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THE CALCULUS OF BOUNDARY PROCESSES

By Jean-Michel BISMUT

ABSTRACT. — This paper is a systematic study of the transition probabilities of the boundary processes associated to a class of reflecting diffusions. The main tools are the theory of stochastic flows, the Malliavin calculus of variations on diffusions, the calculus of variations on jump processes, the Itô theory of excursions and the stochastic calculus on continuous and non continuous semi-martingales. The smoothness of the boundary semi-group is related to the degree of degeneracy of the second-order differential operator defining the diffusion at the boundary.

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The purpose of this paper is to study the semi-groups associated to a class of jump processes, which are the boundary processes of reflecting diffusions. The main technique is the stochastic calculus of variations. Since we are using this calculus on diffusions and on Poisson point processes, we start by giving a brief history of this technique.

Consider the stochastic differential equation:

$$dx = X_0(x)\,dt + \sum_{i=1}^{m} X_i(x)\,.\,dw^i, \qquad x(0) = x_0, \tag{0.1}$$

where $X_0, X_1, \ldots, X_m$ are smooth vector fields, and $w = (w^1, w^2, \ldots, w^m)$ is a Brownian motion. Here (0.1) is taken in the sense of Stratonovitch [31], so that its infinitesimal generator is the second order differential operator $\mathscr{L}^0$ given by:

$$\mathscr{L}^0 = X_0 + 1/2 \sum_{i=1}^{m} X_i^2, \tag{0.2}$$

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$\mathscr{L}^0$ is written in the form of Hörmander [15]. If $x \in \mathrm{R}^d$, the transition probability $p_t(x, dy)$ is a solution (in the sense of distributions) of the forward Fokker-Planck equation:

$$\frac{\partial p}{\partial t} - \mathscr{L}_y^{0*} p = 0, \qquad p_0 = \delta_x,$$

where $\mathscr{L}^{0*}$ is the adjoint of $\mathscr{L}^0$ with respect to the Lebesgue measure.

The existence of smooth densities for $p_t(x, dy)$ can be obtained by using Hörmander'-theorem on the hypoellipticity of second-order differential operators ([15], [24], [42]).

In [29] and [30], Malliavin described a purely probabilistic method to prove the smoothness of $p_t(x, dy)$. If (Ω, P) is the probability space of the Brownian motion w, he showed that it was possible to obtain an integration by parts formula on (Ω, P). Malliavin used as a main tool the Ornstein-Uhlenbeck operator $\mathscr{A}$, which is an unbounded self-adjoint operator on $\mathrm{L}_2(\Omega, \mathrm{P})$, and the associated Ornstein-Uhlenbeck process. Still using the Ornstein-Uhlenbeck operator, Shigekawa [34], Stroock ([36], [37], [38]), Ikeda-Watanabe [17] simplified and extended Malliavin's original approach. In particular the estimates which give the smoothness of $p_t(x, dy)$ were obtained in Malliavin [30], Ikeda-Watanabe [17] and improved in Kusuoka-Stroock ([26], [38]) where the full Hörmander Theorem was in fact obtained.

Another approach to the Malliavin calculus was suggested by us in [7]. Instead of relying on the Ornstein-Uhlenbeck operator, it uses the Girsanov transformation on diffusions [39]. An integration by parts formula is then derived, which is also a consequence of a result of Haussmann [14] concerning the representation of Fréchet differentiable functionals of the trajectory x as stochastic integrals with respect to the Brownian motion w.

The Malliavin calculus on diffusions gave the result that if $\mathscr{L}^0$ is well behaved on a neighborhood of the starting point x, then for $t > 0$, $p_t(x, dy)$ is smooth on R^d, which is a result which is not a consequence of Hörmander's theorem. By using a localization procedure, Stroock [36] was able to use this first result to prove that if $\mathscr{L}^0$ is well behaved on a neighborhood of y, then $p_t(x, dy)$ is smooth on this neighborhood, this last result being a consequence of Hörmander's theorem. As we shall later see, for boundary processes—which have non-local generators—smoothness does not propagate in a similar way from the starting point [*see* section 1 (f)].

Among the applications of the Malliavin calculus which will be useful to us, let us mention the work of Bismut-Michel [10] on conditional diffusions. In [10], results on conditional diffusions are obtained by doing the variation only on certain components of the Brownian motion w.

In [8], we developed a calculus of variations for jump processes. Namely, we considered in [8] the equation:

$$(0.3) \qquad x_t = x + \int_0^t \mathrm{X}_0(x_s)\, ds + y_t,$$

where y_t is an independent increment jump process, whose probability law is modified using the Girsanov transformation on jump processes in Jacod [19]. An integration by parts formula on an infinite dimensional non Gaussian probability space is obtained in [8]. The estimates which are necessary to study the regularity of the semi-groups associated to such jump processes are very different from the corresponding ones for classical diffusions. In particular it can take a strictly positive time for the transition probability $p_t(x, dy)$ to get C^0, then later C^1...

We do four remarks on the results of [8].

(*a*) Although the law of the process $x.$ given by (0.3) is modified by a Girsanov transformation in such a way that $y.$ is no longer an independent increment process, still the basic work is done on a probability space where $y.$ has independent increments. This makes that the Lévy measure $M(x, dy)$ depends in a "weak" way of x.

(*b*) In [8], advantage is taken of the vector space structure of R^d, so that the various jumps are "added" to each other. This prevents us from working on a manifold, or to work with a Lévy kernel $M(x, dy)$ strongly depending on x.

(*c*) In principle it would be possible to use the technique of [8] to study more general stochastic differential equations with jumps introduced by Skorokhod and studied in Jacod [19]. However technical difficulties do arise, essentially because the jumps destroy the local differential structure of R^d.

(*d*) Even working as in (*c*), it would be difficult to describe non trivial interactions between a vector field X_0 and a non-local operator $\mathscr{M}$ so that the Markov process whose generator is $X_0+\mathscr{M}$ would be given by densities, while the Lévy measure $M(x, dy)$ associated to $\mathscr{M}$ would be concentrated on submanifolds (depending on x).

However there is a large class of jump processes which are naturally associated to continuous diffusions. Namely let D be an open domain in R^d with a smooth boundary ∂D. Let x_t be a diffusion in R^d which is either non-reflecting or reflecting on ∂D. If L_t is one local time of x on ∂D, if A_t is its right-continuous inverse, then x_{A_t} and (A_t, x_{A_t}) are strong Markov processes, which are in fact jump processes. Such boundary processes were used by Stroock-Varadhan [39] to prove uniqueness for certain diffusions with boundary conditions.

In this paper, we study the semi-groups of a class of such boundary processes. Namely in the first five Sections, we consider the stochastic differential equation:

$$(0.4) \qquad dx = X_0(x, z)\,dt + D(x)\,dL + \sum_{i=1}^{m} X_i(x, z) . dw^i, \qquad x(0) = x_0$$

where z is a reflecting Brownian motion, $w=(w^1 \ldots w^m)$ is a Brownian motion independent of z, L is the local time at 0 of z, D, $X_0 \ldots X_m$ are smooth vector fields. A Girsanov transformation is also performed on z so as to introduce a drift on z. If A_t is the right-continuous inverse of L, we study the semi-group associated to the Markov process (A_t, x_{A_t}).

The typical model for such a problem is the case where $X_0 \ldots X_m$ do not depend on z. If $\mathscr{L}^0$ is the operator given by (0.2), the formal generator of the boundary process is given in this case by:

$$D - \sqrt{-2\left(\frac{\partial}{\partial t} + \mathscr{L}^0\right)}. \tag{0.5}$$

If instead of (0.4) we had considered the stochastic differential equation:

$$dx = z^{(1/\beta)-2} X_0(x)\, dt + D(x)\, dL + (\sqrt{z})^{(1/\beta)-2} \sum_{i=1}^{m} X_i(x) . dw^i, \tag{0.6}$$

where $0<\beta<1$, using the results in Itô-McKean [18] (p. 226), the formal generator associated to the process (A_t, x_{A_t}) is:

$$D - k_\beta \left[-\frac{\partial}{\partial t} - \mathscr{L}^0 \right]^\beta,$$

where k_β is a given >0 Const. However we would have been forced to work with non-smooth coefficients in the variable z. Although this would not be a serious difficulty, we have prefered to work in the whole text with smooth vector fields.

Section 1 is devoted to the explicit construction of the diffusion (0.4). For this construction, we closely follow Ikeda-Watanabe [17]. To (0.4), we associate a continuous flow of diffeomorphisms such that in (0.4), $x_t = \varphi_t(\omega, x_0)$, by using the techniques in Bismut ([5], [6]), Kunita [25]. In particular it is shown that it is of critical importance to study the process (A_t, x_{A_t}) and not x_{A_t}, if we want that smoothness propagates in a nice way.

In section 2, the calculus of variations is performed on the Brownian motion w, so that the reflecting Brownian motion z does not vary. The technique is very close to what is done in Bismut [7], Bismut-Michel [10]. Explicit computations are very similar to [7]. As in Malliavin ([29], [30]), we make appear a process $C_t^{x_0}$ valued in the set of (d, d) symmetric nonnegative matrices. Contrary to what happens for the semi-groups associated to hypoelliptic diffusions, the boundary semi-groups may well be slowly regularizing, in the same way as the classical jump processes studied in [8]. This last case is labelled "non localizable" for reasons which will clearly appear in section 5. We consider two cases:

(*a*) The case where for any $t>0$, $T \geqq 0$, $1_{A_t \leqq T} |[C_{A_t}^{x_0}]^{-1}|$ is in all the $L_p (1 \leqq p < +\infty)$. Theorems 2.4 and 4.9 show then that the boundary semi-group is given by C^∞ densities. This is the "localizable" case.

(*b*) The case where for a given $t>0$, for any $T \geqq 0$, x_0, $1_{A_t \leqq T} |[C_{A_t}^{x_0}]^{-1}|$ is in one given $L_q (q>2)$. Theorems 2.5, 4.11 and 4.12 show that the boundary semi-group is slowly regularizing. This is the "non localizable" case.

The conclusion of section 2 is that we know how to control the derivatives (in the sense of distributions) in the variable y of the transition probability $p_t(da, dy)$.

Section 3 is preparing for section 4, where the derivatives of $p_t(da, dy)$ in the variable a are controlled. Section 3 also contains results of independent interest. In fact by the Itô's theory of excursions (*see* Itô-McKean [18], Ikeda-Watanabe [17]), we know that the excursions of z out of 0 and the corresponding trajectory of w define a Poisson point process. If $\{\bar{F}_t\}_{t\geqq 0}$ is the natural filtration of (w, z), $\{\bar{F}_{A_t}\}_{t\geqq 0}$ is the natural filtration of this point process. In section 3, we "embed" the stochastic calculus on $\{\bar{F}_{A_t}\}_{t\geqq 0}$ martingales—which are purely discontinuous martingales—in the calculus on $\{\bar{F}_t\}_{t\geqq 0}$ martingales—which are continuous. The effect of a Girsanov transformation is considered on the $\{\bar{F}_t\}_{t\geqq 0}$ and the $\{\bar{F}_{A_t}\}_{t\geqq 0}$ stochastic calculus, so that the results in Jacod [19] on the effect of the Girsanov transformation on point processes, and more classical results on the Girsanov transformation for the Brownian motion [40] are shown to be deeply related. Section 3 also sheds some light on the computations of section 4.

In section 4, we develop a calculus of variations on the reflecting Brownian motion z, in order to control the differentials of $p_t(da, dy)$ in the variable a. This calculus is based on the characterization by Skorokhod of the reflecting Brownian motion. Even if we still use the Itô stochastic calculus, we show that what we do is in fact a variation on each excursion of the Poisson point process associated to z, i. e. something very similar to what we did in our work [8]. Still the possibility of using the continuous time stochastic calculus considerably simplifies the computations in comparison with an earlier version of this paper, where the stochastic calculus on jump processes was explicitly used. The main consequence of section 4 is to show that to each sort of random variable corresponds one possible calculus of variations, so that several differentiable structures can be put on the same probability space, in order to study different random variables.

In section 5, we start giving conditions under which $C_t^{x_0}$ is a. s. invertible, which implies that the boundary semi-group has densities. Non trivial interactions between D and $(X_0 \ldots X_m)$ are exhibited so that densities exist even if the Levy kernels are fully degenerate. It is a remarkable feature of the problem that the interaction between D and the Lévy kernel of the boundary process is expressed through the differential operator which in fact defines the Lévy kernel, and not by just looking at the global behavior of the Lévy kernel. This explains the difficulty we had in [8] to exhibit such an interaction by direct methods, i. e. by constructing from scratch a vector field D and a Lévy kernel $M(x, dy)$ such that such an interaction would appear.

The regularity of the boundary semi-group is also studied. The critical degeneracy of the diffusion $x.$ on the boundary is found so as to ensure that if on a neighborhood of x, the diffusion $x.$ is strictly less degenerate than the critical degeneracy, the boundary semi-group is C^∞, while if $x.$ is everywhere degenerate at the critical degeneracy level, the boundary semi-group is slowly regularizing. The estimates of Malliavin [30], Ikeda-Watanabe [17] and Kusuoka-Stroock ([26], [38]) for standard diffusions are used in the whole section.

In section 6, the reflecting Brownian motion is changed into a standard Brownian motion, so that the diffusion $x.$ is governed by the differential operator $\mathscr{L}$ in the region $(z>0)$, by the operator $\mathscr{L}'$ in the region $(z<0)$. If $\mathscr{L}$, $\mathscr{L}'$ do not depend on z, the formal generator of the boundary process is exactly:

$$D-\frac{1}{2}\left[-2\left(\frac{\alpha\,\partial}{\partial t}+\mathscr{L}\right)\right]^{1/2}-\frac{1}{2}\left[-2\left(\frac{\alpha'\,\partial}{\partial t}+\mathscr{L}'\right)\right]^{1/2},$$

where α, α' are >0 constants.

The estimates are drastically changed by the introduction of two sorts of excursions of z. In particular the non local nature of the perturbation introduced by the negative excursions can destroy the smoothness of the semi-group when only positive excursions appear. The Arcsine law of P. Lévy (Itô-McKean [18], p. 57) gives us a good illustration of this phenomenon.

When the union of some Hörmander-like distributions associated to $\mathscr{L}$ and $\mathscr{L}'$ span R^d in a uniform way, we still prove that the boundary semi-group is smooth. Apparently, the classical $\{\bar{F}_t\}_{t\geq 0}$ stochastic calculus is not good enough to obtain the necessary estimates. We have to rely on estimates on each individual excursion of z, the global effect of piling up the excursions being analysed using the stochastic calculus on Poisson point processes.

Some of the problems considered in this paper apparently fall out of the reach of existing methods in analysis for two reasons:

(*a*) They are very degenerate.

(*b*) The generators of the boundary processes which we consider are not necessarily pseudo-differential operators (*see* [42]) since they may well be not smooth out of the diagonal (this is the case in section 6).

The techniques given here would apply without much change to study the harmonic measures of a diffusion. Let us just say that the regularization effects are more interesting to study on boundary processes. The case where the reflecting diffusion also diffuses on the boundary has also been left aside.

In the whole text $C_b^\infty(R^k)$ [resp. $C_c^\infty(R^k)$] is the space of real functions defined on R^k which are C^∞ with bounded differentials (resp. which are C^∞ and have compact support). The spaces L_p are only considered for $1\leq p<+\infty$ (i. e. $p=+\infty$ is systematically excluded). The constants which appear in *a priori* bounds will be written C, even when they vary from place to place.

The results of this paper have been announced and commented in [50].

1. The boundary process

In this section, we define the boundary process, which will be the object of our study.

In paragraph (*a*), the main notations are introduced. In (*b*), the theory of stochastic flows (Bismut [5], Kunita [25]) is applied to the considered reflecting diffusion. In (*c*), a Girsanov transformation is performed on the diffusion; related technical problems are discussed in (*d*). In (*e*), the boundary process is defined, and technical details are discussed in (f). Connections with the theory of hypoelliptic second-order differential operators are underlined in (g).

(a) NOTATIONS AND ASSUMPTIONS. — m is $a>0$ integer.

Ω (resp. Ω') is the space $\mathscr{C}(\mathrm{R}^+; \mathrm{R}^m)$ [resp. $\mathscr{C}(\mathrm{R}^+; \mathrm{R}^+)$] of continuous functions defined on R^+ with values in R^m (resp. R^+). The trajectory of $\omega \in \Omega$ (resp. $\omega' \in \Omega'$) is written $w_t = (w_t^1, \ldots, w_t^m)$ (resp. z_t).

The σ-field F_t in Ω (resp. F'_t in Ω') is defined by $\mathrm{F}_t = \mathscr{B}(w_s \mid s \leqq t)$ [resp. $\mathrm{F}'_t = \mathscr{B}(z_s \mid s \leqq t)$]. Ω (resp. Ω') is endowed with the filtration $\{\mathrm{F}_t\}_{t \geqq 0}$ (resp. $\{\mathrm{F}'_t\}_{t \geqq 0}$).

$\bar{\Omega}$ is the space $\Omega \times \Omega'$, whose standard element is $\bar{\omega} = (\omega, \omega')$. $\bar{\mathrm{F}}_t$ is the σ-field $\mathrm{F}_t \otimes \mathrm{F}'_t$, and $\{\bar{\mathrm{F}}_t\}_{t \geqq 0}$ is the associated filtration.

All the filtrations considered in this paper will be eventually regularized on the right and completed as in Dellacherie-Meyer [11] without further mention and with no explicit notation. Difficulties which may arise in this respect will be underlined when necessary.

For $s \in \mathrm{R}^+$, θ_s (resp. θ'_s) is the mapping from Ω into Ω (resp. from Ω' into Ω') defined by:

$$\omega = (w_t) \to \theta_s \omega = (w_{s+t} - w_s)$$

[resp. $\omega' = (z_t) \to \theta'_s \omega' = (z_{s+t})$].

$\bar{\theta}_s$ is the mapping from $\bar{\Omega}$ into $\bar{\Omega}$ given by:

$$\bar{\theta}_s(\omega, \omega') = (\theta_s(\omega), \theta'_s(\omega')).$$

P is the Brownian measure on Ω, such that $\mathrm{P}(w_0 = 0) = 1$.

For $z \in \mathrm{R}^+$, P'_z is the probability measure on Ω' associated to the reflecting Brownian motion on $[0, +\infty[$ starting at z, i.e. such that $\mathrm{P}'_z(z_0 = z) = 1$ (Itô-McKean [18], p. 40, Ikeda-Watanabe [17], p. 119). For notational convenience, when $z = 0$, we shall write P' instead of P'_0.

On $(\Omega', \{\mathrm{F}'_t\}_{t \geqq 0}, \mathrm{P}'_z)$, L_t denotes the local time at 0 of z_t. By [17], p. 120, we know that the process B_t defined by:

$$(1.1) \qquad \mathrm{B}_t = z_t - z - \mathrm{L}_t,$$

is a Brownian martingale such that $\mathrm{B}_0 = 0$.

Moreover it is standard ([17], p. 122) that B_t generates the same filtration as z_t. In particular, on (Ω', P'), we have:

$$(1.2) \qquad \mathrm{L}_t = \sup_{0 \leqq s \leqq t} (-\mathrm{B}_s).$$

d is $a>0$ integer. $y = (x, z)$ is the standard element in R^{d+1}, with $x \in \mathrm{R}^d$, $z \in \mathrm{R}$. In the sequel, R^d will be identified to the subspace $\mathrm{R}^d \times \{0\}$ in R^{d+1}, i.e. $\mathrm{X} \in \mathrm{R}^d$ is identified to $(\mathrm{X}, 0)$ in R^{d+1}. π is the projection operator $(x, z) \in \mathrm{R}^{d+1} \to x \in \mathrm{R}^d$.

$\mathrm{X}_0(x, z)$, $\mathrm{X}_1(x, z) \ldots \mathrm{X}_m(x, z)$ are $m+1$ vector fields defined on R^{d+1} with values in R^d, which are C^∞, bounded, whose all differentials are bounded.

$D(x)$ is a vector field defined on R^d with values in R^d, which is C^∞, bounded, whose all differentials are bounded.

$b(x, z)$ is a function defined on R^{d+1} with values in R, which belongs to $C_b^\infty(R^{d+1})$.

If X_t is a continuous semi-martingale (defined on any given filtered probability space), dX denotes its differential in the sense of Stratonovitch, and δX its differential in the sense of Itô (*see* Meyer [31]).

If h is a C^∞ diffeomorphism of R^d onto R^d and if $K(x)$ is a tensor field on R^d, $(h^{*-1}K)(x)$ denotes the tensor-field on R^d obtained by taking the pull-back at x of $K(h(x))$ through the differential $\partial h/\partial x(x)$ (for this notation *see* [5], [7], [10]). In particular if $Y(x)$ is a vector field:

$$(1.3) \qquad (h^{*-1}Y)(x)=\left[\frac{\partial h}{\partial x}(x)\right]^{-1} Y(h(x)).$$

(*b*) *The reflecting process and its associated flow.*

We now build a reflecting process as in Ikeda [16], Watanabe [43], Ikeda-Watanabe [17], p. 203.

Fix $(x_0, z_0)\in R^d\times R^+$. On $(\bar\Omega, P\otimes P'_{z_0})$, we consider the stochastic differential equation:

$$(1.4) \qquad dx=X_0(x, z)\,dt+D(x)\,dL+\sum_1^m X_i(x, z).\,dw^i, \qquad x(0)=x_0$$

(where dw^i is the Stratonovitch differential of w^i).

(1.4) can be written in the equivalent Itô's form:

$$(1.4') \qquad dx=\left(X_0+\frac{1}{2}\frac{\partial X_i}{\partial x}X_i(x, z)\right)dt+D(x)\,dL+X_i(x, z).\,\delta w^i, \qquad x(0)=x_0$$

$\left(\text{from now on, all the summation signs } \sum_1^m \text{ will be omitted}\right)$.

Of course, (1.4) has an essentially unique solution. But more can be said. Namely:

THEOREM 1.1. — *There exists a mapping defined on $\bar\Omega\times R^+\times R^d$ with values in R^d $(\bar\omega, t, x)\to\varphi_t(\bar\omega, x)$ having the following properties.*

(*a*) *For every $(t,x)\in R^+\times R^d$, $\bar\omega\to\varphi_t(\bar\omega, x)$ is measurable, and for every $\bar\omega\in\bar\Omega$, $(t, x)\to\varphi_t(\bar\omega, x)$ is continuous.*

(*b*) *For any $\bar\omega\in\bar\Omega$, $\varphi_0(\bar\omega, .)$ is the identity mapping of R^d.*

(*c*) *For any $\bar\omega\in\bar\Omega$, $t\to\varphi_t(\bar\omega, .)$ is a family of C^∞ diffeomorphisms of R^d onto R^d, which depends continuously on $t\in R^+$ for the topology of uniform convergence of C^∞ functions and their derivatives on the compact sets of R^d.*

(*d*) *For any $z_0\in R^+$, on $(\bar\Omega, P\otimes P'_{z_0})$, for any $x_0\in R^d$, $\varphi_t(\bar\omega, x_0)$ is the essentially unique solution of equation* (1.4).

(*e*) *For any* $z_0 \in R^+$, *any compact set* K *in* $R^+ \times R^d$, *any multi-index m, for any* $n \in N$ *and any* $p \geqq 1$, *the random variables:*

$$(1.5)\qquad 1_{L_t \leqq n} \sup_{(t,\,x) \in K} \left| \frac{\partial^m \varphi_t}{\partial x^m}(\bar{\omega}, x) \right|, \qquad 1_{L_t \leqq n} \sup_{(t,\,x) \in K} \left| \left[\frac{\partial \varphi_t}{\partial x}(\bar{\omega}, x) \right]^{-1} \right|,$$

are in $L_p(\bar{\Omega}, P \otimes P'_{z_0})$, *and their norms in* $L_p(\bar{\Omega}, P \otimes P'_{z_0})$ *may be bounded independently of* $z_0 \in R^+$.

On $(\bar{\Omega}, P \otimes P'_{z_0})$, $\varphi_.(\bar{\omega}, .)$ *is essentially uniquely defined by properties* (*a*) *and* (*d*).

Proof. — Consider the differential equation in R^d:

$$(1.6)\qquad \frac{dx'}{dt} = D(x'), \qquad x'(0) = x'_0$$

and the associated group of diffeomorphisms of R^d $h_t: x'_0 \to h_t(x'_0) = x'_t$. It is trivial to see that for any $n > 0$:

$$\frac{\partial^m h_t}{\partial x^m}(x), \qquad \left[\frac{\partial h_t(x)}{\partial x} \right]^{-1}$$

are uniformly bounded on $[0, n] \times R^d$. Let $z_t \in \mathscr{C}(R^+; R^+)$, and L_t be a given continuous increasing process. For $x_0 \in R^d$, consider the stochastic differential equation on (Ω, P):

$$(1.7)\qquad d\bar{x} = (h^{*-1}_{L_t} X_0)(\bar{x}, z)\,dt + (h^{*-1}_{L_t} X_i)(\bar{x}, z).\,dw^i, \qquad \bar{x}(0) = x_0.$$

Using Theorem I.1.2 and I.2.1 in Bismut [5] we know that it is possible to associate to (1.7) a flow $\bar{\varphi}^{z,\,L}_t(\omega, .)$ of diffeomorphisms of R^d onto R^d, depending continuously on t for the topology of the uniform convergence over compact sets of C^∞ functions and their differentials. From [5], it is easy to see that $\bar{\varphi}^{z,\,L}_.(\omega, .)$ may be made to depend measurably on (z, L, ω). Moreover by [5], Theorems I.1.2 and I.2.1, we know that:

$$(1.8)\qquad \sup_{(t,\,x) \in K} \left| \frac{\partial^m \bar{\varphi}^{z,\,L}_t}{\partial x^m}(\omega, x) \right|, \qquad \sup_{(t,\,x) \in K} \left| \left[\frac{\partial \bar{\varphi}^{z,\,L}_t}{\partial x}(\omega, x) \right]^{-1} \right|,$$

are in $L_p(\Omega, P)$ and that the norms in $L_p(\Omega, P)$ of the random variables in (1.8) may be uniformly bounded in $L_p(\Omega, P)$ as long as the vector fields $(h^{*-1}_{L_t} X_0)(., z_t) \ldots (h^{*-1}_{L_t} X_m)(., z_t)$ and their differentials remain uniformly bounded.

We now set for $\bar{\omega} = (\omega, \omega')$:

$$(1.9)\qquad \varphi_t(\bar{\omega}, x_0) = h_{L_t(\omega')}[\bar{\varphi}^{(z,\,L)(\omega')}_t(\omega, x_0)].$$

The argument in Bismut-Michel [10], Theorem 1.6 on stochastic differential equations which depend on a parameter and the formula of Stratonovitch shows that (*d*) is verified. (*e*) is a consequence of (1.8). □

Remark 1. — The existence of $\varphi_.(\bar{\omega}, .)$ having the properties (*a*)-(*d*) also follows from the results of Kunita [25].

COROLLARY. — *Let* S *be a stopping time on* $(\bar{\Omega}, \{\bar{F}_t\})$. *For any* $z_0 \in R^+$, *on* $(S < +\infty)$, $P \otimes P'_{z_0}$ *a. s.:*

$$\varphi_{S+t}(\bar{\omega}, .) = \varphi_t(\bar{\theta}_S \bar{\omega}, \varphi_S(\bar{\omega}, .)) \quad \textit{for any } t \geqq 0. \tag{1.10}$$

Proof. — On $(S < +\infty)$, the conditional law of $\bar{\theta}_S \bar{\omega}$ given $\bar{F}_S$ is equal to $P \otimes P'_{z_S}$. Moreover the local time L_t is an additive functional of the strong Markov process z_t.

From the essential uniqueness of the solution of (1.4), it is then clear that on $(S < +\infty)$, for any $x_0 \in R^d$, $P \otimes P'_{z_0}$ a. s.:

$$\varphi_{S+t}(\bar{\omega}, x_0) = \varphi_t(\bar{\theta}_S \bar{\omega}, \varphi_S(\bar{\omega}, x_0)) \quad \text{for any } t \geqq 0. \tag{1.11}$$

The corollary follows from the $P \otimes P'_{z_0}$ a. s. continuity of both sides of (1.11) in (t, x_0). □

Remark 2. — It follows from Bismut [5], Kunita [25] that the usual rules of variations of parameters on ordinary differential equations can be extended to stochastic differential equations. For example, for:

$$x_0 \in R^d, \quad Z_t = \frac{\partial \varphi_t}{\partial x}(\bar{\omega}, x_0) \qquad \text{and} \qquad Z'_t = \left[\frac{\partial \varphi_t}{\partial x}(\bar{\omega}, x_0)\right]^{-1},$$

are the solutions of the stochastic differential equations:

$$\left\{\begin{aligned} dZ &= \frac{\partial X_0}{\partial x}(x, z)\, Z\, dt + \frac{\partial D}{\partial x}(x)\, Z\, dL + \frac{\partial X_i}{\partial x}(x, z)\, Z . dw^i, && Z(0) = I,\\ dZ' &= -Z' \frac{\partial X_0}{\partial x}(x, z)\, dt - Z' \frac{\partial D}{\partial x}(x)\, dL - Z' \frac{\partial X_i}{\partial x}(x, z) . dw^i, && Z'(0) = I. \end{aligned}\right. \tag{1.12}$$

In (1.12), x_t is of course the process $\varphi_t(\bar{\omega}, x_0)$. We will use these facts without further mention.

Remark 3. — The situation considered here is very similar to the situation studied in Bismut-Michel [10]. As in [10], z_t and (x_t, z_t) are Markov processes. The analogy will be better illustrated in the sequel.

(*c*) *The Girsanov transformation.*

Take $(z_0, z_0) \in R^d \times R^+$. On $(\bar{\Omega}, P \otimes P'_{z_0})$, consider the stochastic differential equation:

$$\left\{\begin{aligned} dx &= X_0(x, z)\, dt + D(x)\, dL + X_i(x, z) . dw^i, && x(0) = x_0,\\ du &= -\frac{1}{2}[b_z + b^2](x, z)\, dt + b(x, z) . dB, && u(0) = 0. \end{aligned}\right. \tag{1.13}$$

It is of course possible to apply to the system (1.13) the same techniques as to the smaller system (1.4). In particular, there is a function $u_t(\bar{\omega}, x)$ defined on $\bar{\Omega} \times R^+ \times R^d$ with values in R, having the following properties:

(*a*) For each (t, x), $\bar{\omega} \to u_t(\bar{\omega}, x)$ is measurable and for each $\bar{\omega}$, $(t, x) \to u_t(\bar{\omega}, x)$ is continuous.

(*b*) For any $\bar{\omega}$, $u_0(\bar{\omega}, .) = 0$.

(*c*) For any $\bar{\omega}$, $u_t(\bar{\omega}, x)$ is C^∞ in the variable x, and its differentials are jointly continuous in (t, x).

(*d*) On $(\bar{\Omega}, P \otimes P'_{z_0})$, for any $x_0 \in R^d$, $u_t(\bar{\omega}, x_0)$ coincides with the process u in (1.13).

These facts will be used in section 2.

(x_0, z_0) is now kept fixed.

DEFINITION 1.2. — On $(\bar{\Omega}, P \otimes P'_{z_0})$, if x_t is the process $\varphi_t(\bar{\omega}, x_0)$, M_t is the >0 continuous martingale:

$$M_t = \exp\left\{\int_0^t b(x_s, z_s)\,\delta B_s - \frac{1}{2}\int_0^t b^2(x_s, z_s)\,ds\right\}. \tag{1.14}$$

Using Itô's calculus, it is clear that M_t is the unique solution of the stochastic differential equation:

$$dM = M_s\, b(x_s, z_s)\,\delta B, \qquad M(0) = 1. \tag{1.15}$$

Since b is bounded, it is easy to see that M_t is in all the $L_p(\bar{\Omega}, P \otimes P'_{z_0})$. Also note that:

$$M_t = \exp\ u_t(\bar{\omega}, x_0).$$

Proceeding as in Bismut-Michel [10] (and of course as in Ikeda-Watanabe [17]) we now define a new probability measure on $\bar{\Omega}$.

DEFINITION 1.3. — For $(x_0, z_0) \in R^d \times R^+$, $Q_{(x_0, z_0)}$ is the probability measure on $\bar{\Omega}$ whose density relative to P on each $\bar{F}_t$ is M_t, i. e.:

$$\left.\frac{dQ_{(x_0, z_0)}}{d(P \otimes P'_{z_0})}\right| \bar{F}_t = M_t. \tag{1.16}$$

By the fundamental property of the Girsanov transformation ([17], p. 178, [40]-6), under $Q_{(x_0, z_0)}$, the process:

$$B'_t = B_t - \int_0^t b(x_s\, z_s)\,ds,$$

is a Brownian martingale, and $(w_t^1, \ldots, w_t^m, B'_t)$ is a $m+1$ dimensional Brownian martingale.

(*d*) *The Girsanov transformation at infinity.*

The discussion which follows is based on the ideas of Föllmer [13] (*see* also Azéma-Jeulin [2]). Since we do not need the full development of the theory of the

Föllmer measure, we will just illustrate a few points in the particular case which we are treating.

In what follows, the filtration $\{\bar{F}_t\}_{t \geqq 0}$ is made right-continuous, but is *not* completed, since we will deal with several probability measures on $\bar{\Omega}$ which are not necessarily equivalent on $\bar{F}_\infty$. (x_0, z_0) are as in (*c*).

PROPOSITION 1.4. — *Let* T *be a* $\{\bar{F}_t\}_{t \geqq 0}$ *stopping time, and* $A \in \bar{F}_T$. *Then:*

$$Q_{(x_0, z_0)}(A \cap (T < +\infty)) = E^{P \otimes P'_{z_0}}[1_{A \cap (T < +\infty)} M_T]. \tag{1.17}$$

Proof. — For any $n \in N$, $A \cap (T \leqq n) \in \bar{F}_{T \wedge n}$, so that:

$$Q_{(x_0, z_0)}(A \cap (T \leqq n)) = E^{P \otimes P'_{z_0}}[M_{T\,9\,n} 1_{A \cap (T \leqq n)}] = E^{P \otimes P'_{z_0}}[M_T 1_{A \cap (T \leqq n)}]. \tag{1.18}$$

Making $n \to +\infty$, and using Fatou's lemma, (1.17) follows. □

In particular for any stopping time T:

$$E^{P \otimes P'_{z_0}}[1_{T < +\infty} M_T] \leqq 1 \tag{1.19}$$

(which of course also follows from the fact that M_t is a $\geqq 0$ martingale). It can also be proved that for any stopping time T:

$$dQ = M_T\,dP + (dQ - M_T\,dP),$$

is exactly the Lebesgue decomposition of Q relative to P on $\bar{F}_T$ ([13], [2]).

We will now illustrate these various facts.

DEFINITION 1.5. — A_t is the right-continuous inverse of L_t, i.e.:

$$A_t = \inf\{A \geqq 0;\ L_A > t\}. \tag{1.20}$$

Of course for $t > 0$:

$$A_{t^-} = \inf\{A \geqq 0;\ L_A \geqq t\}. \tag{1.21}$$

For any $t \geqq 0$, A_t is a $\{\bar{F}_t\}_{t \geqq 0}$ stopping time. Moreover it is classical that for any $t \geqq 0$:

$$A_t < +\infty, \quad P \otimes P'_{z_0} \text{ a.s.} \tag{1.22}$$

In general (1.22) does not hold for the probability measure $Q_{(x_0, z_0)}$.

We give a trivial example of this fact.

Example. — Assume that b is equal to a constant $\delta \neq 0$. Then:

$$M_s = \exp\left(\delta B_s - \frac{1}{2}\delta^2 s\right).$$

Using (1.1) and the fact that for any $t \geqq 0$, $z_{A_t} = 0$, we find that:

$$M_{A_t} = \exp\left(-\delta z_0 - \delta t - \frac{1}{2}\delta^2 A_t\right). \tag{1.23}$$

(*a*) If $\delta < 0$, for $s \leqq A_t$, $B_s \geqq -z_0 - t$ so that $\delta B_s \leqq \delta(z_0 + t)$. $M_{s \wedge A_t}$ is then a bounded martingale, so that:

$$E^{P \otimes P'}\left[\exp - \delta t - \frac{1}{2}\delta^2 A_t\right] = 1. \tag{1.24}$$

(1.24) illustrates the well-known fact [18] that A_t is a stable process with exponent 1/2 and rate $\sqrt{2}$. Due to (1.17), we find that:

$$Q_{(x_0,\, 0)}(A_t < +\infty) = 1, \tag{1.25}$$

$Q_{(x_0,\, 0)}$ is then equivalent to $P \otimes P'$ on $\bar{F}_{A_t}$ and its density on $\bar{F}_{A_t}$ relative to $P \otimes P'$ is M_{A_t}.

(*b*) If $\delta > 0$, M_{A_t} is < 1. Using (1.17), we find that:

$$Q_{(x_0,\, 0)}(A_t < +\infty) = \exp - (2\,\delta t). \tag{1.26}$$

Let T_0 be the stopping time:

$$T_0 = \inf\{t > 0;\ z_s = 0\}.$$

Clearly:

$$M_{T_0} = \exp\left(-\delta z_0 - \frac{1}{2}\delta^2 T_0\right).$$

Using (1.17) again, we find that:

$$Q_{(x_0,\, z_0)}(T_0 < +\infty) = \exp - 2\,\delta z_0 \tag{1.27}$$

(here we use the known fact [18] that $E^{P \otimes P'_{z_0}} e^{-pT_0} = e^{-\sqrt{2p}\, z_0}$).

Let L^0 be the last exit time of 0 for the process z i. e.:

$$L^0 = \sup\{s \geqq 0;\ z_s = 0\}. \tag{1.28}$$

We may write:

$$M_{A_t} = \exp\left(\delta t - \frac{1}{2}\delta^2 A_t\right)\exp - 2\,\delta t. \tag{1.29}$$

Using (1.29) and the argument in (*a*), it is clear that under $Q_{(x_0,\, 0)}$, the law of z_s ($s \leqq L^0$) is equal to the law of the reflecting Brownian motion z with drift $-\delta$ with a killing associated to the additive functional $2\delta L_s$. This is made clear in Jeulin-Yor [22]. Finally note that (1.26) is equivalent to:

$$Q_{(x_0,\,0)}(L_\infty > t) = \exp - 2\delta t. \tag{1.30}$$

Under $Q_{(x_0,\,0)}$, L_∞ follows an exponential law (this also follows directly from Itô's excursion theory).

Of course (1.26), (1.27), (1.30) may be obtained without effort using the methods of [18]-[22]. However this example illustrates the fact that even if on $(\bar\Omega, P\otimes P'_{z_0})$, M_t is $\{\bar F_t\}_{t\geqq 0}$-martingale, M_{A_t} is in general a $\{\bar F_{A_t}\}_{t\geqq 0}$ supermartingale and can even be a decreasing process. As shown by (1.29), the process (x_s, z_s) $(s \leqq L^0)$ (which is really what will interest us) can be eventually described as a process for which $L^0 = +\infty$ (associated to a new Girsanov exponential), and killed at a rate which is a multiple of L_t.

(*e*) *The boundary process.*

We now define the boundary process which is the object of our study.

Δ is a cemetery point, so that $R^+ \times R^d \cup \{\Delta\}$ is the state space of the boundary process.

DEFINITION 1.6. — D is the set of functions (a_t, y_t) defined on R^+ with values in $(R^+ \times R^d) \cup \{\Delta\}$ which are right-continuous with left hand limits such that if ζ is the function defined on D by:

$$\zeta = \inf\{t \geqq 0;\ (a_t, y_t) = \Delta\}, \tag{1.31}$$

then if $\zeta < +\infty$, for $s \geqq \zeta$, $(a_s, y_s) = \Delta$.

D is of course endowed with the Skorokhod topology (*see* Billingsley [4]) so that it is a Polish space.

DEFINITION 1.7. — Let $(a_0, x_0) \in R^+ \times R^d$. On $(\bar\Omega, Q_{(x_0,\,0)})$, the boundary process (a_t, y_t) with values in $(R^+ \times R^d) \cup \{\Delta\}$ is defined by:

$$\begin{cases} (a_t, y_t) = (a_0 + A_t, \varphi_{A_t}(\bar\omega, x_0)), & t < L_\infty, \\ \Delta, & t \geqq L_\infty. \end{cases} \tag{1.32}$$

Note that on $(\bar\Omega, Q_{(x_0,0)})$, $A_0 = 0$, so that $(a(0), y(0)) = (a_0, y_0)$.

Since L_t is an additive functional of the strong Markov process $(\varphi_t(\bar\omega, x_0), z_t)$, and since for any $t > 0$, on $(A_t < +\infty)$, $z_{A_t} = 0$, it is easy to see that (a_t, y_t) is a strong Markov process.

DEFINITION 1.8. — Take (a_0, x_0) as in definition 1.7. $R_{(a_0, x_0)}$ is the probability law on D of the process (a_t, y_t) under $Q_{(x_0, 0)}$.

The system of probability measures $\{R_{(a_0, x_0)}\}$ on D defines a strong Markov process, which is the object of our study. More precisely, we shall study the smoothness of the probability laws of (a_t, y_t) $(t > 0)$.

(*f*) *A few remarks on the boundary process.*

Let $Y(x, z)$ be a C^∞ vector field having the same properties as $X_0, X_1 \ldots X_m$. Consider the more general stochastic differential equation on $(\bar{\Omega}, P \otimes P')$:

$$(1.33) \quad \begin{cases} dx' = X_0(x', z)\,dt + D(x')\,dL + X_i(x', z)\,dw^i + Y(x', z).\,dB, \\ x(0) = x_0. \end{cases}$$

Rewrite (1.33) in the form:

$$(1.34) \quad \begin{cases} dx' = X_0(x', z)\,dt + (D(x') - Y(x', 0))\,dL + X_i(x', z).\,dw^i + Y(x', z).\,dz, \\ x'(0) = x_0. \end{cases}$$

Consider the differential equation:

$$(1.35) \quad \begin{cases} \dfrac{d\bar{x}'}{dt} = Y(\bar{x}', \bar{z}'); \qquad \bar{x}'(0) = x_0, \\ \dfrac{d\bar{z}'}{dt} = 1; \qquad \bar{z}'(0) = z_0. \end{cases}$$

Let k_t be the associated group of diffeomorphisms of R^{d+1} onto itself. Consider the stochastic differential equation:

$$(1.36) \quad dx'' = (k_{z_t}^{*-1} X_0)(x'', 0)\,dt + (D(x'') - Y(x'', 0))\,dL + (k_{z_t}^{*-1} X_i)(x'', 0).\,dw^i, \quad x''(0) = x_0,$$

[of course the differential of k_t sends R^d into itself, so that $(k_{z_t}^{*-1} X_0)(x'', 0), \ldots, (k_{z_t}^{*-1} X_m)(x'', 0)$ take their values in R^d]. Clearly $x'_t = \pi\, k_{z_t}(x''_t, 0)$ is the unique solution of (1.34). Moreover for any $t > 0$, on $(A_t < +\infty)$, $x'_{A_t} = x''_{A_t}$. Since (1.36) is of the type (1.4) we see it is equivalent to study the boundary processes associated to (1.4) or to (1.33).

Similarly, let $m(z)$ be a function defined on R^+ with values in R^+, which belongs to $C_b^\infty(R^+)$, such that there exists $\alpha > 0$ for which $m \geqq \alpha$. Assume that instead of (1.32), we want to study:

$$\left(a_0 + \int_0^{A_t} m(z_s)\,ds,\ x_{A_t} \right)$$

[where $x_s = \varphi_s(\bar{\omega}, x_0)$].

Let τ_t the time change associated to $\int_0^t m(z_s)\,dz$ i. e.:

$$\tau_t = \inf \left\{ \tau;\ \int_0^\tau m(z_s)\,ds = t \right\}.$$

Set:

$$G(z)=\int_0^z [m(u)]^{1/2}\,du.$$

Clearly G is a diffeomorphism of R^+ onto itself.

Now on $(\bar{\Omega}, P\otimes P')$ if $z'_s=z_{\tau_s}$, $L'_s=L_{\tau_s}$, $x'_s=x_{\tau_s}$ is the solution of the stochastic differential equation:

$$dx'=\frac{1}{m(z'_s)}X_0(x'_s, z'_s)\,ds+D(x')\,dL'+\frac{1}{\sqrt{m(z'_s)}}X_i(x'_s, z'_s).d\bar{w}^i,$$
$$x'(0)=x_0$$

(where $\bar{w}=(\bar{w}^1\ldots\bar{w}^m)$ is a new Brownian motion).

Similarly it is clear that on $(\bar{\Omega}, P\otimes P')$:

$$z'_s=L'_s+\int_0^t\frac{\delta B'}{\sqrt{m(z'_s)}},$$

where B′ is a Brownian motion independent of $\bar{w}$. Now:

$$G(z'_t)=[m(0)]^{1/2}L'_t+\frac{1}{2}\int_0^t\frac{G''(z'_u)}{m(z'_u)}\,du+B'_t.$$

By lemma 4.2 in [17], $[m(0)]^{1/2}L'_t$ is exactly the local time at 0 of $G(z'_s)$. Moreover by a trivial Girsanov transformation, $G(z'_t)$ is transformed back to a reflecting Brownian motion on $[0,+\infty[$. Now the inverse of L'_t is exactly:

$$A'_t=\int_0^{A_t} m(z_s)\,ds,$$

so that $x_{A_t}=x'_{A'_t}$. Of course the inverse of $[m(0)]^{1/2}L'_t$ is $A'_{t/[m(0)]^{1/2}}$.

By doing in succession a time change, the change of variables $z\to G(z)$ and a Girsanov transformation, we are back to the situation previously described. These transformations will be studied from the point of view of differential operators in paragraph (*g*).

The arch-typical example is the case where $d=1$, $m=1$, $X_0=0$, $X_1=1$, $D=0$. If $x(0)=0$, the probability law of x_{A_t} under $P\otimes P'$ is the Cauchy law:

$$\frac{t\,dx}{\pi(t^2+x^2)}. \tag{1.37}$$

Now assume that $X_1(x)$ is a one dimensional vector field which is in $C_b^\infty(R)$, >0 on $]-\infty, 1[$ and which is equal to $1-x$ on a neighborhood of 1. Clearly, if $x(0)=0$, the law of x_{A_t} is the image law of (1.37) through the mapping $s\to y_s$, where y_s is the solution of the differential equation:

$$\frac{dy}{ds}=X_1(y), \qquad y(0)=0.$$

i. e.

(1.38)
$$\frac{1_{y<1}\, t\, dy}{\pi X_1(y)\left[t^2+\left[\int_0^y dx/X_1(x)\right]^2\right]}.$$

Since for $y<1$, $y \to 1$, $\int_0^y dx/X_1(x) \sim -\text{Log}(1-y)$, it is trivial to see that the left-hand limit at $y=1$ of (1.38) is $+\infty$. The law of x_{A_t} is not smooth. Such a phenomenon does not happen if (1.37) is replaced by a Gaussian law, and so, the explicit form of the law of A_t ([18], p. 25) shows that the law of (A_t, x_{A_t}) is still smooth.

In this example, the non smoothness of the law of x_{A_t} comes from the integration on the possibly large values of A_t. The introduction of the supplementary component A_t has the effect of smoothing out the considered probability law.

(*g*) *Some analytical properties of the partial differential operators connected with the boundary process.*

Let $\mathscr{L}$ be the second order differential operator acting on $f(t, x, z) \in C_b^\infty(R^+ \times R^{d+1})$:

(1.39)
$$\mathscr{L} f(t, x, z)=\left(\frac{\partial}{\partial t}+X_0+b\frac{\partial}{\partial z}+\frac{1}{2}\sum_1^m X_i^2+\frac{1}{2}\frac{\partial^2}{\partial z^2}\right)f.$$

Let $\mathscr{D}$ be the first order differential operator defined on the boundary $(z=0)$ acting on $f \in C_b^\infty(R^+ \times R^{d+1})$ by:

(1.40)
$$(\mathscr{D} f)(t, x)=\left(D+\frac{\partial}{\partial z}\right)f(t, x, 0).$$

Clearly, if $f \in C_b^\infty(R^+ \times R^{d+1})$, under the probability law $Q_{(x_0, 0)}$, if $x_t=\varphi_t(\bar{\omega}, x_0)$:

(1.41)
$$f(t, x_t, z_t)-\int_0^t 1_{z\neq 0}\, \mathscr{L} f(s, x_s, z_s)\, ds-\int_0^t (\mathscr{D} f)(s, x_s)\, dL_s,$$

is a martingale. In fact by Ikeda-Watanabe [17], p. 203, $Q_{(x_0, 0)}$ can be fully characterized by this property.

The arguments which follow will be analytically formal but can be made rigorous in some cases, like the elliptic case (*see* Stroock-Varadhan [39]). Let $g \in C_b^\infty(R^+ \times R^d)$, and consider the Dirichlet problem on $f \in C_b^\infty(R^+ \times R^{d+1})$:

(1.42)
$$\begin{cases} \mathscr{L} f=0 & \text{on } (z\geqq 0), \\ f=g & \text{on } (z=0). \end{cases}$$

Assume that (1.42) has in fact a solution. Using (1.41) we see that:

$$g(A_t, x_{A_t}) - \int_0^t (\mathscr{D} f)(A_s, x_{A_s})\, ds, \tag{1.43}$$

is a martingale, so that the operator $\mathscr{A}$ defined by:

$$(\mathscr{A} g)(s, x) = (\mathscr{D} f)(s, x),$$

appears to be (formally) the "generator" associated to the Markov process (A_t, x_{A_t}). Of course in (1.42), the condition $\mathscr{L} f=0$ is equivalent to $a(t, x, z)\mathscr{L} f=0$ when a is a >0 C^∞ function. This makes clear that the boundary process (A_t, x_{A_t}) is invariant under time change on the "inner" process on $(z>0)$, i.e. under time change on the excursions of z_t out of 0. Of course this can be directly (and rigorously) proved using the results in Ikeda-Watanabe [17].

We now consider a general second order operator $\bar{\mathscr{L}}$ on R^{d+2} written in Hörmander's form:

$$\bar{\mathscr{L}} = \bar{X}_0 + \frac{1}{2}\sum_1^{m'} \bar{X}_i^2$$

(where $\bar{X}_0, \bar{X}_1 \ldots \bar{X}_{m'}$ are smooth vector fields). Let D be an open set in R^{d+2} with a smooth boundary ∂D. Let γ be a smooth vector field defined on ∂D pointing inward D (of course γ is $\neq 0$ on ∂D).

Let δ be the 1-differential form defined on ∂D by the following two conditions:

$$\text{If } x \in \partial D, \quad X \in T_x(\partial D), \qquad \delta(X)=0, \qquad \langle \delta, \gamma \rangle = 1 \quad \text{on } \partial D. \tag{1.44}$$

Let $g \in C_b^\infty(\partial D)$. Consider the Dirichlet problem on $f \in C_b^\infty(\bar{D})$ $(\bar{D} = D \cup \partial D)$:

$$\begin{cases} \bar{\mathscr{L}} f = 0 & \text{on } D, \\ f = g & \text{on } \partial D \end{cases} \tag{1.45}$$

(we assume that the problem (1.45) is well-defined and has a unique solution). For g, f as in (1.45), we define the function $\bar{\mathscr{A}} g$ on ∂D by:

$$\bar{\mathscr{A}} g = \gamma f.$$

We want to study the smoothness of the transition probabilities associated to the semi-group $e^{t\bar{\mathscr{A}}}$ acting on $C_b^\infty(\partial D)$.

A natural assumption is that ∂D is non characteristic for $\bar{\mathscr{L}}$ (*see* Treves [42]). Namely let $S(\bar{x}, \bar{p})$ be the principal symbol of $\bar{\mathscr{L}}$ i.e. the function defined on $T^* R^{d+2}$ by:

$$(\bar{x}, \bar{p}) \in T^* R^{d+2} \to S(\bar{x}, \bar{p}) = \sum_1^{m'} \frac{\langle \bar{p}, \bar{X}_i(\bar{x}) \rangle^2}{2}.$$

We then assume that, on ∂D, $S(x, \delta) > 0$. Using the construction of Ikeda-Watanabe [17], it is possitble to construct the boundary process associated to (1.45)-(1.46) explicitly.

We now will consider some of the assumptions which can be done on $(\bar{\mathscr{L}}, \gamma)$, which will be later be proved to imply the existence of densities or their smoothness, and we will underline the difficulties.

DEFINITION 1.9. — For $\bar{x} \in R^{d+2}$, $\bar{T}_{\bar{x}}$ is the vector space in $T_{\bar{x}}(R^{d+2})$ spanned by $\bar{X}_0(\bar{x}), \bar{X}_1(\bar{x}) \ldots \bar{X}_{m'}(\bar{x})$ and their Lie brackets at $\bar{x}$.

It is natural to assume that $\bar{\mathscr{L}}$ verifies the hypoellipticity assumptions of Hörmander [15]; namely we will assume that $x_0 \in \partial D$ is such that $\bar{T}_{\bar{x}_0}$ is equal to $T_{\bar{x}_0}(R^{d+2})$.

Let $(\mathscr{V}, \varphi)$ be a local chart as $\bar{x}_0$ so that in the corresponding coordinate system $(\bar{x}^1 \ldots \bar{x}^{d+2})$, $\partial D \cap \mathscr{V}$ is represented by $(\bar{x}^{d+2} = 0)$.

For $\bar{x} \in \mathscr{V}$, let $e(\bar{x}) \in R^{m'}$ be defined by:

$$e(\bar{x}) = (\langle d\bar{x}^{d+2}, \bar{X}_1(\bar{x}) \rangle, \ldots, \langle d\bar{x}^{d+2}, \bar{X}_{m'}(\bar{x}) \rangle). \tag{1.46}$$

Since ∂D is non characteristic for $\bar{\mathscr{L}}$, $e(\bar{x}_0)$ is $\neq 0$. We can assume than $\mathscr{V}$ is such that if $\bar{x} \in \mathscr{V}$, $e(\bar{x}) \neq 0$. On a neighborhood of x_0 in $\mathscr{V}$—which we can assume to be equal to $\mathscr{V}$—it is possible to define a C^∞ mapping $\bar{x} \to A(\bar{x}) \in 0(m')$ such that on $\mathscr{V}$, $A(x)e(x) = (0, 0, \ldots, h(x))$. If $A(\bar{x}) = (a_i^j(\bar{x}))$, set for $\bar{x} \in \mathscr{V}$:

$$\bar{Y}_i(\bar{x}) = \sum_{j=1}^{m'} a_i^j(x)\, \bar{X}_j(\bar{x}). \tag{1.47}$$

Clearly:

$$\langle d\bar{x}^{d+2}, \bar{Y}_i(\bar{x}) \rangle = 0 \quad \text{for} \quad 1 \leqq i \leqq m'-1, \qquad \langle d\bar{x}^{d+2}, \bar{Y}_{m'}(\bar{x}) \rangle \neq 0. \tag{1.48}$$

Now $\bar{\mathscr{L}}$ can be rewritten in the form:

$$\bar{\mathscr{L}} = (\bar{Y}_0 + b\, \bar{Y}_{m'}) + \frac{1}{2} \sum_{1}^{m'-1} \bar{Y}_i^2 + \frac{1}{2} \bar{Y}_{m'}^2, \tag{1.49}$$

with:

$$\langle d\bar{x}^{d+2}, \bar{Y}_0 \rangle = 0. \tag{1.50}$$

On $\mathscr{V}$, $\bar{Y}_0, \bar{Y}_1, \ldots, \bar{Y}_{m'-1}$ are tangent to the fibration $(\bar{x}^{d+2} = \text{Const.})$.

Consider now the following assumption.

A 1 : $x_0 \in \partial D$ is such that the vector space $\bar{T}'_{\bar{x}_0}(\partial D)$ in $T_{\bar{x}_0}(\partial D)$ spanned by $\bar{Y}_0, \bar{Y}_1 \ldots \bar{Y}_{m'-1}$ and their Lie brackets is equal to $T_{\bar{x}_0}(\partial D)$.

Or course A 1 is stronger than the hypoellipticity assumption on $\bar{\mathscr{L}}$.

We claim that A 1 only depends on $\bar{\mathscr{L}}$ and D, and is invariant under multiplication of $\bar{\mathscr{L}}$ by a C^∞ and >0. To see this, we define the following vector distributions.

DEFINITION 1.10. — For $\bar{x}\in D$, $\bar{T}''_{\bar{x}}(\partial D)$ is the intersection of $T_{\bar{x}}(\partial D)$ and of the vector space spanned by $\bar{X}_0(\bar{x}), \bar{X}_1(\bar{x})\ldots\bar{X}_{m'}(\bar{x})$.

Clearly $\bar{T}''_{\bar{x}}(\partial D)$ depends only on $\bar{\mathscr{L}}$ and ∂D. It is then easy to see that $\bar{T}''(\partial D)$ is locally generated by $\bar{Y}_0, \bar{Y}_1, \ldots, \bar{Y}_{m'-1}$.

The assumption A 1 will be somewhat stronger than the assumptions which we will later do. In fact under A 1, the brackets of $\bar{Y}_0, \bar{Y}_1\ldots\bar{Y}'_{m-1}$ with $\bar{Y}_{m'}$ are unnecessary to generate $T_{\bar{x}_0}(R^{d+2})$.

Let us assume that the coordinate system $(\bar{x}^1, \ldots, \bar{x}^{d+2})$ is chosen in such a way that for $\bar{x}\in\partial D\cap\mathscr{V}$, $d\bar{x}^{d+2}=\delta$ (this is *always* possible). On $\mathscr{V}$, replace $\bar{\mathscr{L}}$ by:

$$\bar{\mathscr{L}}'=\frac{\bar{\mathscr{L}}}{2\,S(\bar{x}, d\bar{x}^{d+2})}.$$

If $\bar{S}'(\bar{x}, d\bar{x}^{d+2})$ is the principal symbol of $\bar{\mathscr{L}}'$, we have that on $\mathscr{V}$:

$$S'(\bar{x}, d\bar{x}^{d+2})=\frac{1}{2}. \tag{1.51}$$

Rewrite $\bar{\mathscr{L}}'$ in the form:

$$\bar{\mathscr{L}}'=\bar{Y}'_0+b'\,\bar{Y}'_{m'}+\frac{1}{2}\sum_1^{m'-1}\bar{Y}'^2_i+\frac{1}{2}\bar{Y}'^2_{m'}, \tag{1.52}$$

by proceeding on $\bar{\mathscr{L}}'$ the way we did on $\bar{\mathscr{L}}$ in (1.47)-(1.49). By eventually replacing $\bar{Y}'_{m'}$ by $-\bar{Y}'_{m'}$, we have:

$$\langle d\bar{x}^{d+2}, \bar{Y}'_i\rangle=0,\qquad 0\leq i\leq m'-1;\quad \langle d\bar{x}^{d+2}, \bar{Y}'_{m'}\rangle=1. \tag{1.53}$$

For $\bar{x}\in\partial D$, consider the differential equation:

$$\frac{d\bar{x}}{dt}=\bar{Y}'_{m'}(\bar{x}),\qquad \bar{x}(0)=\bar{x}. \tag{1.54}$$

In (1.54) $\bar{x}_t=k_t(\bar{x})$. Since $\bar{Y}'_{m'}$ is transversal to the fibration ($\bar{x}^{d+2}=$Const.), it is clear that $(t, \bar{x})\to k_t(\bar{x})$ defines a local chart of R^{d+2} at $\bar{x}_0$. Using (1.53), it is obvious that $t=\bar{x}^{d+2}$. In this chart, we have:

$$\bar{Y}'_{m'}=\frac{\partial}{\partial\bar{x}^{d+2}}, \tag{1.55}$$

so that:

$$\bar{\mathscr{L}}'=\bar{Y}'_0+b'\frac{\partial}{\partial\bar{x}^{d+2}}+\frac{1}{2}\sum_1^{m'-1}\bar{Y}'^2_i+\frac{1}{2}\frac{\partial^2}{(\partial\bar{x}^{d+2})^2}. \tag{1.56}$$

Since $d\bar{x}^{d+2}=\delta$ on ∂D, it is easy to see that at least locally, for the process associated to $(\mathscr{L}', \gamma)$, $\bar{x}_t^{d+2}$ is a reflecting Brownian motion on $[0, +\infty[$ with drift b'.

Using (1.55) [or (1.53)], it is easy to see that the brackets of $\bar{Y}'_{m'}$ with $\bar{Y}'_0, \ldots, \bar{Y}'_{m-1}$ whose length is $\geqq 2$ are tangent to the fibration ($\bar{x}^{d+2}=$Const.). Since $\mathscr{L}'$ verifies the assumption of Hörmander [15], we know that at $\bar{x}_0$, the vector space spanned by $\bar{Y}'_0(\bar{x}_0), \ldots, \bar{Y}'_{m'-1}(\bar{x}_0)$ and the Lie brackets at $\bar{x}'_0$ of length $\geqq 2$ of $\bar{Y}'_0, \ldots, \bar{Y}'_{m'-1}, \bar{Y}'_{m'}$ is equal to $T_{\bar{x}_0}(\partial D)$.

However, this will not be enough to allow us to prove the existence and regularity of the densities for the boundary semi-group. We will work under an assumption somewhat weaker than A 1, but stronger than Hörmander's, namely, we will do an assumption of the type:

A 2 : The vector space spanned by $\bar{Y}'_0(\bar{x}_0), \bar{Y}'_1(\bar{x}_0), \ldots, \bar{Y}'_{m'-1}(x_0)$ and the Lie brackets at $\bar{x}_0$ of $\bar{Y}'_0, \bar{Y}'_1 \ldots \bar{Y}'_{m'-1}, \bar{Y}'_{m'}$ of length $\geqq 2$ in which at least one of the vector fields $\bar{Y}'_1, \ldots, \bar{Y}'_{m'-1}$ appears is equal to $T_{\bar{x}_0}(\partial D)$.

Now in general, A 2 depends on the fibration $\bar{x}^{d+2}$, i. e. it is not invariant under time change on the inner process.

Observe that if in $\bar{\mathscr{L}}$, $\bar{X}_0$ is a linear combination (with C^∞ coefficients) of $\bar{X}_1, \ldots, \bar{X}_{m'}$ (this is an invariant condition), A 2 is again equivalent to the hypoellipticity of $\bar{\mathscr{L}}$. However, as made clear in (1.38), we need to have a purely "parabolic" component in order to prove the regularity of the boundary semi-group.

The situation is then not entirely satisfactory from the point of view of partial differential operators.

2. The calculus of variations on the Brownian motion *w*

In this section, we develop the calculus of variations on the Brownian motion w, i. e. the reflecting Brownian motion z – and the Brownian martingale B – are kept fixed. The spirit of this section is then very close to what has been done in Bismut-Michel [10], where results on conditional laws of diffusions were sought, in the spirit of the theory of filtering.

Of course the computations are very similar to what appears in the classical Malliavin calculus of variations ([29], [30], [36], [7]), but special care must be given to integrability conditions. Moreover different techniques are presented to deal with "localizable" or "non localizable" conditions (the full explanation of such a terminology appears in section 5). In particular step by step integration by parts, as developed in [8] is used in the "non localizable" case.

In (*a*), integration by parts on the Brownian motion w is presented as in [7]. In (*b*), truncated integration by parts on x_{A_t} is obtained. In (*c*), integration by parts on x_{A_t} is done under adequate assumptions in the "localizable" case. In (*d*), the "non localizable" case is considered.

(*a*) *Integration by parts on the Brownian motion w.*

We will now apply the techniques and the results of [7] to perform an integration by parts on the Brownian motion w. The reader is referred for more details to [7].

T is $a>0$ finite real number.

(x_0, z_0) is an element of $R^d \times R^+$.

$h(\omega', x)$ is a function defined on $\Omega' \times \mathscr{C}([0, T]; R^d)$ with values in $T^*_{x_0}(R^d)$, which has the following properties:

(*a*) h is bounded; for any $x \in \mathscr{C}([0, T]; R^d)$, $\omega' \to h(\omega', x)$ is measurable; for any $\omega' \in \Omega'$, $x \to h(\omega', x)$ is continuous.

(*b*) For every $\omega' \in \Omega'$, $x \to h(\omega', x)$ is Fréchet-differentiable on $\mathscr{C}([0, T]; R^d)$, and its differential $d_x h(\omega', x)$ is uniformly bounded.

Of course $\mathscr{C}([0, T]; R^d)$ is considered as a Banach space endowed with the norm:

$$\| x \| = \sup_{0 \leq t \leq T} | x_t |.$$

The dual $\mathscr{M}$ of $\mathscr{C}([0, T]; R^d)$ is the set of bounded measures μ on $[0, T]$ with values in R^d, so that if $x \in \mathscr{C}([0, T]; R^d)$:

$$\langle \mu, x \rangle = \int_{[0,T]} \langle x_t, d\mu(t) \rangle.$$

If $x \in \mathscr{C}([0, T]; R^d)$, the derivative $dh_x(\omega', x)$ can be identified to a bounded measure $d\nu^{\omega', x}(t)$ on $[0, T]$ with values in $R^d \otimes R^d$, so that if $y \in \mathscr{C}([0, T]; R^d)$:

$$(2.1) \qquad \langle dh_x(\omega', x), y \rangle = \int_{[0,T]} d\nu^{\omega', x}(t)(y_t).$$

From the point of view of differential geometry, $d\nu^{\omega', x}(t)$ can be identified to a generalized linear mapping from $T_{x_t}(R^d)$ into $T^*_{x_0}(R^d)$. In particular if $l \in T_{x_0}(R^d)$, the action of:

$$\int_{[0,T]} \varphi_t^{*-1}(\bar{\omega}, x_0)\, d\nu^{\omega', \varphi_\cdot(\bar{\omega}, x_0)}(t)$$

on l is defined by:

$$(2.2) \quad \int_{[0,T]} \varphi_t^{*-1}(\bar{\omega}, x_0)\, d\nu^{\omega', \varphi_\cdot(\bar{\omega}, x_0)}(t)(l) = \int_{[0,T]} d\nu^{\omega', \varphi_\cdot(\bar{\omega}, x_0)}(t)\,[\varphi_t^*(\bar{\omega}, x_0)\, l].$$

DEFINITION 2.1. — On $\bar{\Omega}$, the process $C_t(\bar{\omega})$ with values in the linear mappings from $T^*_{x_0}(R^d)$ into $T_{x_0}(R^d)$ is defined by:

$$(2.3) \quad p \in T^*_{x_0}(R^d) \to C_t(\bar{\omega})\, p = \sum_{i=1}^{m} \int_0^t \langle p, (\varphi_s^{*-1} X_i)(x_0) \rangle\, \varphi_s^{*-1} X_i(x_0)\, ds.$$

Of course if $p, q \in T^*_{x_0}(R^d)$:

$$(2.4)\qquad \langle C_t(\bar\omega)\,p, q\rangle = \sum_{i=1}^{m}\int_0^t \langle p, (\varphi_s^{*-1} X_i)(x_0)\rangle \langle q, (\varphi_s^{*-1} X_i)(x_0)\rangle\, ds$$

so that $C_t(\bar\omega)$ defines a non negative symmetric bilinear form on $T^*_{x_0}(R^d)$.

Of course $C_t(\bar\omega)$ also depends on (x_0, z_0), but we drop this dependence for simplicity.

We then have the following result:

THEOREM 2.2. — *Let $f \in C_c^\infty(R^+ \times R^d)$, whose support is included in $[0, T] \times R^d$. Then if x_s is the process $\varphi_s(\bar\omega, x_0)$, for any $t, t' \geqq 0$, $t' \leqq t$, the following relation holds:*

$$(2.5)\quad E^{P\otimes P'_{z_0}}[M_{A_t} \langle d_x f(A_t, x_{A_t}), \varphi^*_{A_t}(\bar\omega, x_0)\, C_{A_{t'}}\, h(\omega', x_{.\wedge A_t})\rangle]$$

$$+E^{P\otimes P'_{z_0}}\left[M_{A_t}\left\{\int_0^{A_{t'}} \langle \varphi_s^{*-1} X_i)(x_0), \int_{[s,T]} [\varphi^{*-1}_{v\wedge A_t}\, dv^{\omega', x_{.\wedge A_t}}(v)] \times (\varphi_s^{*-1} X_i)(x_0)\rangle\, ds\right\} f(A_t, x_{A_t})\right]$$

$$+E^{P\otimes P'_{z_0}}\left[M_{A_t}\left\langle \int_0^{A_t} C_{s\wedge A_{t'}}(\varphi_s^{*-1} b_x)(x_0))\,\delta B_s - b(x_s, z_s)\, ds), h(\omega', x_{.\wedge A_t})\right\rangle f(A_t, x_{A_t})\right]$$

$$=E^{P\otimes P'_{z_0}}\left[M_{A_t}\left\langle \int_0^{A_{t'}} (\varphi_s^{*-1} X_i)(x_0)\,\delta w^i, h(\omega', x_{.\wedge A_t})\right\rangle f(A_t, x_{A_t})\right].$$

Proof. — Observe that (2.5) makes sense. In fact $f(A_t, x_{A_t})$ or $d_x f(A_t, x_{A_t})$ are $\neq 0$ only for $A_t \leqq T$. Of course for $s \leqq A_t$, $L_s \leqq t$.

From Theorem 1.1, we know that:

$$(2.6)\qquad 1_{A_t \leqq T} \sup_{0\leqq s\leqq A_t}\left|\frac{\partial\varphi}{\partial x} t(\bar\omega, x_0)\right|, \qquad 1_{A_t \leqq T} \sup_{0\leqq s\leqq A_t}\left|\left[\frac{\partial\varphi}{\partial x} t(\bar\omega, x_0)\right]^{-1}\right|,$$

belong to all the $L_p(\bar\Omega, P\otimes P'_{z_0})$. As observed in section 1, M_T is in all the $L_p(\bar\Omega, P\otimes P'_{z_0})$, and so by Doob's inequality $1_{A_t \leqq T} M_{A_t}$ is also in all the $L_p(\bar\Omega, P\otimes P'_{z_0})$. Using (2.6) and Burkholder-Davis-Gundy's inequalities, the same result holds for:

$$(2.7)\qquad \left|\int_0^{A_t\wedge T} C_{u\wedge A_{t'}}(\varphi_u^{*-1} b_x)(x_0)\,\delta B\right|, \qquad \left|\int_0^{A_t\wedge T} \varphi_s^{*-1} X_i(x_0)\,\delta w^i\right|.$$

It also holds for:

$$(2.8)\qquad \left|\int_0^{A_t\wedge T} C_{u\wedge A_{t'}}(\varphi_u^{*-1} b_x)(x_0)\, b(x_u, z_u)\, du\right|.$$

The principle of the proof is very close to the proof of Theorems 2.1 and 3.1 in Bismut [7], as used by Bismut-Michel for the proofs of Theorems 1.9 and 2.11 in [10]. We only sketch the proof, the reader being referred to [7]-[10] for further details.

We first assume that $X_0, X_1, \ldots, X_m$ and b have compact support. We then proceed as in Haussmann [14], Bismut [7] as modified by Williams [47]. Let $u=(u^1 \ldots u^m)$ be a bounded predictable process on $(\bar{\Omega}, P\otimes P'_{z_0})$, which is 0 for $t \geqq T$. For $l \in R$, let Z_T^l be the Girsanov exponential [40]-6:

$$Z_T^l = \exp\left\{ -l \int_0^T u^i \, \delta w^i - \frac{1}{2} l^2 \int_0^T |u|^2 \, ds \right\}. \tag{2.9}$$

Clearly Z_T^l is in all the $L_p(\bar{\Omega}, P\otimes P'_{z_0})$ and moreover $E^{P\otimes P'_{z_0}} Z_T^l = 1$.

Let R^l be the probability measure on $\bar{\Omega}$ given by:

$$R^l = Z_T^l \, d\,(P\otimes P'_{z_0}). \tag{2.10}$$

Now, by the fundamental property of the Girsanov transformation [40]-6 we know that if $w^{l,i}$ is the process:

$$w_t^{l,i} = w_t^i + \int_0^t l u^i \, ds \tag{2.11}$$

then under R^l, $(w^{l,1}, w^{l,2}, \ldots, w^{l,m}, B)$ is a $\{\bar{F}_t\}_{t \geq 0}$ Brownian martingale. In particular, under R^l, z_t is still a reflecting Brownian motion on $[0, +\infty[$, which is independent of w^l.

On $(\bar{\Omega}, P\otimes P'_{z_0})$, consider the stochastic differential equation:

$$\begin{cases} dx^l = X_0(x^l, z)\, dt + X_i(x^l, z) \, . \, dw_t^{l,i} + D(x^l) \, . \, dL, \\ x^l(0) = x_0. \end{cases} \tag{2.12}$$

It is then clear that under R^l, the probability law of (x^l, z) is identical to the law of (x, z) under $P\otimes P'_{z_0}$.

Set:

$$M_t^l = \exp\left\{ \int_0^t b(x_s^l, z_s) \, \delta B_s - \frac{1}{2} \int_0^t b(x_s^l, z_s)^2 \, ds \right\}. \tag{2.13}$$

It is clear that:

$$E^{P\otimes P'_{z_0}} [Z_T^l M_{A_t}^l f(A_t, x_{A_t}^l) \, h(\omega', x^l_{.\wedge A_t})] = E^{P\otimes P'_{z_0}} [M_{A_t} f(A_t, x_{A_t}) \, h(\omega', x_{.\wedge A_t})]. \tag{2.14}$$

[where equality is taken in $T^*_{x_0}(R^d)$]. Of course, (2.14) makes sense because of the support condition on f.

Since the l. h. s. of (2.14) is constant, its differential at $l=0$ is 0. Consider the differential equation:

$$dy^l=(\varphi_s^{*-1}\,X_i)\,(y^l,\,z)\,l u^i\,ds,\qquad y^l(0)=x_0. \tag{2.15}$$

Since $X_0, \ldots, X_m$ have compact support, if y does not belong to the projection of the supports of $X_0, \ldots, X_m$ onto R^d, $\varphi_t(\bar\omega, y)=y$ for any $t \geqq 0$. The vector fields $\varphi_t^{*-1}X_0, \ldots, \varphi_t^{*-1}X_m$ are then uniformly bounded (with an upper bound depending on $\bar\omega$), so that for each $\bar\omega$, (2.15) has a unique solution. By Theorem 4.1 in [6], we know that $x_t^l=\varphi_t(\bar\omega, y_t^l)$.

Now a standard argument shows that in (2.15), $l \to y^l_{.} \in \mathscr{C}([0, T]; R^d)$ is differentiable, and the differential at $l=0$ is given by:

$$\left.\frac{\partial y_t^l}{\partial l}\right|_{l=0}=\int_0^t(\varphi_s^{*-1}\,X_i)\,(x_0)\,u^i\,ds, \tag{2.16}$$

$l \to x^l_{.} \in \mathscr{C}([0, T]; R^d)$ is then differentiable and moreover:

$$\left.\frac{\partial x_t^l}{\partial l}\right|_{l=0}=\varphi_t^*\int_0^t(\varphi_s^{*-1}\,X_i)\,(x_0)\,u^i\,ds. \tag{2.17}$$

The same argument applies to $M^l_{A_t}$. In fact if $u_t(\bar\omega, .)$ is defined as in 1 (c), we have for $s \geqq 0$:

$$M_s^l=\exp\left[u_s(\bar\omega, y_s^l)-\int_0^s\frac{\partial u}{\partial y}(\bar\omega, y_u^l)\,.\,dy_u^l\right] \tag{2.18}$$

[of course the r. h. s. is a regularized version of (2.18) in the variable l]. $l \to M_s^l$ is then differentiable at $l=0$. The explicit computation of the differential shows that:

$$\left.\frac{\partial M^l_{A_t}}{\partial l}\right|_{l=0}=M_{A_t}\int_0^{A_t}\left\langle b_x(x_s, z_s),\,\varphi_s^*\int_0^s\varphi_v^{*-1}\,X_i\,u^i\,dv\right\rangle(\delta B-b(x_s, z_s)\,ds). \tag{2.19}$$

Using (2.17), it is clear that:

$$\frac{\partial}{\partial l}[f(A_t, x^l_{A_t})]_{l=0}=\left\langle d_x f(A_t, x_{A_t}),\,\varphi^*_{A_t}\int_0^{A_t}\varphi_s^{*-1}\,X_i\,u^i\,ds\right\rangle \tag{2.20}$$

$$\left[\frac{\partial}{\partial l}h(\omega', x^l_{.\wedge A_t})\right]_{l=0}=\int_{[0,T]}dv^{\omega', x_{.\wedge A_t}}(v)\,\varphi^*_{v\wedge A_t}\left[\int_0^{v\wedge A_t}\varphi_s^{*-1}\,X_i\,u^i\,ds\right]$$

$$\left[\frac{\partial}{\partial l}Z_T^l\right]_{l=0}=-\int_0^T u^i\,\delta w^i.$$

By reasoning as in [7], Theorem 2.1 and [10], Theorem 2.11—and of course using the fact that $f(A_t, x^i_{A_t})=0$ if $A_t \geqq T$—it can easily be seen that differentiation under the expectation sign is possible in the *l.h.s.* of (2.14), so that the following equality holds.

$$(2.21)\quad E^{P\otimes P'_{z_0}}\left[M_{A_t} \left\langle d_x f(A_t, x_{A_t}), \varphi^*_{A_t} \int_0^{T\wedge A_t} \varphi^{*-1}_s X_i u^i ds \right\rangle h(\omega', x_{.\wedge A_t}) \right]$$

$$+E^{P\otimes P'_{z_0}}\left[M_{A_t} f(A_t, x_{A_t}) \int_0^{T\wedge A_t} \left[\int_{[s,T]} \varphi^{*-1}_{v\wedge A_t} d\nu^{\omega', x_{.\wedge A_t}}(v) \right] \varphi^{*-1}_s X_i u^i ds \right]$$

$$+E^{P\otimes P'_{z_0}}\left[M_{A_t} \left[\int_0^{A_t} \left\langle \varphi^{*-1}_s b_x, \int_0^s \varphi^{*-1}_v X_i u^i dv \right\rangle \right.\right.$$

$$\left. (\delta B_s - b(x_s, z_s) ds) f(A_t, x_{A_t}) h(\omega', x_{.\wedge A_t}) \right]$$

$$= E^{P\otimes P'_{z_0}}\left[M_{A_t} \int_0^T u^i \delta w^i f(A_t, x_{A_t}) h(\omega', x_{.\wedge A_t}) \right].$$

As in [7], Theorem 2.1, it is easy to extend (2.21) to the case where $X_0, \ldots, X_m$, b do not necessarily have compact support. Using the integrability conditions pointed out at the beginning of the proof, (2.21) can also be extended to predictable processes $u=u^1 \ldots u^m$, which are 0 for $t \geqq T$, such that $E^{P\otimes P'_{z_0}} \int_0^T |u|^2 ds < +\infty$.

We now proceed as in [7], Theorem 3.1. Observe that in (2.21), the scalar processes $u^i_t (1 \leqq i \leqq m)$ may be replaced by processes with values in $T_{x_0}(R^d)$, so that equality (2.21) becomes an equality in $T_{x_0}(R^d) \otimes T^*_{x_0}(R^d)$. Take then in (2.21):

$$(2.22)\qquad u^i_s = 1_{s' \leqq T \wedge A_{t'}} (\varphi^*_s - 1 X_i)(x_0)$$

and take the trace in $T_{x_0}(R^d) \otimes T^*_{x_0}(R^d)$ of the corresponding equality. Using the fact that $f(A_t, x_{A_t})$ and $d_x f(A_t, x_{A_t})$ are 0 for $A_t \geqq T$, we obtain (2.5). □

Remark 1. — The fact that f has compact support is crucial for (2.5) to be true. In this respect, we now refer to (1.38), since in fact it is at this stage that the introduction of the supplementary component A_t is seen to be necessary from the point of view of the calculus of variations. However if one of the components of x, say x^d is strictly parabolic, i.e. we have that:

$$dx^d = X^d_0(x, z)\, dt,$$

where $X^d_0 \geqq \alpha > 0$, the introduction of A_t is unnecessary for (2.5) to make sense since if $f \in C^\infty_c(R^d)$, $f(x_{A_t})$ would be $\neq 0$ only for uniformly bounded A_t.

Also note that for $p \geqq 2$, if for a given (z, L), Z, Z' are given by (1.12), an obvious application of Itô's formula and Gronwall's lemma to the processes $|Z_s|^p$ and $|Z'_s|^p$ shows that:

$$(2.23)\qquad E^P |Z_s|^p \leqq C e^{C'(s+L_s)}; \qquad E^P |Z'_s|^p \leqq C e^{C'(s+L_s)}$$

(where C, C′ are fixed >0 constants). It is then feasible to take $s=A_t(\omega')$ in (2.23). However, since for $t>0$, $E^{P'} e^{C' A_t}=+\infty$, we find no adequate bound for $E^{P\otimes P'}|Z_{A_t}|^p$ and $E^{P\otimes P'}|Z'_{A_t}|^p$. This is another explanation of the necessity of introducing the component A_t, strongly connected with the result in (1.38).

However assume that b is equal to a constant $\delta\neq 0$. If $\delta^2\geqq 2C'$, then $E^{P'}\exp(C'-\delta^2/2)A_t<+\infty$. It is then obvious that for any $p\geqq 1$, if δ is large enough, Z_{A_t} and Z'_{A_t} are in $L_p(Q_{x_0,z_0})$. We then find that the size of the drift b has a direct influence on the regularity of the law of x_{A_t}. For other problems connected with the regularity of the law of x_{A_t}, we refer to Remark 3 in section 5.

Remark 2. — If $h(\omega', x)$ is F'_{A_t} measurable in the variable ω', we can now use Proposition 1.4 to replace everywhere $E^{P\otimes P'_{z_0}}[M_{A_t}\ldots]$ by $E^{Q(x_0,z_0)}[\ldots]$.

(*b*) *Truncated integration by parts on the variable x.*

Let σ be a C^∞ function defined on $R^d\otimes R^d$ with values in [0, 1] such that $\sigma(D)=1$ if $\|D\|\leqq 1$, and $\sigma(D)=0$ if $\|D\|\geqq 2$.

For $N\geqq 1$, the function ρ_N is defined on $R^d\otimes R^d$ by:

$$\text{D invertible}\quad \rho_N(D)=\sigma\left(\frac{D^{-1}}{N}\right);$$

$$\text{D non invertible}\quad \rho_N(D)=0.$$

ρ_N is clearly a C^∞ function with bounded differentials. Also observe that the functions $D\to\partial^k\rho_N/\partial D^k(D)(D^{-1})^l$ can be everywhere defined on $R^d\otimes R^d$ by setting:

$$\frac{\partial^k\rho_N}{\partial D^k}(D)(D^{-1})^l=0\quad\text{if D non invertible.}\tag{2.24}$$

These functions are of course C^∞ bounded with bounded differentials on the whole $R^d\otimes R^d$.

Using the convention (2.24), we now have the following result.

THEOREM 2.3. — *Let $f\in C_c^\infty(R^+\times R^d)$. Let $Y(x)$ be a bounded C^∞ vector field defined on R^d with values in R^d, whose components belong to $C_b^\infty(R^d)$. Then for any $N\geqq 1$, $t, t'>0$ with $t'\leqq t$, $x_0\in R^d$, if x_s is the process $\varphi_s(\bar\omega, x_0)$, the following relations holds:*

$$\begin{aligned}(2.25)\quad & E^{P\otimes P'_{z_0}}[\rho_N(C_{A_{t'}})M_{A_t}(Y_x f)(A_t, x_{A_t})]\\ &=E^{P\otimes P'_{z_0}}[\rho_N(C_{A_{t'}})M_{A_t}f(A_t, x_{A_t})\left\{\left\langle\varphi_{A_t}^{*-1}Y, C_{A_{t'}}^{-1}\int_0^{A_{t'}}\varphi_s^{*-1}X_i\,\delta w^i\right\rangle\right.\\ &\quad-\int_0^{A_{t'}}\langle C_{A_{t'}}^{-1}[\varphi_s^{*-1}X_i, \varphi_{A_t}^{*-1}Y], \varphi_s^{*-1}X_i\rangle\,ds\\ &+\int_0^{A_{t'}}ds\,\langle C_{A_{t'}}^{-1}\varphi_s^{*-1}X_i, \varphi_{A_t}^{*-1}Y\rangle\int_0^s\langle C_{A_{t'}}^{-1}[\varphi_v^{*-1}X_j, \varphi_s^{*-1}X_i], \varphi_v^{*-1}X_j\rangle\,dv\end{aligned}$$

$$+\int_0^{A_{t'}} ds \left\langle C_{A_{t'}}^{-1}\, \varphi_s^{*-1} X_i, \int_0^s \langle [\varphi_v^{*-1} X_j, \varphi_s^{*-1} X_i], C_{A_{t'}}^{-1}\, \varphi_{A_t}^{*-1} Y \rangle\, \varphi_v^{*-1} X_j \right\rangle dv$$

$$-\left\langle C_{A_{t'}}^{-1}\, \varphi_{A_t}^{*-1} Y, \int_0^{A_t} C_{s\wedge A_{t'}}\, \varphi_s^{*-1}\, b_x (\delta B_s - b(x_s, z_s)\, ds) \right\rangle \Big\}\Big]$$

$$-E^{P\otimes P'_{z_0}}\Big[M_{A_t} f(A_t, x_{A_t}) \Big\langle \frac{\partial \rho_N}{\partial C}(C_{A_{t'}}), \int_0^{A_{t'}} ds\, \varphi_s^{*-1} X_i$$

$$\otimes \int_0^s [\varphi_v^{*-1} X_j, \varphi_s^{*-1} X_i] (\langle C_{A_{t'}}^{-1}\, \varphi_{A_t}^{*-1} Y, \varphi_v^{*-1} X_j \rangle)\, dv$$

$$+\int_0^{A_{t'}} ds \int_0^s \langle C_{A_{t'}}^{-1}\, \varphi_{A_t}^{*-1} Y, \varphi_v^{*-1} X_j \rangle [\varphi_v^{*-1} X_j, \varphi_s^{*-1} X_i]\, dv \otimes \varphi_s^{*-1} X_i \Big\rangle \Big].$$

Proof. — Observe that (2.25) still makes sense because of the crucial fact that f has compact support. In comparison with the arguments given in the proof of Theorem 2.2, the only difference comes the introduction of $C_{A_{t'}}^{-1}$, but in fact, since $\rho_N(C_{A_{t'}})$ or $\partial\rho_N/\partial C(C_{A_{t'}})$ also appear, they make the corresponding expressions to be bounded.

To obtain (2.25), it suffices to choose h given by:

$$h = \rho_N(C_{A_{t'}})\, C_{A_{t'}}^{-1}\, \varphi_{A_t}^{*-1} Y. \tag{2.26}$$

Of course h does not satisfy the assumptions of Theorem 2.2, since it also depends on the trajectory of $\partial\varphi_t/\partial x(\bar{\omega}, x_0)$, and moreover it is not bounded. Using the argument in [7], it is in fact easy to extend Theorem 2.2 to such a h. The explicit computations being very similar to what is done in Theorem 4.2 of [7] and in [10], Theorem 2.11, we refer to [7]-[10] for the details. □

(*c*) *Integration by parts in the variable x: the localizable case.*

We proceed here as in Malliavin ([29]-[30]), Stroock ([36]-[37]), Bismut [7].

We will in fact obtain a formula of integration by parts on the boundary semi-group in the "localizable" case. This is in fact the case where the estimates which we will later do are "localizable" (*see* section 5).

THEOREM 2.4. — *Assume that* $(x_0, z_0) \in R^d \times R^+$ *and* $t' > 0$ *are such that:*

(*a*) $P\otimes P'_{z_0}$ *a. s.,* $C_{A_{t'}}$ *is a. s. invertible.*

(*b*) *For any* $T \geq 0$, *and any* $p \geq 1$, $1_{A_{t'}\leq T} \,|\, C_{A_{t'}}^{-1} \,|$ *is in* $L_p(\bar{\Omega}, P\otimes P'_{z_0})$.

Then for any multi-index m, and any $t \geq t'$, *there exists a random variable* B_t^m *such that:*

(*a*) *For any* $T > 0$ *and any* $p \geq 1$, $1_{A_t \leq T} B_t^m$ *is in* $L_p(\bar{\Omega}, P\otimes P'_{z_0})$.

(*b*) *For any* $f \in C_c^\infty(R\times R^d)$, *if* x_s *is the process* $\varphi_s(\bar{\omega}, x_0)$:

$$E^{P\otimes P'_{z_0}}\left[M_{A_t} \frac{\partial^m f}{\partial x^m}(A_t, x_{A_t}) \right] = E^{P\otimes P'_{z_0}}[f(A_t, x_{A_t})\, B_t^m]. \tag{2.27}$$

Proof. — For $t \geqq t'$, we first consider formula (2.25). Observe that if D is invertible, we have the uniform bound:

$$\left| \frac{\partial \rho_N}{\partial D} \right| \leqq k \, \| D^{-1} \|. \tag{2.28}$$

Make then $N \to +\infty$ in (2.25). Clearly, since $C_{A_{t'}}$ is a. s. invertible, $\rho_N(C_{A_{t'}}) \to 1$ a. s. Recall that f has compact support; namely there is $T \geqq 0$ such that $f(a, x) = 0$ for $a \geqq T$. In the r. h. s. of (2.25), we can then write everywhere $E^{P \otimes P'_{z_0}} [1_{A_t \leqq T} \ldots]$ instead of $E^{P \otimes P'_{z_0}} [\ldots]$. Now all the terms appearing in the r. h. s. of (2.25) are in all the $L_p(\bar{\Omega}, P \otimes P'_{z_0})$. Using the uniform bound (2.28), it is then possible to take the limit in (2.25) and obtain the formula corresponding to $\rho_N = 1$. (2.27) has then been proved for $|m| = 1$.

Using the fact that $1_{A_{t'} \leqq T} | C_{A_{t'}}^{-1} |$ is in all the $L_p(\bar{\Omega}, P \otimes P'_{z_0})$, it is possible to iterate the procedure as in [29]-[30]-[36]-[37]-[7] so as to get (2.27). □

Remark 3. — Any function f defined on $R^+ \times R^n$ can be extended to Δ by setting $f(\Delta) = 0$. If in Theorem 2.4, $z_0 = 0$, we find that, under the assumptions of this theorem and using Proposition 1.4, for any $a_0 \geqq 0$:

$$E^{Q(x_0, 0)} \left[\frac{\partial^m f}{\partial x^m} (a_0 + A_t, x_{A_t}) \right] = E^{P \otimes P'} [f(a_0 + A_t, x_{A_t}) B_t^m]. \tag{2.29}$$

(*d*) *Integration by parts in the variable x: the non localizable case*

We will now treat the case where the necessary estimates are in general not localizable. This will be especially useful when the boundary semi-group is slowly regularizing [*see* section 5 (*e*)].

We will use here a procedure of step by step integration by parts, which reflects the Markov property of the system in a stronger way than what has been done before (of course the fact that C_t increases with t—in the sense of quadratic forms—is related to the Markov property of the considered processes). This procedure had been developed by us in [8]-[9] for the calculus of variations on jump processes, and it is no suprise that it should appear again here, since the boundary processes are jump processes.

We will write now $C_s^{x_0}$ instead of C_s, since the explicit dependence of $C_s^{x_0}$ on x_0 will be needed.

THEOREM 2.5. — *Assume that $t' > 0$ is such that:*

(*a*) *For every $x \in R^d$, $C_{A_{t'}}^x$ is $P \otimes P'$ a. s. invertible.*

(*b*) *For every $T \geqq 0$, there is $q > 2$ such that for any $x \in R^d$, $1_{A_{t'} \leqq T} | [C_{A_{t'}}^x]^{-1} |$ is in $L_q(\bar{\Omega}, P \otimes P')$, and its norm in $L_q(\bar{\Omega}, P \otimes P')$ can be bounded independently of $x \in R^d$.*

Then for any $(x_0, z_0) \in R^d \times R^+$, any multi-index m, and any $t \geqq |m| t'$, on $(\bar{\Omega}, P \otimes P'_{z_0})$, there exists a random variable D_t^m having the following properties:

(*a*) *For any $T \geqq 0$, $1_{A_t \leqq T} D_t^m$ is $P \otimes P'_{z_0}$ integrable.*

(*b*) *For any* $f \in C_c^\infty(R \times R^d)$, *if* x_s *is the process* $\varphi_s(\bar{\omega}, x_0)$:

$$E^{P \otimes P'_{z_0}}\left[M_{A_t} \frac{\partial^m f}{\partial x^m}(A_t, x_{A_t}) \right] = E^{P \otimes P'_{z_0}}[f(A_t, x_{A_t}) D_t^m]. \tag{2.30}$$

Proof. — First assume that $z_0 = 0$, and $|m| = 1$. We can then proceed as in Theorem 2.4. In fact since f has compact support, everything in (2.25), except the terms where $C_{A_{t'}}$ appears are in all the $L_p(\bar{\Omega}, P \otimes P')$. Now $|C_{A_{t'}}^{-1}|$ is in $L_q(\bar{\Omega}, P \otimes P')$ with $q > 2$. Noting that at most $|C_{A_{t'}}^{-1}|^2$ appears in (2.25), and still using the uniform bound (2.28) we obtain formula (2.25) with $\rho_N = 1$.

The critical step will now come from a different iteration procedure. We will develop the argument for $|m| = 2$. Set:

$$Y^1 = \frac{\partial}{\partial x^{l_1}}, \qquad Y^2 = \frac{\partial}{\partial x^{l_2}}, \quad 1 \leqq l^1, \quad l^2 \leqq d.$$

We rewrite formula (2.25)—with $\rho_N = 1$—with $Y = Y^1$ and f replaced by $Y^2 f$ in a slightly different way. We have for $t \geqq t'$:

$$\begin{aligned}
&E^{P \otimes P'}[M_{A_t} Y_x^1 Y_x^2 f(A_t, x_{A_t})] \\
&= E^{P \otimes P'}\Bigg[M_{A_{t'}} \frac{M_{A_t}}{M_{A_{t'}}} (Y_x^2 f)(A_{t'} + A_t - A_{t'}, (\varphi_{A_t} \circ \varphi_{A_{t'}}^{-1})(x_{A_{t'}})) \\
&\quad \Bigg\{ \left\langle \varphi_{A_{t'}}^{*-1} (\varphi_{A_t} \circ \varphi_{A_{t'}}^{-1})^{*-1} Y^1, C_{A_{t'}}^{-1} \int_0^{A_{t'}} \varphi_s^{*-1} X_i \, \delta w^i \right\rangle \\
&\quad - \int_0^{A_{t'}} \langle C_{A_{t'}}^{-1} \varphi_{A_{t'}}^{*-1} [(\varphi_s \circ \varphi_{A_{t'}}^{-1})^{*-1} X_i, (\varphi_{A_t} \circ \varphi_{A_{t'}}^{-1})^{*-1} Y^1], \varphi_s^{*-1} X_i \rangle \, ds \\
&\quad + \int_0^{A_{t'}} ds \langle C_{A_{t'}}^{-1} \varphi_s^{*-1} X_i, \varphi_{A_{t'}}^{*-1} (\varphi_{A_t} \circ \varphi_{A_{t'}}^{-1})^{*-1} Y^1 \rangle \int_0^s \langle C_{A_{t'}}^{-1} [\varphi_v^{*-1} X_j, \varphi_s^{*-1} X_i], \\
&\quad \varphi_v^{*-1} X_j \rangle \, dv + \int_0^{A_{t'}} ds \left\langle C_{A_{t'}}^{-1} \varphi_s^{*-1} X_i, \int_0^s \langle [\varphi_v^{*-1} X_j, \varphi_s^{*-1} X_i], \right. \\
&\quad \left. C_{A_{t'}}^{-1} \varphi_{A_{t'}}^{*-1} (\varphi_{A_t} \circ \varphi_{A_{t'}}^{-1})^{*-1} Y^1 \rangle \varphi_v^{*-1} X_j \right\rangle dv \\
&\quad - \left\langle C_{A_{t'}}^{-1} \varphi_{A_{t'}}^{*-1} (\varphi_{A_t} \circ \varphi_{A_{t'}}^{-1})^{*-1} Y^1, \int_0^{A_{t'}} C_s \varphi_s^{*-1} b_x (\delta B_s - b(x_s, z_s) \, ds) \right. \\
&\quad \left. + C_{A_{t'}} \varphi_{A_{t'}}^{*-1} \int_{A_{t'}}^{A_t} (\varphi_{A_t} \circ \varphi_{A_{t'}}^{-1})^{*-1} b_x(x_s, z_s)(\delta B - b(x_s, z_s) \, ds) \right\rangle \Bigg\} \Bigg].
\end{aligned} \tag{2.31}$$

Now since L is an additive functional for z, it is clear that:

$$A_t - A_{t'} = A_{t-t'} \circ \bar{\theta}_{A_{t'}}. \tag{2.32}$$

Using the corollary of Theorem 1.1, we find that:

$$\varphi_{A_t}(\bar{\omega}, .)=(\varphi_{A_{t-t'}})(\bar{\theta}_{A_{t'}}\bar{\omega})\circ\varphi_{A_{t'}}(\bar{\omega}, .). \tag{2.33}$$

In the r. h. s. of (2.31), we can take conditional expectations with respect to $\bar{F}_{A_{t'}}$ of the terms which are *not* $\bar{F}_{A_{t'}}$-measurable.

Moreover since $z_{A_{t'}}=0$, it is obvious that the conditional law of $\bar{\theta}_{A_{t'}}\bar{\omega}$ given $\bar{F}_{A_{t'}}$ is equal to $P\otimes P'$. We will use these facts to rewrite the r. h. s. in a still different way. Due to the length of the equation, we write explicitly only the first term (we use M^{x_0} instead of M since the starting point is now important):

$$\begin{aligned}\int d(P\otimes P')(\bar{\omega})\otimes d(P\otimes P')(\bar{\omega}')\Bigg[& M^{x_0}_{A_{t'}}(\bar{\omega})\,M^{x_{A_{t'}}(\bar{\omega})}_{A_t-A_{t'}}(\bar{\omega}')\\ &\times\Big\langle (Y^2_x f)(A_{t'}(\bar{\omega})+A_{t-t'}(\bar{\omega}'), \varphi_{A_{t-t'}(\bar{\omega}')}(\bar{\omega}', x_{A_{t'}}(\bar{\omega})))\\ &\Big\{\Big\langle \varphi^{*-1}_{A_{t'}}(\bar{\omega}, .)(\varphi^{*-1}_{A_{t-t'}}(\bar{\omega}', .)\,Y^1)(x_{A_{t'}}(\bar{\omega})), C^{-1}_{A_{t'}}(\bar{\omega})\\ &\qquad\times\int_0^{A_{t'}(\bar{\omega})}(\varphi^*_s X_i)(\bar{\omega})\,\delta w^i(\bar{\omega})\Big\rangle-\ldots\Big\}\Bigg].\end{aligned} \tag{2.34}$$

Now observe that for each fixed $\bar{\omega}\in\bar{\Omega}$, we can restart the calculus of variations in the variable $\bar{\omega}'$, since as functions of $\bar{\omega}'$, all the random variables which appear in (2.34) are of the type already met in the proof of Theorem 2.4. If $t-t'\geqq t'$, i. e. if $t\geqq 2t'$, for each $\bar{\omega}$, we can then produce a variation of $\bar{\omega}'$ on the time interval $[0, t']$ so as to express (2.34) in terms of:

$$f(A_{t'}(\bar{\omega})+A_{t-t'}(\bar{\omega}'), \qquad \varphi_{A_{t-t'(\bar{\omega}')}}(\bar{\omega}', x_{A_{t'}}(\bar{\omega})),$$

i. e. the differentials of f disappear.

Or course $[C^{x_{A_{t'}}(\bar{\omega})}_{A_{t-t'}}(\bar{\omega}')]^{-1}$ appears (at most in square form), but this causes no difficulty because of assumption (*b*) in Theorem 2.5, and of the fact that f has compact support.

Using (1.12), it is obvious that if in (1.5), K is equal to $[0, T]\times\{x\}$ $(x\in R^d)$, the bounds in $L_p(\bar{\Omega}, P\otimes P'_{z_0})$ of the random variables appearing in (1.5) can be made independent of x, z_0. Using the fact that in the assumption (*b*) of our theorem the bounds can be made independent of x, it is then clear that for each $\bar{\omega}$, the random variables appearing in the integration by parts process in the variable ω', as functions of $\bar{\omega}'$, are integrable in $\bar{\omega}'$, with a L_1-norm bounded independently of $\bar{\omega}$. Using Fubini's theorem and deconditioning the obtained formula, we obtain (2.30) with $z_0=0$, $|m|=2$.

Another way of formulating what we have done would be to start again with formula (2.25) for $\rho_N=1$ with $Y=Y^1$ and f replaced by $Y^2 f$ and to make a calculus of variations on the interval $[A_{t'}, A_{2t'}]$ (which means that in the proof of Theorem 2.2, u would be $\neq 0$ only on the interval $[A_{t'}, A_{2t'}]$).

Also observe that in the final formula the fact that $(C_{A_{t'}}^{x_0})^{-1}$ and $(C_{A_{t'}-A_t}^{x_{A_{t'}}})^{-1}$ appear both in square form is no obstacle. In fact for any $T \geqq 0$, $t \geqq 2t'$:

$$1_{A_t \leqq T} \,|\, [C_{A_{t'}}^{x_0}]^{-1} \,|\, \,|\, [C_{A_{2t'}}^{x_{A_{t'}}} - A_{t'}]^{-1} \,|,$$

is in $L_q(\bar{\Omega}, P \otimes P')$, because of the Markov property of the considered processes (do *not* use Hölder's inequality).

The case of a general m is obtained by applying repeatedly the Markov property, or equivalently by applying the calculus of variations on $[0, A_{t'}]$, $[A_{t'}, A_{2t'}]$, ..., $[A_{(|m|-1)t'}, A_{|m|t'}]$.

When z_0 is $\neq 0$, the same procedure is applied by starting the variation after A_0 (which is $\neq 0$), i. e. doing the corresponding calculus on $[A_0, A_{t'}]$, $[A'_t, A_{2t'}]$... □

Remark 4. — Step by step integration by parts is described in detail in [8]-section 4. As indicated in the proof of Theorem 2.5, the basic idea is to use the calculus of variations repeated by on $[0, A_{t'}]$, $[A_{t'}, A_{2t'}]$, ..., $[A_{(|m|-1)t'}, A_{|m|t'}]$. Details can be worked out easily.

Remark 5. — So far, we have been able to obtain an integration by parts formula in the variable x. If the assumptions of Remark 1 are verified, this would be enough to study the regularity of the semi-group associated to the Markov process x_{A_t}. In this case, the reader can directly read section 5. If we are interested in (A_t, x_{A_t})—as we must be if the assumptions of Remark 1 are not verified—we must find a way of developing a calculus of variations on the variable A_t. This is what we will do in the next two sections.

3. The stochastic calculus on the excursions of the reflecting Brownian motion

This section is somewhat independent of the remainder of the text. The reader who is essentially interested in the analytical aspects of the calculus of variations only needs to read paragraphs (*a*), (*b*) and (*c*). However a complete understanding of the results of section 4 is easied by a quick look at (*d*), (*e*).

In (*a*), we recall the main results of the Itô theory of excursions when applied to the reflecting Brownian notion. Special attention is given to the results of Itô-McKean [18] and Williams ([45], [46]).

In (*b*), a point process description of (w_t, z_t) is given using (*a*) and also Ikeda-Watanabe [17]. A Poisson point process is then defined, whose natural filtration is $\{\bar{F}_{A_t}\}_{t \geqq 0}$.

In (*c*) the main results on the stochastic calculus on point processes are recalled (Meyer [31], Jacod [19], Ikeda-Watanabe [17]). The relation between the $\{\bar{F}_t\}_{t \geqq 0}$ stochastic calculus and the $\{\bar{F}_{A_t}\}_{t \geqq 0}$ stochastic calculus is developed. In particular the quadratic variation of two $\{\bar{F}_{A_t}\}_{t \geqq 0}$ martingales N', $\tilde{N}'$, which is written $[N', \tilde{N}']_t$ is computed in terms of the representation of N', $\tilde{N}'$ as $\{\bar{F}_t\}_{t \geqq 0}$ stochastic integrals.

In (*d*), the effect of a Girsanov transformation is studied on the filtrations $\{\bar{F}_t\}_{t \geqq 0}$ and $\{\bar{F}_{A_t}\}_{t \geqq 0}$. On $\{\bar{F}_t\}_{t \geqq 0}$, the situation is well known (Stroock-Varadhan [40]-6). On $\{\bar{F}_{A_t}\}_{t \geqq 0}$, using the results of Jacod [19], the effect of the Girsanov transformations is to produce an absolutely continuous transformation on the Lévy measure of the considered point process. The corresponding Doléans-Dade equation [19] is exhibited.

In (*e*), special attention is given to the relation between the effects of the Girsanov transformation on $\{\bar{F}_t\}_{t \geqq 0}$ and $\{\bar{F}_{A_t}\}_{t \geqq 0}$ stochastic calculus.

In the whole section, we use the results of Ikeda-Watanabe [17] (p. 307-320), who give us the essentiel tool which is the excursion stochastic integral.

(*a*) *The Itô's theory of excursions on the reflecting Brownian motion.*

We first recall a few facts concerning the Itô's theory of excursions applied to the reflecting Brownian motion on $[0, +\infty[$. Our main sources are Itô-McKean [18], Williams ([45]-[46]), Ikeda-Watanabe [17], and Jeulin [21].

We use the notations in Ikeda-Watanabe [17].

DEFINITION 3.1. — $\mathscr{W}^+$ is the set of continuous functions $e(s)$ defined on R^+ with values in R^+ such that:

(*a*) $e(0)=0$;

(*b*) There exists $\sigma(e)$ such that $0<\sigma(e)\leqq +\infty$, for which

$$0<s<\sigma(e), \qquad e(s)>0,$$

$$s\geqq \sigma(e), \qquad e(s)=0.$$

$\mathscr{W}^+$ will be the set of excursions of the reflecting Brownian motion z_t.

For notational convenience, we take δ to be a point isolated from $\mathscr{W}^+$, corresponding to the empty excursion.

DEFINITION 3.2. — On (Ω', P'), we define the process e_t adapted to $\{F'_{A_t}\}_{t \geqq 0}$ with values in $\mathscr{W}^+ \cup \{\delta\}$ in the following way:

(*a*) If $A_{t^-}<A_t$, e_t is the element of $\mathscr{W}^+$ defined by:

$$\begin{aligned} e_t(s) &= z_{s+A_{t^-}} \quad \text{for} \quad s\leqq A_t - A_{t^-}, \\ &= 0 \quad \text{for} \quad s > A_t - A_{t^-}. \end{aligned}$$

(*b*) If $A_{t^-}=A_t$, $e_t=\delta$.

Now the Itô's theory of excursions tells us that e_t is a stationary Poisson point process on $(\Omega', \{F'_{A_t}\}_{t \geqq 0}, P')$ (*see* [17], p. 123). Its characteristic measure n^+ is a σ-finite $\geqq 0$ measure on $\mathscr{W}^+$.

Here are a few facts about n^+.

DEFINITION 3.3. — B denotes the probability law on $\mathscr{W}^+$ of the Bes (3) process starting at 0, conditioned on $e_1=0$, and stopped at time 1. C denotes the probability law on Ω' of the Bes (3) process starting at 0.

For Bessel diffusions, we refer to Itô-McKean [18], and Ikeda-Watanabe [17].

THEOREM 3.4. — *Let f be the mapping defined on $\mathrm{R}^+ \times \mathscr{W}^+$ with values in $\mathscr{W}^+$ defined by:*

$$(t, \bar{r}) \in \mathrm{R}^+ \times \mathscr{W}^+ \to e(s)=t^{1/2}\, \bar{r}(s/t), \qquad 0 \leqq s \leqq t. \tag{3.1}$$

Then n^+ is the image by f of the σ-finite measure F on $\mathrm{R}^+ \times \mathscr{W}^+$ given by:

$$d\mathrm{F}(t, \bar{r}) = 1_{t \geqq 0} \frac{dt}{\sqrt{2\pi t^3}} \otimes d\mathrm{B}(\bar{r}). \tag{3.2}$$

Proof: This result is contained in implicit form in Ikeda-Watanabe [17], p. 123 and 224 (also *see* Itô-McKean [18], p. 75-81). □

Observe that the mapping (3.1) is one-to-one, since in fact in (3.1), $t=\sigma(e)$. For $e \in \mathscr{W}^+$, $\bar{r}$ is its scaled excursion i. e.:

$$\bar{r}(s) = \frac{1}{[\sigma(e)]^{1/2}}\, e(s\, \sigma(e)) \qquad 0 \leqq s \leqq 1. \tag{3.3}$$

We will use the notation $\bar{r}$ without further mention.

Also observe that (3.2) implies that under P', A_t is a stable process of exponent 1/2 and rate $\sqrt{2}$ ([18], p. 27).

A second useful description of n^+ is provided by Williams ([45]-[46]).

DEFINITION 3.5. — For $(b, r) \in \mathrm{R}^+ \times \Omega'$, we define $\mathrm{T}(b, r)$ by:

$$\mathrm{T}(b, r) = \inf\{ t \geqq 0;\ r(t) > b \}. \tag{3.4}$$

We then have:

THEOREM 3.6. — *Let g be the mapping defined on $\mathrm{R}^+ \times \Omega' \times \Omega'$ with values in $\mathscr{W}^+$:*

$$\begin{aligned}(b, r, r') \to e(s) &= r(s), && 0 \leqq s \leqq \mathrm{T}(b, r)\\ &= r'_{\mathrm{T}(b,r)+\mathrm{T}(b,r')-s}, && \mathrm{T}(b, r) < s \leqq \mathrm{T}(b, r) + \mathrm{T}(b, r').\end{aligned} \tag{3.5}$$

Then n^+ is the image by g of the σ-finite measure G on $\mathrm{R}^+ \times \Omega' \times \Omega'$ given by:

$$1_{b \geqq 0} \frac{db}{b^2} \otimes d\mathrm{C}(r) \otimes d\mathrm{C}(r'). \tag{3.6}$$

Proof. — The full proof of this result appears in Rogers [33]. □

Obviously (3.6) also reflects the known fact that the law of $\| e \| = \sup_{0 \leqq t \leqq \sigma(e)} | e(t) |$ under n^+ is $1_{b \geqq 0}\, db/b^2$.

Of course Theorems 3.5 and 3.6 imply each other. In the sequel, we shall use these two descriptions of n^+ in the most convenient form.

(*b*) *A point process description of the process* (w_t, z_t).

We still follow Ikeda-Watanabe [17].

DEFINITION 3.7. — $\mathscr{W}_0$ is the subset of $\Omega \times \mathscr{W}^+$ consisting of the element $(\varepsilon, e) \in \Omega \times \mathscr{W}^+$ such that for any $s \geqq 0$, $\varepsilon(s) = \varepsilon(s \wedge \sigma(e))$.

DEFINITION 3.8. — On $(\bar{\Omega}, P \otimes P')$, we define the process (ε_t, e_t) adapted to $\{\bar{F}_{A_t}\}_{t \geqq 0}$ with values in $\mathscr{W}_0 \cup \{\delta\}$ in the following way:

(*a*) If $A_{t^-} < A_t$, then:

$$\begin{aligned} &\varepsilon_t(s) = w_{A_{t^-}+s} - w_{A_{t^-}}; && e_t(s) = z_{A_{t^-}+s} && \text{for } s \leqq A_t - A_{t^-}, \\ &\varepsilon_t(s) = w_{A_t} - w_{A_{t^-}}; && e_t(s) = 0 && \text{for } s > A_t - A_{t^-}; \end{aligned}$$

(*b*) If $A_{t^-} = A_t$, $(\varepsilon_t, e_t) = \delta$.

We then have:

THEOREM 3.9. — *The process* (ε_t, e_t) *is a Poisson point process on* $(\bar{\Omega}, \{\bar{F}_{A_t}\}_{t \geqq 0}, P \otimes P')$. *Its characteristic measure* n *on* $\mathscr{W}_0$ *is the image measure by the mapping* i:

$$(w, e) \in \Omega \times \mathscr{W}^+ \to (w_{.\wedge \sigma(e)}, e_.) \tag{3.8}$$

of the σ-finite measure I *on* $\Omega \times \mathscr{W}^+$:

$$dP(w) \otimes dn^+(e). \tag{3.9}$$

Proof. — Since for any $t \geqq 0$, the law under P of $(w_{t+s} - w_t)_{s \geqq 0}$ is still equal to P, using the independence of w and z under $P \otimes P'$, the result obviously follows from the corresponding result on the process z. For more details, *see* [17], p. 215, p. 307). □

Let P^0 be the law of $w_{.\wedge 1}$ under P. We have:

COROLLARY. — *The measure* n *on* $\mathscr{W}_0$ *is the image measure by the mapping* j:

$$\begin{aligned} (t, \bar{w}, \bar{r}) \in R^+ \times \mathscr{W}_0 \to (\varepsilon(s), e(s)) &= (t^{1/2}\, \bar{w}(s/t),\ t^{1/2}\, \bar{r}(s/t)), && s \leqq t, \\ &= (t^{1/2}\, \bar{w}(1), 0), && s > t, \end{aligned} \tag{3.10}$$

of the σ-finite measure $\mathscr{J}$ *on* $R^+ \times \Omega \times \mathscr{W}^+$:

$$1_{t \geqq 0} \frac{dt}{\sqrt{2 \pi t^3}} \otimes dP^0(\bar{w}) \otimes dB(\bar{r}). \tag{3.11}$$

Proof. — Using Theorems 3.4 and 3.9, and the scaling invariance of P (3.11) is obvious. ■

Remark 1. — $(\bar{w}, \bar{r})$ represents the scaled excursion of (w, z). In the sequel, we will use this notation without further mention.

DEFINITION 3.10. — On Ω', the stochastic processes D_t and $\bar{L}_t$ are defined by:

$$\left\{\begin{array}{l} D_t=\inf \{ s>t;\ z_s=0 \}, \\ \bar{L}_t=\sup \{ s<t;\ z_s=0 \}. \end{array}\right. \tag{3.12}$$

Clearly D_t is a right continuous increasing process and $\bar{L}_t$ is a left continuous $\{ \bar{F}_t \}_{t \geqq 0}$-predictable increasing process.

We have the following elementary result.

PROPOSITION 3.11. — *On* $(\bar{\Omega}, P \otimes P')$, *the filtration generated by the point process* (ε_t, e_t) *is equal to* $\{ \bar{F}_{A_t} \}_{t \geqq 0}$.

Proof. — For $A_{s^-} \leqq u < A_s$, we know that $z_t = e_s(u - A_{s^-})$. Moreover on the complementary set of $\bigcup_s [A_{s^-}, A_s[$, z is equal to 0. For each $t>0$, the trajectory $z_{s \wedge A_t}$ is then a function of the excursions $e_s (s \leqq t)$.

Moreover on $(\bar{\Omega}, P \otimes P')$, w is a $\{ F_t \otimes F'_\infty \}_{t \geqq 0}$ martingale. Since $P \otimes P'$ a. s., the complementary set of $\bigcup_s [A_{s^-}, A_s[$ in R^+ is dt-negligible, we see, using Doob's inequality, that:

$$w_s = \lim_{\varepsilon \downarrow 0} \int_0^s 1_{D_u - \bar{L}_u > \varepsilon}\, \delta w_u, \tag{3.13}$$

where in (3.13) the limit is taken in probability uniformly on every compact set in R^+. Now the r. h. s. of (3.13) is equal to:

$$\underset{\substack{v < L_s \\ A_v - A_{v^-} > \varepsilon}}{S} [\varepsilon_v (A_v - A_{v^-})] + 1_{D_s - \bar{L}_s > \varepsilon}\, \varepsilon_{L_s} (s - \bar{L}_s), \tag{3.14}$$

Now the set $(z_s = 0)$ is the complementary of $\bigcup_s]A_{s^-}, A_s[$. It is then easy to see that if $s \leqq A_t$, (3.14) only depends on the trajectory of $A_v (v \leqq t)$ and the excursions $\varepsilon_v (v \leqq t)$.

The proposition is proved. ■

(*c*) *The stochastic calculus on the point process associated to* (w_t, z_t).

At this stage, we start using systematically some general concepts of the theory of stochastic integration. The reader is referred to Jacod [19], Meyer [31], Ikeda-Watanabe [17] for a complete information.

Recall that for each $t \geqq 0$, D_t is a $\{ \bar{F}_t \}_{t \geqq 0}$ stopping time. The filtration $\{ \bar{F}_{D_t} \}_{t \geqq 0}$ is then well defined.

The filtration $\{ \bar{F}_{D_t} \}_{t \geqq 0}$ has been studied — in the frameworks of regenerative sets — by Maisonneuve [27] (also *see* Maisonneuve-Meyer [28]).

DEFINITION 3.12. — $\mathscr{P}, \mathscr{P}', \mathscr{P}''$ (resp. $\mathscr{O}, \mathscr{O}', \mathscr{O}''$) denote the predictable (resp. optional) σ-fields on $R^+ \times \bar{\Omega}$ associated to the filtrations $\{\bar{F}_t\}_{t \geqq 0}$, $\{\bar{F}_{A_t}\}_{t \geqq 0}$, $\{\bar{F}_{D_t}\}_{t \geqq 0}$.

The filtration $\{\bar{F}_{D_t}\}_{t \geqq 0}$ corresponds to the filtration $\{\bar{F}_{A_t}\}_{t \geqq 0}$ in natural time-scale. Namely, we have the following well-known result:

PROPOSITION 3.13. — *If* H_t *is a predictable* (*resp. optional*) *process on* $(\bar{\Omega}, \{\bar{F}_{A_t}\}_{t \geqq 0}, P \otimes P')$ *then* H_{L_t} *is predictable* (*resp. optional*) *on* $(\bar{\Omega}, \{\bar{F}_t\}_{t \geqq 0}, P \otimes P')$ [*resp.* $(\bar{\Omega}, \{\bar{F}_{D_t}\}_{t \geqq 0}, P \otimes P')$]. *Conversely, if* H'_t *is predictable* (*resp. optional*) *on* $(\bar{\Omega}, \{\bar{F}_{D_t}\}_{t \geqq 0}, P \otimes P')$ *then* H'_{A_t-} (*resp.* H'_{A_t}) *is predictable* (*resp. optional*) *on* $(\bar{\Omega}, \{\bar{F}_{A_t}\}_{t \geqq 0}, P \otimes P')$, *and* $H'_{\bar{L}_t}$ *is predictable on* $(\bar{\Omega}, \{\bar{F}_t\}_{t \geqq 0}, P \otimes P')$.

Proof. — We only sketch the proof. It is easily verified that the mappings:

$$
(3.15) \qquad \left\{
\begin{array}{l}
(t, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{P}) \to (L_t, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{P}'), \\
(t, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{O}'') \to (L_t, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{O}'), \\
(t, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{P}') \to (A_{t-}, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{P}''), \\
(t, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{O}') \to (A_t, \bar{\omega}) \in (R^+ \times \bar{\Omega}, \mathscr{O}''),
\end{array}
\right.
$$

are measurable (this is plain algebra). Since $\bar{L}_t = A_{(L_t)-}$, the end of the proposition is obvious. ■

Since $D_t = A_{L_t}$, if H' is optional on $(R^+ \times \bar{\Omega}, \{\bar{F}_{D_t}\}_{t \geqq 0})$ (i.e. $\mathscr{O}''$ measurable), then H'_{D_t} is still optional on $(R^+ \times \bar{\Omega}, \{\bar{F}_{D_t}\}_{t \geqq 0})$. The $\mathscr{O}''$-measurable processes H' which can be written in the form H_{L_t}, with H measurable, are exactly those for which $H'_t = H'_{D_t}$, and H can be taken to be equal to H'_{A_t}. Similarly $\mathscr{P}''$-measurable processes H' which can be written in the form $H'_t = H_{L_t}$ with H measurable are exactly the processes H' such that $H'_t = H'_{\bar{L}_t}$. H can then be taken to be equal to H'_{A_t-}, and H' is then also $\mathscr{P}$-measurable.

Observe that if N''_t is a $\{\bar{F}_{D_t}\}_{t \geqq 0}$-martingale, $N''_t = N''_{D_t}$. Moreover all the natural sets of $\{\bar{F}_{D_t}\}_{t \geqq 0}$ martingales (like H_p for $1 \leqq p \leqq +\infty$, BMO) are in one-to-one correspondance with the corresponding sets of $\{\bar{F}_{A_t}\}_{t \geqq 0}$-martingales. To each $\{\bar{F}_{D_t}\}_{t \geqq 0}$ martingale N''_t, we can in fact associate the $\{\bar{F}_{A_t}\}_{t \geqq 0}$-martingale $N'_t = N''_{A_t}$, and the reciprocal mapping is $N'_t \to N''_t = N'_{L_t}$. The stochastic calculus on $\{\bar{F}_{D_t}\}_{t \geqq 0}$ and $\{\bar{F}_{A_t}\}_{t \geqq 0}$ martingales are in fact formally the same, so that we will concentrate on the calculus on $\{\bar{F}_{A_t}\}_{t \geqq 0}$ martingales and its relation to the calculus on $\{\bar{F}_t\}_{t \geqq 0}$ martingales.

We will now describe the stochastic calculus on $\{\bar{F}_{A_t}\}_{t \geqq 0}$ martingales.

DEFINITION 3.14. — Let $H_s(\bar{\omega}, \varepsilon, e)$ be a function defined on $(R^+ \times \bar{\Omega}) \times \mathscr{W}_0$ with values in R, which is $\mathscr{P}' \otimes \mathscr{B}(\mathscr{W}_0)$ measurable, and such that for any $t \geqq 0$:

$$
(3.16) \qquad E^{P \otimes P'} \left[\int_0^t \left[\int_{\mathscr{W}_0} | H_s(\bar{\omega}, \varepsilon, e) | \, dn(\varepsilon, e) \right] ds \right] < +\infty,
$$

(resp.

$$(3.17)\qquad E^{P\otimes P'}\left[\int_0^t\left[\int_{\mathscr{W}_0}|H_s(\bar{\omega},\varepsilon,e)|^2\,dn(\varepsilon,e)\right]ds\right]<+\infty\Bigg).$$

Then $S_{s\leqq t}H_s(\bar{\omega},\varepsilon_s,e_s)$ [resp. $\overset{c}{S}_{s\leqq t}H_s(\bar{\omega},\varepsilon_s,e_s)$] denotes the $\mathcal{O}'$ measurable right-continuous bounded variation process (resp. $\{\bar{F}_{A_t}\}_{t\geqq 0}$ square-integrable martingale) which is the sum (resp. the compensated sum) of the jumps $H_s(\bar{\omega},\varepsilon(s),e(s))$ before t.

Recall—*see* Jacod [19], Meyer [31], Ikeda-Watanabe [17], p. 61—that:

$$(3.18)\qquad S_{s\leqq t}H_s(\bar{\omega},\varepsilon_s,e_s)=\lim_{\eta\downarrow 0}S_{\substack{s\leqq t\\|A_s-A_{s^-}|\geqq\eta}}H_s(\bar{\omega},\varepsilon_s,e_s)$$

(resp.

$$(3.19)\qquad \overset{c}{S}_{s\leqq t}H_s(\bar{\omega},\varepsilon_s,e_s)=\lim_{\eta\downarrow 0}\Bigg[S_{\substack{s\leqq t\\|A_s-A_{s^-}|\geqq\eta}}H_s(\bar{\omega},\varepsilon_s,e_s) -\int_0^t ds\int_{\mathscr{W}_0}1_{\sigma(e)\geqq\eta}H_s(\bar{\omega},\varepsilon,e)\,dn(\varepsilon,e)\Bigg]\Bigg).$$

where in (3.18) and (3.19), the limit is taken in probability uniformly on the compact sets of R^+.

We now have the fundamental result of Dellacherie, Jacod and Yor [20].

THEOREM 3.15. — *Any square integrable martingale* N'_t on $(\bar{\Omega},\{\bar{F}_{A_t}\}_{t\geqq 0},P\otimes P')$ *such that* $N'_0=0$ *can be represented in the form:*

$$(3.20)\qquad N'_t=\overset{c}{S}_{s\leqq t}H_s(\bar{\omega},\varepsilon_s,e_s),$$

where H *is taken as in Definition* 3.14 *and verifies* (3.17). H *is* $dt\otimes dP\otimes dn$ *essentially unique.*

Proof. — This follows immediately from [20], the Poisson point process characterization of the process (ε_t, e_t), and from proposition 3.11. ■

Theorem 3.15 shows in particular that the $\{\bar{F}_{A_t}\}_{t\geqq 0}$-martingales are pure jump martingales. It is then feasible to set the following definition.

DEFINITION 3.16. — Let N'_t, $\tilde{N}'_t$ be two $\{\bar{F}_{A_t}\}_{t\geqq 0}$-martingales on $(\bar{\Omega}, P\otimes P')$ such that $N'_0=N''_0=0$. $[N',\tilde{N}']_t$ is the bounded-variation $\mathcal{O}'$-measurable process defined by:

$$(3.21)\qquad [N',\tilde{N}']_t=\sum_{s\leqq t}(\Delta N')_s(\Delta\tilde{N}')_s.$$

Recall that by [31], if N′, N″ are taken as in Definition 3.16, then:

$$(3.22)\qquad N'_t\,\tilde{N}'_t=\int_0^t N'_{s^-}\,\delta\tilde{N}'_{s^-}+\int_0^t \tilde{N}'_{s^-}\,\delta N'_s+[N',\tilde{N}']_t.$$

Moreover, by Burkholder-Davis-Gundy inequalities, for any p such that $1<p<+\infty$ the following norms on the $\{\bar{F}_{A_t}\}_{t\geq 0}$-martingales N′ defined by:

$$(3.23)\qquad \begin{cases} \|N'\|'_p=[E^{P\otimes P'}[N'^p_\infty]]^{1/p},\\ \|N'\|'^*=[E^{P\otimes P'}[\sup\limits_{0\leq s\leq +\infty}|N'_s|^p]]^{1/p},\\ \|N'\|'^{\sim}_p=[E^{P\otimes P'}[N',N']^{p/2}_\infty]^{1/p},\end{cases}$$

are equivalent and exactly define the set H_p of $\{\bar{F}_{A_t}\}_{t\geq 0}$-martingale which are 0 at $t=0$ and whose terminal value is in $L_p(\bar{\Omega}, P\otimes P')$.

Similarly, recall that if N_t is a square integrable martingale on $(\bar{\Omega}, P\otimes P')$ with respect to $\{\bar{F}_t\}_{t\geq 0}$, such that $N_0=0$, there is $I_1\ldots I_m$, $\mathscr{I}$ defined on $R^+\times\bar{\Omega}$ with values in R, which are predictable with respect to $\{\bar{F}_t\}_{t\geq 0}$ such that:

(*a*) For any $t\in R^+$:

$$(3.24)\qquad E^{P\otimes P'}\int_0^t\left[\sum_{i=1}^m|I_i|^2+|\mathscr{I}|^2\right]ds<+\infty;$$

(*b*) $P\otimes P'$ a. s.:

$$(3.25)\qquad N_t=\sum_{i=1}^m\int_0^t I_i\,\delta w^i+\int_0^t\mathscr{I}\,\delta B.$$

Moreover by Burkholder-Davis-Gundy inequalities [31], for any $p>1$ the following norms on the $\{\bar{F}_t\}_{t\geq 0}$-martingales N defined by:

$$(3.26)\qquad \begin{cases} \|N\|_p=[E^{P\otimes P'}[N^p_\infty]]^{1/p},\\ \|N\|^*_p=[E^{P\otimes P'}[\sup\limits_{0\leq s\leq +\infty}|N_s|^p]]^{1/p},\\ \|N\|^*_p=\left\{E^{P\otimes P'}\left[\int_0^{+\infty}(\sum\limits_i|I_i|^2+|\mathscr{I}|^2)\,ds\right]^{p/2}\right\}^{1/p},\end{cases}$$

are equivalent and exactly define the set H_p of $\{\bar{F}_t\}_{t\geq 0}$-martingales which are 0 at $t=0$ and whose terminal value is in $L_p(\bar{\Omega}, P\otimes P')$.

PROPOSITION 3.17. — *For any* $p\,(1<p<+\infty)$, *the mapping:*

$$(3.27)\qquad N_t\in H_p \underset{\pi}{\rightarrow} N'_t=N_{A_t}\in H'_p,$$

is a Banach space isomorphism.

Proof. — This result is obvious since (3.27) corresponds exactly to the identification of the terminal values [in $L_p(\bar{\Omega}, P\otimes P')$] of the elements of H_p and H'_p. □

Remark 2. — If $p=1$, π is only continuous and $\pi(H_1)$ is dense in H'_1, but is not H'_1.

In the sequel, we write N′ instead of $\pi(N)$.

We then have a technical result.

THEOREM 3.18. — *If* $N\in H_p(1<p<+\infty)$, *if* K *is a bounded* $\{\bar{F}_t\}_{t\geqq 0}$ *predictable process, then* $P\otimes P'$ *a. s., for any* $t\geqq 0$:

$$\int_0^{A_t} K_{L_s}^- \,\delta N_s = \int_0^t K_{A_s^-}\,\delta N'_s. \tag{3.28}$$

Proof. — $K_{L_s}^-$ is a bounded $\{\bar{F}_t\}_{t\geqq 0}$ predictable process, so that the martingale $\int_0^t K_{L_s}^-\,\delta N_s\in H_p$. This implies that $\int_0^{A_t} K_{L_s}^-\,\delta N_s$ is a $\{\bar{F}_{A_t}\}_{t\geqq 0}$-martingale, which is in H'_p. Similarly, $K_{A_s^-}$ is a bounded $\{\bar{F}_{A_t}\}_{t\geqq 0}$-predictable process, and so $\int_0^t K_{A_s^-}\,\delta N'_s$ is a $\{\bar{F}_{A_t}\}_{t\geqq 0}$-martingale which is in H'_p.

To prove (3.28), we only need to show that both sides of (3.28) have the same jumps.

First assume that $K_s = 1_{s>T}$, where T is a $\{\bar{F}_t\}_{t\geqq 0}$ stopping time. We must prove that for any $k\in N$, $P\otimes P'$ a. s.:

$$1_{|A_t-A_{t^-}|\geqq 1/k}\int_{A_{t^-}}^{A_t} K_{L_s}^-\,\delta N_s = 1_{|A_t-A_{t^-}|\geqq 1/k}\,K_{A_{t^-}}(N_{A_t}-N_{A_{t^-}}). \tag{3.29}$$

Let $\{S_i^k\}_{i\in N}$ be the $\{\bar{F}_{A_t}\}_{t\geqq 0}$ stopping times:

$$\begin{cases} S_0^k = 0, \\ S_{i+1}^k = \inf\{t\geqq S_i^k,\ |A_t - A_{t^-}|\geqq 1/k\}. \end{cases} \tag{3.30}$$

Then as $i\uparrow+\infty$, $S_i^k\uparrow+\infty$ $P\otimes P'$ a. s.

Writing S instead of S_i^k, we are then left to prove that:

$$\int_{A_{S^-}}^{A_S} K_{L_s}^-\,\delta N_s = K_{A_{S^-}}(N_{A_S}-N_{A_{S^-}}), \tag{3.31}$$

Now $K_{L_s}^- = 1_{s>D_T}$, and D_T is a $\{\bar{F}_t\}_{t\geqq 0}$ stopping time. Clearly:

$$\int_0^s K_{L_u}^-\,\delta N_u = 1_{s>D_T}(N_s - N_{D_T}) = 1_{s\geqq D_T}(N_s-N_{D_T}). \tag{3.32}$$

If $A_{S^-}\leqq T$ since $A_S = D_{A_{S^-}}$, $A_S\leqq D_T$. Using (3.32), we see that on $(A_{S^-}\leqq T)$, both sides of (3.31) are 0.

If $A_{S^-} > T$, then $A_{S^-} \geqq D_T$. If $A_{S^-} > D_T$, $A_S > D_T$, and so using (3.32), we see that (3.31) still holds. We claim that $(A_{S^-} = D_T)$ is negligible. In fact since D_T is a $\{\bar{F}_t\}_{t \geqq 0}$ stopping time, the zeros of z accumulate on the right of D_T. But recall that $A_S = D_{A_{S^-}} > A_{S^-}$. (3.31) has then been proved.

(3.28) holds when $K = 1_{s > T}$. (3.28) follows easily using the monotone class Theorem [11]. □

We then have the following result, which expresses $[N', \tilde{N}']$ in terms of $\{\bar{F}_t\}_{t \geqq 0}$-stochastic integrals:

THEOREM 3.19. — *If* $N \in H_p$, $\tilde{N} \in H_{p'}$ $(1 < p, p' < +\infty)$, *then:*

$$[N', \tilde{N}']_t = N_{A_t} \tilde{N}_{A_t} - \int_0^{A_t} N^-_{L_s} \delta \tilde{N}_s - \int_0^{A_t} \tilde{N}^-_{L_s} \delta N_s. \tag{3.33}$$

Proof. — For $n \in \mathbb{N}$, let T'^n be the $\{\bar{F}_{A_t}\}_{t \geqq 0}$ stopping time:

$$T'^n = \inf \{ t \geqq 0; |N'_t| \vee |\tilde{N}'_t| \geqq n \}.$$

Then $A_{T'^n}$ is a $\{\bar{F}_t\}_{t \geqq 0}$ stopping time. Since $T'^n \nearrow +\infty$ $P \otimes P'$ a. s. it suffices to check that:

$$[N', \tilde{N}']_{t \wedge T'_n} = N_{A_t \wedge A_{T'_n}} \tilde{N}_{A_t \wedge A_{T'_n}} - \int_0^{A_t \wedge A_{T'_n}} N^-_{L_s} \delta \tilde{N}_s - \int_0^{A_t \wedge A_{T'_n}} \tilde{N}^-_{L_s} \delta N_s, \tag{3.34}$$

or equivalently to check (3.33) when N_t, $\tilde{N}_t$ are replaced by $N_{t \wedge A_{T'_n}}$, $\tilde{N}_{t \wedge A_{T'_n}}$.

We can then suppose that on $[0, +\infty[$, $|N'_{t^-}|$, $|\tilde{N}'_{t^-}|$ are uniformly bounded processes.

Since $L_s = A_{L_s^-}$, $N^-_{L_s}$ and $\tilde{N}^-_{L_s}$ are bounded $\{\bar{F}_t\}_{t \geqq 0}$-predictable processes. (3.33) follows from (3.22) and (3.28). ■

Remark 3. — Of course (3.33) is true in general if N and N' are only supposed to be uniformly integrable $\{\bar{F}_t\}_{t \geqq 0}$-martingales.

Note that (3.33) gives us $[N', \tilde{N}']_t$ by using the classical Itô calculus on the martingales N_t, N'_t. Practically, this expression is not very useful. Moreover the Burkholder-Davis-Gundy inequalities corresponding to the equivalent norms (3.23) are not "obvious" consequences of (3.33).

(*d*) *The Girsanov transformation on the filtrations* $\{\bar{F}_t\}_{t \geqq 0}$ *and* $\{\bar{F}_{A_t}\}_{t \geqq 0}$.

For obvious reasons, we limit ourselves to studying the Girsanov transformation on B, which leaves w untouched. The Girsanov transformation on w would in fact be essentially trivial to study.

Let c_t be a $\{\bar{F}_t\}_{t \geqq 0}$ predictable bounded process on $(\bar{\Omega}, P \otimes P')$. To simplify the discussion, we will assume there exists $T > 0$ such that if $t > T$, $c_t = 0$.

The process:

$$G_t = \exp \left[\int_0^t c_s \, \delta B_s - \frac{1}{2} \int_0^t c_s^2 \, ds \right], \tag{3.35}$$

is a $\{\bar{F}_t\}_{t\geqq 0}$-martingale stopped at T. Moreover it is easy to see that the random variables:

$$(3.36)\qquad \sup_{0\leqq t<+\infty}\left|\int_0^t c_s\,\delta B_s\right|,\qquad \sup_{0\leqq t<+\infty} G_t,\qquad \sup_{0\leqq t<+\infty} G_t^{-1},$$

are in all the $L_p(\bar{\Omega}, P\otimes P')$ $(1\leqq p<+\infty)$.

Let $\bar{P}$ be the probability measure on $\bar{\Omega}$ defined by:

$$(3.37)\qquad d\bar{P}=G_\infty\,d(P\otimes P').$$

Clearly, for any $t\geqq 0$:

$$(3.38)\qquad \frac{d\bar{P}}{dP\otimes P'}\bar{F}_t=G_t,\qquad \frac{d\bar{P}}{dP\otimes P'}\bar{F}_{A_t}=G_{A_t}.$$

By the fundamental property of the Girsanov transformation on the Brownian motion [40]-6, we know that if $\tilde{B}_t$ is the process:

$$(3.39)\qquad \tilde{B}_t=B_t-\int_0^t c_s\,ds;$$

then $(w_t^1\dots w_t^m, \tilde{B}_t)$ is a $\{\bar{F}_t\}_{t\geqq 0}$ Brownian martingale under $\bar{P}$.

Now the second equality in (3.38) expresses the fact that the probability law of the point process (ε_t, e_t) under $\bar{P}$ is equivalent to P on each $\bar{F}_{A_t}$ (by Proposition 3.11, $\{\bar{F}_{A_t}\}_{t\geqq 0}$ is the natural filtration of this point process). We will now use the results relative to the extended Girsanov transformation on point process given by Jacod [19].

Definition 3.20. — k_t denotes the process:

$$(3.41)\qquad k_t=\int_0^t \frac{\delta G_s}{G_{L_s}^-}.$$

Of course, we also have:

$$(3.42)\qquad k_t=\int_0^t \frac{G_s}{G_{L_s}^-}\,c_s\,\delta B_s.$$

Using (3.36), we see that k_t is a $\{\bar{F}_t\}_{t\geqq 0}$-martingale stopped at T, which is in all the H_p $(1\leqq p<+\infty)$.

Set:

$$(3.43)\qquad G'_t=G_{A_t},\qquad k'_t=k_{A_t}.$$

G'_t, k'_t are $\{\bar{F}_{A_t}\}_{t\geqq 0}$-martingales which are in all the H'_p $(1\leqq p<+\infty)$.

We now have the fundamental.

THEOREM 3.21. — *G'_t is the unique solution of the stochastic differential equation on* $(\bar{\Omega}, \{\bar{F}_{A_t}\}_{t \geqq 0}, P \otimes P')$:

$$G'_t = 1 + \int_0^t G'_{s^-} \, \delta k'_s. \tag{3.44}$$

Proof. — By the result of Doléans-Dade (*see* Jacod [19]) we know that (3.44) has one and only one solution. We must prove that G'_t is in fact solution of (3.44). Theorem 3.18 shows that:

$$\int_0^t G'_{s^-} \, \delta k'_s = \int_0^{A_t} G_{\bar{L}_s}^- \, \delta k_s. \tag{3.45}$$

Using (3.41), (3.44) follows. ■

COROLLARY. — *k'_t is a $\{\bar{F}_{A_t}\}_{t \geqq 0}$-martingale, which belongs to $H'_p (1 \leqq p < +\infty)$, and is the compensated sum of the jumps:*

$$\exp\left\{ \int_{A_t^-}^{A_t} c \, \delta B - \frac{1}{2} \int_{A_t^-}^{A_t} c^2 \, ds \right\} - 1. \tag{3.46}$$

Proof. — Since $k \in H_p$, $k' \in H'_p$. Moreover, using (3.44), we see that the jumps $\Delta k'$ of k' are such that:

$$\Delta k'_t = \frac{\Delta G'_t}{G'_{t^-}}. \tag{3.47}$$

Now:

$$\left\{ \begin{aligned} & (\Delta G')_t = G_{A_t} - G_{A_t^-} = G_{A_t^-} \left[\exp\left\{ \int_{A_t^-}^{A_t} c \, \delta B - \frac{1}{2} \int_{A_t^-}^{A_t} c^2 \, ds \right\} - 1 \right], \\ & G'_{t^-} = G_{A_t^-}. \end{aligned} \right. \tag{3.48}$$

(3.46) follows. ■

Now if $\bar{\omega} \in \bar{\Omega}$, we can write for any s:

$$\bar{\omega} = (\bar{\omega} \mid \bar{L}_s \mid \theta_{\bar{L}_s}^- \bar{\omega}), \tag{3.49}$$

i.e. the trajectory $\bar{\omega} = (w, z)$ is decomposed into two parts: the part before $\bar{L}_s$ and the part after $\bar{L}_s$.

Recall that in Ikeda-Watanabe [17], p. 209, stochastic integrals on $(\mathscr{W}_0, dn)$ are naturally defined. Namely, observe that $\mathscr{W}_0$ is endowed with the natural filtration $\{\mathscr{G}_t\}_{t \geqq 0}$ given by:

$$\mathscr{G}_t = \mathscr{B}(\varepsilon_s, e_s \mid s \leqq t). \tag{3.50}$$

Then if Φ_s is a $\{\mathcal{G}_t\}$ predictable process on $R^+ \times \mathcal{W}_0$ such that:

$$\int_{\mathcal{W}_0} \left[\int_0^\sigma |\Phi_s|^2 \, ds \right] dn < +\infty, \tag{3.51}$$

the stochastic integral:

$$\int_0^\sigma \Phi_s \, \delta e,$$

can be defined and moreover:

$$\int_{\mathcal{W}_0} \left[\int_0^\sigma \Phi_s \, \delta e \right]^2 dn = \int_{\mathcal{W}_0} \left[\int_0^\sigma \Phi_s^2 \, ds \right] dn, \tag{3.52}$$

Of course this is not surprising since by Theorems 3.4 and 3.9, on $(\mathcal{W}_0, dn)$, conditionally on σ, e is a $\{\mathcal{G}_t\}_{t \geqq 0}$ semi-martingale.

DEFINITION 3.22. — The function $d_s(\bar{\omega}, \varepsilon, e)$ is defined on $R^+ \times \bar{\Omega} \times \mathcal{W}_0$ by:

$$d_s(\bar{\omega}, \varepsilon, e) = \exp\left[\int_0^{\sigma(e)} c_{\bar{L}_s + u}(\bar{\omega} \mid \bar{L}_s \mid (\varepsilon, e)) \, \delta e(u) - \frac{1}{2} \int_0^{\sigma(e)} | c_{\bar{L}_s + u}(\bar{\omega} \mid \bar{L}_s \mid (\varepsilon, e)) |^2 \, du \right]. \tag{3.53}$$

(3.53) makes sense. In fact, it is not hard to prove that for each $(s, \bar{\omega}) \in R^+ \times \bar{\Omega}$,

$$(u, (\varepsilon, e)) \in R^+ \times \mathcal{W}_0 \to c_{\bar{L}_s + u}(\bar{\omega} \mid \bar{L}_s \mid (\varepsilon, e))$$

is $\{\mathcal{G}_u\}_{u \geqq 0}$-predictable. Moreover, for each $v > 0$:

$$\int_{\mathcal{W}_0} \sigma \wedge v \, dn(\varepsilon, e) = \int_0^{+\infty} \frac{t \wedge v \, dt}{\sqrt{2 \pi t^3}} < +\infty.$$

Since c is bounded for each $(s, \bar{\omega})$, we can define on $(\mathcal{W}_0, dn)$:

$$\int_0^{\sigma \wedge v} c_{\bar{L}_s + u}(\bar{\omega} \mid \bar{L}_s \mid (\varepsilon, e)) \, \delta e(u). \tag{3.54}$$

Finally by making $v \to +\infty$ in (3.54), the integral:

$$\int_0^\sigma c_{\bar{L}_s + u}(\bar{\omega} \mid \bar{L}_s \mid (\varepsilon, e)) \, \delta e(u)$$

is well-defined.

Also observe that by standard results on stochastic integrals depending on a parameter, d can be defined so as to be $\mathcal{P} \otimes \mathcal{B}(\mathcal{W}_0)$-measurable.

DEFINITION 3.23. — The function $d'_s(\bar{\omega}, \varepsilon, e)$ is defined on $R^+ \times \bar{\Omega} \times \mathscr{W}_0$ by:

$$d'_s(\bar{\omega}, \varepsilon, e) = d_{A_s-}(\bar{\omega}, \varepsilon, e). \tag{3.55}$$

Using Proposition 3.13, we see that d' is $\mathscr{P}' \otimes \mathscr{B}(\mathscr{W}_0)$ measurable.

PROPOSITION 3.24. — *We have the following inequality:*

$$E^{P\otimes P'} \int_0^{+\infty} \left[\int | d'_s(\bar{\omega}, \varepsilon, e) - 1 |^2 \, dn(\varepsilon, e) \right] ds < +\infty, \tag{3.56}$$

k'_t is given by:

$$k'_t = \overset{c}{S}_{s \leq t} [d'_s(\bar{\omega}, \varepsilon_s, e_s) - 1]. \tag{3.57}$$

Proof. — By the Corollary of Theorem 3.21, k' is the compensated sum of the jumps (3.46). Using (3.53) and (3.55), we see that k' is the compensated sum of the jumps $d'_s(\bar{\omega}, \varepsilon_s, e_s) - 1$. Since k' is in H'_2, (3.56) is verified. □

Remark 4. — Inequality (3.56) will be reproved in a much stronger form in Proposition 3.26.

THEOREM 3.25. — *Under $\bar{P}$, (ε_t, e_t) is a point process whose Lévy measure is given by:*

$$d'_s(\bar{\omega}, \varepsilon, e) \, dn(\varepsilon, e). \tag{3.58}$$

In particular, if $H_s(\bar{\omega}, \varepsilon, e)$ is taken as in (3.17), the process $\overset{c'}{S}_{s \leq t} H_s(\bar{\omega}, \varepsilon_s, e_s)$, defined by:

$$\overset{c'}{S}_{s \leq t} H_s(\bar{\omega}, \varepsilon_s, e_s) = \overset{c}{S}_{s \leq t} H_s(\bar{\omega}, \varepsilon_s, e_s) - \int_0^t ds \int [d'_s(\bar{\omega}, \varepsilon, e) - 1] \, H_s(\bar{\omega}, \varepsilon, e) \, dn(\varepsilon, e), \tag{3.59}$$

is a $\{\bar{F}_{A_t}\}_{t \geq 0}$ local martingale on $(\bar{\Omega}, \bar{P})$.

Proof. — Using (3.44) and (3.57), (3.58) is a direct consequence of Jacod [19]. Observe that the r. h. s. of (3.59) makes sense because of (3.17) and (3.56). The end of the theorem is also a consequence of Jacod [19]. ■

Remark 5. — Theorem 3.25 shows that the effect of the Girsanov transformation $P \otimes P' \to \bar{P} = G_\infty \, dP$ is to create a "Girsanov transformation" $n \to d'_s \, dn$ at the level of each excursion.

The analogy is in fact obvious by formula (3.53).

(*e*) *The Girsanov transformation and its effect on the stochastic calculus.*

We will now show how (3.39) — which gives the effect of the Girsanov transformation on $\{\bar{F}_t\}_{t \geq 0}$-martingales — and (3.59) — which gives the corresponding effect on $\{\bar{F}_{A_t}\}_{t \geq 0}$ martingales — are related.

In fact let $N \in H_2$. We can represent the $\{\bar{F}_t\}_{t \geqq 0}$-martingale N_t as in (3.25). Now under $\bar{P}$ we know that:

$$\tilde{N}_t = N_t - \int_0^t \mathscr{I}_s\, c_s\, ds, \tag{3.60}$$

is a $\{\bar{F}_t\}$-martingale. But:

$$E^{\bar{P}}\left[\int_0^{+\infty} |\mathscr{I}_s\, c_s|\, ds\right] \leqq C\left[E^{P\otimes P'} \int_0^T |\mathscr{I}_s|^2\, ds\right]^{1/2}.$$

Since $G_t \in H_p (1 \leqq p < +\infty)$, it follows easily that $\tilde{N}_t$ is a uniformly integrable $\{\bar{F}_t\}_{t \geqq 0}$-martingale on $(\bar{\Omega}, \bar{P})$.

$\tilde{N}_{A_t}$ is then a $\{\bar{F}_{A_t}\}_{t \geqq 0}$-martingale on $(\bar{\Omega}, \bar{P})$.

Now we know that $N_{A_t} \in H'_2$. Then:

$$N_{A_t} = \overset{c}{S}_{s \leqq t}\, H_s(\bar{\omega}, \varepsilon_s, e_s), \tag{3.61}$$

where:

$$H_s(\bar{\omega}, \varepsilon, e) = \int_0^\sigma I_{i, A_s- + u}(\bar{\omega} \mid A_s- \mid (\varepsilon, e))\, \delta\varepsilon^i + \int_0^\sigma \mathscr{I}_{A_s- + u}(\bar{\omega} \mid A_s- \mid (\varepsilon, e))\, \delta e.$$

From now on we will simplify the notations as much as we can, i. e. $\bar{\omega}$, L_s, ε, e will be generally omitted.

Now for fixed $(s, \bar{\omega}) \in R^+ \times \bar{\Omega}$, on $(\mathscr{W}_0, n)$, we know that if $d'_s(u)$ is defined by:

$$d'_s(u) = \exp\left\{\int_{A_s-}^{A_s- + u} c\, \delta e - \frac{1}{2}\int_{A_s-}^{A_s- + u} c^2\, dv\right\}, \tag{3.62}$$

then:

$$\begin{cases} dd'_s(u) = d'_s(u)\, c_{A_s- + u}\, \delta e(u), \\ d'_s(0) = 1. \end{cases} \tag{3.63}$$

In fact (3.63) is obvious, because $e(u)$ is a semi-martingale on $(\mathscr{W}_0, n)$ with respect to $\{\mathscr{G}_u\}_{u \geqq 0}$ (in the sense than conditionally on σ, $e(u)$ is a semi-martingale). Since:

$$E^{P\otimes P'} \int_0^{+\infty} (|I_i|^2 + |\mathscr{I}|^2)\, ds < +\infty, \tag{3.64}$$

we find that:

$$E^{P\otimes P'} \int_0^{+\infty} ds \int_{\mathscr{W}_0} \left[\int_0^\sigma (|I_{i, A_s- + u}|^2 + |\mathscr{I}_{A_s- + u}|^2)\, du\right] dn < +\infty. \tag{3.65}$$

Using Theorem 3.25, we see that:

$$(3.66)\qquad N_{A_t}-\int_0^t ds\left[\int_{\mathscr{W}_0}(d_s'(\bar{\omega},\varepsilon,e)-1)\,H_s(\bar{\omega},\varepsilon,e)\,dn(e)\right],$$

is a $\{\bar{F}_{A_t}\}_{t\geqq 0}$ local martingale for $\bar{P}$.

PROPOSITION 3.26. — $\int_{\mathscr{W}_0}|d_s'-1|^2\,dn$ *is uniformly bounded on* $R^+\times\bar{\Omega}$, *and moreover:*

$$(3.67)\qquad \int_{\mathscr{W}_0}|d_s'-1|^2\,dn=\left[\int_{\mathscr{W}_0}\int_0^\sigma|d_s'(u)\,c_{A_{s^-}+u}|^2\,du\right]dn.$$

Proof. — Set:

$$f_s'(u)=d_s'(u)-1.$$

Then by (3.63):

$$(3.68)\qquad \begin{cases} df_s'(u)=(f_s'(u)+1)\,c_{A_{s^-}+u}\,\delta e(u)\\ f_s'(0)=0.\end{cases}$$

Let σ_k be the $\{\mathscr{G}_t\}_{t\geqq 0}$ stopping time:

$$(3.69)\qquad \sigma_k=\inf\{u\geqq 0;\ |f_s'(u)|\geqq k\}\wedge\sigma.$$

Using (3.52), we get for $t\geqq 0$:

$$(3.70)\qquad \int_{\mathscr{W}_0}|f_s'(\sigma_k\wedge t)|^2\,dn\leqq C\left[\int_{\mathscr{W}_0}\left[\int_0^t|f_s'(\sigma_k\wedge u)|^2\,du+\sigma\wedge t\right]dn\right].$$

Now:

$$\int_{\mathscr{W}_0}(\sigma\wedge t)\,dn=\frac{4}{\sqrt{2\pi}}\,t^{1/2}$$

and moreover both sides of (3.70) are finite.

Using Gronwall's lemma, we find that if t remains in a compact set, $\int_{\mathscr{W}_0}|f_s'(\sigma_k\wedge t)|^2\,dn$ remains uniformly bounded. Since f' is stopped at T (because $c_{A_{s^-}+u}=0$ for $u\geqq T$), we find that $\int_{\mathscr{W}_0}|f_s'(t)|^2\,dn$ is uniformly bounded. Now:

$$(3.71)\qquad \int_{\mathscr{W}_0}\left[\int_0^\sigma|d_s'(u)\,c_{A_{s^-}+u}|^2\,du\right]dn$$

$$\leqq C\left[\int_{\mathscr{W}_0}\left[\int_0^{\sigma\wedge T}|f_s'(u)|^2\,du+\sigma\wedge T\right]dn\right]<+\infty.$$

Using (3.52), (3.63) and (3.71), (3.67) follows. ■

We then find:

PROPOSITION 3.27. — *On* $\mathrm{R}^+\times\bar{\Omega}$, $ds\otimes d\mathrm{P}\otimes d\mathrm{P}'$ *a. s.:*

$$\int_{\mathscr{W}_0}(d'_s-1)\,\mathrm{H}_s\,dn=\int_{\mathscr{W}_0}\left[\int_0^\sigma d'_s(u)\,c_{\mathrm{A}_s^-+u}\,\mathscr{I}_{\mathrm{A}_s^-+u}\,du\right]dn. \tag{3.72}$$

Proof. — Since $\mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\int_0^{+\infty}|\mathscr{I}_s|^2\,ds<+\infty$, we know that:

$$\mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\int_0^{+\infty}ds\int_{\mathscr{W}_0}\int_0^\sigma|\mathscr{I}_{\mathrm{A}_s^-+u}|^2\,du\,dn<+\infty. \tag{3.73}$$

Then:

$$\int_{\mathscr{W}_0}\int_0^\sigma|\mathscr{I}_{\mathrm{A}_s^-+u}|^2\,du\,dn<+\infty,\qquad ds\otimes d\mathrm{P}\otimes d\mathrm{P}'\ \text{a. s.} \tag{3.74}$$

Using (3.74), Proposition 3.26 and (3.52), (3.72) follows. ■

We now have the key result.

THEOREM 3.28. — *On* $\mathrm{R}^+\times\bar{\Omega}$, $ds\otimes d\mathrm{P}\otimes d\mathrm{P}'$ *a. s.:*

$$\int_{\mathscr{W}_0}(d'_s-1)\,\mathrm{H}_s\,dn=\int_{\mathscr{W}_0}\left[\int_0^\sigma c_{\mathrm{A}_s^-+u}\,\mathscr{I}_{\mathrm{A}_s^-+u}\,du\right]d'_s\,dn. \tag{3.75}$$

Proof. — If $(s,\bar{\omega})$ is chosen in such a way that:

$$\int_{\mathscr{W}_0}dn\left[\int_0^\sigma|\mathscr{I}_{\mathrm{A}_s^-+u}|^2\,du\right]<+\infty, \tag{3.76}$$

using Proposition 3.26 and the Lebesgue Theorem, we find that:

$$\int_{\mathscr{W}_0}\left[\int_0^\sigma d'_s(u)\,c_{\mathrm{A}_s^-+u}\,\mathscr{I}_{\mathrm{A}_s^-+u}\,du\right]dn =\lim_{\eta\downarrow 0}\int_{\mathscr{W}_0}\left[1_{\sigma\geqq\eta}\int_\eta^\sigma d'_s(u)\,c_{\mathrm{A}_s^-+u}\,\mathscr{I}_{\mathrm{A}_s^-+u}\,du\right]dn. \tag{3.77}$$

For $\eta>0$, let n^η be the probability law $n(\,.\,/\sigma\geqq\eta)$. By [17], p. 309, under n^η, $e_{\eta+u}-e_\eta$ is a $\{\mathscr{G}_{\eta+u}\}_{u\geqq 0}$ continuous martingale, whose quadratic variation is $t\wedge(\sigma-\eta)$ (i. e. it is a Brownian motion stopped at the time where it first hits $-e_\eta$). Since $c_{\mathrm{A}_s^-+u}$ is 0 for $u\geqq\mathrm{T}$, (3.63) immediately shows that under n^η, $d'_s(\eta+u)$ is a uniformly integrable $\{\mathscr{G}_{\eta+u}\}_{u\geqq 0}$-martingale, stopped a σ. This shows that:

$$\int_{\mathscr{W}_0}\left[\int_\eta^\sigma d'_s(u)\,c_{\mathrm{A}_s^-+u}\,\mathscr{I}_{\mathrm{A}_s^-+u}\,du\right]dn^\eta=\int_{\mathscr{W}_0}\left[\int_\eta^\sigma c_{\mathrm{A}_s^-+u}\,\mathscr{I}_{\mathrm{A}_s^-+u}\,du\right]d'_s\,dn^\eta, \tag{3.78}$$

so that:

$$(3.79)\quad \int_{\mathscr{W}_0} 1_{\sigma \geqq \eta} \left[\int_\eta^\sigma d_s'(u)\, c_{A_s-+u}\, \mathscr{I}_{A_s-+u}\, du \right] dn = \int_{\mathscr{W}_0} 1_{\sigma \geqq \eta} \left[\int_\eta^\sigma c_{A_s-+u}\, \mathscr{I}_{A_s-+u}\, du \right] d_s'\, dn.$$

Now since c_{A_s-+u} is bounded and is 0 for $u \geqq T$:

$$(3.80)\quad \int_{\mathscr{W}_0} \int_0^\sigma | c_{A_s-+u}\, \mathscr{I}_{A_s-+u} |\, du\, dn \leqq \left[\int_{\mathscr{W}_0} \int_0^\sigma | c_{A_s-+u} |^2\, du\, dn \right]^{1/2} \times \left[\int_{\mathscr{W}_0} \int_0^\sigma | \mathscr{I}_{A_s-+u} |^2\, du\, dn \right]^{1/2}$$
$$\leqq C \left[\int_0^{+\infty} t \wedge T \frac{dt}{\sqrt{2\pi t^3}} \right]^{1/2} \left[\int_{\mathscr{W}_0} \int_0^\sigma | \mathscr{I}_{A_s-+u} |^2\, du\, dn \right]^{1/2} < +\infty.$$

and moreover:

$$(3.81)\quad \int_{\mathscr{W}_0} \left[\int_0^\sigma | c_{A_s-+u}\, \mathscr{I}_{A_s-+u} |\, du \right]^2 dn \leqq CT \int_{\mathscr{W}_0} \left[\int_0^\sigma | \mathscr{I}_{A_s-+u} |^2\, du \right] dn < +\infty.$$

Using the identity:

$$(3.82)\quad \int_{\mathscr{W}_0} 1_{\sigma \geqq \eta} \int_\eta^\sigma c_{A_s-+u}\, \mathscr{I}_{A_s-+u}\, du\, d_s'\, dn = \int_{\mathscr{W}_0} 1_{\sigma \geqq \eta} \int_\eta^\sigma c_{A_s-+u}\, \mathscr{I}_{A_s-+u}\, du\, (d_s' - 1)\, dn + \int_{\mathscr{W}_0} 1_{\sigma \geqq \eta} \int_\eta^\sigma c_{A_s-+u}\, \mathscr{I}_{A_s-+u}\, du\, dn,$$

(3.80), (3.81) and Lebesgue's theorem, (3.75) follows. ■

Remark 6. — In the proof of Theorem 3.28, we have essentially used the "martingale" property of $d_s'(u)$ on $(\mathscr{W}_0, \{\mathscr{G}_u\}_{u \geqq 0}, n)$. (3.75) shows that the effect of the "Girsanov transformation" $n \to d_s' dn$ on the "excursion martingales"—which are stochastic integrales with respect to $(e, \varepsilon^1 \ldots \varepsilon^m)$—is formally identical to the effect of the Girsanov transformation $P \otimes P' \to \bar{P}$ on the $\{\bar{F}_t\}_{t \geqq 0}$ martingales.

A consequence of Theorem 3.28 is the following result:

THEOREM 3.29. — *Under* $\bar{P}$:

$$N_{A_t} - \int_0^t ds \int_{\mathscr{W}_0} \int_0^\sigma (c_{A_s^- + u}\, \mathscr{I}_{A_s^- + u})\, du\, d'_s\, dn, \tag{3.83}$$

is a $\{\bar{F}_{A_t}\}_{t \geqq 0}$*-martingale.*

Proof. — First observe that (3.83) is well defined. In fact using the same decomposition as in (3.82), we get:

$$E^{P \otimes P'} \int_0^t ds \int_{\mathscr{W}_0} \int_0^\sigma |c_{A_s^- + u}\, \mathscr{I}_{A_s^- + u}|\, du\, d'_s\, dn \leqq E^{P \otimes P'} \int_0^{A_t \wedge T} |c_s\, \mathscr{I}_s|\, ds$$
$$+ E^{P \otimes P'} \int_0^t ds \int_{\mathscr{W}_0} \int_0^\sigma |c_{A_s^- + u}\, \mathscr{I}_{A_s^- + u}|\, du\, |(d'_s - 1)|\, dn. \tag{3.84}$$

By (3.64), the first integral in the r. h. s. of (3.84) is finite. Using Proposition 3.26 and (3.81), the second integral can be bounded by:

$$C\, t^{1/2} \left[E^{P \otimes P'} \int_0^t ds \int_{\mathscr{W}_0} \int_0^\sigma |\mathscr{I}_{A_s^- + u}|^2\, du\, dn \right]^{1/2}. \tag{3.85}$$

The r. h. s. of (3.85) is $< +\infty$ by (3.73).

Theorem 3.29 is then a consequence of Theorem 3.25 and Proposition 3.26 [which imply that (3.66) is a $\{\bar{F}_{A_t}\}_{t \geqq 0}$-martingale for $\bar{P}$] and of Theorem 3.28. ■

COROLLARY. — *Under* $\bar{P}$

$$\int_0^{A_t} c_s\, \mathscr{I}_s\, ds - \int_0^t ds \int_{\mathscr{W}_0} \int_0^\sigma c_{A_s^- + u}\, \mathscr{I}_{A_s^- + u}\, du\, d'_s\, dn, \tag{3.86}$$

is a $\{\bar{F}_{A_t}\}_{t \geqq 0}$*-martingale.*

Proof. — We know that $\tilde{N}_{A_t}$ is a $\{\bar{F}_{A_t}\}_{t \geqq 0}$-martingale under $\bar{P}$. Using Theorem 3.29, the corollary follows. ■

Remark 7. — As should be expected, the corollary of Theorem 3.29 is a *triviality*. In fact we know that:

$$E^{\bar{P}} \int_0^{A_t} |c_s\, \mathscr{I}_s|\, ds = E^{P \otimes P'}\, G_\infty \int_0^{A_t \wedge T} |c_s\, \mathscr{I}_s|\, ds$$
$$\leqq C T^{1/2}\, [E^{P \otimes P'}\, G_\infty^2]^{1/2} \left[E^{P \otimes P'} \int_0^{+\infty} |\mathscr{I}_s|^2\, ds \right]^{1/2}, \tag{3.87}$$

which is $< +\infty$ since $G \in H_2$.

(3.86) is then an obvious consequence of Theorem 3.25. In fact, by the effect of the Girsanov transformation on $\{\bar{F}_{A_t}\}_{t \geqq 0}$ martingales, $N_{A_t} - \int_0^{A_t} c_s\, \mathscr{I}_s\, ds$ is a

$\{\bar{F}_{A_t}\}_{t \geqq 0}$-martingale. But $\int_0^{A_t} c_s \mathscr{I}_s \, ds$ is a $\{\bar{F}_{A_t}\}_{t \geqq 0}$-optional process, while point processes are characterized by their predictable compensator. Another compensation of $\int_0^{A_t} c_s \mathscr{I}_s \, ds$ is needed to obtain the $\{\bar{F}_{A_t}\}_{t \geqq 0}$ predictable compensator:

$$\int_0^{A_t} ds \int_{\mathscr{W}_0} \int_0^{\sigma} c_{A_s - +u} \mathscr{I}_{A_s - +u} \, du \, d'_s \, dn.$$

4. The calculus of variations on the reflecting Brownian motion

In this section, we develop a calculus of variations on the reflecting Brownian motion z in order to obtain the adequate estimates on the component A_t of the boundary process (A_t, x_{A_t}).

The main difficulty is that the local time $L_.$ is not "naturally" a differentiable function of the trajectories of $z_.$ or of $B_.$, so that the calculus which we developed in section 2, and which was based on the Girsanov transformation, fails.

In an earlier version of this paper, we used the point process description of (z, w) in section 3, and in particular the description of the characteristic measure n in the corollary of Theorem 3.9. Each excursion of z was renormalized so as to have length 1, the length of the excursion σ being made a component of the point process, and the calculus of variations on the point process σ_t was developped as in [8]. Although this calculus was simple and intuitively appealing in its principle, the computations were made difficult because jump martingales had to be explicitly used.

Later on, we discovered that the stochastic calculus in natural time scale could be "twisted" in such a way that an integration by parts could be proved without explicitly using the Itô theory of excursions. This point of view is developed in subsections (*a*), (*b*), (*c*), rather independently of its later applications to the calculus of variations.

In (*a*), we define a class of transformations on the reflecting Brownian motion z under which the law of z is quasi-invariant. After the adequate inclusion of $(w^1 \ldots w^m)$ in this transformation, it appears that the "right" class of perturbations for the reflecting Brownian motion has been found. These transformations are interpreted as transformations of the point process (ε_t, e_t) described in section 3, under which this point process is quasi-invariant. They are the infinite dimensional analogue of what we did in [8] for finite dimensional jump processes.

In (*b*), we obtain the fundamental equality which extends (2.14). In (*c*), an integration by parts formula on the reflecting Brownian motion is proved. This formula is given a direct short proof based on Itô-Tanaka's calculus when simple functionals are considered. Its relation to Ray-Knight's theorem on the local time process (Itô-McKean [18], p. 65, Jeulin-Yor [22]) are underlined.

In (*d*), a truncated integration by parts formula is derived on the component A_t of the boundary process (A_t, x_{A_t}). In (*f*) and (*g*), the full integration by parts formulas on the component A_t are given and the corresponding implications on the regularity of the boundary semi-group are exhibited in the localizable and the non localizable case. In (*h*), the existence of non necessarily regular densities for the boundary semi-group is considered.

In the whole section, we have taken a more naive approach than in section 2, i. e. we first prove integration by parts formulas on "simple" functionals, and later guess for what functionals such formulas should be extended so as to get the desired result on the boundary semi-group.

The results and notations of the previous sections (and especially of section 3) are used.

(*a*) *Quasi-invariance properties of the reflecting Brownian motion.*

On $(\bar{\Omega}, \{\bar{F}_t\}_{t \geqq 0}, P \otimes P')$ we consider a continuous semi-martingale H_t having the following properties:

(*a*) It is bounded and $\geqq 0$.

(*b*) H is 0 on $(z=0)$.

(*c*) The Itô decomposition of H is:

$$H_t = \int_0^t K\, ds + \int_0^t R\, dL + \int_0^t E\, \delta B, \tag{4.1}$$

where K, R, E, are bounded $\{\bar{F}_t\}_{t \geqq 0}$-predictable processes.

Since the support of dL is $(z=0)$, and since by Proposition 3.13, $R_{L_t}^-$ is still $\{\bar{F}_t\}_{t \geqq 0}$ predictable, we may, and we will assume that:

$$R_t = R_{L_t}^-. \tag{4.2}$$

Using a resultat of Ikeda-Watanabe [17], p. 306-307, it can be proved that if E is continuous, then $R_t = E_{L_t}^-$. We will not need this result.

Let L_t^H be the standard local time at 0 of H. Since H is $\geqq 0$, we know by Meyer [31] that:

$$L_t^H = \int_0^t 1_{H=0}\, \delta H. \tag{4.3}$$

Since L_t^H is an increasing process, $\int_0^t 1_{z=0}\, dL^H$ is also increasing. Since $(z=0)$ is negligible for the Lebesgue measure, and since H is 0 on $(z=0)$, we find from (4.1) that:

$$\int_0^t 1_{z=0}\, dL^H = \int_0^t R\, dL. \tag{4.4}$$

So we may assume that R is $\geqq 0$.

Finally, we will assume for technical reasons that:

(4.5) $$E \geqq -\frac{1}{2}.$$

DEFINITION 4.1. — N_t is the >0 $\{\bar{F}_t\}_{t\geqq 0}$-martingale:

(4.6) $$N_t=\exp\left\{-\int_0^t \frac{K}{1+E}\,\delta B-\frac{1}{2}\int_0^t\left(\frac{K}{1+E}\right)^2 ds\right\},$$

S is the probability measure on $\bar{\Omega}$ whose density with respect to $P\otimes P'$ on $\bar{F}_t$ is N_t, i. e.:

(4.7) $$\left.\frac{dS}{d(P\otimes P')}\right|_{\bar{F}_t}=N_t.$$

τ_t is the time change:

(4.8) $$\tau_t=\inf\left\{\tau\geqq 0;\ \int_0^\tau\left(\frac{1+E}{1+R}\right)^2 ds>t\right\}.$$

Since $1+E\geqq 1/2$, $K/(1+E)$ is bounded, and N_t is indeed a $\{\bar{F}_t\}_{t\geqq 0}$-martingale.

DEFINITION 4.2. — The following processes are defined by:

(4.9) $$\begin{cases} z'_t=\left(\dfrac{z+H}{1+R}\right)_{\tau_t}, & L'_t=L_{\tau_t}, \\ w'^1_t=\displaystyle\int_0^{\tau_t}\frac{1+E}{1+R}\,\delta w^1,\ \ldots,\ w'^m_t=\int_0^{\tau_t}\frac{1+E}{1+R}\,\delta w^m. \end{cases}$$

Since $R_t=R_{L_t}^-$, and $z+H$ is 0 on $(z=0)$, $(z_t+H_t)/(1+R_t)$ is clearly a continuous process. z'_t, L'_t, w'^1_t, $\ldots$, w'^m_t are then continuous processes, which are adapted to $\{\bar{F}_{\tau_t}\}_{t\geqq 0}$.

We now have the fundamental result:

THEOREM 4.3. — *Under the probability measure* S, $(z'_t-L'_t, w'^1_t, \ldots, w'^m_t)$ *is a* $m+1$-*dimensional Brownian* $\{\bar{F}_{\tau_t}\}_{t\geqq 0}$-*martingale, and* z'_t *is a reflecting Brownian motion, whose local time at* 0 *is* L'_t.

Proof. — Let $\bar{B}'_t$ be the process:

(4.10) $$\bar{B}'_t=B_t+\int_0^t\frac{K}{1+E}\,ds.$$

Under S, $(\bar{B}', w^1, \ldots, w^m)$ is a $m+1$-dimensional Brownian $\{\bar{F}_t\}_{t\geqq 0}$-martingale.

Clearly:

(4.11) $$z_t+H_t=\int_0^t(1+E)\,\delta\bar{B}'+\int_0^t(1+R)\,dL.$$

Using Azéma-Yor's formula [49], we know that since $R_t = R_{L_t}^-$ and since $z_t + H_t = 0$ on $(z=0)$, $(z_t + H_t)/(1+R_t)$ is a continuous semi-martingale, whose Itô decomposition is given by:

$$\text{(4.12)} \qquad \frac{z_t + H_t}{1+R_t} = \int_0^t \frac{1+E}{1+R}\,\delta\bar{B}' + L_t,$$

Under S, $\left(\int_0^{\tau_t} (1+E)/(1+R)\,\delta\bar{B}', w_t'^1, \ldots, w_t'^m\right)$ is a $m+1$-dimensional Brownian $\{\bar{F}_{\tau_t}\}_{t\geq 0}$-martingale. Obviously:

$$\text{(4.13)} \qquad z_t' = \int_0^{\tau_t} \frac{1+E}{1+R}\,\delta\bar{B}' + L_t'.$$

The support of dL is included in $(z=0)$, and $(z=0)$ coincides with $(z+H=0)$. The support of dL' is then included in $(z'=0)$, and z' is a ≥ 0 process.

Since $\int_0^{\tau_t} (1+E)/(1+R)\,\delta\bar{B}'$ is a Brownian motion, we know by a result of Skorokhod given in Ikeda-Watanabe [17], p. 120 that under S, z' is a reflecting Brownian motion whose local time at 0 is L'. The proof is finished. □

Remark 1. — The mapping $(w^1, \ldots, w^m, z) \to (w'^1, \ldots, w'^m, z')$ induces a natural transformation of the corresponding excursion point processes.

Namely let (ε_t, e_t) be the point process associated to $(w^1, \ldots, w^m, z)$ which has been defined in Definition 3.8. Similarly let (ε_t', e_t') be the point process associated to $(w'^1, \ldots, w'^m, z')$; this process is still defined as in Definition 3.8, using the local time $L'_.$ and its right-continuous inverse $A'_.$ instead of $L_.$ and $A_.$.

Both processes (ε_t, e_t) and (ε_t', e_t') are adapted to $\{\bar{F}_{A_t}\}_{t\geq 0}$. Moreover the jump times of these two processes—which take place at the times t where (ε_t, e_t) or (ε_t', e_t') are $\neq \delta$—are the same. To see this, it suffices to note that the jumps of (ε_t, e_t) coincide with the image of $(z\neq 0)$ by $L_.$. An obvious time change argument shows that this is also the image of $(z'\neq 0)$ by $L'_.$, which gives the jump times of (ε_t', e_t').

We now express (ε_t', e_t') in terms of (ε_t, e_t). For $t\in R^+$, $(\bar{\omega}, \varepsilon, e)\in\bar{\Omega}\times\mathscr{W}_0$, consider the excursion time change:

$$\text{(4.14)} \qquad \rho_s^t = \inf\left\{\rho\geq 0;\ \int_0^{\rho} \left[\frac{1+E_{A_{t^-}+u}(\bar{\omega}\,|\,A_{t^-}\,|\,(\varepsilon, e))}{1+R_{A_{t^-}}(\bar{\omega})}\right]^2 du > s\right\}$$

[for simplicity, we did not write explicitly that ρ depends on $(\bar{\omega}, \varepsilon, e)$].

The reader can easily check that (ε_t', e_t') is given by the following formulas:

If $(\varepsilon_t, e_t)\neq\delta$, $(\varepsilon_t', e_t')(s)$ is stopped at σ_t' where:

$$\sigma_t' = \int_0^{A_t - A_{t^-}} \left[\frac{1+E_{A_{t^-}+u}(\bar{\omega}\,|\,A_{t^-}\,|\,(\varepsilon_t, e_t))}{1+R_{A_{t^-}}}\right]^2 du$$

and for $0 \leqq s \leqq \sigma'_t$:

$$(4.15)\quad \begin{cases} \varepsilon'_t(s) = \displaystyle\int_0^{\rho_s^t} \frac{1+E_{A_t-+u}(\bar{\omega} \mid A_t- \mid (\varepsilon_t, e_t))}{1+R_{A_t-}(\bar{\omega})}\, \delta\varepsilon_t(u), \\ e'_t(s) = \dfrac{e_t(\rho_s^t) + H_{A_t-+\rho_s^t}(\bar{\omega} \mid A_t- \mid (\varepsilon_t, e_t))}{1+R_{A_t-}(\bar{\omega})}. \end{cases}$$

The transformation $(\bar{\omega}, \varepsilon_t, e_t) \to (\varepsilon'_t, e'_t)$ is clearly $\mathscr{P}' \otimes \mathscr{G}_\infty$-measurable. This transformation is closely connected with the transformation which we used in [8] for finite-dimensional jump processes.

A'_t is given by:

$$(4.16)\quad A'_t = \inf\{ A' \geqq 0;\ L'_{A'} > t \},$$

Clearly:

$$(4.17)\quad A'_t = \int_0^{A_t} \left[\frac{1+E}{1+R} \right]^2 ds.$$

Since under S, $z'_.$ is a reflecting Brownian motion, we know that $A'_t < +\infty$ S a. s. Since $1+E \geqq 1/2$, and since R is bounded, we find that S a. s., $A_t < +\infty$. From Proposition 1.4, we find that:

$$(4.18)\quad \left.\frac{dS}{d(P \otimes P')}\right|_{\bar{F}_{A_t}} = N_{A_t}.$$

Because of (4.18), the result stated in Theorem 3.25 is still true, i. e., under S, (ε_t, e_t) is a point process whose Lévy measure $dn'_{(\bar{\omega},t)}(\varepsilon, e)$ is given by:

$$(4.19)\quad dn'_{(\bar{\omega},t)}(\varepsilon, e) = \left[\exp\left\{ -\int_0^{A_t - A_t-} \left(\frac{K}{1+E} \right)_{A_t-+u} (\bar{\omega} \mid A_t- \mid (\varepsilon, e))\, \delta\varepsilon_t(u) \right.\right.$$
$$\left.\left. -\frac{1}{2} \int_0^{A_t - A_t-} \left(\frac{K}{1+E} \right)^2_{A_t-+u} (\bar{\omega} \mid A_t- \mid (\varepsilon, e))\, du \right\} \right] dn(\varepsilon, e).$$

An equivalent formulation of Theorem 4.3 is that under $dn'_{(\bar{\omega},t)}(\varepsilon, e)$, the law of (ε', e') is exactly n. This result can be better formulated by assuming that R is constant (i. e. not random), and that K, E only depend on (ε, e) (i. e. do not depend on $\bar{\omega}$ before A_t-) so that $\bar{\omega}$, t can be considered as parameters.

(*b*) *The basic equality.*

$x_0 \in R^d$ is now fixed.

On $(\bar{\Omega}, \{\bar{F}_t\}_{t \geqq 0}, P \otimes P')$, we consider the stochastic differential equation:

$$(4.20)\quad d\bar{x} = \left(\frac{1+E}{1+R} \right)^2 X_0\left(\bar{x}, \frac{z+H}{1+R} \right) dt + X_i\left(\bar{x}, \frac{z+H}{1+R} \right) . d\int_0^t \frac{1+E}{1+R}\, \delta w^i + D(\bar{x})\, dL,$$
$$\bar{x}(0) = x_0.$$

Here $d\int_0^t (1+E)/(1+R)\,\delta w^i$ denotes the Stratonovitch differential of the Itô integral $\int_0^t (1+E)/(1+R)\,\delta w^i$.

Set:

$$(4.21)\quad \bar{M}_t=\exp\left\{\int_0^t\left(\frac{1+E}{1+R}\,b\left(\bar{x},\frac{z+H}{1+R}\right)-\frac{K}{1+E}\right)\delta B -\frac{1}{2}\int_0^t\left(\frac{1+E}{1+R}\,b\left(\bar{x},\frac{z+H}{1+R}\right)-\frac{K}{1+E}\right)^2 ds\right\}.$$

THEOREM 4.4. — *Let x_s be the process $\varphi_s(\bar{\omega}, x_0)$. Then for any $t>0$, any $f\in C_c^\infty(R\otimes R^d)$, the following equality holds:*

$$(4.22)\quad E^{P\otimes P'}\left[\bar{M}_{A_t} f\left(\int_0^{A_t}\left(\frac{1+E}{1+R}\right)^2 ds,\ \bar{x}_{A_t}\right)\right]=E^{P\otimes P'}[M_{A_t} f(A_t, x_{A_t})].$$

Proof. — As in section 2, it is essential that f has compact support. Since $1+E\geqq 1/2$ and R is bounded, there is $T\geqq 0$ such that if $A_t\geqq T$, $f\left(\int_0^{A_t}((1+E)/(1+R))^2\,ds,\ \bar{x}_{A_t}\right)$ and $f(A_t, x_{A_t})$ are 0.

Set $x'_t=\bar{x}_{\tau_t}$. It is clear that:

$$(4.23)\quad \begin{cases} dx'=X_0(x',z')\,dt+X_i(x',z')\,.\,dw'^i+D(x')\,dL',\\ x'(0)=x_0.\end{cases}$$

From (4.17), we know that $\tau_{A'_t}=A_t$, and so:

$$(4.24)\quad x'_{A'_t}=\bar{x}_{A_t}.$$

Moreover, if $\bar{B}'_t$ is defined by (4.10), we have:

$$(4.25)\quad \bar{M}_t=N_t\exp\left\{\int_0^t b\left(\bar{x},\frac{z+H}{1+R}\right)\frac{1+E}{1+R}\,\delta\bar{B}'-\frac{1}{2}\int_0^t\left[b\left(\bar{x},\frac{z+H}{1+R}\right)\frac{1+E}{1+R}\right]^2 ds\right\}$$

(4.22) is then an obvious consequence of (4.23)-(4.25) and of Theorem 4.3. □

(*c*) *An integration by parts formula on the reflecting Brownian motion.*

We will now use (4.22) in the same way we used (2.14) in the proof of Theorem 2.2, i. e. we will differentiate (4.22).

We have:

THEOREM 4.5. — *Take* $x_0 \in R^d$. *Let* x_s *be the process* $\varphi_s(\bar{\omega}, x_0)$, *and let* U_s *be the process with values in* $T_{x_0}(R^d)$:

$$(4.26)\quad U_s = \int_0^s \varphi_v^{*-1}\left(2(E-R)X_0(x_v, z_v) + (H-R\,z)\frac{\partial X_0}{\partial z}(x_v, z_v)\right)dv + \int_0^s \varphi_v^{*-1} X_i(x_v, z_v)\, d\int_0^v (E-R)_u\, \delta w_u^i + \int_0^s \varphi_v^{*-1}(H_v - R\,z_v)\frac{\partial X_i}{\partial z}(x_v, z_v)\,.\,dw^i.$$

Then for any $t>0$, *and any* $f \in C_c^\infty(R\times R^d)$, *the following equality holds:*

$$(4.27)\quad E^{P\otimes P'}\left[M_{A_t}\, d_a f(A_t, x_{A_t})\int_0^{A_t} 2(E-R)\,ds\right] + E^{P\otimes P'}\left[M_{A_t}\langle d_x f(A_t, x_{A_t}), \varphi_{A_t}^* U_{A_t}\rangle\right] + E^{P\otimes P'}\left[M_{A_t} f(A_t, x_{A_t})\int_0^{A_t}(-K+(E-R)\,b(x_s, z_s) + \langle b_x(x_s, z_s), \varphi_s^* U_s\rangle + (H-R\,z)\,b_z(x_s, z_s))(\delta B_s - b(x_s, z_s)\,ds)\right] = 0.$$

Proof. — First observe that (4.27) makes sense. In fact f has compact support, so that T exists such that if $A_t \geqq T$, $f(A_t, x_{A_t}) = 0$. Now $1_{A_t \leqq T} M_{A_t}$ is in all the $L_p(\bar{\Omega}, P\otimes P')$. Moreover $\sup\limits_{0\leqq s\leqq T} z_s$ is in all the $L_p(\bar{\Omega}, P\otimes P')$. Using Theorem 1.1 (*e*) it is clear that $1_{A_t\leqq T}\sup\limits_{0\leqq s\leqq A_t}|\varphi_s^* U_s|$ is in all the $L_p(\bar{\Omega}, P\otimes P')$.

This shows that:

$$(4.28)\quad 1_{A_t\leqq T}\int_0^{A_t}(-K+(E-R)\,b + \langle b_x, \varphi_s^* U_s\rangle + (H-R\,z)\,b_z)(\delta B - b\,ds),$$

is also in all the $L_p(\bar{\Omega}, P\otimes P')$.

For $l \in R^+$, $l \leqq 1$, consider the stochastic differential equation:

$$(4.29)\quad \begin{cases} dx^l = \left(\dfrac{1+l\,E}{1+l\,R}\right)^2 X_0\left(x^l, \dfrac{z+l\,H}{1+l\,R}\right)ds + X_i\left(x^l, \dfrac{z+l\,H}{1+l\,R}\right).\,d\displaystyle\int_0^t\left(\dfrac{1+l\,E}{1+l\,R}\right)\delta w^i + D(x^l)\,.\,dL, \\ x^l(0) = x_0. \end{cases}$$

Set:

$$(4.30)\quad M_t^l=\exp\left\{\int_0^t\left(\frac{1+l\,E}{1+l\,R}\,b\left(x^l,\frac{z+l\,H}{1+l\,R}\right)-\frac{l\,K}{1+l\,E}\right)\delta B-\frac{1}{2}\int_0^t\left(\frac{1+l\,E}{1+l\,R}\,b\left(x^l,\frac{z+l\,H}{1+l\,R}\right)-\frac{l\,K}{1+l\,E}\right)^2 ds\right\}.$$

From Theorem 4.4, we know that:

$$(4.31)\quad E^{P\otimes P'}\left[M_{A_t}^l f\left(\int_0^{A_t}\left(\frac{1+l\,E}{1+l\,R}\right)^2 ds,\, x_{A_t}^l\right)\right]=E^{P\otimes P'}[M_{A_t} f(A_t, x_{A_t})],$$

so that the l. h. s. of (4.31) does not depend on l. The differential of (4.31) at $l=0$ is then 0.

Using the results of Bismut [5], Kunita [25], we know that it is possible to define $x^l_{\cdot}(\bar\omega)$, $M^l_{\cdot}(\bar\omega)$, so that a. s., $l\to(x^l_{\cdot}(\bar\omega), M^l_{\cdot}(\bar\omega))$ is a C^∞ mapping from $[0,1[$ into $C([0,T];R^{d+1})$, and moreover the standard rules of variations of parameters also apply to stochastic differential equations. An easy computation shows that:

$$(4.32)\quad \left\{\begin{aligned} &\left[\frac{\partial x^l}{\partial l}\,t\right]_{l=0}=\varphi_t^*\,U_t,\\ &\left[\frac{\partial M^l}{\partial l}\,t\right]_{l=0}=M_t\int_0^t(-K+(E-R)\,b+\langle b_x,\varphi_s^*\,U_s\rangle\\ &\qquad\qquad +(H-R\,z)\,b_z)\,(\delta B-b\,(x_s,z_s)\,ds).\end{aligned}\right.$$

By reasoning as in [7], Theorem 2.1 and [10], Theorem 2.11, still using Theorem 1.1 (*e*) and the fact that f has compact support, it can be easily proved that differentiation of the l. h. s. of (4.31) under the expectation sign is possible. The Theorem follows. □

Remark 2. — By Proposition 1.4, in (4.27), $E^{P\otimes P'}\,M_{A_t}[\ldots]$ can be replaced by $E^{Q_{(x_0,0)}}[\ldots]$. Also recall that under $Q_{(x_0,0)}$, $B_t-\int_0^t b\,(x_s,z_s)\,ds$ is a Brownian $\{\bar F_t\}_{t\geq 0}$-martingale. Using the results of section 3, the reader will check that formula (4.27) is strikingly identical to the result given in Theorem 2.5 in [8]. Namely, for $(x,\varepsilon,e)\in R^d\times\mathscr{W}_0$, set:

$$(4.33)\quad m^1(x,\varepsilon,e)=\exp\left\{\int_0^\sigma b\,(\varphi_s(\varepsilon,e,x),e)\,\delta e-\frac{1}{2}\int_0^\sigma b^2\,(\varphi_s(\varepsilon,e,x),e)\,ds\right\}.$$

It suffices to use the fact that at least when b has compact support, under $Q_{(x_0,0)}$, the Lévy measure of the point process (ε_t, e_t) is $m^1(x_{A_t-},\varepsilon,e)\,dn(\varepsilon,e)$, where $x_s=\varphi_s(\bar\omega,x_0)$. The comparison with [8] is then easy.

Also observe that the condition $E\geq -1/2$ can be dropped (it suffices to replace H by λ H, with λ small enough).

Also (4.27) extends to semi-martingales which are the difference of two $\geqq 0$ semi-martingales having the same properties as H. If H′ is a non necessarily $\geqq 0$ semi-martingale which has all the other properties of H, and is $\leqq C z_t$, we may write:

$$(4.34)\qquad H'_t = \frac{C' z_t}{z_t+1} - \left(\frac{C' z_t}{z_t+1} - H'_t\right).$$

For C′ large enough, $H'_t \leqq C' z_t/(z_t+1)$ and so (4.27) applies to H′.

Remark 3. — Let m be an element of $C_b^\infty(R)$ which is such that $m \geqq \alpha > 0$. If $g \in C_c^\infty(R)$, Theorem 4.5 shows that:

$$(4.35)\quad E^{P\otimes P'}\left[g'\left(\int_0^{A_t} m(z)\,ds\right)\int_0^{A_t}(2(E-R)\,m(z)+(H-R\,z)\,m'(z))\,ds\right] - E^{P\otimes P'}\left[g\left(\int_0^{A_t} m(z)\,ds\right)\int_0^{A_t} K\,\delta B\right]=0.$$

It is most instructive to give a direct proof of (4.35) using the usual stochastic calculus. By eventually replacing H_t by $h(t)\,H_t$ [where $h \in C_c^\infty(R)$ and is $\geqq 0$], we will assume that for $t \geqq T$, H, K, R, E are 0.

We will then prove (4.35) for $g(a)=e^{-a}$ (g does not have compact support but this is irrelevant). Replacing m by βm ($\beta>0$), (4.35) will then be proved for $g(a)=e^{-\beta a}$ and readily extends to $g \in C_c^\infty(R)$.

Let u be the only decreasing $\geqq 0$ C^∞ function which is the solution of the Sturm-Liouville problem:

$$(4.36)\qquad \frac{u''}{2} - mu = 0, \qquad u(0)=1.$$

Let G_t be the $\{\bar F_t\}_{t\geqq 0}$-martingale:

$$(4.37)\qquad G_t = u(z_t)\exp\left\{-\int_0^t m(z)\,ds - u'(0)\,L_t\right\}.$$

It is easy to see that G_{A_t} is a $\{\bar F_{A_t}\}_{t\geqq 0}$-martingale and moreover:

$$(4.38)\qquad dG = G\,\frac{u'(z)}{u(z)}\,\delta B,$$

Using (4.38) and Itô's calculus, we find that:

$$E^{P\otimes P'}\left[G_{A_t}\int_0^{A_t} K\,\delta B\right] = E^{P\otimes P'}\left[G_{A_t}\int_0^{A_t} K\,\frac{u'}{u}(z)\,ds\right].$$

Now using the fact that H is 0 on $(z=0)$, and Itô-Tanaka and Azéma-Yor's formulas, we have:

$$(4.39)\quad (\mathrm{H}_t-\mathrm{R}_t\, z_t)\left(\frac{u'}{u}\right)(z_t)=\int_0^t \mathrm{K}\,\frac{u'}{u}(z)\,ds + \int_0^t \left[\frac{1}{2}(\mathrm{H}-\mathrm{R}\,z)\left(\frac{u'}{u}\right)''(z)+(\mathrm{E}-\mathrm{R})\left(\frac{u'}{u}\right)'(z)\right]ds + \int_0^t \left[(\mathrm{E}-\mathrm{R})\,\frac{u'}{u}(z)+(\mathrm{H}-\mathrm{R}\,z)\left(\frac{u'}{u}\right)'(z)\right]\delta \mathrm{B}.$$

Another application of Itô's calculus shows that:

$$(4.40)\quad \mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\mathrm{G}_{\mathrm{A}_t}\int_0^{\mathrm{A}_t}\left[(\mathrm{E}-\mathrm{R})\,\frac{u'}{u}(z)+(\mathrm{H}-\mathrm{R}\,z)\left(\frac{u'}{u}\right)'(z)\right]\delta\mathrm{B}\right] = \mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\mathrm{G}_{\mathrm{A}_t}\int_0^{\mathrm{A}_t}\left[(\mathrm{E}-\mathrm{R})\left(\frac{u'}{u}\right)^2(z)+(\mathrm{H}-\mathrm{R}\,z)\left(\frac{u'}{u}\right)'(z)\left(\frac{u'}{u}\right)(z)\right]ds\right].$$

Using (4.36)-(4.40) and the fact that $\mathrm{H}_{\mathrm{A}_t}-\mathrm{R}_{\mathrm{A}_t}\, z_{\mathrm{A}_t}=0$, we find easily that:

$$(4.41)\quad \mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\mathrm{G}_{\mathrm{A}_t}\int_0^{\mathrm{A}_t}\mathrm{K}\,\delta\mathrm{B}\right] = -\mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\mathrm{G}_{\mathrm{A}_t}\int_0^{\mathrm{A}_t}[2(\mathrm{E}-\mathrm{R})\,m(z)+(\mathrm{H}-\mathrm{R}\,z)\,m'(z)]\,ds\right],$$

which is exactly (4.35) with $g(a)=e^{-a}$.

In the case where $\mathrm{H}_t=h(z_t)$ (with $h\in \mathrm{C}_b^\infty(\mathrm{R})$, $h(0)=0$), (4.35) writes:

$$(4.42)\quad \mathrm{E}^{\mathrm{P}'}\left[g'\left(\int_0^{\mathrm{A}_t} m(z)\,ds\right)\int_0^{\mathrm{A}_t}[2(h'(z)-h'(0))\,m(z) + (h(z)-h'(0)\,z)\,m'(z)]\,ds\right] - \mathrm{E}^{\mathrm{P}'}\left[g\left(\int_0^{\mathrm{A}_t} m(z)\,ds\right)\int_0^{\mathrm{A}_t}\frac{1}{2}h''(z)\,\delta\mathrm{B}\right]=0.$$

(4.42) is also a consequence of Ray-Knight's theorem, which states that if $\mathrm{L}^t(a)$ $(a\geqq 0)$ is the standard local time of z. at a at time A_t, then as a function of a, $\mathrm{L}^t(a)$ is half of a $\mathrm{Bes}^2(0)$ process starting from $2t$ at time 0 (*see* Itô-McKean [18], p. 65, Jeulin-Yor [22]). Namely, we have:

$$(4.43)\quad \int_0^{\mathrm{A}_t} h''(z)\,\delta\mathrm{B}=\int_0^{\mathrm{A}_t} h''(z)\,dz-\frac{1}{2}\int_0^{\mathrm{A}_t} h'''(z)\,ds - \int_0^{\mathrm{A}_t} h''(0)\,d\mathrm{L}=-\int_0^{+\infty} h'''(a)\,\mathrm{L}^t(a)\,da-h''(0)\,t,$$

so that (4.42) writes:

$$(4.44)\quad E^{P'}\left[g'\left(2\int_0^{+\infty} m(a)\, L^t(a)\, da\right)\int_0^{+\infty}[4(h'(a)-h'(0))\, m(a) + 2(h(a)-h'(0)\,a)\, m'(a)]\, L^t(a)\, da\right] + E^{P'}\left[g\left(2\int_0^{+\infty} m(a)\, L^t(a)\, da\right)\left(\int_0^{+\infty}\frac{h'''(a)\, L^t(a)}{2}\, da + \frac{h''(0)\, t}{2}\right)\right] = 0.$$

Let β_t be an auxiliary Brownian motion, and (Ω'', P'') be its probability space. Using a result of Yamada (Ikeda-Watanabe [17], p. 168), we know that in law, $L^t(a)$ is the unique solution of:

$$(4.45)\qquad dL^t(a) = \sqrt{2\, L^t(a)}\, \delta\beta(a), \qquad L^t(0) = t.$$

Let $k \in C_c^\infty(\mathbb{R})$ be such that $k(0)=0$, $|k| \leqq 1/2$. For $l \leqq 1$, we have:

$$(4.46)\qquad \begin{cases} d(1+lk(a))\, L^t(a) = (1+lk(a))\sqrt{2\, L^t(a)}\, \delta\beta(a) + lk'(a)\, L^t(a)\, da, \\ (1+lk(0))\, L^t(0) = t. \end{cases}$$

Let 0^l be the Girsanov exponential:

$$(4.47)\quad 0^l = \exp\left[-\int_0^{+\infty}\frac{lk'(a)}{1+lk(a)}\sqrt{\frac{L^t(a)}{2}}\, \delta\beta(a) - \frac{1}{4}\int_0^{+\infty}\left[\frac{lk'(a)}{1+lk(a)}\right]^2 L^t(a)\, da\right\} = \exp\left\{\frac{ltk'(0)}{2} + \int_0^{+\infty}\left[\frac{lk''(a)}{2(1+lk(a))} - \frac{3}{4}\left(\frac{lk'(a)}{1+lk(a)}\right)^2\right] L^t(a)\, da\right\}$$

and let γ_a be the time change:

$$(4.48)\qquad \gamma_a = \inf\left\{\gamma \geqq 0;\ \int_0^{\gamma}(1+lk(b))\, db > a\right\}.$$

Using (4.45)-(4.47), we see that under the probability measure $0^l\, dP''$, the process $(1+lk(\gamma_a))\, L^t(\gamma_a)$ has the same law as $L^t(a)$ under P''.

It is then clear that:

$$(4.49)\quad E^{P''}\left[0^l\, g\left(2\int_0^{+\infty} m\left(\int_0^a(1+lk(b))\, db\right)(1+lk(a))^2\, L^t(a)\, da\right)\right] = E^{P''}\left[g\left(2\int_0^{+\infty} m(a)\, L^t(a)\, da\right)\right].$$

By differentiating (4.49) at $l=0$, we find:

$$(4.50)\quad E^{P''}\left[g'\left(2\int_0^{+\infty} m(a)\,L^t(a)\,da\right)\int_0^{+\infty}\left(4\,k(a)\,m(a)+2\int_0^a k(b)\,db\,m'(a)\right)L^t(a)\,da\right]$$
$$+E^{P''}\left[g\left(2\int_0^{+\infty} m(a)\,L^t(a)\,da\right)\left(\int_0^{+\infty}\frac{k''(a)\,L^t(a)}{2}\,da+\frac{k'(0)\,t}{2}\right)\right]=0.$$

If $k(a)=h'(a)-h'(0)$, since $h(0)=0$, we find that (4.50) is exactly (4.44).

(4.44) can also be derived using Itô's calculus on $L^t(a)$. We will prove (4.44) with $g(b)=e^{-b}$. If u is given by (4.36), then it immediately follows from (4.45) that Q_a given by:

$$(4.51)\qquad Q_a=\exp\left\{-2\int_0^a m(b)\,L^t(b)\,db+\frac{u'(a)}{u(a)}\,L^t(a)\right\},$$

is a martingale on (Ω'', P'') (this is the basis of Jeulin-Yor [22]) and moreover:

$$(4.52)\qquad dQ_a=Q_a\,\frac{u'(a)}{u(a)}\,\delta L^t(a).$$

Now:

$$(4.53)\qquad \int_0^{+\infty}\frac{h'''(a)\,L^t(a)\,da}{2}+\frac{h''(0)\,t}{2}=-\int_0^{+\infty}\frac{h''(a)\,\delta L^t(a)}{2}.$$

Using Itô's stochastic calculus and (4.52), we find:

$$(4.54)\quad -E^{P''}\left[Q_\infty\int_0^{+\infty}\frac{h''(a)\,\delta L^t(a)}{2}\right]=-E^{P''}\left[Q_\infty\int_0^{\infty} h''(a)\,\frac{u'}{u}(a)\,L^t(a)\,da\right].$$

Now:

$$(4.55)\quad -\int_0^{+\infty} h''(a)\,\frac{u'}{u}\,L^t(a)\,da=\int_0^{+\infty}(h'(a)-h'(0))\left(\frac{u''}{u}-\frac{u'^2}{u^2}\right)L^t(a)\,da$$
$$+\int_0^{+\infty}(h'(a)-h'(0))\,\frac{u'}{u}\,\delta L^t(a).$$

Using (4.52) again we find from (4.54) and (4.55) that:

$$(4.56)\quad -E^{P''}\left[Q_\infty\int_0^{+\infty}\frac{h''(a)\,\delta L^t(a)}{2}\right]$$

$$=\mathrm{E}^{\mathrm{P}''}\left[\mathrm{Q}_\infty\int_0^{+\infty}(h'(a)-h'(0))\left(\frac{u''}{u}+\frac{u'^2}{u^2}\right)(a)\,\mathrm{L}^t(a)\,da\right]$$

$$=\mathrm{E}^{\mathrm{P}''}\left[\mathrm{Q}_\infty\left(\int_0^{+\infty}4(h'(a)-h'(0))\,m(a)\,\mathrm{L}^t(a)\,da\right.\right.$$

$$\left.\left.-\int_0^{+\infty}(h'(a)-h'(0))\left(\frac{u'}{u}\right)'\mathrm{L}^t(a)\,da\right)\right].$$

Using the fact that $h(0)=0$, we have:

$$(4.57)\quad -\mathrm{E}^{\mathrm{P}''}\left[\mathrm{Q}_\infty\int_0^{+\infty}(h'(a)-h'(0))\left(\frac{u'}{u}\right)'\mathrm{L}^t(a)\,da\right]$$

$$=\mathrm{E}^{\mathrm{P}''}\,\mathrm{Q}_\infty\int_0^{+\infty}(h(a)-h'(0)\,a)\left(\left(\frac{u'}{u}\right)''+2\left(\frac{u'}{u}\right)'\left(\frac{u'}{u}\right)(a)\right)\mathrm{L}^t(a)\,da$$

$$=\mathrm{E}^{\mathrm{P}''}\,\mathrm{Q}_\infty\int_0^{+\infty}2(h(a)-h'(0)\,a)\,m'(a)\,\mathrm{L}^t(a)\,da.$$

From (4.51)-(4.57), (4.44) holds with $g(b)=e^{-b}$.

(*d*) *Truncated integration by parts in the variable* A.

Recall that we want to obtain a formula of the type:

$$(4.58)\qquad \mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}[\mathrm{M}_{\mathrm{A}_t}\,d_a f(\mathrm{A}_t, x_{\mathrm{A}_t})]=\mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}[f(\mathrm{A}_t, x_{\mathrm{A}_t})\,\bar{\mathrm{D}}_t^1].$$

From (4.27), it is clear that we must try to apply the calculus of variations to $\mathrm{M}_{\mathrm{A}_t} f(\mathrm{A}_t, x_{\mathrm{A}_t})\Big/\int_0^{\mathrm{A}_t}2(\mathrm{E}-\mathrm{R})\,ds$ instead of applying it only to $\mathrm{M}_{\mathrm{A}_t} f(\mathrm{A}_t, x_{\mathrm{A}_t})$.

In this section we will—somewhat arbitrarily—select one special H_t, which will be z_t^2 (which is unbounded, but this will be irrelevant).

Let τ be an element of $\mathrm{C}_c^\infty(\mathrm{R})$, with values in [0, 1], such that $\tau(u)=1$ if $|u|\leqq 1$, and $\tau(u)=0$ if $|u|\geqq 2$.

For $\mathrm{N}\geqq 1$, the function χ_N is defined by:

$$\chi_\mathrm{N}(u)=1-\tau(\mathrm{N}u).$$

For $k, l\in\mathrm{N}$, the functions $u\to[d^k\chi_\mathrm{N}(u)/du^k]\,u^{-l}$ can be extended to $u=0$ by setting:

$$\left[\frac{d^k\chi_\mathrm{N}}{du^k}\right]u^{-l}=0\qquad\text{if}\quad u=0.$$

These functions are in $\mathrm{C}_b^\infty(\mathrm{R})$.

We have the analogue of Theorem 2.3.

THEOREM 4.6. — *Take* $x_0 \in \mathrm{R}^d$. *Let* x_s *be the process* $\varphi_s(\bar{\omega}, x_0)$ *and let* V_s *be the process with values in* $\mathrm{T}_{x_0}(\mathrm{R}^d)$:

$$(4.59)\quad \mathrm{V}_s = \int_0^s \varphi_v^{*-1}\left(4 z_v \mathrm{X}_0(x_v, z_v) + z_v^2 \frac{\partial \mathrm{X}_0}{\partial z}(x_0, z_v)\right) dv + \int_0^s \varphi_v^{*-1}\left(2 z_v \mathrm{X}_i(x_v, z_v) + z_v^2 \frac{\partial \mathrm{X}_i}{\partial z}(x_v, z_v)\right) . dw^i.$$

Then for any $\mathrm{N} \geq 1$, $t, t'' \geq 0$, $t \geq t''$, *and any* $f \in \mathrm{C}_c^\infty(\mathrm{R} \times \mathrm{R}^d)$, *the following equality holds:*

$$(4.60)\quad 4\,\mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\chi_{\mathrm{N}}\left(\int_0^{\mathrm{A}_{t''}} z\, ds\right) \mathrm{M}_{\mathrm{A}_t}\, d_a f(\mathrm{A}_t, x_{\mathrm{A}_t})\right] + \mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\frac{\chi_{\mathrm{N}}\left(\int_0^{\mathrm{A}_{t''}} z\, ds\right)}{\int_0^{\mathrm{A}_{t''}} z\, ds} \mathrm{M}_{\mathrm{A}_t}\left\{\langle d_x f(\mathrm{A}_t, x_{\mathrm{A}_t}), \varphi_{\mathrm{A}_t}^* \mathrm{V}_{\mathrm{A}_{t''}}\rangle + f(\mathrm{A}_t, x_{\mathrm{A}_t}) \right.\right.$$
$$\times\left(\int_0^{\mathrm{A}_{t''}} (-1 + 2 z b(x_s, z_s) + z_s^2 b_z(x_s, z_s) + \langle b_x(x_s, z_s), \varphi_s^* \mathrm{V}_s\rangle)\right.$$
$$\left.\left.\left.(\delta \mathrm{B}_s - b(x_s, z_s)\, ds) + \left\langle \int_{\mathrm{A}_{t''}}^{\mathrm{A}_t} \varphi_s^* b_x(x_s, z_s)(\delta \mathrm{B} - b(x_s, z_s)\, ds), \mathrm{V}_{\mathrm{A}_{t''}}\right\rangle\right)\right\}\right]$$
$$+ \mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\mathrm{M}_{\mathrm{A}_t} f(\mathrm{A}_t, x_{\mathrm{A}_t})\left(\frac{\chi'_{\mathrm{N}}\left(\int_0^{\mathrm{A}_{t''}} z\, ds\right)}{\int_0^{\mathrm{A}_{t''}} z\, ds} - \frac{\chi_{\mathrm{N}}\left(\int_0^{\mathrm{A}_{t''}} z\, ds\right)}{\left(\int_0^{\mathrm{A}_{t''}} z\, ds\right)^2}\right) \int_0^{\mathrm{A}_{t''}} 5 z^2\, ds\right] = 0.$$

Proof. — Observe that for any $\mathrm{T} \geq 0$, $\sup_{0 \leq s \leq \mathrm{T}} z_s$ is in all the $\mathrm{L}_p(\bar{\Omega}, \mathrm{P}\otimes\mathrm{P}')$. Moreover since χ_{N} or χ'_{N} appear everywhere, the fact that $\int_0^{\mathrm{A}_t} z\, ds$ is in the denominator of the various expressions does not raise any difficulty. (4.60) makes sense for the same reason as (4.27). Also recall that as pointed out in section 1, $\varphi_s^* b_x(x_s, z_s)$ is the element of $\mathrm{T}_{x_0}^*(\mathrm{R}^d)$ given by $\widetilde{\partial\varphi_s/\partial x}(\bar{\omega}, x_0)\, b_x(x_s, z_s)$ ($\sim$ is the sign for transposition).

We now prove (4.60). We first take H as in (*a*) with $\mathrm{R}=0$. Using the notations in the proof of Theorem 4.5, we claim that:

$$(4.61)\quad \mathrm{E}^{\mathrm{P}\otimes\mathrm{P}'}\left[\mathrm{M}_{\mathrm{A}_t}^l \frac{\chi_{\mathrm{N}}\left(\int_0^{\mathrm{A}_{t''}} (1 + l\,\mathrm{E})^2 (z + l\,\mathrm{H})\, ds\right)}{\int_0^{\mathrm{A}_{t''}} (1 + l\,\mathrm{E})^2 (z + l\,\mathrm{H})\, ds} f\left(\int_0^{\mathrm{A}_t} (1 + l\,\mathrm{E})^2\, ds, x_{\mathrm{A}_t}^l\right)\right]$$

$$= E^{P \otimes P'} \left[M_{A_t} \frac{\chi_N \left(\int_0^{A_{t''}} z \, ds \right)}{\int_0^{A_{t''}} z \, ds} f(A_t, x_{A_t}) \right],$$

(4.61) is an obvious consequence of Theorems 4.3 and 4.4. If we differentiate (4.61) at $l=0$ the way we did in the proof of Theorem 4.5, we get:

$$(4.62) \quad E^{P \otimes P'} \left[\frac{\chi_N \left(\int_0^{A_{t''}} z \, ds \right)}{\int_0^{A_{t''}} z \, ds} M_{A_t} \, d_a f(A_t, x_{A_t}) \int_0^{A_t} 2 \, E \, ds \right]$$

$$+ E^{P \otimes P'} \left[\frac{\chi_N \left(\int_0^{A_{t''}} z \, ds \right)}{\int_0^{A_{t''}} z \, ds} M_{A_t} \Big(\langle d_x f(A_t, x_{A_t}), \varphi^*_{A_t} U_{A_t} \rangle + f(A_t, x_{A_t}) \right.$$

$$\times \int_0^{A_t} (-K_s + E_s \, b(x_s, z_s)$$

$$\left. + H_s \, b_z(x_s, z_s) + \langle b_x(x_s, z_s), \varphi^*_s U_s \rangle (\delta B_s - b(x_s, z_s) \, ds) \Big) \right]$$

$$+ E^{P \otimes P'} \left[M_{A_t} f(A_t, x_{A_t}) \left(\frac{\chi'_N \left(\int_0^{A_{t''}} z \, ds \right)}{\left(\int_0^{A_{t''}} z \, ds \right)} \right. \right.$$

$$\left. \left. - \frac{\chi_N \left(\int_0^{A_{t''}} z \, ds \right)}{\left(\int_0^{A_{t''}} z \, ds \right)^2} \right) \int_0^{A_{t''}} (2 \, E \, z + H) \, ds \right] = 0.$$

For $k \in N$, we apply (4.62) to $H_s = H^k_s$ where H^k_s is given by:

$$H^k_s = \tau \left(\frac{z_{s \wedge A_{t''}}}{k} \right) z^2_{s \wedge A_{t''}}.$$

Clearly:

$$(4.63) \quad H^k_s = \int_0^{s \wedge A_{t''}} \left(\tau \left(\frac{z}{k} \right) + \frac{2}{k} \tau' \left(\frac{z}{k} \right) z + \frac{1}{2k^2} \tau'' \left(\frac{z}{k} \right) z^2 \right) du$$

$$+ \int_0^{s \wedge A_{t''}} \left(2 \tau \left(\frac{z}{k} \right) z + \frac{1}{k} \tau' \left(\frac{z}{k} \right) z^2 \right) \delta B.$$

so that the corresponding K^k_s, E^k_s are given by:

$$(4.64)\quad \begin{cases} K^k_s = \left(\tau\left(\frac{z_s}{k}\right) + \frac{2}{k}\tau'\left(\frac{z_s}{k}\right)z_s + \frac{1}{2k^2}\tau''\left(\frac{z_s}{k}\right)z^2_s\right)1_{s<A_{t''}}, \\ E^k_s = \left(2\tau\left(\frac{z_s}{k}\right)z_s + \frac{1}{k}\tau'\left(\frac{z_s}{k}\right)z^2_s\right)1_{s<A_{t''}}. \end{cases}$$

We now make $k \to +\infty$ in (4.62) calculated with H^k, K^k, E^k. Clearly as $k \to +\infty$,

$$(4.65)\quad H^k_s \to z^2_{s\wedge A_{t''}}, \qquad K^k_s \to 1_{s<A_{t''}}, \qquad E^k_s \to 2 1_{s<A_{t''}} z.$$

and moreover H^k_s, K^k_s, E^k_s may be uniformly bounded by $C(1+z^2_t)$. Using Theorem 1.1 (*e*), the fact that $\sup_{0\leq s\leq T} z_s$ is in all the $L_p(\bar{\Omega}, P\otimes P')$, and since f has compact support, we can take the obvious limit of (4.62) calculated with H^k, K^k, E^k and obtain (4.60). □

Remark 4. — Since $\langle z, w^i\rangle = 0$, the integrals $\int_0^t z\,dw^i$ and $\int_0^t z\,\delta w^i$ are equal, so that the difficulty we had in writting U_s disappears in V_s.

(*e*) *Integration by parts in the variable* A.

We will now show that χ_N may be replaced by 1 in (4.60).

We have the easy result.

PROPOSITION 4.7. — *For any* $t>0$, *the random variable:*

$$(4.66)\quad \frac{1}{\int_0^{A_t} z\,ds},$$

is in all the $L_p(\bar{\Omega}, P\otimes P')$.

Proof. — Since $tz_{\cdot}/t^2$ has the same law as $z_{\cdot}$, $\int_0^{A_t} z\,ds$ has the same law as $t^3\int_0^{A_1} z\,ds$. Since under P', $\int_0^{A_t} z\,ds$ is a stationary independent increment process, it is clear that $\int_0^{A_t} z\,ds$ is a one-sided stable process with exponent 1/3. Now:

$$(4.67)\quad E^{P'}\left[\left(\int_0^{A_t} z\,ds\right)^{-p}\right] = \frac{1}{\Gamma(p)}\int_0^{+\infty} \beta^{p-1}\,E^{P'}\,e^{-\beta\int_0^{A_t} z\,ds}\,d\beta$$
$$= \frac{1}{\Gamma(p)}\int_0^{+\infty} \beta^{p-1}\,e^{-k\beta^{1/3}}\,d\beta < +\infty.$$

The proof is finished. □

We now claim:

THEOREM 4.8. — *Formula* (4.60) *still holds when the function* χ_N *is replaced by the constant function* 1.

Proof. — Since $\tau'(u)=0$ if $|u|\geqq 2$, we have:

$$\left|\frac{d\chi_N(u)}{du}\right| = N\,|\tau'(N u)| \leqq \frac{C}{u}. \tag{4.68}$$

We now make $N \to +\infty$ in (4.60). Since $\int_0^{A_{t''}} z\,ds>0$ a. s., it is clear that:

$$\left\{\begin{aligned} &\chi_N\left(\int_0^{A_{t''}} z\,ds\right) \to 1 \quad \text{a. s.,}\\ &\frac{d\chi_N}{du}\left(\int_0^{A_{t''}} z\,ds\right) \to 0 \quad \text{a. s.} \end{aligned}\right. \tag{4.69}$$

Using again Theorem 1.1 (*e*), the fact that f has compact support, Proposition 4.7 and the uniform bound (4.68), it is easy to take the limit of (4.60) by using Lebesgue's theorem. Theorem 4.8 is proved. □

(*f*) *Regularity of the boundary semi-group: the localizable case.*

We now give the result of integration by parts of any order on the variable A when the assumptions of section 2 (*c*) are verified.

THEOREM 4.9. — *Assume that* $(x_0, z_0)\in R^d\times R^+$ and $t'>0$ *are such that:*

(*a*) $P\otimes P'_{z_0}$, $C_{A_{t'}}$ *is invertible.*

(*b*) *For any* $T\geqq 0$, $p\geqq 1$, $1_{A_{t'}\leqq T}\,|\,C_{A_{t'}}^{-1}\,|$ *is in* $L_p(\bar\Omega, P\otimes P'_{z_0})$.

Then for any $n\in N$, *any* $t\geqq t'$, *there exists a random variable* $\bar D_t^n$ *such that:*

(*a*) *For any* $T>0$, *and any* $p\geqq 1$, $1_{A_t\leqq T}\,\bar D_t^n$ *is in* $L_p(\bar\Omega, P\otimes P'_{z_0})$.

(*b*) *For any* $f\in C_c^\infty(R\times R^d)$, *if* x_s *is the process* $\varphi_s(\bar\omega, x_0)$, *then:*

$$E^{P\otimes P'_{z_0}}\left[M_{A_t}\frac{\partial^n f}{\partial a^n}(A_t, x_{A_t})\right] = E^{P\otimes P'_{z_0}}\left[f(A_t, x_{A_t})\,\bar D_t^n\right]. \tag{4.70}$$

Proof. — We first consider the case where $z_0=0$. We prove (4.70) with $n=1$. By Theorem 4.8, we know that for $0<t''\leqq t$, (4.60) holds with $\chi_N=1$. (4.60) can be rewritten:

$$\begin{aligned} 4\,E^{P\otimes P'}[M_{A_t}\,d_a f(A_t, x_{A_t})] + E^{P\otimes P'}[M_{A_t} f(A_t, x_{A_t})\,J^1]&\\ + E^{P\otimes P'}\left[\frac{M_{A_t}}{\int_0^{A_{t''}} z\,ds}\langle d_x f(A_t, x_{A_t}), \varphi^*_{A_t} V_{A_{t''}}\rangle\right] &= 0. \end{aligned} \tag{4.71}$$

Now we claim that the calculus of variations of section 2 can be restarted on the third term of the *l.* h. s. of (4.71) so as to equal it to:

$$E^{P\otimes P'}[M_{A_t} f(A_t, x_{A_t}) J^2] \tag{4.72}$$

and so (4.70) will be proved for $n=1$. Indeed, recall that in section 2, only $(w^1 \ldots w^m)$ is made to vary, while z is unchanged. Since $\left[\int_0^{A_{t''}} z\, ds\right]^{-1}$ is in all the $L_p(\bar{\Omega}, P\otimes P')$, and since $\varphi^*_{A_t} V_{A_{t''}}$ is a term of the sort we already met in section 2, there is no major difficulty to use the technique of section 2—as long as $1_{A_{t'}\leq T} |C^{-1}_{A_{t'}}|$ is in all the $L_p(\bar{\Omega}, P\otimes P')$—and obtain (4.72).

We now prove (4.70) for $z_0=0$ and for a general n.

Take $g\in C^\infty_c(R\times R^d)$ and consider formula (4.60) with:

$$\chi_N=1, \qquad t''=\frac{t}{n}, \qquad f(a,x)=\frac{\partial^{n-1} g}{\partial a^{n-1}}(a,x),$$

(4.60) writes:

$$4\, E^{P\otimes P'}\left[M_{A_t}\frac{\partial^n g}{\partial a^n}(A_t, x_{A_t})\right]+E^{P\otimes P'}\left[M_{A_t}\frac{\partial^{n-1} g}{\partial a^{n-1}}(A_t, x_{A_t}) J^1\right]$$
$$+E^{P\otimes P'}\left[M_{A_t}\left\langle \frac{\partial^{n-1}}{\partial a^{n-1}}\frac{\partial g}{\partial x}(A_t, x_{A_t}), K^1\right\rangle\right]=0. \tag{4.73}$$

We now reapply the step by step integration by parts procedure described in the proof of Theorem 2.5. Namely we will do on the last two terms of the *l.* h. s. of (4.73) a variation of z on the time interval $A_{t/n}<s\leqq A_{2t/n}$.

Observe that:

$$\left\{\begin{aligned} J^1 &= J'^1_{A_{t/n}}+\left(\int_0^{A_{t/n}} z\, ds\right)^{-1}\left\langle\int_{A_{t/n}}^{A_t}\varphi^*_s\, b_x(x_s, z_s)(\delta B_s - b(x_s, z_s)\, ds), V_{A_{t/n}}\right\rangle \\ K^1 &= \left(\int_0^{A_{t/n}} z\, ds\right)^{-1}\varphi^*_{A_t} V_{A_{t/n}}. \end{aligned}\right. \tag{4.74}$$

where J'^1, V are $\{\bar{F}_t\}_{t\geqq 0}$-optional processes. $J'^1_{A_{t/n}}, V_{A_{t/n}}$ are then left invariant by this new variation of z. It is easily checked that the previous procedure of integration by parts can be repeatedly applied on the remaining time intervals $]A_{2t/n}, A_{3t/n}], \ldots,]A_{(n-1)t/n}, A_t]$, so that we make all the differentials of g in the variable A disappear. We finally arrive at:

$$4\, E^{P\otimes P'}\left[M_{A_t}\frac{\partial^n g}{\partial a^n}(A_t, x_{A_t})\right]+\sum_{0\leq|m|\leq n} E^{P\otimes P'}\left[M_{A_t}\frac{\partial^m g}{\partial x^m}(A_t, x_{A_t})\,\xi^m\right]=0 \tag{4.75}$$

and moreover the variables $1_{A_t\leq T}\,\xi^m$ are in all the $L_p(\bar{\Omega}, P\otimes P')$.

Now we claim that using the techniques of section 2, we have:

$$(4.76)\qquad E^{P\otimes P'}\left[M_{A_t}\frac{\partial^m g}{\partial x^m}(A_t, x_{A_t})\,\xi^m\right]=E^{P\otimes P'}[g(A_t, x_{A_t})\,\eta^m],$$

where η^m has the required integrability properties. Indeed as clearly shown by (4.60), the random variables ξ^m which are produced by the calculus of variations in z can be submitted to the calculus of variations in $(w^1, \ldots, w^m)$. Since $1_{A_{t'}\leqq T}\,|\,C_{A_{t'}}^{-1}\,|$ is in all the $L_p(\bar{\Omega}, P\otimes P')$, it is not difficult to obtain (4.76).

When $z_0\neq 0$, the proof is almost identical. In fact, formula (4.60) remains true if the integrals $\int_0^{\cdot}$ are replaced by $\int_{A_0}^{\cdot}$. The way to see this is to note that A_0 is the hitting time of 0 by z. (4.60) can then be written in conditional form (with x_0 replaced by x_{A_0}) and then interpreted so as to get the new formula. A direct procedure is also possible. □

Remark 5. — Under the previous assumptions, to integrate by parts in the variable z, we could as well use a technique similar to what is done in Theorem 2.4 instead of using a step by step procedure. In this case the step by step procedure is useful when the vector fields $X_0(x, z)\ldots X_m(x, z)$ are not C^∞ in the variable z [as in (0.6)].

We have the fundamental.

THEOREM 4.10. — *Under the assumptions of Theorem* 4.9, *for any* $t\geqq t'$, *the law under* $Q_{(x_0, z_0)}$ *of* (A_t, x_{A_t}) *is given by* $p_t(a, y)\,da\,dy$, *where* $p_t(a, y)\in C^\infty(R\times R^d)$.

Proof. — Let $h(a, y)\in C_c^\infty(R\times R^d)$. For $f\in C_b^\infty(R\times R^d)$ and for a given multi-index m, we have classicaly:

$$(4.77)\qquad \left(\frac{\partial^m f}{\partial x^m}\right)h=\sum_{|m'|\leqq m}(-1)^{|m'|}\binom{m}{m'}\frac{\partial^{m-m'}}{\partial x^{m-m'}}\left[f\frac{\partial^{m'} h}{\partial x^{m'}}\right].$$

Using (2.27) in Theorem 2.4 and (4.77) we find that since $f\partial^{m'}h/\partial x^{m'}\in C_c^\infty(R\times R^d)$:

$$(4.78)\qquad E^{P\otimes P'_{z_0}}\left[M_{A_t}\left(\frac{\partial^m f}{\partial x^m}h\right)(A_t, x_{A_t})\right]\leqq C_h^m\sup_{(a,x)\in R^+\times R^d}|f(a, x)|.$$

Applying (4.78) to the function:

$$(4.79)\qquad f(a, x)=e^{i(\alpha a+\langle\beta, x\rangle)},$$

we find that if $p_t(da, dy)$ is the law of (A_t, x_{A_t}) under the measure $M_{A_t}\,d(P\otimes P'_{z_0})$, the Fourier transform $\psi_t^h(\alpha, \beta)$ of $h(a, y)\,p_t(da, dy)$ is such that:

$$(4.80)\qquad |\beta^m|\,|\psi_t^h(\alpha, \beta)|\leqq C_h^m.$$

Similarly, by using (4.70) we find that for any $n\in N$:

$$(4.81)\qquad |\alpha^n|\,|\psi_t^h(\alpha, \beta)\leqq C_h'^n.$$

Using (4.80), (4.81), we find that for any $n \in \mathrm{N}$:

$$[|\alpha|^2+|\beta|^2]^{n/2}\,|\psi_t^h(\alpha,\beta)| \leqq C_h''^{n}. \tag{4.82}$$

Using (4.82), we see that $h(a,y)p_t(da,dy)$ is in $C_b^\infty(R\times R^d)$. Since this is true for any $h\in C_c^\infty(R\times R^d)$, the result follows. □

(*g*) *Regularity of the boundary semi-group: the non localizable case.*

We now have the result which corresponds to Theorem 2.5.

THEOREM 4.11. — *If the assumptions in Theorem 2.5 are verified, for every* $(x_0,z_0)\in R^d\times R^+$, *any* $n\in N$, *and any* $t\geqq nt'$, *on* $(\bar\Omega, P\otimes P'_{z_0})$, *there exists a random variable* $\bar D_t^n$ *such that:*

(*a*) *For any* $T\geqq 0$, $1_{A_t\leqq T}\,\bar D_t^n$ *is* $P\otimes P'_{z_0}$ *integrable.*

(*b*) *For any* $f\in C_c^\infty(R\times R^d)$, *if* x_s *is the process* $\varphi_s(\bar\omega, x_0)$*:*

$$E^{P\otimes P'_{z_0}}\left[M_{A_t}\frac{\partial^n f}{\partial a^n}(A_t, x_{A_t})\right]=E^{P\otimes P'_{z_0}}[f(A_t,x_{A_t})\,\bar D_t^n]. \tag{4.83}$$

Proof. — The proof is identical to the proof of Theorem 4.9 except that the step by step integration procedure of Theorem 2.5 will be used to get rid of the differentials $\partial^m g/\partial x^m$ in (4.76). This is no problem since $1_{A_t\leqq T}\,\xi^m$ is in all the $L_p(\bar\Omega, P\otimes P')$ $(1\leqq p<+\infty)$. □

We now claim:

THEOREM 4.12. — *Under the assumptions of Theorem 4.11, for any* $k\in N$, *and any* $t\geqq(k+d+2)t'$, *the law under* $Q_{(x_0,z_0)}$ *of* (A_t, x_{A_t}) *is given by* $p_t(a,y)\,da\,dy$, *where* $p_t(a,y)\in C^k(R\times R^d)$. □

Proof. — Proceeding as in the proof of Theorem 4.10 and using the same notations, we have for $t\geqq(k+d+2)\,t'$:

$$[|\alpha|^2+|\beta|^2]^{(k+d+2)/2}\,|\psi_t^h(\alpha,\beta)| \leqq C_h''^{k}. \tag{4.84}$$

Since $(a,x)\in R^{d+1}$, a trivial exercise in Fourier transform shows that $h(a,y)p_t(da,dy)\in C_c^k(R\times R^d)$. □

Remark 6. — As pointed out in section 1(*d*), $p_t(a,y)\,da\,dy$ is not necessarily a probability measure, i. e. its integral is in general $\leqq 1$.

(*h*) *Existence of densities for the boundary semi-group.*

We now give the final result in this section, in the manner of Malliavin [29].

THEOREM 4.13. — *If* $(x_0,z_0)\in R^d\times R^+$, *and* $t'>0$ *are such that* $P\otimes P'_{z_0}$ *a. s.*, $C_{A_{t'}}^{x_0}$ *is invertible, then for any* $t\geqq t'$, *the law under* $Q_{(x_0,z_0)}$ *of* (A_t, x_{A_t}) *is given by* $p_t(a,y)\,da\,dy$.

Proof. — Using (2.25), we find that if h is taken as in the proof of Theorem 4.10, then for any $f \in C_\infty^b(R \times R^d)$:

$$(4.85)\qquad \left| E^{P \otimes P'_{z_0}}\left[\rho_N(C_{A_{t'}}^{x_0}) M_{A_t}\left(h \frac{\partial}{\partial x^l} f \right)(A_t, x_{A_t}) \right] \right| \leqq C^h \sup |f(a, x)|, \qquad 1 \leqq l \leqq d.$$

Similarly, by introducing the mollifier $\rho_N(C_{A_{t'}}^{x_0})$ in formula (4.60) with $\chi_N = 1$, its is not hard to get:

$$(4.86)\qquad \left| E^{P \otimes P'_{z_0}}\left[\rho_N(C_{A_{t'}}^{x_0}) M_{A_t}\left(h \frac{\partial f}{\partial a} \right)(A_t, x_{A_t}) \right] \right| \leqq C'^h \sup |f(a, x)|$$

[of course a new application of a formula similar to (2.25) is needed to get rid of $\partial f/\partial x$ in (4.60)].

Now using a result of harmonic analysis [29], [37], (4.85), (4.86) show that the law of (A_t, x_{A_t}) under $\rho_N(C_{A_{t'}}^{x_0}) M_{A_t} h(A_t, x_{A_t}) dP \otimes P'$ has a density with respect to the Lebesgue measure. Since $C_{A_{t'}}^{x_0}$ is a. s. invertible, as $N \to +\infty$, $\rho_N(C_{A_{t'}}^{x_0}) \to 1$ a. s. The Theorem follows. □

5. The analysis of boundary semi-groups

In this section, we describe conditions under which the boundary semi-group is given by absolutely continuous measures, and we study the smoothness of the corresponding densities.

As pointed out in Theorem 4.13, a. s. invertibility of $C_{A_t}^{x_0}$ implies the existence of densities. In (*a*), we give conditions under which $C_{A_t}^{x_0}$ is a. s. invertible. Surprisingly enough, we show that such a property may hold even if the support of the Lévy measure of the boundary process does not span R^d, essentially because of the possible interaction between D and $\mathscr{L}$. To prove the a. s. invertibility of $C_{A_t}^{x_0}$ under non standard conditions, we use a result which we proved in [51] on the zeros of certain semi-martingales. In (*b*) estimates of Malliavin [30], Ikeda-Watanabe [17], Kusuoka-Stroock ([26]-[38]) on standard hypoelliptic diffusions are recalled. In (*c*) conditions are given under which for every $t > 0$, $T \geqq 0$, $1_{A_t \leqq T} |[C_{A_t}^{x_0}]^{-1}|$ is in all the $L_p(\bar{\Omega}, P \otimes P')$, so that by Theorem 4.9, the boundary semi-group is smooth. This is true even for diffusions such that $\mathscr{L}$ is very degenerate at the boundary, i. e. does not verify Hörmander's conditions [15] at the boundary. In (*d*) the localization of such conditions is studied, so that they only need to be verified at the terminal — i. e. not the initial — value of the boundary process. In (*f*) the limit case of slowly regularizing boundary semi-groups is studied, which corresponds to a critical degeneracy of $\mathscr{L}$ on the boundary.

(*a*) *Invertibility of* $C_t^{x_0}$.

We first give conditions under which for $t' > 0$, $C_{t'}^{x_0}$ is $P \otimes P'$ a. s. invertible. As we have seen in Theorem 4.13, this will imply that the boundary semi-group has a density relative to the Lebesgue measure.

We will concentrate on conditions which exhibit in the most striking way the possible interaction between the operators $\mathscr{L}$ and $\mathscr{D}$ defined in (1.39)-(1.40).

Recall that if $A(x, z)$ is any vector field with values in R^d, then $[\partial/\partial z, A]$ is also a vector field with values in R^d.

If $E=(e_i)_{i\in I}$ is a family of vector fields on R^{d+1}, if $y\in R^{d+1}$, $E(y)$ is the family of vectors of R^{d+1} given by $E(y)=(e_i(y))_{i\in I}$.

If $E=(e_i)_{i\in I}$, $F=(f_j)_{j\in J}$ are two family of vector fields, $[E, F]$ denotes the family of vector fields:

$$[E, F]=([e_i, f_j])_{(i,j)\in I\times J}.$$

DEFINITION 5.1. — For $l\in N$, E_l, F_l are the families of vector fields in R^d defined by:

$$(5.1)\qquad \left\{\begin{array}{l} E_1=(X_1, X_2, \ldots, X_m); \qquad F_1=\{0\}, \\ E_{l+1}=\left[\left(X_0, X_1, \ldots, X_m, \dfrac{\partial}{\partial z}\right), E_l\right]; \\ F_{l+1}=[D, E_l]\cup[(D, X_1, \ldots, X_m), F_l]. \end{array}\right.$$

We have then the following result, which was first proved in [50] using a different technique.

THEOREM 5.2. — *If x_0 is such that $\bigcup_1^{+\infty}(E_l\cup F_l)(x_0, 0)$ spans R^d, then $P\otimes P'$ a. s., for any $t>0$, $C_t^{x_0}$ is invertible.*

Proof. — Let U_s be the vector space in $T_{x_0}(R^d)$ spanned by $(\varphi_s^{*-1}X_i)(x_0)\,(1\leqq i\leqq m)$ and V_t the vector space spanned by $\bigcup_{s\leqq t}(U_s)$. We define V_t^+ by:

$$V_t^+=\bigcap_{s>t}V_s.$$

By the zero-one law, we know that $P\otimes P'$ a. s., V_0^+ is a fixed space, not depending on $\bar{\omega}$. Let us assume that $V_0^+\neq T_{x_0}(R^d)$. If S is the $\{\bar{F}_t\}_{t\geqq 0}$ stopping time:

$$(5.2)\qquad S=\inf\{t>0;\ V_t\neq V_0^+\},$$

then a. s. S is >0. Let f be a non-zero element in $T_{x_0}^*(R^d)$ orthogonal to V_0^+. Then:

$$(5.3)\qquad \langle f, (\varphi_s^{*-1}X_i)(x_0)\rangle=0 \qquad \text{for}\quad s\leqq S.$$

Now using equation (1.12) (*see* [5]), we know that:

$$(5.4)\quad \varphi_t^{*-1}X_i=X_i+\int_0^t\varphi_s^{*-1}[X_0, X_i]\,ds+\int_0^t\varphi_s^{*-1}\left[D+\frac{\partial}{\partial z}, X_i\right].dL$$
$$+\int_0^t\varphi_s^{*-1}[X_j, X_i].dw^j+\int_0^t\varphi_s^{*-1}\left[\frac{\partial}{\partial z}, X_i\right].dB,$$

or equivalently:

$$(5.5)\quad \varphi_t^{*-1} X_i = X_i + \int_0^t \varphi_s^{*-1}\left([X_0, X_i] + \frac{1}{2}[X_j, [X_j, X_i]] + \frac{1}{2}\left[\frac{\partial}{\partial z}, \left[\frac{\partial}{\partial z}, X_i\right]\right]\right) ds + \int_0^t \varphi_s^{*-1}\left[D + \frac{\partial}{\partial z}, X_i\right] . dL + \int_0^t \varphi_s^{*-1} [X_j, X_i] . \delta w^j + \int_0^t \varphi_s^{*-1}\left[\frac{\partial}{\partial z}, X_i\right] . \delta B.$$

Now (5.5) gives the Itô-Meyer decomposition of the $\{\bar{F}_t\}_{t\geq 0}$ semi-martingale $\langle f, \varphi_s^{*-1} X_i\rangle$, which is 0 for $s \leqq S$. By canceling the martingale terms, we find that for $s \leqq S$:

$$(5.6)\quad \begin{cases} \displaystyle\int_0^s \langle f, \varphi_u^{*-1}[X_j, X_i]\rangle \, \delta w^j = 0; \qquad 1 \leqq i, j \leqq m, \\ \displaystyle\int_0^s \left\langle f, \varphi_u^{*-1}\left[\frac{\partial}{\partial z}, X_i\right]\right\rangle \delta B = 0. \end{cases}$$

An elementary reasoning on the quadratic variation of the local martingales (5.6) and the continuity of the processes $\varphi_s^{*-1}[X_j, X_i]$, $\varphi_s^{*-1}[\partial/\partial z, X_i]$ (*see* [7], Theorem 5.2) show that $P \otimes P'$ a. s., for $s \leqq S$:

$$(5.7)\quad \begin{cases} \langle f, \varphi_s^{*-1}[X_j, X_i]\rangle = 0; \qquad 1 \leqq i, j \leqq m, \\ \left\langle f, \varphi_s^{*-1}\left[\dfrac{\partial}{\partial z}, X_i\right]\right\rangle = 0. \end{cases}$$

Reapplying (5.5) on (5.7), we find that for $s \leqq S$:

$$(5.8)\quad \begin{cases} \langle f, \varphi_s^{*-1}[X_j, [X_j, X_i]]\rangle = 0, \\ \left\langle f, \varphi_s^{*-1}\left[\dfrac{\partial}{\partial z}, \left[\dfrac{\partial}{\partial z}, X_i\right]\right]\right\rangle = 0. \end{cases}$$

We now cancel the bounded variation process in the Meyer decomposition of $\langle f, \varphi_s^{*-1} X_i\rangle$ $(s \leqq S)$, i. e. using (5.7)-(5.8), we get for $s \leqq S$:

$$(5.9)\quad \left\langle f, \int_0^s \varphi_u^{*-1}[X_0, X_i]\, du \right\rangle + \left\langle f, \int_0^s \varphi_u^{*-1}[D, X_i]\, dL \right\rangle = 0.$$

Since $P \otimes P'$ a. s., the support of the measure dL is exactly the closed set $(z_s = 0)$ which is negligible for the Lebesgue measure dt ([18], p. 44), from (5.9) we deduce that for $s \leqq S$:

$$(5.10)\quad \begin{cases} \left\langle f, \displaystyle\int_0^s \varphi_u^{*-1}[X_0, X_i]\, du \right\rangle = 0, \\ \left\langle f, \displaystyle\int_0^s \varphi_u^{*-1}[D, X_i]\, dL \right\rangle = 0. \end{cases}$$

and so using the continuity of $\varphi_u^{*-1}[X_0, X_i]$, $\varphi_u^{*-1}[D, X_i]$ and the support property of dL, we get from (5.10):

$$(5.11) \qquad \begin{cases} \langle f, \varphi_s^{*-1}[X_0, X_i]\rangle = 0 & \text{for } s \leqq S, \\ \langle f, \varphi_s^{*-1}[D, X_i]\rangle = 0 & \text{on } (z_s=0) \cap [0, S[. \end{cases}$$

By iteration of the previous procedure on (5.7) and on the first line in (5.11) we find that for any $l \in N$, if $Y_1, \dots Y_l$ are taken among $(X_0, X_1, \dots, X_m, \partial/\partial z)$, then $P \otimes P'$ a. s., for $1 \leqq i \leqq m$:

$$(5.12) \qquad \begin{cases} \langle f, \varphi_s^{*-1}[Y_l, [Y_{l-1}, \dots, [Y_1, X_i]] \dots]\rangle = 0, & s \leqq S, \\ \langle f, \varphi_s^{*-1}[D, [Y_{l-1}, \dots, [Y_1, X_i]] \dots]\rangle = 0 & \text{on } (z_s=0) \cap [0, S[, \end{cases}$$

so that in particular at $s=0$, we get:

$$(5.13) \qquad \begin{cases} \langle f, [Y_l, [Y_{l-1}, \dots, [Y_1, X_i]] \dots](x_0, 0)\rangle = 0, & 1 \leqq i \leqq m, \\ \langle f, [D, [Y_{l-1}[, \dots, [Y, X_i]] \dots](x_0, 0)\rangle = 0, & 1 \leqq i \leqq m \end{cases}$$

and so f is orthogonal to $\left(\left(\bigcup_1^{+\infty} E_l\right) \cup \left[D, \bigcup_1^{+\infty} E_l\right]\right)(x_0, 0)$.

We will now exploit the second line of (5.11). Let $H(x)$ be a C^∞ vector field defined on R^d with values in R^d such that:

$$(5.14) \qquad \langle f, \varphi_s^{*-1} H\rangle = 0 \quad \text{on } (z_s=0) \cap [0, S[.$$

This is the case for $H = X_i(x, 0)$ or $H = [D, X_i](x, 0)$.

We claim that:

$$(5.15) \qquad \begin{cases} \langle f, \varphi_s^{*-1}[D, H]\rangle = 0 & \text{on } (z_s=0) \cap [0, S[, \\ \langle f, \varphi_s^{*-1}[X_j, H]\rangle = 0 & \text{on } (z_s=0) \cap [0, S[, \quad 1 \leqq j \leqq m. \end{cases}$$

Note that in (5.15), we may as well assume that $[D, H] = [D, H](x)$, $[X_j, H] = [X_j, H](x, 0)$, so that the previous procedure can be iterated.

We have:

$$(5.16) \quad (\varphi_t^{*-1} H) = H(x_0) + \int_0^t \varphi_s^{*-1}\left([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\right) ds + \int_0^t \varphi_s^{*-1}[D, H]\, dL + \int_0^t \varphi_s^{*-1}[X_j, H] . \delta w^j$$

(it is crucial at this stage H does *not* depend on z so that no stochastic integral $\int_0^t \dots \delta B$ appears).

Set:

$$(5.17)\quad \left\{\begin{aligned} &J_s=\langle f, \varphi_s^{*-1}[D, H]\rangle,\\ &K_s=\left\langle f, \varphi_s^{*-1}\left([X_0, H]+\frac{1}{2}[X_j, [X_j, H]]\right)(x_0, 0)\right\rangle,\\ &M_s=\sum_{j=1}^{m}\langle f, \varphi_s^{*-1}([X_j, H])(x_0, 0)\rangle^2. \end{aligned}\right.$$

By eventually enlarging the filtration $\{\bar{F}_t\}_{t\geqq 0}$, we know that there exists a Brownian martingale β_t orthogonal to the Brownian martingale B_t such that:

$$(5.18)\quad \langle f, (\varphi_t^{*-1} H)(x_0)\rangle=\langle f, H(x_0)\rangle+\int_0^t K_s\,ds+\int_0^t J_s\,dL+\int_0^t \sqrt{M_s}\,\delta\beta_s.$$

From (5.14), (5.18) and Theorem 2.1 in Bismut [51], we find that:

$$(5.19)\quad \left\{\begin{aligned} &J_s=0 \quad \text{on } (z_s=0)\cap [0, S[,\\ &M_s=0 \quad \text{on } (z_s=0)\cap [0, S[. \end{aligned}\right.$$

(5.19) is equivalent to (5.15).

By iterating the same procedure on (5.15), and by taking $s=0$ on all the analogues of (5.15), we find that f is orthogonal to $\bigcup_1^{+\infty}(E_l\cup F_l)(x_0, 0)$. The assumption which is done in the Theorem shows that $f=0$. This is a contradiction to $S>0$. □

Remark 1. — It should be pointed out that if $X_1\ldots X_m$ do *not* depend on z, in Definition 5.1, F_{l+1} can be enlarged to be:

$$(5.20)\quad F_{l+1}=[D, E_l]\cup[(D, X_0, X_1, \ldots, X_m), F_l].$$

The proof is as follows. We will show that if $H(x)$ is such that (5.14) holds, then:

$$(5.21)\quad \langle f, \varphi_s^{*-1}[X_0, H]\rangle=0 \quad \text{on } (z_s=0)\cap[0, S[,$$

For $1\leqq j\leqq m$, set $Y_j(x)=[X_j, H](x)$. Now $\varphi_t^{*-1}Y_j$ is a semi-martingale whose Itô decomposition is of the same type as in (5.16); in particular no stochastic integral with respect to B appears in the decomposition. From (5.16) and from Theorem 2.3 in [51], we immediately find that (5.21) holds. This procedure can be also iterated, so that we may take F_{l+1} as in (5.20).

Note that it is here crucial that $X_1, \ldots, X_m$ do not depend on z in order to use the result of [51].

(*b*) *The basic estimates.*

We first recall the basic estimates of Malliavin [30], Ikeda-Watanabe [17] and Kusuoka-Stroock ([26]-[38]) in the form given by Stroock in [38]. Recall that Kusuoka-Stroock ([26]-[38]) obtained the most general result on hypoelliptic semi-groups for standard—i. e. non reflecting diffusions.

$X'_0(x'), X'_1(x') \ldots X'_{m'}(x')$ are $m'+1$ vector fields defined on R^n with values in R^n, whose components are elements of $C_b^\infty(R^n)$.

On (Ω, P), we consider the stochastic differential equation:

$$\left\{ \begin{array}{c} dx' = X'_0(x')\,dt + X'_i(x')\,dw^i, \\ x(0) = x_0 \end{array} \right. \tag{5.22}$$

and its associated flow of diffeomorphisms of R^n $\varphi'_t(\omega, .)$. [*See* Bismut ([6], [7]), Kunita [25]].

DEFINITION 5.3. — $C'^{x_0}_t$ is the process of linear mappings:

$$f \in T^*_{x_0}(R^n) \to C'^{x_0}_t f = \sum_{i=1}^{m} \int_0^t \langle f, \varphi'^{*-1}_s X'_i(x_0) \rangle \varphi'^{*-1}_s X'_i(x_0)\,ds \in T_{x_0}(R^n). \tag{5.23}$$

DEFINITION 5.4. — For $l \in N$, E'_l is the family of vector fields defined by:

$$\left\{ \begin{array}{l} E'_1 = (X'_1, X'_2 \ldots X'_{m'}), \\ E'_{l+1} = [(X'_0, X'_1, \ldots, X'_{m'}), E'_l]. \end{array} \right. \tag{5.24}$$

DEFINITION 5.5. — If $x_0 \in R^n$, $f \in T^*_{x_0}(R^n)$, $l \in N$, f_t^{l, x_0} denotes the continuous process:

$$f_t^{l, x_0} = \sum_{n=1}^{l} \sum_{Y \in E'_n} \langle (\varphi'^{*-1}_t Y)(x_0), f \rangle^2, \tag{5.25}$$

σ is the stopping time:

$$\sigma = \inf \left\{ t \geqq 0;\ \left| \left[\frac{\partial \varphi'_s}{\partial x}(\bar{\omega}, x_0) \right]^{-1} - I \right| \geqq \frac{1}{2} \right\}. \tag{5.26}$$

α is the constant $21/2^{11}$.

We have:

THEOREM 5.6. — *For any $l \in N$, there are constants δ, k, K in $]0, +\infty[$ depending only on l and on:*

$$\sup_{1 \leqq i \leqq l+3} \sup_{\substack{Y \in E_i \\ x \in R^n}} |Y(x)|,$$

such that for any $x_0 \in R^n$, $\eta > 0$, and $N \in R^+$ such that:

$$N^3 \geqq 2(m+1)^{l-1}\,\delta/\eta, \tag{5.27}$$

then if $m_l = 20^{l-1} \times 6$, for any $f \in R^d$ such that $\|f\| = 1$:

$$P\left[\langle C'^{x_0}_{1/N^3} f, f \rangle \leqq \frac{2\delta}{N^{m_l}},\ f_t^{l, x_0} \geqq \eta \text{ on } \left[0, \frac{1}{N^3}\right],\ \sigma \geqq \frac{1}{N^3} \right] \leqq K \exp(-k N^\alpha). \tag{5.28}$$

Proof. — This result is contained in the proof of Theorem 8.31 of the result of Kusuoka-Stroock in Stroock [38]. To make the comparison possible, note that our l would be $l-1$ in [38], and in [38] (8.37) and (8.42), m_{l_0} should read m_0. Observe that the condition (8.39) in [38] has been absorbed in the constant K. □

We will use (5.28) in a different form. Set:

$$D_1 = \frac{1}{(2\,\delta)^{3/m_l}}, \qquad D_2 = \frac{2\,(m+1)^{l-1}\,\delta}{(2\,\delta)^{3/m_l}}, \qquad D_3 = k\,(2\,\delta)^{\alpha/m_l}.$$

From (5.28), we get that for any $\varepsilon > 0$:

$$(5.29)\quad P[\langle C'^{x_0}_{D_1 \varepsilon^{3/m_l}} f, f\rangle \leqq \varepsilon;\ f_t^{l, x_0} \geqq D_2\,\varepsilon^{3/m_l} \text{ on } [0, D_1\,\varepsilon^{3/m_l}];\ \sigma \geqq D_1\,\varepsilon^{3/m_l}] \leqq K \exp - [D_3\,\varepsilon^{-\alpha/m_l}].$$

(*c*) *Regularity of the boundary semi-group: the localizable case.*

We now prove the regularity of the boundary semi-group under some assumptions on the vector fields $X_0, \ldots, X_m$.

Recall that the families of vector fields E_l have been defined in Definition 5.1.

DEFINITION 5.7. — For $l \in \mathbb{N}$, the function $k^l(x, z)$ is defined by:

$$(5.31)\qquad k^l(x, z) = \inf_{f \in \mathbb{R}^d, \|f\| = 1} \left\{ \sum_{j=1}^{l} \sum_{Y \in E_j} \langle Y(x, z), f\rangle^2 \right\}.$$

We have the key result:

THEOREM 5.8. — *If $x_0 \in \mathbb{R}^d$ is such that for a given $l \in \mathbb{N}$, $\theta > 0$:*

$$(5.32)\qquad \lim_{z > 0\, z \to 0} z \log [\inf_{|x - x_0| \leqq \theta} k^l(x, z)] = 0,$$

then for any $t > 0$, $T \geqq 0$, $1_{A_t \leqq T} \,|\, [C^{x_0}_{A_t}]^{-1} \,|$ is in all the $L_p(\bar{\Omega}, P \otimes P')$.

Proof. — For simplicity, we write $\bar{P}$ instead of $P \otimes P'$, C_s instead of $C_s^{x_0}$. $\bar{\omega}$ will also be omitted. For $s \leqq s'$, $C^s_{s'}$ is the mapping:

$$(5.33)\quad f \in \mathbb{R}^d \to C^s_{s'} f = \sum_{i=1}^{m} \int_s^{s'} \langle [(\varphi_t \circ \varphi_s^{-1})^{*-1} X_i] (\varphi_s(x_0)), f\rangle (\varphi_t \circ \varphi_s^{-1})^{*-1} X_i (\varphi_s(x_0))\, dt.$$

We can of course take t as small as we want, since if the result is proved for t, it is proved for any $t' \geqq t$.

λ is a >0 real, which will tend to $+\infty$, γ is a >0 real number, depending on λ, which will become arbitrary small as $\lambda \to +\infty$. We will determine γ at the end of the proof. We have:

$$(5.34)\quad \bar{P}[|\, C_{A_t}^{-1} \,| \geqq \lambda;\ A_t \leqq T] \leqq \bar{P}[A_t \leqq 2\,t\,\gamma\,\sqrt{2}] + \bar{P}[|\, C_{A_t}^{-1} \,| \geqq \lambda;\ 2\,t\,\gamma\,\sqrt{2} \leqq A_t \leqq T].$$

Now L. has the same law as $\sup_{0 \leqq s \leqq .} w_s^1$ — [18], p. 41 — so that:

$$(5.35)\qquad \bar{P}[A_t \leqq 2t\gamma\sqrt{2}] = \bar{P}[\sup_{0 \leqq s \leqq 2t\gamma\sqrt{2}} w_s^1 \geqq t] \leqq 2\exp\left(-\frac{t}{4\gamma\sqrt{2}}\right).$$

Let T_0 to be the stopping time:

$$(5.36)\qquad T_0 = \inf\{t \geqq 0;\ |x_t - x_0| \geqq \theta\}.$$

Using equation (1.4), since on $(A_t \geqq 2t\gamma\sqrt{2})$, on $[0, 2t\gamma\sqrt{2}]$, $L_s \leqq t$, we have classically [40]:

$$(5.37)\qquad \bar{P}[T_0 \leqq 2t\gamma\sqrt{2} \leqq A_t] \leqq 2d\exp - \frac{(\theta - E(t + 2t\gamma\sqrt{2}))^{+2}}{4Ft\gamma\sqrt{2}},$$

where:

$$(5.38)\qquad \left\{\begin{aligned} E &= \sup_{(x,z)}\left|X_0(x,z) + \frac{1}{2}\frac{\partial X_j}{\partial x}X_j(x,z)\right| \vee \sup_x |D(x)|, \\ F &= m \sup_{\substack{1 \leqq i \leqq m \\ (x,z) \in R^{d+1}}} |X_i(x,z)|^2, \end{aligned}\right.$$

t will be chosen such that:

$$(5.39)\qquad t \leqq \frac{\theta}{8E}$$

and of course γ is "small" in the r. h. s. of (5.37).

We have then:

$$(5.40)\qquad \bar{P}[|C_{A_t}^{-1}| \geqq \lambda,\ 2t\gamma\sqrt{2} \leqq A_t \leqq T] \leqq 2d\exp\left[-\frac{\theta^2}{C_\theta t\gamma}\right] + \bar{P}[|C_{2t\gamma\sqrt{2}}^{-1}| \geqq \lambda;\ A_t \wedge T_0 \geqq 2t\gamma\sqrt{2}].$$

Let T_1^γ be the stopping time:

$$(5.41)\qquad T_1^\gamma = \inf\{t \geqq 0;\ z_t = \gamma\}.$$

We now have the *key estimate:*

$$(5.42)\qquad \bar{P}[T_1^\gamma \geqq t\gamma\sqrt{2}] \leqq \sqrt{2}\exp - \frac{t\gamma\sqrt{2}}{8\gamma^2} = \sqrt{2}\exp - \frac{t\sqrt{2}}{8\gamma}$$

(this can be proved using the well-known equality ([18], p. 205):

$$(5.43)\qquad E^{\bar{P}}\left[\exp - \frac{\alpha^2}{2}\int_0^T (w_s^1)^2\, ds\right] = \frac{1}{[\operatorname{ch} \alpha T]^{1/2}}.$$

and Čebyšev's inequality). Moreover if T_2^γ is the stopping time:

$$(5.44)\qquad T_2^\gamma=\inf\left\{t\geqq T_1^\gamma,\ |z_t-z_{T_1^\gamma}|=\frac{\gamma}{2}\right\},$$

we have clearly:

$$(5.45)\qquad \bar{P}\,[T_2^\gamma-T_1^\gamma\leqq D_1\,[2/\sqrt{\lambda}]^{3/m_l}]\leqq 2\exp-\left[\frac{\gamma^2}{8\,D_1\,2^{3/m_l}}\,\lambda^{3/2m_l}\right].$$

We will choose γ so that:

$$(5.46)\qquad t\,\gamma\,\sqrt{2}\geqq D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}.$$

Now:

$$(5.47)\qquad \bar{P}\,[|C_{2t\gamma\sqrt{2}}^{-1}|\geqq\lambda;\ A_t\wedge T_0\geqq 2\,t\,\gamma\,\sqrt{2}]\leqq\sqrt{2}\,\exp\left(-\frac{t\sqrt{2}}{8\,\gamma}\right)$$
$$+2\exp\left(\frac{-\gamma^2\,\lambda^{3/2m_l}}{8\,D_1\,2^{3/m_l}}\right)+\bar{P}\Bigg[\,|C_{2t\gamma\sqrt{2}}^{-1}|\geqq\lambda;\ A_t\wedge T_0\geqq 2\,t\,\gamma\,\sqrt{2},$$
$$T_1^\gamma\leqq t\,\gamma\,\sqrt{2},\ T_2^\gamma-T_1^\gamma\geqq D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\Bigg].$$

Now if $T_1^\gamma\leqq t\,\gamma\,\sqrt{2}$, using (5.46), we have:

$$(5.48)\qquad C_{2t\gamma\sqrt{2}}\geqq\left[\frac{\partial\varphi_{T_1^\gamma}}{\partial x}\right]^{-1}C^{T_1^\gamma}_{T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}}\widetilde{\left[\frac{\partial\varphi_{T_1^\gamma}}{\partial x}\right]^{-1}}$$

[where the inequality (5.48) is taken in the sense of nonnegative quadratic forms].

So using (5.48), we obtain:

$$(5.49)\qquad \bar{P}\Bigg[\,|C_{2t\gamma\sqrt{2}}^{-1}|\geqq\lambda;\ A_t\wedge T_0\geqq 2\,t\,\gamma\,\sqrt{2},\ T_1^\gamma\leqq t\,\gamma\,\sqrt{2},$$
$$T_2^\gamma-T_1^\gamma\geqq D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\Bigg]\leqq\bar{P}\left[\,\left|\frac{\partial\varphi_{T_1^\gamma}}{\partial x}(\bar{\omega},\,x_0)\right|\geqq\lambda^{1/4},\ T_1^\gamma\leqq t\,\gamma\,\sqrt{2}\leqq A_t\right]$$
$$+\bar{P}\left[\,|[C^{T_1^\gamma}_{T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}}]^{-1}|\geqq\lambda^{1/2};\ T_0\wedge T_2^\gamma\geqq T_1^\gamma+D_1^\gamma\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right].$$

Using Theorem 1.1 (*e*), we know that for any $p\geqq 1$:

$$(5.50)\qquad \bar{P}\left[\,\left|\frac{\partial\varphi_{T_1^\gamma}}{\partial x}(\bar{\omega},\,x_0)\right|\geqq\lambda^{1/4},\ T_1^\gamma\leqq t\,\gamma\,\sqrt{2}\leqq A_t\right]\leqq\frac{A}{\lambda^p}.$$

Let T_3^γ be the stopping time:

$$(5.51)\quad T_3^\gamma=\inf\left\{t\geqq T_1^\gamma;\ \left|\left[\frac{\partial\varphi_{t-T_1^\gamma}}{\partial x}(\theta_{T_1^\gamma}\bar\omega,\ \varphi_{T_1^\gamma}(\bar\omega,x_0))\right]^{-1}-I\right|\geqq\frac{1}{2}\right\}$$
$$=\inf\left\{t\geqq T_1^\gamma,\ \left|\left[\frac{\partial\varphi_t}{\partial x}(\bar\omega,x_0)\frac{\partial\varphi_{T_1^\gamma}^{-1}}{\partial x}(\bar\omega,x_0)\right]^{-1}-I\right|\geqq\frac{1}{2}\right\}.$$

When $T_2^\gamma\geqq T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}$, L does not increase on $[T_1^\gamma, T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}]$ and so using the Markov property of the flow $\varphi.(\bar\omega, .)$ we get:

$$(5.52)\quad \bar P\left[T_3^\gamma\leqq T_1^\gamma+D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l},\ T_2^\gamma\geqq T_1^\gamma+D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right]\leqq C\exp(-C'\lambda^{3/2m_l}).$$

Moreover by lemma V.8.4 in [17], we know that given $\varepsilon>0$, $R>0$, there are $N(R,\varepsilon)$ elements of the unit sphere of R^d $f_1\ldots f_{N(R,\varepsilon)}$ such that if A is a symmetric nonnegative (d,d) matrix such that $|A|\leqq R$, if for every f_i, $\langle A f_i, f_i\rangle\geqq\varepsilon$, then A^{-1} exists and $|A^{-1}|\leqq 2/\varepsilon$. Moreover $N(R,\varepsilon)\leqq C(R/\varepsilon)^{d-1}$.

We find then that:

$$(5.53)\quad \bar P\left[\,|[C_{T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}}^{T_1^\gamma}]^{-1}|\geqq\lambda^{1/2},\right.$$
$$\left.T_0\wedge T_2^\gamma\geqq T_1^\gamma+D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right]\leqq C\exp(-C'\lambda^{3/2m_l})$$
$$+\sum_{i=1}^{N}\bar P\left[\langle C_{T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}}^{T_1^\gamma}f_i,f_i\rangle\leqq\frac{2}{\sqrt{\lambda}},\ T_0\wedge T_2^\gamma\wedge T_3^\gamma\geqq T_1^\gamma+D_1\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}\right]$$

and:

$$(5.54)\quad N\leqq C\lambda^{(d-1)/2}.$$

Now assume that γ is chosen in such a way that if $|x-x_0|\leqq\theta$, $\gamma/2\leqq z\leqq 3\gamma/2$, then:

$$(5.55)\quad k^l(x,z)\geqq 4D_2\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}.$$

Clearly on $(T_0\wedge T_2^\gamma\wedge T_3^\gamma\geqq T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l})$, using (5.55) we see that on $[T_1^\gamma, T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}]$:

$$(5.56)\quad k^l(x_t,z_t)\geqq 4D_2\left[\frac{2}{\sqrt{\lambda}}\right]^{3/m_l}$$

and moreover on the interval $[T_1^\gamma, T_1^\gamma+D_1[2/\sqrt{\lambda}]^{3/m_l}]$, L does *not* increase, so that the flow $\varphi_t(\theta_{T_1^\gamma}\bar\omega, .)$ behaves like an ordinary flow of the sort described in (*b*). It is then elementary to use the estimate (5.29) to conclude that for one $\beta>0$:

$$(5.57)\quad \sum_{i=1}^{N} P\Big[\langle C_{T_1^\gamma + D_1[2/\sqrt{\lambda}]^{3/m_l}}^{T_1^\gamma} f_i, f_i \rangle \leqq \frac{2}{\sqrt{\lambda}}, \quad T_0 \wedge T_2^\gamma \wedge T_3^\gamma \geqq T_1^\gamma + D_1 \Big[\frac{2}{\sqrt{\lambda}}\Big]^{3/m_l} \Big] \leqq C\, \lambda^{(d-1)/2} \exp - \frac{D_3\, \lambda^{\beta/2}}{2^\beta}.$$

We now will choose γ so that (5.55) is verified. Using (5.32), we know that for any $\delta>0$, there is $\eta_\delta>0$ such that for $z \leqq \eta_\delta$, then for any x such that $|x-x_0|<\theta$:

$$(5.58)\quad \operatorname{Log} k^l(x, z) \geqq -\frac{\delta}{z}.$$

For (5.55) to hold, it is then enough that:

$$(5.59)\quad z \leqq \eta_\delta, \qquad z \geqq \frac{m_l\, \delta}{\operatorname{Log} \lambda}.$$

We choose:

$$(5.60)\quad \gamma = \frac{2\, m_l\, \delta}{\operatorname{Log} \lambda}$$

(5.46) is clearly verified.

For λ large enough, $3\gamma/2 \leqq \eta_\delta$, and moreover (5.55) holds.

Using the estimates (5.34)-(5.57), we find that for λ large enough:

$$(5.61)\quad \bar{P}[\,|C_{A_t}^{-1}| \geqq \lambda;\ A_t \leqq T] \leqq \frac{2}{[\lambda]^{t\sqrt{2}/16 m_l \delta}} + \frac{2\,d}{\lambda^{\theta^2/2C_\theta t m_l \delta}} + \frac{\sqrt{2}}{\lambda^{t\sqrt{2}/16 m_l \delta}} + 2 \exp(-\lambda^{1/m_l}) + \frac{A}{\lambda^p} + C \exp(-C'\, \lambda^{3/2m_l}) + C\, \lambda^{(d-1)/2} \exp\left(\frac{-D_3\, \lambda^{\beta/2}}{2^\beta}\right).$$

Since δ is arbitrary small, we find that for any $p \geqq 1$, for λ large enough:

$$(5.62)\quad \bar{P}[\,|C_{A_t}^{-1}| \geqq \lambda;\ A_t \leqq T] \leqq \frac{A_p}{\lambda^p}.$$

The Theorem follows. □

Remark 2. — Due to Theorem 4.9, this result implies the smoothness of the boundary semi-group. The importance of the estimate (5.42) will appear in section 6.

(*d*) *Localization of the estimates.*

Following the ideas of Stroock in [36], we will show how the previous estimates can be localized.

We will show that under adequate assumptions, the regularity of the boundary semi-group associated to the process (A_t, x_{A_t}) at $y \in R^d$ depends only of the behavior of $\mathscr{L}$ at $y \in R^d$. We will also show in what cases the law of x_{A_t} itself is smooth in a neighborhood of $y \in R^d$.

THEOREM 5.9. — *If $y_0 \in R^d$ is such that for $l \in N$, $\theta > 0$, there exists a $\geqq 0$ increasing function $z \in [0, \theta] \to h(z)$ such that:*

$$(5.63)\qquad \left\{\begin{array}{l} \inf\limits_{|y-y_0| \leqq \theta} k^l(y, z) \geqq h(z), \qquad z \in [0, \theta], \\ \lim\limits_{\substack{z>0 \\ z \to 0}} z \log h(z) = 0. \end{array}\right.$$

then for any $x_0 \in R^d$ such that $|x_0 - y_0| \geqq \theta$, if x_s is the process $\varphi_s(\bar{\omega}, x_0)$, the law of (A_t, x_{A_t}) under $Q_{(x_0, 0)}$, when restricted to $R \times B(y_0, \theta)$ is given by a C^∞ density with respect to the Lebesgue measure.

Proof. — Take $\theta_1, \theta_2 > 0$ such that $0 < \theta_2 < \theta_1 < \theta$. Let S_1, S_2 be the stopping times:

$$S_1 = \inf\{t \geqq 0; |x_s - y_0| + z_s = \theta_1\},$$
$$S_2 = \inf\{t \geqq S_1; |x_s - y_0| + z_s = \theta_2\} \wedge \inf\{t \geqq S_1; |x_s - y_0| + z_s = \theta\}.$$

Theorems 2.4 and 4.9 show that we must prove that $1_{S_2 \leqq A_t \leqq T} |C_{A_t}^{-1}|$ is in all the $L_p(\bar{\Omega}, P \otimes P')$. If $\bar{P} = P \otimes P'$, by reasoning as in (5.48)-(5.50) we have:

$$(5.64)\qquad \left\{\begin{array}{l} \bar{P}[|C_{A_t}^{-1}| \geqq \lambda; S_2 \leqq A_t \leqq T] \leqq \bar{P}\left[\left|\dfrac{\partial \varphi_{S_1}}{\partial x}(\bar{\omega}, x_0\right| \geqq \lambda^{1/4}, \right. \\ \left. S_1 \leqq A_t \leqq T\right] + \bar{P}[|[C_{S_2}^{S_1}]^{-1}| \geqq \lambda^{1/2}; S_2 \leqq A_t \leqq T]. \end{array}\right.$$

Now by Theorem 1.1 (*e*):

$$(5.65)\qquad \bar{P}\left[\left|\frac{\partial \varphi_{S_1}}{\partial x}(\bar{\omega}, x_0)\right| \geqq \lambda^{1/4}; S_1 \leqq A_t \leqq T\right] \leqq \frac{A_p}{\lambda^p}.$$

Set for $\delta > 0$:

$$(5.66)\qquad \gamma = \frac{4 m_l \delta}{\operatorname{Log} \lambda}.$$

Clearly:

$$(5.67)\quad \bar{P}[|C_{S_2}^{S_1}]^{-1}| \geqq \lambda^{1/2}; S_2 \leqq A_t \leqq T]$$
$$\leqq \bar{P}[S_2 - S_1 \leqq 2\gamma] + \bar{P}[|[C_{S_1+2\gamma}^{S_1}]^{-1}| \geqq \lambda^{1/2}; z_{S_1} \leqq \gamma; S_2 - S_1 \geqq 2\gamma; S_2 \leqq A_t \leqq T]$$
$$+ \bar{P}[|[C_{S_1+2\gamma}^{S_1}]^{-1}| \geqq \lambda^{1/2}; z_{S_1} > \gamma; S_2 - S_1 \geqq 2\gamma; S_2 \leqq A_t \leqq T].$$

Since L. has the same law as $\sup_{0 \leqq s \leqq .} w_s^1$, we have:

$$(5.68) \qquad \bar{P}[S_2 - S_1 \leqq 2\gamma] \leqq \exp - \frac{C}{\gamma}.$$

If $z_{S_1} \geqq \gamma$, we know that since h is increasing on $[0, \theta]$, and since $z_{S_1} \leqq \theta$, $|x_{S_1} - y_0| \leqq \theta$, we have:

$$(5.69) \qquad k^l(x_{S_1}, z_{S_1}) \geqq h(z_{S_1}) \geqq h(\gamma).$$

and moreover if $|x' - y_0| \leqq \theta$:

$$k^l\left(x', z_{S_1} - \frac{\gamma}{2}\right) \geqq h(z_{S_1} - \gamma/2) \geqq h(\gamma/2).$$

The estimation of the last term in the r. h. s. of (5.67) can then be done in the same way as after (5.44) with T_2^γ replaced by:

$$(5.70) \qquad T_2'^\gamma = \inf\{s \geqq S_1;\ |z_s - z_{S_1}| = \gamma/2\}.$$

Let $T_3'^\gamma$ be the stopping time:

$$(5.71) \qquad T_3'^\gamma = \inf\{s \geqq S_1;\ z_s = \gamma\}.$$

Using the reflection principle and (5.42), we have:

$$(5.72) \qquad \bar{P}[z_{S_1} \leqq \gamma;\ T_3'^\gamma - S_1 \geqq \gamma] \leqq \sqrt{2} \exp - \frac{1}{32\gamma}$$

and so:

$$(5.73) \qquad \bar{P}[|[C_{S_1+S_1+2\gamma}^{S_1}]^{-1}| \geqq \lambda^{1/2},$$
$$z_{S_1} \leqq \gamma;\ S_2 - S_1 \geqq 2\gamma;\ S_2 \leqq A_t \leqq T] \leqq \sqrt{2} \exp - \frac{1}{32\gamma} + \frac{A_p}{\lambda^p}$$
$$+ \bar{P}[|[C_{T_3'^\gamma+\gamma}^{T_3'^\gamma}]^{-1}| \geqq \lambda^{1/4};\ S_1 \leqq T_3'^\gamma \leqq T_3'^\gamma + \gamma \leqq S_2 \leqq A_t].$$

After (5.73), we can restart the procedure as after (5.53) and so we obtain the final estimate:

$$(5.74) \qquad \bar{P}|C_{A_t}^{-1}| \geqq \lambda;\ S_2 \leqq A_t \leqq T) \leqq \frac{C}{\lambda^p}. \quad \square$$

Remark 3. — The result of Theorem 5.9 is not exactly of the type given by Stroock in [36]. Such a result would state that the smoothness of $p_t(da, dy)$ on the neighborhood of (a_0, y_0) would depend only on the behaviour of $\mathscr{L}$ [given by (1.39)] on this neighborhood.

The local smoothness of the law of x_{A_t} raises other difficult questions. It can be easily proved that under the global condition.

$$(5.75)\qquad k^1(x, z) \geqq \alpha > 0 \quad \text{on } \mathbf{R}^d \times [0, \theta] \qquad (\alpha > 0)$$

and if $b=0$, for any $t>0$, the law of x_{A_t} is smooth. To see this, it suffices to modify the proof of Theorem 2.2. In fact (2.14) is still true if u is taken to be $\{F_t \otimes F'_\infty\}_{t \geqq 0}$-predictable. It is then feasible to choose instead of (2.22):

$$(5.76)\qquad u^i = 1_{A_t \leqq 1} 1_{s \leqq A_t} (\varphi_s^{*-1}(\bar{\omega}) X_i)(x_0) + 1_{A_t > 1} 1_{A_t - 1 \leqq s \leqq A_t} (\varphi_{s-A_t+1}^{*-1}(\bar{\theta}_{A_t-1}\bar{\omega}) X_i(x_{A_t-1}).$$

The problem of the necessary boudedness of A_t in (2.5) disappears, since if $A_t \geqq 1$, conditionally on $F_{A_t-1} \otimes F'_\infty$, we are back to a bounded interval. $C_t^{x_0}$ is replaced by:

$$(5.77)\qquad 1_{A_t \leqq 1} C_t^{x_0}(\bar{\omega}) + 1_{A_t > 1} C_1^{x_{A_t-1}}(\bar{\theta}_{A_t-1}\bar{\omega}).$$

It is then a trivial matter to prove the estimates on (5.77) under (5.75).

The introduction of b raises a first difficulty since it is not possible to bound adequately the "anticipating" stochastic integral $\int_{A_t-1}^{A_t} b(x_s, z_s)\,\delta B$. Moreover if the condition (5.75) is only local, new difficulties arise.

(*e*) *Regularity of the boundary semi-group: the non localizable case.*

We now give a sufficient condition under which the assumptions of Theorem 2.5 are verified.

THEOREM 5.10. — *Assume that for a given* $l \in \mathbf{N}$, *there exists a constant* $C>0$ *such that:*

$$(5.78)\qquad \varliminf_{z>0, z\to 0} z \operatorname{Log} \inf_{x \in \mathbf{R}^d} k_l(x, z) = -C.$$

Then for any $t > 16\sqrt{2}\, m_l C$, $T \geqq 0$, *there is* $q>2$ *such that for any* $x_0 \in \mathbf{R}^d$,

$$1_{A_t \leqq T} \left| [C_{A_t}^{x_0}]^{-1} \right| \in L_q(\bar{\Omega}, P \otimes P')$$

with a norm in $L_q(\bar{\Omega}, P \otimes P')$ *bounded independently of* x_0.

Proof. — The proof is identical to the proof of Theorem 5.8 with a few changes. Observe that $\theta = +\infty$ in (5.37), and that no *restriction* on t exists any more.

Take $\varepsilon>0$. By (5.78), we can choose $\delta = C + \varepsilon$ in (5.58). (5.61) still holds for each given $t \in \mathbf{R}^+$, and for λ large enough. From (5.61), we get:

$$(5.79)\qquad \bar{P}[|C_{A_t}^{-1}| \geqq \lambda;\ A_t \leqq T] \leqq \frac{C}{[\lambda]^{t\sqrt{2}/16 m_l (C+\varepsilon)}}.$$

If $t\sqrt{2}/16\, m_l\,(C+\varepsilon)>2$, $1_{A_t\leq T}\,|\,C_{A_t}^{-1}\,|$ is clearly in one given $L_q(\bar{\Omega}, P\otimes P')$ with $q>2$. Since ε is arbitrary, the result is proved. □

COROLLARY. — *Under the assumptions of Theorem* 5.10, *for any* $t>0$, *the law under* $Q_{(x_0,z_0)}$ *of* (A_t, x_{A_t}) *is given by* $p_t(a,y)\,da\,dy$ *which is such that:*

(*a*) *For any* $t>0$, $p_t(a,y)$ *is* C^∞ *on* $]0,+\infty[\times R^d$;

(*b*) *For any* $k\in N$, *and any* $t>(k+d+2)\,16\sqrt{2}\,m_l\,C$, $p_t(a,y)\in C^k(R\times R^d)$.

In particular the singular support of $p_t(a,y)$ *is included in* $\{0\}\times R^d$.

Proof. — (*b*) is an obvious consequence of Theorems 4.12 and 5.10. To prove (*a*), we only need to estimate for any given $\varepsilon>0$:

$$P[|\,C_{A_t}^{-1}\,|\geq\lambda;\ 2\,\varepsilon\leq A_t\leq T]. \tag{5.80}$$

The estimate (5.35) is no longer needed, since only $(A_t\geq 2\,\varepsilon)$ is considered. The estimate (5.37) is still valid with $\theta=+\infty$, for any $t>0$, i.e. it is not needed. (5.42) is replaced by:

$$\bar{P}(T_1^\gamma\geq\varepsilon)\leq\sqrt{2}\,\exp-\frac{\varepsilon}{8\,\gamma^2}. \tag{5.81}$$

Taking $\delta=C+\varepsilon$ and choosing γ as in (5.60), it is clear that the leading term in the estimation of (5.80) is the r.h.s. of (5.81), i.e.:

$$\sqrt{2}\,\exp-\frac{\varepsilon\,(\mathrm{Log}\,\lambda)^2}{32\,(m_l\,\delta)^2}, \tag{5.82}$$

which is $\leq A/\lambda^p$. $1_{2\varepsilon\leq A_t\leq T}\,|\,C_{A_t}^{-1}\,|$ is then in all the L_p. (*a*) has then been proved. □

Remark 5. — Assume that $(X_0, X_1\ldots X_m)$ do not depend on z, and that moreover $D=0$, $b=0$. x_t and z_t are independent processes. C_t is then independent of z. If the assumptions of the previous Corollary are verified, it is easily seen, by reasoning as in (5.80)-(5.82) that for any $s>0$:

$$\bar{\bar{P}}[|\,C_s^{-1}\,|\geq\lambda]\leq C\,\exp-\frac{s\,(\mathrm{Log}\,\lambda)^2}{64\,(m_l\,\delta)^2},+\frac{C}{\lambda^p}, \tag{5.83}$$

so that for $q>2$:

$$E^{\bar{P}}[|\,C_s^{-1}\,|^q]\leq C+C'\int_0^{+\infty}\lambda^{q-1}\,\exp-ks\,(\mathrm{Log}\,\lambda)^2\,d\lambda. \tag{5.84}$$

By doing the change of variable $\lambda=e^u$, we find that as $s\to 0$:

$$E^{\bar{P}}[|\,C_s^{-1}\,|^q]\leq C+\frac{C'}{\sqrt{2\,ks}}\exp\frac{q^2}{4\,ks}. \tag{5.85}$$

Now by [18] (p. 25), the law of A_t under P′ is:

$$(5.86)\qquad \frac{1_{s\geqq 0}}{\sqrt{2\pi s^3}}\, t \exp -\frac{t^2}{2s}\, ds.$$

Now using the general results on the Malliavin calculus, we know that if $p_s(y)\,dy$ is the law of x_s, the norm of $p_s(y)$ in $C^k(R^d)$ can be adequately estimated by $E^{\overline{P}}[|C_s^{-1}|^q]$ (with q depending on k). Since the law of (A_t, x_{A_t}) is now:

$$(5.87)\qquad \frac{1_{s\geqq 0}}{\sqrt{2\pi s^3}}\, t \exp -\frac{t^2}{2s}\, p_s(y)\, ds\, dy,$$

from (5.85)-(5.87), we see that for t small enough no adequate bound on (5.87) exists as $s \to 0$.

Let us finally remark that the condition (5.78) is in a sense minimal. In fact by Theorems 3.4 and 3.6, for $k>0$, $\alpha>0$:

$$(5.88)\qquad n^+\left(\int_0^\sigma e^{-k/z_s}\, ds \geqq \alpha\right) \leqq C + n^+\left(\int_0^\sigma e^{-k/z_s}\, ds \geqq \alpha,\ \sigma \leqq 1\right)$$
$$\leqq C + n^+\left[\sup_{0\leqq s\leqq \sigma} z_s \geqq \frac{k}{\text{Log}(1/\alpha)}\right] = C + \frac{\text{Log}(1/\alpha)}{k}.$$

It then follows that for $\beta>1$:

$$(5.89)\qquad \int_0^{+\infty}\left[\exp -\left(\beta\int_0^\sigma e^{-k/z_s}\, ds\right) - 1\right] dn^+ \geqq C$$
$$-\beta\int_0^1 e^{-\beta\alpha}\left(C + \frac{\text{Log}(1/\alpha)}{k}\right) d\alpha \geqq C - C'\,\text{Log}\,\beta,$$

with $C'>0$. From (5.89), we get:

$$(5.90)\qquad E^{P'} \exp -\beta\int_0^{A_t} e^{-k/z_s}\, ds \geqq \frac{D}{\beta^{C't}}.$$

Consider now the stochastic differential equation:

$$(5.91)\qquad dx = \exp\left(-\frac{1}{z_s}\right) dw_s^1, \qquad x(0)=0.$$

If C_t is the process:

$$(5.92)\qquad C_t = \int_0^t \exp -\frac{2}{z_s}\, ds,$$

it is clear that conditionally on z, the law of x_{A_t} is a centered gaussian whose variance is C_{A_t}. Since:

$$(5.93)\qquad E^{P'}[C_{A_t}^{-1/2}]=\frac{1}{\Gamma(1/2)}\int_0^{+\infty}\frac{1}{\beta^{1/2}}E^{P'}e^{-\beta C_{A_t}}d\beta$$

we see from (5.90) that for $t>0$ small enough:

$$(5.94)\qquad E^{P'}[C_{A_t}^{-1/2}]=+\infty.$$

Since the law of x_{A_t} under $P\otimes P'$ is $h_t(x)\,dx$ where:

$$(5.95)\qquad h_t(x)=\int_{\Omega'}\frac{1}{\sqrt{2\pi C_{A_t}}}\exp-\frac{x^2}{2C_{A_t}}dP'$$

it is clear from (5.94)-(5.95) that for t small enough $\lim_{x\to 0}h_t(x)=+\infty$. For t small enough, h_t is not even continuous.

(5.78) is then seen to be minimal.

6. The analysis of two-sided boundary processes

In this section we assume that the reflecting Brownian motion z is replaced by a standard Brownian motion, which is still written z. The diffusion process x is now governed by two second order differential operators $\mathscr{L}$ and $\mathscr{L}'$ in the regions $(z>0)$ and $(z<0)$.

(*a*) and (*b*) are devoted to a quick definition of the two-sided diffusion and its associated boundary process. In (*c*) we do some remarks very similar in their spirit to what has been done in section 1 (*f*). In (*d*) the principle of the calculus of variations is briefly sketched, and the key quadratic form $C_t^{x_0}$ is again exhibited.

In (*e*), conditions of a. s. invertibility of $C_t^{x_0}$ are given. Non trivial interactions of $\mathscr{L}$ and $\mathscr{L}'$ are exhibited, which imply that Hörmander-like interactions of Lévy kernels are possible.

In (*f*) the crucial problem of the possible localization of the condition of regularity for the boundary semi-group is considered. In fact in section 5 (Theorems 5.8 and 5.9), we had seen that for a diffusion with one type of reflection, a local condition on the diffusion operator $\mathscr{L}$ could ensure the smoothness of the boundary process. The introduction of a second type of excursion, associated to a new type operator $\mathscr{L}'$ drastically modifies the situation. In fact if $\mathscr{L}'$ is badly behaved, it is shown that the process may go out of the region of regularization before getting "regularized" enough by the excursion governed by $\mathscr{L}$. The Arcsine law of P. Lévy ([18], p. 57) helps us to give a striking illustration of this phenomenon.

In (*g*), the case where the Hörmander's distributions associated to the operators $\mathscr{L}$ *and* $\mathscr{L}'$ span the whole R^d at each point or the boundary ($z=0$) is considered. Under a non-local condition, and using the stochastic calculus on Poisson point processes, the boundary semi-group is shown to be smooth.

(*a*) *The two-sided process and its associated flow.*

We first do a change of notations with respect to section 1.

Ω' is now the space $\mathscr{C}(R^+, R)$. The trajectory of $\omega' \in \Omega'$ is still z. The σ-field F'_t on Ω' is $\mathscr{B}(z_s \mid s \leqq t)$. Ω' is endowed with the filtration $\{F'_t\}_{t \geqq 0}$.

For $z \in R$, P'_z is the standard Brownian measure on Ω' such that $P'_z(z(0)=z)=1$. P' is the measure P'_0.

All the notations of section 1 are otherwise preserved, including the definition of $\Omega, \tilde{\Omega}, P, \theta, \bar{\theta}, \ldots$

L_t denotes the local time at 0 of the reflecting Brownian motion $|z_t|$ (i.e. L is twice the standard local time at 0 of z), so that:

$$(6.1) \qquad |z_t| = |z_0| + \int_0^t \operatorname{sgn} z_s \, \delta z_s + L_t,$$

z^+, z^- are defined by:

$$(6.2) \qquad z_t^+ = z_t \vee 0, \quad z_t^- = z_t \wedge 0.$$

$X_0(x, z), \ldots, X_m(x, z)$, $D(x)$ are taken as in section 1. $X'_0(x, z) \ldots X'_m(x, z)$ is another family of $m+1$ vector fields, having the same properties as $X_0(x, z) \ldots X_m(x, z)$.

α, α' are >0 real numbers.

On $(\tilde{\Omega}, P \otimes P'_{z_0})$ consider the stochastic differential equation:

$$(6.3) \qquad \left\{ \begin{aligned} dx &= 1_{z>0} [X_0(x, z)\, dt + X_i(x, z)\, dw^i] \\ &\qquad + 1_{z<0} [X'_0(x, z)\, dt + X'_i(x, z)\, dw^i] + D(x)\, dL, \\ x(0) &= x_0, \end{aligned} \right.$$

(6.3) still defines a flow $\varphi_.(\bar{\omega}, .)$ of diffeomorphisms of R^d so that in (6.3), $\varphi_t(\bar{\omega}, x_0) = x_t$, and the results of section 1 (*b*) hold without any change.

$b(x, z)$ is taken as in section 1, and $b'(x, z)$ is another function having the same properties as $b(x, z)$. In (1.13), the first equation is replaced by (6.3) and the second equation by:

$$(6.4)\quad du=1_{z>0}\left[b(x,z)\,\delta z-\frac{1}{2}b^2(x,z)\,dt\right]+1_{z<0}\left[b'(x,z)\,\delta z-\frac{1}{2}b'^2(x,z)\,dt\right].$$

Of course, if we use Stratonovitch integrals, the reader will check that (6.4) is equivalent to:

$$(6.5)\quad du=b(x,z)\,dz^+ +b'(x,z)\,dz^- + \frac{(b'-b)(x,0)}{2}\,d\mathrm{L}-\frac{1_{z>0}}{2}(b_z+b^2)(x,z)\,dt-\frac{1_{z<0}}{2}(b'_z+b'^2)(x,z)\,dt.$$

The results of 1 (*c*) still hold. (1.14) is replaced by:

$$\mathrm{M}_t=\exp\left[\int_0^t 1_{z>0}\left(b(x,z)\,\delta z-\frac{1}{2}b^2(x,z)\,dt\right)+\int_0^t 1_{z<0}\left(b'(x,z)\,\delta z-\frac{1}{2}b'^2(x,z)\,dt\right)\right].$$

For $(x_0, z_0)\in \mathrm{R}^d\times\mathrm{R}$, the probability measure $\mathrm{Q}_{(x_0,z_0)}$ is still defined by:

$$\left.\frac{d\mathrm{Q}_{(x_0,z_0)}}{d\,(\mathrm{P}\otimes\mathrm{P}'_{z_0})}\right|_{\bar{\mathrm{F}}_t}=\mathrm{M}_t.$$

(*b*) *The boundary process.*

A_t is the right continuous inverse of L_t. Definition 1.6 is unchanged.

DEFINITION 6.1. — Take $(a_0, x_0)\in\mathrm{R}^+\times\mathrm{R}^d$. On $(\bar{\Omega}, \mathrm{Q}_{(x_0,0)})$, the boundary process (a_t, y_t) with values in $(\mathrm{R}^+\times\mathrm{R}^d)\cup\{\Delta\}$ is defined by:

$$(6.6)\quad \begin{cases}(a_t, y_t)=\left(a_0+\alpha\displaystyle\int_0^{\mathrm{A}_t}1_{z>0}\,ds+\alpha'\int_0^{\mathrm{A}_t}1_{z<0}\,ds,\ \varphi_{\mathrm{A}_t}(\bar{\omega}, x_0)\right), & t<\mathrm{L}_\infty,\\ \Delta,\ t\geqq \mathrm{L}_\infty.\end{cases}$$

We will still study the semi-group associated to the boundary process (a_t, y_t) which is a strong Markov process.

(*c*) *Some remarks on the boundary process.*

Let $\mathrm{Y}(x,z)$, $\mathrm{Y}'(x,z)$ be two vector fields which have the same properties as $\mathrm{X}_0(x,z)$. Consider the more general stochastic differential equation on $(\bar{\Omega}, \mathrm{P}\otimes\mathrm{P}')$:

$$(6.7)\quad dx=1_{z>0}[\mathrm{X}_0(x,z)\,dt+\mathrm{X}_i(x,z)\,dw^i] +1_{z<0}[\mathrm{X}'_0(x,z)\,dt+\mathrm{X}'_i(x,z)\,dw^i]+\mathrm{Y}(x,z)\,dz^+ +\mathrm{Y}'(x,z)\,dz^- +\mathrm{D}(x)\,d\mathrm{L}.$$

For $z_0^+, z_0^- \in \mathrm{R}$ consider the differential equations:

$$(6.8)\quad \left\{\begin{aligned} &\frac{dx^+}{dt}=\mathrm{Y}(x^+, z^+), \qquad x^+(0)=x_0;\\ &\frac{dz^+}{dt}=1, \qquad z^+(0)=z_0^+;\\ &\frac{dx^-}{dt}=\mathrm{Y}'(x^-, z^-), \qquad x^-(0)=x_0;\\ &\frac{dz^-}{dt}=1, \qquad z^-(0)=z_0^-.\end{aligned}\right.$$

Let k_t^+, k_t^- be the groups of diffeomorphisms of R^{d+1} onto itself associated to (6.8). Of course $\partial k_t^+/\partial x$, $\partial k_t^-/\partial x$ map R^d onto itself.

Consider the stochastic differential equation on $(\bar{\Omega}, \mathrm{P}\otimes\mathrm{P}')$:

$$(6.9)\quad \left\{\begin{aligned} dx''=&\,1_{z>0}\,[(k_{z_t}^{+*-1}\mathrm{X}_0)(x_t'', 0)\,dt+(k_{z_t}^{+*-1}\mathrm{X}_i)(x_t'', 0)\,dw^i]\\ &+1_{z<0}\,[(k_{z_t}^{-*-1}\mathrm{X}_0')(x_t'', 0)\,dt+(k_{z_t}^{-*-1}\mathrm{X}_i)(x_t'', 0)\,dw^i]+\mathrm{D}(x'').d\mathrm{L},\\ &x''(0)=x_0.\end{aligned}\right.$$

Recall that π is the projection operator $(x, z)\in\mathrm{R}^{d+1}\to x\in\mathrm{R}^d$. We claim that in (6.7):

$$(6.10)\qquad x_t=\pi\, k_{z_t^+}^+\circ k_{z_t^-}^-(x_t'', 0).$$

Of course if $z_t^+>0$, $z_t^-=0$, and if $z_t^-<0$, $z_t^+=0$, so that in (6.10), we can write:

$$(6.11)\qquad x_t=\pi\, k_{z^-}^-\circ k_{z_t^+}^+(x_t'', 0).$$

We check that x_t given by (6.10) is a solution of (6.7). In fact:

$$\begin{aligned}(6.12)\quad d\pi\, k_{z_t^+}^+\circ k_{z_t^-}^-(x_t'', 0)=&\,\mathrm{Y}(\pi\, k_{z_t^+}^+\circ k_{z_t^-}^-(x_t'', 0),\, z_t^+ + z_t^-)\,dz^+\\ &+\frac{\partial k_{z_t^+}^+}{\partial x}(k_{z_t^-}^-(x_t'', 0))\Bigg[\mathrm{Y}'(\pi\, k_{z_t^-}^-(x_t'', 0),\, z_t^-)\,dz_t^-\\ &+\frac{\partial k_{z_t^-}^-}{\partial x}(x_t'', 0)\Bigg\{1_{z>0}\left[\frac{\partial k_{z_t}^+}{\partial x}(x_t'', 0)\right]^{-1}\\ &(\mathrm{X}_0(\pi\, k_{z_t}^+(x_t'', 0), z_t)\,dt+\mathrm{X}_i(\pi\, k_{z_t}^+(x_t'', 0), z_t).dw^i\\ &+1_{z<0}\left[\frac{\partial k_{z_t}^-}{\partial x}(x_t'', 0)\right]^{-1}(\mathrm{X}_0'(\pi\, k_{z_t}^-(x_t'', 0), z_t)\,dt\\ &+\mathrm{X}_i'(\pi\, k_{z_t}^-(x_t'', 0), z_t)\,dw^i)+\mathrm{D}(x'').d\mathrm{L}\Bigg\}\Bigg]+\pi\,\frac{\partial k_{z_t^+}^+}{\partial z}(k_{z_t^-}^-(x_t'', 0)).dz^-.\end{aligned}$$

Now $z_t^+ + z_t^- = z_t$, and $\pi\, \partial k_0^+/\partial z = 0$, $\pi\, \partial^2 k_0^+/\partial x\, \partial z = 0$, $\pi\, \partial^2 k_0^+/\partial z^2 = 0$. Moreover:

$$(6.13)\quad \int_0^s \frac{\partial k^+_{z_t^+}}{\partial x}(k^-_{z_t^-}(x_t'', 0))\, Y'(\pi\, k^-_{z_t^-}(x_t'', 0), z_t^-)\, dz_t^- = \int_0^s Y'(\pi\, k^+_{z_t^+} \circ k^-_{z_t^-}(x_t'', 0), z_t)\,.\, dz_t^-.$$

essentially because $\langle z^+, z^- \rangle = 0$ ($\langle z^+, z^- \rangle$ is the quadratic variation of z^+ and z^-) and because on $(z_t \leqq 0)$ $\partial^2 k^+_{z_t^+}/\partial x\, \partial z = 0$. Moreover:

$$(6.14)\quad \begin{cases} 1_{z>0} \dfrac{\partial k^+_{z_t^+}}{\partial x}(k^-_{z_t^-}(x_t'', 0))\, \dfrac{\partial k^-_{z_t^-}}{\partial x}(x_t'', 0) \left[\dfrac{\partial k^+_{z_t}}{\partial x}(x_t'', 0) \right]^{-1} = 1_{z>0}\, \mathrm{I}, \\ 1_{z<0} \dfrac{\partial k^+_{z_t^+}}{\partial x}(k^-_{z_t^-}(x_t'', 0))\, \dfrac{\partial k^-_{z_t^-}}{\partial x}(x_t'', 0) \left[\dfrac{\partial k^-_{z_t}}{\partial x}(x_t'', 0) \right]^{-1} = 1_{z<0}\, \mathrm{I}. \end{cases}$$

It is then obvious that x_t given by (6.11) is the solution of (6.7) on $(\bar{\Omega}, P \otimes P')$. Now clearly, since $z_{A_t} = 0$, we have $x_{A_t} = x''_{A_t}$. Equation (6.9) is of the type (6.3), and defines the same boundary process as (6.7). It is then not a restriction to study equation (6.3) instead of (6.7).

Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be the second order differential operator acting on $f \in C_b^\infty(R^+ \times R^d \times R^+)$ [resp. $C_b^\infty(R^+ \times R^d \times R^-)$]:

$$(6.15)\quad \mathscr{L} f = \left(\alpha \frac{\partial}{\partial t} + X_0 + b \frac{\partial}{\partial z} + \frac{1}{2} \sum_{i=1}^{m'} X_i^2 + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right) f$$

$$\left(\text{resp. } \mathscr{L}' f = \left(\alpha' \frac{\partial}{\partial t} + X_0' + b' \frac{\partial}{\partial z} + \frac{1}{2} \sum_{i=1}^{m'} X_i'^2 + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right) f \right).$$

DEFINITION 6.2. — $C_b^{\infty\pm}(R^+ \times R^d \times R)$ is the set of functions $f(t, x, z)$ defined on $R^+ \times R^d \times R$ with values in R whose restriction to $R^+ \times R^d \times R^+$ and $R^+ \times R^d \times R^-$ are in $C_b^\infty(R^+ \times R^d \times R^+)$ and $C_b^\infty(R^+ \times R^d \times R^-)$.

When $z = 0$, f has generally distinct right and left first derivatives in the variable z, which we write $\partial f/\partial z^+$ and $\partial f/\partial z^-$.

$\mathscr{D}$ is now the differential operator defined on the boundary $(z = 0)$ acting on $f \in C_b^{\infty\pm}(R^+ \times R^d \times R)$:

$$(6.16)\quad \mathscr{D} f = \left[D + \frac{1}{2} \left(\frac{\partial}{\partial z^+} - \frac{\partial}{\partial z^-} \right) \right] f(t, x, 0).$$

Then if $f \in C_b^{\infty\pm}(R^+ \times R^d \times R)$, if:

$$(6.17)\quad x_t = \varphi_t(\bar{\omega}, x_0), \qquad s_t = \int_0^t (\alpha\, 1_{z>0} + \alpha'\, 1_{z<0})\, ds,$$

then under $Q_{(x_0, 0)}$:

(6.18) $$f(s_t, x_t, z_t) - \int_0^t (1_{z>0} \mathscr{L} f + 1_{z<0} \mathscr{L}' f)(s_u, x_u, z_u)\, du - \int_0^t (\mathscr{D} f)(s_u, x_u, 0)\, dL_u$$

is a martingale. To obtain (6.18), it suffices to apply Tanaka's formula to the process $f(s_t, x_t, z_t)$.

We now proceed as in Section 1 (g). Take $g \in C_b^\infty (R^+ \times R^d)$. Consider the two Dirichlet problems on $f^+ \in C_b^\infty (R^+ \times R^d \times R^+)$, $f^- \in C_b^\infty (R^+ \times R^d \times R^-)$:

(6.19) $$\begin{cases} \mathscr{L} f^+ = 0 & \text{on } z \geqq 0, \\ \mathscr{L}' f^- = 0 & \text{on } z \leqq 0, \\ f^+ = f^- = g & \text{on } z = 0. \end{cases}$$

Assume that (6.19) has a solution. Let f be the element of $C_b^{\infty \pm} (R^+ \times R^d \times R)$ which coincides with f^+ on $(z \geqq 0)$, with f^- on $(f \leqq 0)$.

Using (6.18), we see that:

$$g(s_{A_t}, x_{A_t}) - \int_0^{A_t} (\mathscr{D} f)(s_{A_u}, x_{A_u}\, 0)\, du,$$

is a $\{ \bar{F}_{A_t} \}_{t \geqq 0}$-martingale. By setting:

(6.20) $$(\mathscr{A} g)(s, x) = (\mathscr{D} f)(s, x, 0),$$

we see that at least formally, $\mathscr{A}$ is the infinitesimal generator of the process (s_{A_t}, x_{A_t}).

Of course the same discussion as after (1.44) applies. In particular let $\mathscr{L}$ and $\mathscr{L}'$ be two second order operators written in Hörmander's form as before (1.44). D is an open set in R^{d+2} with smooth boundary ∂D. γ and γ' are two smooth vector fields which are pointing inward and outward D (they could as well point both inward). δ and δ' are the 1 differential forms associated to γ and γ' as in (1.44).

Take $g \in C_b^\infty (\partial D)$. Consider the Dirichlet problem on $f^+ \in C_b^\infty (\bar{D})$, $f^- \in C_b^\infty ({}^c D)$:

(6.21) $$\begin{cases} \mathscr{L} f^+ = 0 & \text{on D.} \\ \mathscr{L}' f^- = 0 & \text{on } {}^c\bar{D}, \\ f^+ = f^- = g & \text{on } \partial D. \end{cases}$$

(If γ and γ' point inward, the condition $\mathscr{L}' f^- = 0$ on ${}^c\bar{D}$ should be replaced by $\mathscr{L}' f^- = 0$ on D.) Set:

(6.22) $$\mathscr{A} g = \frac{\gamma f^+ + \gamma' f^-}{2} \quad \text{on } \partial D.$$

We want to study the smoothness of the transition probabilities associated to the semi-group $e^{t\mathscr{A}}$.

A discussion very similar to what has been done after (1.44) is interesting but we leave it to the reader.

The coordinate systems $(\bar{x}^1 \ldots \bar{x}^{d+2})$ and $(\bar{x}'^1 \ldots \bar{x}'^{d+2})$ associated to $(\mathscr{L}, \gamma)$ and $(\mathscr{L}', \gamma')$ are in general *distinct*.

Some computations very similar to (6.9), (6.12) show that, to build the semi-group $e^{t\mathscr{A}}$, at least locally, a stochastic differential equation living on the boundary ∂D of the type (6.3) must be solved, whose solution is x_t. If k_t, k'_t are the flows of diffeomorphisms associated to $Y_{m'}$, $Y'_{m'}$ (which are obtained in the reduction of $\mathscr{L}$ and $\mathscr{L}'$ to the form (1.52)), then $(k_{z_t^+} \circ k'_{z_t^-}(x_t), z_t)$ is the true solution in R^{d+1} of the stochastic differential equation from which the boundary process is built.

Assume temporarily that $X_0, X_1 \ldots X_m, X'_0 \ldots X'_m$ do not depend on z, and that $b = b' = 0$. Let $\mathscr{L}^0, \mathscr{L}'^0$ be the differential operators acting on $C_b^\infty(R^d)$:

$$\mathscr{L}^0 = X_0 + \frac{1}{2}\sum_1^m X_i^2, \qquad \mathscr{L}'^0 = X'_0 + \frac{1}{2}\sum_1^m X_i'^2.$$

In this case $\mathscr{A}$ is given by:

$$\mathscr{A} = D - \frac{1}{2}\sqrt{-2\left(\alpha\frac{\partial}{\partial t} + \mathscr{L}^0\right)} - \frac{1}{2}\sqrt{-2\left(\alpha'\frac{\partial}{\partial t} + \mathscr{L}'^0\right)}. \tag{6.23}$$

At this stage, the reader can ask how to construct the process associated to:

$$\mathscr{A} = D - \sqrt{-\alpha\frac{\partial}{\partial t} - \mathscr{L}^0} - \sqrt{-\alpha'\frac{\partial}{\partial t} - \mathscr{L}'^0} - \sqrt{-\alpha''\frac{\partial}{\partial t} - \mathscr{L}''^0} \tag{6.24}$$

where $\mathscr{L}''^0$ is similar to $\mathscr{L}^0, \mathscr{L}'^0$.

The complete answer lies in the Poisson point process properties of the reflecting Brownian motion and the standard Brownian motion. To see this, we give a few definitions.

DEFINITION 6.3. — $\mathscr{W}^-$ is the space $-\mathscr{W}^+$.

We now have the basic result.

THEOREM 6.4. — *On (Ω', P'), let e_t be the process adapted to $\{F'_{A_t}\}_{t \geqq 0}$ taking its values in $\mathscr{W}^+ \cup \mathscr{W}^- \cup \{\delta\}$ defined by:*

(*a*) If:

$$A_t^- < A_t, \qquad e_t(s) = z_{s+A_t^-}, \qquad 0 \leqq s \leqq A_t - A_{t^-},$$
$$0 \quad \text{for} \quad s > A_t - A_{t^-};$$

(*b*) If:

$$A_{t^-} = A_t, \qquad e_t(s) = \delta,$$

Let n^- be the image on $\mathscr{W}^-$ of the measure n^+ on $\mathscr{W}^+$ by the mapping $e \to -e$.

Then e_t is a Poisson point process whose characteristic measure $n^{\pm}$ is given by:

$$n^{\pm} = \frac{1}{2}(n^+ + n^-), \tag{6.25}$$

$\{ F'_{A_t} \}_{t>0}$ *is the natural filtration of* e_t.

Proof. — This result is contained in Itô-McKean [18], p. 75 and Ikeda-Watanabe [17], p. 123. The proof of the result on the filtration $\{ F'_{A_t} \}_{t \geqq 0}$ is the same as in proposition 3.11. □

Theorem 6.4 tells us in fact that the Poisson point process $e_t(s)$ of the excursion of z_t is obtained by marking with the marks $(+, -)$ and the weights $(1/2, 1/2)$ the excursions of the reflecting Brownian motion $|z_t|$. To study (6.24) as well as more complex systems extending (6.3), the excursions of a reflecting Brownian motion must be marked with three (or more) marks and equal weights (taking unequal weights does not change anything, since by time change on the excursions, we can equalize the weights). Such a construction is elementary and left to the reader.

However, we will take much advantage of the fact that estimates do exist on the standard Brownian motion, which would be hard to obtain on an abstract marked reflected Brownian motion.

(*d*) *The calculus of variations on the two-sided process.*

The calculus of variations is identical to what we have done in sections 2 and 4. The key process of linear mappings from $T^*_{x_0}(R^d)$ into $T_{x_0}(R^d)$ is now:

$$p \to C_t^{x_0} p = \int_0^t 1_{z>0} \langle (\varphi_u^{*-1} X_i)(x_0), p \rangle \varphi_u^{*-1} X_i(x_0)\, du + \int_0^t 1_{z<0} \langle \varphi_u^{*-1} X'_i(x_0), p \rangle \varphi_u^{*-1} X'_i(x_0)\, du. \tag{6.26}$$

To control the differentials of the law of A_t, the technique of section 4 can be adapted. We now consider semi-martingales H which have the following properties:

(*a*) H is bounded and $\geqq 0$;

(*b*) H is 0 on $(z \leqq 0)$;

(*c*) The Itô decomposition of H is:

$$H_t = \int_0^t 1_{z>0} K\, ds + \int_0^t 1_{z>0} E\, \delta z, \tag{6.27}$$

where K, E are bounded $\{ \bar{F}_t \}_{t \geqq 0}$-predictable processes and $E \geqq -1/2$ [to simplify, we do not assume that L appears in (6.27)].

Except for the boundedness conditions, $H_t = z_t^{+2}$ is an adequate choice.

The arguments of section 4 can be reproduced almost identically. In section 4 (*a*), the exponential martingale is now changed into:

$$N_t = \exp\left\{ -\int_0^t 1_{z>0} \frac{K}{1+E} \delta z - \frac{1}{2} \int_0^t 1_{z>0} \left(\frac{K}{1+E} \right)^2 ds \right\},$$

τ_t will be the time change:

$$\tau_t = \inf\left\{ \tau > 0; \int_0^\tau (1_{z>0} (1+E)^2 + 1_{z<0})\, ds > t \right\}.$$

Then if S is defined as in (4.7), it is easily proved that under S, if z'_t, L'_t, $w'^t_t \ldots w'^m_t$ are the processes given by:

$$\left\{ \begin{array}{c} z'_t = (z+H)_{\tau_t}, \qquad L'_t = L_{\tau_t}, \\ w'^1_t = \int_0^{\tau_t} (1_{z>0} (1+E) + 1_{z<0})\, \delta w^1, \ldots, w'^m_t = \int_0^{\tau_t} (1_{z>0} (1+E) + 1_{z<0})\, \delta w^m \end{array} \right.$$

then $(z', w'^1 \ldots w'^m)$ is a $\{ \bar{F}_{\tau_t} \}_{t \geq 0}$-Brownian martingale, and L′ is the local time at 0 of $|z'|$. The proof that under S, $(z', w'^1 \ldots w'^m)$ is a Brownian $\{ \bar{F}_{\tau_t} \}_{t \geq 0}$-martingale is trivial. Moreover since H is $\geqq 0$ and is 0 on $(z \leqq 0)$, we have the obvious:

$$\left\{ \begin{array}{l} |z_t + H_t| = |z_t| + H_t, \\ \operatorname{sgn}(z+H) = \operatorname{sgn} z, \\ (\operatorname{sgn} z)\, 1_{z>0} = 1_{z>0}, \end{array} \right.$$

so that:

$$(6.28) \qquad |z_t + H_t| = \int_0^t \operatorname{sgn}(z+H)(\delta z + \delta H) + L_t.$$

From (6.28), we deduce easily that L′ is the local time at 0 of z'.

The calculus of variations on s_{A_t} is then done in the same way as in section 4, using $H_t = z_t^{+2}$, and the fact that since $\int_0^{A_t} z_s^+ \, ds$ is a stable process whose exponent is 1/3, for any $t > 0$, $\left[\int_0^{A_t} z_s^+ \, ds \right]^{-1}$ is in all the $L_p(\bar{\Omega}, P \otimes P')$.

Statements strictly similar to Theorems 4.9-4.13 are easily proved. Details are left to the reader.

However the estimates of section 5 must be in general drastically modified.

(*e*) *Existence of densities for the boundary semi-group.*

We first study the a. s. invertibility of $C_t^{x_0}$.

DEFINITION 6.5. — For $l \in \mathbf{N}$, E_l, E'_l, F_l are the family of vector fields defined by:

$$\left\{\begin{aligned} &E_1=(X_1, \ldots, X_m), \qquad E'_1=(X'_1, \ldots, X'_m), \qquad F_1=\{0\},\\ &E_{l+1}=\left[\left(X_0, X_1, \ldots, X_m, \frac{\partial}{\partial z}\right), E_l\right],\\ &E'_{l+1}=\left[\left(X'_0, X'_1 \ldots X'_m, \frac{\partial}{\partial z}\right), E'_l\right],\\ &F_{l+1}=[(X'_1 \ldots X'_m, D), E_l] \cup [(X_1, \ldots, X_m, D), E'_l]\\ &\qquad\qquad \cup [(X_1, \ldots, X_m, X'_1 \ldots X'_m, D), F_l]. \end{aligned}\right.$$

We then have the following result, which was first given in [50], Theorem 2.19.

THEOREM 6.6. — *If* $\bigcup_{l=1}^{+\infty} (E_l \cup E'_l \cup F_l)(x_0, 0)$ *spans* $\mathbf{R}^d$, *then* $P \otimes P'$ *a. s., for any* $t>0$, $C_t^{x_0}$ *is invertible.*

Proof. — U_s is the vector space spanned by:

$$1_{z_s>0}\,(\varphi_s^{*-1} X_i)(x_0) \quad (1 \leqq i \leqq m) \qquad \text{and} \qquad 1_{z_s<0}\,(\varphi_s^{*-1} X'_i)(x_0) \quad (1 \leqq i \leqq m),$$

V_t is the vector space spanned by $\bigcup_{s \leqq t} U_s$ and V_{t^+} is defined by:

$$V_{t^+} = \bigcap_{s>t} V_s.$$

We then proceed as in the proof of Theorem 5.2. Namely assume that V_{0^+} (which is a non random vector space) is $\neq T_{x_0}(\mathbf{R}^d)$. Then if S is the stopping time:

$$S=\inf\{t>0;\ V_t \neq V_{0^+}\},$$

S is >0 a. s. Let f be a non-zero element of $T^*_{x_0}(\mathbf{R}^d)$ orthogonal to V_0^+. Then:

$$\left\{\begin{aligned} &\langle f, (\varphi_u^{*-1} X_i)(x_0)\rangle=0 \quad \text{on } (z_u>0) \cap [0, S],\\ &\langle f, (\varphi_u^{*-1} X'_i)(x_0)\rangle=0 \quad \text{on } (z_u<0) \cap [0, S]. \end{aligned}\right. \tag{6.29}$$

Using the optional selection Theorem [11]-IV-84, it is easily proved that $(z=0)$ is included in both closures of $(z>0)$ and $(z<0)$. (6.29) can be replaced by:

$$\left\{\begin{aligned} &\langle f, (\varphi_u^{*-1} X_i(x_0)\rangle=0 \quad \text{on } (z_u \geqq 0) \cap [0, S[,\\ &\langle f, (\varphi_u^{*-1} X'_i)(x_0)\rangle=0 \quad \text{on } (z_u \leqq 0) \cap [0, S[. \end{aligned}\right. \tag{6.30}$$

From (6.29), we find that:

$$z_t^+ \langle f, \varphi_u^{*-1} X_i)(x_0)\rangle=0 \quad \text{on } [0, S]. \tag{6.31}$$

From Itô-Tanaka's formula, we know that:

$$(6.32)\quad z_t^+ \varphi_t^{*-1} X_i(x_0) = \int_0^t z_s^+ \varphi_s^{*-1}\Big([X_0, X_i] + \frac{1}{2}[X_j, [X_j, X_i]] + \frac{1}{2}\Big[\frac{\partial}{\partial z}, \Big[\frac{\partial}{\partial z}, X_i\Big]\Big]\Big) ds + \int_0^t z_s^+ \varphi_s^{*-1}[X_j, X_i]\,\delta w^j + \int_0^t 1_{z>0}\,\varphi_s^{*-1}\Big(X_i + z^+\Big[\frac{\partial}{\partial z}, X_i\Big]\Big)\delta z + \int_0^t 1_{z>0}\,\varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big] ds$$

(there is no integral $\int_0^t \ldots dL$ because the support of dL is $(z=0)$, and of (6.30)). From (6.32), we find easily that for $1 \leqq j \leqq m$:

$$(6.33)\quad \begin{cases} \langle f, \varphi_s^{*-1}[X_j, X_i]\rangle = 0 & \text{on } (z_s>0) \cap [0, S], \\ \Big\langle f, \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\Big\rangle = 0 & \text{on } (z_s>0) \cap [0, S]. \end{cases}$$

By iteration, using (6.33) again as in (5.8), and reasoning as in (6.30), we find that for $0 \leqq j \leqq m$:

$$(6.34)\quad \begin{cases} \langle f, \varphi_s^{*-1}[X_j, X_i]\rangle = 0 & \text{on } (z_s \geqq 0) \cap [0, S[, \\ \Big\langle f, \varphi_s^{*-1}\Big[\frac{\partial}{\partial z}, X_i\Big]\Big\rangle = 0 & \text{on } (z_s \geqq 0) \cap [0, S[. \end{cases}$$

We now will use the following result in Ikeda-Watanabe [17], p. 307. Namely if g is $\{\bar{F}_t\}_{t \geqq 0}$ predictable right-continuous process with left hand limits, then for any $t \geqq 0$:

$$(6.35)\quad \begin{cases} \lim\limits_{\varepsilon \downarrow\downarrow 0}\Big[\sum\limits_{A_u - A_{u^-} > \varepsilon} \int_{A_u^- \wedge t}^{A_u \wedge t} 1_{z>0}\, g\, \delta w^i\Big] = \int_0^t 1_{z>0}\, g\, \delta w^i, \\ \lim\limits_{\varepsilon \downarrow\downarrow 0}\Big[\sum\limits_{A_u - A_{u^-} > \varepsilon} \int_{A_{u^-} \wedge t}^{A_u \wedge t} 1_{z<0}\, g\, \delta w^i\Big] = \int_0^t 1_{z<0}\, g\, \delta w^i, \end{cases}$$

where the limits as $\varepsilon \downarrow\downarrow 0$ are taken in probability.

In [17] such a result is proved in the case of a reflecting Brownian motion. The proof of [17] can be mimicked so that (6.35) holds. Also note that if c_t, κ_t are the processes:

$$(6.36)\quad \begin{cases} c_t = \int_0^t 1_{z>0}\, ds, \\ \kappa_t = \inf\{\gamma, c_\gamma > t\}, \end{cases}$$

then by [17], p. 123, z_{κ_t} is a reflecting Brownian motion, and moreover $\int_0^{\kappa_t} 1_{z>0}\,\delta w^i$ $(1 \leqq i \leqq m)$ are also Brownian motions independent of z_{κ_t}.

Let $H(x)$ be a C^∞ vector field defined on R^d with values in R^d. Assume that:

$$(6.37)\qquad \langle f, \varphi_t^{*-1} H\rangle = 0 \quad \text{on } (z_t=0)\cap [0, S[,$$

We claim that:

$$(6.38)\qquad \begin{cases} \langle f, \varphi_t^{*-1}[D, H]\rangle = 0 \quad \text{on } (z_t=0)\cap [0, S[, \\ \langle f, \varphi_t^{*-1}[X_j, H]\rangle = 0 \quad \text{on } (z_t=0)\cap [0, S[, \qquad 1\leqq j\leqq m \\ \langle f, \varphi_t^{*-1}[X'_j, H]\rangle = 0 \quad \text{on } (z_t=0)\cap [0, S[, \qquad 1\leqq j\leqq m. \end{cases}$$

We first prove the first line of (6.38). We have:

$$(6.39)\qquad \varphi_t^{*-1} H = H(x_0) + \int_0^t 1_{z>0}\, \varphi_u^{*-1}\left([X_0, H] + \frac{1}{2}[X_j, [X_j, H]]\right) du$$
$$+ \int_0^t 1_{z<0}\, \varphi_u^{*-1}\left([X'_0, H] + \frac{1}{2}[X'_j, [X'_j, H]]\right) du$$
$$+ \int_0^t \varphi_u^{*-1}[D, H]\, dL + \int_0^t 1_{z>0}\, \varphi_u^{*-1}[X_j, H]\, \delta w^j$$
$$+ \int_0^t 1_{z<0}\, \varphi_u^{*-1}[X'_j, H]\, \delta w^j.$$

Let G_t, G'_t be the $\{\bar F_t\}_{t\geqq 0}$ predictable processes [11]-IV-90:

$$(6.40)\qquad \begin{cases} G_t = \overline{\lim}_{s\uparrow\uparrow t}\, 1_{z_s>0}, \\ G'_t = \overline{\lim}_{s\uparrow\uparrow t}\, 1_{z_s<0}. \end{cases}$$

We claim that for any $t\geqq 0$:

$$(6.41)\qquad \sum \int_{A_{u^-}\wedge t\wedge S}^{A_u\wedge t\wedge S} 1_{z>0}\, \delta\langle f, \varphi_u^{*-1} H\rangle = G_{t\wedge S}\langle f, \varphi_{t\wedge S}^{*-1} H\rangle.$$

In fact:

• if $t<S$, if $z_t>0$, using (6.37), the sum is $\langle f, \varphi_t^{*-1} H\rangle$, $(z_t=0)$ is negligible, and if $z_t<0$, the sum is 0.

• if $t\geqq S$, if $z_S>0$, the sum is $\langle f, \varphi_S^{*-1} H\rangle$.

If $z_S=0$, and if S is a left cluster point of $(z=0)$, the sum is still 0, and moreover by (6.37), $\langle f, \varphi_S^{*-1}\cdot H\rangle=0$. If $z_S=0$ and z is >0 on a left neighborhood of S, the sum is $<f, \varphi_S^{*-1} H\rangle$ and $f_S=1$. If $z_S=0$ and if z is <0 on a left neighborhood of S, both sides of (6.41) are 0. Finally if $z_S<0$, the l. h. s. of (6.41) is 0 and $G_S=0$.

Let E_t, E'_t be the processes:

$$
(6.42)\quad \left\{\begin{aligned}
E_t &= \int_0^t 1_{z>0}\left(\left\langle f,\ \varphi_u^{*-1}\left([X_0, H]+\frac{1}{2}[X_j, [X_j, H]]\right)\right\rangle du\right. \\
&\qquad + \int_0^t 1_{z>0}\langle f,\ \varphi_u^{*-1}[X_j, H]\rangle\, \delta w^j, \\
E'_t &= \int_0^t 1_{z<0}\left(\left\langle f,\ \varphi_u^{*-1}\left([X'_0, H]+\frac{1}{2}[X'_j, [X'_j, H]]\right)\right\rangle du\right. \\
&\qquad + \int_0^t 1_{z<0}\langle f,\ \varphi_u^{*-1}[X'_j, H]\rangle\, \delta w^j.
\end{aligned}\right.
$$

By using line 1 in (6.35) as well as (6.41), we find that for any $t \geqq 0$, a. s.:

$$
(6.43)\qquad E_{t\wedge S} = G_{t\wedge S}\langle f,\ \varphi_{t\wedge S}^{*-1} H\rangle.
$$

Now the process $G_{t\wedge S}\langle f,\ \varphi_{t\wedge S}^{*-1} H\rangle$ is continuous. This is clear if $t<S$, by using (6.37). If S is a left cluster point of $(z=0)$, $\langle f,\ \varphi_S^{*-1} X_i\rangle = 0$ and continuity at S still holds, while if S is isolated on the left from $(z=0)$, G will be continuous at S. From (6.43), we find that a. s.:

$$
(6.44)\qquad E_t = G_t\langle f,\ \varphi_t^{*-1} H\rangle \quad \text{on } [0, S].
$$

Similarly:

$$
(6.45)\qquad E'_t = G'_t\langle f,\ \varphi_t^{*-1} H\rangle \quad \text{on } [0, S].
$$

We claim that for $t \leqq S$:

$$
(6.46)\qquad (G_t + G'_t)\langle f,\ \varphi_t^{*-1} H\rangle = \langle f,\ \varphi_t^{*-1} H\rangle.
$$

We only need to prove (6.46) if $z_t = 0$. If $t<S$, $\langle f,\ \varphi_t^{*-1} H\rangle = 0$ and (6.46) is true. If $t=S$, and S is a cluster point on the left of $(z=0)$, the same reasoning applies. If S is not a cluster point on the left of $(z=0)$, $G_S + G'_S = 1$, and (6.46) still holds. From (6.44)-(6.46), we see that:

$$
(6.47)\qquad \langle f,\ \varphi_t^{*-1} H\rangle = E_t + E'_t \quad \text{on } [0, S].
$$

Comparing with (6.39), we find that:

$$
(6.48)\qquad \left\langle f,\ \int_0^t \varphi_u^{*-1}[D, H]\, dL \quad = 0 \quad \text{on } [0, S].\right\rangle
$$

so that:

$$
(6.49)\qquad \langle f,\ \varphi_t^{*-1}[D, H]\rangle = 0 \quad \text{on } (z=0) \cap [0, S[.
$$

The first line in (6.38) has been proved.

Now from (6.37), (6.44), it is clear that:

$$E_t=0 \quad \text{on } (z=0)\cap [0, S[. \tag{6.50}$$

c_t, κ_t have been defined in (6.36). Set:

$$(6.51)\quad \left\{ \begin{aligned} & K_s=\langle f, \varphi_s^{*-1}\left([X_0, H]+\frac{1}{2}[X_j, [X_j, H]]\right)\rangle, \\ & \bar{z}_t=z_{\kappa_t}, \quad \bar{w}_t^i=\int_0^{\kappa_t} 1_{z>0}\,\delta w^i, \quad \bar{B}_t=\int_0^{\kappa_t} 1_{z>0}\,\delta z, \\ & \bar{K}_t=K_{\kappa_t}, \quad \bar{E}_t=E_{\kappa_t}. \end{aligned} \right.$$

We know that $\bar{z}$ is a reflecting Brownian motion, and that $(\bar{w}^1, \ldots, \bar{w}^m, \bar{B})$ is a $\{\bar{F}_{\kappa_t}\}_{t\geq 0}$ Brownian martingale. Moreover c_S is a $\{\bar{F}_{\kappa_t}\}_{t\geq 0}$ stopping time. Using (6.50), we have:

$$\bar{E}_t=0 \quad \text{on } (\bar{z}=0)\cap [0, c_S[. \tag{6.52}$$

Moreover using standard results on semi-martingales we know that $\bar{E}_t$ is a continuous process. Using (6.42), we find that:

$$\bar{E}_t=\int_0^t \bar{K}_s\,ds+\int_0^t \langle f, \varphi_{\kappa_s}^{*-1}[X_j, H]\rangle\,\delta\bar{w}^j. \tag{6.53}$$

We can now proceed as in the proof of Theorem 5.2.

Using (6.52), (6.53) and Theorem 2.1 in [51], we know that for $1\leqq j\leqq m$:

$$\langle f, \varphi_{\kappa_s}^{*-1}[X_j, H]\rangle=0 \quad \text{on } (\bar{z}=0)\cap [0, c_S[. \tag{6.54}$$

Note that the result of [51] is proved under the assumption that all the integrands are continuous processes, but the proof adapts without any change when they are only right-continuous.

It is easy to see that (6.54) implies the second line in (6.38). The third line of (6.38) is proved in the same way.

Of course we may take H to be equal to $X_i(x, 0)$, $X_i'(x, 0)$ or to any Lie bracket appearing in the iteration of (6.34), (6.38).

It is now easy to proceed as in the proof of Theorem 5.2 and prove the Theorem. □

Remark 1. — If $X_1 \ldots X_m$, $X_1' \ldots X_m'$ do not depend on z, F_{l+1} can be enlarged to be:

$$F_{l+1}=[(X_0', \ldots, X_m', D), E_l]\cup[(X_0, \ldots, X_m, D), E_l'] \cup[(X_0, \ldots, X_m, X_0', \ldots, X_m', D), F_l].$$

In fact note that from Theorem 2.3 in [51], we find that because of (6.52), (6.53):

$$\bar{K}_s=0 \quad \text{on } (\bar{z}=0)\cap [0, c_S[.$$

It is then easy to proceed as in the proof of Theorem 6.6 with the enlarged F_l.

Of course, under the assumptions of Theorem 6.6, the analogue of Theorem 4.13 holds.

(*f*) *Local and non local regularity of the boundary semi-group.*

We now study systematically the regularity of the boundary semi-group associated to the two-sided reflecting diffusion.

E_l, E'_l have been defined in Definition 6.5. k_l has been defined in Definition 5.7 (for the family of vector fields E_l). Similarly we can define k'_l calculated on E'_l.

THEOREM 6.7. — *If $x_0 \in R^d$ is such that for a given $l \in N$, $\theta > 0$:*

$$(6.55) \qquad \begin{cases} \lim_{z>0, z\to 0} z \operatorname{Log}[\inf_{|x-x_0|\leqq\theta} k^l(x, z)] = 0, \\ \lim_{z<0, z\to 0} z \operatorname{Log}[\inf_{|x-x_0|\leqq\theta} k^l(x, z)] = 0, \end{cases}$$

then for any $t>0$, $T \geqq 0$, $1_{A_t \leqq T} \mid [C^{x_0}_{A_t}]^{-1} \mid$ is in all the $L_p(\bar{\Omega}, P \otimes P')$.

Proof. — The proof is strictly identical to the proof of Theorem 5.8. Of course the definition of T^{γ}_1 in (5.41) is changed into:

$$(6.56) \qquad T^{\gamma}_1 = \inf\{ t \geqq 0; \mid z_t \mid = \gamma \}$$

and the estimate (5.42) is still valid. The proof can proceed as the proof of Theorem 5.8 because (6.55) is a two-sided assumption, so that in both cases $(z_{T^{\gamma}_1} = \gamma)$ and $(z_{T^{\gamma}_1} = -\gamma)$, the estimates of the proof of Theorem 5.8 are valid. □

Assume now that $x_0 \in R^d$ is such that for a given $\theta > 0$, $\eta > 0$ exists such that:

$$(6.57) \qquad \inf_{|x-x_0|\leqq\theta} k^1(x, z) \geqq \eta,$$

which means that on $(z_s > 0)$, x_s is an elliptic diffusion for s small enough. By Theorem 6.6, we know that the boundary semi-group has densities.

However, we are going to show that if $\theta < +\infty$ — i.e. the assumption (6.57) is *local* — and if on $(z < 0)$, the diffusion x_s is very degenerate, then the boundary semi-group may well be not smooth. The introduction of badly behaved negative excursions destroys then the regularity result of Theorem 5.8.

The reader can assume that $X'_0 \neq 0$, $X'_1 = \ldots = X'_m = 0$.

The estimates on $C^{x_0}_t$ are then only possible on the positive excursions of z. The stopping time T^{γ}_1 should now be replaced by:

$$(6.58) \qquad T'^{\gamma}_1 = \inf\{ t \geqq 0, z_t = \gamma \},$$

where z is a standard Brownian motion. Now contrary to the sharp estimate (5.42), we have:

$$\bar{P}[T_1^{\gamma} \geqq t\,\gamma\,\sqrt{2}] \leqq \frac{C\,\gamma}{(t\,\gamma\,\sqrt{2})^{1/2}} = \frac{C\,\gamma^{1/2}}{(t\,\sqrt{2})^{1/2}}, \tag{6.59}$$

This follows from the fact that $\sup_{0 \leqq s \leqq h} z_s$ has the same law as $|z_h|$.

(6.59) is an insufficient bound. In fact, to dominate $\bar{P}(|[C_{A_t}^{-1}]| \geqq \lambda;\ A_t \leqq T)$ by A/λ^p we should take $\gamma \leqq A/\lambda^{2p}$, but this makes the estimation (5.45) lousy.

The phenomenon which can happen is that the negative excursions of z push x far from the region $|x-x_0| \leqq \theta$ fast enough so that the semi-group is prevented from getting smooth.

Although it is difficult to construct an explicit counter-example, we build something which is very close to that.

T is $a>0$ real. Consider first the stochastic differential equation:

$$\left\{\begin{aligned} dx &= 1_{t \leqq T}\, dw_t^1, \\ x(0) &= 0, \end{aligned}\right. \tag{6.60}$$

which can be put in the equivalent form:

$$\left\{\begin{aligned} dx &= 1_{h \leqq T}\, dw^1, \\ x(0) &= 0, \\ dh &= dt, \\ h(0) &= 0. \end{aligned}\right. \tag{6.61}$$

Of course the system (6.61) is not smooth in the variable h. However the calculus of variations of section 2 can be applied to (6.61). $\partial\varphi_t/\partial x(\omega, 0)$ is of course equal to I, so that we can analyse the process x_{A_t}, without needing to look at the component A_t. Of course, C_t^0 is given by:

$$C_t^0 = t \wedge T. \tag{6.62}$$

Since the law of A_t is classically ([18], p. 26):

$$1_{s \geqq 0}\, \frac{t}{\sqrt{2\,\pi\, s^3}}\, e^{-t^2/2s}\, ds, \tag{6.63}$$

we find that for any $t>0$, $1/C_{A_t}^0$ is in all the $L_p(\bar{\Omega}, P \otimes P')$.

The calculus of variations tells us that x_{A_t} has a smooth law. Of course this is entirely obvious because $x_{A_t} = w_{A_t \wedge T}^1$, and the law of $w_{A_t \wedge T}^1$ can be explicitly calculated.

Now consider the two-sided equation:

$$\left\{\begin{aligned} dx &= 1_{t \leqq T}\, 1_{z>0}\, dw_t^1, \\ x(0) &= 0, \end{aligned}\right. \tag{6.64}$$

which can be rewritten:

$$\left\{\begin{array}{ll} dx = 1_{z>0}\, 1_{h \leqq T}\, dw_t^1, & x(0)=0, \\ dh = 1_{z>0}\, dt + 1_{z<0}\, dt, & h(0)=0 \end{array}\right. \tag{6.65}$$

(6.65) appears as a two-sided perturbation of (6.61). We claim that the law of x_{A_t} is not smooth. In fact C_t^0 is now:

$$C_t^0 = \int_0^{t \wedge T} 1_{z>0}\, ds.$$

Conditionaly on z, the law of x_{A_t} is a centered gaussian whose variance is $C_{A_t}^0$, i. e. is given by:

$$\frac{1}{\sqrt{2\pi C_{A_t}^0}}\, e\, \frac{-x^2}{2 C_{A_t}^0}\, dx.$$

The law of x_{A_t} is then given by $k(x)\,dx$, where:

$$k(x) = \int_{\Omega'} \frac{1}{\sqrt{2\pi C_{A_t}^0}} \exp \frac{-x^2}{2 C_{A_t}^0}\, dP'(\omega').$$

Now k is clearly C^∞ for $x \neq 0$, and moreover $k(0) = \lim_{\substack{x \to 0 \\ x \neq 0}} k(x)$, so that:

$$k(0) = \int_{\Omega'} \frac{1}{\sqrt{2\pi C_{A_t}^0}}\, dP'(\omega').$$

Of course $C_{A_t}^0 \leqq C_T^0$. Moreover by Lévy's Arcsine law (Itô-McKean [18], p. 57), the law of C_T^0 is given by:

$$1_{0 \leqq s \leqq T} \frac{ds}{\pi [s(T-s)]^{1/2}},$$

so that:

$$k(0) \geqq \frac{1}{\sqrt{2\pi}} \int_0^T \frac{ds}{\pi\, s\, (T-s)^{1/2}} = +\infty$$

$k(x)$ is then not even continuous at 0! In fact $1/C_{A_t}^0$ does not belong to any $L_p(\bar{\Omega}, P \otimes P')$ $(1 \leqq p < +\infty)$ because $\int_{\Omega'} 1/(C_{A_t}^0)^{1/2}\, dP' = +\infty$.

The effect of introducing negative excursions of z prevents the process x_t to take "advantage" of the region $t \leqq T$ to have a regular density, since the negative excursions are pushing it far enough from this region.

Remark 2. — If $X'_0=X'_1=\ldots=X'_n=0$, we come back to the situation studied in sections 1-5. In fact on $(z<0)$, x does not move. Using the fact that if τ_t is the time change:

$$\tau_t=\inf\left\{\tau;\ \int_0^\tau 1_{z>0}\,ds>t\right\}$$

z_{τ_t} is a reflecting Brownian motion whose local time is L_{τ_t} (*see* Ikeda-Watanabe [17], p. 123), we would be back to the situation studied in sections 1-5.

(*g*) *Non-local regularity of the boundary process.*

The results of section 6-*f* show that when the two sides of the diffusion x_t are not equally regularizing, then the regularity of the boundary semi-group is in general *a non local* property.

We will now show that under a condition which states basically that if $l\in N$ exists such that for any $x_0\in R^d$, $\bigcup_1^l (E_n\cup E'_n)(x_0, 0)$ spans R^d, the boundary semi-group is smooth.

The reasoning in the proof of Theorem 5.8 does not work any more. Note first that the estimate (5.42) is useless here. In fact if $T_1''^\gamma$ is the hitting time of γ by $|z_t|$, it may well be that if $z_{T_1''^\gamma}=-\gamma$:

$$\sum_{j=1}^{l}\sum_{Y'\in E'_j}\langle f, \varphi^{*-1}_{T_1''^\gamma}\,Y'(x_0)\rangle^2,$$

is small if for instance f is orthogonal to $\bigcup_1^l E'_n(x_0)$. Of course in this case:

$$\sum_{j=1}^{l}\sum_{Y\in E_j}\langle f, Y(x_0)\rangle^2,$$

will be large, and it would be "better" to choose the stopping time $T_1'^\gamma$ defined in (6.58) instead of $T_1''^\gamma$. However the estimate (6.59) is lousy.

The reason for which the arguments of section 5 do not work any more is that the stochastic calculus in real time-scale is not good enough to take into account the fact that the regularity of the boundary semi-group comes from piling up heterogeneous excursions. This fact could not be clearly seen in Theorem 5.10, where there were excursions of only one type. We will then do an analysis of the individual excursions of the two-sided process.

DEFINITION 6.8. — For $e\in\mathscr{W}^+$ (resp. $\mathscr{W}^-$), $\Psi_{\cdot}(\varepsilon, e, .)$ [resp. $\Psi'_{\cdot}(\varepsilon, e, .)$] is the flow of diffeomorphisms of R^d associated to the stochastic differential equation on (Ω, P):

(6.66) $$dx=X_0(x, e)\,dt+X_i(x, e)\,.\,d\varepsilon^i.$$

[resp. (6.66') $dx=X'_0(x, e)\,dt+X'_i(x, e)\,.\,d\varepsilon^i$].

Of course the notation (ε, e) is used to underline the fact that we are working on excursions.

DEFINITION 6.9. — $\bar{C}_t^x(\varepsilon, e)$ [resp. $\bar{C}_t'^x(\varepsilon, e)$] is the linear mapping from $T_x^*(R^d)$ into $T_x(R^d)$:

$$(6.67) \qquad f \in T_x^* R^d \to \bar{C}_t^x(\varepsilon, e) f = \sum_{i=1}^{m} \int_0^t \langle (\Psi_s^{*-1} X_i)(x), f \rangle (\Psi_s^{*-1} X_i)(x)\, ds$$

$\Bigg($ resp:

$$(6.67') \qquad f \in T_x^* R^d \to \bar{C}_t'^x f = \sum_{i=1}^{m} \int_0^t \langle (\Psi_s'^* X_i')(x), f \rangle (\Psi_s'^{*-1} X_i')(x)\, ds \Bigg).$$

Recall that $m_l = 20^{l-1} \times 6$. η is a >0 real number.

We have then the key result.

THEOREM 6.10. — *For any $\eta > 0$, there exists $C > 0$, $\mu > 0$ such that for any $x \in R^d$, any $f \in T_x^* R^d$ with $\|f\| = 1$, then if:*

$$(6.68) \qquad \sum_{n=1}^{l} \sum_{Y \in E_n} \langle f, Y(x, 0) \rangle^2 \geqq \frac{\eta}{2}$$

$\Bigg($ *resp.:*

$$(6.68') \qquad \sum_{n=1}^{l} \sum_{Y' \in E_n'} \langle f, Y'(x, 0) \rangle^2 \geqq \frac{\eta}{2} \Bigg),$$

then for $0 < \rho < \mu$:

$$(6.69) \qquad n(\langle \bar{C}_{\sigma(e)}^x(\varepsilon, e) f, f \rangle \geqq \rho) \geqq \frac{C}{\rho^{1/m_l}}$$

$\Bigg($ *resp.:*

$$(6.69') \qquad n(\langle \bar{C}_{\sigma(e)}'^x(\varepsilon, -e) f, f \rangle \geqq \rho) \geqq \frac{C}{\rho^{1/m_l}} \Bigg).$$

Proof. — We will only prove (6.69). We will use the description of n^+ by Williams ([45]-[46]) and Rogers [33] given in Theorem 3.6. The measure n has been described in Theorem 3.9.

χ is a >0 real number such that $\chi \leqq 1/2$. The value of χ will be chosen later. Let T_0, T_1^ρ be the stopping times:

$$(6.70)\quad T_0=\inf\{t\geqq 0;\ |\Psi_t(\varepsilon, e, x)-x|\geqq\chi\}\ |\wedge\inf\{t\geqq 0;\ \left\|\left[\frac{\partial\Psi_t}{\partial x}(\varepsilon, e, x)\right]^{-1}-I\,|\geqq\chi\right\|.$$

$$T_1^\rho(e)=\inf\{t\geqq 0;\ e_t=\rho^{1/m_l}\}.$$

By fixing temporarily e in (6.66), we have:

$$(6.71)\qquad P[T_0(\varepsilon, e)\leqq T_1^\rho(e)]\leqq C\exp-\frac{(\chi-CT_1^\rho(e))^{+2}}{CT_1^\rho(e)}\leqq C_\chi(T_1^\rho(e)\wedge 1).$$

By Theorem 3.6, under n and conditionally on $(T_1^\rho(e)<+\infty)$, $e_t(0\leqq t\leqq T_1^\rho(e))$ is a Bes (3) process starting at 0, stopped when it hits ρ^{1/m_l}. Since we have the obvious:

$$(6.72)\qquad E^{Bes(3)}\,T_1^\rho(r)=\frac{\rho^{2/m_l}}{3},$$

we get from (6.71) that:

$$(6.73)\qquad n(T_0(\varepsilon, e)\leqq T_1^\rho(e)\,|\,T_1^\rho(e)<+\infty)\leqq C_\chi\,\rho^{2/m_l}.$$

We define $\bar{C}_{s'}^s(\varepsilon, e)(s\leqq s')$ in the same way as in (5.33) (we drop the dependence on x for simplicity).

Set:

$$(6.74)\qquad g(\varepsilon, e)=\widetilde{\left[\frac{\partial\Psi_{T_1^\rho}}{\delta x}(\varepsilon, e, x)\right]^{-1}f}.$$

Clearly:

$$(6.75)\quad n(\langle\bar{C}_{\sigma(e)}^x f, f\rangle\leqq\rho\,|\,T_1^\rho<+\infty)$$
$$\leqq C_\chi\,\rho^{2/m_l}+n(\langle\bar{C}_\sigma^{T_1^\rho}g, g\rangle\leqq\rho,\ T_1^\rho\leqq T_0\,|\,T_1^\rho<+\infty).$$

Recall that the components of $X_0, X_1\ldots X_m$ are in $C_b^\infty(R^{d+1})$, so that for $\rho^{1/m_l}\leqq\chi$, if $T_1^\rho\leqq T_0$:

$$(6.76)\qquad \sum_{n=1}^{l}\sum_{Y\in E_n}\langle g(\varepsilon, e), Y(\Psi_{T_1^\rho}(\varepsilon, e, x), \rho^{1/m_l})\rangle^2\geqq\frac{\eta}{2}-C\,\chi^2.$$

We choose χ so that $C\chi^2\leqq\eta/4$, and so the l. h. s. of (6.76) is $\geqq\eta/4$.

Set:

$$(6.77)\qquad g'(\varepsilon, e)=\frac{g(\varepsilon, e)}{\|g(\varepsilon, e)\|}.$$

Of course if $T_1^\rho \leqq T_0$, since $\chi \leqq 1/2$:

$$\frac{1}{2} \leqq \| g(\varepsilon, e) \| \leqq \frac{3}{2}, \tag{6.78}$$

so that:

$$\sum_{n=1}^{l} \sum_{Y \in E_n} \langle g'(\varepsilon, e), Y(\Psi_{T_1^\rho}(\varepsilon, e, x), \rho^{1/m_l}) \rangle^2 \geqq \frac{\eta}{9} \tag{6.79}$$

and moreover:

$$n(\langle \bar{C}_\sigma^{T_1^\rho} g, g \rangle \leqq \rho, T_1^\rho \leqq T_0 \mid T_1^\rho < +\infty) \leqq n(\langle \bar{C}_\sigma^{T_1^\rho} g', g' \rangle \leqq 4\rho, T_1^\rho \leqq T_0 \mid T_1^\rho < +\infty). \tag{6.80}$$

Recall that the canonical filtration $\{\mathscr{G}_t\}_{t \geqq 0}$ on $\mathscr{W}_0$ has been defined in (3.50).

We know that for the conditional law $n(. \mid T_1^\rho < +\infty)$, conditionally on $\mathscr{G}_{T_1^\rho}$, for $t \geqq T_1^\rho$, (ε_t, e_t) is a $m+1$-dimensional Brownian motion stopped when ε hits 0 (the result on ε_t is the basis of Williams ([45]-[46]), Rogers [33]).

On $(\bar{\Omega}, P \otimes P'_{\rho^{1/m_l}})$ let S be the stopping time:

$$S = \inf \{ t \geqq 0; \mid z_t - \rho^{1/m_l} \mid = \rho^{1/m_l} \}. \tag{6.81}$$

Let $x' \in R^d$, and $h \in T_{x'}^*(R^d)$ such that $\| h \| = 1$, and moreover:

$$\sum_{n=1}^{l} \sum_{Y \in E_n} \langle h, Y(x', \rho^{1/m_l}) \rangle^2 \geqq \frac{\eta}{9}. \tag{6.82}$$

We are then led to estimate:

$$(P \otimes P'_{\rho^{1/m_l}})(\langle \bar{C}_S^{x'}(\bar{\omega}) h, h \rangle \leqq 4\rho). \tag{6.83}$$

Of course the estimation of (6.83) will be used in (6.80), with $x' = \Psi_{T_1^\rho}(\varepsilon, e, x)$, $h = g'(\varepsilon, e)$.

Let χ' be a real number such that $0 < \chi' \leqq 1/2$.

Let U be the stopping time:

$$U = \inf \{ t \geqq 0; \mid \Psi_t(\bar{\omega}, x') - x' \mid \geqq \chi' \} \wedge \inf \left\{ t \geqq 0; \left| \left[\frac{\partial \Psi_t}{\partial x}(\bar{\omega}, x') \right]^{-1} - I \right| \geqq \chi' \right\} \wedge S. \tag{6.84}$$

For $t \leqq U$ and $2\rho^{1/m_l} \leqq \chi'$, we get, still using the fact that the components of $X_0, X_1 \dots X_m$ are in $C_b^\infty(R^{d+1})$:

$$\sum_{n=1}^{l} \sum_{Y \in E_n} \langle \Psi_t^{*-1} Y(x', z_t), h \rangle^2 \geqq \frac{\eta}{9} - C\chi'^2. \tag{6.85}$$

We choose χ' so that $C\chi'^2 \leqq \eta/18$.

Now for ρ small enough:

$$(6.86)\quad (P\otimes P'_{\rho^{1/m_l}})(U\leqq D_1(4\rho)^{3/m_l})\leqq C\exp\left[-\frac{C'\rho^{2/m_l}}{(4\rho)^{3/m_l}}\right]\leqq C\exp\left[-\frac{C''}{\rho^{1/m_l}}\right],$$

so that:

$$(6.87)\quad (P\otimes P'_{\rho^{1/m_l}})(<\bar{C}_S^{x'}(\bar{\omega})h, h>\leqq 4\rho)\leqq C\exp\left[-\frac{C''}{\rho^{1/m_l}}\right] + (P\otimes P'_{\rho^{1/m_l}})(\langle \bar{C}^{x'}_{D_1(4\rho)^{3/m_l}}(\bar{\omega})h, h\rangle \leqq 4\rho;\ U\geqq D_1(4\rho)^{3/m_l}).$$

Using the estimate (5.29) and (6.85), we know that for ρ small enough, the second term in the r. h. s. of (6.87) is dominated by $K\exp[-D_3(4\rho)^{-\alpha/m_l}]$.

So for ρ small enough:

$$(6.88)\quad (P\otimes P'_{\rho^{1/m_l}})(\langle \bar{C}_S^{x'}(\bar{\omega})h, h\rangle\leqq 4\rho)\leqq C\left[\exp-\frac{C''}{\rho^{1/m_l}}+\exp-\frac{C''}{\rho^{\alpha/m_l}}\right].$$

Note that the estimate (6.88) is uniform in $x'\in R^d$ as long as (6.85) is satisfied, essentially because the components of $X_0, \ldots, X_m$ are in $C_b^\infty(R^{d+1})$.

Now since $(T_1^\rho\leqq T_0)$, $\Psi_{T_1^\rho}(\varepsilon, e, x)$, $g'(\varepsilon, e)$ are $\mathscr{G}_{T_1^\rho}$-measurable; since moreover $\|g'(\varepsilon, e)\|=1$, we can use the estimate (6.88) in conditional form as previously indicated, so that:

$$(6.89)\quad n(\langle \bar{C}_\sigma^{T_1^\rho}g', g'\rangle\leqq 4\rho,\ T_1^\rho\leqq T_0\,|\,T_1^\rho<+\infty)\leqq C\left(\exp-\frac{C''}{\rho^{1/m_l}}+\exp-\frac{C''}{\rho^{\alpha/m_l}}\right).$$

Using (6.75), (6.80), (6.89), we get for ρ small enough:

$$(6.90)\quad n(\langle \bar{C}^x_{\sigma(e)}f, f\rangle\geqq\rho\,|\,T_1^\rho<+\infty)\geqq 1-C\rho^{2/m_l}-C\left(\exp-\frac{C''}{\rho^{1/m_l}}+\exp-\frac{C''}{\rho^{\alpha/m_l}}\right).$$

By Theorem 3.6, we know that:

$$(6.91)\qquad n(T_1^\rho<+\infty)=\frac{1}{\rho^{1/m_l}}$$

(6.69) follows from (6.90) and (6.91). □

Definition 6.11. — For $l\in N$, $\chi^l(x)$ is the function defined on R^d by:

$$(6.92)\quad \chi^l(x)=\inf_{\|f\|=1}\left[\sum_{n=1}^{l}\left(\sum_{Y\in E_n}\langle f, Y(x,0)\rangle^2+\sum_{Y'\in E'_n}\langle f, Y'(x,0)\rangle^2\right)\right].$$

Note that if $x \in \mathrm{R}^d$ is such that $\chi^l(x) > 0$, this means exactly that $\bigcup_{n=1}^{l} (\mathrm{E}_n \cup \mathrm{E}'_n)(x, 0)$ spans R^d.

We have then the essential result of this section.

THEOREM 6.12. — *Assume that* $l \in \mathrm{N}$, $\eta > 0$ *exist so that for any* $x \in \mathrm{R}^d$:

$$\chi^l(x) \geqq \eta. \tag{6.93}$$

Then for any $x_0 \in \mathrm{R}^d$, $\mathrm{T} \geqq 0$, $t > 0$, $1_{\mathrm{A}_t \leqq \mathrm{T}} \,|\, [\mathrm{C}^{x_0}_{\mathrm{A}_t}]^{-1} \,|$ *is in all the* $\mathrm{L}_p(\bar{\Omega}, \mathrm{P} \otimes \mathrm{P}'_{z_0})$.

Proof. — We will write $\bar{\mathrm{P}}$ instead of $\mathrm{P} \otimes \mathrm{P}'$. Take $f \in \mathrm{T}^*_{x_0}(\mathrm{R}^d)$ such that $\|f\| = 1$. We estimate first for $\delta > 0$:

$$\bar{\mathrm{P}}[\langle \mathrm{C}^{x_0}_{\mathrm{A}_t} f, f \rangle \leqq \delta, \mathrm{A}_t \leqq \mathrm{T}]. \tag{6.94}$$

x_s is again the process $\varphi_s(\bar{\omega}, x_0)$. Let T_δ be the $\{\bar{\mathrm{F}}_t\}_{t \geq 0}$ stopping time:

$$\mathrm{T}_\delta = \inf\left\{ t \geqq 0; \left[\left| \frac{\partial \varphi_t}{\partial x}(\bar{\omega}, x_0) \right| \vee \left| \left[\frac{\partial \varphi_t}{\partial x}(\bar{\omega}, x_0) \right]^{-1} \right| \right] \geqq \frac{1}{\delta^{1/4}} \right\}. \tag{6.95}$$

Now $\mathrm{L}_{\mathrm{T}_\delta}$ is a $\{\bar{\mathrm{F}}_{\mathrm{A}_t}\}_{t \geq 0}$ stopping time. Moreover by Proposition 3.13, $\partial \varphi_{\mathrm{A}_t-}/\partial x\,(\bar{\omega}, x_0)$ is a $\{\bar{\mathrm{F}}_{\mathrm{A}_t}\}_{t \geq 0}$ predictable process. Obviously, for $t \leqq \mathrm{L}_{\mathrm{T}_\delta}$:

$$\left[\left| \frac{\partial \varphi_{\mathrm{A}_t-}}{\partial x}(\bar{\omega}, x_0) \right| \vee \left| \left[\frac{\partial \varphi_{\mathrm{A}_t-}}{\partial x}(\bar{\omega}, x_0) \right]^{-1} \right| \right] \leqq \frac{1}{\delta^{1/4}}. \tag{6.96}$$

Observe that since:

$$\langle \mathrm{C}^{x_0}_{\mathrm{A}_t} f, f \rangle = \int_0^{\mathrm{A}_t} 1_{z>0} \langle (\varphi_s^{*-1} \mathrm{X}_i)(x_0), f \rangle^2 \, ds + \int_0^{\mathrm{A}_t} 1_{z<0} \langle (\varphi_s^{*-1} \mathrm{X}'_i)(x_0), f \rangle^2 \, ds, \tag{6.97}$$

then we have:

$$\langle \mathrm{C}^{x_0}_{\mathrm{A}_t} f, f \rangle = \mathrm{S}_{s \leqq t} 1_{e_s \in \mathscr{W}^+} \langle \bar{\mathrm{C}}^{x_{\mathrm{A}_s-}}_{\sigma_s}(\varepsilon_s, e_s) \varphi^*_{\mathrm{A}_s-}(\bar{\omega}, x_0) f, \varphi^*_{\mathrm{A}_s-}(\bar{\omega}, x_0) f \rangle + \mathrm{S}_{s \leqq t} 1_{e_s \in \mathscr{W}^-} \langle \bar{\mathrm{C}}'^{x_{\mathrm{A}_s}}_{\sigma_s}(\varepsilon_s, e_s) \varphi^*_{\mathrm{A}_s-}(\bar{\omega}, x_0) f, \varphi^*_{\mathrm{A}_s-}(\bar{\omega}, x_0) f \rangle. \tag{6.98}$$

In (6.98), we use the notation:

$$\varphi^*_{\mathrm{A}_s-}(\bar{\omega}, x_0) f = \overbrace{\left[\frac{\partial \varphi_{\mathrm{A}_s-}}{\partial x}(\bar{\omega}, x_0) \right]^{-1} f}. \tag{6.99}$$

To prove (6.98), take $\varepsilon' > 0$. Let S_i be the sequence of $\{\mathrm{F}'_{\mathrm{A}_t}\}_{t \geq 0}$ stopping times defined by:

$$\mathrm{S}_0 = 0,$$
$$\mathrm{S}_{i+1} = \inf\{ t \geqq \mathrm{S}_i, \mathrm{A}_t - \mathrm{A}_{t-} > \varepsilon' \}.$$

For every $i \in \mathbb{N}$, $A_{S_i^-}$ is a $\{F'_{D_t}\}_{t \geqq 0}$ stopping time. Since w is a $\{F_t \otimes F'_{D_t}\}_{\geqq 0}$ martingale, the Markov property of φ can be used at the $\{\bar{F}_{D_t}\}_{t \geqq 0}$ stopping time $A_{S_i^-}$, so that, if $e_{S_i} \in \mathscr{W}^+$:

$$(6.100) \qquad \varphi_{A_{S_i}}(\bar{\omega}, x_0) = \Psi_{\sigma S_i}(\varepsilon_{S_i}, e_{S_i}, \varphi_{A_{S_i^-}}(\bar{\omega}, x_0)).$$

A similar relation holds when $e_{S_i} \in \mathscr{W}^-$. Now clearly:

$$(6.101) \quad \langle C_{A_t}^{x_0} f, f \rangle = \lim_{\varepsilon' \downarrow 0} \sum_{\substack{S_i \leqq t \\ e \in \mathscr{W}^+}} \left[\int_{A_{S_i^-}}^{A_{S_i}} \langle \varphi_s^{*-1} X_k, f \rangle^2 \, ds \right] + \sum_{\substack{S_i \leqq t \\ e \in \mathscr{W}^-}} \left[\int_{A_{S_i^-}}^{A_{S_i}} \langle \varphi_s^{*-1} X'_k, f \rangle^2 \, ds \right].$$

Using (6.101), (6.98) is now obvious.

Now by Theorems I.1.2 and I.2.1 in [5], we know that for every $e \in \mathscr{W}^+$, and every $p \geqq 1$:

$$(6.102) \qquad \int_\Omega \sup_{0 \leqq t \leqq 1} \left| \left[\frac{\partial \Psi_t}{\partial x}(\varepsilon, e, x) \right] \right|^p dP(\varepsilon),$$

is uniformly bounded by a constant not depending on x, e. It follows that using Theorems 3.4 and 3.9:

$$(6.103) \quad \int_{\mathscr{W}_0} 1_{\sigma \leqq 1} \left[\int_0^\sigma |(\Psi_s^{*-1}(\varepsilon, e) X_k)(x)|^2 \, ds \right] dn(\varepsilon, e) \leqq C \int_0^1 \frac{v \, dv}{\sqrt{2 \pi v^3}} < +\infty.$$

For $\beta > 0$, let $\tau_s^f(\bar{\omega}, \beta)$ be the function defined by:

$$(6.104) \quad \tau_s^f(\bar{\omega}, \beta) = \int_{\mathscr{W}_0} 1_{\sigma \leqq 1} \times \left\{ \exp - \beta \left[\langle \bar{C}_{\sigma(e)}^{x_{A_s^-}(\bar{\omega})}(\varepsilon, e) \, \varphi_{A_s^-}^*(\bar{\omega}, x_0) f, \varphi_{A_s^-}^*(\bar{\omega}, x_0) f \rangle \right] - 1 \right\} dn(\varepsilon, e).$$

Using (6.103), (6.104) is finite and moreover if $s \leqq L_{T_\delta}$, we see by (6.96), (6.103) that $\tau_s^f(\bar{\omega}, \beta)$ is uniformly bounded. Similarly if $\tau_s'^f(\bar{\omega}, \beta)$ is the function:

$$(6.105) \quad \tau_s'^f(\bar{\omega}, \beta) = \int_{\mathscr{W}_0} 1_{\sigma \leqq 1} \times \{ \exp - \beta [\langle \bar{C}_{\sigma(e)}'^{x_{A_s^-}(\bar{\omega})}(\varepsilon, -e) \, \varphi_{A_s^-}^*(\bar{\omega}, x_0) f, \varphi_{A_s^-}^*(\bar{\omega}, x_0) f \rangle] - 1 \} \, dn(\varepsilon, e).$$

then for $s \leqslant L_{T_\delta}$, $\tau_s'^f(\bar{\omega}, \beta)$ is uniformly bounded.

Using Theorem 6.4, (6.98), the stochastic calculus on point process in Jacod [19], and the boundedness of $\tau^f_{s \wedge L_{T_\delta}}(\bar{\omega}, \beta)$, $\tau'^f_{s \wedge L_{T_\delta}}(\bar{\omega}, \beta)$, we know that if $N_t(\bar{\omega}, \beta)$ is the process:

$$N_t(\bar{\omega}, \beta) = \exp\left[-\beta \langle C^{x_0}_{A_t} f, f \rangle - \frac{1}{2} \int_0^t (\tau^f_s(\bar{\omega}, \beta) + \tau'^f_s(\bar{\omega}, \beta))\, ds \right], \tag{6.106}$$

then $N_{t \wedge L_{T_\delta}}(\bar{\omega}, \beta)$ is a $\{\bar{F}_{A_t}\}_{t \geq 0}$ supermartingale, and so:

$$E^{\bar{P}}[N_{t \wedge L_{T_\delta}}(\bar{\omega}, \beta)] \leqq 1. \tag{6.107}$$

As $\delta \to +\infty$, $T_\delta \to +\infty$, $L_{T_\delta} \to +\infty$, so that by Fatou's lemma:

$$E^{\bar{P}}[N_t] \leqq 1. \tag{6.108}$$

For $x' \in R^d$, $g \in T^*_x R^d$ with $\|g\| = 1$, $\beta > 0$, we define:

$$\left\{ \begin{aligned} \theta^g(x', \beta) &= \int_{\mathscr{W}_0} 1_{\sigma \leq 1} \{ \exp(-\beta \langle \bar{C}^{x'}_{\sigma(e)}(\varepsilon, e) g, g \rangle) - 1 \}\, dn(\varepsilon, e), \\ \theta'^g(x', \beta) &= \int_{\mathscr{W}_0} 1_{\sigma \leq 1} \{ \exp(-\beta \langle \bar{C}'^{x'}_{\sigma(e)}(\varepsilon, -e) g, g \rangle) - 1 \}\, dn(\varepsilon, e). \end{aligned} \right. \tag{6.109}$$

We now do the key observation that due to (6.93) either:

$$\sum_{n=1}^{l} \sum_{Y \in E_n} \langle g, Y(x', 0) \rangle^2 \geqq \eta/2, \tag{6.110}$$

or:

$$\sum_{n=1}^{l} \sum_{Y' \in E'_n} \langle g, Y'(x', 0) \rangle^2 \geqq \eta/2 \tag{6.111}$$

[of course (6.110) and (6.111) can be simultaneously verified]. If (6.110) is verified, by Theorem 6.10, we know that for $\rho \leqq \mu$:

$$n(\langle \bar{C}^{x'}_{\sigma(e)}(\varepsilon, e) g, g \rangle \geqq \rho) \geqq \frac{C}{\rho^{1/m_l}}. \tag{6.112}$$

Since $n(\sigma > 1) < +\infty$, we find that (by taking eventually smaller constants μ and C) if $\rho \leqq \mu$:

$$n(\langle \bar{C}^{x'}_{\sigma(e)}(\varepsilon, e) g, g \rangle \geqq \rho, \sigma \leqq 1) \geqq \frac{C}{\rho^{1/m_l}}. \tag{6.113}$$

Now we have the trivial:

$$(6.114)\quad \theta^g(x',\beta)=-\beta\int_0^{+\infty} e^{-\beta\rho}\, n(\langle \bar{C}^{x'}_{\sigma(e)}(\varepsilon,e)\,g,g\rangle\geqq\rho,\ \sigma\leqq 1)\,d\rho$$

$$\leqq -C\,\beta\int_0^{\mu}\frac{e^{-\beta\rho}}{\rho^{1/m_l}}\,d\rho=-C\,\beta^{1/m_l}\int_0^{\beta\mu}\frac{e^{-u}}{u^{1/m_l}}\,du.$$

If $\beta\geqq 1/\mu$, we get from (6.114) that for a given $D>0$:

$$(6.115)\quad \theta^g(x',\beta)\leqq -D\,\beta^{1/m_l}.$$

If (6.111) is verified, we obtain similary for $\beta\geqq 1/\mu$:

$$(6.116)\quad \theta'^g(x',\beta)\leqq -D\,\beta^{1/m_l}.$$

Recall that $m_l\geqq 6$. Set:

$$(6.117)\quad \beta(\delta)=\left[\frac{D\,t}{2\,m_l\,\delta^{1-(1/2m_l)}}\right]^{m_l/(m_l-1)}.$$

Clearly as $\delta\to 0$, $\beta(\delta)\to +\infty$. We now have:

$$(6.118)\quad \bar{P}[\langle C^{x_0}_{A_t} f,f\rangle\leqq\delta;\ A_t\leqq T]\leqq\bar{P}[T_\delta\leqq A_t\leqq T)$$

$$+\bar{P}[N_t(\bar{\omega},\beta(\delta))\geqq e^{-\beta(\delta)\delta-(1/2)\int_0^t[\tau_s^f(\bar{\omega},\beta(\delta))+\tau_s'^f(\bar{\omega},\beta(\delta))]\,ds},\ A_t<T_\delta].$$

Now by Theorem 1.1 (*e*) :

$$(6.119)\quad \bar{P}(T_\delta\leqq A_t\leqq T)\leqq C\,\delta^p.$$

Set:

$$(6.120)\quad g_s(\bar{\omega})=\frac{\varphi^*_{A_s^-}(\bar{\omega},x_0)\,f}{\|\varphi^*_{A_s^-}(\bar{\omega},x_0)\,f\|},\qquad h_s(\bar{\omega})=\|\varphi^*_{A_s^-}(\bar{\omega},x_0)\,f\|.$$

We then have the trivial:

$$(6.121)\quad \begin{cases}\tau_s^f(\bar{\omega},\beta)=\theta^{g_s(\bar{\omega})}(x_{A_s^-}(\bar{\omega}),\beta\,h_s^2(\bar{\omega})),\\ \tau_s'^f(\bar{\omega},\beta)=\theta^{g_s(\bar{\omega})}(x_{A_s^-}(\bar{\omega}),\beta\,h_s^2(\bar{\omega})).\end{cases}$$

Now if $T_\delta>A_t$, for $s\leqq t$:

$$(6.122)\quad h_s(\bar{\omega})\geqq\frac{1}{\|\partial\varphi_{A_s^-}/\partial x(\bar{\omega},x_0)\|}\geqq\delta^{1/4}$$

so that as $\delta\to 0$:

$$(6.123)\quad \beta(\delta)\,h_s^2(\bar{\omega})\geqq\frac{1}{\delta^{[m_l/2(m_l-1)]}}\left[\frac{D\,t}{2\,m_l}\right]^{m_l/(m_l-1)}.$$

Of course as $\delta \to 0$, the r. h. s. of (6.123) tends to $+\infty$. For δ small enough, if $A_t < T_\delta$, for $s \leqq t$ we get using (6.115), (6.116), (6.121), (6.123):

$$(6.124) \qquad \tau_s^f(\bar{\omega}, \beta(\delta)) + \tau_s'^f(\bar{\omega}, \beta(\delta)) \leqq -D\left[\frac{Dt}{2m_l}\right]^{1/(m_l-1)} \frac{1}{\delta^{1/2(m_l-1)}}.$$

From (6.124), we find that the second term in the r. h. s. of (6.118) is dominated by:

$$(6.125) \qquad \bar{P}\left[N_t(\bar{\omega}, \beta(\delta)) \geqq \exp\left(-\beta(\delta)\,\delta + \left[\frac{Dt}{2m_l}\right]^{m_l/(m_l-1)} \frac{m_l}{\delta^{1/2(m_l-1)}}\right)\right].$$

Now:

$$(6.126) \qquad -\beta(\delta)\,\delta + \left[\frac{Dt}{2m_l}\right]^{m_l/(m_l-1)} \frac{m_l}{\delta^{1/2(m_l-1)}} = (m_l-1)\left[\frac{Dt}{2m_l}\right]^{m_l/(m_l-1)} \frac{1}{\delta^{1/2(m_l-1)}}.$$

Using (6.108), (6.126) and Čebyšev's inequality, we obtain that (6.125) is dominated by:

$$(6.127) \qquad \exp\left[-(m_l-1)\left[\frac{Dt}{2m_l}\right]^{m_l/(m_l-1)} \frac{1}{\delta^{1/2(m_l-1)}}\right].$$

From (6.118), (6.119), (6.125), (6.127) we obtain as $\delta \to 0$:

$$(6.128) \qquad \bar{P}[\langle C_{A_t}^{x_0} f, f\rangle \leqq \delta;\ A_t \leqq T] \leqq C\,\delta^p + \exp\left[-(m_l-1)\left[\frac{Dt}{2m_l}\right]^{m_l/(m_l-1)} \frac{1}{\delta^{1/2(m_l-1)}}\right].$$

We now use the result stated after (5.52). Namely as $\lambda \to +\infty$, we get:

$$(6.129) \qquad \bar{P}[\,|[C_{A_t}^{x_0}]^{-1}| \geqq \lambda;\ A_t \leqq T] \leqq \bar{P}[\,|C_{A_t}^{x_0}| \geqq \lambda^{1/2};\ A_t \leqq T] + \sum_{j=1}^{N(\lambda^{1/2}, 2/\lambda)} P\left[\langle C_{A_t}^{x_0} f_j, f_j\rangle \leqq \frac{2}{\lambda};\ A_t \leqq T\right],$$

where $f_j \in T_{x_0}^*(R^d)$ with $\|f_j\| = 1$ and $N(\lambda^{1/2}, 2/\lambda) \leqq C\,\lambda^{3/2(d-1)}$. Now by Theorem 1.1 (*e*), for any $p \geqq 1$:

$$(6.130) \qquad \bar{P}[\,|C_{A_t}^{x_0}| \geqq \lambda^{1/2};\ A_t \leqq T] \leqq \frac{C}{\lambda^p}.$$

Using (6.128), we find that:

$$(6.131) \qquad \bar{P}[\,|[C_{A_t}^{x_0}]^{-1}| \geqq \lambda;\ A_t \leqq T] \leqq \frac{A}{\lambda^p}.$$

The proof is finished. □

Remark 3. — As we have seen after Definition 6.11, (6.93) implies that at any $x_0 \in \mathrm{R}^d$, the process x_{A_s} "tends" to jump in a family of directions which in a loose sense span the whole space. Under the assumptions of Theorem 6.12 the boundary semi-group will have C^∞ densities.

We have vainly tried to prove Theorem 6.12 using the normal time scale, that is the classical $\{\bar{\mathrm{F}}_t\}_{t \geqq 0}$ stochastic calculus instead of the $\{\bar{\mathrm{F}}_{\mathrm{A}_t}\}_{t \geqq 0}$ stochastic calculus, as we did in Theorem 5.8. The reason of the failure is that the behavior of the Brownian motion z at 0 becomes here of critical importance.

As made clear in (6.110), (6.111), (6.115), (6.116), the excursions in $\mathscr{W}^+, \mathscr{W}^-$ can both contribute to the smoothness of the boundary semi-group. It would be apparently quite hard to keep track in normal time scale of the two types of excursions. This is why we have prefered to slower the computations that is:

— estimate the contribution to regularity of each type of excursion;

— compute the effect of piling up the excursions using the stochastic calculus on Poisson point processes.

Remark 4. — There is no "natural" assumption under which Theorem 6.12 would hold when $\mathscr{L}$ and $\mathscr{L}'$ are degenerate on the boundary in the same way as in section 5.

However, it is interesting to use the technique of Theorems 6.10-6.12 to give another proof of Theorem 5.10 in the one-sided case. Under the same assumption as in Theorem 5.10 we would find that if $f \in \mathrm{T}_x^* \mathrm{R}^d$, $\|f\| = 1$, then for ρ small enough:

$$n(\langle \mathrm{C}^x_{\sigma(e)} f, f \rangle \geqq \rho) \geqq -\frac{\log \rho}{m_l \mathrm{C}} - 1. \tag{6.132}$$

The estimates follow then the same line for Theorem 6.12. In particular, instead of (6.117) we will take $\beta = 1/\delta$.

Of course (6.132) is also useful when $\mathscr{L}$ verifies the conditions of Theorem 5.10, $\mathscr{L}'$ does not verify any special condition, to obtain the equivalent of Theorem 5.10 for the two-sided process. Details are left to the reader.

We have vainly tried to find an "excursion" proof of Theorems 5.8 and 6.7. This is rather natural due to the results of section 6(f).

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