomorphisms of $\mathfrak{s}$. Let $\mathrm{ad}\,\tilde{\mathfrak{s}}$ denote the algebraic hull of $\mathrm{ad}\,\mathfrak{s}$ (*see* [3], II, § 14, for a definition). Then $\mathrm{ad}\,\tilde{\mathfrak{s}}$ leaves invariant each ideal of $\mathfrak{s}$ and we have

$$[\mathrm{ad}\,\tilde{\mathfrak{s}}, \mathrm{ad}\,\tilde{\mathfrak{s}}] = [\mathrm{ad}\,\mathfrak{s}, \mathrm{ad}\,\mathfrak{s}] = \mathrm{ad}\,[\mathfrak{s}, \mathfrak{s}].$$

From [3], V, § 3.5, we know that $\mathrm{ad}\,\tilde{\mathfrak{s}}$ is solvable and $\mathrm{ad}\,\tilde{\mathfrak{s}} = \mathfrak{a} + \mathfrak{n}$ where $\mathfrak{n}$ is the ideal of $\mathrm{ad}\,\tilde{\mathfrak{s}}$ consisting of all nilpotent elements of $\mathrm{ad}\,\tilde{\mathfrak{s}}$ and $\mathfrak{a}$ is an abelian algebraic Lie algebra consisting of semisimple endomorphisms. Moreover, $\mathfrak{a} = \mathfrak{a}_{\mathbb{R}} + \mathfrak{a}_i$ where $\mathfrak{a}_{\mathbb{R}}$ (resp. $\mathfrak{a}_i$) consists of the elements of $\mathfrak{a}$ having only real (resp. purely imaginary) eigenvalues [3], II, § 13. From the above it follows that $\mathfrak{r}$ contains a subspace $\mathfrak{q}$ so that $\mathfrak{n}.\mathfrak{q} = 0$, $T|\mathfrak{q} = \lambda(T)\,\mathrm{Id}$ for $T| \in \mathfrak{a}_{\mathbb{R}}$ and $T|\mathfrak{q} = \mu(T)\,I$ for $T \in \mathfrak{a}_i$ where $I^2 = -\mathrm{Id}$. In case $T|\mathfrak{q} = 0$ for all $T \in \mathfrak{a}_i$ we may choose $\mu \equiv 0$ and $\mathfrak{q}$ one dimensional. In general we only know $\dim \mathfrak{q} \leqq 2$. Of course, $\mathfrak{q}$ is an abelian ideal of $\mathfrak{s}$. Moreover, $\mathfrak{q} \cap j\mathfrak{q} = 0$ or $\mathfrak{q} \cap j\mathfrak{q} = \mathfrak{q}$. In the second case we may write $\mathfrak{s} = \mathfrak{q} + \mathfrak{q}^{\perp}$ with a solvable Kähler algebra $\mathfrak{q}^{\perp}$. We will see in section 4 that this implies the fundamental conjecture of [8]. We assume therefore in what follows $\mathfrak{q} \cap j\mathfrak{q} = 0$. Clearly, in paragraph 1 we may put $\mathfrak{r} = \mathfrak{q}$.

2.2. We assume $\mathfrak{q}\cap j\mathfrak{q}=0$. Put $\mathfrak{q}_1=\overline{U}\cap(-\overline{U})$ where $\overline{U}$ denotes the closure of U in $\mathfrak{q}$. We set $\mathfrak{q}_2=\mathfrak{q}_1^{\perp}$. Hence $\mathfrak{q}=\mathfrak{q}_1+\mathfrak{q}_2$ and $U=\mathfrak{q}_1+U_2$ where $U_2=U\cap\mathfrak{q}_2$. Moreover, U_2 is an open convex domain in $\mathfrak{q}_2$ not containing a straight line.

LEMMA. — $\mathfrak{q}_1+j\mathfrak{q}_1$ *is an ideal of* $\mathfrak{q}+j\mathfrak{q}$.

Proof. — Let $h\in\mathfrak{q}$ be arbitrary and $W_t=\exp t C_{jh}$. Then it is easy to see that the linear part of W_t leaves q_1 invariant, i. e. $[jh, \mathfrak{q}_1]\subset\mathfrak{q}_1$. For all $h\in\mathfrak{q}$. We also know $[C_{jh}, C_{jq}]=C_{[jh, jq]}$. From the definition of $\mathfrak{q}$ we get that this commutator is just a translation with $a=-j[jh, jq]$. But then $ta+x\in U$ for all $x\in U$ whence $a\in\mathfrak{q}_1$. This implies—using (1.2)$-[jh, q]+[h, jq]\in\mathfrak{q}_1$ for all $h, q\in\mathfrak{q}$. We choose $q\in\mathfrak{q}_1$ and get $[jq_1, \mathfrak{q}]\subset\mathfrak{q}_1$ for all $q_1\in\mathfrak{q}_1$. Now it is straightforward to show that $\mathfrak{q}_1+j\mathfrak{q}_1$ is an ideal in $\mathfrak{q}+j\mathfrak{q}$.

COROLLARY. — $\mathfrak{q}_2+j\mathfrak{q}_2$ *is a subalgebra of* $\mathfrak{q}+j\mathfrak{q}$.

2.3. We retain the notation of the last section.

LEMMA. — *Assume* $\mathfrak{q}\cap j\mathfrak{q}=0$, *then* $\mathfrak{q}_1+j\mathfrak{q}_1$ *is abelian.*

Proof. — We clearly have to consider the cases $\dim\mathfrak{q}_1=0, 1, 2$. It is clear that we only have to prove $[j\mathfrak{q}_1, \mathfrak{q}_1]=0$. If $\dim\mathfrak{q}_1=1$, then $\mathfrak{q}_1=\mathbb{R}q$ and $[jq, q]=\lambda q$. By the definition of $\mathfrak{q}$, this implies $\operatorname{ad} jq \mid \mathfrak{q}=\lambda\,\mathrm{Id}$, whence $[jg, x_2]=\lambda x_2$ for all $x_2\in q_2$. But $\mathfrak{q}_1+j\mathfrak{q}_1$ is an ideal in $\mathfrak{q}+j\mathfrak{q}$ whence $\lambda=0$. Therefore $\mathfrak{q}_1+j\mathfrak{q}_1$ is abelian and commutes with $\mathfrak{q}_2$.

Assume now $\dim\mathfrak{q}_1=2$. Thus $\mathfrak{q}_1=\mathfrak{q}$. Suppose there exists $q\in\mathfrak{q}$ so that $\operatorname{ad}|jh_1\,\mathfrak{q}=\alpha\,\mathrm{Id}+\beta\,\mathrm{I}$, $\alpha\neq 0$, then we can find $h_2\in\mathfrak{q}$, for which $\alpha=0$. Thus $\operatorname{ad} jh_2=\beta\,\mathrm{I}$. We choose h_1 and h_2 so that

$$\operatorname{ad}|jh_2\,\mathfrak{q}=\beta\,\mathrm{I},\qquad \operatorname{ad}|jh_1\,\mathfrak{q}=\mathrm{Id}+\gamma\,\mathrm{I},\qquad \mathrm{I}\,h_1=h_2,\ \mathrm{I}\,h_2=-h_1.$$

Therefore

$$[jh_1, jh_2]=j[jh_1, h_2]-j[jh_2, h_1]=j(h_2-\gamma h_1)-j\beta h_2=-\gamma jh_1+(1-\beta)jh_2.$$

This element acts trivially on $\mathfrak{q}$ only if $\gamma=0$ and $(1-\beta)\beta=0$. If $\beta=0$, then from (1.3) we get

$$\begin{aligned}0&=\prec[jh_1, jh_2], jh_2\succ+\prec[jh_2, h_2], jjh_1\succ+\prec[h_2, jh_1], jjh_2\succ\\&=\prec h_2, h_2\succ+\prec -h_1, -h_1\succ+\prec -h_2, -h_2\succ,\end{aligned}$$

a contradiction.

If $\beta=1$, then $[jh_1, jh_2]=0$ and (1.3) shows

$$\begin{aligned}0&=\prec[jh_1, jh_2], jh_2\succ+\prec[jh_2, h_2], -h_1\succ+\prec[h_2, jh_1], -h_2\succ\\&=\prec h_1, h_1\succ+\prec h_2, h_2\succ,\end{aligned}$$

a contradiction.

Therefore $\operatorname{ad} jh \mid \mathfrak{q} = \alpha(h)\mathrm{I}$ for all $h \in \mathfrak{q}$. We choose $h_2 \in \mathfrak{q}$ so that $\alpha(h)=0$ and we choose $h_1 \in \mathfrak{q}$ so that $\mathrm{I} h_1 = h_2$, $\mathrm{I} h_2 = -h_1$ holds. Then $\operatorname{ad} jh_1 \mid \mathfrak{q} = \alpha \mathrm{I}$ and

$$[jh_1, jh_2] = j[jh_1, h_2] + j[h_1, jh_2] = -\alpha jh_1.$$

As above we know that this commutator acts trivially on $\mathfrak{q}$. Hence $\alpha = 0$ and the lemma is proven.

2.4. We retain the notation of the previous sections.

LEMMA. — *If* $\mathfrak{q} \cap j\mathfrak{q} = 0$, *then either* $\dim \mathfrak{q} = 1$ *and there exists* $r \in \mathfrak{q}$ *satisfying* $[jr, r] = r$ *or* $\mathfrak{q} + j\mathfrak{q}$ *is abelian.*

Proof. — From 2.2 we get $\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_2$ and $\mathrm{U} = \mathfrak{q}_1 + \mathrm{U}_2$. Suppose $\dim \mathfrak{q} = 2$. By the last lemma we may assume $\dim \mathfrak{q}_1 \leqq 1$.

Assume first $\dim \mathfrak{q}_1 = 1$. Let $x \in \mathfrak{q}_1$, $x \neq 0$; then $\mathbb{R} x = \mathfrak{q}_1$ and $[jx, x] = 0$. By the definition of $\mathfrak{q}$ this implies $[jx, \mathfrak{q}] = 0$. From Corollary 2.2 we know that $\mathfrak{q}_2 + j\mathfrak{q}_2$ is a subalgebra of $\mathfrak{q} + j\mathfrak{q}$. Because $\dim \mathfrak{q}_2 = 1$ and $[j\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}$ we know $[jr, r] = \lambda r$ for $r \in \mathfrak{q}_2$. But (1.3) now gives

$$\begin{aligned} 0 &= \prec [jr, jx], jx \succ + \prec [jx, x], jjr \succ + \prec [x, jr], jjx \succ \\ &= \prec j[jr, x] + j[r, jx], jx \succ + 0 + \prec [jr, x], x \succ = 2\lambda \prec x, x \succ. \end{aligned}$$

This implies $\lambda = 0$. Hence $[j q, q] = 0$ for all $q \in \mathfrak{q}$ and a straightforward computation shows that $\mathfrak{q} + j\mathfrak{q}$ is abelian.

It remains to consider the case $\dim \mathfrak{q}_1 = 0$. From paragraph 1 we know that the Lie group $\exp \mathfrak{g}_0$ generated by C_{jq}, $q \in \mathfrak{q}$, has an open convex orbit U in $\mathfrak{q}$ which contains no straight line. We therefore may apply the results of [15], Chap. II. Hence

$$\mathfrak{q} = \mathfrak{q}_1 + \mathfrak{q}_{1/2} \qquad \text{and} \qquad \mathrm{U} = \left\{ x_1 + x_{1/2};\ x_1 - \frac{1}{2} x_{1/2} \Delta x_{1/2} \in \mathrm{K} \right\},$$

where K is a convex regular cone in $\mathfrak{q}_1$. By construction, $\mathrm{C}_{jq} \in \operatorname{Lie}\operatorname{Aut}\mathrm{U}$ for all $q \in \mathfrak{q}$. Because $\mathfrak{q}_1 \neq 0$ and $\dim \mathfrak{q} \leqq 2$ we only have the possibilities $\dim \mathfrak{q}_1 = 2$, $\mathfrak{q}_{1/2} = 0$ or $\dim \mathfrak{q}_1 = \dim \mathfrak{q}_{1/2} = 1$. It is easy to see that in both cases the isotropy algebra in Lie Aut U of any point in U is trivial. Therefore $\{\mathrm{C}_{jq};\ q \in \mathfrak{q}\} = \operatorname{Lie}\operatorname{Aut}\mathrm{U}$. If $\mathfrak{q}_{1/2} = 0$ and $\dim \mathfrak{q}_1 = 2$ then the action of $\mathrm{T} \in \operatorname{Lie}\operatorname{Aut}\mathrm{U}$ is not of type $\lambda \mathrm{Id} + \mu \mathrm{I}$ on $\mathfrak{q} = \mathfrak{q}_1$. Hence $\mathfrak{q}_{1/2} = 0$ implies $\dim \mathfrak{q}_1 = 1$. It remains to consider the case $\dim \mathfrak{q}_1 = \dim \mathfrak{q}_{1/2} = 1$. But then a comparison with [15], Chap. II, shows that $\operatorname{ad} x\, \mathfrak{q}$ is nilpotent for $x \in \mathfrak{q}_{1/2}$. This contradicts the choice of $\mathfrak{q}$.

3.1. In the next sections of this paper we prove that solvable Kähler algebras are "modifications" of "semi-direct products $\mathfrak{a} \oplus \mathfrak{b}$ where $\mathfrak{a}$ is an abelian ideal and $\mathfrak{b}$ is a normal j-algebra" (*) (*see* [12], Chap. 2.3, for *a* definition). Let $\mathfrak{s}$ be a solvable Kähler algebra with scalar product $\prec .\,, .\succ$ and complex structure j. Let $\mathrm{D}: \mathfrak{s} \to \operatorname{Der}\mathfrak{s}$ be a

(*) in this paper such an expression is exclusively used for split solvable $\mathfrak{a} \oplus \mathfrak{b}$

linear map satisfying for all $x, y \in \mathfrak{s}$

(3.1) $$D(x) \text{ is skew adjoint relative to } \prec .,. \succ,$$

(3.2) $$[D(x), j]=0,$$

(3.3) $$[D(x), D(y)]=0,$$

(3.4) $$D([x, y])=0,$$

(3.5) $$D(D(x)y-D(y)x)=0.$$

We call such a map a "weak modification map". We would like to mention that modification maps and modifications have been used before ([2], [5]).

LEMMA. — *Let* D *be a weak modification map of the solvable Kähler algebra* $\mathfrak{s}$. *Then the product* $(x, y):=[x, y]+D(x)y-D(y)x$ *defines on* $\mathfrak{s}$ *the structure of a solvable Kähler algebra.*

Proof. — Obviously, $(x, x)=0$. A straight forward computation shows that (x, y) defines the structure of a Lie algebra on the vector space $\mathfrak{s}$ and that $\mathfrak{s}$ remains solvable. Using (3.1) and (3.2) one easily verifies that (1.2) and (1.3) are satisfied for $(.,.)$. This proves the lemma.

3.2. Let M be a simply connected h. k. m. (-homogeneous Kähler manifold) and S a connected solvable group of automorphisms of M. Assume moreover that S acts simply transitive on M. Hence S is simply connected.

Let $\mathfrak{s}=\text{Lie } S$ and D a weak modification map of $\mathfrak{s}$. We put

$$\mathfrak{k}:=\{D(x);\ x\in\mathfrak{s}\},\qquad \mathfrak{g}:=\mathfrak{k}\oplus\mathfrak{s}.$$

Then $\mathfrak{g}$ is a Lie algebra with product

$$[D+x, D'+x']=[x, x']+Dx'-D'x.$$

We extend j to $\mathfrak{g}$ by putting $j|\mathfrak{k}=0$. Moreover the skew form $\rho(x, y)=\prec x, jy\succ$ on $\mathfrak{s}$ will be extended trivially to $\mathfrak{g}$. Then it is easy to verify that $\mathfrak{g}$ is a (general) Kähler algebra of infintesimal automorphism of M [7], Part II, 1. Let G denote the connected group of automorphisms of M having Lie algebra $\mathfrak{g}$, then $M\cong G/K$ and Lie $K=\mathfrak{k}$. Moreover, K is connected and closed in G and we have $S\subset G$.

We consider $\tilde{D}: \mathfrak{s}\to\mathfrak{g},\ x\mapsto D(x)+x$. Then it is clear that $\tilde{D}$ is an injective homomorphism of the "modified" Lie algebra on the vector space $\mathfrak{s}$ onto the sub algebra $\tilde{\mathfrak{s}}=\tilde{D}(\mathfrak{s})$ of $\mathfrak{g}$.

Let $\tilde{S}$ denote the connected subgroup of G with Lie $\tilde{S}=\tilde{\mathfrak{s}}$. Then $\tilde{S}$ is solvable and it is easy to see that $\tilde{S}$ acts transitive on M. But $\dim\mathfrak{s}=\dim\tilde{\mathfrak{s}}$, therefore S has discrete isotropy subgroup. We have assumed that M is simply connected, so S acts simply transitive on M. Hence, a modification corresponds to a different choice of a simply transitive group of automorphisms of M. We also note that the complex structure $\tilde{j}$ on $\tilde{\mathfrak{s}}$ is given by $\tilde{j}(D(x)+x)=D(jx)+jx$ where j denotes the complex structure of $\mathfrak{s}$.

3.3. In this section we investigate weak modifications of particularly simple Kähler algebras.

LEMMA 1. — *Let* $\mathfrak{a}$ *be an abelian Kähler algebra and* D *a weak modification map for* $\mathfrak{a}$. *Then* $\mathfrak{a}$ *is the orthogonal sum of j-invariant subspaces* $\mathfrak{a}=\hat{\mathfrak{a}}_0+\hat{\mathfrak{a}}_1=\hat{\mathfrak{a}}_0+\hat{\mathfrak{a}}_{10}+\hat{\mathfrak{a}}_{11}$ *where:*

(*a*) $\hat{\mathfrak{a}}_0$ *is spanned by* $\{D(x)y;\ x,\ y\in\mathfrak{a}\}$.

$$\hat{\mathfrak{a}}_{10}=\{x\in\mathfrak{a};\ D(x)=0,\ D(jx)=0 \text{ and } D(y)x=0 \text{ for all } y\in\mathfrak{a}\}.$$

$$\hat{\mathfrak{a}}_{11}\subset\{x\in\mathfrak{a};\ D(y)x=0 \text{ for all } x,\ y\in\mathfrak{a}\}.$$

(*b*) $D(D(x)y)=0$ *for all* $x,\ y\in\mathfrak{a}$.

(*c*) *In the modified algebra* $\hat{\mathfrak{a}}_0$ *is an abelian ideal,* $\hat{\mathfrak{a}}_1$ *is an abelian subalgebra and we have* $[\hat{\mathfrak{a}}_{10},\ \mathfrak{a}]=0$ *and* $[\mathfrak{a},\ \mathfrak{a}]=\hat{\mathfrak{a}}_0$.

Proof. — We know that $\{D(x);\ x\in\mathfrak{a}\}$ is abelian and consists of skew-adjoint endomorphisms of the complex space $\mathfrak{a}$. Hence there exists a basis of $\mathfrak{a}$ consisting of common eigenvectors for all $D(x)$. Moreover $D(x)$ acts on such a basis vector by multiplication with a purely imaginary number. We can therefore identify $\mathfrak{a}$ with $\mathbb{C}^n$ so that the canonical basis $e_1,\ \ldots,\ e_n$ consists of common eigenvectors for all $D(x)$. Then $D(x)$ is a diagonal matrix with purely imaginary eigenvalues. Hence there exist $\mathbb{R}$-linear maps $\lambda_m:\ \mathbb{C}^n\to\mathbb{R}$, $m=1,\ \ldots,\ n$, so that $D(x)e_m=\lambda_m(x)ie_m$ for all $x\in\mathfrak{a}$, $1\leqq m\leqq n$. The last condition for a modification map requires $D(D(x)y-D(y)x)=0$, i.e. $\lambda_m(D(x)y-D(y)x)=0$ for all $x,\ y\in\mathfrak{a}$, $m=1,\ \ldots,\ n$. Choosing $x=\alpha_r e_r$ and $y=\beta_s e_s$ this means $0=\lambda_m(\lambda_s(\alpha_r e_r)\beta_s ie_s-\lambda_r(\beta_s e_s)\alpha_r ie_r)$. Hence

$$(*)\qquad \lambda_s(\alpha_r e_r)\lambda_m(\beta_s ie_s)=\lambda_r(\beta_s e_s)\lambda_m(\alpha_r ie_r).$$

Choosing $s=r=m$ in $(*)$ we have $\lambda_m(\alpha_m e_m)\lambda_m(\beta_m ie_m)=\lambda_m(\beta_m e_m)\lambda_m(\alpha_m ie_m)$. We know that $\lambda_m:\ \mathbb{C}e_m\to\mathbb{R}$ is $\mathbb{R}$-linear and has a nontrivial kernel. Let $\lambda_m(\alpha_m e_m)=0$. Then $0=\lambda_m(\beta_m e_m)\lambda_m(\alpha_m ie_m)$ for all $\beta_m\in\mathbb{C}$. If $\lambda_m(\beta_m e_m)\neq 0$ for some β_m, then $\lambda_m(\alpha_m ie_m)=0$. But $\alpha_m e_m$ and $i\alpha_m e_m$ span $\mathbb{C}e_m$ over $\mathbb{R}$. Hence $\lambda_m(\gamma e_m)=0$ for all $\gamma\in\mathbb{C}$, a contradiction. This implies

$$(*)\qquad \lambda_m(\alpha e_m)=0 \quad \text{for all } \alpha\in\mathbb{C},\ 1\leqq m\leqq n.$$

Next we choose $r=m$ in $(*)$. Then by $(**)$ we have

$$0=\lambda_s(\alpha_m e_m)\lambda_m(\beta_s ie_s) \quad \text{for all } 1\leqq s\leqq n \text{ and all } \alpha_m,\ \beta_s\in\mathbb{C}.$$

This identity is surely satisfied if, for a certain m, $\lambda_s(\mathbb{C}e_m)=0$ for all s. Assume $m=1,\ \ldots,\ l$ are exactly these m's. At any rate

$$(***)\qquad \lambda_s(\mathbb{C}e_m)=0 \quad \text{or} \quad \lambda_m(\mathbb{C}e_s)=0 \quad \text{for all } 1\leqq m,\ s\leqq n.$$

Assume now $r>l$ and let s be arbitrary. If $\lambda_s(\mathbb{C}e_r)=0$, then in $(*)$ we have $0=\lambda_r(\beta_s e_s)\lambda_m(\alpha_r ie_r)$. But $r>l$ and there exists m and α_r so that

$\lambda_m(\alpha_r ie_r) \neq 0$. Therefore $\lambda_r(\mathbb{C}e_s)=0$. The second possibility is anyway $\lambda_r(\mathbb{C}e_s)=0$. Therefore $\lambda_r(\mathbb{C}e_s)=0$ for all $r>l$, $1 \leqq s \leqq n$. As a consequence, $\lambda_1, \ldots, \lambda_l$ depend only on $\mathbb{C}e_{l+1}, \ldots, \mathbb{C}e_m$ and $\lambda_{l+1}=\ldots=\lambda_m=0$. Some of $\lambda_1, \ldots, \lambda_l$ may also be 0. Say $\lambda_1=\ldots=\lambda_k=0$ and $\lambda_m \neq 0$ for $k+1 \leqq m < l$. We put $\hat{\mathfrak{a}}_{10}=\mathbb{C}e_1+\ldots+\mathbb{C}e_k$, $\hat{\mathfrak{a}}_0=\mathbb{C}e_{k+1}+\ldots+\mathbb{C}e_l$ and $\hat{\mathfrak{a}}_{11}=\mathbb{C}e_{l+1}+\ldots+\mathbb{C}e_n$. For $x, y \in \mathfrak{a}$ we have $[\mathrm{D}(x)y]_m=\lambda_m(x)y_m ie_m$. This is zero if $m \notin \{k+1, \ldots, l\}$. For $m \in \{k+1, \ldots, l\}$ we have $\lambda_m(x)=\lambda_m\left(\sum_{r=l+1}^{n} x_r e_r\right)$. By the choice of m there exist $r>l$ and x_r so that $\lambda_m(x_r e_r) \neq 0$. Hence $\mathbb{C}e_m \subset \{\mathrm{D}(x)y;\ x, y \in \mathfrak{a}\} \subset \hat{\mathfrak{a}}_0$. This proves (*a*). To verify (*b*) we choose r and m as above then

$$\mathrm{D}(x_r e_r) y_m e_m - \mathrm{D}(y_m e_m) x_r e_r = \mathrm{D}(x_r e_r) y_m e_m.$$

Hence $\mathrm{D}(\hat{\mathfrak{a}}_0)=0$. The last statement of the assertion follows from the definition of the subspaces and the properties of D.

LEMMA 2. — *Let $\mathfrak{s}=\mathfrak{a}+\mathfrak{s}_\mathrm{D}$ be the semi-direct product of an abelian ideal $\mathfrak{a}$ of $\mathfrak{s}$ and a normal j-algebra $\mathfrak{s}_\mathrm{D}$. We assume that $\mathfrak{s}$ is a Kähler algebra and that $\mathfrak{a}$ and $\mathfrak{s}_\mathrm{D}$ are orthogonal and j-invariant. Let $\mathrm{D}: \mathfrak{s} \to \mathrm{Der}\,\mathfrak{s}$ be a weak modification map. Then:*

(*a*) *For all $x \in \mathfrak{s}$, $\mathrm{D}(x)$ leaves $\mathfrak{a}$ and $\mathfrak{s}_\mathrm{D}$ invariant.*

(*b*) *$\mathrm{D}_a: \mathfrak{a} \to \mathrm{Der}\,\mathfrak{a}$, $\mathrm{D}_a(x):=\mathrm{D}(x)\,|\,\mathfrak{a}$ is a weak modification map of $\mathfrak{a}$.*

(*c*) *$\mathrm{D}_\mathrm{D}: s_\mathrm{D} \to \mathrm{Der}\,\mathfrak{s}_\mathrm{D}$, $\mathrm{D}_\mathrm{D}(x)\,|\,\mathfrak{s}_\mathrm{D}$ is a weak modification map of $\mathfrak{a}$.*

(*d*) *Let $\mathfrak{a}=\mathfrak{a}_0+\mathfrak{a}_1$ where $\mathfrak{a}_0=\{x \in \mathfrak{a};\ \mathrm{D}(x)\,|\,\mathfrak{s}_\mathrm{D}=0\}$ and $\mathfrak{a}_1=\mathfrak{a} \ominus \mathfrak{a}_0$. Then $\mathfrak{a}_0$ is an ideal of the modification of $\mathfrak{s}$ via D and $\mathfrak{a}_1$ is an abelian subalgebra. Moreover, in the modification Lie algebra $(\mathfrak{a}_1, \mathfrak{s}_\mathrm{D}) \subset \mathfrak{a}_0+(\mathfrak{s}_\mathrm{D}, \mathfrak{s}_\mathrm{D})$.*

(*e*) *We have $\mathrm{D}(x)y \in (\mathfrak{s}, \mathfrak{s})$ (for all $x, y \in \mathfrak{s}$). In particular $\mathrm{D}(\mathrm{D}(x)y)=0$.*

Proof. — (*a*) Let $\mathrm{W}=\exp t\mathrm{D}(x)$, then W is an automorphism of $\mathfrak{s}$ which commutes with j. Hence, for all $a \in \mathfrak{a}$ we get $[j\mathrm{W}a, \mathrm{W}a]=\mathrm{W}[ja, a]=0$. We write $\mathrm{W}a=u+x$, $u \in \mathfrak{a}$, $x \in \mathfrak{s}_\mathrm{D}$ and get $0=[ju, x]+[jx, u]+[jx, x]$. Here the first two summands are in $\mathfrak{a}$ whereas the last is in $\mathfrak{s}_\mathrm{D}$. This implies $0=[jx, x]$. From [4], Lemma 3.5. We get $x=0$.

(*b*) and (*c*) are clear.

(*d*) Let $a \in \mathfrak{a}_0$ and $x \in \mathfrak{s}$, then in the modified algebra we have $(a, x)=[a, x]+\mathrm{D}(a)x-\mathrm{D}(x)a$.

For $x \in \mathfrak{a}$ we see $(a, x) \in \mathfrak{a}$ and $\mathrm{D}((a, x))=0$. If $x \in \mathfrak{s}_\mathrm{D}$, then $\mathrm{D}(a)x=0$ whence $(a, x) \in \mathfrak{a}$. Obviously $\mathrm{D}((a, x))=0$. Next we consider the weak modification map D_a and split $\mathfrak{a}=\hat{\mathfrak{a}}_0+\hat{\mathfrak{a}}_1$ as in Lemma 1. Clearly $\mathfrak{a}_1 \subset \hat{\mathfrak{a}}_1$ where $(\mathfrak{a}_1, \mathfrak{a}_1)=0$. To see that $(\mathfrak{s}_1, \mathfrak{s}_\mathrm{D}) \subset \mathfrak{a}_0+(\mathfrak{s}_\mathrm{D}, \mathfrak{s}_\mathrm{D})$ holds we first note $(x_1, x_\mathrm{D})=[x_1, x_\mathrm{D}]+\mathrm{D}(x_1)x_\mathrm{D}-\mathrm{D}(x_\mathrm{D})x_1$ where $[.,\ .]$ denotes the Lie product of the unmodified algebra. By definition, $\mathrm{D}([x_1, x_\mathrm{D}])=0$ and because $\mathrm{D}(x_1)x_\mathrm{D} \in (\mathfrak{s}_\mathrm{D}, \mathfrak{s}_\mathrm{D})$ we also have $\mathrm{D}(\mathrm{D}(x_1)\,x_\mathrm{D})=0$. Hence, using this and (3.4) and (3.5) we get $\mathrm{D}(\mathrm{D}(x_\mathrm{D})\,x_1)=0$. Therefore $(x_1, x_\mathrm{D}) \in \mathfrak{a}_0+(\mathfrak{s}_\mathrm{D}, \mathfrak{s}_\mathrm{D})$. Finally let $x=x_a+x_a+x_\mathrm{D}$ and $y=y_a+y_\mathrm{D}$. Then

$$\mathrm{D}(x)y=\mathrm{D}(x)y_a+\mathrm{D}(x)y_\mathrm{D}=\mathrm{D}_a(x_a)y_a+\mathrm{D}(x_\mathrm{D})y_a+\mathrm{D}(x)y_\mathrm{D}.$$

We know that $D(x)$ is a skew symmetric derivation of $\mathfrak{s}_D$, hence $D(x)y_D \in [\mathfrak{s}_D, \mathfrak{s}_D]$ by [4], Lemma 3.5. Therefore we have only to consider $D_a(x_a)y_a$ and $D(x_D)y_a$. From the last Lemma we get $D_a(x_a)y_a \in (\mathfrak{a}, \mathfrak{a}) \subset \hat{\mathfrak{a}}_0$. Hence $D(D_a(x_a)y_a)=0$. Now we consider $D(x_D)y_0$. Here we may assume $x_D \perp [\mathfrak{s}_D, \mathfrak{s}_D]$. Then $\operatorname{ad} x_D$ acts self adjoint on $\mathfrak{a}$. We also know $[D(x_D), \operatorname{ad} x_D] = \operatorname{ad} D(x_D)x_D = 0$, hence $D(x_D)$ leaves the eigenspaces of $\operatorname{ad} x_D$ invariant. If $\lambda \neq 0$ is an eigenvalue of $\operatorname{ad} x_D$ on $\mathfrak{a}$, then $\operatorname{ad} x_D + D(x_D)$ is invertible on the corresponding eigenspace. Therefore this eigenspace is contained in $(\mathfrak{s}_D, \mathfrak{a})$. In particular $D(x_D)a \in (\mathfrak{s}_D, a)$ for all $a \in \mathfrak{a}$ which lie in an eigenspace for $\operatorname{ad} x_D$ for a nonzero eigenvalue. If $\operatorname{ad} x_D a = 0$, then $(x_D, a) = [x_D, a] + D(x_D)a = D(x_D)a$. Hence the assertion.

3.4. The results of 3.3 indicate that condition (3.5) can be sharpened.

A map $D: \mathfrak{s} \to \operatorname{Der} \mathfrak{s}$ satisfying (3.1), ..., (3.4) as in 3.1 is called a modification map if D satisfies

$$(3.5^*) \qquad D(D(x)y)=0 \quad \text{for all } x, y \in \mathfrak{s}.$$

Remarks. — (*a*) (3.5*) is automatically satisfied if $D(x)\mathfrak{s} \subset [\mathfrak{s}, \mathfrak{s}]$ for all $x \in \mathfrak{s}$.

(*b*) By 3.3 we have: let $\mathfrak{s}$ be the semi-direct product of an abelian ideal and the Lie algebra of a simply transitive group of a bounded homogeneous domain. Assume D is a weak modification map of $\mathfrak{s}$. Then D is a modification map.

LEMMA 1. — *Let $\mathfrak{s}$ be a solvable Kähler algebra with product $[.,.]$ and $D: s \to \operatorname{Der}(\mathfrak{s}[.,.])$ a modification map. Denote by $(.,.)$ the product in the modified algebra. Then:*

(*a*) $D(x) \in \operatorname{Der}(\mathfrak{s}, (.,.))$ *for all* $x \in \mathfrak{s}$.

(*b*) $-D: \mathfrak{s} \to \operatorname{Der}(\mathfrak{s}, (.,.))$ *is a modification map.*

(*c*) *The composition of the modifications corresponding to* D *and* $-D$ *reproduces* $(\mathfrak{s}, [.,.])$.

Proof. — Straightforward computation.

The above results show that modifications are reversible. The following result proves that modifications subtract parts of the adjoint representation.

LEMMA 2. — *Let $\mathfrak{s}$ be a solvable Kähler algebra and* D *a modification map of* $\mathfrak{s}$. *Let* $u_1 = \{x \in \mathfrak{s};\ D(x)=0\}$ *and* $u_2 = \mathfrak{s} \ominus \mathfrak{u}_1$. *Then:*

(*a*) $D(x)\mathfrak{s} \subset \mathfrak{u}_1$ *for all* $x \in \mathfrak{s}$.

(*b*) $D(x)\mathfrak{u}_2 = 0$, $x \in \mathfrak{s}$.

(*c*) *In the modified algebra the adjoint representation is given by* $\operatorname{ad} u + D(u)$ *for* $u \in \mathfrak{u}_2$.

Proof. — (*a*) Follows from (3.5*).

(*b*) As $D(x)$ leaves $\mathfrak{u}_1$ invariant and is skew-adjoint, it also leaves $\mathfrak{u}_2$ invariant. Hence $D(x)\mathfrak{u}_2 = 0$ because of (*a*).

(*c*) From $(u, x) = [u, x] + D(u)x - D(x)u$ the assertion follows.

Occasionaly we will use the following Lemma 3.

LEMMA 3. — *Let* $\mathfrak{s}$ *be solvable Kähler algebra. Assume* $\mathfrak{s}$ *is the modification of the semidirect product of an abelian ideal* $\mathfrak{a}$ *and a normal j-algebra* $\mathfrak{s}_D$. *Denote by* D *the modification map of* $\mathfrak{s}$ *which leads back to the original semidirect product. Then the maps* $D_a(a+x)=D(a)$ *and* $D_D(a+x)=D(x)$ *are modification maps of* $\mathfrak{s}$.

Proof. — We have to verify only (3.4) and (3.5*). We denote the Lie product of the semidirect product by $(.,.)$. Then

$$[a+x,\ b+y]=[a,\ b]+[a,\ y]+[x,\ b]+[x,\ y].$$

It is clear that both maps vanish on $[a, b]$ and $[x, y]$. Hence it suffices to show $D_a([a, y])=0$ for all $a\in\mathfrak{a}$, $y\in\mathfrak{s}_D$. But $[a, y]=(a, y)-D(a)y+D(y)a$ whence the $\mathfrak{a}$-component of $[a, y]$ is $(a, y)+D(y)a=[a, y]+D(a)y$. We know $D(a)\mathfrak{s}_D\subset[\mathfrak{s}_D, \mathfrak{s}_D]$; therefore the $\mathfrak{a}$-component of $[a, y]$ is in $[\mathfrak{s}, \mathfrak{s}]$. Hence $D_a([a, y])=0$. Next we consider $D_a(a+x)(b+y)=D(a)b+D(a)y$. From Lemma 3.3.1 we know $D(a)b\in[\mathfrak{a}, \mathfrak{a}]$ hence $D_a(D_a(a+x)(b+y))=D(D(a)b)=0$. Analogously we have

$$D_D(a+x)(b+y)=D(x)b+D(x)y \text{ and } D(x)y\in[\mathfrak{s}_D,\ \mathfrak{s}_D].$$

Hence (3.5*) for D_D.

LEMMA 4. — *Under the assumptions of Lemma* 3 *we have* $[jx, x]=0$ *only if* $x\in\mathfrak{a}$.

Proof. — Split $x=a+x_D$, then $0=[j(a+x_D),\ a+x_D]$ implies

$$0=[jx_D,\ x_D]-D(ja)x_D+D(a)jx_D.$$

In the normal j-algebra underlying $\mathfrak{s}_D$ we have ω as in [4], Lemma 3.5. But then it is easy to see $x_D=0$.

LEMMA 5. — *Let* $\mathfrak{s}=\mathfrak{a}+\mathfrak{s}_D$ *be the semidirect product of an abelian ideal and a normal j-algebra. Assume that* $\mathfrak{s}$ *is a Kähler algebra relative to* $[.,.]$, ρ *and* j. *Let* D *be a modification map for* $\mathfrak{s}$; *then* $\mathfrak{s}$ *is a Kähler algebra relative to* $(.,.)$, ρ *and* j *where* $(u, v)=[u, v]+D(u)v-D(v)u$.

Assume $\mathfrak{s}$ *is also a Kähler algebra relative to* $(.,.)$, $\tilde{\rho}$ *and* j. *Then* $\tilde{\rho}(\mathfrak{a}, \mathfrak{s}_D)=0$.

Proof. — We follow the proof of [7], Part III, Lemma 3. Let s be the principal idempotent of $(\mathfrak{s}_D, [.,.])$, then $\mathfrak{a}=\mathfrak{a}_0+\mathfrak{a}_{1/2}+\mathfrak{a}_{-1/2}$ and $\mathfrak{s}_D=\mathfrak{s}_{D_0}+\mathfrak{s}_{D_{1/2}}+\mathfrak{s}_{D_1}$ relative to $[js, .]$. We have

$$\begin{aligned}\tilde{\rho}(s,\ a_{-1/2})&=\tilde{\rho}(js,\ a_{-1/2})=\tilde{\rho}(js,\ [s,\ a_{-1/2}])\\&=\tilde{\rho}(js,\ (s,\ a_{-1/2})-D(s)a_{-1/2}+D(a_{-1/2})s)=\tilde{\rho}(js,\ (s,\ a_{-1/2}))\end{aligned}$$

because $s,\ a_{-1/2}\in[\mathfrak{s}, \mathfrak{s}]$. We apply (1.3) and get

$$\begin{aligned}&\tilde{\rho}(s,\ a_{-1/2})-\tilde{\rho}(s,\ (a_{-1/2},\ js))-\tilde{\rho}(a_{-1/2},\ (js,\ s))\\&\quad=\tilde{\rho}(s,\ [js,\ a_{-1/2}]+D(js)a_{-1/2}-D(a_{-1/2})js)-\tilde{\rho}(a_{-1/2},\ s)\\&\quad=-\frac{1}{2}\tilde{\rho}(s,\ a_{-1/2})+\tilde{\rho}(s,\ D(js)a_{-1/2})+\tilde{\rho}(s,\ a_{-1/2}).\end{aligned}$$

Hence

$$\frac{1}{2}\tilde{\rho}(s,\ a_{-1/2})=\tilde{\rho}(s,\ \mathrm{D}(js)\,a_{-1/2}).$$

Therefore

$$\tilde{\rho}(s,\ \exp t\,\mathrm{D}(js)\,a_{-1/2})=e^{(1/2)t}\,\tilde{\rho}(s,\ a_{-1/2})$$

whence $\tilde{\rho}(s,\ a_{-1/2})=0$, because $\exp t\,\mathrm{D}(js)$ is bounded and $e^{(1/2)t}$ unbounded. This implies

$$(*)\qquad \tilde{\rho}(js,\ \mathfrak{a}_{1/2})=0.$$

Let $u\in\mathfrak{a}_\zeta$ and $v\in\mathfrak{s}_\sigma$ where $\zeta<\sigma$. Hence

$$-\zeta+\sigma=\frac{1}{2},\qquad ju\in\mathfrak{a}_{-\zeta}\qquad\text{and}\qquad [ju,\ v]\in\mathfrak{a}_{-\zeta+\sigma}=\mathfrak{a}_{1/2}.$$

By $(*)$ we have

$$0=\tilde{\rho}(js,\ [ju,\ v])=\tilde{\rho}(js,\ (ju,\ v)-\mathrm{D}(ju)\,v+\mathrm{D}(v)\,ju).$$

If $\sigma=1/2,\ 1$, then $\mathrm{D}(v)=0$ because $\mathfrak{s}_\sigma\subset[\mathfrak{s},\ \mathfrak{s}]$ and $\mathrm{D}(ju)\,v\in\mathfrak{s}_\sigma$. If $\sigma=1$, then $\mathrm{D}(ju)\,v=0$ and if $\sigma=1/2$ then $\tilde{\rho}(js,\ \mathrm{D}(ju)\,v)=0$ by [4], Lemma 3.5. Hence $\tilde{\rho}(js,\ (ju,\ v))=0$ if $\sigma=1/2,\ 1$. Assume now $\sigma=0$. Then $\zeta=-1/2$ and $[ju,\ v]\in\mathfrak{a}_{1/2}$ whence

$$(ju,\ v)=[ju,\ v]+\mathrm{D}(ju)\,v-\mathrm{D}(v)\,ju=[ju,\ v]-\mathrm{D}(v)\,ju\in\mathfrak{a}_{1/2}$$

because $\mathrm{D}(ju)=0$. Thus we get from $(*)$ again $\tilde{\rho}(js,\ (ju,\ v))=0$. Therefore from [7], Part III, Lemma 9, we get

$$\frac{d}{dt}\tilde{\rho}(e^{t\,\mathrm{Ad}\,js}ju,\ e^{t\,\mathrm{Ad}\,js}v)=\tilde{\rho}(js,\ e^{t\,\mathrm{Ad}\,js}(ju,\ v))=0$$

where $\mathrm{Ad}\,js\,x=(js,\ x)$. Hence

$$\tilde{\rho}(e^{t\,\mathrm{Ad}\,js}ju,\ e^{t\,\mathrm{Ad}\,js}v)=\tilde{\rho}(ju,\ v).$$

But $\mathrm{Ad}\,js$ has on $j\,\mathfrak{a}_\zeta$ real part $-\zeta\,\mathrm{Id}$ and on $\mathfrak{s}_\sigma$ real part $\sigma\,\mathrm{Id}$. Therefore the left handside grows as $e^{(-\zeta+\sigma)t}$, but the right hand side is constant, whence

$$(*)\qquad \tilde{\rho}(ju,\ v)=0\qquad\text{for}\quad u\in\mathfrak{a}_\zeta,\quad v\in\mathfrak{s}_\sigma\quad\text{and}\quad\zeta<\sigma.$$

Assume now $\sigma\leqq\zeta$. Then $-\zeta+\sigma\leqq 0<1$ whence $-\zeta<1-\sigma$. Therefore $\tilde{\rho}(ju,\ v)=\tilde{\rho}(jju,\ jv)=0$ for all $u\in\mathfrak{a}_\zeta$ and $v\in\mathfrak{s}_\sigma$ by $(**)$. This proves the Lemma.

LEMMA 6. — *Let $\mathfrak{s}=\mathfrak{a}+\mathfrak{s}_{\mathrm{D}}$ be the semidirect product of the abelian ideal $\mathfrak{a}$ and the normal j-algebra $\mathfrak{s}_{\mathrm{D}}$. Let D be a modification map of $\mathfrak{s}$. Then there exists an abelian* *subalgebra* $\mathfrak{u}$ *of* $\mathfrak{s}$ *so that* $D(x)$, $x \in \mathfrak{u}$, *is the purely imaginary part of the adjoint representation of* x *in the modified algebra. Moreover,* $D(x)=0$ *in the complement of* $\mathfrak{u}$ *in* $\mathfrak{s}$ *and* $D(x)\mathfrak{u}=0$ *for all* $x \in \mathfrak{s}$.

Proof. — We put $\mathfrak{u}_D = \mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$ and $\mathfrak{u}_a = \mathfrak{a} \ominus \tilde{\mathfrak{a}}_0$ where $\tilde{\mathfrak{a}}_0 = \{a \in \mathfrak{a};\ D(a)=0\}$. It is clear that D vanishes on the orthogonal complement of $\mathfrak{u} = \mathfrak{u}_a + \mathfrak{u}_D$. It is clear that $\mathfrak{u}_D$ is abelian and $\mathfrak{u}_a$ is abelian by Lemma 3.3.1. Let $a \in \mathfrak{u}_a$ and $h \in \mathfrak{u}_D$ then

$$[a, h] + D(a)h - D(h)a = [a, h] - D(h)a \in \mathfrak{a}.$$

Now $\operatorname{ad} h \mid \mathfrak{a}$ is self adjoint; therefore the eigenspaces for nonzero eigenvalue are contained in $\tilde{\mathfrak{a}}_0$. This implies $[a, h]=0$ for $h \in \mathfrak{u}_D$ and $a \in \mathfrak{u}_a$. Finally, $D(h) \mid \mathfrak{a}$ acts on the complex vector space $\mathfrak{a}$ by $\mathbb{C}$-linear skew-adjoint endomorphisms. This shows that $\mathfrak{u}_a$ is orthogonal to all eigenvectors for non zero eigenvalue of $D(h)$. Moreover, we have shown $D(\mathfrak{u})\mathfrak{u}=0$. Hence $[u, x] + D(u)x - D(x)u = [u, x] + D(u)x$ because $D(x)u = D(u')u = 0$ for some $u' \in \mathfrak{u}$. We see that $D(u)$ commutes with $[u, .]$, $u \in \mathfrak{u}$, and $[u, .]$ has only real eigenvalues. Hence the assertion.

3.5. Let M be a flat h. k. m. and $\mathfrak{s}$ a solvable Kähler algebra for M. It is well-known that the universal cover space for M is the affine space $\mathbb{C}^n$. The action of $\mathfrak{s}$ on M can be lifted to $\mathbb{C}^n$. Therefore $\mathfrak{s}$ acts on $\mathbb{C}^n$ by affine transformations with skew-adjoint linear parts. Because $\mathfrak{s}$ is solvable, the linear parts of any two elements of $\mathfrak{s}$ commute.

We denote the elements of $\mathfrak{s}$ by $X(a)$ where $a \in \mathbb{C}^n$ and $X(a)z = a + A(a)z$, $z \in \mathbb{C}^n$.

The homogeneity implies that all $a \in \mathbb{C}^n$ occur. We also note $jX(a) = X(ia)$. Hence $\mathbb{C}^n$ and $\mathfrak{s}$ are isomorphic as complex vector spaces.

LEMMA 1. — *There exists a base* $z_1, \ldots, z_n$ *of* $\mathbb{C}^n$ *so that*

$$[X(z_k), jX(z_k)] = 0 \qquad \textit{for} \quad 1 \leqq k \leqq n.$$

Proof. — As mentioned above, $\{A(a);\ a \in \mathbb{C}^n\}$ is a commuting family of skew-adjoint endomorphisms of $\mathbb{C}^n$. Let $w_1, \ldots, w_n$ be *a* base of $\mathbb{C}^n$ consisting of common eigenvectors. Let $\lambda_r : \mathbb{C}^n \to \mathbb{R}$ be $\mathbb{R}$-linear so that $A(a)w_r = \lambda_r(a)iw_r$ for all $a \in \mathbb{C}^n$. Then

$$[X(w_r), jX(w_r)] = [w_r + A(w_r)z,\ iw_r + A(iw_r)z]$$
$$= A(w_r)iw_r - A(iw_r)w_r = -(\lambda_r(w_r) + i\lambda_r(iw_r))w_r.$$

If this is non-zero then put

$$z_r = -(\lambda_r(w_r) + i\lambda_r(iw_r))w_r,$$

otherwise set $z_r = w_r$. In the first case we have $A(z_r)=0$. Hence

$$[X(z_r), jX(z_r)] = [z_r,\ iz_r + A(iz_r)z] = -A(iz_r)z_r = -\lambda_r(iz_r)iz_r.$$

If $\lambda_r(iz_r) \neq 0$, then $A(iz_r)=0$ whence $[X(z_r), jX(z_r)]=0$, a contradiction. Therefore $\lambda_r(iz_r)=0$ and the Lemma is proven.

We give $\mathbb{C}^n$ the Lie algebra structure of $\mathfrak{s}$ via $a \mapsto X(a)$.

LEMMA 2. — *The map* $y \mapsto A(y)$ *is a modification map of the abelian Lie algebra on* $\mathbb{C}^n$. *The given structure on* $\mathfrak{s}$ *is the corresponding modification of* $\mathbb{C}^n$.

Proof. — From the remarks preceding Lemma 1 we get that only (3.5*) has to be verified. By Lemma 3.3.1 it suffices to prove (3.5). But

$$[a+A(a)x,\ b+A(b)x]=A(a)b-A(b)a$$

implies (3.5) because $\mathfrak{s}$ has a trivial isotropy subalgebra. The last statement of the assertion is just the equation above.

We apply Lemma 3.3.1 and get:

LEMMA 3. — (*a*) *If* ad x *acts nilpotent on* $\mathfrak{s}$, *then* x *corresponds to a translation.*

(*b*) $A(x)$ *is the semisimple part of* ad x *in* $\mathfrak{s}$.

Proof. — Let $\mathfrak{s}=\hat{\mathfrak{s}}_0+\hat{\mathfrak{s}}_1$ as in Lemma 3.3.1. Then $\operatorname{ad} x y=\operatorname{ad} x\hat{y}_0=A(x)\hat{y}_0$ by the last Lemma. Hence the assertion.

3.6. In this section we prove that a sequence of modifications can be represented by one modification.

LEMMA. — *Let* $\mathfrak{s}$ *be a solvable Kähler algebra and* $D_1, \ldots, D_r$ *a sequence of successive modifications resulting in the semidirect product of an abelian ideal and a normal j-algebra. Then* $D_1+\ldots+D_r$ *is a modification map.*

Proof. — By Lemma 3.4.1 each of the modifications D_m is reversible. It therefore suffices to prove that for two successive modifications D_1, D_2 starting from the semidirect product $\mathfrak{s}=\mathfrak{a}+\mathfrak{s}_D$ of an abelian ideal $\mathfrak{a}$ and *a* normal *j*-algebra $\mathfrak{s}_D$ also D_1+D_2 is a modification map. The Lie product in the semidirect product will be denoted by $[.,.]$. The product after modification by D_1 will be $(.,.)$.

First we note that $D_1(x)$ leaves invariant $\mathfrak{a}$. The modified algebra is still associated to a flat h. k. m. by 3.2. Hence we can apply 3.5. From Lemma 3.5.1 we get a basis of $\mathfrak{a}$ satisfying $(jx, x)=0$. Let $W_t=\exp tD_2(y)$, then $(jW_tx, W_tx)=0$. We decompose $W_tx=a+z_D$ and get in particular, $(jz_D, z_D)+D_1(ja)z_D-D_1(a)jz_D$. From [4], Lemma 3.5, we get $z_D=0$ hence $W_t\mathfrak{a}=\mathfrak{a}$. Therefore $D_2(y)$ leaves $\mathfrak{a}$ invariant. This implies that $\mathfrak{a}$ and $\mathfrak{s}_D$ are subalgebras of $\mathfrak{s}$ after carrying out both modifications. By Lemma 3.5.2 we know that the algebra structure on $\mathfrak{a}$ is given via a modification map $D_a\colon \mathfrak{a}\mapsto \operatorname{End}\mathfrak{a}$ from the abelian Lie algebra $\mathfrak{a}$. The "D-component" of ad a, $a\in\mathfrak{a}$, is given by $D_D(a)=D_1(a)+D_2(a)\mid \mathfrak{s}_D$. It is clear that $D_1(a)$ is a skew-adjoint derivation of the normal *j*-algebra $\mathfrak{s}_D$ which commutes with *j*. The same holds for $D_2(a)$ by [4], Lemma 3.5. We claim that $D(a)=D_a(a)+D_D(a)$ is a modification map of the original $\mathfrak{a}+\mathfrak{s}_D$. Here (3.1), (3.2) are clear. To prove (3.3) for D we have to show $[D_D(a), D_D(b)]=0$ on $\mathfrak{s}_D$ for all $a, b\in\mathfrak{a}$. It suffices to show $[D_1(a), D_2(b)]=0$ on $\mathfrak{s}_D$ for all $a, b\in\mathfrak{a}$. We know that $D_2(b)$ is a derivation of the intermediate Lie algebra. Comparing the $\mathfrak{s}_D$ component of (a, x) gives

$$D_2(b)D_1(a)x=D_1(a)D_2(b)x+D_1(D_2(b)a)x.$$

We know that the family $\{D_1(a); a \in \mathfrak{a}\}$ commutes and consists of skew-adjoint j-linear endomorphisms of $\mathfrak{s}_D$. We choose a basis of common eigenvectors and get easily that they are also eigenvectors for $D_2(b)$. Hence $[D_2(b), D_1(a)] = 0$ on $\mathfrak{s}_D$. To verify (3.4) we have to show $D([\mathfrak{a}, \mathfrak{s}_D]) = 0$. But $D_a(a) = D_1(a) + D_2(a)$ and $D_1([\mathfrak{a}, \mathfrak{s}_D]) = 0$. Using the intermediate modification we get $D_2([a, x_D] + D_1(a)x_D - D_1(x_D)a) = 0$ for all $a \in \mathfrak{a}$, $x_D \in \mathfrak{s}_D$. We know $D_1(a)x_D \in [\mathfrak{s}_D, \mathfrak{s}_D]$ whence $D_2(D_1(a)x_D) = 0$. If $x_D \in [\mathfrak{s}_D, \mathfrak{s}_D]$, then $D_1(x_D) = 0$ hence $D_2([a, x_D]) = 0$. So let $x_D \in \mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$. But $\operatorname{ad} x_D \mid \mathfrak{a}$ is self-adjoint on $\mathfrak{a}$ and $D_1(x_D)$ leaves the eigenspaces invariant. For eigenvalues $\neq 0$ $\operatorname{ad} x_D \mid \mathfrak{a} - D_1(x_D) \mid \mathfrak{a}$ is invertible, whence $D_2(a) = 0$ for all eigenvectors of $\operatorname{ad} x_D$ for non-zero eigenvalues. In particular $D_2([a, x_D]) = 0$ for all $a \in \mathfrak{a}$, $x_D \in \mathfrak{s}_D$. To prove (3.5*) it suffices to prove (3.5). Splitting $x = x_a + x_D$ and $y = y_a + y_D$ we get

$$D(x)y - D(y)x = D(x_a)y_a - D(y_a)x_a + D(x_a)y_D - D(y_a)x_D.$$

Because D is a modification of $\mathfrak{a}$ and has $\mathfrak{s}_D$ in its kernel we get (3.5). Finally we note that $D(a) = D_1(a) + D_2(a)$. By definition, $D_1(a)$ is a derivation of the original $\mathfrak{s}$. So we only have to prove that $D_2(a)$ is also a derivation. This is clear on $\mathfrak{a}$ and on $\mathfrak{s}_D$. So we only have to show

$$D_2(a)[u, x_D] = [D_2(a)u, x_D] + [u, D_2(a)x_D].$$

For $L = D_2(a)$ we know

$$L([u, x_D] + D_1(u)x_D - D_1(x_D)u)$$
$$= [Lu, x_D] + D_1(Lu)x_D - D_1(x_D)Lu + [u, Lx_D] + D_1(u)Lx_D - D_1(Lx_D)u.$$

We are only interested in the $\mathfrak{a}$-component and we know $Lx_D \in [\mathfrak{s}_D, \mathfrak{s}_D]$. Hence

$$L([u, x_D] - D_1(x_D)u) = [Lu, x_D] - D_1(x_D)Lu + [u, Lx_D].$$

For $x_D \in [\mathfrak{s}_D, \mathfrak{s}_D]$ we have $D_1(x_D) = 0$ hence L satisfies the required identity. If $x_D \in \mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$, then $Lx = 0$ and we have $[\operatorname{ad} x_D \mid \mathfrak{a} + D_1(x_D), L] = 0$. But $\operatorname{ad} x_D \mid \mathfrak{a}$ is self-adjoint and $D_1(x_D)$ is skew-adjoint on $\mathfrak{a}$, therefore $[\operatorname{ad} x_D \mathfrak{a}, L] = 0$ on $\mathfrak{a}$. This finishes the proof that $D_2(a)$ is a derivation of the original algebra $\mathfrak{s}$.

Now we consider the action of $\mathfrak{s}_D$ after the second modification. We know that $\mathfrak{s}_D$ is a subalgebra. Hence, by [4], Theorem 3.3.2, $\mathfrak{s}_D$ is a modification of the underlying normal j-algebra. It is easy to see that the modification map is $D_D(x) = D_1(x) + D_2(x) \mid \mathfrak{s}_D$. We know that D_D is non zero only on $\mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$. We put $D_a(x) = D_1(x) + D_2(x) \mid \mathfrak{a}$ and $D(x) = D_a(x) + D_D(x)$. It is clear that $D(x)$ is skew-adjoint, commutes with j and we have $D([\mathfrak{s}_D, \mathfrak{s}_D]) = 0$. To see that $D(x)$ is a derivation of the orginal $\mathfrak{s}$ it suffices to prove that it acts as a derivation on $[\mathfrak{a}, \mathfrak{s}_D]$. This is clear for $D_1(x)$ and for $L = D_2(x)$ we know

$$L([a, y] + D_1(a)y - D_1(y)a)$$
$$= [La, y] + D_1(La)y - D_1(y)La + [a, Ly] + D_1(a)Ly - D_1(Ly)a.$$

Arguing as above shows the assertion. It remains to verify (3.3) and (3.5). As above one shows $[\operatorname{ad} x_D \mid \mathfrak{a} + D_1(x_D), D_2(y_D)] = 0$. Hence $D_1(x_D)$ commutes with $D_2(y_D)$ on $\mathfrak{a}$. On $\mathfrak{s}_D$ we use the intermediate Lie algebra and the fact that $D_2(y)$ is a derivation of the normal j-algebra $\mathfrak{s}_D$ by [4], Lemma 3.5. Hence

$$D_2(y)(D_1(x)z - D_1(z)x) = D_1(D_2(y)x)z - D_1(z)D_2(y)x + D_1(x)D_2(y)z - D_1(D_2(y)z)x.$$

But $D_2(y)\mathfrak{s}_D \subset [\mathfrak{s}_D, \mathfrak{s}_D]$ and

$$[D_2(y), D_1(x)]z = [D_2(y), D_1(z)]x \quad \text{for all } x, y, z \in \mathfrak{s}_D$$

follows. Hence

$$0 = \prec [D_2(y), D_1(z)]x, x \succ = \prec [D_2(y), D_1(x)]z, x \succ \qquad \text{and} \qquad [D_2(y), D_1(x)]x = 0$$

follows for all $x, y \in \mathfrak{s}_D$. Linearizing in x and comparing with the original identity shows $[D_2(y), D_1(x)]z = 0$ for all $y, x, z \in \mathfrak{s}_D$. Hence $D_1(x)$ and $D_2(y)$ commute on $\mathfrak{s}_D$. This implies (3.3). Next we consider

$$D(x_a + x_D)(y_a + y_D) - D(y_a + y_D)(x_a + x_D) = D(x_D)y_a - D(y_D)x_a + D(x_D)y_D - D(y_D)x_D.$$

As $\mathfrak{a}$ is in the kernel of D and $D \mid \mathfrak{s}_D$ is a modification map, (3.5) follows.

Finally, we consider the map $Q(x) = D_1(x) + D_2(x)$. It is clear that $[x, y] + Q(x)y - Q(y)x$ is the Lie product after the second modification. Clearly $Q(x) = Q(x_a) + Q(x_D)$. Both maps are modifications of the original Lie algebra $\mathfrak{s}$. So we only have to verify (3.3) and (3.5) for Q. We have

$$Q(x)y - Q(y)x = Q(x)y_a - Q(y)x_a + Q(x)y_D - Q(y)x_D.$$

The last two summands lie in $[\mathfrak{s}_D, \mathfrak{s}_D]$, hence are annihilated by Q. Because $Q \mid \mathfrak{a}$ is a modification map as shown above it remains to prove that Q annihilates $Q(x_D)y_a$. We have shown above rhat $Q \mid \mathfrak{s}_D$ is a modification map. Therefore we may assume $x_D \in \mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$. Using the fact that D_1 and D_2 are modification maps we only have to show

$$D_1(D_2(x_D)y_a) = 0 \qquad \text{and} \qquad D_2(D_1(x_D)y_a) = 0.$$

By definition

$$D_2([x_D, y_a] + D_1(x_D)y_a - D_1(y_a)x_D) = 0.$$

By the choice of x_D we have $D_1(y_a)x_D = 0$ and because D_2 annihilates the eigenspaces of $\operatorname{ad} x_D \mid \mathfrak{a}$ for non-zero eigenvalue we have $D_2(D_1(x_D)y_a) = 0$. Next we use that $L = D_2(h)$ is a derivation of the intermediate Lie algebra. We note here that the product in $\mathfrak{a}$ is just $D_1(a)b - D_1(b)a$. Therefore

$$L(D_1(a)b - D_1(b)a) = D_1(La)b - D_1(b)La + D_1(a)Lb - D_1(Lb)a.$$

This implies

$$\{[L, D_1(a)] - D_1(La)\} b = \{[L, D_1(b)] - D_1(Lb)\} a.$$

We abbreviate this by $U(a)b = U(b)a$ where $U(b)$ is skew-adjoint. Then

$$0 = \langle U(a)b, b \rangle = \langle U(b)a, b \rangle = -\langle a, U(b)b \rangle \quad \text{for all } a \in \mathfrak{a}.$$

Therefore $U(b)b = 0$ for all $b \in \mathfrak{a}$. Replacing b by $a+b$ gives

$$0 = U(a)b + U(b)a = 2U(a)b.$$

Hence $U(a) = 0$. This means $[D_2(h), D_1(a)] = D_1(D_2(h)a)$ on $\mathfrak{a}$. We know that the family $\{D_1(a);\ a \in a\}$ is abelian. On the common eigenvectors we have

$$i\lambda(a) D_2(h) b - D_1(a) D_2(b) b = \lambda(D_2(h)a) ib.$$

Decomposing $D_2(h)b$ too shows $D_2(h)b = \mu b$. Hence $[D_2(h), D_1(a)] = 0$ on $\mathfrak{a}$ and $D_1(D_2(h)a) = 0$ on $\mathfrak{a}$. Applying $D_2(h)$ to the product of $x \in \mathfrak{s}_D$ and $a \in \mathfrak{a}$ in the intermediate Lie algebra gives for the $\mathfrak{s}_D$ component

$$D_2(h) D_1(a) x = D_1(a) D_2(h) x + D_1(D_2(h)a).$$

A argument as above shows $[D_2(h), D_1(a)] = 0$ on $\mathfrak{s}_D$ and $D_1(D_2(h)a) = 0$ on $\mathfrak{s}_D$. This finishes the proof of (3.5*) and gives half of (3.3). The second half of (3.3) is $[D_1(x), D_2(a)] = 0$. On $\mathfrak{a}$ this has been shown above. On $\mathfrak{s}_D$ we use gain the $\mathfrak{s}_D$-components from the product of $x, y \in \mathfrak{s}_D$. We note that $D_2(a)$ is a derivation of the normal j-algebra $\mathfrak{s}_D$, hence, in the modification via D_1, we get

$$D_2(a)(D_1(x)y - D_1(y)x)$$
$$= D_1(D_2(a)x)y - D_1(y)D_2(a)x + D_1(x)D_2(a)y - D_1(D_2(a)y)x.$$

Because $D_2(a)\mathfrak{s}_D \subset [\mathfrak{s}_D, \mathfrak{s}_D]$ this implies

$$[D_2(a), D_1(x)]y = [D_2(a), D_1(y)]x.$$

This gives, as above, $[D_2(a), D_1(x)] = 0$ on $\mathfrak{s}_D$. This finishes the proof of the Lemma.

3.7. In this section we state the algebraic main result of this paper.

THEOREM. — *Each solvable Kähler algebra is a modification of the semidirect product of an abelian ideal and a normal j-algebra.*

The proof of this Theorem will be carried out by induction on the dimension of the Kähler algebra in sections 4, 5 and 6. In case the dimension is 2, the Theorem is trivially verified.

We also note that, by Lemma 3.4.1 and Lemma 3.6, for the proof of the Theorem it suffices to find a (finite) sequence of modifications starting at the given Kähler algebra and leading to the semidirect product of an abelian ideal and a normal j-algebra.

We point out that we will frequently use the results on modifications which are listed in this section 3.

4.1. In the following sections we prove (by induction) that to each solvable Kähler algebra there exists *a* (finite) sequence of modifications leading to the semidirect product of an abelian ideal and a normal j-algebra.

4.2. We construct $\mathfrak{q}$ as in 2.1. There are three cases that we have to consider. The first is $\mathfrak{q} \cap j\mathfrak{q} = \mathfrak{q}$. The other two cases are listed in Lemma 2.4.

Case 1. — $\mathfrak{q} \cap j\mathfrak{q} = \mathfrak{q}$. Here $\mathfrak{q}$ is a Kähler ideal in $\mathfrak{s}$. Therefore the orthogonal complement $\mathfrak{s}'$ of $\mathfrak{q}$ in $\mathfrak{s}$ is a Kähler algebra, $\mathfrak{s} = \mathfrak{q} + \mathfrak{s}'$. As $\mathfrak{q} \neq 0$ we may apply the induction hypothesis to $\mathfrak{s}'$. Hence $\mathfrak{s}' = \mathfrak{s}'_0 + \mathfrak{s}'_1 + \mathfrak{s}'_D$ and Lemma 3.3.2 applies. We consider the action of $\mathfrak{s}'$ on $\mathfrak{q}$. Put $A(x) = \operatorname{ad} x \mid \mathfrak{q}$, $x \in \mathfrak{s}'$. We write $A(x) = A_1(x) + A_2(x)$ where $A_1(x)$ commutes with j and $A_2(x)$ anticommutes with j. Then it is straightforward to see that (1.2) implies $A_2(jx) = jA_2(x)$ for all $x \in \mathfrak{s}'$. Next we choose $y, z \in \mathfrak{q}$ and $x \in \mathfrak{s}'$, then (1.3) means that $jA(x)$ is self-adjoint on $\mathfrak{q}$. We note that by the definition of $\mathfrak{q}$ we have $A([x, y]) = 0$ for all $x, y \in \mathfrak{s}'$. Therefore the family $\{A(x);\ x \in \mathfrak{s}'\}$ is abelian and as in [7], Part II, Lemma 3, one shows that $A(x) = A_1(x)$ is skew-adjoint and commutes with j.

Let D be the modification map of $\mathfrak{s}'$ as in Lemma 3.3.2. We extend D by putting $D(x) \mid \mathfrak{q} = A(x)$. Then $D(x)$ is a commuting family of skew-adjoint endomorphisms of $\mathfrak{s}$. Each $D(x)$ commutes with j. This shows (3.1), (3.2) and (3.3). We compute

$$[q_1 + x'_1,\ q_2 + x'_2] = [q_1, x'_2] + [x'_1, q_2] + [x'_1, x'_2];$$

here the first two summands are in $\mathfrak{q}$, the last is in $\mathfrak{s}'$. Therefore $D([u, v]) = 0$ for all $u, v \in \mathfrak{s}$. By Lemma 3.3.2 we have $D(x)y \in [\mathfrak{s}', \mathfrak{s}']$ for all $x, y \in \mathfrak{s}'$. This implies (3.5*). It remains to show that $D(a)$ is a derivation of $\mathfrak{s}$. We only have to verify

$$A(a)[q, x] = [A(a)q, x] + [q, D(a)x] \quad \text{for} \quad x \in \mathfrak{s}', \quad q \in \mathfrak{q}.$$

But $A(a) \mid \mathfrak{q} = \operatorname{ad} a \mid \mathfrak{q}$ whence

$$A(a)[q, x] = [a, [q, x]] = [[a, q], x] + [q, [a, x]] = [A(a)q, x] + [q, [a, x]].$$

If suffices to show $0 = [[a, x] - D(a)x, \mathfrak{q}]$. But $[a, x] - D(a)x \in [\mathfrak{s}', \mathfrak{s}']$ by Lemma 3.3.2 and the assertion follows.

This implies that D is a modification map of $\mathfrak{s}$. The modified algebra is the sum of the abelian ideal $\mathfrak{q} + \mathfrak{s}'_0 + \mathfrak{s}'_1$ and the normal j-algebra $\mathfrak{s}'_D$. This finishes the first case.

5.1. We consider the case where $\mathfrak{q} + j\mathfrak{q}$ is not abelian. Then, by Lemma 2.4, $\dim \mathfrak{q} = 1$ and we can choose $r \in \mathfrak{q}$ so that $[jr, r] = r$.

Put $\mathfrak{p} := \{x \in \mathfrak{s};\ [x, r] = 0,\ [jx, r] = 0\}$. Then as in [7], Part III, Lemma 8, one proves that $\mathfrak{p}$ is invariant under j and $\operatorname{ad} jr$ and that the operator $\operatorname{ad} jr \mid \mathfrak{p}$ commutes with j. Moreover, [7], Part III, Lemma 9, shows:

$$(1) \qquad \frac{d}{dt}\rho(e^{t \operatorname{ad} jr}u,\ e^{t \operatorname{ad} jr}v) = \rho(jr,\ e^{t \operatorname{ad} jr}[u, v]) \quad \text{for all } u, v \in \mathfrak{s},$$

where $\rho(x, y) := \prec x, jy \succ$. The following proposition together with its proof is almost identical with [7], Part III, Proposition 2.

PROPOSITION. — *(a) The operator* $\operatorname{ad} jr \mid \mathfrak{p}$ *is semisimple and has eigenvalues* λ *with* $\operatorname{Re}\lambda \in \{0, 1/2\}$.

(b) $\mathfrak{s} = \mathbb{R} r + \mathbb{R} jr + \mathfrak{u} + \mathfrak{s}'$ *where the eigenvalues of* $\operatorname{ad} jr \mid \mathfrak{u}$ *(resp.* $\operatorname{ad} jr \mid \mathfrak{s}'$*) have real part* $1/2$ *(resp.* 0*). Moreover,* $\mathfrak{v} := \mathbb{R} r + \mathbb{R} jr + \mathfrak{u}$ *is a j-invariant subalgebra of* $\mathfrak{s}$, $\mathfrak{s}'$ *is a j-invariant subalgebra of* $\mathfrak{s}$ *and* r, jr, $\mathfrak{u}$ *and* $\mathfrak{s}'$ *are pairwise orthogonal.*

Proof. — We modify the proof of [7], Chap. III, Proposition 2. Let $x \in \mathfrak{s}$, then we determine $a, b \in \mathbb{R}$ so that $x - ajr - br \in \mathfrak{p}$ by

$$0 = [x, r] - a[jr, r] = \alpha r - ar \qquad \text{and} \qquad 0 = [jx, r] - b[jr, r] = \beta r - br.$$

Hence

$$\mathfrak{s} = \mathbb{R} r + \mathbb{R} jr + \mathfrak{p}. \tag{2}$$

As in [7] we get from (1)

$$\rho(r, e^{t \operatorname{ad} jr} v) = a(v) e^{-t} \quad \text{for all } v \in \mathfrak{p}. \tag{3}$$

As j commutes with $\operatorname{ad} jr$ on $\mathfrak{p}$ we also get

$$\rho(jr, e^{t \operatorname{ad} jr} v) = -a(jv) e^{-t} \quad \text{for all } v \in \mathfrak{p}. \tag{4}$$

Let now $u, v \in \mathfrak{s}$ be arbitrary, then $[u, v] = \alpha r + \beta jr + p$ and the right-hand side of (1) equals

$$\rho(jr, e^{t \operatorname{ad} jr}[u, r]) = \alpha\rho(jr, e^t r) + \beta\rho(jr, jr) + \rho(jr, e^{t \operatorname{ad} jr} p) = be^t - a(jp) e^{-t}$$

where we have used $\rho(jr, jr) = -\prec jr, r \succ = 0$. An integration gives for all $u, v \in \mathfrak{s}$

$$\rho(e^{t \operatorname{ad} jr} u, e^{t \operatorname{ad} jr} v) = a(jp) e^{-t} + \beta e^t + \gamma. \tag{5}$$

Clearly, the coefficients depend on u and v. If $u \in \mathfrak{p}$, then also $ju \in \mathfrak{p}$ and (5) yields—because $\operatorname{ad} jr$ and j commute on $\mathfrak{p}$—for $u \in \mathfrak{p}$, $v \in \mathfrak{s}$.

$$\prec e^{t \operatorname{ad} jr} u, e^{t \operatorname{ad} jr} v \succ = a(jp) e^{-t} + \beta e^t + \gamma. \tag{6}$$

We consider the endomorphism $A := \operatorname{ad} jr \mid \mathfrak{p}$ more closely. Let $\lambda + \mu i$ be an eigenvalue of A. Then there exists a subspace $\mathfrak{u}$ of $\mathfrak{p}$ such that $A \mid \mathfrak{u} = \lambda \operatorname{Id} + \mu I$ where $I^2 = -\operatorname{Id}$. For $u \in \mathfrak{u}$ we get

$$e^{t \operatorname{ad} jr} u = \exp A\, u = e^{t\lambda}(\cos \mu\, tu + \sin \mu\, t\, I\, u).$$

Hence

$$\prec e^{t \operatorname{ad} jr} u, e^{t \operatorname{ad} jr} u \succ = e^{2t\lambda} \left| (\cos \mu t) u + (\sin \mu t) I u \right|^2 = a(jp) e^{-t} + \beta e^t + \gamma. \tag{7}$$

As a consequence we have

(8) $$\lambda \in \{0, \pm 1/2\},$$

(9) $$|(\cos \mu t) u + (\sin \mu t) \mathrm{I} u|^2 = |u|^2 \quad \text{for all } t \in \mathbb{R}.$$

Expanding (9) gives

(9′) $$\langle \mathrm{I} u, \mathrm{I} u \rangle = \langle u, u \rangle, \ \langle \mathrm{I} u, u \rangle = 0.$$

We next prove that A is semisimple (whence $\mathrm{ad}\,jr$ is semisimple on $\mathfrak{s}$). Let $\mathrm{A} = \mathrm{A}_h + \mathrm{A}_n$ where A_h (resp. A_m) is the semisimple (resp. nilpotent) part of A. We may choose a vector u so that

$$\mathrm{A}\, u = \lambda u + \mu \mathrm{I} u + \mathrm{A}_n u \qquad \text{where} \quad \mathrm{I}^2 = -\mathrm{Id}, \quad \mathrm{A}_n u \neq 0$$

and

$$\exp t \mathrm{A}\, u = e^{t\lambda}((\cos t \mu)\, \mathrm{Id} + (\sin t \mu)\, \mathrm{I}) \exp t \mathrm{A}_n u.$$

But now (7) shows that

$$|((\cos t \mu)\mathrm{Id} + (\sin t \mu)\, \mathrm{I}) \exp t \mathrm{A}_n u|^2 = |u|^2 \quad \text{for all } t \in \mathbb{R}.$$

This implies that $|\exp t \mathrm{A}_n u|^2$ is bounded. Therefore $\mathrm{A}_n u = 0$, a contradiction. To prove (*a*) it suffices to show that in (8) $\lambda = -1/2$ does not occur. Here we proceed as in (*c*) of the proof of [7], Chap. III, Prop. 2. We derive from (3) and (8) that $a(v) = 0$ for all $v \in \mathfrak{p}$. Therefore the right-hand side of (7) is $\beta e^t + \gamma$. This implies that in (8) only $\lambda \in \{0, 1/2\}$ is possible. This proves (*a*) and the first part of (*b*). As $\mathrm{ad}\,jr \mid \mathfrak{p}$ commutes with j we see that $\mathfrak{u}$ and $\mathfrak{s}'$ are j-invariant, hence $\mathfrak{v}$ and $\mathfrak{s}'$ are j-invariant. Writing $\mathrm{ad}\,jr\, v = \lambda v + \mu \mathrm{I} v$ one derives from (3) and (4) that $\mathfrak{u}$ and $\mathfrak{s}'$ are orthogonal to r and jr. Moreover, $u \in \mathfrak{u}$ and $v \in \mathfrak{s}'$ in (5) imply $\rho(u, v) = 0$ whence $\mathfrak{u}$ and $\mathfrak{s}'$ are orthogonal. As in [7] one notes now $\mathfrak{s}' \subset \mathfrak{p}$ whence $[r, \mathfrak{s}'] = 0$. As a consequence, (1.3) shows $[\mathfrak{s}', \mathfrak{s}'] \perp jr$.

Finally, considering the real parts of the eigenvalues of $\mathrm{ad}\,jr$, one shows that $\mathfrak{v}$ and $\mathfrak{s}'$ are subalgebras of $\mathfrak{s}$. This finishes the proof.

5.2. We are now prepared to finish the case $\dim \mathfrak{q} = 1$. Using 5.1 we write $\mathfrak{s} = \mathfrak{v} + \mathfrak{s}'$ where $\mathfrak{v} = \mathbb{R} r + \mathbb{R} jr + \mathfrak{u}$. We consider the map

$$\mathrm{D}(\alpha r + \beta jr + u + x') := [jr, u] - 1/2\, u + [jr, x'].$$

It is easy to see that D is a derivation of $\mathfrak{s}$. From the definition of $\mathfrak{p} = \mathfrak{u} + \mathfrak{s}'$ we get that $\mathrm{ad}\,jr$ commutes with j on $\mathfrak{u}$ and on $\mathfrak{s}'$. Therefore D commutes with j. We choose $x', y' \in \mathfrak{s}'$ and get from (1.3) the equation

$$0 = \langle [x', y'], jjr \rangle \equiv \langle [y', jr], jx' \rangle \equiv \langle [jr, x'], jy' \rangle.$$

This implies that $j\mathrm{D}$ is self-adjoint. Therefore D is skew-adjoint on $\mathfrak{s}'$. From (9)′ of the proof of Proposition 5.1 we see that D is skew-adjoint on $\mathfrak{u}$. Altogether D is a skew-adjoint derivation of $\mathfrak{s}$ commuting with j. It is now straightforward to show that the map $\alpha r+\beta jr+u+x'\mapsto\beta \mathrm{D}$ is a modification map of $\mathfrak{s}$. We therefore may and will assume from now on that $\operatorname{ad} jr$ is self-adjoint and that $\mathfrak{v}$ is an ideal of $\mathfrak{s}$. We apply the induction hypothesis to $\mathfrak{s}'$ and split $\mathfrak{s}'=\mathfrak{s}'_0+\mathfrak{s}'_1+\mathfrak{s}'_{\mathrm{D}}$ according to Lemma 3.3.2. Then $\mathfrak{s}'_{\mathrm{D}}$ is the Lie algebra of a simply transitive solvable group on a homogeneous bounded domain. Let D' denote the modification of $\mathfrak{s}'$. We consider the map $\mathrm{D}_v(a)=\operatorname{ad} a \mid \mathfrak{v}$, $a\in\mathfrak{a}'=\mathfrak{s}'_0+\mathfrak{s}'_1$. From Lemma 3.5.1 we get a basis $z_1, \ldots, z_n$ of the complex vector space $\mathfrak{a}'$ so that $[jz_r, z_r]=0$ for $1\leqq r\leqq n$. Hence by [7], Part III, Lemma 3, we get that $\operatorname{ad} jz_r \mid \mathfrak{v}$ and $\operatorname{ad} z_r \mid \mathfrak{v}$ are skew-adjoint and commute with j. Therefore $\mathrm{D}_v(a)$ is skew-adjoint for all $a\in\mathfrak{a}'$. We have $[\mathrm{D}_v(a), \mathrm{D}_v(b)]=\operatorname{ad}[a, b]\mid\mathfrak{v}$; but this implies $[\mathrm{D}_v(a), \mathrm{D}_v(b)]=0$. We define $\mathrm{D}(v+a'+x')=\mathrm{D}_v(a')+\mathrm{D}'(a')$. We claim that D is a modification map of $\mathfrak{s}$. By Lemma 3.4.3 we know that $\mathrm{D}\mid\mathfrak{s}'$ is a modification map of $\mathfrak{s}'$. So (3.1), (3.2) and (3.3) are satisfied. We know that $\mathfrak{v}$ is an ideal of $\mathfrak{s}$, orthogonal to $\mathfrak{a}'$. Hence it suffices to prove $\mathrm{D}([x, y])=0$ for $x, y\in\mathfrak{s}'$. But

$$[a+x_{\mathrm{D}}, b+y_{\mathrm{D}}]=[a, b]+[a, y_{\mathrm{D}}]+[x_{\mathrm{D}}, b]+[x_{\mathrm{D}}, y_{\mathrm{D}}]$$

and it suffices to prove $\mathrm{D}([a, x])=0$ for all $a\in\mathfrak{a}'$, $x\in\mathfrak{s}'_{\mathrm{D}}$. But $[a, x]=(a, x)+\mathrm{D}'(a)x-\mathrm{D}'(x)a$. Therefore the $\mathfrak{a}$-component f of $[a, x]$ is $f=[a, x]-\mathrm{D}'(a)x\in[\mathfrak{s}', \mathfrak{s}']$. Hence $\operatorname{ad} f\mid\mathfrak{v}$ is nilpotent; but we have seen above that it is skew-adjoint. Therefore $\mathrm{D}([a, x])=0$. Next we consider

$$\mathrm{D}(v+x')(\tilde{v}+\tilde{x}')=\mathrm{D}(x')\tilde{v}+\mathrm{D}(x')\tilde{x}'.$$

To verify (3.5*) we may assume $v=\tilde{v}=0$. Now

$$\mathrm{D}(a+x_{\mathrm{D}})(\mathrm{b}+\mathrm{y}_{\mathrm{D}})=\mathrm{D}'(a)b+\mathrm{D}(a)y_{\mathrm{D}}.$$

Hence it suffices to prove $\operatorname{ad}\mathrm{D}'(a)b\,\mathfrak{v}=0$. But from Lemma 3.3.1 we know $[\mathfrak{a}', \mathfrak{a}']=\{\mathrm{D}'(a)b;\ a, b\in\mathfrak{a}'\}$. Hence the assertion. It remains to show that $\mathrm{D}(x)\in\operatorname{Der}\mathfrak{s}$ for all $x\in\mathfrak{s}$. It is easy to see that we only have to show

$$\mathrm{D}(a)[v, x']=[\mathrm{D}(a)v, x']+[v, \mathrm{D}'(a)x'].$$

But this is equivalent to showing $[v, [a, x']-\mathrm{D}'(a)x']=0$. But

$$[a, x']-\mathrm{D}'(a)x'=(a, x')-\mathrm{D}'(x)a\in\mathfrak{a}\cap[\mathfrak{s}', \mathfrak{s}']$$

whence the assertion.

Carrying out the modification D gives $\mathfrak{s}=\mathfrak{a}'+\mathfrak{v}+\mathfrak{s}'_{\mathrm{D}}$ where $\mathfrak{a}'$ commutes with $\mathfrak{v}$ and is an abelian Kähler ideal of $\mathfrak{a}'+\mathfrak{s}'_{\mathrm{D}}$. Hence $\mathfrak{a}'$ is an abelian Kähler ideal of $\mathfrak{s}$. Hence $\mathfrak{v}+\mathfrak{s}'_{\mathrm{D}}$ is a Kähler subalgebra of $\mathfrak{s}$. From the induction hypothesis we get that $\mathfrak{v}$ and $\mathfrak{s}'_{\mathrm{D}}$ are "simple j-algebras" in the sense of [11], § 1. Moreover, the restriction of the adjoint representation of $\mathfrak{s}'_{\mathrm{D}}$ to $\mathfrak{v}$ is a symplectic representation in the sense of [11], § 2, p. 313. We apply the remark at the end of [11], § 2, with $\Theta=0$ and $\Psi=0$ and see that $\mathfrak{v}+\mathfrak{s}'_{\mathrm{D}}$ is a simple j-algebra, hence it is associated with a bounded homogeneous domain by [11], § 3. But then, by [4], Theorem 3.3.2, there exists a modification map D of $\mathfrak{v}+\mathfrak{s}'_{\mathrm{D}}$. Now we consider $\mathfrak{a}'+\mathfrak{s}'_{\mathrm{D}}$. Again by induction we get a modification map $\tilde{\mathrm{D}}$. It is easy to verify that these two modifications are consistent, i. e. define a modification of $\mathfrak{s}$. After carrying out this modification $\mathfrak{s}$ is the semidirect product of an abelian ideal and a normal j-algebra. This finishes the case where $\mathfrak{q}+j\mathfrak{q}$ is not abelian.

6.0. In this section we consider the last remaining case: $\mathfrak{q}+j\mathfrak{q}$ abelian. Here we modify [7], Part III, § 3, and prove that—after several modifications of $\mathfrak{s}$—the ideal $\mathfrak{q}$ is contained in a j-invariant ideal of $\mathfrak{s}$.

6.1. Pick $0 \neq r \in \mathfrak{q}$.

(1) $$[jr, \mathfrak{s}] \subset \mathfrak{p} = \{x \in \mathfrak{s};\ [x, \mathfrak{q}]=0,\ [jx, \mathfrak{q}]=0\}.$$

The proof of (1) is as in [7]. We next prove

(2) $\operatorname{ad} jr = \mathrm{D}+\mathrm{N}$ where D is skew-adjoint, N nilpotent and $[\mathrm{D}, \mathrm{N}]=0$ holds.

We may copy (*a*) and (*b*) from the proof of *loc. cit.* Lemma 11. Hence

(∗) $$\prec e^{t \operatorname{ad} jr} u,\ e^{t \operatorname{ad} jr} v \succ = at^2+bt+c$$

for all $u \in \mathfrak{p}$, $v \in \mathfrak{s}$. We split $\operatorname{ad} jr$ into its semisimple part D and its nilpotent part N. From (∗) we get that the eigenvalues of D are purely imaginary. Then $\mathfrak{s}$ splits into a direct sum of subspaces $\mathfrak{s}_k$, $\mathfrak{s}_k \subset \mathfrak{p}$ if $k \neq 0$, invariant under D and N, so that on each of these spaces $\mathfrak{s}_k$ we have $\mathrm{D}v = \alpha \mathrm{I} v + \mathrm{N} v$ where $\mathrm{I}^2 = -\mathrm{Id}$. Then

$$e^{t\mathrm{D}} v = ((\cos \alpha t) + (\sin \alpha t)\, \mathrm{I})\, e^{t\mathrm{N}} v.$$

From (∗) we now derive that the spaces $\mathfrak{s}_k$ are pairwise orthogonal and that I is orthogonal on $\mathfrak{s}_k$. This proves that D is skew-adjoint on $\mathfrak{s}$. As j commutes with $\operatorname{ad} jr$ on p so j and D commute.

Now we construct a modification map $\mathrm{F}: \mathfrak{s} \to \operatorname{Der} \mathfrak{s}$ so that $\operatorname{ad} jr$ acts nilpotent on $\mathfrak{s}$ in the new algebra. We choose $r_1, r_2 \in \mathfrak{q}$ so that trace $\operatorname{ad} jr_n \operatorname{ad} jr_m = -\delta_{nm}$ and put $\alpha_n(x) := \operatorname{trace} \operatorname{ad} jr_n \operatorname{ad} x$. Let D_1 and D_2 denote the semisimple parts of $\operatorname{ad} jr_1$ and $\operatorname{ad} jr_2$ respectively. Then $\mathrm{F}(x) := \alpha_1(x)\mathrm{D}_1 + \alpha_2(x)\mathrm{D}_2$ maps $\mathfrak{s}$ into $\operatorname{Der} \mathfrak{s}$. It is easy to check that F is a modification map. Hence after carrying this modification:

(3) We may assume that $\operatorname{ad} jr$ is nilpotent for all $r \in \mathfrak{q}$.

But now the proof of *loc. cit* Lemma 11 goes through without any change and we get

(4) $$(\operatorname{ad} jr)^2 = 0 \quad \text{for all } r \in \mathfrak{q}.$$

6.2. We define

$$\mathfrak{s}^{(-1)} := \mathfrak{s}, \quad \mathfrak{s}^{(0)} := \{x \in \mathfrak{s};\ [j\mathfrak{q}, x] \in \mathfrak{q}+j\mathfrak{q}\}, \quad \mathfrak{s}^{(1)} := [jr_1, \mathfrak{s}] + [jr_2, \mathfrak{s}] + \mathfrak{q}+j\mathfrak{q}, \quad \mathfrak{s}^{(2)} = \mathfrak{q}+j\mathfrak{q}.$$

The subspaces $\mathfrak{s}^{(i)}$ form a j-invariant filtration of the Lie algebra $\mathfrak{s}$. Furthermore

(5) $$[\mathfrak{s}^{(1)}, \mathfrak{s}^{(1)}] = 0.$$

Proof. — We modify the proof of *loc. cit.* Lemma 12. First it is clear that $\mathfrak{s}^{(-1)} \supset \mathfrak{s}^{(0)} \supset \mathfrak{s}^{(1)} \supset \mathfrak{s}^{(2)}$ holds. It is also easy to see that all $\mathfrak{s}^{(k)}$ are j-invariant. By the definition of $\mathfrak{s}^{(k)}$ we get $[\mathfrak{s}^{(-1)}, \mathfrak{s}^{(2)}] \subset \mathfrak{s}^{(1)}$ and $[\mathfrak{s}^{(0)}, \mathfrak{s}^{(2)}] \subset \mathfrak{s}^{(2)}$. One proves next $[\mathfrak{s}^{(0)}, \mathfrak{s}^{(0)}] \subset \mathfrak{s}^{(0)}$ as in *loc. cit.* Therefore $\mathfrak{s}^0$ is a Kähler algebra. If $\mathfrak{s}^{(0)}=\mathfrak{s}$ then we continue with 6.6. Otherwise we decompose $\mathfrak{s}=\mathfrak{s}_0+\mathfrak{s}_1+\mathfrak{s}_D$ as in 3.3 by induction hypotheses. Clearly $[j\mathfrak{q}, \mathfrak{q}]=0$ for all $\mathfrak{q}\in\mathfrak{q}+j\mathfrak{q}$. Hence $\mathfrak{q}+j\mathfrak{q}\subset\mathfrak{s}_0+\mathfrak{s}_1$ by Lemma 3.4.4. We claim $[jx, x]=0$ if $x=[jq, y]$ with some $q\in\mathfrak{q}$, $y\in\mathfrak{s}$. We first show as in *loc. cit.* that $[[jq, u], [jq, v]]=0$ holds for all $q\in\mathfrak{q}$, u, $v\in\mathfrak{s}$. But then

$$[jx, x]=[j\,[jq, y], [jq, y]]=[[jq, jy]-j\,[q, jy]-[q, y], [jq, y]]=-[j\,[q, jy]+[q, y], [jq, y]].$$

Now we note that $j\,[q, jy]+[g, y]\in\mathfrak{q}+j\,\mathfrak{q}$ and $[j\,\mathfrak{q}, y]\subset\mathfrak{p}$ by (1). Therefore $[jx, x]=[ju, [jq, y]]$ with $u=-[q, jy]\in\mathfrak{q}$. But we know

$$0=[\operatorname{ad}(ju+jq)]^2=(\operatorname{ad}ju)(\operatorname{ad}jq)+(\operatorname{ad}jq)(\operatorname{ad}ju)$$

from (4) and

$$[\operatorname{ad}ju, \operatorname{ad}jq]=\operatorname{ad}[ju, jq]=0,$$

therefore $(\operatorname{ad}ju)(\operatorname{ad}jq)=0$ and $[jx, x]=0$ follows. But this implies $[jq, \mathfrak{s}]\subset\mathfrak{s}_0+\mathfrak{s}_1$ for all $q\in\mathfrak{q}$. Hence

(6) $$\mathfrak{s}^{(1)}\subset\mathfrak{s}_0+\mathfrak{s}_1\subset\mathfrak{s}^{(0)}.$$

We know that $[jr, \mathfrak{s}]$ acts on $\mathfrak{s}$ by nilpotent endomorphisms, therefore $[jr, \mathfrak{s}]\in\mathfrak{s}_0+\mathfrak{s}_1$ is a translation (thus an affine transformation without linear part). By definition, $\mathfrak{q}$ acts by nilpotent endomorphisms and $\operatorname{ad} j\,\mathfrak{q}$ is nilpotent by (4), hence also $\mathfrak{q}$ and $j\,\mathfrak{q}$ are translations in $\mathfrak{s}_0+\mathfrak{s}_1$. Therefore

(7) $\mathfrak{s}^{(1)}$ is abelian and acts nilpotent on $\mathfrak{s}$.

We finish the proof of (5) by showing $[\mathfrak{s}^{(0)}, \mathfrak{s}^{(1)}]\subset\mathfrak{s}^{(1)}$ and $[\mathfrak{s}^{(-1)}, \mathfrak{s}^{(1)}]\subset\mathfrak{s}^{(0)}$ as in *loc. cit.*

Next we show

(8) let $x\in\mathfrak{s}_0+\mathfrak{s}_1$, $\operatorname{ad}x=S+N$, where S is the semisimple part of $\operatorname{ad}x$ and N its nilpotent part. Then $N\mathfrak{s}\subset\mathfrak{s}^{(0)}$.

Proof. — Let $r\in\mathfrak{q}$ then $[jr, Nx]=N[jr, x]-[Njr, x]$ for all $x\in\mathfrak{s}$. Because $[jr, x]$, $jr\in\mathfrak{s}^{(1)}$ it suffices to show $N\mid\mathfrak{s}^{(1)}=0$. But $a\in\mathfrak{s}^{(1)}$ is represented by a translation, $x\in\mathfrak{s}_0+\mathfrak{s}_1$ by an affine transformation $x+A(x)z$ hence

$$[x, a]\equiv[x+A(x)z, a]=-A(x)a.$$

Therefore $\operatorname{ad}x\mid\mathfrak{s}^{(1)}=-A(x)$ whence semisimple. Thus $N\mid\mathfrak{s}^{(1)}=0$.

Let $\mathfrak{a}=\mathfrak{s}_0+\mathfrak{s}_1$ and $\mathfrak{h}=\mathfrak{s}_D$. Let s denote the principal idempotent of the normal j-algebra underlying $\mathfrak{h}$ [4], Theorem 3.3.2. We split $\operatorname{ad}js=H+N$ into semisimple and nilpotent part and denote by H_0 the real part of H. Then $H_0\in\operatorname{Der}\mathfrak{s}$. We denote by

$\mathfrak{s}_\lambda$ the eigenspace of H_0 for the (real) eigenvalue λ. Then $\mathfrak{s}=\oplus\mathfrak{s}_\lambda$ and $[\mathfrak{s}_\lambda, \mathfrak{s}_\mu]\subset\mathfrak{s}_{\lambda+\mu}$. Moreover, defining $\mathfrak{s}_\lambda^{(0)}$, $\mathfrak{a}_\lambda$ and $\mathfrak{h}_\lambda$ analogously, we get $\mathfrak{s}_\lambda\cap\mathfrak{s}^{(0)}=\mathfrak{s}_\lambda^{(0)}$, $\mathfrak{s}_\lambda^{(0)}=\mathfrak{a}_\lambda+\mathfrak{h}_\lambda$.

Now we go over [7], Chap. III, § 3, Sect. 5. We define $\bar{\mathfrak{s}}^{(j)}:=\mathfrak{s}^{(j)}/\mathfrak{s}^{(j+1)}$, $\mathfrak{s}^{(j)}=0$ for $j=3$. On $\bar{\mathfrak{s}}=\bigoplus_{j=-1}^{2}\bar{\mathfrak{s}}^{(j)}$ we put $[\overline{x^{(j)}}, \overline{x^{(k)}}]=[x^{(j)}, x^{(k)}]\bmod\mathfrak{s}^{(j+k+1)}$.

Then $\bar{\mathfrak{s}}$ is a graded Lie algebra. We define on $\bar{\mathfrak{s}}^{(-1)}$ (for a fixed $r\in\mathfrak{q}$)

$$\{abc\}=[[[\overline{jr}, a], b], c]. \tag{9}$$

Note that $\{abc\}\in\bar{\mathfrak{s}}^{(-1)}$, hence, it is determined modulo $\mathfrak{s}^{(0)}$.

As in *loc. cit.* one proves that $\{abc\}$ is invariant under permutation of a, b and c. Also

$$[\overline{jq}, \{abc\}]=[[[\overline{jr}, a], b], [\overline{jq}, c]] \quad \text{for all } q\in\mathfrak{q}. \tag{10}$$

Our next goal is to prove $\{abc\}=0$ for all $a, b, c\in\bar{\mathfrak{s}}^{(-1)}$. We know $\mathfrak{q}\subset\mathfrak{s}^{(1)}\subset\mathfrak{a}$. By the construction of $\mathfrak{q}$ we also have $H_0\mid\mathfrak{q}=\alpha\,\mathrm{Id}$. From the classification of "symplectic representations" [12], p. 234, we know $\alpha\in\{0, \pm 1/2\}$. If $\alpha=-1/2$, then $[s, r]=jr$, but $jr\notin\mathfrak{q}$ whence $\alpha\neq -1/2$. Moreover, $r\in\mathfrak{a}_\alpha$, therefore $jr\in\mathfrak{a}_{-\alpha}$. This implies in $\bar{\mathfrak{s}}$

$$\overline{H}_0\overline{jr}=-\alpha\overline{jr} \quad \text{where} \quad \alpha\in\{0, 1/2\}. \tag{11}$$

We show

$$[jr, \mathfrak{s}_\lambda^{(-1)}]\subset\mathfrak{s}_{\lambda-\alpha}^{(1)}. \tag{12}$$

Proof. — For $x_\lambda\in\mathfrak{s}_\lambda^{(-1)}$ we have

$$H_0[jr, x_\lambda]=[H_0 jr, x_\lambda]+[jr, H_0 x_\lambda]=-\alpha[jr, x_\lambda]+\lambda[jr, x_\lambda].$$

We know from the description of "symplectic representations" [12], p. 234, that H_0 has only the eigenvalues 0, $\pm 1/2$ on $\mathfrak{s}^{(1)}$. Hence

$$\lambda=\alpha, \quad \alpha\pm 1/2 \quad \text{where} \quad \alpha\in\{0, 1/2\}, \quad \text{if} \quad \mathfrak{s}_\lambda^{(-1)}\neq 0. \tag{13}$$

Choose $a\in\mathfrak{s}_\lambda^{(-1)}$, $b\in\bar{\mathfrak{s}}_\mu^{(-1)}$, $c\in\bar{\mathfrak{s}}_\nu^{(-1)}$ so that $\{abc\}\neq 0$. Then $[\overline{jq}, \{abc\}]\neq 0$ for some $q\in\mathfrak{q}$. If $[\overline{jq}, \{abc\}]=0$ for all $q\in\mathfrak{q}$, then all representatives of $\{abc\}$ are in $\mathfrak{s}^{(0)}$. Therefore $\{abc\}=0$.

From the above we know

$$\lambda, \mu, \nu\in\{\alpha, \alpha\pm 1/2\} \quad \text{where} \quad \alpha\in\{0, 1/2\}. \tag{13'}$$

Furthermore

$$[[\overline{jr}, a], b]\in\bar{\mathfrak{s}}_{\lambda+\mu-\alpha}\cap\bar{\mathfrak{s}}^{(0)}=\bar{\mathfrak{a}}_{\lambda+\mu-\alpha}+\bar{\mathfrak{h}}_{\lambda+\mu-\alpha}.$$

If $\lambda+\mu-\alpha\neq 0$, 1 or 1/2 then $\bar{\mathfrak{h}}_{\lambda+\mu-\alpha}=0$ because H_0 has on $\mathfrak{h}$ only the eigenvalues 0, 1, 1/2 whence $\bar{y}=[[\overline{jr}, a], b]\in\bar{\mathfrak{a}}$ and $\operatorname{ad} y$ acts nilpotent. Therefore (8) implies that each representative of $[\bar{y}, c]$ is in $\mathfrak{s}^{(0)}$ whence $[\bar{y}, c]=0$. But $[\bar{y}, c]=\{abc\}$, a contradiction. This implies (using the symmetry of $\{abc\}$)

(14) $$\lambda+\mu,\ \mu+\nu,\ \nu+\lambda\in\{\alpha,\ 1+\alpha,\ 1/2+\alpha\},\qquad \alpha\in\{0,\ 1/2\}.$$

Now we show

$$[\mathfrak{s},\ \mathfrak{s}^{(1)}]\subset\mathfrak{s}_0\cap[\mathfrak{s},\ \mathfrak{s}]+(\mathfrak{s}_D\cap[\mathfrak{s},\ \mathfrak{s}]).$$

Proof. — We know $[\mathfrak{s},\ \mathfrak{s}^{(1)}]\subset\mathfrak{s}^{(0)}$, therefore each element of $[\mathfrak{s},\ \mathfrak{s}^{(1)}]$ is of type $x=x_0+x_1+x_D$ with $x_0\in\mathfrak{s}_0$, $x_1\in\mathfrak{s}_1$ and $x_D\in\mathfrak{s}_D$. Moreover, $\operatorname{ad}x$ acts nilpotent on $\mathfrak{s}$. From the properties of $\mathfrak{s}^{(0)}=\mathfrak{s}_0+\mathfrak{s}_1+\mathfrak{s}_D$ we see that the $\mathfrak{s}_D$-part of $\operatorname{ad}x\mid\mathfrak{s}_D$ is $(\operatorname{ad}x_1+\operatorname{ad}x_D)\mid\mathfrak{s}_D$ modulo $\mathfrak{s}_0$. This endomorphism acts nilpotent on $\mathfrak{s}_D$. In particular, $x_D\in[\mathfrak{s}_D,\ \mathfrak{s}_D]$. But then $\operatorname{ad}x_D$ acts nilpotent on $\mathfrak{s}_D$. This implies that $\operatorname{ad}x_1\mid\mathfrak{s}_D$ is nilpotent, therefore $\operatorname{ad}x_1\mid\mathfrak{s}_D=0$. From the above we derive that $x-x_D$ acts nilpotent on $\mathfrak{a}$. Hence $x_1=0$ and (15) is verified.

As a consequence of (15) we note that the $\mathfrak{a}$—components of $[\mathfrak{s},\ \mathfrak{s}^{(1)}]$ consist of translations.

Next we prove

(16) Let $g_\nu\in\mathfrak{s}_\nu$, $u_\eta\in\mathfrak{s}_\eta^{(1)}$, $v_\xi\in\mathfrak{s}_\xi^{(1)}$ and assume $\eta+\xi>0$ or $\eta=\xi=0$, then $[[g_\nu,\ u_\eta],\ v_\xi]=0$.

Proof. — Put $y=j[[g_\nu,\ u_\eta],\ v_\xi]$, then $y\in\mathfrak{s}^{(1)}_{-(\nu+\eta+\xi)}$ and we have

$$-\prec y,\ y\succ=\rho([[g_\nu,\ u_\eta],\ v_\xi],\ y)=-\rho([v_\xi,\ y],\ [g_\nu,\ u_\eta])-\rho([y,\ [g_\nu,\ u_\eta]],\ v_\xi).$$

Here the first summand on the right-hand side vanishes because $v_\xi,\ y\in\mathfrak{s}^{(1)}$ and (7). It therefore suffices to prove $[[g_\nu,\ u_\eta],\ y]=0$. By the Jacobi identity and (7) this is equivalent to $[[g_\nu,\ y],\ u_\eta]=0$.

If $\eta+\xi>0$, then $[g_\nu,\ y]\in\mathfrak{s}^{(0)}_{-(\eta+\xi)}\cap\mathfrak{a}$ because H_0 has only non-negative eigenvalues on $\mathfrak{s}_D$. Obviously, $[g_\nu,\ y]$ acts nilpotent on $\mathfrak{s}^{(1)}$; therefore it is represented in $\mathfrak{a}$ by a translation. Consequently $[[g_\nu,\ y],\ u_\eta]=0$ because $u_\eta\in\mathfrak{s}^{(1)}$.

Now we assume $\eta=\xi=0$. Then $y\in\mathfrak{s}^{(1)}_{-\nu}$ and $[g_\nu,\ y]\in\mathfrak{s}_0^{(0)}$. We know from (15) that the $\mathfrak{a}$-component of $[g_\nu,\ y]$ is a translation; it therefore commutes with $u_\eta\in\mathfrak{s}^{(1)}$. It therefore suffices to prove

(*) $$\mathfrak{n}_0=[\mathfrak{s}_D,\ \mathfrak{s}_D]\cap\mathfrak{s}_0^{(0)}\text{ acts trivial on }\mathfrak{s}_0^{(1)}.$$

Proof. — Let $n\in\mathfrak{n}_0$ and put $A=\operatorname{ad}n\mid\mathfrak{s}_0^{(1)}$. We have

$$[jn,\ jx]=j\,[jn,\ x]+j\,[n,\ jx]+[n,\ x]\qquad\text{for}\quad x\in\mathfrak{s}_0^{(1)}.$$

Because

$$[\mathfrak{s}_1^{(0)},\ \mathfrak{s}_0^{(1)}]\subset\mathfrak{s}_1^{(1)}=0\qquad\text{and}\qquad j\,\mathfrak{s}_0^{(1)}=\mathfrak{s}_0^{(1)}$$

we get $0=0+j[n, jx]+[n, x]$, i. e. $[A, j]=0$. Next we consider

$$\rho(Ax, z)=\rho([n, x], z)=-\rho([x, z], n)-\rho([z, n], x)$$
$$=\rho([n, z], x)=-\rho(x, [n, z])=-\rho(x, Az)$$

because $x, z\in\mathfrak{s}^{(1)}$ and $\mathfrak{s}^{(1)}$ is abelian. Using that A commutes with j we conclude that A is skew-adjoint. On the other hand A acts nilpotent on $\mathfrak{s}$ whence $A=0$.

As a consequence of (16) we get

$$(17)\qquad [[\bar{\mathfrak{s}}^{(-1)}, \bar{\mathfrak{s}}^{(1)}_{\eta}], \bar{\mathfrak{s}}^{(1)}_{\xi}]=0 \qquad \text{if} \quad \eta+\xi>0 \quad \text{or} \quad \eta=\xi=0.$$

We resume the assumptions on a, b, c made after (13) and note $[\overline{jr}, a]\in\bar{\mathfrak{s}}^{(1)}_{\lambda-\alpha}$, $[\overline{jq}, c]\in\bar{\mathfrak{s}}^{(1)}_{\nu-\alpha}$ by (12). Using the remark preceding (13′) and (10) we get that $\{abc\}\neq 0$ is possible only in the case where

(18) $\lambda+\nu-2\alpha\leqq 0$, and $\lambda-\alpha$ and $\nu-\alpha$ are not simultaneously zero.

Using the symmetry of $\{abc\}$ we obtain

(19) $\lambda+\mu$, $\mu+\nu$, $\lambda+\nu\leqq 2\alpha$, and at most one of the numbers λ, μ, ν is equal to α.

It is easy to check that for both cases, $\alpha=0$ and $\alpha=1/2$, it is impossible to satisfy simultaneously the conditions (13)′, (14) and (19). Therefore $\{abc\}=0$.

This implies $[[\mathfrak{s}^{(1)}, \mathfrak{s}], \mathfrak{s}]\subset\mathfrak{s}^{(0)}$. Hence

$$[jq, [[\mathfrak{s}^{(1)}, \mathfrak{s}], \mathfrak{s}]]\subset[jr, \mathfrak{s}^{(0)}]\subset\mathfrak{s}^{(2)} \quad \text{for all } q\in\mathfrak{q}.$$

Using the Jacobi identity and the commutativity of $\mathfrak{s}^{(1)}$ we see that the left-hand side of this inclusion equals $[[\mathfrak{s}^{(1)}, \mathfrak{s}], [j\mathfrak{q}, \mathfrak{s}]]$. Hence $[[\mathfrak{s}^{(1)}, \mathfrak{s}], [j\mathfrak{q}, \mathfrak{s}]]\subset\mathfrak{s}^{(2)}$. Replacing $[j\mathfrak{q}, \mathfrak{s}]$ by $\mathfrak{q}+j\mathfrak{q}$ we get $[[\mathfrak{s}^{(1)}, \mathfrak{s}], \mathfrak{q}+j\mathfrak{q}]\subset\mathfrak{s}^{(2)}$. Therefore altogether $[[\mathfrak{s}^{(1)}, \mathfrak{s}], \mathfrak{s}^{(1)}]\subset\mathfrak{s}^{(2)}$. It is easy to show $\rho([[\mathfrak{s}^{(1)}, \mathfrak{s}], \mathfrak{s}^{(1)}], \mathfrak{s}^{(2)})=0$. As $\mathfrak{s}^{(2)}$ is j-invariant we may replace ρ by the inner product on $\mathfrak{s}$. Therefore

$$(20)\qquad [[\mathfrak{s}^{(1)}, \mathfrak{s}], \mathfrak{s}^{(1)}]=0.$$

Now we define

$$Z(\mathfrak{s}^{(1)})=\{x\in\mathfrak{s};\ [x, \mathfrak{s}^{(1)}]=0\}.$$

Because $j\mathfrak{q}\subset\mathfrak{s}^{(1)}$ this implies $Z(\mathfrak{s}^{(1)})\subset\mathfrak{s}^{(0)}$. We can take over the first part of the proof of [7], Part III, Lemma 18, without change and get

$$(21)\qquad Z(\mathfrak{s}^{(1)})\subset\mathfrak{s}^{(0)} \quad \text{is an ideal of } \mathfrak{s}.$$

Eventually we want to show that $Z(\mathfrak{s}^{(1)})$ is a Kähler ideal.

We pick $z\in Z(\mathfrak{s}^{(1)})$ arbitrary. Because $Z(\mathfrak{s}^{(1)})\subset\mathfrak{s}^{(0)}$ we know $jz\in\mathfrak{s}^{(0)}$. Therefore $[jz, \mathfrak{s}^{(1)}]\subset\mathfrak{s}^{(1)}$. We put $A=\operatorname{ad} jz \mid \mathfrak{s}^{(1)}$. Then

$$\rho(Ax, y)=\rho([jz, x], y)=-\rho([x, y], jz)-\rho([y, jz], x)=\rho(Ay, x)=-\rho(x, Ay)$$

where we used that $\mathfrak{s}^{(1)}$ is commutative.

From the integrability condition we get

$$[jz, jx]=j[jz, x]+j[z, jx]+[z, x]=j[jz, x]$$

for $z\in Z(\mathfrak{s}^{(1)})$ and $x\in\mathfrak{s}^{(1)}$. This implies $[A, j]=0$. Therefore

$$\prec Ax, y\succ=\rho(Ax, jy)=-\rho(x, Ajy)=-\prec x, Ay\succ.$$

Hence A is skew-adjoint.

(22) $\operatorname{ad} jz \mid \mathfrak{s}^{(1)}$ is skew-adjoint for all $z\in Z(\mathfrak{s}^{(1)})$ and commutes with j.

Hence, to be able to show that $Z(\mathfrak{s}^{(1)})$ is a Kähler ideal we have to modify the algebra $\mathfrak{s}$.

6.3. For $x\in\mathfrak{s}^{(0)}$ we denote by x_D the component of x in $\mathfrak{s}_D\subset\mathfrak{s}^{(0)}$. Let $\mathfrak{h}_0$ denote the orthogonal complement of $\mathfrak{n}_D=[\mathfrak{s}_D, \mathfrak{s}_D]$ in $\mathfrak{s}_D$ and $\mathfrak{h}_\alpha$ the weight spaces of $\mathfrak{h}_0$ in $\mathfrak{n}_D$. It is known that the projections onto $\mathfrak{h}$ are polynomials with no constant coefficient in elements of $\mathfrak{h}_0$.

We put $\mathfrak{z}_D=\{x_D;\ x\in\mathfrak{z}\}$, $\mathfrak{z}=Z(\mathfrak{s}^{(1)})$.

$$\mathfrak{z}_D=(\mathfrak{z}_D\cap\mathfrak{h}_0)\oplus\bigoplus_{\alpha\neq 0}(\mathfrak{z}_D\cap\mathfrak{h}_\alpha). \tag{23}$$

Let $x\in\mathfrak{z}$ and $h\in\mathfrak{h}_0$; we know $[\mathfrak{s}_1, h]\subset\mathfrak{a}_0$ hence the $\mathfrak{s}_D$-component of $[x, h]$ is $[x_D, h]$. This implies that the $\mathfrak{n}_D$ component of x_D splits into the weight spaces. Hence (23).

(24) Let $x\in\mathfrak{z}$ and $x_D\in\mathfrak{z}_D\cap\mathfrak{h}_\alpha$, $\alpha\neq 0$. Then $x=x_a+x_D$, x_a, $x_D\in\mathfrak{z}$.

Proof. — Because $\mathfrak{s}^{(1)}$ is an ideal of $\mathfrak{s}^{(0)}$ we know that x_D acts nilpotent on $\mathfrak{s}^{(1)}$. But $\operatorname{ad} x\mid\mathfrak{s}^{(1)}=0$, therefore $\operatorname{ad} x_a\mid\mathfrak{s}^{(1)}$ has no semisimple part. This implies $\operatorname{ad} x_a\mid\mathfrak{s}^{(1)}=0$, i.e. $x_a\in\mathfrak{z}$. Consequently $x_D\in\mathfrak{z}$.

6.4. In this subsection we assume $\mathfrak{z}_D\neq 0$.

We consider $z\in\mathfrak{z}_D\cap j\mathfrak{h}_0$. Choosing z from a weight space we may assume in addition $[jz, z]=z$. Let ${}_\gamma\mathfrak{s}$, $\gamma\in\mathbb{R}$, be the weight spaces (of the real part) of $\operatorname{ad} jz$ in $\mathfrak{s}$. We know $[{}_\gamma\mathfrak{s}, {}_\beta\mathfrak{s}]\subset{}_{(\beta+\gamma)}\mathfrak{s}$. Moreover, because $\operatorname{ad} jz$ leaves invariant $\mathfrak{s}^{(0)}$ we may find a complementary subspace $\mathfrak{u}$ of $\mathfrak{s}^{(0)}$ in $\mathfrak{s}$ that splits into weight spaces relative to (the real part of) $\operatorname{ad} jz$, $\mathfrak{u}=\oplus(\mathfrak{u}\cap{}_\gamma\mathfrak{s})$. We note that jz commutes with all jr, $r\in\mathfrak{q}$: We have

$$[jz, jr]=j[jz, r]+j[z, jr]+[z, r];$$

here the last two summands vanish because $z\in\mathfrak{z}$ and jr, $r\in\mathfrak{s}^{(1)}$. Finally, jz acts skew-adjoint and $\mathbb{C}$-linear on $\mathfrak{s}^{(1)}$ and leaves the ideal $\mathfrak{q}$ invariant. Therefore $[jz, r]\in j\mathfrak{q}\cap\mathfrak{q}=0$. This implies that $\operatorname{ad} jz$ has on $\mathfrak{u}$ and $[jr, \mathfrak{u}]\subset\mathfrak{s}^{(1)}$ the same weights. But we know that $\operatorname{ad} jz$ is skew-adjoint on $\mathfrak{s}^{(1)}$ whence $\operatorname{ad} jz$ has only real weight 0 on $\mathfrak{u}$. Therefore, $\operatorname{ad} jz$ has non-zero weight only in $\mathfrak{s}^{(0)}$. We know that it has possibly the weights 0, $\pm 1/2$ in $\mathfrak{a}=\mathfrak{s}^{(0)}\ominus\mathfrak{s}_D$ and 0, 1/2, 1 in $\mathfrak{s}_D$. It is clear that $\mathfrak{w}={}_0\mathfrak{s}+{}_1\mathfrak{s}$ is a subalgebra of $\mathfrak{s}$. We claim that $\mathfrak{w}$ is j-invariant. It suffices to show that ${}_\beta\mathfrak{s}$ is orthogonal to ${}_0\mathfrak{s}+{}_1\mathfrak{s}$ for all $\beta \neq 0, 1$. But we know

$$\frac{d}{dt}\rho(e^{t\,\mathrm{ad}\,jz}x_\beta, e^{t\,\mathrm{ad}\,jz}x_0)=\rho(jz, e^{t\,\mathrm{ad}\,jz}[x_\beta, x_0]).$$

Here the last term vanishes because $e^{t\,\mathrm{ad}\,jz}[x_\beta, x_0]\in{}_\beta\mathfrak{s}\cap\mathfrak{s}^{(0)}$ and ${}_\beta\mathfrak{s}\cap\mathfrak{s}^{(0)}$ is orthogonal to ${}_0\mathfrak{s}^{(0)}+{}_1s^{(0)}$. This implies

$$\rho(e^{t\,\mathrm{ad}\,jz}x_\beta, e^{t\,\mathrm{ad}\,jz}x_0)=\rho(x_\beta, x_0).$$

But the left-hand side behaves like $e^{t\beta}$ whence $\rho(x_\beta, x_0)=0$. Replacing $x_0\in{}_0\mathfrak{s}$ by $x_1\in{}_1\mathfrak{s}$ in the argument above shows $\rho(x_\beta, x_1)=0$ as well. It now suffices to note that $\bigoplus_{\beta\neq 0,1}{}_\beta\mathfrak{s}={}_{1/2}\mathfrak{s}^{(0)}+{}_{-1/2}\mathfrak{s}^{(0)}$ is invariant under j. This shows that $\bigoplus_{\beta\neq 0,1}{}_\beta\mathfrak{s}$ is orthogonal to $\mathfrak{w}$ and $\mathfrak{w}$ is j-invariant. Hence $\mathfrak{w}$ is a Kähler algebra.

If $\mathfrak{w}=\mathfrak{s}$, then ${}_1\mathfrak{s}$ is a commutative ideal of $\mathfrak{s}$ and it is easy to see that paragraph 5 applies to this case. Hence $\mathfrak{s}$ can be modified to become a semidirect product of an abelian ideal and *a* normal j-algebra. Assume now $\mathfrak{w}\neq\mathfrak{s}$; then we can apply the induction hypothesis and get $\mathfrak{w}=\mathfrak{a}^w+\mathfrak{w}_D$ as well as a modification map D for $\mathfrak{w}$. We put $\mathfrak{h}_0^w=\mathfrak{w}_D\ominus[\mathfrak{w}_D, \mathfrak{w}_D]$. By $\mathfrak{a}_0^w$ we denote the set of elements of $\mathfrak{a}^w$ which are annihilated by the real parts of all $h\in\mathfrak{h}_0^w$. We know that D vanishes on the orthogonal complement of $\mathfrak{a}_0^w+\mathfrak{h}_0^w$. Moreover, by Lemma 3.5.1 we get a basis $a_1, \ldots, a_l$ of $\mathfrak{a}^w$ satisfying $[ja_r, a_r]=0$ for $1\leqq r\leqq l$. Now we use that $\mathfrak{w}$ leaves invariant $\mathfrak{v}$. Hence, by [7], Part III, Lemma 3, we know that $\mathfrak{a}_0^w$ acts on $\mathfrak{v}$ by skew-adjoint endomorphisms which commute with j. Now we consider $\mathfrak{h}_0^w$. It is easy to see that $\mathfrak{h}_0$ is contained in $\mathfrak{h}_0^w$. Because $\mathfrak{h}_0+\mathfrak{v}\subset\mathfrak{s}^0$ and $[\mathfrak{h}_0, \mathfrak{v}]\subset\mathfrak{v}$ we know that the purely imaginary parts of $\mathrm{ad}\,\mathfrak{h}_0\mid\mathfrak{v}$ are skew-adjoint and commute with j. Let $h\in\mathfrak{h}_0^w\ominus\mathfrak{h}_0$. Then $[h, jz]=0$ and $\mathrm{ad}\,h$ leaves ${}_{1/2}\mathfrak{s}$ and ${}_{-1/2}\mathfrak{s}$ invariant. We claim that h is orthogonal to $[{}_{1/2}\mathfrak{s}, {}_{-1/2}\mathfrak{s}]$. From above we know that all these spaces are contained in $\mathfrak{s}^{(0)}$. Moreover, $\mathfrak{a}=\mathfrak{a}_0+\mathfrak{a}_{1/2}+\mathfrak{a}_{-1/2}$ and $\mathfrak{s}_D=\mathfrak{s}_{D0}+\mathfrak{s}_{D(1/2)}+\mathfrak{s}_{D1}$ relative to the real part of $\mathrm{ad}\,jz$. Hence

$$[{}_{1/2}\mathfrak{s}, {}_{-1/2}\mathfrak{s}]=[\mathfrak{a}_{1/2}+\mathfrak{s}_{D(1/2)}, \mathfrak{a}_{-1/2}]\subset\mathfrak{a}_0$$

because the modification map of $\mathfrak{s}^0$ vanishes on all occurring spaces. We know from the description of sympletic representations that $\mathfrak{a}_0$ is j-invariant. Hence, by Lemma 3.5.1 we have a basis $b_1, \ldots, b_m$ of $\mathfrak{a}$ so that $[jb_r, b_r]=0$ holds. But this implies $\mathfrak{a}_0\subset\mathfrak{a}^w$. In particular

$$\prec\mathbb{R}h+\mathbb{R}jh, [{}_{1/2}\mathfrak{s}, {}_{-1/2}\mathfrak{s}]\succ=0.$$

Therefore

$$\frac{d}{dt}\rho(e^{t\,\mathrm{ad}\,h}u_{1/2}, e^{t\,\mathrm{ad}\,h}v_{-1/2})=\rho(h, e^{t\,\mathrm{ad}\,h}[u_{1/2}, v_{-1/2}])=0.$$

This shows

$$\rho(e^{t\,\mathrm{ad}\,h}u_{1/2}, e^{t\,\mathrm{ad}\,h}v_{-1/2})=\rho(u_{1/2}, v_{-1/2})$$

whence $\operatorname{ad} h \mid \mathfrak{v}$ and $\operatorname{ad} jh \mid \mathfrak{v}$ are symplectic transformations of $\mathfrak{v}$. Hence, by [12], p. 234, $\mathfrak{v}$ decomposes into eigenspaces $\mathfrak{v}^h_{1/2}$, $\mathfrak{v}^h_{-1/2}$ and $\mathfrak{v}^h_0$ for the real part of $\operatorname{ad} h \mid \mathfrak{v}$. The fact that $\operatorname{ad} h$ is symplectic and satisfies (12) implies that the purely imaginary part of $\operatorname{ad} h \mid \mathfrak{v}$ is skew-adjoint and commutes with j. For $w \in \mathfrak{w}$ we denote by $D(w)$ the skew-adjoint part of $\operatorname{ad} w$ on $\mathfrak{s}$. From the remarks above we see that the family $\{D(w);\ w \in \mathfrak{w}\}$ consists of skew-adjoint derivations of $\mathfrak{s}$ which commute with j. By the definition of $D(w)$ it is clear that these endomorphisms are contained in the algebraic hull of $\operatorname{ad}(\mathfrak{a}^w + \mathfrak{h}^w_0)$. But $\mathfrak{a}^w + \mathfrak{h}^w_0$ is a solvable Lie algebra whence the purely imaginary parts of $\operatorname{ad}(\mathfrak{a}^w + \mathfrak{h}^w_0)$ commute. This proves (3.3).

From the definition of $D(w)$ we get $D([\mathfrak{w}\ \mathfrak{w}]) = 0$ and $D([\mathfrak{w}\ \mathfrak{v}]) = 0$. We know $\mathfrak{v} = {}_{1/2}\mathfrak{s} + {}_{-1/2}\mathfrak{s}$; hence

$$[\mathfrak{v}, \mathfrak{v}] = [{}_{1/2}\mathfrak{s}, {}_{-1/2}\mathfrak{s}] \subset \mathfrak{w}.$$

The elements of $[\mathfrak{v}\ \mathfrak{v}]$ act nilpotent on $\mathfrak{s}$, therefore $D([\mathfrak{v}\ \mathfrak{v}]) = 0$. This proves (3.4). Finally, (3.5*) follows by Lemma 3.3.2. This shows that D is a modification map. Carrying out this modification we get a solvable Kähler algebra for which each $\operatorname{ad} x$, $x \in \mathfrak{s}$, has only real eigenvalues. Hence, by [7], $\mathfrak{s}$ is the semidirect product of an abelian ideal and a normal j-algebra.

6.5. In this section we assume $\mathfrak{z}_D = 0$. Hence $\mathfrak{z} \subset \mathfrak{a}$. We apply Lemma 3.3.1 to $\mathfrak{a}$ and get $\mathfrak{a} = \hat{\mathfrak{a}}_0 + \hat{\mathfrak{a}}_{10} + \hat{\mathfrak{a}}_{11}$ where $\hat{\mathfrak{a}}_0 = [\mathfrak{a}, \mathfrak{a}]$, $A(\hat{\mathfrak{a}}_0 + \hat{\mathfrak{a}}_{10}) = 0$ and $A(\mathfrak{a})(\hat{\mathfrak{a}}_{10} + \hat{\mathfrak{a}}_{11}) = 0$, where $A(a)$ denotes as usual the linear part of the affine transformation associated with $a \in \mathfrak{a}$. Because $\mathfrak{s}^{(1)}$ and $\hat{\mathfrak{a}}_0 + \hat{\mathfrak{a}}_{10}$ consist of translations we have $\hat{\mathfrak{a}}_0 + \hat{\mathfrak{a}}_{10} \subset \mathfrak{z}$. Because $\mathfrak{z}$ is an ideal $\mathfrak{s}$ we see that $\operatorname{ad} z$ has the same eigenvalues as $\operatorname{ad} z \mid \mathfrak{a}$. Hence $\operatorname{ad} z$ has only purely imaginary eigenvalues. The eigenspaces for non-zero eigenvalue are contained in $\hat{\mathfrak{a}}_0$ and are invariant under j. Let ${}_\alpha\mathfrak{s}$ be the eigenspaces for $\operatorname{ad} z$. Then $[{}_0\mathfrak{s}, {}_\alpha\mathfrak{s}] \subset \hat{\mathfrak{a}}_0$ if $\alpha \neq 0$ because $H^2 = -\alpha^2 \operatorname{Id}$ on $[{}_0\mathfrak{s}, {}_\alpha\mathfrak{s}]$ where H denotes the semisimple part of $\operatorname{ad} z$.

Let now $z \in \hat{\mathfrak{a}}_{10} + \hat{\mathfrak{a}}_0$, then z and $[z, x]$, $x \in \mathfrak{s}$, are translations whence $(\operatorname{ad} z)^2 = 0$. If $z \in \hat{\mathfrak{a}}_{11}$ then $jz \in \hat{\mathfrak{a}}_{11}$ and $\rho(z, [{}_0\mathfrak{s}, {}_\alpha\mathfrak{s}]) = 0$ follows. Hence from [7], Part III, Lemma 9, we get

$$\frac{d}{dt} \rho(e^{t \operatorname{ad} z}\, {}_0x,\ e^{t \operatorname{ad} z}\, {}_\alpha x) = \rho(z,\ e^{t \operatorname{ad} z}[{}_0x, {}_\alpha x]) = 0.$$

We know $\operatorname{ad} z \mid \hat{\mathfrak{a}}_0 = A(z) \mid \hat{\mathfrak{a}}_0$ and $\operatorname{ad} z$ commutes with j on $\hat{\mathfrak{a}}_0$. Replacing ${}_\alpha x$ by $j_\alpha x$ above shows that we can replace ρ by the inner product. Hence

$$\prec e^{t \operatorname{ad} z}\, {}_0x,\ e^{t \operatorname{ad} z}\, {}_\alpha x \succ\ =\ \prec {}_0x, {}_\alpha x \succ.$$

This implies $\prec {}_0x, {}_\alpha x \succ = 0$. Therefore ${}_0\mathfrak{s} \perp {}_\alpha\mathfrak{s}$ for all $\alpha \neq 0$ and ${}_0\mathfrak{s}$ is invariant under j. As a consequence we get that the semisimple part of $\operatorname{ad} z$ is skew-adjoint and commutes with j.

If ${}_0\mathfrak{s} = \mathfrak{s}$ for all $z \in \mathfrak{z}$, then $\operatorname{ad} \mathfrak{z}$ consists of nilpotent endomorphisms. Otherwise we fix an orthonormal basis $D_1, \ldots, D_m$ relative to trace AB^* in the vectorspace of semisimple parts of $\{\operatorname{ad} z;\ z \in \mathfrak{z}\}$. We define $D(x) = \sum_{r=1}^{m} (\operatorname{trace} \operatorname{ad} x\, D_r)\, D_r$. It is clear that these maps are skew-adjoint derivations of $\mathfrak{s}$ which commute with j. Because $\hat{\mathfrak{a}}_{11}$ is abelian we have (3.3) satisfied. Finally, (3.4) and (3.5*) are verified using the fact that the algebraic hull $(\operatorname{ad} \mathfrak{s})^*$ of $\operatorname{ad} \mathfrak{s}$ is solvable whence trace $[A, B]\,C = 0$ for $A, B, C \in (\operatorname{ad} \mathfrak{s})^*$. Therefore D is a modification map of $\mathfrak{s}$. We note that D does not change the adjoint action of $\hat{\mathfrak{a}}_0 + \hat{\mathfrak{a}}_{10}$.

Carrying out this modification we can assume that $\operatorname{ad} z$ is nilpotent for all $z \in \mathfrak{z}$.

We put $\mathfrak{t} = \mathfrak{z} \ominus (\mathfrak{z} \cap j\mathfrak{z})$. Then $\mathfrak{t} \subset \hat{\mathfrak{a}}_{11}$, $\mathfrak{t} \cap j\,\mathfrak{t} = 0$ and $A(\mathfrak{a})\,\mathfrak{t} = 0$. Because $\operatorname{ad} \mathfrak{t}$ consists of nilpotent endomorphisms we also have $A(\mathfrak{t}) = 0$. Hence in particular $[\mathfrak{a}, \mathfrak{t}] = 0$.

We have

$$(25) \qquad \begin{aligned} &[jx, jr] = 0 \quad \text{for all} \quad x \in \mathfrak{t}, \quad r \in \mathfrak{q}, \\ &[jx, jy] = 0 \quad \text{for all} \quad x, y \in \mathfrak{t}, \\ &[jx, y] = 0 \quad \text{for all} \quad x, y \in \mathfrak{t}. \end{aligned}$$

Proof:

$$[jx, jr] = j\,[x, jr] + j\,[jx, r] + [x, r] = 0$$

because $x \in \mathfrak{t} \subset \mathfrak{z}$ implies $[x, \mathfrak{q} + j\,\mathfrak{q}] = 0$ and $[jx, r] = A(ix)\, r \in \mathfrak{q} \cap j\,\mathfrak{q} = 0$. Similarly, $[jx, jy] = 0$ because x, y act as translations, $[jx, y] = A(ix)\, y = 0$ and $[jy, x] = A(iy)\, x = 0$. This also proves the last assertion.

Let H_x, $x \in \mathfrak{t}$, denote the real part of the semisimple part of $\operatorname{ad} jx$. Then (25) implies $[H_x, \operatorname{ad} jr] = 0$. Therefore

$$\beta\,[jr, x_\beta] = [jr, H_x\, x_\beta] = H_x\,[jr, x_\beta].$$

But we know that $\operatorname{ad} x$ has no real eigenvalue in $\mathfrak{s}^{(1)}$. This implies $\beta = 0$ or $[jr, x_\beta] = 0$ for all $r \in \mathfrak{q}$. But the latter case implies $x_\beta \in \mathfrak{s}^0$. Hence

(26) $\operatorname{ad} jx$, $x \in \mathfrak{t}$, has only purely imaginary eigenvalues in $\mathfrak{s}$.

Next we note

$$(27) \qquad (\operatorname{ad} x)^2 = 0 \quad \text{for all} \quad x \in \mathfrak{z}.$$

Proof. — For any $u \in \mathfrak{s}$ we get $\operatorname{ad} x(u) \in \mathfrak{z}$. Hence $(\operatorname{ad} x)^2 u = \operatorname{ad} x(\operatorname{ad} xu) = 0$ because $\mathfrak{z} \subset \mathfrak{a}$ consists of translations.

We consider $x \in \mathfrak{t}$ and denote the j-linear part of $\operatorname{ad} x$ by q_1 and the j-anti-linear part by q_2. Similarly we define p_1 and p_2 for $\operatorname{ad} jx$. We copy the proof of [7], Lemma 3, so far as to get

$$p_2 = jq_2, \qquad j\,[p_1, q_1] = 2\, q_2^2 \qquad \text{and} \qquad [p_1, q_2] + [p_2, q_1] = 0.$$

Clearly, $[j, \operatorname{ad} x] = 2jq_2$ and a computation shows

$$[j, \operatorname{ad} x]\, u = -[x, ju] + j\,[x, u] \quad \text{for all} \quad u \in \mathfrak{s}.$$

Hence

$$4q_2^2 = j[x, j[x, u] - [x, ju]] - [x, -j[x, ju] - [x, u]] = j[x, j[x, u]] + [x, j[x, ju]]$$

by (27). We note $[x, ju] \in z \subset \mathfrak{a}$ whence $j[x, ju] \in \mathfrak{a}$. This implies

$$[x, j[x, ju]] = \mathrm{A}(j[x, ju])\, x = 0$$

by the definition of $\mathfrak{t}$. Therefore $q_2^2 = 0$. In particular we get $[p_1, q_1] = 0$. An argument as above shows even $q_1 q_2 = q_2 q_1 = 0$. Hence the remaining equation gives

$$0 = -[p_1, jp_2] + [jq_2, q_1] = -jp_1 p_2 + jp_2 p_1 + jq_2 q_1 - jq_1 q_2 = -j\,[p_1, p_2].$$

Therefore $[p_1, p_2] = 0$. Because $q_2^2 = 0$ we also have $p_2^2 = 0$. This implies that the semisimple part of $\operatorname{ad} jx$ is just the semisimple part of p_1; hence it commutes with j. Moreover, by (26) it has only imaginary eigenvalues.

By (25) we know that the family $\mathscr{D}$ of semisimple parts of $\operatorname{ad} jx$, $x \in \mathfrak{t}$, is abelian and commutes with j. Moreover, each element of $\mathscr{D}$ has only imaginary eigenvalues.

Let K denote the closed Lie group generated by $\exp \mathrm{D}$, $\mathrm{D} \in \mathscr{D}$. Then K is compact, $\mathrm{K} \subset \operatorname{Aut} \mathfrak{s}$ and $\mathrm{K} j = j \mathrm{K}$.

We define

$$\tilde{\rho}(u, v) = \int_{\mathrm{K}} \rho(\mathrm{W} u, \mathrm{W} v)\, d\mathrm{W}, \tag{28}$$

where $d\mathrm{W}$ denotes the normalized (right- and left-invariant) Haar measure on K.

We easily get

$$\begin{aligned} &\tilde{\rho}(x, y) \quad \text{is skew-symmetric,} \\ &\tilde{\rho}(x, jx) > 0 \quad \text{for all } x \neq 0, \\ &\tilde{\rho}([x, y], z) + \tilde{\rho}([y, z], x) + \tilde{\rho}([z, x], y) = 0, \\ &\tilde{\rho}(\mathrm{U} x, \mathrm{U} y) = \tilde{\rho}(x, y) \quad \text{for all } \mathrm{U} \in \mathrm{K}, \\ &\tilde{\rho}(x, y) = \rho(x, y) \quad \text{for all } x, y \in \mathfrak{s}^{(0)}. \end{aligned} \tag{29}$$

As a result we see that $\mathscr{D}$ consists of skew-adjoint endomorphisms relative to $\tilde{\rho}(x, jy)$ and $(\mathfrak{s}, \tilde{\rho})$ is a solvable Kähler algebra. We define $\mathrm{D}: \mathfrak{s} \to \operatorname{Der} \mathfrak{s}$ by $\mathrm{D}(x) = \sum_n (\operatorname{trace} \operatorname{ad} x\, \mathrm{D}_n)\, \mathrm{D}_n$ where D_n is an orthonormal base of $\mathscr{D}$ relative to trace AB^*. As above we verify that D is a modification map for $(\mathfrak{s}, \tilde{\rho})$.

We would like to point out that $\mathrm{D}(x) = 0$ if $\operatorname{ad} x$ is nilpotent.

W denote the new algebra by $\tilde{\mathfrak{s}}$ and its product by (x, y). After this modification we define $\tilde{\mathfrak{s}}^{(j)}$ as before. We have

$$\tilde{\mathfrak{s}}^{(0)} = \{x \in \tilde{\mathfrak{s}};\ [jq, x] + \mathrm{D}(jq)\, x - \mathrm{D}(x)\, jq \in \mathfrak{q} + j\,\mathfrak{q} \text{ for all } a \in \mathfrak{q}\}.$$

But $\operatorname{ad} jq$ is nilpotent whence $D(jq)=0$. Moreover, by definition of $D(x)$, $D(x) \mid \mathfrak{s}^{(1)}$ is skew-adjoint, therefore

$$D(x) j \mathfrak{q} = j D(x) \mathfrak{q} \subset j(\mathfrak{q} \cap j \mathfrak{q}) = 0.$$

Hence $x \in \tilde{\mathfrak{s}}^{(0)}$ iff $x \in \mathfrak{s}^{(0)}$.

Next we consider $[jr, x] + D(jr)x - D(x)jr$. As above we have $D(jr)=0$ and $D(x)jr=0$. Therefore $\tilde{\mathfrak{s}}^{(1)} = \mathfrak{s}^{(1)}$.

Of course, $\mathfrak{s}^{(-1)} = \tilde{\mathfrak{s}}^{(-1)}$ and $\mathfrak{s}^{(2)} = \tilde{\mathfrak{s}}^{(2)}$.

Now we consider $\tilde{\mathfrak{z}} = \{x \in \tilde{\mathfrak{s}};\ (x, \mathfrak{s}^{(1)}) = 0\}$. It is easy to see $\mathfrak{z} + j\mathfrak{z} \subset \tilde{\mathfrak{z}}$. Moreover, $0 = (x, \mathfrak{s}^{(1)})$ is equivalent to

$$0 = [x, y] + D(x)y - D(y)x \quad \text{for all } y \in \mathfrak{s}^{(1)}.$$

We know that $\operatorname{ad} y$ acts nilpotent on $\mathfrak{s}$ whence $D(y)=0$. It is clear that there exists $z \in \mathfrak{z}$ so that $D(x) \mid \mathfrak{s}^{(1)} = \operatorname{ad} jz \mid \mathfrak{s}^{(1)}$. Hence $[x - jz, y] = 0$ and $x - jz \in \mathfrak{z}$. This implies $x \in \mathfrak{z} + j\mathfrak{z}$ and therefore $\tilde{\mathfrak{z}} = \mathfrak{z} + j\mathfrak{z}$. It follows that $\mathfrak{z} + j\mathfrak{z}$ is a j-invariant ideal of $\tilde{\mathfrak{s}}$. As a consequence $\tilde{\mathfrak{s}} = \tilde{\mathfrak{z}} + \tilde{\mathfrak{w}}$ where $\tilde{\mathfrak{w}}$ is the orthogonal complement of $\tilde{\mathfrak{z}}$ in $\tilde{\mathfrak{s}}$ relative to $\tilde{\rho}$. We know that $D(x)$ leaves invariant $\tilde{\mathfrak{z}} = \mathfrak{z} + j\mathfrak{z}$ and is skew-adjoint relative to $\tilde{\rho}$. It therefore also leaves invariant $\tilde{\mathfrak{w}}$. Moreover, it is clear from Lemma 3.4.1 that $D(x)$ is a derivation of $\mathfrak{s}$ and of $\tilde{\mathfrak{s}}$.

We define a map $\tilde{D}: \tilde{\mathfrak{s}} \to \operatorname{Der} \tilde{\mathfrak{s}}$ by $\tilde{D} \mid \tilde{\mathfrak{z}} = 0$, $\tilde{D} \mid \tilde{\mathfrak{w}} = -D \mid \tilde{\mathfrak{w}}$. It is easy to see that $\tilde{D}$ is a modification map of $\tilde{\mathfrak{s}}$. As a matter of fact one can carry out the two modifications D and $\tilde{D}$ in one step by Lemma 3.6. Hence we can assume $\tilde{\mathfrak{z}} = \mathfrak{z} + j\mathfrak{z}$ is a Kähler ideal and $D(\tilde{\mathfrak{w}}) = 0$. This implies in particular that $\tilde{\mathfrak{w}}$ is a subalgebra of $\mathfrak{s}$. As a result $D(x) \mid \tilde{\mathfrak{w}}$ is skew-symmetric relative to ρ. We split $\tilde{\mathfrak{w}} = \tilde{\mathfrak{w}}_a + \tilde{\mathfrak{w}}_D$ according to our induction hypothesis. By Lemma 3.5.1 we have a basis $w_1, \ldots, w_l$ of $\tilde{\mathfrak{w}}_a$ so that $[jw_r, w_r] = 0$ for all r. Let $U = \exp t D(x)$ then $[U, j] = 0$ and $[j U w_r, U w_r] = 0$. Hence, by Lemma 3.4.4, we get $U \tilde{\mathfrak{w}}_a = \tilde{\mathfrak{w}}_a$ whence $U \tilde{\mathfrak{w}}_D = \tilde{\mathfrak{w}}_D$. Therefore $\tilde{\mathfrak{s}} = \tilde{\mathfrak{a}} + \tilde{\mathfrak{w}}_D$ where $\tilde{\mathfrak{a}} = \tilde{\mathfrak{z}} + \tilde{\mathfrak{w}}_a$. Because $D(x)$ leaves $\tilde{\mathfrak{z}}$, $\tilde{\mathfrak{w}}_a$ and $\tilde{\mathfrak{w}}_D$ invariant we see that $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{w}}_D$ are subalgebras of $\mathfrak{s}$. Moreover, $\tilde{\mathfrak{a}}$ is a modification of an abelian Lie algebra and $\tilde{\mathfrak{w}}_D$ is a modification of a normal j-algebra. Altogether $(\mathfrak{s}, \tilde{\rho})$ is the modification of the semidirect product of an abelian ideal and a normal j-algebra. But $(\mathfrak{s}, \rho)$ is also a Kähler algebra. Therefore, by Lemma 3.4.5, we get $\rho(\tilde{\mathfrak{a}}, \tilde{\mathfrak{w}}_D) = 0$. But because $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{w}}_D$ are subalgebras of $(\mathfrak{s}, \rho)$ we know that $D(x) \mid \tilde{\mathfrak{a}}$ *and* $D(x) \mid \tilde{\mathfrak{w}}_D$ are skew-adjoint. Therefore $D(x)$ is skew-adjoint relative to ρ whence $\rho = \tilde{\rho}$. Hence, after a modification, $\mathfrak{z}$ is a Kähler ideal of $\mathfrak{s}$ (which is a modification of an abelian Kähler algebra).

6.6. It is now very easy to finish the proof of the algebraic main theorem. In sections 4, 5 and 6 we have shown that—after carrying out a modification—the given Kähler algebra is the semidirect product of an abelian ideal and a normal j-algebra or it has a non-trivial Kähler ideal. In the last case we have $\mathfrak{s} = \mathfrak{z} + \mathfrak{w}$ where $\mathfrak{z}$ is a Kähler ideal which is a modification of an abelian Kähler algebra and $\mathfrak{w}$ a Kähler subalgebra, $\mathfrak{z} \perp \mathfrak{w}$. We write $\mathfrak{w} = \mathfrak{w}_a + \mathfrak{w}_D$ as usual. Then $\mathfrak{s} = (\mathfrak{z} + \mathfrak{w}_a) + \mathfrak{w}_D$ and it is easy to see now that $\mathfrak{s}$ is the modification of an abelian Kähler ideal on $\mathfrak{z} + \mathfrak{w}_a$ and a normal j-algebra on $\mathfrak{w}_D$. This finishes the proof of the main theorem.

7.1. In this section we collect some facts on solvable general Kähler algebras. For the notion of an effective Kähler algebra we refer to [7], Part II, 1.

LEMMA 1. — *Let* $\mathfrak{s}$ be an effective solvable general Kähler algebra with isotropy subalgebra $\mathfrak{k}$.

Then $\mathfrak{s}=\mathfrak{k}+\mathfrak{s}'$ *where* $\mathfrak{k}\cap\mathfrak{s}'=0$, $\mathfrak{k}$ *is abelian and* $\mathfrak{s}'$ *is an ideal of* $\mathfrak{s}$. Moreover, $\mathfrak{s}'$ *is naturally a Kähler algebra.*

Proof. — First note $\mathfrak{w}=\mathfrak{k}\cap[\mathfrak{s},\mathfrak{s}]=0$. We know that $\mathfrak{k}$ acts by semisimple endorphisms and $[\mathfrak{s},\mathfrak{s}]$ by nilpotent endomorphisms on $\mathfrak{s}$. Hence $\mathfrak{w}$ is contained in the center of $\mathfrak{s}$. But then $\mathfrak{w}$ is an ideal of $\mathfrak{s}$ which is contained in $\mathfrak{k}$. By assumption, $\mathfrak{s}$ is effective whence $\mathfrak{w}=0$. It is clear that $\mathfrak{k}+[\mathfrak{s},\mathfrak{s}]$ is $\mathfrak{k}$ invariant. Therefore there exists a $\mathfrak{k}$ invariant complementary subspace $\mathfrak{a}$ of $\mathfrak{s}$. Clearly $[\mathfrak{k}\,\mathfrak{k}]\subset\mathfrak{k}\cap[\mathfrak{s},\mathfrak{s}]=0$ and also $[\mathfrak{k},\mathfrak{a}]\subset\mathfrak{a}\cap[\mathfrak{s},\mathfrak{s}]=0$. We put $\mathfrak{s}'=\mathfrak{a}+[\mathfrak{s},\mathfrak{s}]$. Then $\mathfrak{s}'$ satisfies the first part of the lemma. We define a complex structure j' on $\mathfrak{s}'$ by projecting jx, $x\in\mathfrak{s}'$, to $\mathfrak{s}'$ along $\mathfrak{k}$. Thus $j'x=jx+k$ for some $k\in\mathfrak{k}$. We use (KA1) to (KA7) of [7], Part II, 1, to verify (1.1), (1.2) and (1.3). We put $\prec x, y\succ=\rho(jx, y)$, then $\prec .\,, .\succ$ is an inner product on $\mathfrak{s}'$. We note $\rho(jx, y)=\rho(j'x, y)$ for $x, y\in\mathfrak{s}'$ by (KA4). To verify (1.1) we compute $j'x=jx+k$ and

$$j'j'x=j'(jx+k)=j(jx+k)+k'=-x+jk+k'+k''.$$

By definition $j'x\in s'$ whence $jk+k'+k''=0$ and $(j')^2=-\text{id}$. Next we consider (1.2). Here we have

$$\begin{aligned}[j'x, j'y]=[jx+k, jy+k']&=[jx, jy]+[k, jy]+[jx, k']\\ &=j[jx, y]+j[x, jy]+[x, y]+j[k, y]+k_1+j[x, k']+k_2\\ &=j[jx+k, y]+j[x, jy+k']+[x, y]+k_1+k_2\\ &=j[j'x, y]+j[x, j'y]+[x, y]+k_1+k_2\\ &=j'[j'x, y]+j'[x, j'y]+[x, y]+k_1+k_2+k_3+k_4.\end{aligned}$$

Comparing terms in $\mathfrak{s}'$ *and* $\mathfrak{k}$ we see $k_1+k_2+k_3+k_4=0$. Finally, we note that (1.3) is trivially verified by (KA4) of [7].

In case $\mathfrak{s}$ acts on a homogeneous bounded domain we can sharpen the above result.

LEMMA 2. — *Let* $\mathfrak{s}$ *be a solvable general Kähler algebra with isotropy algebra* $\mathfrak{k}$ *and complex structure* j. *Assume that* (1.1) *and* (1.2) *hold and* $\mathfrak{s}$ *is the Lie algebra of a transitive group on a homogeneous bounded domain. Then in Lemma* 1 *we can choose* $\mathfrak{s}'$ *to be j-invariant.*

Proof. — First choose $\tilde{\mathfrak{s}}'$ as in Lemma 1. By construction we have $j'x=jx+k(x)$ for all $x\in\tilde{\mathfrak{s}}'$ where $k(x)\in\mathfrak{k}$. From (1.2) we get

$$\begin{aligned}[j'x, j'y]=[jx+k(x), jy+k(y)]&=[jx, jy]+[jx, k(y)]+[k(x), jy]\\ &=j[jx, y]+j[x, jy]+[x, y]+j[x, k(y)]+j[k(x), y]\\ &=j[j'x, y]+j[x, j'y]+[x, y]\\ &=j'[j'x, y]+j'[x, j'y]+[x, y]-k([j'x, y])-k([x, j'y]).\end{aligned}$$

Hence $k([j'x, y]+[x, j'y])=0$ for all $x, y \in \tilde{\mathfrak{s}}'$. From [4] we know that $\tilde{\mathfrak{s}}'=\tilde{\mathfrak{s}}'_0+\tilde{\mathfrak{s}}'_{-1/2}+\tilde{\mathfrak{s}}'_{-1}$ and the subspaces $\tilde{\mathfrak{s}}'_{-r}$, $r=0$, 1/2, 1, split into root spaces like a "normal j-algebra" relative to an algebra $\mathfrak{a} \subset \tilde{\mathfrak{s}}'_{-1}$ (note, however, that $j'a$, $a \in \mathfrak{a}$, does not necessarily have only real eigenvalues). Moreover, for $y \subset \tilde{\mathfrak{s}}'_{-1, \lambda}$, $y \notin \mathfrak{a}$, we have $k(\tilde{\mathfrak{s}}'_{-1, \lambda})=0$, *and also* $k(\tilde{\mathfrak{s}}'_{0, \lambda})=0$ if $\mathfrak{s}'_0 \cap j\mathfrak{a}=0$ and $k(\mathfrak{s}'_{-1/2})=0$. Let $\mathfrak{n}'$ be the sum of all root spaces of $\tilde{\mathfrak{s}}'$ that are not contained in $\mathfrak{a}+j'\mathfrak{a}$. Put $\mathfrak{s}'=\mathfrak{n}'+\mathfrak{a}+j\mathfrak{a}$; then $\mathfrak{s}'$ is an ideal in $\mathfrak{s}$ because $[\tilde{\mathfrak{s}}', \tilde{\mathfrak{s}}']=\mathfrak{n}'+\mathfrak{a}$ and $j'\mathfrak{a}+\mathfrak{k}$ is abelian and leaves $[\tilde{\mathfrak{s}}', \tilde{\mathfrak{s}}']$ invariant. Moreover, $\mathfrak{n}'$ and $\mathfrak{a}+j\mathfrak{a}$ are j-invariant, hence $\mathfrak{s}'$ is a j-invariant ideal of $\mathfrak{s}$.

7.2. We apply the results of 7.1 to Kähler manifolds.

LEMMA. — *Let* M *be a connected Kähler manifold and* S *a solvable transitive Lie group of holomorphic isometries. Then there exists a subgroup* S′ *of* S *so that* $M \cong S'/K$ *where* K *is discrete in* S′.

Proof. — Put $\mathfrak{s}=\mathrm{Lie}\,S$; then $\mathfrak{s}=\mathfrak{k}+\mathfrak{s}'$ with $\mathfrak{k}$ and $\mathfrak{s}'$ as in Lemma 7.1. Let S′ be the connected subgroup of S with $\mathrm{Lie}\,S'=\mathfrak{s}'$; then S′ and its isotropy subgroup K of some point of M have the desired properties.

COROLLARY. — *Let* M *be a connected, simply connected Kähler manifold admitting a transitive solvable group* S *of holomorphic isometries. Then* S *contains a simply transitive, simply connected, connected subgroup* S′.

7.3. We prove the geometric main result of this paper, the "fundamental conjecture" of [8] for manifolds admitting a solvable transitive group.

THEOREM. — *Let* M *be a connected Kähler manifold admitting a solvable transitive group of holomorphic isometries. Then* M *admits a holomorphic fibering the base of which is analytically isomorphic with a bounded homogeneous domain and each fiber is, with the induced Kähler structure, a locally flat Kähler manifold and it is homogeneous relative to the subgroup of* Aut M *leaving the fiber invariant.*

Proof. — By Lemma 7.2 we may assume that the isotropy subgroup of S is discrete. By S^* we denote the connected, simply connected Lie group with Lie algebra $\mathfrak{s}=\mathrm{Lie}\,S$. Then S^* is a h. k. m. (which we will denote by M^*) and also the universal cover of M. From 3.2 and Theorem 3.7 we get that M^* admits a split solvable transitive Lie group $\tilde{S}$. Hence by [7] we know that M^* admits an equivariant holomorphic fibering onto a bounded homogeneous domain B, $\pi: M^* \to B$ satisfying the assertions of the theorem. We know further that $\mathfrak{s}$ is the modification of $\tilde{\mathfrak{s}}=\mathrm{Lie}\,\tilde{S}$ via a modification map D.

By the choice of S and S^* we know $M=S^*/A_0^*$ where A_0^* is discrete in S^*. The natural projection of S^* onto M will be denoted by φ. We consider the isotropy representation $\lambda(a^*)$ of $a^* \in A_0^*$ in the tangent space $T_{\varphi(e)}M$ where e is the neutral element of S^*. By definition

$$\lambda(a^*)u=\lambda(a^*)\,d_e\varphi\,x=d_e a^*\,d_e\varphi\,x$$

$$=\left.\frac{d}{dt}\right|_0 a^*.\varphi(\exp tx)=\left.\frac{d}{dt}\right|_0 \varphi(a^*\exp tx\,a^{*-1})=d_e\varphi\,\mathrm{Ad}\,a^*x.$$

Hence only such $a^* \in S^*$ are admissible, for which $\mathrm{Ad}\, a^*$ acts orthogonal on $\mathfrak{s}$ and commutes with j. It is clear that $\mathrm{Ad}\, a^*$ is an automorphism of $\mathfrak{s}$. We decompose $\mathfrak{s} = \mathfrak{a} + \mathfrak{s}_D$ as usual. Then $0 = [j \,\mathrm{Ad}\, a^* Z_r, \mathrm{Ad}\, a^* Z_r]$ for Z_r as in Lemma 3.5.1. We write $\mathrm{Ad}\, a^* Z_r = a + x_D$ and get $[jx_D, x_D] + D(ja)x_D - D(a) jx_D = 0$. From [4], Lemma 3.5, we derive $x_D = 0$ whence $\mathrm{Ad}\, a^*$ leaves invariant $\mathfrak{a}$ and $\mathfrak{s}_D$. Next we prove that for each $g \in S^*$ we have $g = ahn$ where a is a product of $\exp a'$, $a' \in \mathfrak{a}$, h is a product of $\exp h'$, $h' \in \mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$ and $n = \exp n'$, $n' \in [\mathfrak{s}_D, \mathfrak{s}_D]$. We note that S^* is generated by $\{\exp x,\ x \in \mathfrak{a} \cup \mathfrak{h} \cup \mathfrak{n}\}$. We know $\exp h \exp a\, exp(-h) = \exp(\exp(\mathrm{ad}\, h)\, a)$ and $\exp(\mathrm{ad}\, h)\, a \in \mathfrak{a}$ because $D(a) h = 0$. Hence $\exp h \exp a = \exp a' \exp h$. Next, we consider

$$\exp(-a) \exp n \exp a = \exp(\exp(-\mathrm{ad}\, a)\, n) = \exp(a' + n')$$

where $a' \in \mathfrak{a}$, $n' \in [\mathfrak{s}_D, \mathfrak{s}_D]$ and $D(a') = 0$, $D(n') = 0$. Hence the Lie products of n' and $a' + n'$ in $\mathfrak{s}$ are the same as in the unmodified Lie algebra. But there $\mathfrak{a}$ is an ideal whence $\exp(a' + n') \exp(-n') = \exp a''$, $a'' \in \mathfrak{a}$. Althogether we have shown $\exp n \exp a = \exp a \exp a'' \exp n'$. Therefore we are able to bring all $\exp a$, $a \in \mathfrak{a}$, in the position as is claimed above. Finally, because $[\mathfrak{s}_D, \mathfrak{s}_D]$ is an ideal of $\mathfrak{s}_D$ we can move all $\exp n$ across $\exp h$. Hence we can represent each g as stated above. We apply this result to $\mathrm{Ad}\, a^*$. Hence $\mathrm{Ad}\, a^* = \mathrm{Ad}\, a\, \mathrm{Ad}\, h\, \mathrm{Ad}\, n$. But $\mathrm{Ad}\, a \mid \mathfrak{s}_D$ is orthogonal whence $\mathrm{Ad}\, h = \mathrm{Id}$. Now $\mathrm{Ad}\, a\, \mathrm{Ad}\, n$ is orthogonal; but $\mathrm{Ad}\, a$ gives only orthogonal contributions over $\mathfrak{s}_D$ and $\mathrm{Ad}\, n$ is unipotent. This implies $\mathrm{Ad}\, n = \mathrm{Id}$. Hence $\mathrm{Ad}\, a^* = \mathrm{Ad}\, a$. We know $\mathrm{Ad}\, h \mid \mathfrak{s}_D = \mathrm{Id}$ implies $h = \mathrm{Id}$ and also $\mathrm{Ad}\, n \mid \mathfrak{s}_D = Id$ *implies* $n = \mathrm{Id}$. Therefore $a^* = \exp a$, $a \in \mathfrak{a}$.

Now we consider A_0^* more closely. As in 3.2 we consider $\mathfrak{g} = \mathfrak{k} + \mathfrak{s} = \mathfrak{k} + \tilde{\mathfrak{s}}$ and split $\mathfrak{s} = \mathfrak{a} + \mathfrak{s}_D$ and $\mathfrak{s} = \tilde{\mathfrak{a}} + \tilde{\mathfrak{s}}_D$. The Lie product in $\mathfrak{g}$ will be denoted by $[.,.]$. We know $\mathfrak{s} = \{D(x) + x;\ x \in \tilde{\mathfrak{s}}\}$ and $[\mathfrak{g}, \mathfrak{g}] \subset \tilde{\mathfrak{s}}$. In the Lie transformation group G on M^* corresponding to $\mathfrak{g}$ we have $a^* = \tilde{a}k$ where $\tilde{a} = \exp \mathrm{ad}\, a$, $a \in \tilde{\mathfrak{a}}$, and $k \in \exp \mathfrak{k}$. Then $\mathrm{Ad}\, a^* \mid \mathfrak{s} = \mathrm{Ad}\, \tilde{a}\, \mathrm{Ad}\, k \mid \mathfrak{s}$. We know that $\mathrm{Ad}\, k$ leaves $\mathfrak{a}$ and $\mathfrak{s}_D$ invariant. Hence $\mathrm{Ad}\, \tilde{a}\, \mathfrak{s}_D \subset \mathfrak{s}_D$. But

$$\mathrm{Ad}(\exp \mathrm{ad}\, a)(D(x_D) + x_D) = D(x_D) + x_D - D(x_D)\, a + [a, x_D]$$

whence $[a, x_D] = D(x_D)\, a$. This implies $[a, x_D] = 0$ for $x_D \in [\mathfrak{s}_D, \mathfrak{s}_D]$ and for $x_D \in \mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$ we split $\tilde{\mathfrak{a}}$ into the eigenspaces relative to $\mathrm{ad}\, x_D \mid \tilde{\mathfrak{a}}$. We get $\mathrm{ad}\, a x_D = 0$ and $D(x_D)\, a = 0$. This shows $a \in \mathrm{center}\ \tilde{\mathfrak{s}}$.

We identify S^* with M^* via the base point e^*. Then the action of $\tilde{S}$ on M^* is given on S^* by $\tilde{x}.y = \tilde{x}y\tilde{k} = \tilde{x}\tilde{k}\tilde{k}^{-1}y\tilde{k}$ where $\tilde{k} \in \exp \mathfrak{k}$ is such that $\tilde{x}\tilde{k} \in S$. Hence for $a^* \in A^*$ we have $\tilde{x}.ya^* = \tilde{x}y\tilde{a}k\tilde{k} = \tilde{x}y\tilde{k}\tilde{a}\tilde{k} = \tilde{x}y\tilde{k}a^*$ where we have used that $\tilde{a}$ is in the center of $\tilde{S}$ and $\exp \mathfrak{k}$ is commutative. This shows that $\tilde{S}$ leaves the right cosets of S^* modulo A_0^* invariant. Hence the group action of $\tilde{S}$ can be pushed down to M. The isotropy group $\tilde{A}_0$ of this action is given by

$$\tilde{A}_0 = \{\tilde{x} \in \tilde{S};\ \tilde{x}.A_0^* \subset A_0^*\} = \{\tilde{x} \in \tilde{S};\ \tilde{x}\tilde{k} \in A_0^*\}.$$

Finally, we consider the semidirect product $\tilde{S}_D \times \tilde{A}$ where $\tilde{S}_D$ and $\tilde{A}$ correspond to $\tilde{\mathfrak{s}}_D$ and $\tilde{\mathfrak{a}}$. It is easy to see that the natural map into the semidirect product $\tilde{S}_D \times \tilde{A}/\tilde{A}_0$ is a homomorphism which induces the identity map on $\tilde{\mathfrak{s}}$. The kernel is clearly $\{id\} \times \tilde{A}_0$. Therefore $M = \tilde{S}/\tilde{A}_0 \cong \tilde{S}_D \times \tilde{A}/\tilde{A}_0$. But the last group is split solvable and we can apply [7], Part II, Theorem 2. Hence M satisfies the fundamental conjecture.

7.4. In this section we apply our results to Kählerian NC-algebras. Let $(\mathfrak{s}, j, \prec ., . \succ$ be a solvable Kähler algebra. We decompose $\mathfrak{s}$ as usual $\mathfrak{s} = \mathfrak{a}^* + \mathfrak{s}_D^*$. The Lie product in $\mathfrak{s}$ will be denoted by $[., .]$ and the Lie product in the unmodified Lie algebra will be denoted by $(., .)$. Hence $[x, y] = (x, y) + D(x) y - D(y) x$ where D is a modification map in the sense of section 3.

We want to determine when $\mathfrak{s}$ is an NC-algebra. For a definition we refer to [2], 3.4. We will use the notation of [2]. We point out that now $\mathfrak{a}$ denotes the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$ and $\mathfrak{a}^*$ corresponds to the abelian part of the Kähler algebra $\mathfrak{s}$.

LEMMA. — *The solvable Kähler algebra* $(\mathfrak{s}, j, \prec ., . \succ)$ *is an* NC-*algebra and* $\prec ., . \succ$ *is an admissible inner product if and only if* $(\mathfrak{a}^*, \mathfrak{s}) = 0$. *Its flat part is* $\mathfrak{a}^*$. *The corresponding simply connected, connected h. k. m. is a product of* $\mathbb{C}^n$ *with a homogeneous bounded domain.*

Proof. — First we consider $\mathfrak{a} = \mathfrak{s} \ominus \mathfrak{n}$. It is clear that we have $\mathfrak{a} \subset \mathfrak{h} + a^*$ where $\mathfrak{h} = \mathfrak{s}_D \ominus [\mathfrak{s}_D, \mathfrak{s}_D]$. The weight spaces of $\mathfrak{h}$ on $\mathfrak{a}$ for weights with non-zero real part are contained in $[\mathfrak{s}, \mathfrak{s}]$. Hence $\mathfrak{a} \subset \mathfrak{h} + \mathfrak{a}_0^*$ where

$$\mathfrak{a}_0^* = \{a \in \mathfrak{a}^*;\ (h, a) = 0 \text{ for all } h \in \mathfrak{h}\}.$$

For $a \in \mathfrak{a}_0^*$ we have $[h, a] = D(h) a$. Hence $\mathfrak{a} \subset \mathfrak{h} + \mathfrak{a}_{00}^*$ where

$$\mathfrak{a}_{00}^* = \{a \in \mathfrak{a}^*;\ (h, a),\ D(h) a = 0\}.$$

In particular, we have $[\mathfrak{h}, \mathfrak{a}_{00}^*] = 0$. We define $\hat{\mathfrak{a}}_0^*$ and $\hat{\mathfrak{a}}_1^*$ as in Lemma 3.3.1. We get $\mathfrak{a} = \mathfrak{h} + \mathfrak{a}_{00}^* \cap \hat{\mathfrak{a}}_1^* \cap [\mathfrak{s}, \mathfrak{s}]^{\perp}$. This proves that $\mathfrak{a}$ is abelian. Now we consider the generalized root space decomposition defined in [2], 3.3. For $a \in \mathfrak{a}$ we have

$$[a, b+n] = [a, n] = (a, n) + D(a) n.$$

Hence $\alpha(a) \neq 0$ only if $a = b + h$, $b \in \mathfrak{a}^*$, $h \in \mathfrak{h}$ and $h \neq 0$. In particular $\alpha(a) = 0$ for all $a \in \mathfrak{a} \cap \mathfrak{a}^*$. Hence (ii) of [2], 3.4, is satisfied if and only if $(a, \mathfrak{s}_D) = 0$. This implies that $\mathfrak{a} \ominus \mathfrak{h}$ is contained in the center of the unmodified Lie algebra. We consider (iii). It is clear that we may assume $H_0 \in \mathfrak{h}$. Moreover, the condition $\alpha(H_0) > 0$ for all $\alpha \neq 0$ for which $\alpha + i\beta$ is a root implies $H_0 = \sum_{r=1}^{l} \sigma_r H_r$ where $H_1, \ldots, H_l$ is a complete set of "minimal orthogonal idempotents", i.e. $[H_r, j H_s] = \delta_{rs} H_r$ and $H_1 + \ldots + H_l = js$ where s is the principal idempotent of $\mathfrak{s}_D$. Let $a \in \mathfrak{a}^*$ be a common eigenvector for $\mathfrak{h}$ relative to $(., .)$. Assume $(H_m, a) = -1/2\, a$. We apply the classification of symplectic representations [7], p. 234, to H_m, H_r and $H_m + H_r$, $r \neq m$. Hence $(H_r, a) = \lambda a$ and $\lambda \in \{0, 1/2\}$. If $\lambda = 1/2$ then $(H_m + H_r, a) = 0$ whence $(j H_m, a) = ja$, $(j H_r, a) = 0$ and $(j(H_m + H_r), a) = 0$, a contradiction. Hence $(H_m, a) = -1/2\, a$ implies $(H_r, a) = 0$ for all

$r \neq m$. Therefore $(\mathrm{H}_0, a) = -1/2\,\sigma_m$. But $\sigma_m = (\mathrm{H}_0, j\,\mathrm{H}_m) > 0$ by assumption. This shows that (iii) is satisfied only if each H_m has only non-negative eigenvalues on $\mathfrak{a}^*$. But j interchanges the eigenspaces for the eigenvalues $1/2$ and $-1/2$ and $(\mathrm{H}_m, \mathfrak{a}^*) = 0$ for all $1 \leqq m \leqq l$ follows. From [7], Part III, Lemma 3, we now derive $(\mathfrak{a}^*, \mathfrak{s}_\mathrm{D}) = 0$. Hence $\mathfrak{a}^*$ is the center of the unmodified Lie algebra. Let us now assume $(\mathfrak{a}^*, \mathfrak{s}) = 0$. Then (iii) is clearly satisfied. From the above it is also clear that (iii) implies (ii). Next we consider (iv). But from the definition of $\mathfrak{n}_0$ it is clear $\mathfrak{n}_0 \subset \mathfrak{a}^*$. Because $\mathrm{D}(\mathfrak{n}_0) = 0$ and $\mathrm{D}(\mathfrak{n}) = 0$ we have $[\mathfrak{n}_0, \mathfrak{n}] = (\mathfrak{n}_0, \mathfrak{n}) \subset (\mathfrak{a}^*, \mathfrak{n}) = 0$. To verify (v) it suffices to note that $n^0_{\alpha, \beta}$ is $\mathfrak{a}$-invariant and that $\mathrm{ad}\, a$, $a \in \mathfrak{a}$, is the sum of a self-adjoint and a skew-adjoint endomorphism. This proves incidentally that $\prec .\,, . \succ$ is admissible [2], 3.7. Now it is easy to see that $\mathfrak{n}_0 + \mathfrak{a}_0 = \mathfrak{a}^*$ holds where $\mathfrak{n}_0$ and $\mathfrak{a}_0$ are defined as in [2], 4.3. Finally, let M be the connected, simply connected Kähler manifold associated with $\mathfrak{s}$. From 3.2 and Theorem 3.7 we know that the Lie group S which is generated by the unmodified Lie algebra acts simply transitive on M. But the unmodified Lie algebra is the direct product of $\mathfrak{a}^*$ with the normal j-algebra $\mathfrak{s}_\mathrm{D}$. This finishes the proof.

COROLLARY 1. — *Each Kählerian* NC-*algebra without flat part is a modification of a normal j-algebra.*

COROLLARY 2. — *Let* M *be a connected homogeneous Kähler manifold. Assume that* M *has non-positive sectional curvature. Then* $\mathrm{M} \cong \mathbb{C}^n/\mathrm{L} \times \mathrm{D}$ *where* L *is a discrete* $\mathbb{Z}$*-module in* $\mathbb{C}^n$ *and* D *is a bounded homogeneous domain.*

Proof. — Let M* be the universal cover of M. Then M* is a homogeneous riemannian manifold with non-positive curvature. Then by [1], Proposition 2.5, we get a solvable simply transitive group S of holomorphic isometries for M*. Hence $\mathfrak{s} = \mathrm{Lie}\,\mathrm{S}$ is a solvable Kähler algebra; but [1], 6.3, shows that it is also an NC-algebra. Hence $\mathrm{M}^* = \mathbb{C}^n \times \mathrm{D}$ by the lemma above. From the proof of Theorem 7.3 we get $\mathrm{M} \cong \tilde{\mathrm{S}}_\mathrm{D} \times \tilde{\mathrm{A}}/\tilde{\mathrm{A}}_0$ where the right handside is under our assumptions here a product of groups which correspond to unmodified Lie algebras and the assertion follows.

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Added in proof: 1. Theorem 1.4 is also contained in K. Nakajima, J-algebras and homogeneous Kähler manifolds (to appear).

2. Professor Xu Yichao Kindly sent me a different (and somewhat more direct) proof for the results of section 2.