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FILTRATIONS OF COHOMOLOGY MODULES FOR CHEVALLEY GROUPS

BY HENNING HAAHR ANDERSEN

Let G denote a Chevalley group and let B be a Borel subgroup of G. Suppose k is a field of characteristic $p>0$ and let G_k denote the algebraic group over k obtained from G. In this paper we construct filtrations of certain subquotients of the sheaf cohomology groups $H^i(G_k/B_k, \mathscr{L}(\lambda))$, where $\mathscr{L}(\lambda)$ is the line bundle on G_k/B_k induced by the character λ of B. The Weyl modules for G_k occur as the top cohomology groups of certain such line bundles and in this case our filtrations coincide with the filtrations constructed by J. C. Jantzen [13].

From our construction it is easy to obtain a bound on the length of the filtrations, *see* Proposition 4.6. Moreover, we show that the formal characters of the filtrations satisfy a nice "sum formula" which in the Weyl module case is Jantzen's sum formula. Thus we prove (and extend) Jantzen's formula in arbitrary characteristic (J. C. Jantzen proved his sum formula (in the Weyl module case) under some mild restrictions on p [13] and he conjectured it to hold for all primes [14]).

Finally we consider the behaviour of homomorphisms between two "neighbouring" Weyl modules with respect to the filtrations. We conjecture that when the highest weights involved belong to the lowest p^2-alcove and are not close to its upper wall then these intertwining homomorphisms respects the filtrations (up to shift by 1). (Though not stated anywhere (known to this author) this conjecture can be traced back to J. C. Jantzen. It is analogous to his characteristic zero conjecture for intertwining homomorphisms between Verma modules, *see* [15]). Following O. Gabber and A. Joseph's approach for Verma modules [10] we prove that this conjecture implies the conjecture of G. Lusztig on the modular characters of simple modules for G [17].

In order to construct our filtrations we need to consider both sheaf cohomology on the $\mathbb{Z}$-variety G/B and Hochschild cohomology over the integers. A key result here is a universal coefficient theorem, Theorem 1.18 below, which allows us to compare characteristic zero cohomology with characteristic p cohomology. As a byproduct we prove a conjecture of J. E. Humphreys concerning weight multiplicities in the cohomology modules $H^i(G_k/B_k, \mathscr{L}(\lambda))$, *see* Corollary 2.9.ii.

The paper is organized as follows. In section 1 we derive the results we need on cohomology over the integers (actually in this section we work over any principal ideal domain and G may be any affine group scheme whose coordinate ring is flat). Then in section 2 we combine these general results with the Borel-Weit-Bott theorem and with the strong linkage principle to obtain information about the cohomology modules $H^i(G/B, \mathscr{L}(\lambda))$. Section 3 is devoted to the study of the semi-simple rank 1 case and the results here are used in section 4 to construct the above mentioned filtrations and to prove their basic properties. Finally in section 5 we show that it is possible to define the concept of translation over a discrete valuation ring and we use this in section 6 to compare the filtrations of two neighbouring Weyl modules.

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1. Cohomology over a principal ideal domain

In this section A will denote a principal ideal domain and G will be any affine group scheme over A whose coordinate ring A [G] is flat (i. e. torsion free) over A. We denote the comultiplication in A [G] by $\mu : A[G] \to A[G] \otimes A[G]$ and the coidentity by $\varepsilon : A[G] \to A$.

DEFINITION 1.1. — Let V be an A-module. Then V is called a rational G-module if V has the structure of a comodule for A [G].

When V is a rational G-module then we let $\Delta_V : V \to V \otimes A[G]$ denote the map defining its comodule structure (occasionally we shall need also to consider right comodules).

J.-P. Serre [18] has proved that rational G-modules have the following properties

(1.2) The category of rational G-modules is an abelian category.

(1.3) Any rational G-module is a filtered union of submodules of finite type.

(1.4) Every rational G-module of finite type is the quotient of a free rational G-module of finite type. In particular any such module V has a resolution

$$0 \to P_1 \to P_0 \to V \to 0,$$

with P_i, $i=0, 1$ rational G-modules which are free of finite type.

The words "free", "of finite type" refer to the structure as A-modules.

Moreover, we have

(1.5) The category of rational G-modules has enough injectives.

This statement is very easy to prove once we have available the concept of induction:

DEFINITION 1.6. — Let H be a closed subgroup of G. Then if E is a rational H-module we define the induced module $E|_H^G$ as follows

$$E|_H^G = (E \otimes A[G])^H.$$

Here upper H means fixed points under the action of H (recall that if F is a rational H-module then the set of fixed points in F are given by $F^H = \{x \in F \mid \Delta_F(x) = x \otimes 1\}$). The action of H on $E \otimes A[G]$ is the right diagonal action and $E|_H^G$ becomes a G-module through the left action of G on A[G]. We will only consider subgroups H for which A[H] is flat.

Recall that the induction functor $|_H^G$ has the following properties, *see* [7]

(1.7) (Universal mapping). Let $Ev_E : E|_H^G \to E$ be the map induced by the counit ε. If $f: V \to W$ is an H-homomorphism from a rational G-module V to a rational H-module W then there exists a unique G-homomorphism $\tilde{f}: V \to W|_H^G$ such that $f = Ev_E \circ \tilde{f}$.

(1.8) (Reciprocity). With V and W as in (1.7) we have

$$\mathrm{Hom}_G(V, W|_H^G) \simeq \mathrm{Hom}_H(V, W).$$

(1.9) (Transitivity). If K is a closed subgroup contained in H and F is a rational K-module then

$$F|_K^H|_H^G \simeq F|_K^G.$$

(1.10) (Tensor identity). If V and W are as in (1.7) then we have

$$W|_H^G \otimes V \simeq (W \otimes V)|_H^G.$$

(1.11) If I is a rational H-module which is injective in the category of rational H-modules then $I|_H^G$ is injective in the category of rational G-modules.

Now we can prove (1.15) easily: let 1_A denote the trivial group scheme over A and suppose V is a rational G-module. Then if I is an injective A-module containing V then by (1.7) and (1.11) we see that $I|_{1_A}^G$ is an injective G-module containing V.

Q.E.D.

The properties (1.2) and (1.5) of the category of rational G-modules (resp. H-modules) allow us to define the right derived functors

$$H^n(G, -) \text{ of the fixed point functor } V \to V^G$$

and

$$H^n(G/H, -) \text{ of the induction functor } E \to E|_H^G.$$

The relation between these two functors is (just as in the case where A is a field, *see* [8]).

PROPOSITION 1.12. — *Let* E *denote a rational* H-*module. Then we have*

$$H^n(G/H, E) \simeq H^n(H, E \otimes A[G]),$$

for all $n>0$.

To prove this we need a useful

LEMMA 1.13. — *If* E *is any rational* H-*module then both* $H^n(G/H, E \otimes A[H])$ *and* $H^n(H, E \otimes A[H])$ *vanish for* $n>0$.

Proof. — Let $E \to I.$ be a resolution of the A-module E by injective A-modules I_i. Since A[H] is flat the complex $I. \otimes A[H] = I. |_{1_A}^H$ is a resolution of $E \otimes A[H] = E|_{1_A}^H$ by injective H-modules (1.11). Now by reciprocity (1.8) we have

$$(I_i|_{1_A}^H)^H \simeq \operatorname{Hom}_H(A, I_i|_{1_A}^H) \simeq \operatorname{Hom}_A(A, I_i) \simeq I_i.$$

It follows that $H^n(H, E \otimes A[H]) = 0$ for $n>0$. On the other hand we have by transitivity (1.9)

$$(I_i|_{1_A}^H)|_H^G \simeq I_i|_{1_A}^G \simeq I_i \otimes A[G]$$

and since A[G] is flat we conclude that $H^n(G/H, E \otimes A[H]) = 0$ for $n>0$.

Q.E.D.

Remark 1.14. — The above lemma allows us to use the standard resolution

$$0 \to A \to A[H] \to A[H] \otimes A[H] \to A[H] \otimes A[H] \otimes A[H] \to \dots$$

tensored by E to compute both $H^n(G/H, E)$ and $H^n(H, E)$.

Proof of Proposition 1.12. — For $n=0$ the statement is just the definition of $H^0(G/H, E)$. Now by Remark 1.14 we may compute $H^n(G/H, E)$ as the n-th cohomology of the complex

$$0 \to (E \otimes A[H])|_H^G \to (E \otimes A[H] \otimes A[H])|_H^G \to \dots$$

while $H^n(H, E \otimes A[G])$ is the n-th cohomology of the complex

$$0 \to (E \otimes A[G] \otimes A[H])^H \to (E \otimes A[G] \otimes A[H] \otimes A[H])^H \to \dots$$

However, these two complexes are identical.

Q.E.D.

As an immediate consequence of (1.9) and (1.11) we get

PROPOSITION 1.15. — *If* K *is a closed subgroup contained in* H *and* E *is a rational* K-*module then we have the spectral sequence*

$$H^i(G/H, H^j(H/K, E)) \Rightarrow H^{i+j}(G/K, E).$$

We can also generalize (1.10)

PROPOSITION 1.16 (*Generalized tensor identity*). — *Let* V *be a rational* G*-module which is flat over* A. *Then for any rational* H*-module* E *we have*

$$H^n(G/H, E \otimes V) \simeq H^n(G/H, E) \otimes V$$

for $n \geqq 0$.

Proof. — Let I. denote the standard complex

$$0 \to A[H] \to A[H] \otimes A[H] \to A[H] \otimes A[H] \otimes A[H] \to \ldots$$

Then using (1.14) and (1.10) we get

$$\begin{aligned} H^n(G/H, E \otimes V) &\simeq H^n((I.\otimes E \otimes V)|_H^G) \simeq H^n((I.\otimes E)|_H^G \otimes V) \\ &\simeq H^n((I.\otimes E)|_H^G) \otimes V \simeq H^n(G/H, E) \otimes V. \end{aligned}$$

Q.E.D.

We will now examine how the cohomology behaves under base change. We let $A \to R$ denote a homomorphism into a commutative ring R and we denote by G_R the extension of G to R (i.e. G_R is the affine group scheme over R with coordinate ring $R[G_R] = A[G] \otimes R$).

LEMMA 1.17. — *Let* V *be a rational* G*-module which is flat over* A. *Then*

$$V^G \otimes R \simeq (V \otimes R)^{G_R}.$$

Proof. — V^G is the kernel of the map $V \to V \otimes A[G]$ which takes v into $\Delta_V(v) - v \otimes 1$. As $V \otimes A[G]$ is torsionfree so is the image of this map. Hence $V^G \otimes R$ is the kernel of the induced map $V \otimes R \to (V \otimes R) \otimes_R R[G_R]$ which by definition is $(V \otimes R)^{G_R}$.

Q.E.D.

THEOREM 1.18 (*Universal coefficients*).

(i) *Let* V *be a rational* G*-module which is flat over* A. *For each* $i \geqq 0$ *we have a short exact sequence of* R*-modules*

$$0 \to H^i(G, V) \otimes R \to H^i(G_R, V \otimes R) \to \mathrm{Tor}_1^A(H^{i+1}(G, V), R) \to 0.$$

(ii) *Let* E *be a rational* H*-module which is flat over* A. *For each* $i \geqq 0$ *we have a short exact sequence of rational* G_R*-modules*

$$0 \to H^i(G/H, E) \otimes R \to H^i(G_R/H_R, E \otimes R) \to \mathrm{Tor}_1^A(H^{i+1}(G/H, E), R) \to 0.$$

Proof. — (i) Let $I_j = (V \otimes A[G]^{\otimes j})^G$ and set B_j (resp. C_j) equal to the image of the natural map $I_{j-1} \to I_j$ (resp. the kernel of $I_j \to I_{j+1}$). Then B_j and C_j are torsionfree and we get the diagram

$$\begin{tikzcd}
 & & & 0 & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}_1^A(H^{j+1}(I.), R) \arrow[rr] & & B_{j+1} \otimes R \arrow[r] \arrow[u] & C_{j+1} \otimes R \arrow[d] & \\
 & & I_{j-1} \otimes R \arrow[r] \arrow[d] & I_j \otimes R \arrow[r] \arrow[u] & I_{j+1} \otimes R & \\
 & & B_j \otimes R \arrow[r] & C_j \otimes R \arrow[r] \arrow[u] & H^j(I.) \otimes R \arrow[r] & 0 \\
 & & & 0 \arrow[u] & &
\end{tikzcd}$$

By Lemma 1.17 we know $H^j(G_R, V \otimes R) = H^j(I. \otimes R)$ and hence we easily get (i) from this diagram. Now (ii) follows from (i) by Proposition 1.12.

Q.E.D.

Let now E and F be rational H-modules. There is a natural G-homomorphism

$$E|_H^G \otimes F|_H^G \to (E \otimes F)|_H^G$$

coming from the H-homomorphism

$$Ev_E \otimes Ev_F : E|_H^G \otimes F|_H^G \to E \otimes F.$$

This homomorphism is functorial in both E and F and gives rise to natural G-homomorphisms

$$(1.19) \qquad H^i(G/H, E) \otimes H^j(G/H, F) \to H^{i+j}(G/H, E \otimes F).$$

We end this section by considering the case where the quotient G/H exists (as a scheme). Then we let $\pi : G \to G/H$ denote the quotient map and we will assume that π is locally trivial. In this case every rational H-module E induces a locally free sheaf $\mathscr{L}(E)$ on G/H. The sections of $\mathscr{L}(E)$ over an open subset $U \subset G/H$ are

$$\Gamma(U, \mathscr{L}(E)) = (\Gamma(\pi^{-1}(U), \mathscr{O}_G) \otimes E)^H,$$

where the action of H on $\Gamma(\pi^{-1}(U)\ \mathscr{O}_G) \otimes E$ is the right diagonal action.

Note that the set of global sections of $\mathscr{L}(E)$ is just the induced module $E|_H^G$. In fact we find

Proposition 1.20. — *In the above situation we have*

$$H^i(G/H, E) \simeq H^i(G/H, \mathscr{L}(E))$$

for all $i \geqq 0$.

Proof. — It is enough to prove that for any rational H-module F we have

$$H^i(G/H, \mathscr{L}(F \otimes A[H])) = 0 \qquad \text{for} \quad i > 0.$$

Note that $\mathscr{L}(A[H]) \simeq \pi_* \mathcal{O}_G$ and hence $\mathscr{L}(F \otimes A[H]) \simeq \mathscr{L}(F) \otimes \pi_* \mathcal{O}_G \simeq \pi_*(\pi^* \mathscr{L}(F))$ where the last identification is the projection formula. Since π is an affine homomorphism we have $R^i \pi_*(\pi^* \mathscr{L}(F)) = 0$ for $i > 0$ and hence

$$H^i(G/H, \mathscr{L}(F \otimes A[H])) \simeq H^i(G/H, \pi_*(\pi^* \mathscr{L}(F))) \simeq H^i(G, \pi^* \mathscr{L}(F)).$$

The proposition now follows *via* the fact that G is affine.

Q.E.D.

Remark 1.21. — From this proposition it follows that $H^i(G/H, E) = 0$ for $i > \dim G/H$ and any rational H-module E. If E is finitely generated over A and G/H is projective then it also follows that $H^i(G/H, E)$ is finitely generated over A for all i.

2. Induction from a Borel subgroup of a Chevalley group

Let G be a Chevalley group. Thus G is an affine group scheme over $\mathbb{Z}$ and if k is an algebraically closed field then G_k is a connected reductive linear algebraic group over k. We fix a (split) maximal torus T in G and a Borel subgroup B containing T. Denote by R the root system of G with respect to T and choose a basis S in R such that the roots of B are the negative roots $R_- = -R_+$. We let W denote the Weyl group and we set X(T) equal to the character group of T. In addition to the usual action of W on X(T) which is given by $s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$ when $\alpha \in R$, s_α is the corresponding reflection, $\alpha^\vee$ the coroot, $\lambda \in X(T)$ we also consider the "dot action" given by $w . \lambda = w(\lambda + \rho) - \rho$. Here ρ is half the sum of the positive roots. We assume $\rho \in X(T)$.

From now on we will write $H^i(E)$ instead of $H^i(G/B, E)$ and if R is a commutative ring we set $H^i_R(E) = H^i(G_R/B_R, E \otimes R)$. By Remark 1.21 $H^i_R(E) = 0$ for $i > \dim G/B$ and if E is of finite type then $H^i_R(E)$ is a finitely generated R-module for all i.

Set $X(T)_+ = \{ \lambda \in X(T) \mid \langle \alpha^\vee, \lambda \rangle \geqq 0 \}$ and recall the following result on the cohomology of line bundles on G/B.

(2.1) (Borel-Weil-Bott) ([6], [9]). Let $\lambda + \rho \in X(T)_+$ and $w \in W$.

Then

$$H^i_{\mathbb{Q}}(w . \lambda) \simeq \begin{cases} H^0_{\mathbb{Q}}(\lambda), & i = l(w) \\ 0 & \text{otherwise.} \end{cases}$$

Here l denotes the length function on W.

(2.2) (Kempf's vanishing) ([16], [3]). Suppose k is a field of arbitrary characteristic. If $\lambda \in X(T)_+$ then

$$H^i_k(\lambda) = 0 \qquad \text{for} \quad i > 0.$$

Fix now a prime p and let in the following k denote an algebraically closed field of characteristic p. Denote by W_p the affine Weyl group, i.e. the group generated by $\{s_{\alpha, m} \mid \alpha \in R_+, m \in \mathbb{Z}\}$ where $s_{\alpha, m}$ is the reflection given by $s_{\alpha, m} . \lambda = s_\alpha . \lambda + mp\alpha$. Recall that the irreducible representations of G_k are parametrized by $X(T)_+$. We let $L(\lambda)$ denote the irreducible G_k-module with highest weight $\lambda \in X(T)_+$.

(2.3) (The linkage principle) [1]. Let $\mu \in X(T)$, $\lambda \in X(T)_+$. If $L(\lambda)$ is a composition factor in $H_k^i(\mu)$ for some i then $\lambda \in W_p . \mu$.

The proof of the strong linkage principle shows also

(2.4) Suppose λ is minimal in $X(T)_+$ (with respect to the ordering induced by R_+). Then for $w \in W$ we have

$$H_k^i(w . \lambda) \simeq \begin{cases} H_k^0(\lambda) & \text{if } i = l(w) \\ 0 & \text{otherwise.} \end{cases}$$

In particular if $N = |R_+| = \dim G/B$ then $H_k^N(-2\rho) \simeq k$.

Remembering that $H_k^\cdot(\lambda)$ is the cohomology of the line bundle $\mathscr{L}(\lambda)$, Proposition 1.20, and noting that $\mathscr{L}(-2\rho)$ is the canonical sheaf on G_k/B_k we have

(2.5) (Serre duality). Let E be a rational B_k-module. Then for all i

$$H_k^i(E) \simeq H_k^{N-i}(E^* \otimes (-2\rho))^*.$$

Combining these results with Theorem 1.18 we obtain

COROLLARY 2.6. — *Let* $\mu, \lambda \in X(T)_+$ *and* $w \in W$:

(i) $H^i(w . \lambda)$ *is a torsion module for* $i \neq l(w)$.

(ii) $H^0(\lambda)$ *and* $H^N(-\lambda - 2\rho)$ *are free* $\mathbb{Z}$*-modules.*

(iii) *If* $L(\mu)$ *is a composition factor in* $H^i(w . \lambda) \otimes k$ *for some* i *then* $\mu \in W_p . \lambda$.

(iv) $H^N(-2\rho) \simeq \mathbb{Z}$.

Suppose V is a rational T-module of finite type. Recall that $V = \bigoplus_{\mu \in X(T)} V_\mu$ where $V_\mu = \{v \in V \mid \Delta_V(v) = v \otimes \mu\}$ (note that an element in $X(T)$ corresponds to a homomorphism $\mathbb{Z}[X, X^{-1}] \to \mathbb{Z}[T]$ and thus may be identified with an element in $\mathbb{Z}[T]$). If V is free as $\mathbb{Z}$-module we define

$$\operatorname{ch} V = \sum_{\mu \in X(T)} \operatorname{rank}(V_\mu) e^\mu \in \mathbb{Z}[X(T)].$$

Then $\operatorname{ch} V = \operatorname{ch}(V \otimes k) = \operatorname{ch}(V \otimes \mathbb{Q})$ where the last two expressions denote the usual character of a T_k, resp. $T_{\mathbb{Q}}$, -module.

We set for $\mu \in X(T)$

$$\chi(\mu) = \sum_i (-1)^i \operatorname{ch} H_k^i(\mu)$$

(i.e. $\chi(\mu)$ is the Euler character of the line bundle $\mathscr{L}(\mu)$ on G_k/B_k).

COROLLARY 2.7. — *Let* $\lambda \in X(T)_+$ *and* $w \in W$. *Then*

(i) $\chi(w.\lambda)=(-1)^{l(w)} \operatorname{ch} H_{\mathbb{Q}}^{l(w)}(w.\lambda)=(-1)^{l(w)} \operatorname{ch} H_{\mathbb{Q}}^{0}(\lambda)=(-1)^{l(w)}\chi(\lambda)$.

(ii) $\operatorname{ch}(H^{l(w)}(w.\lambda)_{\text{free}})=\chi(\lambda)$, *where* $H^{l(w)}(w.\lambda)_{\text{free}}$ *denotes the free quotient of* $H^{l(w)}(w.\lambda)$, *i.e. the quotient of* $H^{l(w)}(w.\lambda)$ *modulo the torsion submodule* $H^{l(w)}(w.\lambda)_t$.

Proof. — (i) is a consequence of the invariance of the Euler character and (ii) follows from (i) *via* Theorem 1.18. Alternatively the following lemma combined with Theorem 1.18 show that

$$\begin{aligned}\sum_i(-1)^i \operatorname{ch} H_k^i(w.\lambda)&=\sum_i(-1)^i(\operatorname{ch} H^i(w.\lambda)\otimes k+\operatorname{ch} H^{i+1}(w.\lambda)_t\otimes k)\\ &=(-1)^{l(w)} \operatorname{ch} H^{l(w)}(w.\lambda)_{\text{free}}+\sum_i(-1)^i(\operatorname{ch} H^i(w.\lambda)_t\otimes k+\operatorname{ch} H^{i+1}(w.\lambda)_t\otimes k)\\ &=(-1)^{l(w)} \operatorname{ch} H^{l(w)}(w.\lambda)_{\text{free}}.\end{aligned}$$

Q.E.D.

LEMMA 2.8. — *Let* V *be a finite rational* T-*module. Then*

$$\operatorname{ch} V\otimes k=\operatorname{ch} \operatorname{Tor}_1^{\mathbb{Z}}(V, k).$$

Proof. — Use the resolution $0\to\mathbb{Z}\xrightarrow{p}\mathbb{Z}\to\mathbb{Z}/p\mathbb{Z}\to 0$ to see that the $T_{\mathbb{Z}/p\mathbb{Z}}$-modules $V\otimes\mathbb{Z}/p\mathbb{Z}$ and $\operatorname{Tor}_1^{\mathbb{Z}}(V, \mathbb{Z}/p\mathbb{Z})$ have the same characters.

Q.E.D.

If W is a finite dimensional G_k-module and $\lambda\in X(T)_+$ then we let $[W : L(\lambda)]$ denote the composition factor multiplicity of $L(\lambda)$ in W.

As a consequence of Theorem 1.18, Corollary 2.6 (i) and the above lemma we get the following

COROLLARY 2.9. — *Let* $\mu, \lambda\in X(T)_+$ *and* $w\in W$. *Then*

(i) $$\begin{aligned}[H_k^{l(w)}(w.\lambda) : L(\mu)]&=[H^{l(w)}(w.\lambda)\otimes k : L(\mu)]+[H^{l(w)+1}(w.\lambda)\otimes k : L(\mu)]\\ &=[H_k^0(\lambda) : L(\mu)]+[H^{l(w)}(w.\lambda)_t\otimes k : L(\mu)]+[H^{l(w)+1}(w.\lambda)\otimes k : L(\mu)];\end{aligned}$$

(ii) $$\dim H_k^{l(w)}(w.\lambda)_\nu\geqq\dim H_{\mathbb{Q}}^{l(w)}(w.\lambda)_\nu=\dim H_{\mathbb{Q}}^0(\lambda)_\nu=\dim H_k^0(\lambda)_\nu.$$

Here $H^{l(w)}(w.\lambda)_t$ denotes the torsion part of $H^{l(w)}(w.\lambda)$.

Part (ii) was conjectured by J. E. Humphreys.

Finally Serre duality gives:

PROPOSITION 2.10. — *Let* E *be a rational* B-*module which is free of finite type. Then the natural homomorphism* (1.19)

$$H^i(E)\otimes H^{N-i}(E^*\otimes(-2\rho))\to H^N(E\otimes E^*\otimes(-2\rho)),$$

combined with the homomorphism induced by the canonical map $E\otimes E^*\to\mathbb{Z}$ *gives a non-singular pairing*

$$H^i(E)_{\text{free}}\times H^{N-i}(E^*\otimes(-2\rho))_{\text{free}}\to H^N(-2\rho)\simeq\mathbb{Z}.$$

Proof. — We shall prove that the homomorphism $H^i(E)_{free} \to H^{N-i}(E^* \otimes (-2\rho))^*_{free}$ induced by this pairing is an isomorphism. The injectivity follows by taking $k=\mathbb{Q}$ in (1.18) and (2.5). To prove the surjectivity we consider first the case $i=0$.

If the rank of E is 1 then it follows from (2.2), (2.5) and (1.18) that for any field k have $H^0(E)\otimes k \simeq H^0_k(E)$ and $H^N(E^* \otimes (-2\rho))\otimes k \simeq H^N_k(E^* \otimes (-2\rho))$. So in this case the proposition follows immediately from (2.5). An easy induction on the rank of E gives then the case where all weights of E are in $X(T)_+$. For a general E pick n so big that $n\rho+\nu \in X(T)_+$ for all weights ν of E. As $-n\rho$ is the smallest weight in $H^0(n\rho)$ we have a short exact sequence of free rational B-modules

$$0 \to \mathbb{Z} \to H^0(n\rho)\otimes(n\rho) \to Q \to 0 \tag{2.11}$$

(with B acting trivially on $\mathbb{Z}$). Tensoring this sequence by E [resp. dualizing this sequence and tensoring it by $E^* \otimes (-2\rho)$] we obtain the commutative diagram where for the middle terms we have used (1.16)

$$\begin{tikzcd}
0 \arrow[r] & H^0(E) \arrow[r] \arrow[d] & H^0(n\rho)\otimes H^0(E\otimes(n\rho)) \arrow[r] \arrow[d] & H^0(Q\otimes E) \arrow[d] \\
& H^N(E^*\otimes(-2\rho))^*_{free} \arrow[r] & H^0(n\rho)\otimes H^N(E^*\otimes(-(n+2)\rho))^* \arrow[r] & H^N(Q^*\otimes E^*\otimes(-2\rho))^*_{free}
\end{tikzcd}$$

(note that all the H^0's are free and that so is $H^N(E^* \otimes -(n+2)\rho)$ because all weights in $E^* \otimes (-n\rho)$ are in $-X(T)_+$). By the above the middle vertical map is an isomorphism and hence so is the left vertical map.

Next, consider the case $i>0$. Choosing n as before we have $H^i(E\otimes n\rho)=H^{N-i}(E^* \otimes -(n+2)\rho)=0$
(2.2). Hence we get *via* the exact sequence (2.11) the commutative diagram

$$\begin{tikzcd}
H^{i-1}(Q\otimes E)_{free} \arrow[r] \arrow[d] & H^i(E)_{free} \arrow[r] \arrow[d] & 0 \\
H^{N-i+1}(Q^*\otimes E^*\otimes(-2\rho))^*_{free} \arrow[r] & H^{N-i}(E^*\otimes(-2\rho))^*_{free} \arrow[r] & 0
\end{tikzcd}$$

By induction on i we may assume that the left vertical map is surjective and the proposition follows.

Q.E.D.

3. Filtrations. The rank 1 case

We preserve the notation from section 2.

In this section we fix a simple root α and we let P_α denote the minimal parabolic subgroup corresponding to α (i.e. P_α contains B and has α as its only positive root). When E is a rational B-module we will use the notation $H^i_\alpha(E)$, $H^i_{\alpha,p}(E)$ and $H^i_{\alpha,k}(E)$ for the cohomology modules $H^i(P_\alpha/B, E)$, $H^i(P_{\alpha\mathbb{Z}_p}/B_{\mathbb{Z}_p}, E\otimes \mathbb{Z}_p)$ and $H^i_\alpha(P_{\alpha k}/B_k, E\otimes k)$, respectively. Here $\mathbb{Z}_p$ denotes the localization of $\mathbb{Z}$ at the prime p. By abuse of notation we also write p for the generator of the maximal ideal in $\mathbb{Z}_p$.

Recall that $H^i_\alpha(E)=0$ for $i>1$ and that if $\lambda\in X(T)$ then

$$H^0_{\alpha,k}(\lambda)\neq 0 \qquad \text{if and only if} \quad \langle \alpha^\vee, \lambda\rangle \geqq 0$$

and

$$H^1_{\alpha,k}(\lambda)\neq 0 \qquad \text{if and only if} \quad \langle \alpha^\vee, \lambda\rangle < -1.$$

It follows from Theorem 1.18 that $H^i_\alpha(\lambda)$ is always a free $\mathbb{Z}$-module and that $H^i_{\alpha,R}(\lambda)\simeq H^i_\alpha(\lambda)\otimes R$ for any commutative ring R.

Let now $\lambda\in X(T)$ be fixed such that $\langle \alpha^\vee, \lambda\rangle = r \geqq 0$. It is easy to see (compare e. g. with Lemma 2 in [1]) that we may choose a basis $\{v_0, v_1, \ldots, v_r\}$ for $H^0_\alpha(\lambda)$ (resp. $\{v'_0, v'_1, \ldots, v'_r\}$ for $H^1_\alpha(s_\alpha.\lambda)$) such that

$$v_i\in H^0_\alpha(\lambda)_{\lambda-i\alpha}, \qquad \text{resp. } v'_i\in H^1_\alpha(s_\alpha.\lambda)_{s_\alpha.\lambda+(i+1)\alpha}$$

and such that the action of the root subgroup $U_\alpha\subset P_\alpha$ is given by

$$\Delta_{H^0_\alpha(\lambda)}(v_i)=\sum_j \binom{i}{j}(v_j\otimes X^{i-j}), \quad \text{resp. } \Delta_{H^1_\alpha(s_\alpha.\lambda)}v'_i=\sum_j \binom{j}{i}(v'_j\otimes X^{j-i}).$$

Here we have identified $\mathbb{Z}[U_\alpha]$ with $\mathbb{Z}[X]$.

It is now straight forward to check that the two maps

$$T^\lambda_\alpha: \ H^1_\alpha(s_\alpha.\lambda)\to H^0_\alpha(\lambda),$$
$$\tilde{T}^\lambda_\alpha: H^0_\alpha(\lambda)\to H^1_\alpha(s\ .\lambda),$$

defined by

$$T^\lambda_\alpha(v'_i)=\binom{r}{i}v_{r-i}, \qquad i=0, 1, \ldots, r$$

and

$$\tilde{T}^\lambda_\alpha(v_i)=(r-i)!\,i!\,v'_{r-i}, \qquad i=0, 1, \ldots, r,$$

are P_α-homomorphisms. Note that the composite of these two maps in either order is multiplication by $r!$

Let $v_p:\mathbb{Z}\to\mathbb{N}$ denote p-adic evaluation, i. e. $m=p^{v_p(m)}s$ where $(p, s)=1$. We will need the following.

LEMMA 3.1. — *Suppose* $r \in \mathbb{N}$ *has p-adic expansion*
$$r = a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0 (a_i \in \{0, 1, \ldots, p-1\}).$$
Then

(i) $$\nu_p(r\,!) = \left(r - \sum_{i=0}^{n} a_i\right) / p - 1.$$

(ii) *If* $a_n \neq 0$ *and* $r < p^{n+1} - 1$ *then* $\max \left\{ \nu_p \binom{r}{i} \right\} = n - \nu_p(r+1).$

Proof. — This is of course elementary, *see* e. g. ([11], p. 263), with (ii) following from (i) (the maximum is achieved for $i = p^n - 1$).

Q.E.D.

From this lemma and the above discussion we get remembering that $H^i_{\alpha,\,p}(\mu) \simeq H^i_\alpha(\mu) \otimes \mathbb{Z}_p$ for all i and μ.

PROPOSITION 3.2. — *Let* $\lambda \in X(T)$ *such that* $p^n - 1 \leqq \langle \alpha^\vee, \lambda \rangle < p^{n+1} - 1$. *Then we have two homomorphisms of rational* $P_{\alpha \mathbb{Z}_p}$*-modules*
$$T^\lambda_\alpha : \quad H^1_{\alpha,\,p}(s_\alpha . \lambda) \to H^0_{\alpha,\,p}(\lambda)$$
and
$$\tilde{T}^\lambda_\alpha : \quad H^0_{\alpha,\,p}(\lambda) \to H^1_{\alpha,\,p}(s_\alpha . \lambda),$$
such that

(i) *The induced maps* $H^1_{\alpha,\,k}(s_\alpha . \lambda) \to H^0_{\alpha,\,k}(\lambda)$, *resp.* $H^0_{\alpha,\,k}(\lambda) \to H^1_{\alpha,\,k}(s_\alpha . \lambda)$ *are non-zero.*

(ii) *The composites* $T^\lambda_\alpha \circ \tilde{T}^\lambda_\alpha$ *and* $\tilde{T}^\lambda_\alpha \circ T^\lambda_\alpha$ *are both multiplication by* $p^{n - \nu_p(\langle \alpha^\vee, \lambda + \rho \rangle)}$.

These two homomorphisms betwee free $\mathbb{Z}_p$-modukes give rise to filtrations:
$$H^1_{\alpha,\,p}(s_\alpha . \lambda) = H^1_{\alpha,\,p}(s_\alpha . \lambda)^0 \supseteqq H^1_{\alpha,\,p}(s_\alpha . \lambda)^1 \supseteqq \ldots$$
and
$$H^0_{\alpha,\,p}(\lambda) = H^0_{\alpha,\,p}(\lambda)^0 \supseteqq H^0_{\alpha,\,p}(\lambda)^1 \supseteqq \ldots$$
defined by
$$H^1_{\alpha,\,p}(s_\alpha . \lambda)^j = \{ v \in H^1_{\alpha,\,p}(s_\alpha . \lambda) \mid T^\lambda_\alpha v \in p^j H^0_{\alpha,\,p}(\lambda) \}$$
and
$$H^0_{\alpha,\,p}(\lambda)^j = \{ v \in H^0_{\alpha,\,p}(\lambda) \mid \tilde{T}^\lambda_\alpha v \in p^j H^1_{\alpha,\,p}(s_\alpha . \lambda) \}.$$
The image of these filtrations under the natural map
$$H^1_{\alpha,\,p}(s_\alpha . \lambda) \to H^1_{\alpha,\,p}(s_\alpha . \lambda) \otimes k \simeq H^1_{\alpha,\,k}(s_\alpha . \lambda), \quad \text{resp. } H^0_{\alpha,\,p}(\lambda) \to H^0_{\alpha,\,p}(\lambda) \otimes k \simeq H^0_{\alpha,\,k}(\lambda)$$
give filtrations $(H^1_{\alpha,\,k}(s_\alpha . \lambda)^j)_{j \geqq 0}$ and $(H^0_{\alpha,\,k}(\lambda)^j)_{j \geqq 0}$ of $H^1_{\alpha,\,k}(s_\alpha . \lambda)$, resp. $H^0_{\alpha,\,k}(\lambda)$. By Proposition 3.2 (ii) we see that
$$H^1_{\alpha,\,k}(s_\alpha . \lambda)^j = H^0_{\alpha,\,k}(\lambda)^j = 0 \qquad \text{for } j > n - \nu_p(\langle \alpha^\vee, \lambda + \rho \rangle).$$

Moreover, if we by subscripts denote filtration level (e. g. $H^0_{\alpha,k}(\lambda)_j = H^0_{\alpha,k}(\lambda)^j / H^0_{\alpha,k}(\lambda)^{j+1}$) then the map

$$p^{-j} T^\lambda_\alpha : \quad H^1_{\alpha,p}(s_\alpha . \lambda)^j \to H^0_{\alpha,p}(\lambda),$$

induces an isomorphism

$$(3.3) \qquad H^1_{\alpha,k}(s_\alpha . \lambda)_j \simeq H^0_{\alpha,k}(\lambda)_{n - \nu_p(\langle \alpha^\vee, \lambda+\rho \rangle) - j}.$$

If now $\mu \in X(T)$ then we set

$$\chi_\alpha(\mu) = \mathrm{ch}\, H^0_{\alpha,k}(\mu) - \mathrm{ch}\, H^1_{\alpha,k}(\mu)$$

i. e.

$$\chi_\alpha(\mu) = \begin{cases} e^\mu + e^{\mu-\alpha} + \ldots + e^{\mu - n\alpha} & \text{if } \langle \alpha^\vee, \mu \rangle = n \geqq 0, \\ 0 & \text{if } \langle \alpha^\vee, \mu \rangle = -1, \\ -e^{\mu+\alpha} - e^{\mu+2\alpha} - \ldots - e^{\mu+(n-1)\alpha} & \text{if } \langle \alpha^\vee, \mu \rangle = -n > 1 \end{cases}$$

With this notation we have the "sum formulas".

PROPOSITION 3.4. — *Let* λ *be as before. Then*

(i) $$\sum_{j \geqq 1} \mathrm{ch}(H^1_{\alpha,k}(s_\alpha . \lambda)^j) = \sum_{\substack{m \\ 0 < mp < \langle \alpha^\vee, \lambda+\rho \rangle}} \nu_p(mp)\, \chi_\alpha(s_{\alpha,m} . \lambda),$$

(ii) $$\sum_{j \geqq 1} \mathrm{ch}(H^0_{\alpha,k}(\lambda)^j) = \sum_{\substack{m \\ 0 < mp < \langle \alpha^\vee, \lambda+\rho \rangle}} \nu_p(mp)\, \chi_\alpha(\lambda - mp\,\alpha) + (n - \nu_p(\langle \alpha^\vee, \lambda+\rho \rangle))\, \chi_\alpha(\lambda).$$

Proof. — By the construction of T^λ_α we see that the image of v'_i in $H^1_{\alpha,k}(s_\alpha . \lambda)$ belongs to $H^1_{\alpha,k}(s_\alpha . \lambda)^j$ if and only if $\nu_p\binom{r}{i} \geqq j$. Hence the statement in (i) says

$$\nu_p\binom{r}{i} = \sum_{m_1} \nu_p(m_1 p) - \sum_{m_2} \nu_p(m_2 p),$$

where the first sum is extended over all m_1 with $\min\{i, (r+2)/2\} \leqq m_1 p \leqq r$ and the second sum is extended over all m_2 with $0 < m_2 p \leqq \min\{r+1-i, r/2\}$. This equality follows from the identity:

$$\binom{r}{i} = \frac{r(r-1)\ldots(i+1)}{(r-i)(r-i-1)\ldots 2.1}.$$

The proof of (ii) is analogous [one may prefer to obtain (ii) from (i) by using (3.3)].

Q.E.D.

Before we leave the rank 1 case we need to look at how the filtration of $H^0_{\alpha,k}(\lambda)$ relates to the filtration of $H^0_{\alpha,k}(\lambda+\omega)$ when ω is "small".

Suppose $\omega \in X(T)$ such that $\langle \alpha^\vee, \omega \rangle = 1$. Then we have the short exact sequence

$$0 \to \omega - \alpha \to H^0_{\alpha, p}(\omega) \to \omega \to 0,$$

of rational $B_{\mathbb{Z}_p}$-modules. Tensoring this sequence by λ (resp. by $s_\alpha . \lambda$) and using the tensor identity 1.10 [resp. (1.16)] we get the exact cohomology sequence

$$0 \to H^0_{\alpha, p}(\lambda + \omega - \alpha) \xrightarrow{i_0} H^0_{\alpha, p}(\omega) \otimes H^0_{\alpha, p}(\lambda) \xrightarrow{\pi_0} H^0_{\alpha, p}(\lambda + \omega) \to 0$$

[resp.

$$0 \to H^1_{\alpha, p}(s_\alpha . \lambda + \omega - \alpha) \xrightarrow{i_1} H^0_{\alpha, p}(\omega) \otimes H^1_{\alpha, p}(s_\alpha . \lambda) \xrightarrow{\pi_1} H^1_{\alpha, p}(s_\alpha . \lambda + \omega) \to 0).$$

LEMMA 3.5. — *In the above situation there exist homomorphisms of rational $P_{\alpha\mathbb{Z}_p}$-modules*

$$r_0 : \ H^0_{\alpha, p}(\omega) \otimes H^0_{\alpha, p}(\lambda) \to H^0_{\alpha, p}(\lambda + \omega - \alpha),$$
$$s_0 : \ H^0_{\alpha, p}(\lambda + \omega) \to H^0_{\alpha, p}(\omega) \otimes H^0_{\alpha, p}(\lambda),$$
$$r_1 : \ H^0_{\alpha, p}(\omega) \otimes H^1_{\alpha, p}(s_\alpha . \lambda) \to H^1_{\alpha, p}(s_\alpha . \lambda + \omega - \alpha),$$
$$s_1 : \ H^1_{\alpha, p}(s_\alpha . \lambda + \omega) \to H^0_{\alpha, p}(\omega) \otimes H^1_{\alpha, p}(s_\alpha . \lambda),$$

such that the following identities hold

(i) $r_j \circ i_j$, $\pi_j \circ s_j$ *and* $s_j \circ \pi_j + i_j \circ r_j$ *are multiplication by* $p^{\nu_p(\langle \alpha^\vee, \lambda + \rho \rangle)}$ *for* $j = 0, 1$.

(ii) $s_0 \circ T_\alpha^{\lambda + \omega} = p^{\nu_p(\langle \alpha^\vee, \lambda + \rho \rangle)} (T_\alpha^\lambda \otimes 1) \circ i_1$.

(iii) $T_\alpha^{\lambda + \omega - \alpha} \circ \pi_1 = p^{\nu_p(\langle \alpha^\vee, \lambda \rangle)} r_0 \circ (T_\alpha^\lambda \otimes 1)$.

(iv) $(T_\alpha^\lambda \otimes 1) \circ s_1 = p^{\nu_p(\langle \alpha^\vee, \lambda \rangle)} i_0 \circ T_\alpha^{\lambda + \omega - \alpha}$.

(v) $T_\alpha^{\lambda + \omega} \circ r_1 = p^{\nu_p(\langle \alpha^\vee, \lambda + \rho \rangle)} \pi_0 \circ (T_\alpha^\lambda \otimes 1)$.

Proof. — If $\nu \in X(T)$ such that $\langle \alpha^\vee, \nu \rangle \geqq 0$ then let us denote by $\{ v_i^\nu \}$ (resp. $\{ v_i'^\nu \}$) the "standard" basis for $H^0_\alpha(\nu)$ [resp. $H^1_\alpha(s_\alpha . \nu)$] used above. Then r_j and s_j are defined by

$$r_0 (v_i^\lambda \otimes v_0^\omega) = i v_{i-1}^{\lambda + \omega - \alpha}, \qquad r_0 (v_i^\lambda \otimes v_1^\omega) = (i - r) v_i^{\lambda + \omega - \alpha},$$
$$r_1 (v_i'^\lambda \otimes v_0^\omega) = (i + 1) v_{i+1}'^{\lambda + \omega}, \qquad r_1 (v_1'^\lambda \otimes v_1^\omega) = (r + 1 - i) v_i'^{\lambda + \omega},$$
$$s_0 (v_i^{\lambda + \omega}) = (r + 1 - i) v_i^\lambda \otimes v_0^\omega + i v_{i-1}^\lambda \otimes v_1^\omega,$$
$$s_1 (v_i'^{\lambda + \omega - \alpha}) = (r - i)(v_i'^\lambda \otimes v_0^\omega) - (i + 1)(v_{i+1}'^\lambda \otimes v_1^\omega).$$

In the same basis the homomorphisms i_j and π_j are given by

$$i_0 (v_i^{\lambda + \omega - \alpha}) = v_{i+1}^\lambda \otimes v_0^\omega - v_i^\lambda \otimes v_1^\omega,$$
$$i_1 (v_i'^{\lambda + \omega}) = v_{i-1}'^\lambda \otimes v_0^\omega + v_i'^\lambda \otimes v_1^\omega,$$
$$\pi_0 (v_i^\lambda \otimes v_0^\omega) = v_i^{\lambda + \omega}, \qquad \pi_0 (v_i^\lambda \otimes v_1^\omega) = v_{i+1}^{\lambda + \omega},$$
$$\pi_1 (v_1'^\lambda \otimes v_0^\omega) = v_i'^{\lambda + \omega - \alpha}, \qquad \pi_1 (v_i'^\lambda \otimes v_1^\omega) = - v_{i-1}'^{\lambda + \omega - \alpha}.$$

It is now left to the reader to check the above relations.

Q.E.D.

Let us finally record the following result.

PROPOSITION 3.6. — *Let* $\lambda \in X(T)$ *such that* $p^n - 1 \leqq \langle \alpha^\vee, \lambda \rangle < p^{n+1} - 1$. *Then the diagram*

$$\begin{array}{ccc}
H^1_{\alpha, p}(-\lambda - 2\rho) \otimes H^0_{\alpha, p}(\lambda) \to H^1_{\alpha, p}(-2\rho) \simeq \mathbb{Z}_p & & \\
\downarrow{\scriptstyle T_\alpha \otimes \hat{T}_\alpha} & \searrow{\scriptstyle p^{n - v_p(\langle \alpha^\vee, \lambda + \rho \rangle)}} & \\
H^0_{\alpha, p}(-s_\alpha . \lambda - 2\rho) \otimes H^1_{\alpha, p}(s_\alpha . \lambda) \to H^1_{\alpha, p}(-2\rho) \simeq \mathbb{Z}_p & &
\end{array}$$

is commutative [*the horizontal maps are those from* (1.19)].

Proof. — Omitting the weight superscripts on basis elements the horizontal maps are given by

$$v'_i \otimes v_j \to (-1)^i \delta_{i, j} [\text{resp. } v_i \otimes v'_j \to (-1)^i \delta_{i, j}]$$

and the proposition is easily checked.

Q.E.D.

Remark 3.7. — Let us again use (,) to denote the bilinear pairings induced from the horizontal maps in Proposition 3.6. Then the result says

$$(T_\alpha v, \hat{T}_\alpha w) = p^{n - v_p(\langle \alpha^\vee, \lambda + \rho \rangle)} (v, w),$$

for all $v \in H^1_{\alpha, p}(-\lambda - 2\rho)$, $w \in H^0_{\alpha, p}(\lambda)$. Using Proposition 3.2 (ii) this may also be written

$$(T_\alpha v, w) = (v, \hat{T}_\alpha w),$$

for all $v \in H^1_{\alpha, p}(-\lambda - 2\rho)$, $w \in H^1_{\alpha, p}(s_\alpha . \lambda)$.

4. Filtrations. The general case

We preserve the notation from the previous sections and now consider the general case.

PROPOSITION 4.1. — *Let* $\lambda \in X(T)$, $\alpha \in S$ *and suppose* $p^n \leqq \langle \alpha^\vee, \lambda + \rho \rangle < p^{n+1}$. *Then there exist homomorphisms of rational* G-*modules*

$$H^{i+1}_p(s_\alpha . \lambda) \to H^i_p(\lambda) \quad \text{and} \quad H^i_p(\lambda) \to H^{i+1}_p(s_\alpha . \lambda),$$

such the composite (*in either order*) *is multiplication by* $p^{n - v_p(\langle \alpha^\vee, \lambda + \rho \rangle)}$.

Proof. — As $H^i_{\alpha, p}(s_\alpha . \lambda) = 0$ for $i \neq 1$ the spectral sequence (Proposition 1.15)

$$H^i(G_{\mathbb{Z}_p}/P_{\alpha \mathbb{Z}_p}, H^j(P_{\alpha \mathbb{Z}_p}/B_{\mathbb{Z}_p}, s_\alpha . \lambda)) \Rightarrow H^{i+j}(G_{\mathbb{Z}_p}/B_{\mathbb{Z}_p}, s_\alpha . \lambda)$$

degenerates and gives us isomorphisms

$$H_p^{i+1}(s_\alpha . \lambda) \simeq H^i(G_{\mathbb{Z}_p}/P_{\alpha\mathbb{Z}_p}, H^1(P_{\alpha\mathbb{Z}_p}/B_{\mathbb{Z}_p}, s_\alpha . \lambda)),$$

$i=0, 1, \ldots$

Similarly, we have isomorphisms

$$H_p^i(\lambda) \simeq H^i(G_{\mathbb{Z}_p}/P_{\alpha\mathbb{Z}_p}, H^0(P_{\alpha\mathbb{Z}_p/B_{\mathbb{Z}_p}}, \lambda)), \qquad i=0, 1, \ldots$$

The proposition follows now from Proposition 3.2.

Q.E.D.

NOTATION 4.2. — If E is a rational $B_{\mathbb{Z}_p}$-module then we will let $H_f^i(E)$ resp. $H_t^i(E)$ denote the free quotient, resp. torsion submodule of $H_p^i(E)$. Note that for $\mu \in X(T)$ we have because of Corollary 2.6. (i) that $H_f^i(\mu)=0$ unless $w.\mu \in X(T)_+$ for some $w \in W$ with $l(w)=i$.

Fix now $\lambda \in X(T)_+$. For each $w \in W$ Proposition 4.1 gives us a homomorphism

$$H_f^{l(w)}(w . \lambda) \to H_f^{l(s_\alpha w)}(s_\alpha w . \lambda),$$

which we shall denote $T_{s_\alpha}^{w.\lambda}$ or just T_{s_α} when it is clear which λ and w we refer to. (In fact one might prefer to consider T_{s_α} as an endomorphism of $\bigoplus_{w \in W} H_f^{l(w)}(w . \lambda)$).

Let $y \in W$ have reduced expression $y=s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_n}$ and set $T_y=T_{s_{\alpha_1}} T_{s_{\alpha_2}} \ldots T_{s_{\alpha_n}}$.

LEMMA 4.3. — *The homomorphism* $T_y : H_f^{l(w)}(w . \lambda) \to H_f^{l(yw)}(yw . \lambda)$ *is independent (up to a unit in* $\mathbb{Z}_p$*) of the chosen reduced expression for y.*

Proof. — Let $y=s_{\beta_1} s_{\beta_2} \ldots s_{\beta_n}$ be another reduced expression for y and set $T'_y=T_{s_{\beta_1}} T_{s_{\beta_2}} \ldots T_{s_{\beta_n}}$. As $H_f^{l(w)}(w . \lambda) \otimes \mathbb{Q}$ and $H_f^{l(yw)}(yw . \lambda) \otimes \mathbb{Q}$ are isomorphic irreducible $G_{\mathbb{Q}}$-modules we see that T_y and T'_y are proportional. It is therefore enough to show that $T_y(v)=T'_y(v)$ for some $v \in H_f^{l(w)}(w . \lambda) \setminus \{0\}$. To see this recall that by Corollary 2.7 λ is the unique highest weight in $H_f^{l(w)}(w . \lambda)$ occurring with multiplicity 1. We can therefore choose a generator w_w for $H_f^{l(w)}(w . \lambda)_\lambda$. The following lemma implies that (up to unit in $\mathbb{Z}_p$)

$$T_y v_w = p^m v_{yw},$$

where m is independent of the reduced expression for y. (In the notation of Lemma 4.4 we have $m=\sum_\beta n_\beta$ with the sum extended over all $\beta \in R_+$ for which $y(\beta) \in R_-$ and $w^{-1}(\beta) \in R_+$).

Q.E.D.

LEMMA 4.4. — *Suppose* $p^{r_\beta} \leqq \langle \beta^\vee, \lambda+\rho \rangle < p^{r_\beta+1}$, $\beta \in R_+$ *and set*

$$n_\beta = r_\beta - \nu_p(\langle \beta^\vee, \lambda+\rho \rangle).$$

If v_w *is as above and* $\alpha \in S$ *then up to a unit in* $\mathbb{Z}_p$ *we have*

$$T_{s_\alpha} v_w = \begin{cases} v_{s_\alpha w} & \text{if } \; l(s_\alpha w)=l(w)-1, \\ p^{n_\beta} v_{s_\alpha w}, \; \beta=w^{-1}(\alpha) & \text{if } \; l(s_\alpha w)=l(w)+1. \end{cases}$$

Proof. — Suppose $l(s_\alpha w)=l(w)-1$. It follows from the analysis of weights in [2], section 2, that $v_w\otimes 1\in H_p^{l(w)}(w.\lambda)\otimes k\subset H_k^{l(w)}(w.\lambda)$ is not in the kernel of the homomorphism $H_k^{l(w)}(w.\lambda)\to H_k^{l(w)-1}(s_\alpha w.\lambda)$.
The lemma follows from this *via* Proposition 4.1.

Q.E.D.

We are now ready to define the filtrations. By w_0 we denote the longest element in W.

DEFINITION 4.5. — Let $\lambda\in X(T)_+$ and $w\in W$. Then we define

$$H_f^{l(w)}(w.\lambda)^j=\{v\in H_f^{l(w)}(w.\lambda)\,|\,T_{w_0}(v)\in p^j H^{l(w_0w)}(w_0w.\lambda)\}$$

and we set

$$H_k^{l(w)}(w.\lambda)^j=\text{the image of } H_f^{l(w)}(w.\lambda)^j,$$

under the natural map $H_f^{l(w)}(w.\lambda)\to H_f^{l(w)}(w.\lambda)\otimes k$

PROPOSITION 4.6. — *Let* $\lambda\in X(T)_+$ *and define* n_β *as in Lemma* 4.4. *Then for all* $w\in W$ *we have:*

(i) *The homomorphism* $p^{-j}T_{w_0}: H_f^{l(w)}(w.\lambda)^j\to H_f^{l(w_0w)}(w_0w.\lambda)$ *induces an isomorphism*

$$H_k^{l(w)}(w.\lambda)_j\simeq H_k^{l(w_0w)}(w_0w.\lambda)_{m-j},\ m=\sum_{\beta\in R_+}n_\beta,$$

(*where subscript again denotes filtration level*).

(ii) $H_k^{l(w)}(w.\lambda)^j=0$ *for* $j>\sum_{\beta\in R_+}n_\beta$.

Proof. — Immediate consequence of Proposition 4.1.

Q.E.D.

PROPOSITION 4.7. — *Let* $\lambda\in X(T)_+$. *Then*

(i) $H_k^N(w_0.\lambda)^1$ *is the maximal submodule of* $H_k^N(w_0.\lambda)$.

(ii) $H_k^0(w_0.\lambda)^j\neq 0$ *if and only if* $j\leqq\sum_{\beta\in R_+}n_\beta$.

Proof. — (i) By construction $H_k^N(w_0.\lambda)^1$ is the kernel of the map $H_f^N(w_0.\lambda)\otimes k\to H_f^0(\lambda)\otimes k$. However, $H_f^N(w_0.\lambda)\otimes k\simeq H_k^N(w_0.\lambda)$ and $H_f^0(\lambda)\otimes k\simeq H_k^0(\lambda)$ by Corollary 2.6 (ii). Now it is well-known that there is only one non-zero homomorphism $H_k^N(w_0.\lambda)$ into $H_k^0(\lambda)$, and that the image of this is $L(\lambda)$, *see* [1], Lemma 4. Our homomorphism is non-zero by Lemma 4.4.

(ii) Set $m=\sum_{\beta\in R_+}n_\beta$. Then Lemma 4.4 shows that $T_{w_0}v_1=p^m v_{w_0}$ and hence $v_1\otimes 1\in H_k^0(\lambda)^m$.

Q.E.D.

Remark 4.8. — For $w \neq 1$ it is not always true that $\mathrm{H}_k^{l(w)}(w.\lambda)^m \neq 0$. Also it is not always the case that $\mathrm{H}_k^0(\lambda)_j \neq 0$ for all $j \leqq m$. In fact $\mathrm{H}_k^0(\lambda)$ may be irreducible although $m > 0$.

We will now see how the formulas in Proposition 3.4 give us sum formulas in the general case. First we need:

LEMMA 4.9. — *Let* E *denote a rational* B-*module which as a* $\mathbb{Z}$-*module is cyclic of finite order* p^n. *Then*

$$\sum_{\mu} \sum_{i} (-1)^i \nu_p(\operatorname{ord}(\mathrm{H}^i(\mathrm{E})))\, e^{\mu} = n\,\chi(\lambda),$$

where λ *is the weight of* T *on* E.

Proof. — Let $\mathbb{Z}_\lambda$ denote the free rank 1 B-module on which B acts *via* λ. Then we have the short exact sequence of rational B-modules

$$0 \to \mathbb{Z}_\lambda \xrightarrow{p^n} \mathbb{Z}_\lambda \to \mathrm{E} \to 0.$$

Suppose $w \in \mathrm{W}$ such that $w(\lambda+\rho) \in \mathrm{X}(\mathrm{T})_+$. Then $\mathrm{H}^i(\lambda)$ is a torsion module for $i \neq l(w)$ [Corollary 2.6(i)] and hence the long exact cohomology sequence associated to the short exact sequence above shows that

$$\sum_{i} (-1)^i \nu_p(\operatorname{ord} \mathrm{H}^i(\mathrm{E})_\mu) = \nu_p(\det(p^n, \mathrm{H}_f^{l(w)}(\lambda)_\mu)),$$

where we by $\det(p^n, \mathrm{V})$ denotes determinant of multiplication by p^n on the module V. The lemma follows therefore from Corollary 2.7(i).

Q.E.D.

Let $\lambda \in \mathrm{X}(\mathrm{T})_+$. For $w \in \mathrm{W}$ we set

$$\theta_w(\mu) = \sum_{j} (-1)^j \nu_p(\operatorname{ord} \mathrm{H}_t^j(w.\lambda)_\mu) \qquad \text{and} \qquad \theta_w = \sum_{\mu} \theta_w(\mu)\, e^{\mu}.$$

Also let $\operatorname{sgn} : \mathrm{R} \to \{+1\}$ be defined by $\operatorname{sgn}(\mathrm{R}_+) = 1$ and $\operatorname{sgn}(\mathrm{R}_-) = -1$. With this notation and with n_α as in Lemma 4.4 we have

THEOREM 4.10

$$\sum_{j \geqq 1} \operatorname{ch} \mathrm{H}_k^{l(w)}(w.\lambda)^j - (-1)^{l(w)}\theta_w + (-1)^{l(w_0 w)}\theta_{w_0 w}$$

$$\Big(\sum_{\alpha \in \mathrm{R}_+ \cap w^{-1}\mathrm{R}_+} n_\alpha\Big)\chi(\lambda) + \sum_{\alpha \in \mathrm{R}_+} \operatorname{sgn}(w(\alpha)) \sum_{\substack{m \\ 0 < mp < \langle \alpha^{\check{}}, \lambda+\rho \rangle}} \nu_p(mp)\,\chi(\lambda - mp\,\alpha).$$

Proof. — Choose basis $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_m\}$ for $\mathrm{H}_f^{l(w)}(w.\lambda)$ and $\mathrm{H}_f^{l(w_0 w)}(w_0 w.\lambda)$, respectively, in such a way that $\mathrm{T}_{w_0}(v_i) = p^{a_i} w_i$ for some $a_i \in \mathbb{N}$. Then

$H_k^{l(w)}(w.\lambda)^j = \text{span}\,\{v_i \otimes 1 \mid a_i \geqq j\}$ and hence

$$\sum_{j \geqq 1} \operatorname{ch} H_k^{l(w)}(w.\lambda)^j = \sum_\mu \nu_p(\det T_{w_0} \mid_{H_f^{l(w)}(w.\lambda)_\mu})\, e^\mu.$$

Now, if $w_0 = s_{\beta_1} s_{\beta_2} \ldots s_{\beta_N}$ is a reduced expression for w_0 then $T_{w_0} = T_{s_{\beta_N}} \ldots T_{s_{\beta_1}}$ and hence $\det T_{w_0} = \prod_i \det T_{s_{\beta_i}}$.

We set

$$\tau_i(\mu) = (-1)^{l(s_{\beta_i} \ldots s_{\beta_1} w)} (\theta_{s_{\beta_i} \ldots s_{\beta_1} w} + \theta_{s_{\beta_{i-1}} \ldots s_{\beta_1} w})(\mu)$$

and

$$\tau_i = \sum_\mu \tau_i(\mu)\, e^\mu.$$

Then $\sum_i \tau_i = -(-1)^{l(w)} \theta_w + (-1)^{l(w_0 w)} \theta_{w_0 w}$ and hence the theorem follows once we prove the identities

$$(4.11)\quad \tau_i + \sum_\mu \nu_p(\det T_{s_{\beta_i}} \mid \mu) = \begin{cases} \displaystyle\sum_{\substack{m \\ 0 < mp < \langle \check{\alpha}_i, \lambda+\rho \rangle}} \nu_p(mp)\, \chi(s_{\alpha_i, m} . \lambda) & \text{if } \alpha_i \in R_+, \\ n_{-\alpha_i} \chi(\lambda) + \displaystyle\sum_{\substack{m \\ 0 < mp < -\langle \check{\alpha}_i, \lambda+\rho \rangle}} \nu_p(mp)\, \chi(\lambda + mp\,\alpha_i) & \text{if } \alpha_i \in R_-. \end{cases}$$

Here $T_{s_{\beta_i}} \mid_\mu$ denotes the restriction to the μ-weight space of the homomorphism:

$$T_{s_{\beta_i}} :\ H_f^{l(s_{\beta_{i-1}} \ldots s_{\beta_1} w)}(s_{\beta_{i-1}} \ldots s_{\beta_1} w.\lambda) \to H_f^{l(s_{\beta_i} \ldots s_{\beta_1} w)}(s_{\beta_i} \ldots s_{\beta_1} w.\lambda)$$

and we have set $\alpha_i = w^{-1} s_{\beta_1} s_{\beta_2} \ldots s_{\beta_i}(\beta_i)$, $i = 1, 2, \ldots, N$.

To prove this suppose first that $\alpha_i \in R_+$ and note that then $\langle \check{\beta}_i, s_{\beta_i} \ldots s_{\beta_1} w(\lambda+\rho) \rangle = \langle \check{\alpha}_i, \lambda+\rho \rangle \geqq 0$. From section 3 we therefore have the $P_{\beta_i \mathbb{Z}_p}$-homomorphism $T_{\beta_i} : H^1_{\beta_i, p}(s_{\beta_{i-1}} \ldots s_{\beta_1} w.\lambda) \to H^0_{\beta_i, p}(s_{\beta_i} \ldots s_{\beta_1} w.\lambda)$. The cokernel Q of T_{β_i} has weightspaces Q^i_ξ which are cyclic of order p^{m_ξ}, say, and with this notation we have

$$\sum_\xi m_\xi e^\xi = \sum_\xi \nu_p(\operatorname{ord} Q^i_\xi)\, e^\xi = \sum_\xi \nu_p(\det T_{\beta_i} \mid_\xi)\, e^\xi$$

$$= \sum_{j \geqq 1} \operatorname{ch} H^1_{\beta_i, k}(s_{\beta_{i-1}} \ldots s_{\beta_1} w.\lambda)^j$$

$$= \sum_{\substack{m \\ 0 < mp < \langle \check{\beta}_i, s_{\beta_i} \ldots s_{\beta_1} w(\lambda+\rho) \rangle}} \nu_p(mp)\, \chi_{\beta_i}(s_{\beta_i, m} . s_{\beta_i} \ldots s_{\beta_1} w.\lambda),$$

where the last equality follows from Proposition 3.4(i). Now

$$\langle \beta_i^\vee, s_{\beta_i} \dots s_{\beta_1} w(\lambda+\rho) \rangle = \langle \alpha_i^\vee, \lambda+\rho \rangle$$

and

$$s_{\beta_i, m} \cdot s_{\beta_i} \dots s_{\beta_1} w.\lambda = s_{\beta_i} \dots s_{\beta_1} w.(s_{\alpha_i, m}.\lambda),$$

i. e. we have

$$\sum_{\xi} m_{\xi} e^{\xi} = \sum_{\substack{m \\ 0 < mp < \langle \alpha_i^\vee, \lambda+\rho \rangle}} \nu_p(mp) \chi_{\beta_i}(s_{\beta_i} \dots s_{\beta_1} w.(s_{\alpha_i, m}.\lambda)). \tag{4.12}$$

From the short exact sequence

$$0 \to H^1_{\beta_i, p}(s_{\beta_{i-1}} \dots s_{\beta_1} w.\lambda) \to H^0_{\beta_i, p}(s_{\beta_i} \dots s_{\beta_1} w.\lambda) \to Q^i \to 0,$$

we get by observing that $H^i_f(H^1_{\beta_i, p}(s_{\beta_{i-1}} \dots s_{\beta_1} w.\lambda)) \simeq H^{i+1}_f(s_{\beta_{i-1}} \dots s_{\beta_1} w.\lambda) = 0$ for $i \neq l(s_{\beta_i} \dots s_{\beta_1} w)$ and $H^i_f(H^0_{\beta_i, p}(s_{\beta_i} \dots s_{\beta_1} w.\lambda)) \simeq H^i_f(s_{\beta_i} \dots s_{\beta_1} w.\lambda) = 0$ for $i \neq l(s_{\beta_i} \dots s_{\beta_1} w)$ [Corollary 2.6(i)] the following identity

$$\sum_j (-1)^j \nu_p(\operatorname{ord}(H^j_p(Q^i)_\mu) = (-1)^{l(s_{\beta_i} \dots s_{\beta_1} w)} (\nu_p(\det T_{s_{\beta_i}} |_\mu) + \tau_i). \tag{4.13}$$

The B-module Q^i has a filtration with quotients isomorphic to the weight spaces Q^i_ξ and hence the left side of this equality equals

$$\sum_{\xi} \sum_j (-1)^j \nu_p(\operatorname{ord}(H^j_p(Q^i_\xi)_\mu)).$$

Employing now Lemma 4.9 we get

$$\sum_{\mu} \sum_{\zeta} \sum_j (-1)^j \nu_p(\operatorname{ord}(H^j_p(Q^i_\xi)_\mu) e^\mu = \sum_{\xi} m_\xi \chi(\xi).$$

On the other hand applying χ to (4.12) and using that $\chi(\chi_\alpha(\mu)) = \chi(\mu)$ for all $\alpha \in S$, $\mu \in X(T)$ we find

$$\sum_{\xi} m_\xi \chi(\xi) = \sum_{\substack{m \\ 0 < mp < \langle \alpha_i, \lambda+\rho \rangle}} \nu_p(mp) \chi(s_{\beta_i} \dots s_{\beta_1} w.(s_{\alpha_i, m}.\lambda)).$$

Inserting this in (4.13) we obtain the first case of (4.11). The other case is proved in the same way using Proposition 3.4(ii).

Q.E.D.

Remark 4.14(i). — In the generic case $\theta_w=\theta_{w_0w}=0$. This holds for instance always when $w=1$ or $w=w_0$ and more generally whenever $H_k^*(w.\lambda)$ is concentrated in degree $l(w)$. The formula should be compared to the first conjecture in [12], 2.4.

(ii) When we specialize w to w_0 the right hand side coincides with Jantzen's sum formula in [13]. We shall now prove that our filtration $(H_k^N(w_0.\lambda)^j)_j$ is in fact the same as Jantzen's Weyl module filtration constructed *via* a contravariant form.

First we make the following observation: Suppose φ denotes an endomorphism of G which maps a closed subgroup H into itself. For any rational H-module E we get a natural homomorphism of rational G-modules

$$(E|_H^G)^\varphi \to E^\varphi|_H^G,$$

where we by the superscript φ indicates that the action on the module is given as the ordinary action composed with φ. We inherit natural homomorphisms

$$H^n(G/H, E)^\varphi \to H^n(G/H, E^\varphi), \qquad n\geqq 0,$$

which of course are isomorphisms if φ is an automorphism.

Now let $\varphi : G\to G$ denote the automorphism of G corresponding to the root system automorphism $\alpha\to -w_0(\alpha)$, $\alpha\in R$. Note that $\varphi(B)=B$ and that $\varphi(t)=w_0t^{-1}w_0$, $t\in T$. By the above we therefore have isomorphisms (which we also denote by φ)

$$\varphi : \quad H^i(\lambda)^\varphi \simeq H^i(-w_0(\lambda)), \qquad \lambda\in X(T), \qquad i\geqq 0.$$

Fix again $\lambda\in X(T)_+$, $w\in W$ and let (,) denote the bilinear pairing obtained from the natural homomorphism (1.19)

$$H_f^{N-l(w)}(w_0w.\lambda)\otimes H_f^{l(w)}(-w_0w.\lambda-2\rho)\to H_p^N(-2\rho)\simeq\mathbb{Z}_p.$$

Then we define a bilinear form β_w on $H_f^{l(w)}(w.\lambda)$ by

$$\beta_w(v, v')=(T_{w_0}(v), w_0\varphi(v')), \ v, v'\in H_f^{l(w)}(w.\lambda).$$

We check easily that β_w satisfies

$$\beta_w(gv, v')=\beta_w(v, \varphi(w_0g^{-1}w_0)v')$$

for all $g\in G_{\mathbb{Z}_p}$, $v, v'\in H_f^{l(w)}(w.\lambda)$. The map from $G_{\mathbb{Z}_p}$ into $G_{\mathbb{Z}_p}$ which takes g into $\varphi(w_0g^{-1}w_0)$ is an antiautomorphism. It is the identity on T and it takes U_α into $U_{-\alpha}$. In other words β_w is a contravariant form [13]. Moreover, since the pairing (,) is non-singular, Proposition 2.10, we get

$$H_f^{l(w)}(w.\lambda)^j=\{v\in H_f^{l(w)}(w.\lambda)\,|\,\beta_w(v, H_f^{l(w)}(w.\lambda))\in p^j\mathbb{Z}_p\}.$$

Let again $v_w\in H_f^{l(w)}(w.\lambda)$ be a generator of the λ-weight space. Then we have

LEMMA 4.15. — *$H_p^N(w_0.\lambda)$ is generated as a $G_{\mathbb{Z}_p}$-module by v_{w_0}.*

Proof. — Let M be the submodule generated by v_{w_0} and let $Q = H_p^N(w_0 . \lambda)/M$. Then λ is not a weight of Q. On the other hand $Q \otimes k$ is a quotient of $H_p^N(w_0 . \lambda) \otimes k \simeq H_k^N(w_0 . \lambda)$ which implies that $Q \otimes k = 0$ since $L(\lambda)$ is the only simple quotient of $H_k^N(w_0 . \lambda)$.

Q.E.D.

Comparing this lemma with the definition of $V(\lambda)_{\mathbb{Z}}$ in [13] we see that $H_p^N(w_0 . \lambda) \simeq V(\lambda)_{\mathbb{Z}} \otimes \mathbb{Z}_p$ and from the above follows

PROPOSITION 4.16. — *The filtration* $(H_k^N(w_0 . \lambda)^j)_{j \geqq 0}$ *coincides with Jantzen's filtration* $(V(\lambda)^j)_{j \geqq 0}$ *of the Weyl module* $V(\lambda)$.

Let $\alpha \in S$ and set $\beta = -w_0(\alpha)$. Then the automorphism φ takes P_α into P_β and for any rational B-module E we get isomorphisms (of rational P_β-modules)

$$H^i(P_\alpha/B, E)^\varphi \simeq H^i(P_\beta/B, E^\varphi), \qquad i \geqq 0.$$

Moreover, if $\mu \in X(T)$ such that $\langle \alpha\check{}, \mu \rangle \geqq 0$ then we have a commutative diagram

$$\begin{array}{ccc} H^1_{\alpha,\, p}(s_\alpha . \mu)^\varphi & \simeq & H^1_{\beta,\, p}(s_\beta(-w_0 . \mu)) \\ \downarrow{\scriptstyle T_\alpha} & & \downarrow{\scriptstyle T_\alpha} \\ H^0_{\alpha,\, p}(\mu)^\varphi & \simeq & H^0_{\beta,\, p}(-w_0 . \mu) \end{array}$$

If $\langle \alpha\check{}, \mu \rangle < 0$ we have a similar commutative diagram involving $\hat{T}_\alpha$ and $\hat{T}_\beta$. Hence for any $w \in W$ we have a commutative diagram

$$\begin{array}{ccc} H_f^{l(w)}(w . \lambda)^\varphi & \overset{\varphi}{\simeq} & H_f^{l(w)}(-w_0 w . \lambda) \\ \downarrow{\scriptstyle T_{s_\alpha}} & & \downarrow{\scriptstyle T_{s_\beta}} \\ H_f^{l(s_\alpha w)}(s_\alpha w . \lambda)^\varphi & \overset{\varphi}{\simeq} & H_f^{l(s_\alpha w)}(s_\beta . (-w_0 w . \lambda)) \end{array}$$

i. e. $\varphi \circ T_{s_\alpha} = T_{s_\beta} \circ \varphi$.

We now have

LEMMA 4.17. — *Let* $\lambda \in X(T)_+$, $w \in W$ *and* $\alpha \in S$. *Then:*

$$\beta_{s_\alpha w}(T_{s_\alpha} v, z) = \beta_w(v, T_{s_\alpha} z)$$

for all $v \in H_f^{l(w)}(w . \lambda)$, $z \in H_f^{l(s_\alpha w)}(s_\alpha w . \lambda)$.

Proof. — It is an easy consequence of Lemma 4.3 that $T_{w_0} T_{s_\alpha} = T_{s_\beta} T_w$ where $\beta = -w_0(\alpha)$. Hence $\beta_{s_\alpha w}(T_{s_\alpha} v, z) = (T_{w_0} T_{s_\alpha} v, w_0 \varphi(z)) = (T_{s_\beta} T_{w_0} v, w_0 \varphi(z))$. From

Proposition 3.6 we obtain the commutative diagrams

$$\begin{tikzcd}[column sep=small]
\mathrm{H}_f^{l(y)-1}(\mathrm{H}^1_{\beta,\,p}(y,.\lambda))\otimes \mathrm{H}_f^{\mathrm{N}-l(y)}(\mathrm{H}^0_{\beta,\,p}(-y.\lambda-2\rho)) \arrow[r] \arrow[d, "\mathrm{T}_{s_\beta}\otimes \mathrm{T}_{s_\beta}"] & \mathrm{H}_f^{\mathrm{N}-1}(\mathrm{H}^1_{\beta,\,p}(y.\lambda)\otimes \mathrm{H}^0_{\beta,\,p}(-y.\lambda-2\rho)) \arrow[r] & \mathrm{H}^{\mathrm{N}-1}(\mathrm{H}^1_\beta(-2\rho) \arrow[d] \\
\mathrm{H}_f^{l(y)-1}(\mathrm{H}^0_{\beta,\,p}(s_\beta y.\lambda))\otimes \mathrm{H}^{\mathrm{N}-l(y)}(\mathrm{H}^1_{\beta,\,p}(-s_\beta y.\lambda-2\rho)) \arrow[r] & \mathrm{H}_f^{\mathrm{N}-1}(\mathrm{H}^0_{\beta,\,p}(s_\beta y.\lambda)\otimes \mathrm{H}^1_{\beta,\,p}(-s_\beta y.\lambda-2\rho) \arrow[r] & H^{N-1}(\mathrm{H}^1_{\beta,\,p}(-2\rho))
\end{tikzcd}$$

where the right vertical map is multiplication by p^{n_γ}, $\gamma=-y^{-1}(\beta)$. These diagrams are obtained for all y with $y^{-1}(\beta)\in \mathrm{R}_-$ and we have similar diagrams for y's with $y^{-1}(\beta)\in \mathrm{R}_+$. Note that

$$\mathrm{H}_f^{l(y)-1}(\mathrm{H}^1_{\beta,\,p}(y.\lambda))\simeq \mathrm{H}_f^{l(y)}(y.\lambda), \qquad \mathrm{H}_f^{l(y)-1}(\mathrm{H}^0_\beta(s_\beta y.\lambda))\simeq \mathrm{H}^{l(s_\beta y)}(s_\beta y.\lambda),$$

etc. As in Remark 3.7 the commutativity of these diagrams can also be expressed by the formula

$$(\mathrm{T}_{s_\beta}x,\, x')=(x,\, \mathrm{T}_{s_\beta}x'), \qquad x\in \mathrm{H}_f^{l(y)}(y.\lambda), \qquad x'\in \mathrm{H}_f^{\mathrm{N}-l(s_\beta y)}(-s_\beta y.\lambda-2\rho).$$

Using this we find

$$(\mathrm{T}_{s_\beta}\mathrm{T}_{w_0}v,\, w_0\,\varphi(z))=(\mathrm{T}_{w_0}v,\, \mathrm{T}_{s_\beta}w_0\,\varphi(z))=(\mathrm{T}_{w_0}v,\, w_0\,\varphi(\mathrm{T}_{s_\alpha}z))=\beta_w(v,\, \mathrm{T}_{s_\alpha}z),$$

where we have also used that $\varphi\circ \mathrm{T}_{s_\alpha}=\mathrm{T}_{s_\beta}\circ\varphi$ as observed above.

Q.E.D.

5. Translation

In this section we show that it is possible to define translation functors in the category of rational $\mathrm{G}_{\mathbf{Z}_p}$-modules.

Let C (resp. $\overline{\mathrm{C}}$) denote an alcove in X(T), i. e. C (resp. $\overline{\mathrm{C}}$) is a W_p conjugate of the bottom alcove in $\mathrm{X}(\mathrm{T})_+$

$$\mathrm{C}_0=\{\lambda\in \mathrm{X}(\mathrm{T})\,|\,0<\langle \alpha^\vee, \lambda+\rho\rangle<p,\ \alpha\in \mathrm{R}_+\}\ (\text{resp. of } \overline{\mathrm{C}}_0=\{\lambda\in \mathrm{X}(\mathrm{T})\,|\,0\leqq\langle\alpha^\vee, \lambda+\rho\rangle\leqq p,\ \alpha\in \mathrm{R}_+\}).$$

For $\lambda\in\overline{\mathrm{C}}$ we let $\mathscr{M}_\lambda$ denote the category of G_k-modules with the property that all their composition factors have highest weights in $\mathrm{W}_p.\lambda$. If V is a finite dimensional rational G_k-module then it is a consequence of the linkage principle that we may write

$$\mathrm{V}=\bigoplus_{\lambda\in \mathrm{C}} p_\lambda(\mathrm{V}),$$

where $p_\lambda(\mathrm{V})$ denotes the largest submodule of V which belongs to $\mathscr{M}_\lambda$.

LEMMA 5.1. — *Let* $0 \to N \to M \to Q \to 0$ *be an exact sequence of rational* G-*modules of finite type. Then for each* $\lambda \in \overline{C}$ *we have*

$$M \otimes k \in \mathscr{M}_\lambda \quad \textit{if and only if} \quad N \otimes k, \qquad Q \otimes k \in \mathscr{M}_\lambda.$$

Proof. — Suppose $M \otimes k \in \mathscr{M}_\lambda$. Being a quotient of $M \otimes k$ it is clear that $Q \otimes k \in \mathscr{M}_\lambda$ and hence also the submodule $Q_t \otimes k \in \mathscr{M}_\lambda$ (Q_t being the torsion part of Q). The exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}}(Q_t, k) \to N \otimes k \to M \otimes k,$$

together with Lemma 2.8 imply then that $N \otimes k \in \mathscr{M}_\lambda$. The other implication in the lemma is obvious.

Q.E.D.

PROPOSITION 5.2. — *Suppose* $\lambda, \lambda' \in \overline{C}$ *with* $\lambda \neq \lambda'$. *If* M *and* N *are rational* $G_{\mathbb{Z}_p}$-*modules of finite type with* $M \otimes k \in \mathscr{M}_\lambda$, $N \otimes k \in M_\lambda$, *then*

$$\mathrm{Ext}^i_{G_{\mathbb{Z}_p}}(M, N) = 0 \quad \textit{for all } i.$$

Proof. — If M and N are free as $\mathbb{Z}_p$-modules this follows from Theorem 1.18 combined with the fact that $\mathrm{Ext}^i_{G_k}(M \otimes k, N \otimes k) = 0$ for all i [4], 1.5. Before we treat the general case we need the following result.

LEMMA 5.3. — *Suppose* V *is a rational* G-*module of finite type then* V *has a filtration of rational* G-*modules* $(V^i)_{i \geq 0}$ *with* $V^i/V^{i+1} \otimes k \in \mathscr{M}_{\lambda_i}$ *for some* $\lambda_i \in \overline{C}$.

Proof. — We use induction on the number of generators for V. If V is cyclic then clearly $V \otimes k \in \mathscr{M}_0$. To prove the induction step pick a generator $v \in V_\mu$ where μ is maximal among the weights of V. Then we have a surjection $V \to E$, where $E = \mathbb{Z}v$, of rational B-modules. By (1.7) we get an induced homomorphism $V \to H^0(E)$ of rational G-modules. Note that $H^0(E) \otimes k \in \mathscr{M}_\mu$. This follows from Corollary 2.6 (iii) if E is free and also in the finite cyclic case it then follows by taking a free resolution. If I (resp. K) denotes the image (resp. kernel) of the homomorphism $V \to H^0(E)$ it follows from Lemma 5.1 that $I \otimes k \in \mathscr{M}_\mu$. Since K has fewer generators than V the lemma holds for K by induction hypothesis. Hence it holds for V.

Q.E.D.

Proof of Proposition 5.2. (cont.). — Let M be a rational $G_{\mathbb{Z}_p}$-module of finite type such that $M \otimes k \in \mathscr{M}_\lambda$. By (1.4) M is a quotient of a free rational $G_{\mathbb{Z}_p}$-module P of finite type. By Lemma 5.3 (or rather its proof) P has a filtration with quotients contained in certain $H^0_p(\mu)'s$. By Corollary 2.6 (ii) these quotients are free $\mathbb{Z}_p$-modules and hence if P_0 denotes the biggest submodule of P with $P_0 \otimes k \in \mathscr{M}_\lambda$ then by the special case of the proposition treated above (the free case) P_0 is a direct summand of P, i. e. $P = P_0 \oplus P'$ for some submodule P'. Moreover, $\mathrm{Hom}_{G_{\mathbb{Z}_p}}(P', M) = 0$ and hence P_0 maps surjectively onto M so that we get a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0,$$

where P_i is a free rational G_{Z_p}-module with $P_i \otimes k \in \mathcal{M}_\lambda$, $i=0,1$. A similar sequence exists for N and the proposition therefore follows from the free case.

Q.E.D.

For $\lambda \in \overline{C}$ and V a rational G_{Z_p}-module of finite type we will use the notation $p_\lambda(V)$ for the biggest submodule of V which upon tensoring with k belongs to $\mathcal{M}_\lambda$.

THEOREM 5.4. — *Let* V *be a rational* G_{Z_p}*-module of finite type. Then*

$$V = \bigoplus_{\lambda \in \overline{C}} p_\lambda(V).$$

Proof. — Immediate consequence of Proposition 5.2 and Lemma 5.3.

Q.E.D.

This theorem allows us to define translation functors

Suppose $\lambda, \mu \in \overline{C}$ and pick $\tau \in W$ such that $\tau(\mu-\lambda) \in X(T)_+$. Then we define the functor T_λ^μ by

$$T_\lambda^\mu V = p_\mu(V \otimes H_p^0(\tau(\mu-\lambda))),$$

V a rational G_{Z_p}-module of finite type. Note that $(T_\lambda^\mu V) \otimes k \simeq T_\lambda^\mu(V \otimes k)$, where T_λ^μ on the right side denotes the usual translation functor, *see* [4] or [14].

LEMMA 5.5. — (i) *The functor* T_λ^μ *is exact.*

(ii) *When restricted to the subcategory of all rational* G_{Z_p}*-modules* V *with* $V \otimes k \in \mathcal{M}_\lambda$ (*resp.* $V \otimes k \in \mathcal{M}_\mu$) *the two functors* T_λ^μ *and* T_μ^λ *are adjoint.*

Proof. — (i) is clear because $H_p^0(\tau(\mu-\lambda))$ is free and (ii) is proved exactly as in the usual case [note that by (2.4) and (2.5) we have

$$H_p^0(\tau(\mu-\lambda))^* \simeq H_p^N(-\tau(\mu-\lambda)-2\rho) \simeq H_p^0(w_0\tau(\lambda-\mu))].$$

Q.E.D.

As in [4] we set $S_\lambda = \{ s \in W_p \mid s.\lambda = \lambda \}$ and we get

PROPOSITION 5.6. — (i) *If* $S_\lambda = 1$ *or* $S_\lambda = S_\mu$ *then* $T_\lambda^\mu H_p^i(w.\lambda) \simeq H_p^i(w.\mu)$ *for all* $w \in W$, $i \geqq 0$.

(ii) *Suppose* $S_\mu = \{1, s\}$ *where* $s \notin S_\lambda$. *Then for all* $w \in W_p$ *with* $ws.\lambda < w.\lambda$ *we have a long exact sequence*

$$\ldots \to H_p^i(ws.\lambda) \to T_\mu^\lambda H_p^i(w.\mu) \to H_p^i(w.\lambda) \to \ldots$$

Similarly for all $w \in W_p$ *with* $w.\lambda < ws.\lambda$ *we have a long exact sequence*

$$\ldots \to H_p^i(w.\lambda) \to T_\mu^\lambda H_p^i(w.\mu) \to H_p^i(ws.\lambda) \to \ldots$$

Proof. — *See* Proposition 2.1 in [4] (the analogue of [4], Proposition 2.1 (*c*) also holds in our case, but we shall not need it here).

Q.E.D.

6. On Lusztig's conjecture

In this section we assume throughout that $p>h$ (h being the Coxeter number) and we fix an alcove $C\subseteqq X(T)_+$, $\lambda\in C$, $s\in W_p$ a reflection in a wall of C such that $\lambda'=s.\lambda<\lambda$ and $\mu\in\overline{C}$ with $S_\mu=\{1, s\}$. Furthermore we assume that $\langle \alpha^\vee, \lambda+\rho\rangle\leqq p^2$ for all $\alpha\in R_+$ (several of the results below have a straightforward analogue for more general λ).

We also work with a fixed reduced expression for w_0, $w_0=s_{\beta_1}s_{\beta_2}\ldots s_{\beta_N}$ and we set

$$\begin{aligned} \lambda_j&=s_{\beta_j}s_{\beta_{j-1}}\ldots s_{\beta_1}.\lambda, &\quad X(j)&=H^j_f(\lambda_j),\\ \mu_j&=s_{\beta_j}s_{\beta_{j-1}}\ldots s_{\beta_1}.\mu, &\quad Y(j)&=T^\lambda_\mu H^j_f(\mu),\\ \lambda'_j&=s_{\beta_j}s_{\beta_{j-1}}\ldots s_{\beta_1}.\lambda', &\quad Z(j)&=H^j_f(\lambda'_j).\end{aligned}$$

Let α be the positive root for which $s=s_{\alpha,n}$ for some $n\in\mathbb{N}$. Then $\alpha=s_{\beta_1}s_{\beta_2}\ldots s_{\beta_{j_0}}(\beta_{j_0+1})$ for some $j_0\leqq N$. Since $\lambda'=\lambda-d\alpha$ for some $d>0$ we see that

$$\begin{aligned}\lambda'_j&=\lambda_j-ds_{\beta_j}\ldots s_{\beta_1}(\alpha)\\ &=\begin{cases}\lambda_j-ds_{\beta_{j+1}}\ldots s_{\beta_{j_0}}(\beta_{j_0+1})<\lambda_j & \text{for } j\leqq j_0,\\ \lambda_j+ds_{\beta_j}\ldots s_{\beta_{j+2}}(\beta_{j_0+1})>\lambda_j & \text{for } j>j_0.\end{cases}\end{aligned}$$

Hence from Proposition 5.6 (ii) we get complexes

$$0\to X(j)\xrightarrow{i_j} Y(j)\xrightarrow{\pi_j} Z(j)\to 0 \qquad \text{for } j>j_0$$

and

$$0\to Z(j)\xrightarrow{i_j} Y(j)\xrightarrow{\pi_j} X(j)\to 0 \qquad \text{for } j<j_0.$$

It is clear that these complexes are exact for $j=0$ and for $j=N$ and also that i_j is always injective. From section 4 we have the homomorphisms $T^X_j=T^{\lambda_j}_{s_{\beta_j}}: X(j)\to X(j-1)$ and $T^Z_j=T^{\lambda'_j}_{s_{\beta_j}}: Z(j)\to Z(j-1)$. The homomorphism $T^{\mu_j}_{s_{\beta_j}}: H^j_f(\mu_j)\to H^{j-1}_f(\mu_{j-1})$ induces a homomorphism $Y(j)\to Y(j-1)$ which we will denote T^Y_j.

LEMMA 6.1. — (i) *For $j>j_0+1$ we have a commutative diagram (up to a unit in $\mathbb{Z}_p$)*

$$\begin{tikzcd} X(j) \arrow[r, "i_j"] \arrow[d, "T^X_j"] & Y(j) \arrow[r, "\pi_j"] \arrow[d, "T^Y_j"] & Z(j) \arrow[dr, "T^Z_j"] & \\ X(j-1) \arrow[r, "i_{j-1}"] & Y(j-1) \arrow[r, "\pi_{j-1}"] & Z(j-1) & \end{tikzcd}$$

(ii) *For $j\leqq j_0$ we have a commutative diagram (up to a unit in $\mathbb{Z}_p$)*

$$\begin{tikzcd} Z(j) \arrow[r, "i_j"] \arrow[d, "T^Z_j"] & Y(j) \arrow[r, "\pi_j"] \arrow[d, "T^Y_j"] & X(j) \arrow[d, "T^X_j"] \\ Z(j-1) \arrow[r, "i_{j-1}"] & Y(j-1) \arrow[r, "\pi_{j-1}"] & X(j-1). \end{tikzcd}$$

Proof. — Note that by Proposition 5.6(i) we have

$$\begin{aligned}\operatorname{Hom}_{G_{\mathbb{Z}_p}}(X(j), Y(j-1)) &\cong \operatorname{Hom}_{G_{\mathbb{Z}_p}}(T_\lambda^\mu X(j), H_f^{j-1}(\mu_{j-1}))\\ &\cong \operatorname{Hom}_{G_{\mathbb{Z}_p}}(H_f^j(\mu_j), \leftarrow H_f^{j-1}(\mu_{j-1})) \cong \mathbb{Z}_p.\end{aligned}$$

Hence for $j > j_0 + 1$ the homomorphisms $T_j^Y \circ i_j$ and $i_{j-1} \circ T_j^X$ differ only by a constant in $\mathbb{Z}_p$ and since they both are non-zero when tensored by k (trace highest weights) we conclude that this constant is a unit. A similar argument proves the commutativity of the other diagrams.

Q.E.D.

In order to analyse what happens for $j = j_0 + 1$ we assume for simplicity that $\langle \alpha^\vee, \lambda - \mu \rangle = 1$. Then we have

LEMMA 6.2. — *Set* $\beta = \beta_{j_0+1}$. *Then:*

(i) $$T_\mu^\lambda H_p^{j_0+1}(\mu_{j_0+1}) \cong H_p^{j_0}(H_{\beta,p}^1(\mu_{j_0+1}) \otimes H_{\beta,p}^0(\lambda_{j_0} - \mu_{j_0}))$$

and:

$$T_\mu^\lambda H_p^{j_0}(\mu_{j_0}) \cong H_p^{j_0}(H_{\beta,p}^0(\mu_{j_0}) \otimes H_{\beta,p}^0(\lambda_{j_0} - \mu_{j_0})).$$

(ii) *We have the diagram*

$$\begin{tikzcd} X(j_0+1) \arrow[r, shift left, "i_{j_0+1}"] & Y(j_0+1) \arrow[l, shift left, "r_{j_0+1}"] \arrow[r, shift left, "\pi_{j_0+1}"] \arrow[d, "T_{j_0+1}^Y"] & Z(j_0+1) \arrow[l, shift left, "s_{j_0+1}"] \\ Z(j_0) \arrow[r, shift left, "i_{j_0}"] & Y(j_0) \arrow[l, shift left, "r_{j_0}"] \arrow[r, shift left, "\pi_{j_0}"] & X(j_0) \arrow[l, shift left, "s_{j_0}"] \end{tikzcd}$$

where the following identities hold

(*a*) $r_n \circ i_n$, $\pi_n \circ s_n$ *and* $s_n \circ \pi_n + i_n \circ r_n$ *are multiplication by* p, $n = j_0, j_0 + 1$.

(*b*) $s_{j_0} \circ T_{j_0+1}^X = p T_{j_0+1}^Y \circ i_{j_0+1}$.

(*c*) $T_{j_0+1}^Z \circ \pi_{j_0+1} = r_{j_0} \circ T_{j_0+1}^Y$.

(*d*) $T_{j_0+1}^Y \circ s_{j_0+1} = i_{j_0} \circ T_{j_0+1}^Z$.

(*e*) $T_{j_0+1}^X \circ r_{j_0+1} = p \pi_{j_0} \circ T_{j_0+1}^Y$.

Proof. — Let $(F^n)_{n \geq 0}$ be a $B_{\mathbb{Z}_p}$-filtration of $H_p^0(\tau(\lambda - \mu))$ such that $F^n/F^{n+1} \cong \lambda_n$, where the λ_n's are the weights of $H_p^0(\tau(\lambda - \mu))$. Then we get exact sequences of $P_{\beta \mathbb{Z}_p}$-modules

$$0 \to H_{\beta,p}^0(F^{n+1}) \to H_{\beta,p}^0(F^n) \to H_{\beta,p}^0(\lambda_n).$$

As λ_n is a weight in $H_p^0(\tau(\lambda - \mu))$ we must have $\langle \beta^\vee, \lambda_n \rangle < p$ and hence $H_{\beta,p}^0(\lambda_n) \otimes k$ is an irreducible $P_{\beta k}$-module. It follows that if $H_{\beta,p}^0(F^n) \to H_{\beta,p}^0(\lambda_n)$ is non-zero then it is surjective. Hence $H_p^0(\tau(\lambda - \mu))$ has a filtration $(E^n)_{n \geq 0}$ of $P_{\beta \mathbb{Z}_p}$-modules with

$$0 \to E^{n+1} \to E^n \to H_{\beta,p}^0(\lambda_n) \to 0,$$

where the λ_n's are among the weights of $H_p^0(\tau(\lambda - \mu))$.

Now by definition $T^{\lambda}_{\mu} H_p^{j_0+1}(\mu_{j_0+1}) = p_\lambda(H_p^{j_0+1}(\mu_{j_0+1})\otimes H_p^0(\tau(\lambda-\mu)))$ and from (1.16) and (1.15) we get

$$H_p^{j_0+1}(\mu_{j_0+1})\otimes H_p^0(\tau(\lambda-\mu)) \cong H_p^{j_0+1}(\mu_{j_0+1}\otimes H_p^0(\tau(\lambda-\mu)))$$
$$\cong H_p^{j_0}(H^1_{\beta,\,p}(\mu_{j_0+1})\otimes H_p^0(\tau(\lambda-\mu))).$$

If we tensor the above sequence with $H^1_{\beta,\,p}(\mu_{j_0+1})$ and take cohomology then we get the long exact sequence

$$\dots \to H_p^j(H^1_{\beta,\,p}(\mu_{j_0+1})\otimes E^{n+1}) \to H_p^j(H^1_{\beta,\,p}(\mu_{j_0+1})\otimes E^n)$$
$$\to H_p^j(H^1_{\beta,\,p}(\mu_{j_0+1})\otimes H^0_{\beta,\,p}(\nu_n)) \to \dots$$

Now according to [13], Lemma 3, the only ν_n for which $H^1_{\beta,\,p}(\mu_{j_0+1})\otimes H^0_{\beta,\,p}(\nu_n)$ contains a weight linked to λ is $\nu_n=\lambda_{j_0}-\mu_{j_0}$. Hence by Corollary 2.6(iii) we get

$$p_\lambda(H_p^{j_0}(H^1_{\beta,\,p}(\mu_{j_0+1}))\otimes H_p^0(\tau(\lambda-\mu))) \cong p_\lambda(H_p^{j_0}(H^1_{\beta,\,p}(\mu_{j_0+1})\otimes H^0_{\beta,\,p}(\lambda_{j_0}-\mu_{j_0})))$$
$$\cong H_p^{j_0}(H^1_{\beta,\,p}(\mu_{j_0+1})\otimes H^0_{\beta,\,p}(\lambda_{j_0}-\mu_{j_0})),$$

where the last isomorphism comes from the short exact sequence (*see* section 3)

$$0 \to H^1_{\beta,\,p}(\mu_{j_0+1}+\lambda_{j_0}-\mu_{j_0}-\beta_{j_0}) \to H^1_{\beta,\,p}(\mu_{j_0+1})\otimes H^0_{\beta,\,p}(\lambda_{j_0}-\mu_{j_0})) \to H^1_{\beta,\,p}(\mu_{j_0+1}+\lambda_{j_0}-\mu_{j_0}) \to 0$$

and by noting that $\mu_{j_0+1}+\lambda_{j_0}-\mu_{j_0}-\beta_{j_0}=\lambda_{j_0+1}$, $\mu_{j_0+1}+\lambda_{j_0}-\mu_{j_0}=\lambda_{j_0+2}+\beta=\lambda_{j_0+1}$ are both linked to λ. We have thus proved the first isomorphism in (i). The second is proved in the same way and (ii) follows from (i) *via* Lemma 3.5.

Q.E.D.

LEMMA 6.3. — (i) *Let* $j>j_0+1$. *There exist homomorphisms of rational* G_{Z_p}*-modules*

$$r_j:\ Y(j)\to X(j), \qquad s_j:\ Z(j)\to Y(j),$$

such that the diagrams

$$\begin{tikzcd}
X(j) \arrow[d, "T_j^X"] & Y(j) \arrow[l, "r_j"'] \arrow[d, "T_j^Y"] & Z(j) \arrow[l, "s_j"'] \arrow[d, "T_j^Z"] \\
X(j-1) & Y(j-1) \arrow[l, "r_{j-1}"] & Z(j-1) \arrow[l, "s_{j-1}"]
\end{tikzcd}$$

commute and such that $r_j\circ i_j$, $\pi_j\circ s_j$ *and* $i_j\circ r_j+s_j\circ\pi_j$ *are multiplication by* p.

(ii) *Let* $j\leqq j_0$. *Then there exist homomorphisms of rational* G_{Z_p}*-modules*

$$r_j:\ Y(j)\to Z(j), \qquad s_j:\ X(j)\to Y(j),$$

such that the diagrams

$$\begin{tikzcd} Z(j) \arrow[d, "T_j^Z"] & Y(j) \arrow[l, "r_j"'] \arrow[d, "T_j^Y"] & X(j) \arrow[l, "s_{j-1}"'] \arrow[d, "T_j^X"] \\ Z(j-1) & Y(j-1) \arrow[l, "r_{j-1}"'] & X(j-1) \arrow[l, "s_{j-1}"'] \end{tikzcd}$$

commute and such that $r_j \circ i_j$, $\pi_j \circ s_j$ and $i_j \circ r_j + s_j \circ \pi_j$ are multiplication by p.

Proof. — Suppose $j > j_0 + 1$. Note that

$$\mathrm{Hom}_{G_{\mathbb{Z}_p}}(Y(j), X(j)) \cong \mathrm{Hom}_{G_{\mathbb{Z}_p}}(H^j_{j'}(\mu_j), \quad T^\mu_\lambda X(j)) \cong \mathrm{Hom}_{G_{\mathbb{Z}_p}}(H^j_{j'}(\mu_j), H^j_{j'}(\mu_j)) \cong \mathbb{Z}_p.$$

Let r_j be a generator of $\mathrm{Hom}_{G_{\mathbb{Z}_p}}(Y(j), X(j))$. As in Lemma 6.2 it follows that $T^X_j \circ r_j = r_{j-1} \circ T^Y_j$ (up to a unit in $\mathbb{Z}_p$). The commutativity of the other diagram is checked in the same way [s_j is defined as a generator of $\mathrm{Hom}_{G_{\mathbb{Z}_p}}(Z(j), Y(j))$].

By Lemma 6.2 we know that $r_{j_0+1} \circ i_{j_0+1} = p$. Hence

$$T^X_{j_0+2} \circ r_{j_0+2} \circ i_{j_0+2} = r_{j_0+1} \circ T^Y_{j_0+2} \circ i_{j_0+1} = r_{j_0+1} \circ i_{j+1} \circ T^X_{j_0+2} = p\, T^X_{j_0+2},$$

where the second equality comes from Lemma 6.1. It follows that $r_{j_0+2} \circ i_{j_0+2} = p$. Repeating this argument we find $r_j \circ i_j = p$ for all $j > j_0$. The other relations are checked in the same way.

Q.E.D.

Let $T^X = T^X_1 \circ T^X_2 \circ \ldots \circ T^X_N$ and define T^Y and T^Z analogously. Then the above lemmas prove in particular

Proposition 6.4. — *With the above notation and assumptions we have homomorphisms i_j, π_j, r_j and s_j, $j = 1, \ldots, N$ of rational $G_{\mathbb{Z}_p}$-modules satisfying*

(i) *$r_j \circ i_j$, $\pi_j \circ s_j$ and $i_j \circ r_j + s_j \circ \pi_j$ are multiplication by p, $j = 1, \ldots, N$.*

(ii) $s_0 \circ T^X = p\, T^Y \circ i_N$.

(iii) $T^Z \circ \pi_N = r_0 \circ T^Y$.

(iv) $T^Y \circ s_N = i \circ T^Z$.

(v) $T^X \circ r_N = p\, \pi_0 \circ T^Y$.

Let $X(N)^j$, $Y(N)^j$ denote the filtrations defined by T^X, T^Y and T^Z, respectively (e. g. $X(N)^j = \{ v \in X(N) \mid T^X(v) \in p^j X(0) \}$).

Lemma 6.5. — *With the above notation and assumptions we get:*

(i) *$i_N(X(N)^{j+1}) \subseteq Y(N)^j$ for all j.*

(ii) *$\pi_N(Y(N)^j) \subseteq Z(N)^j$ for all j.*

Proof. — This is immediate in view of Proposition 6.4 (ii) and (iii).

Q.E.D.

Set now $\overline{X}(N)=X(N\otimes k)$, $\overline{Y}(N)=Y(N)\otimes k$ and $\overline{Z}(N)=Z(N)\otimes k$. Let $(\overline{X}(N)^j)_{j\geqq 0}$, $(\overline{Y}(N)^j)_{j\geqq 0}$ and $(\overline{Z}(N)^j)_{j\geqq 0}$ denote the images in $\overline{X}(N)$, $\overline{Y}(N)$ and $\overline{Z}(N)$ and $Z(N)$, respectively, of the above filtrations.

We have the following relations between translation and these filtrations

LEMMA 6.6

(i) $$T_\lambda^\mu \overline{X}(N)^{j+1}=H_k^N(\mu_N)^j.$$

(ii) $$T_\lambda^\mu \overline{Z}(N)^j=H_k^N(\mu_N)^j.$$

(iii) *Suppose* $y\in W_p$ *such that* $y.\mu\in X(T)_+$. *If* $y.\lambda<ys.\lambda$ *then:*

$$[X(N)^{j+1} : L(y.\lambda)]=[\overline{Z}(N)^j : L(y.\lambda)] \quad \textit{for all } j.$$

Proof. — For $j\neq j_0+1$ we have the commutative diagram

$$\begin{array}{ccc} T_\lambda^\mu X(j) & \cong & H_f^j(\mu_j) \\ \downarrow & & \downarrow {\scriptstyle T_{s_j}^{\mu_j}} \\ T_\lambda^\mu X(j-1) & \cong & H_f^{j-1}(\mu_{j-1}) \end{array}$$

where the left vertical map is induced by T_j^X and where the horizontal isomorphisms come from Proposition 5.6 (i). For $j=j_0+1$ we have instead the commutative diagram

$$\begin{array}{ccc} T_\lambda^\mu X(j_0+1) & \cong & H_f^{j_0+1}(\mu_{j_0+1}) \\ \downarrow & & \downarrow {\scriptstyle p\,T_{s_{j_0+1}}^{\mu_{j_0+1}}} \\ T_\lambda^\mu X(j_0) & \cong & H_f^{j_0}(\mu_{j_0}) \end{array}$$

All this follows exactly as in the proof of Lemma 6.2 from Lemma 3.5. Hence we see that $T_\lambda^\mu(T^X)=p\,T_{w_0}^{\mu_N}$ and (i) follows. The proof of (ii) is analogous [one shows that $T_\lambda^\mu(T^Z)=T^{\mu_N}$]. Finally (iii) is a consequence of (i) and (ii) by recalling that $T_\lambda^\mu L(y.\lambda)=L(y.\mu)$ when $y.\lambda<ys.\lambda$, [4].

Q.E.D.

The identity $s_N\circ\pi_N=p-i_N\circ r_n$ shows that $s_n(\overline{Z}(N))\subseteqq i_N(\overline{X})$ [by abuse of notation we let the same letter s_N, i_N, etc. denote also the map $\overline{Z}(N)\to\overline{Y}(N)$, $\overline{Y}(N)\to\overline{Y}(N)$, etc induced by s_N, i_N, ...]. In other words, s_N is the intertwining homomorphism between the two Weyl modules $H_k^N(\lambda'_N)$ and $H_k^N(\lambda_N)$ (compare [4]).

CONJECTURE 6.7. — Suppose $\langle\alpha^\vee, \lambda+\rho\rangle\langle p(p-h+2)$ for all $\alpha\in R_+$. Then the intertwining homomorphism

$$H_k^N(\lambda'_N)\to H_k^N(\lambda_N)$$

maps $H_k^N(\lambda'_N)^j$ into $H_k^N(\lambda_N)^{j+1}$ for all j.

From now on we assume that λ satisfies the assumption in this conjecture and we assume that the conjecture holds.

LEMMA 6.8. — (i) $Z(N)^{j+1} \subseteq \pi_N(Y(N)^j)$ *for all j*.

(ii) $\overline{Y}(N)^j \cap i_N(\overline{X}(N)) \subseteq i_N(\overline{X}(N)^j)$ *for all j*.

Proof. — (i) Let $z \in Z(N)^{j+1}$. By Conjecture 6.7 we may write $s_N(z) = i_N(x) + py$ for some $x \in X(N)^{j+2}$, $y \in Y(N)$. Applying π_N to this equation and using that $\pi_N \circ s_N = p$ we see that $\pi_N(y) = z$. Now $p^2 y = ps_N(z) - pi_N(x)$ and hence by Proposition 6.4 (iv) and (ii) we get

$$p^2\, T^Y(y) = pi_0 \circ T^Z(z) - s_0 \circ T^X(x) \in p^{j+2}\, Y(0).$$

Hence $y \in Y(N)^j$.

(ii) Take $x \in X(N)$ such that $i_N(x) = y + py_1$ for some $y \in Y(N)^j$, $y_1 \in Y(N)$. Let $\overline{x}$, $\overline{y}$, etc denote the images of x, y, etc in $\overline{X}(N)$, $\overline{Y}(N)$, ...

We want to show that $i_N(\overline{x}) \in i_N(\overline{X}(N)^j)$.

As $i_N \circ r_N + s_N \circ \pi_N = p$ we have $i_N \circ r_N(\overline{y}_1) + s_N \circ \pi_N(\overline{y}_1) = 0$ and hence

$$i_N(\overline{x}) = -s_N \circ \pi_N(\overline{y}_1) + i_N(\overline{x - r_N(y_1)}).$$

We have $p\,\pi_N(y_1) = -\pi_N(y) \in \pi_N(Y(N)^j) \subseteq Z(N)^j$ by Lemma 6.5 (ii). Hence by Conjecture 6.7 we get $s_N \circ \pi_N(\overline{y}_1) \in s_N(\overline{Z}(N)^{j-1}) \subseteq \overline{X}(N)^j$. We claim that also $x - r_N(y_1) \in X(N)^j$. In fact $px = r_N \circ i_N(x) = r_N(y) + pr_N(y_1)$ and so *via* Proposition 6.4 (v)

$$T^X(p(x - r_N(y_1)) = T^X r_N(y) = p\,\pi_0\, T^Y(y) \in p^{j+1}\, X(0), \text{ i.e. } x - r_N(y_1) \in X(N)^j.$$

Q.E.D.

The lemmas 6.5 and 6.8 allow us to define a 4 step filtration of the filtration levels (as usual denoted by subscripts) of $\overline{Y}(N)$ as follows

We set

$$\overline{X}^a(N)_{j+1} = \overline{X}(N)^{j+1} / i_N^{-1}(\overline{Y}(N)^{j+1}), \qquad \overline{X}^b(N)_j = i_N^{-1}(\overline{Y}(N)^j) / \overline{X}(N)^{j+1},$$
$$\overline{Y}^x(N)_j = i_N^{-1}(\overline{Y}(N)^j) / i_N^{-1}(\overline{Y}(N)^{j+1}),$$
$$\overline{Z}^a(N)_j = \pi_N(\overline{Y}(N)^j) / \overline{Z}(N)^{j+1}, \qquad \overline{Z}^b(N)_j = \overline{Z}(N)^j / \pi_N(\overline{Y}(N)^j),$$
$$\overline{Y}^z(N)_j = \pi_N(\overline{Y}(N)^j) / \pi_N(\overline{Y}(N)^{j+1}),$$

so that we have the 5 exact sequences

$$0 \to \overline{X}^b(N)_{j+1} \to \overline{X}(N)_{j+1} \to \overline{X}^a(N)_{j+1} \to 0,$$
$$0 \to \overline{X}^a(N)_{j+1} \to \overline{Y}^x(N)_j \to \overline{X}^b(N)_j \to 0,$$
$$0 \to \overline{Z}^a(N)_{j+1} \to \overline{Z}(N)_{j+1} \to \overline{Z}^b(N)_{j+1} \to 0,$$
$$0 \to \overline{Z}^b(N)_{j+1} \to \overline{Y}^z(N)_j \to \overline{Z}^a(N)_j \to 0,$$
$$0 \to \overline{Y}^x(N)_j \to \overline{Y}(N)_j \to \overline{Y}^z(N)_j \to 0.$$

LEMMA 6.9. — (i) $T_\lambda^\mu \overline{X}^b(N)_j = T_\lambda^\mu \overline{Z}^b(N)_j = 0$ *for all* j.

(ii) *If* $Q \neq 0$ *is either a quotient of* $\overline{Z}^a(N)_j$ *or a submodule of* $\overline{X}^a(N)_j$ *then* $T_\lambda^\mu Q \neq 0$.

Proof. — By Lemma 6.6 and Lemma 6.8, we see that we may identify $T_\lambda^\mu \overline{Y}(N)^j$ with $T_\lambda^\mu \overline{X}(N)^{j+1} \oplus T_\lambda^\mu \overline{Z}(N)^j$ and (i) follows. To prove (ii) let $Q \neq 0$ be a quotient of $\overline{Z}^a(N)_j$. Then Q is also a quotient of $\overline{Y}(N)_j \simeq T_\mu^\lambda H_k^N(\mu_N)_j$. Hence $0 \neq \mathrm{Hom}_{G_k}(T_\mu^\lambda H_k^N(\mu_N)_j, Q) \simeq \mathrm{Hom}_{G_k}(H_k^N(\mu_N)_j, T_\lambda^\mu Q)$ and so certainly $T_\lambda^\mu Q \neq 0$. A similar argument takes care of the case where Q is a submodule of $\overline{X}^a(N)_j$.

Q.E.D.

LEMMA 6.10. — *Suppose the sequence* $0 \to \overline{Z}^a(N)_j \to \overline{Z}(N)_j \to \overline{Z}^b(N)_j \to 0$ *splits. Then* $\overline{X}^a(N)_{j+1} \simeq \overline{Z}^a(N)_j$.

Proof. — Consider the homomorphism $p^{-j} T^Y : Y(N)^j \to Y(0)$ and note that since $i_N(X(N)^{j+1}) \subseteq Y(N)^j$ (Lemma 6.5) and $\pi_0 p^{-j} T^Y i_N(X(N)^{j+1}) = p^{-j} T^X(X(N)^{j+1}) \subseteq p X(0)$ we get a homomorphism

$$\overline{X}(N)^{j+1} \to \overline{Z}(0).$$

We claim that the image of this is contained in $\overline{Z}(0)^{m-j}$, where $m = \sum_{\beta \in R_+} n_\beta$ as in section 4.

To see this let $x \in X(N)^{j+1}$ and set $y = p^{-j} T^Y i_N(x)$, $x_1 = p^{-1} \pi_0(y)$. Then $x_1 = p^{-j-1} T^X(x)$ and we find that $x_1 \in X(0)^{m-j-1}$. Hence by Conjecture 6.7 (or rather its dual analogue) we have $s_0(\overline{x}_1) \in i_0\ (\overline{Z}(0)^{m-j})$. However, we have $s_0(x_1) = p^{-1} s_0 \pi_0(y) = y - p^{-1} i_0 r_0(y) = y$ since

$$i_0 r_0(y) = p^{-j} i_0 r_0 T^Y i_N(x) = p^{-j} i_0 T^Z \pi_N i_N(x) = 0$$

(Proposition 6.4) and the claim follows.

From this claim, we see that we *get* an induced homomorphism $\overline{X}(N)_{j+1} \to \overline{Z}(0)_{m-j} \simeq \overline{Z}(N)_j$ where we have employed Proposition 4.6 (i). Now it is not hard to check taht when we apply T_λ^μ to this homomorphism we get the identity on $H_k^N(\mu_N)_j$. From Lemma 6.9 follows then first that the composite $\overline{X}(N)_j \to \overline{Z}(N)_j \to \overline{Z}^a(N)_j$ is non-zero and factors through $\overline{X}^a(N)_{j+1}$ and next that both the kernel and the cokernel of the resulting homomorphism $\overline{X}^a(N)_{j+1} \to \overline{Z}^a(N)_j$ are zero.

Q.E.D.

Let d_1 denote the composite $\overline{X}^a(N)_{j+1} \to \overline{Y}^x(N)_j \to \overline{Y}(N)_j$ and d_2 the composite $\overline{Y}(N)_j \to \overline{Y}^Z(N)_j \to \overline{Z}^a(N)_j$. Set $U_j(\lambda) = \ker d_2 / \mathrm{Im}\, d_1$.

LEMMA 6.11. — *Suppose* $\overline{Z}(N)_j$ *is semi-simple. Then we have a short exact sequence*

$$0 \to \overline{X}^b(N)_j \to U_j(\lambda) \to \overline{Z}^b(N)_{j+1} \to 0.$$

Proof. — Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
0 \to & \overline{Z}^b(N)_{j+1} & \to & \overline{Y}^z(N)_j & \to & \overline{Z}^a(N)_j & \to 0 \\
 & & & \uparrow & & \| & \\
0 \to & \overline{X}^a(N)_{j+1} & \xrightarrow{d_1} & \overline{Y}(N)_j & \xrightarrow{d_2} & \overline{Z}^a(N)_j & \\
 & \| & & \uparrow & & & \\
0 \to & \overline{X}^a(N)_{j+1} & \to & \overline{Y}^x(N)_j & \to & \overline{X}^b(N)_j & \to 0 \\
 & & & \uparrow & & & \\
 & & & 0 & & &
\end{array}$$

The only thing we have to show is that $\overline{X}^b(N)_j$ is the kernel of the map $U_j(\lambda) \to \overline{Z}^b(N)_{j+1}$. This is clear because by Lemma 6.9 we have $\mathrm{Hom}_{G_k}(\overline{X}^b(N)_j, \overline{Z}^a(N)_j) = 0$.

Q.E.D.

To formulate the main result in this section let $\lambda \in C_0$, $\mu \in \overline{C}$ be fixed and suppose $S_\mu = \{1, s\}$. If $w \in W_p$ such that $w.\mu \in X(T)_+$ and $w.\lambda > ws.\lambda$ then we let $U(w)$ denote the cohomology of the complex (compare [5])

$$0 \to L(ws.\lambda) \to T_\mu^\lambda L(w.\mu) \to L(ws.\lambda) \to 0.$$

THEOREM 6.12. — *Suppose Conjecture 6.7 holds. Then for all $w \in W_p$ such that $w.\mu \in X(T)_+$, $w.\lambda > ws.\lambda$ and $\langle \alpha^\vee, w(\lambda+\rho) \rangle < p(p-h+2)$ we have*

(i) $U(w)$ *is semi-simple.*

(ii) $H_k^N(w_0 w.\lambda)_j$ *is semi-simple.*

(iii) *For all $y \in W_p$ with $y.\lambda \in X(T)_+$ and $y.\lambda > ys.\lambda$ we have*

$$[T_\mu^\lambda H_k^N(w_0 w.\mu)_j : L(y.\lambda)] = [H_k^N(w_0 ws.\lambda)_{j+1} : L(y.\lambda)] + [H_k^N(w_0 w.\lambda)_j : L(y.\lambda)].$$

Proof. — We use induction on w. If $w=1$ then $H_k^N(w_0 w.\lambda)$ is irreducible and there is nothing to prove. Suppose $w>1$. Then Lemma 6.11 taken relative to $w.\lambda$ gives for $j=0$ the short exact sequence

$$0 \to L(w.\lambda) \to U(w) \to \overline{Z}^b(N)_1 \to 0.$$

In fact, $\overline{Y}(N)_j = T_\mu^\lambda T_\lambda^\mu \overline{Z}(N)_j = T_\mu^\lambda T_\lambda^\mu \overline{Z}^a(N)_j$ so that $U_j(w.\lambda) \simeq \oplus U(y)$ with the direct sum extended over all $y \in W_p$ with $y.\mu \in X(T)_+$, $y.\lambda > ys.\lambda$ and $L(ys.\lambda)$ a composition factor of $\overline{Z}(N)_j$ (combine Lemma 6.10 and 6.11).

As $[U(w.\lambda) : L(w.\lambda)] = 1$ the fact that $L(w.\lambda)$ is contained in $U(w)$ implies that $L(w.\lambda)$ is a direct summand. By induction hypothesis $H_k^N(w_0 ws.\lambda)_1$ is semi-simple and hence so is the submodule $\overline{Z}^b(N)_1$. This proves (i).

To prove (ii) note that since all composition factors of $\overline{Z}(N)_j$ for $j>0$ have the form $L(y.\lambda)$ with $y<ws$ all $U(y)$'s occurring in $U(w.\lambda)_j$ for $j>0$ are semi-simple (by induction hypothesis). From Lemma 6.11 we see that $\overline{X}^b(N)_j$ is semi-simple for $j>0$ and Lemma 6.10 gives that so is $\overline{X}^a(N)_{j+1}$. It easy now to see that $H_k^N(ww_0.\lambda)_j=\overline{X}(N)_j=\overline{X}^a(N)_j \oplus \overline{X}^b(N)_j$ and (ii) follows.

Finally (iii) is an easy consequence of Lemma 6.9 and 6.11.

Q.E.D.

In [5] we proved that Lusztig's conjecture [17], Problem IV, on the characters of irreducible G_k-modules is equivalent to the semi-simplicity of the $U(w)$'s above. Hence Theorem 6.12 (i) proves:

COROLLARY 6.13. — *Conjecture* 6.7 *implies Lusztig's conjecture.*

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