DUSA MCDUFF

Local homology of groups of volume-preserving diffeomorphisms. III

Annales scientifiques de l'É.N.S. 4e série, tome 16, nº 4 (1983), p. 529-540
<http://www.numdam.org/item?id=ASENS_1983_4_16_4_529_0>

Ann. scient. Éc. Norm. Sup.,
4ᵉ série, t. 16, 1983, p. 529 à 540.

LOCAL HOMOLOGY OF GROUPS OF VOLUME-PRESERVING DIFFEOMORPHISMS. III

By Dusa McDUFF (*)

This is the last in a series of papers which study the local homology of groups of volume preserving diffeomorphisms ([10], [11]). However it may be read independently of the others, since it is self-contained apart from quoting some of their results.

Let M be a compact, connected and oriented C^∞-manifold without boundary, and with volume form ω. Thus ω is a non-vanishing n-form, where $n = \dim M$, compatible with the orientation of M. Further, let $\mathscr{D}iff_\omega M$ denote the group of all ω-preserving C^∞-diffeomorphisms of M in the compact-open C^∞-topology. We will be concerned here with the "local homology" of the group $\mathscr{D}iff_\omega M$. As explained by Mather in [7], the local homology of a topological group $\mathscr{G}$ is the homology of the homotopy fiber $\overline{B}\mathscr{G}$ of the natural map $BG \to B\mathscr{G}$, where G is the group $\mathscr{G}$ but considered with the discrete topology. This space $\overline{B}\mathscr{G}$ depends only on the algebraic and topological structure of the germ of $\mathscr{G}$ at the identity element e (that is, of an arbitrarily small neighbourhood of e). In fact, it is not hard to show that if $\mathscr{G}$ is locally contractible the cohomology of $\overline{B}\mathscr{G}$ may be calculated from the complex of Eilenberg-MacLane cochains on this germ. Furthermore, one can define the "continuous" local cohomology of $\mathscr{G}$, which for locally contractible $\mathscr{G}$ is just the cohomology of the complex of continuous Eilenberg-MacLane cochains on the germ of $\mathscr{G}$ at e. When $\mathscr{G}$ is a Lie group, the van Est theorem implies that this is isomorphic to the cohomology of the Lie algebra of $\mathscr{G}$. Similarly, when $\mathscr{G} = \mathscr{D}iff_\omega M$, it is just the cohomology of the Lie algebra of divergence free vector fields on M ([2], [5]).

Mather and Thurston showed that the local homology of the group $\mathscr{D}iff\,M$ of all diffeomorphisms of M is isomorphic to the homology of the space of sections of a certain bundle over M which is associated to the tangent bundle of M. The fiber of this bundle is made from germs of diffeomorphisms of M. It is suggestive, but not quite correct, to say that the fiber at x is made from the set of germs of diffeomorphisms at x. (The trouble is that

(*) Partially supported by NSF grant no MCS 79 05795 A02.

this set has no algebraic structure.) Further, the map from $\overline{\mathrm{B}}\,\mathscr{D}iff\,\mathrm{M}$ to the space of sections is essentially given by thinking of a diffeomorphism as a collection of germs, one at each point of M. Hence one can interpret the Mather-Thurston theorem as saying that the homology of $\mathscr{D}iff\,\mathrm{M}$ localized at the identity may be calculated by localizing the diffeomorphisms spatially. Finally, note that because the elements of $\overline{\mathrm{B}}\,\mathscr{D}iff\,\mathrm{M}$ may be thought of as holonomic or integrable sections of the fiber bundle, this theorem is very close in spirit to Gromov's work in [3] for example.

In this paper we prove the analogous result for $\mathscr{D}iff_{\omega}\,\mathrm{M}$. Besides being of theoretical interest, this result is of great help in the calculation of the local homology of $\mathscr{D}iff_{\omega}\,\mathrm{M}$. *See* [12] and in particular [6], where Hurder proves the existence of an enormous number of non-zero elements in $\mathrm{H}_{*}(\overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}\,\mathrm{M})$. Since all the classes found so far are continuous, they also live on the Lie algebra level.

Here is a precise statement of the main theorem. We state it for $\mathscr{D}iff_{\omega}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$, the group of ω-preserving diffeomorphisms of M which are the identity in some neighbourhood of A. Throughout we assume that the (possibly empty) subset A of M is closed and that $\mathrm{M}-\mathrm{A}$ is connected. (The latter restriction entails no loss of generality since $\mathscr{D}iff_{\omega}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$ decomposes as a product with one factor for each connected component of $\mathrm{M}-\mathrm{A}$.) The canonical M-bundle over $\mathrm{B}\,\mathrm{Diff}_{\omega}\,\mathrm{M}$ has discrete structural group and so is foliated transversely to the fibers. Its pull-back to $\overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}\,\mathrm{M}$ is isomorphic to the product $\overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}\,\mathrm{M}\times\mathrm{M}$. Hence the space $\overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}\,\mathrm{M}\times\mathrm{M}$ has a canonical foliation $\mathscr{F}$ transverse to the fibers $pt\times\mathrm{M}$. One can check that $\mathscr{F}$ is defined by a closed n-form which restricts to ω on the fibers. Moreover the restriction of $\mathscr{F}$ to $\overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}(\mathrm{M}, \mathrm{rel}\ \mathrm{A})\times\mathrm{A}$ has leaves $\overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}(\mathrm{M}, \mathrm{rel}\ \mathrm{A})\times pt$ and so is trivial. (For more detail *see* [10] and [12].)

Now consider the groupoid Γ^n_{sl} of germs of diffeomorphisms of $\mathbb{R}^n$ which preserve the standard volume form $dx_1\wedge\ldots\wedge dx_n$. Give Γ^n_{sl} the sheaf topology. The homomorphism $\Gamma^n_{sl}\to\mathscr{S}\mathscr{L}(n, \mathbb{R})$, which takes the germ g at x to its derivative dg_x, induces a map on classifying spaces $\nu:\mathrm{B}\,\Gamma^n_{sl}\to\mathrm{B}\,\mathscr{S}\mathscr{L}(n, \mathbb{R})$. We will suppose that ν is a Hurewicz fibration and will call its fiber $\overline{\mathrm{B}\,\Gamma}{}^n_{sl}$. It follows from Haefliger's general theory [4] that the foliation $\mathscr{F}$ is classified by a commutative diagram

$$\begin{array}{ccc}
\overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}\,\mathrm{M}\times\mathrm{M} & \xrightarrow{\mathrm{F}} & \mathrm{B}\Gamma^n_{sl}\\
\Big\downarrow{\scriptstyle \mathrm{proj.}} & & \Big\downarrow{\scriptstyle \nu}\\
\mathrm{M} & \xrightarrow{\ \tau\ } & \mathrm{B}\,\mathscr{S}\mathscr{L}(n, \mathbb{R})
\end{array}$$

where τ classifies the tangent bundle to M. Let $\mathrm{E}_{\mathrm{M}}\to\mathrm{M}$ be the pull-back of ν over τ. Then F induces a map

$$f:\quad \overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}\,\mathrm{M}\to S_{\omega}(\mathrm{M}),$$

where $S_{\omega}(\mathrm{M})$ is the space of continuous sections of $\mathrm{E}_{\mathrm{M}}\to\mathrm{M}$ with the compact-open topology. By choosing F carefully, one can ensure that f restricts to give a map

$$f:\quad \overline{\mathrm{B}}\,\mathscr{D}iff_{\omega}(\mathrm{M}, \mathrm{rel}\ \mathrm{A})\to S_{\omega}(\mathrm{M}, \mathrm{rel}\ \mathrm{A}),$$

where $S_\omega(M, \text{rel}\, A)$ is the space of sections which equal a given base section s_0 on A. (*See* proof of Lemma 3.1 below and [9], Appendix.) The section space $S_\omega(M, \text{rel}\, A)$ need not be connected and we write $S_{\omega 0}(M, \text{rel}\, A)$ for the connected component which contains s_0 and the image of F.

The main theorem is

THEOREM 1. — *The map*

$$f: \quad \overline{B}\,\mathcal{D}\mathit{iff}_\omega(M, \text{rel}\, A) \to \; S_{\omega 0}(M, \text{rel}\, A);$$

is a homology equivalence, that is, f induces an isomorphism on homology for all local coefficients coming from $S_{\omega 0}(M, \text{rel}\, A)$.

We will see below that, except in the case $n=2$, $A \neq \emptyset$, $\pi_1(S_{\omega 0}(M, \text{rel}\, A))$ is isomorphic to $H_1(\overline{B}\,\mathcal{D}\mathit{iff}_\omega(M, \text{rel}\, A); \mathbb{Z}) \cong H^{n-1}(M, A; \mathbb{R})$. Theorem 1 is then equivalent to the statement

$$\tilde{f}: \quad \overline{B}\,\mathcal{D}\mathit{iff}^{\Phi}_{\omega 0}(M, \text{rel}\, A) \xrightarrow{H_* \cong} \tilde{S}_{\omega 0}(M, \text{rel}\, A),$$

where $\mathcal{D}\mathit{iff}^{\Phi}_{\omega 0}$ denotes the kernel of the flux homomorphism Φ as defined in §2 below, and where $\tilde{S}$ is the universal cover of S. (When $n=2$ and $A \neq \emptyset$ the appropriate space on the right is a cover of S with fundamental group $\mathbb{R}$.) Corresponding results for non-compact M are given in [10]. For example, if $A = \emptyset$, Theorem 1 holds provided that M is the interior of a compact manifold of dimension ≥ 3 such that each of its ends has infinite ω-volume. Note that we do not treat the case of a non-compact manifold of finite volume.

2. Sketch of proof of Theorem 1

Most of the work of proving Theorem 1 was done in [10] and [11]. Suppose for the moment that A is an n-dimensional compact submanifold of M and let A_0 be A-(open collar nbhd of ∂A). We showed in [10] that

$$f: \quad \overline{B}\,\mathcal{D}\mathit{iff}^c_{\tilde{\omega}}(M - A_0) \xrightarrow{H_* \cong} S^c_{\tilde{\omega} 0}(M - A_0),$$

where $\tilde{\omega}$ is an extension of $\omega \,|\, M - A$ to the non-compact manifold $M - A_0$ such that every end has infinite volume, and where "c" denotes compact support. Also, by [11], we have

$$\overline{B}\,\mathcal{D}\mathit{iff}_{\tilde{\omega}}(M, \text{rel}\, A) \xrightarrow{H_* \cong} \overline{B}\,\mathcal{D}\mathit{iff}^c_{\tilde{\omega}}(M - A_0).$$

Since $\tilde{\omega} = \omega$ on $M - A$, it follows easily that Theorem 1 holds for this A. By taking direct limits, one then proves Theorem 1 for all non-empty A.

Before going further, let us recall some facts about the fundamental groups of $\overline{B}\,\mathcal{D}\mathit{iff}_\omega M$ and $S_{\omega 0}(M)$. Let $\mathcal{D}\mathit{iff}_{\omega 0} M$ be the identity component of $\mathcal{D}\mathit{iff}_\omega M$, and $\widetilde{\text{Diff}}_{\omega 0} M$ be the universal cover of $\mathcal{D}iff_{\omega 0}$ M, but considered as a discrete group. It is easy to see that $\overline{\mathrm{B}}\,\mathcal{D}iff_{\omega 0}\,M \simeq \overline{B}\,\mathcal{D}iff_{\omega}\,\mathrm{M}$ and that $\pi_1\,\overline{\mathrm{B}}\,\mathcal{D}iff_{\omega}\,M \cong \check{\mathrm{D}}\mathrm{iff}_{\omega 0}\,\mathrm{M}$. The flux homomorphism

$$\tilde{\Phi} : \widetilde{\mathcal{D}iff}_{\omega 0}\,\mathrm{M} \to \mathrm{H}^{n-1}(\mathrm{M}; \mathbb{R}),$$

may be defined as follows [16]. An element of $\widetilde{\mathrm{Diff}}_{\omega 0}$ M is a pair $(g, \{g_t\})$, where $g \in \mathrm{Diff}_{\omega 0}$ M and $\{g_t\}$ is a homotopy class of paths joining $g_0 = \mathrm{id}$ to $g_1 = g$. If z is a singular $(n-1)$-cycle in M, then $\{g_t(z)\}$ is a singular n-chain whose ω-volume depends only on the homotopy class $\{g_t\}$ and is zero if z is a boundary. Therefore one may define $\tilde{\Phi}$ by the formula

$$\tilde{\Phi}(g, \{g_t\})(z) = \mathrm{vol}_{\omega}\{g_t(z)\}.$$

One checks that $\tilde{\Phi}$ is a group homomorphism by using the fact that the g_t preserve ω. Note also that $\tilde{\Phi}$ induces a homomorphism

$$\Phi : \quad \mathrm{Diff}_{\omega 0}\,\mathrm{M} \to \mathrm{H}^{n-1}(\mathrm{M}; \mathbb{R})/\tilde{\Phi}(\pi_1\,\mathcal{D}iff_{\omega 0}\,\mathrm{M}).$$

We write $\mathrm{Diff}^{\Phi}_{\omega 0}$ M for the kernel of Φ, and $\mathcal{D}iff^{\Phi}_{\omega 0}$ M for the same group topologized as a subspace of $\mathcal{D}iff_{\omega 0}$ M. (In fact $\mathcal{D}iff^{\Phi}_{\omega 0}$ M is closed in $\mathcal{D}iff_{\omega 0}$ M, since, as one can easily show, $\tilde{\Phi}(\pi_1\,\mathcal{D}iff_{\omega 0}\,\mathrm{M})$ is a discrete subgroup of $\mathrm{H}^{n-1}(\mathrm{M}; \mathbb{R})$.) Clearly $\pi_1\,\overline{\mathrm{B}}\,\mathcal{D}iff^{\Phi}_{\omega 0}\,\mathrm{M} \cong \ker\tilde{\Phi}$. A difficult result of Thurston [16] and Banyaga [1] states that $\ker\tilde{\Phi}$ is perfect. It follows that

$$\mathrm{H}_1(\overline{\mathrm{B}}\,\mathcal{D}iff^{\Phi}_{\omega 0}\,\mathrm{M}; \mathbb{Z}) = 0,$$

and that

$$\mathrm{H}_1(\overline{\mathrm{B}}\,\mathcal{D}iff_{\omega 0}\,\mathrm{M}; \mathbb{Z}) \cong \mathrm{H}^{n-1}(\mathrm{M}; \mathbb{R}).$$

Note also that the map $\overline{\mathrm{B}}\,\mathcal{D}iff^{\Phi}_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A}) \to \overline{\mathrm{B}}\,\mathcal{D}iff_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$, when made into a fibration, is a covering map whose fiber is the discrete abelian group $\mathrm{H}^{n-1}(\mathrm{M}, \mathrm{A}; \mathbb{R})$.

Now consider $\pi_1\,S_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$. We showed in [10] that when $n \geqq 3$, $\pi_n(\overline{\mathrm{B}\Gamma}^n_{sl}) \cong \mathbb{R}$ and $\pi_i(\overline{\mathrm{B}}\Gamma^n_{sl}) = 0$ for $1 \leqq i < n$ and $i = n+1$. Therefore, obstruction theory implies that

$$\pi_1\,S_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A}) \cong \mathrm{H}^{n-1}(\mathrm{M}, \mathrm{A}; \mathbb{R}).$$

When $n = 2$ we have $\pi_1(\overline{\mathrm{B}}\,\Gamma^2_{sl}) = 0$ and $\pi_2(\overline{\mathrm{B}}\,\Gamma^2_{sl}) \cong \pi_3(\overline{\mathrm{B}}\,\Gamma^2_{sl}) \cong \mathbb{R}$. By using obstruction theory or by looking at the fibration obtained by restricting sections to the 1-skeleton of (M, A), one can show that $\pi_1\,S_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$ is an extension of $\mathrm{H}^1(\mathrm{M}, \mathrm{A}; \mathbb{R})$ by a quotient of $\mathbb{R}$. In fact, we showed in [10], § 7 that, when $\mathrm{A} \neq \emptyset$, $\pi_1\,S_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$ is a central extension of $\mathrm{H}^1(\mathrm{M}, \mathrm{A}; \mathbb{R})$ by $\mathbb{R}$ and so is nilpotent. In a moment we will see that $\pi_1\,S_{\omega 0}\,\mathrm{M} \cong \mathrm{H}^1(\mathrm{M}; \mathbb{R})$. For now, however, let $S'_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$ be the covering space of $S_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})$ corresponding to the kernel of the map

$$\pi_1\,(S_{\omega 0}(\mathrm{M}, \mathrm{rel}\,\mathrm{A})) \to \mathrm{H}^{n-1}(\mathrm{M}, \mathrm{A}; \mathbb{R}).$$

Thus $\pi(S')$ is zero if $n \geqq 3$ and is abelian otherwise.

We return to the proof of Theorem 1. Consider the commutative diagram

$$\begin{array}{ccccc} \overline{B}\,\mathcal{D}iff_{\omega 0}(M, \mathrm{rel}\ x_0) & \to & \overline{B}\,\mathcal{D}iff_{\omega 0}\, M & \xrightarrow{\beta} & \overline{B}\Gamma^n_{sl} \\ {\scriptstyle f^c}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ S_{\omega 0}(M, \mathrm{rel}\ x_0) & \to & S_{\omega 0}\, M & \xrightarrow{\varepsilon} & \overline{B}\,\Gamma^n_{sl} \end{array} \qquad (*)$$

where the map ε evaluates sections at a point $x_0 \in M$ and where $\beta = \varepsilon \circ f$. The argument of [10], Lemma 6.1 shows that the restrictions of f^c and f to $\overline{B}\,\mathcal{D}iff^{\Phi}_{\omega 0}$ lift to S'. Therefore there is a commutative diagram

$$\begin{array}{ccccc} \overline{B}\,\mathcal{D}iff^{\Phi}_{\omega 0}(M, \mathrm{rel}\ x_0) & \to & \overline{B}\,\mathcal{D}iff^{\Phi}_{\omega 0}\, M & \xrightarrow{\tilde{\beta}} & \overline{B}\Gamma^n_{sl} \\ {\scriptstyle \tilde{f}^c}\downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \| \\ S'_{\omega 0}(M, \mathrm{rel}\ x_0) & \to & S'_{\omega 0}\, M & \xrightarrow{\tilde{\varepsilon}} & \overline{B}\,\Gamma^n_{sl}. \end{array} \qquad (**)$$

Note the following

(i) The map f^c in diagram ($\star$) is a homology equivalence because Theorem 1 holds for the pair (M, x_0). This immediately implies that its lift $\tilde{f}^c$ is also a homology equivalence.

(ii) The bottom row of ($\star\star$) is a fibration sequence because the bottom row of ($\star$) is, and because $H^{n-1}(M, x_0; \mathbb{R}) \cong H^{n-1}(M; \mathbb{R})$.

(Recall that $F \to E \xrightarrow{\beta} B$ is called a *fibration sequence*, resp. *homology fibration sequence*, if there is an associated inclusion of F into the homotopy fiber of β which is a weak homotopy, resp. $\mathbb{Z}$-homology, equivalence. Further, a $\mathbb{Z}$-homology equivalence is a map which induces an isomorphism on untwisted integer homology.) We will prove in § 3 below that

PROPOSITION 2. — *The top row of* ($\star\star$) *is a homology fibration sequence.*

A comparison of the Leray-Serre spectral sequence for the rows of (**) now shows that $\tilde{f}$ is a $\mathbb{Z}$-homology equivalence. But we saw above that $H_1(\overline{B}\,\mathcal{D}iff^{\Phi}_{\omega 0} M; \mathbb{Z}) = 0$ and $\pi_1(S'_{\omega 0} M)$ is abelian. It follows that $\pi_1(S'_{\omega 0} M) = 0$. Therefore $\tilde{f}$ and f are homology equivalences. This completes the proof of Theorem 1.

3. Proof of Proposition 2

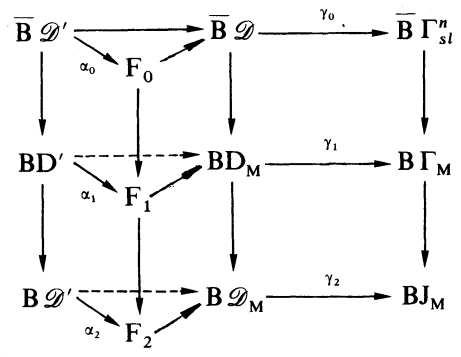
Let $\mathcal{D} = \mathcal{D}iff^{\Phi}_{\omega 0} M$ and $\mathcal{D}' = \mathcal{D}iff^{\Phi}_{\omega 0}(M, \mathrm{rel}\, x_0)$. The corresponding discrete groups are denoted D and D′. We want to show that the sequence

$$\overline{B}\,\mathcal{D}' \to \overline{B}\,\mathcal{D} \xrightarrow{\beta} \overline{B}\,\Gamma^n_{sl},$$

is a homology fibration sequence. As in [9], we do this by considering corresponding sequences for the discrete and topologized groups.

Let D_M be the groupoid whose elements are pairs (g, x), $g \in D$, $x \in M$, topologized as $D \times M$, where D is discrete and M has its usual topology. The partial composition law is $(h, gx).(g, x) = (hg, x)$. **Then BD_M is the total space of the canonical M-bundle over BD, and so $M \to BD_M \to BD$ is a fibration. Note: in [9], § 3 BD_M is** written $D \backslash\backslash M$.) Similarly, if $\mathscr{D}_M$ denotes the groupoid D_M topologized as $\mathscr{D} \times M$, there is a fibration $M \to B\mathscr{D}_M \to B\mathscr{D}$. It follows that the homotopy fiber of $BD_M \to B\mathscr{D}_M$ is homotopy equivalent to $\overline{B}\mathscr{D}$. Further, Let Γ_M be the groupoid of germs of ω-preserving diffeomorphisms of M, with the sheaf topology, and let J_M be the groupoid of 1-jets of elements of Γ_M, with its usual topology. Since $B\Gamma_M$ classifies the same objects as $B\Gamma^n_{sl}$, the spaces $B\Gamma_M$ and $B\Gamma^n_{sl}$ are weakly equivalent. (Another proof of this is given in [8], § 2.) **Similarly $BJ_M \simeq B\mathscr{S}\mathscr{L}(n, \mathbb{R})$. Hence we may identify the homotopy fiber of the differential $\nu : B\Gamma_M \to BJ_M$ with $\overline{B}\Gamma^n_{sl}$.**

We now construct the commutative diagram



as follows. The middle row $BD' \to BD_M \to B\Gamma_M$ consists of the classifying spaces of the exact sequence $D' \to D_M \to \Gamma_M$ of groupoids, where D' is included in D_M as the subobject $\{(g, x_0) : g = \text{id near } x_0\}$ and D_M is mapped to Γ_M by taking (g, x) to the germ of g at x. Further, F_1 is defined to be the homotopy fiber of γ_1 at the point $\star$ in $B\Gamma_M$ which corresponds to the identity germ (id, x_0) in Γ_M. Since D' maps to the base point (id, x_0) of Γ_M, the image of BD' in $B\Gamma_M$ contracts to $\star$. (It is not equal to $\star$ since we have to take thick realizations, see [9], Appendix.) The choice of contraction determines α_1. The bottom row is constructed similarly. Clearly, one can make the square involving α_1, α_2 commute. The spaces in the top row are the homotopy fibers of the corresponding vertical maps and the maps α_0, γ_0 are induced in the obvious way by the α_i, γ_i. Notice that F_0 is the homotopy fiber of both γ_0 and $F_1 \to F_2$.

We will prove:

LEMMA 3.1. — $\gamma_0 \sim \beta$.

LEMMA 3.2. — α_2 *is a homotopy equivalence.*

LEMMA 3.3. — α_1 *is a $\mathbb{Z}$-homology equivalence.*

PROOF OF PROPOSITION 2. — Since $\gamma_0 \sim \beta$, it suffices to show that α_0 is a $\mathbb{Z}$-homology equivalence. But $B\mathscr{D}'$ and F_2 are simply connected. Therefore we may apply the spectral sequence comparison theorem to the columns $\overline{B}\mathcal{D}' \to BD' \to B\mathcal{D}'$ and $F_0 \to F_1 \to F_2$. The result now follows from Lemmas 3.2 and 3.3. □

It remains to prove Lemmas 3.1-3.3. The proofs of 3.1 and 3.2 are straightforward. In 3.3 we replace the groupoids D_M and Γ_M by discrete categories so that we can use Quillen's Theorem B [13]. This is applicable because of the results of [11].

It will be convenient from now on to use the language of categories, rather than groupoids, since it is more flexible and more highly developed. Recall that a groupoid Γ may be thought of as a topological category all of whose morphisms are invertible. The space of objects of $\mathcal{C}(\Gamma)$ is the subspace of Γ formed by the identities, and the space of morphisms of $\mathcal{C}(\Gamma)$ is Γ itself. Groupoid homomorphisms then correspond to continuous functors. We will assume that the reader is familiar with the basic definitions of [14] and [9], § 3.

Proof of Lemma 3.1. – This is just a matter of spelling out definitions.

First consider $\tilde{\beta}$. Let $\mathcal{G} = \mathcal{D}iff_{\omega 0}\,M$ and recall the definition of $f : \overline{B}\mathcal{G} \to S_{\omega 0}\,M$ from [8], § 2. It arises from a homotopy commutative classifying diagram

$$\begin{array}{ccc} \overline{B}\mathcal{G} \times M & \xrightarrow{F} & B\Gamma_M \\ {\scriptstyle \pi = \text{proj.}}\downarrow & & \downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\tau} & BJ_M \circlearrowright H \end{array}$$

for the canonical foliation on $\overline{B}\mathcal{G} \times M$ in the following way. We identify $S_{\omega 0}\,M$ with the space of pairs $(\mathcal{C}, h)$, where $\mathcal{C}$ is a map $M \to B\Gamma_M$ and h is a homotopy from τ to $\nu \circ \mathcal{C}$. Then, given $y \in \overline{B}\mathcal{G}$, we define $f(y) = (F \mid y \times M,\ H \mid y \times M)$, where H is the indicated homotopy from $\tau \circ \pi$ to $\nu \circ F$.

Now diagram (&) is the realization of a diagram of categories and functors

$$\begin{array}{ccc} \mathcal{C}(G \diagdown\!\!\diagdown \mathcal{G} \times M) & \xrightarrow{\hat{F}} & \mathcal{C}(\Gamma_M) \\ {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle \hat{\nu}} \\ \mathcal{C}(\{e\} \diagdown\!\!\diagdown M) & \overset{\hat{\tau}}{\hookrightarrow} & \mathcal{C} \to \mathcal{C}(J_M) \circlearrowright \hat{H} \end{array}$$

Here $\mathcal{C}(G \diagdown\!\!\diagdown \mathcal{G} \times M)$ is made from the action $g : (h, x) \mapsto (gh, x)$ of G on $\mathcal{G} \times M$ as in [9], § 3. Thus its spaces of objects and morphisms are $\mathcal{G} \times M$ and $G \times \mathcal{G} \times M$ respectively. Similarly, $\mathcal{C}(\{e\} \diagdown\!\!\diagdown M)$ has M as space of objects and only identity morphisms. The functor $\hat{\pi}$ is the obvious projection, $\hat{\tau}$ is the inclusion and $\hat{F}$ is given by

$$\hat{F}(g : (h, x) \to (gh, x)) = \text{germ of } g \text{ at } hx.$$

Observe that $\hat{\tau} \circ \hat{\pi} \neq \hat{\nu} \circ \hat{F}$. However there is a natural transformation $\hat{H}$ from $\hat{\tau} \circ \hat{\pi}$ to $\hat{\nu} \circ \hat{F}$. It is a continuous map from the objects $\mathcal{G} \times M$ of $\mathcal{C}(G \diagdown\!\!\diagdown \mathcal{G} \times M)$ to the morphisms J_M of $\mathcal{C}(J_M)$ and is defined by

$$\hat{H}(h, x) = (dh_x, x).$$

It follows from [9], § 3, Appendix that one can realise this diagram so as to get (&). In particular the (thick) realization $G\backslash\backslash\mathcal{G}\times M$ of $\mathcal{C}(G\backslash\backslash\mathcal{G}\times M)$ is homeomorphic to the product $(G\backslash\backslash\mathcal{G})\times M$, and $G\backslash\backslash\mathcal{G}\simeq\overline{B}\mathcal{G}$. Further, by [14], § 1, the realization of the natural transformation $\hat{H}$ is the homotopy H.

This defines f. The map $\beta:\overline{B}\mathcal{G}\to\overline{B}\Gamma^n_{sl}$ is the composite of f with evaluation at the point x_0. Since $\overline{B}\Gamma^n_{sl}$ is the homotopy fiber of ν and $\overline{B}\mathcal{G}\simeq G\backslash\backslash\mathcal{G}$, the map β is given by a pair (β',β''), where $\beta':G\backslash\backslash\mathcal{G}\to B\Gamma_M$ and β'' is a homotopy from the constant map to $\nu\circ\beta'$. Identifying $\mathcal{C}(G\backslash\backslash\mathcal{G})$ with the full subcategory of $\mathcal{C}(G\backslash\backslash\mathcal{G}\times M)$ with objects $\mathcal{G}\times x_0$, one can easily check that β' and β'' are induced by the restrictions of $\hat{F}$ and $\hat{H}$. Finally note that $\hat{\beta}:\overline{B}\mathcal{D}\to\overline{B}\Gamma^n_{sl}$ is just the restriction of β to $\overline{B}\mathcal{D}\subset\overline{B}\mathcal{G}$.

Now consider γ_0. Instead of using the model $D\backslash\backslash\mathcal{D}$ for $\overline{B}\mathcal{D}$ in its definition, we identified $\overline{B}\mathcal{D}$ with the homotopy fiber F′ of $t:D\backslash\backslash M\to\mathcal{D}\backslash\backslash M$. (Recall that $BD_M=D\backslash\backslash M$ and $B\mathcal{D}_M=\mathcal{D}\backslash\backslash M$.) Therefore in order to relate γ_0 to $\hat{\beta}$ we must first describe an explicit homotopy equivalence $i:D\backslash\backslash\mathcal{D}\to F'$. This will be given by a pair (i', i''), where $i':D\backslash\backslash\mathcal{D}\to D\backslash\backslash M$ and i'' is a homotopy from the constant map to $t\circ i'$. As before, we define i' and i'' on the level of categories by a diagram

$$\begin{array}{ccc}\mathcal{C}(D\backslash\backslash\mathcal{D}) & \xrightarrow{\hat{j}} & \mathcal{C}(D\backslash\backslash M)\\ \downarrow & & \downarrow{\scriptstyle\hat{t}}\\ \mathcal{C}(\{e\}\backslash\backslash x_0) & \hookrightarrow & \mathcal{C}(\mathcal{D}\backslash\backslash M) \end{array}\ \supset\ \hat{I}$$

Here $\hat{j}$ is the inclusion given on objects by the evaluation map $h\mapsto h(x_0)$ at x_0, and $\hat{I}$ is the natural transformation from the constant functor to $\hat{t}\circ\hat{j}$ given by $\hat{I}(h)=(h:x_0\to h(x_0))$. (Observe that $\hat{I}$ is a *continuous* map from the objects $\mathcal{D}$ of the category $\mathcal{C}(D\backslash\backslash\mathcal{D})$ to the morphisms $\mathcal{D}\times M$ of $\mathcal{C}(\mathcal{D}\backslash\backslash M)$. Also e denotes the identity element of the group D.)

We claim that the map $i=(i', i'')$ induced by the pair $(\hat{j},\hat{I})$ is a homotopy equivalence. One way to prove this is to recall that there are fibration sequences $M\to D\backslash\backslash M\to BD$, $M\to\mathcal{D}\backslash\backslash M\to B\mathcal{D}$ and to compare the above diagram with the analogous diagram

$$\begin{array}{ccc}\mathcal{C}(D\backslash\backslash\mathcal{D}) & \to & \mathcal{C}(D\backslash\backslash\star)\\ \downarrow & & \downarrow\\ \mathcal{C}(\{e\}\backslash\backslash\star) & \to & \mathcal{C}(\mathcal{D}\backslash\backslash\star)\end{array}\ \supset$$

which expresses $D\backslash\backslash\mathcal{D}$ as the homotopy fiber of $BD\to B\mathcal{D}$.

Finally observe that the composite $D\backslash\backslash\mathcal{D}\xrightarrow{i}F'\xrightarrow{\gamma_0}\overline{B}\Gamma^n_{sl}$ is given by the pair $(\gamma_1\circ i', \gamma_2\circ i'')$. But $\gamma_1\circ i'=\beta'$ and $\gamma_2\circ i''=\beta''$ because the underlying functors and natural transformations are the same. Hence $\hat{\beta}\sim\gamma_0$. □

PROOF OF LEMMA 3.2. — We must show that $B\mathcal{D}' \to B\mathcal{D}_M \to BJ_M$ is a fibration sequence, where $\mathcal{D}' = \mathcal{D}iff^{\Phi}_{\omega 0}(M, \text{rel } x_0)$. Let $\mathcal{D}_0 = \{g \in \mathcal{D} : g(x_0) = x_0\}$ and $\mathcal{D}_1 = \{g \in \mathcal{D}_0 : dg_{x_0} = \text{id}.\}$. Then $\mathcal{D}_1 \to \mathcal{D}_0 \to \mathcal{S}\mathcal{L}(n, \mathbb{R})$ is an exact sequence of groups. Since $\mathcal{D}' \simeq \mathcal{D}_1$, this implies that

$$B\mathcal{D}' \to B\mathcal{D}_0 \to B\mathcal{S}\mathcal{L}(n, \mathbb{R}),$$

is a fibration sequence. By comparing the fibrations $M \to B\mathcal{D}_0 \to B\mathcal{D}$ and $M \to B\mathcal{D}_M \to B\mathcal{D}$ one sees that the obvious inclusion $B\mathcal{D}_0 \subsetneq B\mathcal{D}_M$ is a homotopy equivalence. The result now follows easily. □

PROOF OF LEMMA 3.3. — We must consider the sequence

$$BD' \to BD_M \to B\Gamma_M.$$

Since the groupoid homomorphism $D_M \to \Gamma_M$ is not a fibration and has no other apparent redeeming topological properties, the easiest way to understand the map $BD_M \to B\Gamma_M$ seems to be to replace the groupoids D_M and Γ_M by discrete categories, since then we may use Quillen's Theorem B.

Let $\mathcal{U} = \{U_\alpha\}$, $\alpha \in A$, be the cover of M by the interiors of all smoothly embedded closed discs. Let $\mathcal{C}(D_{\mathcal{U}})$ be the discrete category with objects $\alpha \in A$ and morphisms $\alpha \to \beta$ given by all $g \in D$ such that $gU_\alpha \subseteq U_\beta$. Further, let $\mathcal{C}(E_{\mathcal{U}})$ be the discrete category with the same objects as $\mathcal{C}(D_{\mathcal{U}})$ and with morphisms $\alpha \to \beta$ given by the germs at $\overline{U}_\alpha$ of those $g \in D$ with $gU_\alpha \nsubseteq U_\beta$. There are two related topological categories $\mathcal{C}(D^*_{\mathcal{U}})$ and $\mathcal{C}(E^*_{\mathcal{U}})$ whose spaces of objects consists of all pairs (x, α), $x \in U_\alpha$, topologized as the disjoint union $\coprod_\alpha U_\alpha$. Their morphisms are those morphisms $g : (x, \alpha) \to (y, \beta)$ in $\mathcal{C}(D_{\mathcal{U}})$, resp. $\mathcal{C}(E_{\mathcal{U}})$, which are such that $g(x) = y$ and $gU_\alpha \nsubseteq U_\beta$. The forgetful functors:

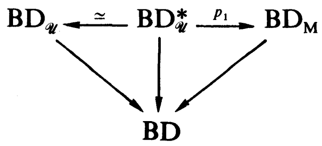
$$\mathcal{C}(D^*_{\mathcal{U}}) \to \mathcal{C}(D_{\mathcal{U}}) \quad \text{and} \quad \mathcal{C}(E^*_{\mathcal{U}}) \to \mathcal{C}(E_{\mathcal{U}})$$

give homotopy equivalences upon realization since they induce homotopy equivalences on the spaces of objects and morphisms. There are also functors:

$$p_1 : \mathcal{C}(D^*_{\mathcal{U}}) \to \mathcal{C}(D_M) \quad \text{and} \quad p_2 : \mathcal{C}(E^*_{\mathcal{U}}) \to \mathcal{C}(\Gamma_M).$$

Now p_2 induces a homotopy equivalence by the argument of [15], § 1.

To understand p_1, consider the diagram



The homotopy fiber of $BD_M \to BD$ is clearly M. We will show that the same is true for $BD_{\mathcal{U}} \to BD$. To do this, we apply

QUILLEN'S THEOREM B [13], § 1. — *Let $f: \mathcal{C} \to \mathcal{C}'$ be a functor between discrete categories. If $Y \in \mathrm{obj}\, \mathcal{C}'$, let $Y \diagdown f$ denote the category whose objects are pairs (X, v), $X \in \mathrm{obj}\, \mathcal{C}$, $v: Y \to fX$, and where a morphism $(X, v) \to (X', v')$ is a morphism $w: X \to X'$ in $\mathcal{C}$ such that $f(w)v = v'$. If for every morphism $Y \to Y'$ in $\mathcal{C}'$ the induced functor $Y' \diagdown f \to Y \diagdown f$ is a homotopy equivalence (resp. $\mathbb{Z}$-homology equivalence) then the sequence*

$$Y \diagdown f \to \mathcal{C} \xrightarrow{f} \mathcal{C}',$$

is a homotopy (resp. homology) fibration sequence.

(Following Quillen, we call a functor a homotopy equivalence, etc. if it is one upon realization.)

Since in our situation $\mathcal{C}' = \mathcal{C}(D)$ has only one object $\star$ and since all its morphisms are invertible, the induced functors $\star \diagdown f \to \star \diagdown f$ have inverses. They therefore give homeomorphisms upon realization. Hence the homotopy fiber of $BD_{\mathcal{U}} \to BD$ is $\| \star \diagdown f \|$. We aim to show that $\| \star \diagdown f \| \simeq M$. Now $\star \diagdown f$ has objects (α, h), $\alpha \in A$, $h \in D$, and a morphism $(\alpha, h) \to (\beta, gh)$ if and only if $g U_\alpha \supseteq U_\beta$. Consider the full subcategory $f^{-1}(\star)$ of $\star \diagdown f$ with objects (α, e). There is a functor $\rho : \star \diagdown f \to f^{-1}(\star)$ defined on objects by $\rho(\alpha, h) = (h^{-1}\alpha, e)$, where $h^{-1}\alpha \in A$ satisfies $U_{h^{-1}\alpha} = h^{-1} U_\alpha$. If $i : f^{-1}(\star) \subsetneq \star \diagdown f$ is the inclusion, then $\rho \circ i = \mathrm{Id}$ and there is a natural transformation from $i \circ \rho$ to Id. Therefore i and ρ are adjoint functors, and so are homotopy equivalences by [14]. But $f^{-1}(\star)$ is the full subcategory of the category of open sets and inclusions of M corresponding to the cover $\mathcal{U}$. Therefore $f^{-1}(\star) \simeq M$ by Segal's covering lemma in [15], Prop. A. 5. Hence the homotopy fiber of $BD_{\mathcal{U}} \to BD$ is M as claimed. It follows that p_1 is an equivalence.

We now have a commutative diagram

$$\begin{array}{ccccc}
BD_{\mathcal{U}} & \xleftarrow[\simeq]{} & BD^*_{\mathcal{U}} & \xrightarrow[\simeq]{p_1} & BD_M \\
\downarrow q & & \downarrow & & \downarrow q_1 \\
BE_{\mathcal{U}} & \xleftarrow[\simeq]{} & BE^*_{\mathcal{U}} & \xrightarrow[\simeq]{p_2} & B\Gamma_M
\end{array}$$

Choose $\alpha \in A$ with $x_0 \in U_\alpha$, and let D'_α be the group $\{g \in D' : g = \mathrm{id} \text{ near } \overline{U}_\alpha\}$. Then $\mathcal{C}(D'_\alpha)$ may be included in $\mathcal{C}(D_{\mathcal{U}})$ as the subcategory with objects (α, g), $g \in D'_\alpha$. Since the inclusion $BD'_\alpha \to BD'$ is a $\mathbb{Z}$-homology equivalence [11], it will clearly suffice to show that:

$$BD'_\alpha \to BD_{\mathcal{U}} \to BE_{\mathcal{U}},$$

is a homology fibration sequence.

To do this we apply Quillen's Theorem B to the functor $q: \mathscr{C}(D_{\mathscr{U}}) \to \mathscr{C}(E_{\mathscr{U}})$. For each object α in $\mathscr{C}(E_{\mathscr{U}})$, the category $\alpha \diagdown q$ has objects (γ, h), where h is a germ of diffeomorphism at $\overline{U}_\alpha$ taking U_α into U_γ, and has a morphism $(\gamma, \bar{h}) \to (\gamma', \overline{gh})$ for all $g: \gamma \to \gamma'$ in $\mathscr{C}(D_{\mathscr{U}})$. Let v be the morphism $\bar{k}: \beta \to \alpha$ in $\mathscr{C}(E_{\mathscr{U}})$, and consider the diagram:

$$\begin{array}{ccc} \alpha \diagdown q & \xrightarrow{v_1} & \beta \diagdown q \\ i \Big\uparrow \Big\downarrow \rho & & i \Big\uparrow \Big\downarrow \rho \\ \mathscr{C}(D'_\alpha) & \xrightarrow{v_2} & \mathscr{C}(D'_\beta), \end{array}$$

where the functors i are the inclusions and v_1 is induced by v in the obvious way. We define $\rho: \alpha \diagdown q \to \mathscr{C}(D'_\alpha)$ on morphisms by:

$$\rho((\gamma, \bar{h}) \xrightarrow{g} (\gamma', \overline{gh})) = (\overline{\overline{gh}})^{-1} g \bar{\bar{h}},$$

where, for each $(\gamma, \bar{h})$, the element $\bar{\bar{h}} \in D$ is chosen to have germ $\bar{h}$ at $\overline{U}_\alpha$. The functor $\rho: \beta \diagdown q \to \mathscr{C}(D'_\beta)$ is defined similarly. Finally v_2 is induced by the group homomorphism $g \mapsto k^{-1} g k$, where $k \in D$ is chosen to have germ $\bar{k}$ at $\overline{U}_\beta$. It is easy to check that i and ρ are adjoint, so that they are homotopy equivalences. Also, since there is a natural transformation from $i \circ v_2$ to $v_1 \circ i$, the diagram is homotopy commutative. Moreover, v_2 is the composite of an isomorphism followed by the inclusion $D'_{k^{-1}\alpha} \subsetneq D'_\beta$. But this inclusion is a $\mathbb{Z}$-homology equivalence by [11]. Hence v_1 is also a $\mathbb{Z}$-homology equivalence. Therefore Quillen's Theorem B applies to show that $\| \alpha \diagdown q \| \to BD_{\mathscr{U}} \to BE_{\mathscr{U}}$ is a homology fibration sequence. Since $BD'_\alpha \simeq \| \alpha \diagdown q \|$, the same is true of $BD'_\alpha \to BD_{\mathscr{U}} \to BE_{\mathscr{U}}$. □

REFERENCES

[1] A. BANYAGA, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique* (*Comm. Math. Helv.*, 53, 1978, pp. 174-227).

[2] R. BROOKS and P. TRAUBER, *The van Est Theorem for Groups of Diffeomorphisms* (*Hadronic Journ.*, 1, 1978, pp. 141-146).

[3] M. L. GROMOV, *Stable Mappings of Foliations into Manifolds* (*Math. U.S.S.R. Izv.* 3, 1969, pp. 671-694).

[4] A. HAEFLIGER, *Homotopy and Integrability*, in *Manifolds-Amsterdam*, 1970 (*Springer Lecture Notes*, # 197, 1971, pp. 133-163).

[5] A. HAEFLIGER, *Differential Cohomology*, Summer School in Varenna, 1976, C.I.M.E., 1979.

[6] S. HURDER, *Global Invariants for Measured Foliations*, preprint, 1982.

[7] J. MATHER, *Foliations and Local Homology of Groups of Diffeomorphisms*, Proceedings of the ICM, Vancouver 1974.

[8] D. MCDUFF, *Foliations and Monoids of Embeddings*, in *Geometric Topology*, Cantrell, Academic Press, 1979, pp. 429-444.

[9] D. McDUFF, *The Homology of Some Groups of Diffeomorphisms* (*Comm. Math. Helv.*, 55, 1980, pp. 97-129).
[10] D. McDUFF, *Local Homology of Groups of Volume-preserving Diffeomorphisms*, I (*Ann. Sc. Éc. Norm. Sup.*, 1982, pp. 609-648).
[11] D. McDUFF, *Local Homology of Groups of Volume-preserving Diffeomorphisms*, II (*Comm. Math. Helv.*, 58, 1983, pp. 135-165).
[12] D. McDUFF, *Some Canonical Cohomology Classes on Groups of Volume-preserving Diffeomorphisms*, *Trans. A.M.S.*, 275, 1983, pp. 345-356.
[13] D. QUILLEN, *Higher Algebraic K-Theory I*, in *Springer Lect. Notes* # 341, 1973, pp. 75-148.
[14] G. B. SEGAL, *Classifying spaces and spectral sequences* (*Publ. Math. I.H.E.S.*, 34, 1968, pp. 105-112.
[15] G. B. SEGAL, *Classifying Spaces Related to Foliations* (*Topology*, 17, 1978, pp. 367-382).
[16] W. THURSTON, *On the Structure of the Group of Volume-preserving Diffeomorphisms*, preprint, 1973.

(Manuscrit reçu le 11 novembre 1982.)

D. McDUFF,
Department of Mathematics,
State University of New York at Stony Brook,
Stony Brook,
NY 11794,
U.S.A.