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ON THE DYNAMICS OF RATIONAL MAPS

BY R. MAÑÉ, P. SAD AND D. SULLIVAN

I. — Introduction

1. It is a remarkable fact that each analytic [1] endomorphism f of the Riemann sphere $\mathbb{C}$ exhibits highly non-trivial dynamical phenomena. In this paper we first describe classical and recent results which give the basic dynamical picture of these mappings. Then we make use of this picture to construct topological conjugacies or partial topological conjugacies between certain nearby endomorphisms in analytic families of endomorphisms.

The partial or global conjugacies we will construct depend analytically on the parameters of the family and these satisfy the interesting geometric property of *quasi-conformality*. One corollary will be — *there is an open dense set* $\mathbb{C}$ *of degree* d *polynomial mappings of* $\mathbb{C}$ *such that all mappings in each connected component of* C *are conjugate by quasi-conformal homeomorphisms.*

Another corollary will be — *an open dense set of polynomial mappings satisfies the Axiom* A *expanding property iff there is an open dense set of polynomial mappings where the Julia set has Lebesgue measure zero.* A precise statement of results comes later in this introduction.

2. Our construction of conjugacies depends on a simple but at first surprising proposition concerning analytic perturbations of the inclusion of an arbitrary subset A of the sphere.

λ-LEMMA. — *Let* A *be a subset of* $\overline{\mathbb{C}}$, D *the open unit disk of* $\overline{\mathbb{C}}$ *and* $i_\lambda : A \to \overline{\mathbb{C}}$ *a family of injections depending analytically on* $\lambda \in D$ (*i.e. the function* $\lambda \to i_\lambda(z)$ *is analytic for all* $z \in A$). *Suppose that* i_0 *is the inclusion map* $A \hookrightarrow \overline{\mathbb{C}}$. *Then every* i_λ *has a quasi-conformal extension* $i_\lambda : \overline{A} \to \overline{\mathbb{C}}$ *which is a topological embedding depending analytically on* $\lambda \in D$ *and so that the map* $D \times A \ni (\lambda, z) \to i_\lambda(z) \in \overline{\mathbb{C}}$ *is continuous.*

Using the λ-Lemma and the general picture of the dynamics of endomorphisms of $\overline{\mathbb{C}}$ the topological conjugacies between endomorphisms are built in two steps. First, one builds analytically varying partial conjugacies on some dynamically easy subset A of the sphere

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(1) Without mention to the contrary analytic here means complex analytic.

(like the countable set of expanding periodic points union the basins of attracting periodic cycles). Second, one saturates by the mapping and applies the λ-Lemma to extend this partial conjugacy to the closure of A.

It is easy to see that from the above version of the λ-Lemma it can be deduced the more general statement obtained by replacing the unit disk in $\mathbb{C}$ by the unit ball in $\mathbb{C}^n$.

After this paper was written, Sullivan and Thurston [14] proved an extension Lemma that states that given a compact set $A \subset \mathbb{C}$ and a family of injections i_λ depending analytically on λ and satisfying the hypothesis of the λ-Lemma, then there exist a family of quasi-conformal homeomorphisms i_λ: $\overline{\mathbb{C}} \circlearrowleft$ extending $i_\lambda: A \to \overline{\mathbb{C}}$ and depending analytically on λ for λ in a disk $D_0 \subset D$, whose radius is a universal constant independent of the original family. Combining this Lemma with the techniques developed in this paper Sullivan obtained in [13] a substantial improvement of the stability Theorem presented here (Theorem D).

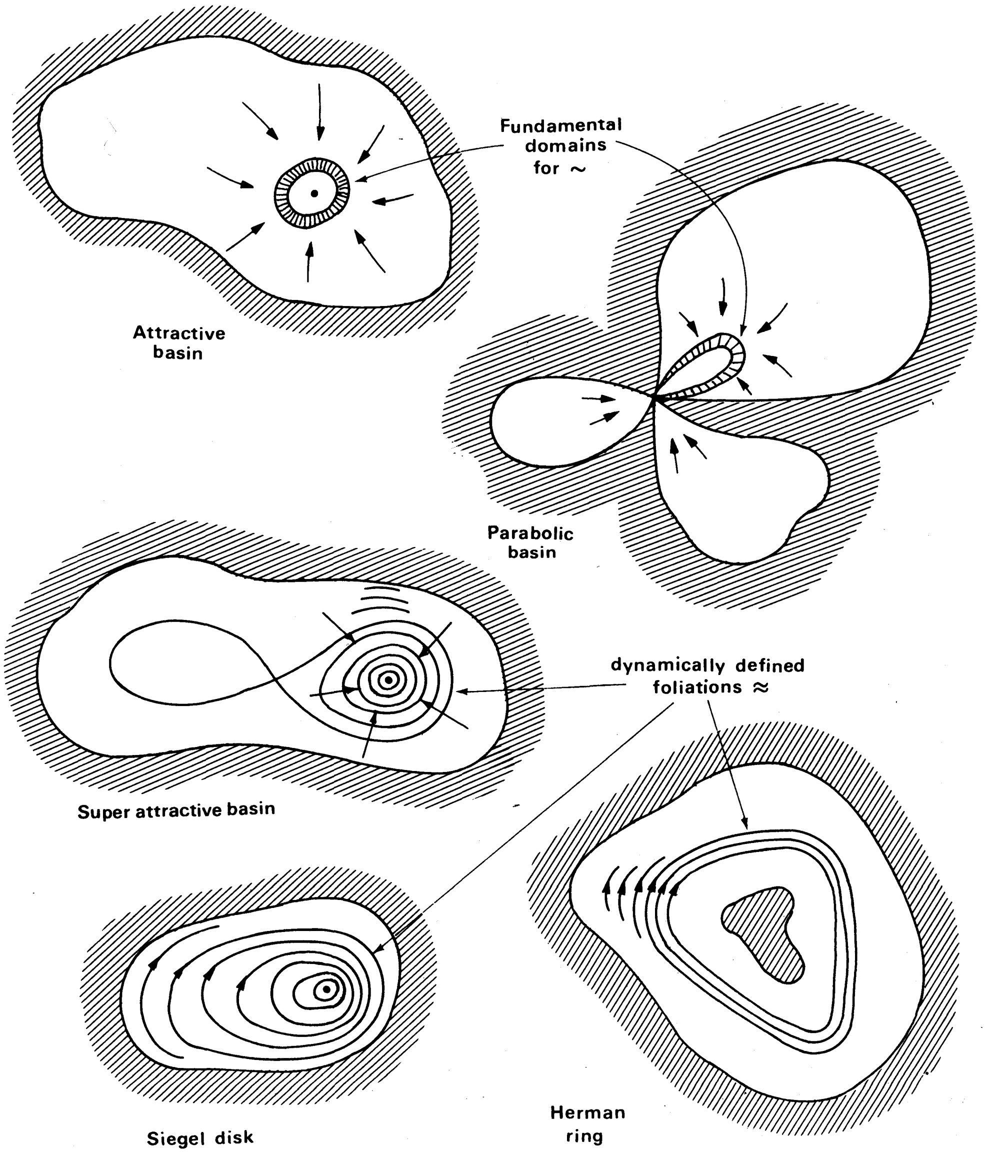
3. We will work with analytic families $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of endomorphisms where W is a connected complex manifold and $f = f(w, z)$ is analytic in two variables. For the global family of all endomorphisms of degree d, $\mathrm{End}_d(\mathbb{C})$, W is an open connected subset of the complex projective space $\mathbb{C}P^{2d+1}$, with the inherited topology equivalent to the C^0 topology on $\mathrm{End}_d(\overline{\mathbb{C}})$. The three-dimensional Moebius group $\{A : z \to (az+b)/(cz+d)\}$ acts by conjugation $f \to A . f . A^{-1}$ on the global family showing the space of analytically inequivalent endomorphisms has dimension $2d-2$. The number $2d-2$ coincides with the number of critical points $\{c \mid f'(c)=0\}$ of any endomorphism f of degree d. Critical points have topological dynamical meaning (f is not locally injective there) and the reader will observe in the course of the discussion below a relationship between the structure of the orbits of critical points and the number of essential analytic parameters for perturbations of f.

Similarly, the analytic family of all polynomial mappings of degree d is parameterized by an open connected dense subset of $\mathbb{C}P^{d+1}$ in which the similarity group $\{z \to az+b\}$ acts by conjugation. The quotient analytic space of analytically inequivalent polynomial mappings has dimension $d-1$ which coincides with the number of critical points of a polynomial mapping of degree d.

4. THE JULIA SET AND THE STABLE REGIONS. — Now we describe the dynamical picture of an analytic endomorphism f of $\overline{\mathbb{C}}$. Say that a point $x \in \overline{\mathbb{C}}$ is *stable* for f if on some neighborhood of x the family of interates $f, f^2, f^3, \ldots$, is an equicontinuous family of mappings of a neighborhood into the sphere. Note that when x is not stable i. e. *unstable*, for any neighborhood the union of images of iterates must cover $\overline{\mathbb{C}}$ except two points at most. Fatou [3] and Julia [6] showed *the set of unstable points* $J(f)$ (now called the Julia set) coincides with the *closure of the expanding periodic points* (sources).

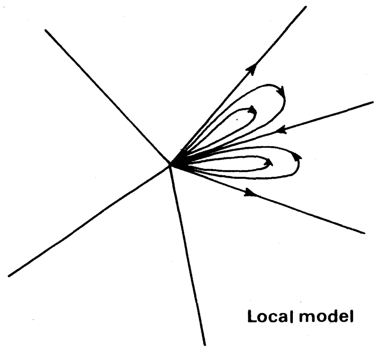
The open set of stable points consists of countably many connected components, *the stable regions of f*, which are transformed among themselves by f. In Sullivan [9] it is shown that under the iteration of *f each stable region is eventually cyclic*. The cycles of stable regions are classified into five types (Sullivan [9]). The first two types *attractive basins* and *parabolic basins* have fundamental domains for the equivalence relation $x \sim y$ iff $f^n x = f^m y$ for some $n, m \geqq 0$. The third type, *superattractive basins* do not, but they are foliated by the closures of the classes of the equivalence relation, $x \approx y$ iff $f^n x = f^n y$ for some $n \geqq 0$. The last two types are rotation domains, *Siegel disks* or *Herman rings*, which are foliated by the closures of forward orbits.

(i) An *attractive basin* D arises from an attractive periodic cycle γ with non zero derivative of modulus less than one, $\gamma = \{z, f(z), \ldots, f^{n-1}(z)\}$, $f^n(z) = z$, $0 < |(f^n)(z)| < 1$, and D consists of the components of $W_s(\gamma) = \bigcup_{x \in \gamma} \{y \mid \lim_{n \to +\infty} \text{distance}(f^n y, f^n x) = 0\}$ containing points of γ. Fatou [3] showed that such a D must contain a critical point of f. Thus there are no more than $2d-2$ attractive basins for an endomorphism of degree d.

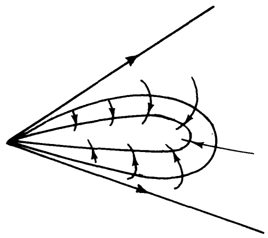


If we remove from D the inverse orbit of γ, $\{\bigcup_{n\geqq 0} f^{-n}\gamma\}$, the set of $\sim$ equivalence classes ($x\sim y$ iff $f^n x=f^m y$) defines a torus with branch points corresponding to the critical points of f. This follows easily from the local model of f near γ, where near a fixed point of a power of f we have $z\to\lambda z$, $0<|\lambda|<1$.

(ii) A *parabolic basin* D arises from a non-hyperbolic periodic cycle γ with derivative a root of unity, $\gamma=\{z, f(z),\ldots, f^{n-1}(z)\}$, $f^n(z)=z$, $((f^n)'(z))^m=1$, γ is contained in the frontier of D, and each compact in D converges to γ under forward iteration of f (Fatou [3]). The local picture of the dynamics consists of parabolic sectors arranged around the fixed point of a power of f which in local coordinates is $z\to z+z^l+\ldots$ and topologically equivalent to $z\to z+z^l$ (Fatou [3], Camacho [2]).



Local model



Fundamental domain

The local model produces a fundamental domain for the global dynamics on D because all orbits in D tend to γ. Looking at the local picture then shows the quotient of D by the $x\sim y$ equivalent classes is a union of twice punctured sphere with branched points coming from the critical points of f lying in D (there must be at least one critical point in D, Fatou [3]).

(iii) *A superattractive basin* D is defined just like an attractive basin but now the derivative of the power of f having a fixed point is zero. Now points arbitrarily near the attracting cycle are identified by f and there is no true fundamental domain for the $\sim$ equivalence classes. The more precise relation $x\approx y$ iff $f^n x=f^n y$ for some $n\geqq 0$ defines a foliation with singularities of $D'=D-$inverse orbit of γ by the closures of the $\approx$ equivalence classes. The leaves are 1-manifolds which are not necessarily compact and which have singularities at the inverse orbit of other critical points in D. The local analytic "linearization" near a superattractive fixed point or, more precisely, its analytical equivalence to $z\to z^m$ for some $m>0$ shows the leaves near γ are nearly concentric closed curves around the points of γ. The rest of the foliation of D′ is obtained by applying f^{-1} to this concentric foliation near γ.

(iv) A *Siegel disk* is a stable region which is cyclic and on which the appropriate power of f is analytically conjugate to a rotation of the standard unit disk. Siegel [1942] proved these occur near a non-hyperbolic periodic point if the argument u of its derivative satisfies the following diophantine condition: there exists $c>0$ and $\nu \geqq 2$ such that $|u-(p/q)| \geqq c/q^{\nu}$ for every relatively prime integers p and q.

Fatou and Julia showed that if such regions existed their frontiers were contained in the union of the ω-limit sets of critical points.

Siegel disks around the origin may occur already in the family $z \to \lambda z + z^2$, $|\lambda| = 1$. However, they do not occur when u is sufficiently Liouville because then there are periodic points tending to zero in this case (an easy calculation).

(v) A *Herman ring* is a stable region similar to a Siegel disk. Now we have a periodic cycle of annuli and a power of f which restricted to any of these annuli is analytically equivalent to an irrational rotation of the standard annulus. Again the frontier is contained in the ω-limit sets of critical points (Fatou [3]). Such regions were found by M. Herman for the map:

$$z \to \frac{e^{i\theta}}{z} \cdot \left(\frac{z-a}{1-\bar{a}z} \right)^2,$$

for appropriate θ and a. Herman uses Arnold's theorem about real analytic conjugations of real analytic diffeomorphisms of the circle to rigid rotations when the rotation number is like a Siegel number. Note that both Siegel disks and Herman rings are foliated by the closures of orbits and the leaves are closed real analytic curves.

5. MORE DYNAMICAL PROPERTIES. — (i) One knows there are only finitely many cyclic stable regions described in 4 (Sullivan [9]). But it is a *problem* to find the sharp upper bound for the number of cycles in terms of the degree. Is it $2d-2$?

(ii) Also for polynomials one knows *each bounded stable region is simply connected* (apply the maximum principle to $f, f^2, \ldots$). *Thus polynomials do not have Herman rings.*

(iii) An amusing corollary of the classification of stable regions in 4 is the following — *if all critical points of f are eventually periodic but none are periodic then the Julia set of f is all of $\overline{\mathbb{C}}$* (because each type of cyclic region besides the superattractive basin requires a critical point with an infinite forward orbit). Examples of this type are $z \to ((z-2)/z)^2$ and the quotient of some higher degree endomorphism of a one-dimensional torus by the equivalence relation $x \sim -x$. See for instance the example due to Lattes [4].

(iv) Fatou and Julia showed that f on $J(f)$ is topologically transitive. In fact, for any z in $J(f)$ the inverse orbit $\bigcup_{n \geqq 0} f^{-n}(z)$ is dense in $J(f)$. If no critical points tend to $J(f)$ or touch it, Fatou showed some power of f is expanding on $J(f)$. He surmised *the dynamical structure was continuous in the coefficients for such examples* (now called Axiom A or hyperbolic systems, see below) *and guessed that this property should be true except for special values of the parameters.*

Even when $J(f)$ is contaminated by critical points one may think of $J(f)$ as the *hyperbolic* part [2] of the dynamics. The Siegel disks and Herman rings are in the *elliptic* part of the dynamics. The attractive basins and the parabolic basins are the *properly discontinuous* part of the dynamics.

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[2] The words "hyperbolic" and "elliptic" are meant to suggest chaotic and rigid structure respectively in the dynamics.

6. Now we state our theorems about partial conjugacies between members of analytic families $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of endomorphisms. First we shall introduce the concept of *persistently non-hyperbolic periodic point*. We shall give first the definition in a particular case where it is easier to understand. If z_0 is a periodic point of f_{w_0}, say $(f_{w_0}^n(z_0)=z_0$, and if $(f_{w_0}^n)'(z_0) \neq 1$, then when w moves in a small neighborhood W_0 of w_0 we can find $z(w)$, depending analytically on w and such that $z(w_0)=z_0$ and $(f_w^n)(z(w))=z(w)$ for every w in the neighborhood. We say that z_0 is a persistently non-hyperbolic periodic point of f_{w_0} if $|(f_w^n)'(z(w))|=1$ for every w in a neighborhood of w_0 i.e. if we cannot destroy the non hyperbolicity of z_0 by moving the parameter. Observe that the condition $|(f_w^n)'(z(w))|=1$ for every w nearby w_0 implies that in fact $(f_w^n)'(z(w))$ is constant in that neighborhood. Unfortunately this definition is not sufficient because we shall need to handle the case $(f_{w_0}^n)'(z_0)=1$, when it is not always possible to find $z(w)$ as before. To state the general definition we first introduce the analytic sets:

$$M_n=\{(w, z) \in W \times \overline{\mathbb{C}} \mid f_w^n(z)=z, f_w^j(z) \neq z, 0 \leqq j<n\}.$$

Define the projection $P_n: M_n \to W$ as $P_n(w, z)=w$ and the eigenvalue function $\lambda_n: M_n \to \mathbb{C}$ by:

$$\lambda_n(w, z)=(f_w^n)'(z).$$

If P_n is injective when restricted to a neighborhood of a point $(w_0, z_0) \in M_n$, then there exists a neighborhood W_0 of w_0 and an analytic function $\varphi: W_0 \to \overline{\mathbb{C}}$, with $\varphi(w_0)=z_0$, such that its graph $\{(w, \varphi(w)) \mid w \in W_0\}$ is a neighborhood of (w_0, z_0) in M_n. We say that a periodic point z_0 of f_{w_0} is persistently non-hyperbolic if it is non-hyperbolic and:

(i) There exists a neighborhood W_0 of w_0 and an analytic function $\varphi: W_0 \to \overline{\mathbb{C}}$ such that $\varphi(w_0)=z_0$ and its graph $\{(w, \varphi(w)) \mid w \in W_0\}$ is a neighborhood of (w_0, z_0) in M_n;

(ii) λ_n is constant in a neighborhood of (w_0, z_0) in M_n.

By the analicity of f we can reformulate (i) in a weaker form: P_n is injective on a neighborhood of (z_0, w_0). In fact, since f is analytic and the function $f_{w_0}^n(z)-z$ is not identically zero, we can find neighborhoods W_0 of w_0 and U_0 of z_0 such that on $W_0 \times U_0$ we can factorize $f_w^n(z)-z$ (assuming $z_0=0$) as:

$$f_w^n(z)-z=(z^k+\sum_{j=0}^{k-1} a_j(w) z^j) g(w, z),$$

where the coefficients a_j are analytic functions of w and g is analytic and $\neq 0$ in $W_0 \times U_0$. If $P_n/((W_0 \times U_0) \cap M_n)$ is injective then for every $w \in W_0$ there exists a unique $\varphi(w) \in U_0$ such that $(w, \varphi(w)) \in M_n$. Then $\varphi(w)$ is the unique element of U_0 such that $f_w^n(\varphi(w))-\varphi(w)=0$. By Rouche's Theorem, if W_0 is small enough, the order of $\varphi(w)$ as zero of $z \to f_w^n(z)-z$ must be the same of $z_0=0$ as root of $z \to f_{w_0}^n(z)-z$, that is k. But $g \neq 0$ in a neighborhood of (w_0, z_0). Hence $\varphi(w)$ is a root of order k of $z^k+\sum_{j=0}^{k-1} a_j(w) z^j$. Hence this polynomial is $(z-\varphi(w))^k$. This means that $\varphi(w)=k a_{k-1}(w)$ and then φ is analytic.

Moreover observe that λ_n is analytic on the analytic set M_n [since it is the restriction of $(w, z) \to (f_w^n)'(z)$]. Then, if λ_n is constant in a neighborhood of a point, it is constant on the whole connected component of M_n containing that point. Since every connected component of M_n projects onto W, it follows that if for some w_0, f_{w_0} has a persistently non-hyperbolic periodic point, then f_w has a non-hyperbolic periodic point for every $w \in W$. Therefore, if for some $w_1 \in W$, all the periodic points of f_{w_1} are hyperbolic, then, for all $w \in W, f_w$ has no persistently non-hyperbolic periodic points.

Now define the set $H(f) \subset W$ as the set of values $w_0 \in W$ that have a neighborhood W_0 such that for every $w \in W$ every periodic point of f_w is either hyperbolic or persistently non-hyperbolic. Clearly this set is open. It is also dense:

THEOREM A. — *For any analytic family* $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $H(f)$ *is an open dense subset of* W.

Thus we have an open dense set of parameters W for which the nature of each periodic point of f_w stays constant. We can show that in this open dense set of parameters the dynamical structure of f_w in its Julia set $J(f)$ remains topologically unchanged.

DEFINITIONS. — Two endomorphisms f, g in End $\overline{\mathbb{C}}$ are J-equivalent if there exists a homeomorphism $h: J(f) \to J(g)$ such that $hf = gh$. Given an analytic family $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ we say that $w_0 \in W$ is J-*stable* if w_0 has a neighborhood W_0 such that f_w is J-equivalent to f_{w_0} for all w in W_0 and $J(f_w)$ depends continuously on $w \in W_0$ in the sense of the Hausdorff distance between two closed sets. Say that a map φ of a subset $X \subset \overline{\mathbb{C}}$ into the sphere is *quasi-conformal* if it is a topological embedding and:

$$\sup_{x \in X} \limsup_{t \to 0} \frac{\sup_{y \in S_t(x)} d(\varphi(y), \varphi(x))}{\inf_{Y \in S_t(x)} d(\varphi(y), \varphi(x))} \leqq \infty,$$

where $S_t(x) = \{ y \in X \mid d(y, x) = t \}$.

We ignore whether a quasi-conformal map $\varphi: X \to \overline{\mathbb{C}}$ can be extended to a quasi-conformal map of a neighborhood of X.

THEOREM B. — *For every analytic family of endomorphisms* $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $H(f) \subset W$ *coincides with the set of* J-*stable values of the parameter. Moreover, if* w_0 *belongs to* $H(f)$ *there exists a neighborhood* W_0 *in* W *of* w_0 *and a continuouu conjugacy function* $h: W_0 \times J(f_{w_0}) \to \overline{\mathbb{C}}$ *so that for all* w *in* W_0;

(i) h_w *is a conjugacy between* f_{w_0} *on* $J(f_{w_0})$ *and* f_w *on* $J(f_w)$ *and* h_{w_0} *is the identity;*

(ii) *for each* z, $h_w(z)$ *is analytic in* w;

(iii) *For each* w, h_w *is quasi-conformal;*

(iv) *the set of* J-*stable points coincide with the interior of the set of parameters where the Julia set moves continuously.*

7. Now we will enlarge our topological conjugacies beyond the Julia set using the structure described in 4. We obtain almost the natural expected result. Whenever Siegel disks or Herman rings are present there is, however, a glueing problem near their frontiers. On this problem, see remark after the statement of Theorem D.

Now we discuss the orbit structure of critical points: we say $a \approx b$ if either $a=b$ or a and b lie in the same leaf of the dynamically defined foliations of superattractive basins, Siegel disks, or Herman rings discussed in 4. We define a subset $C(f) \subset W$ for the analytic family $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by $w_0 \in C(f)$ iff there is a neighborhood $W_0 \subset W$ of w_0 and analytic functions $c_1, \ldots, c_k$ on W_0 so that:

(i) $c_1(w), \ldots, c_k(w)$ are all the critical points of f_w;

(ii) either $f_w^m(c_j(w)) \approx f_w^n(c_i(w))$ for some $m \geqq 0$, $n \geqq 0$ and all $w \in W_0$, or $f_w^m(c_j(w)) \not\approx f_w^n(c_i(w))$ for all $w \in W_0$, $m \geqq 0$, $n \geqq 0$.

Remark. — Note that for w in a component of $H(f)$ the critical points in $J(f)$ satisfy (ii) by Theorem B.

THEOREM C. — *For any analytic family $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $C(f)$ is an open dense subset of $H(f)$ (which is open and dense in W).*

Now we will state global or almost global stability results for endomorphisms of $C(f)$. Given an endomorphism f let $\Lambda(f)$ denote the collection of completely invariant compact sets Λ [i. e. $f(\Lambda)=\Lambda=f^{-1}(\Lambda)$] which intersect each (open) Siegel disk or (open) Herman ring in a compact set. For instance, the closure of all stable regions which are eventually attractive, superattractive or parabolic basins belongs to $\Lambda(f)$.

THEOREM D. — *For any analytic family $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$; if w_0 belongs to the open dense subset $C(f) \subset W$ then for any choice of $\Lambda \in \Lambda(f_{w_0})$ there is a neighborhood W_0 of w_0 and a continuous mapping $h: W_0 \times \Lambda \to \overline{\mathbb{C}}$ satisfying:*

(i) *h_w is a conjugacy between f_{w_0} on Λ and f_w on $h_w(\Lambda)$; and $h_w(\Lambda)$ belongs to $\Lambda(f_w)$;*

(ii) *for each z, $h_w(z)$ is analytic in w;*

(iii) *For each w, h_w is quasi-conformal.*

Conversely, if for any choice $\Lambda \in \Lambda(f_{w_0})$ there exists a neighborhood W_0 of w_0 and a continuous mapping $h: W_0 \times \Lambda \to \overline{\mathbb{C}}$ satisfying (i) *then $w_0 \in C(f)$.*

Observe that when f_{w_0} has no Siegel disks or Herman domains then $\overline{\mathbb{C}} \in \Lambda(f_{w_0})$, and this is the best choice to which Theorem D applies. It always happens for the family of polynomials of a given degree. As we explained in the introduction, in [13] Sullivan, combined the extension Lemma in [14] with the results above to obtain an improvement of Theorem D that states that it holds globally i. e. if $w \in C(f)$ then we can find $h: W_0 \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfying all the properties in Theorem D. Therefore, the set of values $w_0 \in W$ such that f_{w_0} is stable in the family (i. e. topologically equivalent to any f_w with w near to w_0) is open and dense.

8. Say that an endomorphism f is expanding on, the Julia set if for each z in $J(f)$ there is an n so that $|(f^n)'(z)|>1$. An easy argument shows the expanding property is equivalent to the Axiom A property: there exist $c>1$ and $N>0$ such that $|(f^k)'(z)|>c^k$, for all $k>N$ and $z \in J(f)$.

The classification of 4 shows immediately the *Axiom A expanding property implies all critical points are contained in attractive or superattractive basins and these are the only periodic stable regions.* The converse as remarked above follows from Fatou [3].

Note that the Axiom A expanding property implies membership in $H(f)$ for any family f. Thus we have an intrinsic property, Axiom A, which implies a property, membership in $H(f)$, determined by perturbations. For example we don't know if membership in $H(f)$ for the global family f implies the same is true for the iterates. However, it is obvious the Axiom A expanding property passes to iterates.

A second favorable feature of this property is that there is a powerful theory [Anosov, Sinai, Smale, Bowen, ...] for treating the dynamics of these hyperbolic systems. Markov partitions and symbolic dynamics can be used to describe f on $J(f)$ (Jacobson [5]).

Also in the conformal case the Axiom A expanding property for f implies $J(f)$ is a quasi-self similar fractal. It is not hard to show $J(f)$ has finite positive Hausdorff measure in its dimension which is strictly less than 2 (Sullivan [10], see also Bowen [1] and Ruelle [7]).

For all these reasons it is important to be able to verify the Axiom A expanding property. It would be important *to know whether or not the Axiom A expanding property is true for an open dense set of endomorphisms* (in reasonable families). The openess is known (see Jacobson [5]). But the density has defied verification.

Say that an *endomorphism f has a invariant line field on* $J(f)$ if there is a completely invariant subset $A \subset J(f)$ of positive Lebesgue measure, and a measurable family of tangent lines defined a. e. in A invariant by the tangent action of f.

Say that an analytic family of endomorphisms $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is *reduced* if different members cannot be analytically conjugate. Define $r(w)$ for $w \in W$ as the number of equivalence classes of not eventually periodic critical points in $J(f)^c$ under the equivalence relation $c_i \approxeq c_j$ iff $f_w^n(c_i) \approx f_w^m(c_j)$ for some $n \geqq 0$, $m \geqq 0$ (where $\approx$ is defined before the statement of Theorem C) plus the number of critical points in $J(f_w)$.

THEOREM E. — *Suppose $f: W \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a reduced analytic family and w is a point of* $C(f)$ *where $r(w) \leqq$ dimension* W. *Then f_w satisfies the Axiom* A *expanding property iff*:

(i) *all periodic points of f_w are hyperbolic*;

(ii) *f_w has no Herman rings*;

(iii) *f_w has no invariant line fields on the Julia set.*

COROLLARY. — *The Axiom* A *expanding property is true for an open dense set of polynomial mappings of degree d iff the Lebesgue measure of the Julia set is zero for an open dense set of polynomial mappings of degree d.*

II. — Proof of the λ-Lemma and Theorem A and B

The proof of the λ-Lemma is based in the following: any analytic map of the unit λ-disk into the triply punctured sphere $\overline{\mathbb{C}} - \{0, 1, \infty\}$ is distance non increasing for the complete Poincaré metrics on the unit disk and punctured sphere (Schwarz's Lemma). Choose three points from A and renormalize i so their images by i_λ are constantly 0, 1, and ∞.

For any three other distinct finite points x, y, z of A, consider the functions $x(\lambda) = i_\lambda(x)$, $y(\lambda) = i_\lambda(y)$, $z(\lambda) = i_\lambda(z)$, $\lambda \to (y(\lambda) - x(\lambda))/y(\lambda)$. Let $0 < R < 1$ and $0 < m < M$ be given. These functions avoid 0, 1 and ∞. Applying the above to the second function we see that $|y(0)|<M$ implies that $y(\lambda)$ is not too large for $|\lambda|\leqq R<1$ (i.e. bounded by a function of M). Applying the above to the fourth function we see that if $0<m\leqq|y(0)|\leqq M<\infty$ and $|x(0)-y(0)|$ is small then $|(x(0)-v(0))/y(0)|$ is small, which implies $|(x(\lambda)-v(\lambda))/y(\lambda)|$ is small if $|\lambda|\leqq R<1$, which implies $|x(\lambda)-y(\lambda)|$ is small by the first remark.

Thus each i_λ is uniformly continuous on $A\cap\{z\,|\,m\leqq|z|\leqq M\}$.

Such annuli cover the sphere (permuting the roles of $0, 1, \infty$) so i_λ has a continuous extension $i_\lambda : \overline{A}\to\overline{\mathbb{C}}$. Since 0 and any other particular λ_0 play symmetric roles in the hypothesis, the φ_λ have continuous inverses. For each z in $\overline{A}$, $i_\lambda(z)$ is analytic in λ because it is a uniform limit of analytic functions on each disk $|\lambda|\leqq R<1$.

To prove $i_\lambda(x)$ is quasiconformal apply the non-increasing property to the function $g(\lambda)=(x(\lambda)-y(\lambda))/(x(\lambda)-z(\lambda))$ when $|x(0)-y(0)|=|x(0)-z(0)|$ and conclude that $|g(\lambda)|$ is bounded for $|\lambda|\leqq R$. To prove the continuity of $i : D\times\overline{A}\to\overline{\mathbb{C}}$ is sufficient to show that the family of functions $\lambda\to i_\lambda(x)$, $x\in\overline{A}$, is equicontinuous. Again this is a consequence of the non-increasing property that grants $|i_{\lambda_1}(x)-i_{\lambda_2}(x)|=|x(\lambda_1)-x(\lambda_2)|$ is small uniformely in x if $|\lambda_1-\lambda_2|$ is small.

To prove Theorem A take $w_0\in W$ and any open neighborhood W_0 of w_0. We shall show that W_0 contains points of $H(f)$. If $w\in W_0$ denote $\alpha(w)$ the number of attractive and superattractive periodic orbits of f_w, and $\beta(w)$ the number of non-hyperbolic periodic orbits. By Fatou [3], $\alpha(w)+\beta(w)\leqq 4(d-1)$. Choose $w_1\in W_0$ such that $\alpha(w_1)=\max\{\alpha(w)\,|\,w\in W_0\}$. Then, since attractive and superattractive periodic orbits are persistent, there exists an open neighborhood $W_1\subset W_0$ of w_1 such that α/W_1 is constant. Choose $w_2\in W_1$ such that $\beta(w_2)=\max\{\beta(w)\,|\,w\in W_1\}$. Denote $p_1,\ldots, p_l$ the non-hyperbolic periodic orbits of f_{w_2}. We claim there exists a neighborhood $W_2\subset W_1$ of w_2 and neighborhoods U_i of p_i, $1\leqq i\leqq l$, such that if n_i is determined by $(w_2, p_i)\in M_{n_i}$, then $P_n/((W_2\times U_i)\cap M_{n_i})$ is injective and:

$$\bigcup_{i=1}^{l}(\{w\}\times\overline{\mathbb{C}})\cap((W_2\times U_i)\cap M_{n_i}),$$

is the set of non-hyperbolic periodic points of f_w, for all $w\in W_2$. Since the attractive and superattractive periodic points of f_w move analytically with w and their number is constant in W_1, it follows that we can find neighborhoods U_i of p_i and a neighborhood $W_2\subset W_1$ of w_2 such that for every $w\in W_2$ and $1\leqq i\leqq l$, U_i doesn't contain attractive or superattractive periodic points of f_w. Suppose W_2 and the U_i's $i=1, \ldots, l$ so small that $M_{n_i}\cap(W_2\times U_i)$ is connected for every $1\leqq i\leqq l$. The absence of attractive or superattractive periodic points in U_i for all $w\in W_2$ means that the analytic function λ_{n_i} on the analytic set $M_{n_i}\cap(W_2\times U_i)$ satisfies $|\lambda_{n_i}(w, z)|\geqq 1$. Hence $\lambda_{n_i}^{-1}$ is also an analytic function on this analytic set and is bounded by 1. But $|\lambda_{n_i}^{-1}(w_2, p_i)|=1$. Hence it attains its maximum at (w_2, p_i). Therefore it is constant in the connected analytic set $(W_2\times U_i)\cap M_{n_i}$. Then, for every $w\in W_2$, the points of $(\{w\}\times\overline{\mathbb{C}})\cap((W_2\times U_i)\cap M_{n_i})$ are non-hyperbolic periodic points of f_w. Since β attains a maximum at $w=w_2$ it is easy to see that the number of non-hyperbolic periodic points of f_w also attains a maximum at $w=w_2$. Then:

$$\sum_{i=1}^{l} \#(\{w\}\times\overline{\mathbb{C}})\cap((W_2\times U_i)\cap M_{n_i})\leqq\beta(w_2)=l,$$

for every $w\in W_2$. But $\#(\{w\}\times\overline{\mathbb{C}})\cap((W_2\times U_i)\cap M_{n_i})\geqq 1$. Hence it must be equal to one for all $w\in W_2$. This means that $P_n/((W_2\times U_i)\cap M_{n_i})$ is injective for all $1\leqq i\leqq l$. The previous arguments also show that points in the set:

$$S(w)=\bigcup_{i=1}^{l}(\{w\}\times\overline{\mathbb{C}})\cap((W_2\times U_i)\cap M_{n_i}),$$

are non-hyperbolic and that $\# S(w)=l$. But since as we observed above the number of non-hyperbolic periodic points of f_w attains a maximum l at $w=w_2$, it follows that number is bounded by l. Hence $S(w)$ is exactly the set of *all* the non-hyperbolic periodic points of f_w and this completes the Proof of the Claim. Now we claim that $w_2\in H(f)$. If it doesn't, there exists $w_3\in W_2$ such that f_{w_3} has a non-hyperbolic periodic point p that is not persistently non-hyperbolic. But by the Claim, p must have the form $p=(\{w_3\}\times\overline{\mathbb{C}})\cap((W_2\times U_i)\cap M_{n_i})$ for some $1\leqq i\leqq l$. Then P_{n_i} is injective in a neighborhood of (w_2, p) in M_{n_i} [in fact, the neighborhood $(W_2\cap U_i)\cap M_{n_i}$ satisfies this property]. Hence (w_2, p) satisfies condition (i) of the Definition of persistently non-hyperbolic periodic point (in the equivalent formulation that we gave after the Definition). But we proved that λ_{n_i} is constant in $(W_2\times U_i)\cap M_{n_i}$. Then it also satisfies part (ii) of the Definition and p is persistently non-hyperbolic.

To prove Theorem B suppose w_0 belongs to $H(f)\subset W$ for the analytic family f: $W\times\overline{\mathbb{C}}\to\overline{\mathbb{C}}$. Let W_0 containing w_0 be a simply connected neighborhood of w_0 in $H(f)$. We claim each expanding periodic point x_n of f_{w_0} defines an analytic function x_n : $W_0\to\overline{\mathbb{C}}$ such that $x_n(w)$ is a periodic point of f_w of the same period of x_n. The implict function theorem tells us we can analytically continue a transversal fixed point of f_w^n uniquely for some neighborhood of parameters. Thus by following the periodic point determined by $x_n(w_0)$ we locally define $x_n(w)$ on an open set. At a frontier point w_1 of such an open set, the limit of $x_n(w)$ is still a periodic point which is hyperbolic if it has period n because $W_0\subset H(f)$.

Actually, the period can not drop neither when $x=\lim_{w\to w_1} x_n(w)$ has period $k<n$ $(f_{w_1}^k)'(x)$ not a root of unit (if $m\neq k$ is given there exists a neighborhood U of x so that no point in $U-\{x\}$ has period m) nor when $(f_w^k)'(x)$ is a root of unit [use the local model for the dynamics and the fact that $W_0\subset H(f)$].

Thus $x_n(w_1)$ is a hyperbolic periodic point of order n which may be analytically continued on a neighborhood of w_1. This definition agrees with the previous one by the uniqueness. So we can define $x_n(w)$ on all of W_0 which is simply connected.

Since hyperbolic periodic points cannot collide, as we have already remarked, we may apply the λ-Lemma to the set A of expanding periodic points. We obtain $h(w, z)$: $W_0\times J(f_{w_0})\to\overline{\mathbb{C}}$, analytic in w, quasi-conformal in z.

Since the role of w_0 and any particular w_1 in W_1 may be reversed it is clear the $x_n(w_1)$ must be all of the expanding periodic points of f_{w_1}. Thus for each w, $h(w, z)$ defines a homeomorphism between Julia sets. By definition $h(w, z)$ is a conjugacy between sets of expanding periodic points. By continuity $h(w, z)$ is a conjugacy between Julia sets.

This proves the first part of Theorem B.

To prove the converse property, let w_0 be a J-stable value of W. Since the number of periodic cycles of f_w in $J(f_w)$ stays constant for w in a small neighborhood W_0 of w_0, we see that all periodic cycles of f_{w_0} with derivative equal to one have an analytic continuation through W_0. Now observe that the function $W_0 \ni w \to J(f_w)$ is continuous with respect to the Hausdorff metric. We shall show that $w \in H(f)$. Obviously this shows that J-stable values of the parameter belong to $H(f)$. If $w_0 \notin H(f)$, there exists $w_1 \in W_0$ such that for some $n \geqq 0$, $f^n_{w_1}$ has a non hyperbolic fixed point z_1 that is not persistently non hyperbolic. This implies that we can find w_2 near to w_1 such that $f^n_{w_2}$ has a fixed point z_2 (near to z_1) such that $(f^n_{w_2})'(z_2)$ is a Siegel number and z_2 is not a persistently non hyperbolic periodic point. Then $z_2 \notin J(f_{w_2})$.

But since z_2 is not persistently non hyperbolic, we can find w_3 arbitrarily near to w_2 and fixed points z_3 of $f^n_{w_3}$, arbitrarily near to z_2, such that $|(f^n_{w_3})'(z_3)| \geqq 1$. Hence $z_3 \in J(f_{w_3})$. We have thus proved that there exist arbitrarily small perturbations of $w_2 \in W_0$ that make the Julia set reach points (like z_3) that are bounded away from $J(f_{w_2})$ [because z_3 is arbitrarily near to z_2, and $z_2 \notin J(f_{w_2})$]. This concludes the proof of Theorem B.

III. — Proof of Theorem C

First we shall show that $C(f)$ is contained in $H(f)$. We shall use the following standard Lemma:

LEMMA III.1. — *If $W_0 \subset W$ is an open simply connected set and $\varphi : W_0 \to \overline{\mathbb{C}}$ is an analytic function such that for every $w \in W_0$ the point $\varphi(w)$ doesn't belong to any forward f_w-orbit of a critical point of f_w, then there exist analytic functions $\varphi_{n,i} = W_0 \to \overline{\mathbb{C}}$, $n \geqq 0$, $i \leqq i \leqq d^n$, satisfying $f^n_w(\varphi_{n,i}(w)) = \varphi(w)$ and $\varphi_{n,i}(w) \neq \varphi_{n,j}(w)$ for all $n \geqq 0$, $1 \leqq i \leqq d^n$, $w \in W_0$.*

The λ-Lemma now yields the following property:

LEMMA III.2. — *Suppose that W_0 and φ satisfy the hypothesis of Lemma III.1. Moreover suppose that either $\varphi(w)$ is not f_w-periodic for any $w \in W_0$ or that for some $N \geqq 0$ we have $f^N_w(\varphi(w)) = \varphi(w)$ for all $w \in W_0$. Then $W_0 \subset H(f)$.*

Proof. — We shall prove the Lemma only in the case when $\varphi(w)$ is not f_w-periodic for any $w \in W_0$. The other case reduces to this just by replacing φ by $\varphi_{1,i}$, where i is chosen taking any $w_0 \in W_0$ and i such that $f_{w_0}(\varphi_{1,i}(w_0)) \neq f^{N-1}_{w_0}(\varphi(w_0))$. Then the same relation holds for all $w \in W_0$ because by III.1 preimages of $\varphi(w_0)$ don't collide. Therefore $\varphi_{1,i}(w)$ is not f_w-periodic for any $w \in W_0$. Now fix some $w_0 \in W_0$ and set $\Lambda = \bigcup_{n \geqq 0} f^{-n}_{w_0}(\varphi(w_0))$. Define $h : W_0 \times \Lambda \to \overline{\mathbb{C}}$ by:

$$h_w(z) = \varphi_{n,i}(w),$$

if $z=\varphi_{n,i}(w_0)$. Observe that since $\varphi(w_0)$ is not f_{w_0}-periodic, and $\varphi_{n,j}(w_0)\neq\varphi_{n,i}(w_0)$ for all $n>0$, $1\leqq i<j\leqq d^n$, the n and i satisfying $z=\varphi_{n,i}(w_0)$ are unique. This shows that $h_w(z)$ is well defined and depends analytically in w. Moreover, every h_w is injective. In fact if $z_1\neq z_2$ belong to Λ, then:

$z_1=\varphi_{n_1,i_1}(w_0)$, $\quad z_2=\varphi_{n_2,i_2}(w_0)$, where either $n_1\neq n_2$ or $n_1=n_2$ and $i_1\neq i_2$. If $n_1\neq n_2$, the equality $h_w(z_1)=h_w(z_2)$ implies:

$$\varphi_{n_1,i_1}(w)=\varphi_{n_2,i_2}(w)$$

and then:

$$f_w^{n_2}(\varphi_{n_2,i_2}(w))=\varphi(w)=f_w^{n_1}(\varphi_{n_1,i_1}(w))=f_w^{n_1}(\varphi_{n_2,i_2}(w)).$$

Suppose $n_2\geqq n_1$. Then:

$$f_w^{n_2-n_1}(\varphi(w))=f_w^{n_2-n_1}(f_w^{n_1}(\varphi_{n_2,i_2}(w)))=f_w^{n_2}(\varphi_{n_2,i_2}(w))=\varphi(w),$$

thus contradicting that $\varphi(w)$ is not f_w-periodic. If $n_1=n_2$ then:

$$h_w(z_1)=\varphi_{n_1,i_1}(w)\neq\varphi_{n_1,i_2}(w)=h_w(z_2).$$

Now we can apply the λ-Lemma to extend every map $h_w:\Lambda\rightarrow h_w(\Lambda)$ to a homeomorphism $h_w:\overline{\Lambda}\rightarrow\overline{h_w(\Lambda)}$ that obviously satisfies:

$$(\star)\qquad f_w(h_w(z))=h_w(f_w(f_{w_0}(z)),$$

for every $z\in f_{w_0}^{-1}(\Lambda)$. But $\overline{\Lambda}$ contains $J(f_{w_0})$ because it contains the full backward orbit of a point. Therefore, if we show that $h_w(J(f_{w_0}))=J(f_w)$ we shall be done. If $z\in J(f_{w_0})$, it is the limit of a sequence z_n of f_{w_0}-periodic points different from z. Then $h_w(z_n)$ is a sequence of f_w-periodic points different from $h_w(z)$ and converging to $h_w(z)$. Hence $h_w(z)\in J(f_w)$ thus proving $h_w(J(f_{w_0}))\subset J(f_w)$. To prove $J(f_w)\subset h_w(J(f_{w_0}))$ we just observe that interchanging the roles of w and w_0 we obtain h_w^{-1} instead of h_w. Then $h_w^{-1}(J(f_w))\subset J(f_{w_0})$ and $J(f_w)=h_w(h_w^{-1}(J(f_w)))\subset h_w(J(f_{w_0}))$ thus concluding the Proof of Lemma III.2.

To prove $C(f)\subset H(f)$ we start observing than any endomorphism $g:\overline{\mathbb{C}}\hookleftarrow$ not satisfying Axiom A, has a critical point z_0 such that $\bigcup_{n\geqq 0} g^{-n}(z_0)$ doesn't contain critical points. In fact, if such a critical point doesn't exist, it is easy to see that every critical point must be periodic. Therefore g satisfies Axiom A. Now suppose that $w_0\in C(f)$. Then either f_{w_0} satisfies Axiom A (and then $w_0\in H(f)$) or it doesn't, in which case we can take z_0 with the above property. The definition of $C(f)$ grants the existence of a neighborhood W_0 of w_0 and an analytic function $\psi:W_0\rightarrow\overline{\mathbb{C}}$ such that $\psi(w_0)=z_0$ and $(f_w)'(\psi(w))=0$ for all $w\in W_0$. Also from the definition of $C(f)$ we know that $\psi(w)$ doesn't belong to the forward f_w-orbit of any critical point because $z_0=\psi(w_0)$ is not in the f_{w_0}-forward orbit of any critical points. Applying Lemma III.2 to $\psi:W_0\rightarrow\overline{\mathbb{C}}$ it follows that $W_0\subset H(f)$.

The openness of $C(f)$ is trivial. To prove its density we shall produce a dense set $S \subset H(f)$ whose elements can be aproximated by elements in $C(f)$. First define the critical set:

$$C_s = \{(w, z) \mid (f_w)'(z) = 0\}.$$

If $\pi : W \times \overline{\mathbb{C}} \to W$ is the projection, let S_0 be the set of regular values of π/C_s. Then S_0 is open and dense in W. Moreover given any $w_0 \in S_0$ there exist a neighborhood $W_0 \subset S_0$ of w_0 and analytic functions $c_i : W_0 \to \overline{\mathbb{C}}$, $1 \leqq i \leqq k$, such that $\{c_1(w), \ldots, c_k(w)\}$ is the set of critical points for all $w \in W_0$. Define $S(i, j, n, m)$ as the set of values $w_0 \in S_0 \cap H(f)$ such that either $f^n_{w_0}(c_i(w_0)) \approx f^m_{w_0}(c_j(w_0))$ or $f^n_w(c_i(w)) \not\approx f^m_w(c_j(w))$ holds for every w in a neighborhood of w_0. Now we set:

$$S(i, j) = \bigcap_{n, m} S(i, j, n, m)$$

We claim that each $S(i, j, n, m)$ is open and dense and that if $w_0 \in S(i, j)$, it has a neighborhood W_0 such that for all $n \geqq 0$, $m > 0$ either $f^n_w(c_i(w)) \approx f^m_w(c_j(w))$ or $f^n_w(c_i(w)) \not\approx f^m_w(c_j(w))$ holds for every $w \in W_0$. It follows from the claim that the residual (thus dense) set $S = \bigcap_{i, j} S(i, j, n, m)$ is contained in $C(f)$. The Proof of the Claim requires a series of Lemmas that describe how cyclic domains vary with the parameter.

LEMMA III.3. — *Let z_0 be an attractive periodic point of f_{w_0} with period m. Then there exist neighborhoods V_0 and W_0 of z_0 and w_0 and analytic functions $h : W_0 \times V_0 \to \mathbb{C}$, $\varphi : W_0 \to V_0$ such that:*

(*a*) $f^m_w(V_0) \subset V_0$ *for all* $w \in W_0$;

(*b*) $f^m_w(\varphi(w)) = \varphi(w)$ *for all* $w \in W_0$;

(*c*) $h_w(\varphi(w)) = 0$, $w \in W_0$;

(*d*) $(f^m_w)'(\varphi(w))\, h_w(z) = h_w(f^m_w(z))$ *for all* $w \in W_0$, $z \in V_0$.

LEMMA III.4. — *Suppose that for some i and $m > 0$, $f^m_w(c_i(w)) = c_i(w)$ for every w in a neighborhood of some $w_0 \in S_0$. Then there exist neighborhoods V_0 and W_0 of z_0 and w_0 respectively and an analytic function $h : W_0 \times V_0 \to \mathbb{C}$ such that:*

(*a*) $f^m_w(V_0) \subset V_0$ *for all* $w \in W_0$;

(*b*) $h(c_i(w)) = 0$, $w \in W_0$;

(*c*) $h_w(z)^n = h_w(f^m_w(z))$ *for all* $w \in W_0$, $z \in V_0$,

where n is the multiplicity of $c_i(w_0)$ as a critical point of $f^m_{w_0}$.

LEMMA III.5. — *Suppose that $w_0 \in H(f)$ and z_0 is a parabolic periodic point of f_{w_0} of period m. Then there exist neighborhoods V_0 and W_0 of z_0 and w_0 respectively and a continuous map $h : W_0 \times V_0 \to \mathbb{C}$ such that:*

(*a*) $h_w(.) = h(w, .)$ *is continuous and injective for all* $w \in W_0$;

(*b*) *for all* $z \in V_0$, *the map* $w \to h_w(z)$ *is analytic;*

(*c*) $f_w h_w(z) = h_w f_{w_0}(z)$ *for every* $w \in W_0$ *and* $z \in V_0 \cap f^{-1}_{w_0}(V_0)$.

LEMMA III.6. — *Suppose that* $w_0 \in \mathrm{H}(f)$ *and* A *is a Herman ring or Siegel disk satisfying* $f^m(\mathrm{A}) = \mathrm{A}$. *Let* $\mathrm{U} \subset \mathrm{A}$ *be an open* f^m*-invariant (i. e.* $f^m(\mathrm{U}) = \mathrm{U}$*) disk or annulus whose closure is contained in* A *and whose boundary consists in one or two* f^m*-invariant analytic curves. Then there exist a neighborhood* W_0 *of* w_0*, an open disk or ring* $\mathrm{T} \subset \mathbb{C}$ *and an analytic function* $\hbar : \mathrm{W}_0 \times \mathrm{T} \to \mathrm{C}$ *such that every* $\hbar_w$ *is a conformal representation satisfying:*

$$\hbar_w(e^{i\theta} z) = f_w(\hbar_w(z)),$$

for all $z \in \mathrm{U}$*, where* θ *is the rotation number of* $f^m_{w_0}/\mathrm{A}$*, and* $\hbar_0(\mathrm{T}) = \mathrm{U}$.

Lemma III.3 is an analytically parametrized version of Poincaré's linearization Theorem, and its Proof follows immediately from the usual technique used to prove this Theorem. Lemma III.4 is in a similar situation with respect to the analytic linearization Theorem of super attractive periodic points. The Proofs of Lemmas III.5 and III.6 will be presented after completing the Proof of the Claim. First observe the Lemmas above imply easily that each $\mathrm{S}(i, j, n, m)$ is open and dense. Suppose that $w_0 \in \mathrm{S}(i, j)$. The Proof of the Claim will be divided according to the following cases:

(I) There exist $n \geqq 0$, $m > 0$, such that $f^n_{w_0}(c_i(w_0)) \approx f^m_w(c_j(w_0))$. Suppose that n_0 is the minimum positive integer such that the equality above is satisfied for some $m > 0$. Then take as m_0 the minimum integer such that the equality $f^{n_0}_{w_0}(c_i(w_0) \approx f^{m_0}_{w_0}(c_j(w_0))$ holds. Since $w_0 \in \mathrm{S}(i, j, n_0, m_0)$ there exists a neighborhood W_0 such that $f^{n_0}_w(c_i(w)) \approx f^{m_0}_w(c_j(w))$ for all $w \in \mathrm{W}_0$. Restricting W_0 if necessary, we can grant that $f^n_w(c_i(w)) \not\approx f^m_w(c_j(w))$ for all $w \in \mathrm{W}_0$ and $0 \leqq n \leqq n_0$, $0 \leqq m < m_0$. From these facts it follows that W_0 satisfies the property required by the Claim.

(II) $f^n_{w_0}(c_i(w)) \not\approx f^m_w(c_j(w))$ for every $n \geqq 0$, $m > 0$. This case is subdivided in two situations:

(II *a*) $c_i(w_0)$ or $c_j(w_0)$ (perhaps both) belong to $\mathrm{J}(f_{w_0})$. Then it suffices to takes as W_0 the connected component of $\mathrm{H}(f)$ that contains w_0. Then the topological equivalence between the Julia sets of f_{w_0} and f_w, for any $w \in \mathrm{W}_0$, given by Theorem B, plus the fact that the number (counted with multiplicity) of critical points in $\mathrm{J}(f_w)$ is constant on connected components of $\mathrm{H}(f)$ (again a Corollary of Theorem B), implies the Claim.

(II *b*) $c_i(w_0) \notin \mathrm{J}(f_{w_0})$, $c_j(w_0) \notin \mathrm{J}(f_{w_0})$ and they eventually belong to orbits of different cyclic domains of $\mathrm{J}(f_{w_0})^c$. Then Lemmas III.1, 2, 3 or 4 imply that $f^n_w(c_i(w)) \not\approx f^m_w(c_j(w))$ for every $n \geqq 0$, $m \geqq 0$ and all w in a neighborhood of w_0.

(II *c*) $c_i(w_0) \notin \mathrm{J}(f_{w_0})$, $c_j(w_0) \notin \mathrm{J}(f_{w_0})$ and they eventually belong to the orbit of the same cyclic domain. Taking as W_0 the neighborhood given by Lemma I.1, 2, 3 or 4 (according to which type belongs the cyclic domain that eventually contain the orbits of $c_i(w_0)$ and $c_j(W_0)$) the parametrized linearization shows that $f^n_w(c_i(w)) \not\approx f^m_w(c_j(w))$ for every $w \in \mathrm{W}_0$.

Let us prove Lemma III.5. Suppose that $f_w(z_0) = z_0$ to simplify the discussion. Set $\lambda = (f_{w_0})'(z_0)$. If $\lambda^q = 1$ and $\lambda^j \neq 1$ for $0 < j < q$, we can write, after an analytic change of coordinates in a neighborhood of z_0 (*see* [2] for details):

$$f_w(z) = \lambda z + \sum_{j=1}^{\infty} a_j(w) z^{q+j},$$

where the series converge in a fixed disk $B_r(z_0)$ for every $w \in W_0$ (taking W_0 sufficiently small) and the functions a_j are analytic. Observe that we are here using that z_0 is a persistently where the series converge in a fixed disk $B_r(z_0)$ for every $w \in W_0$ (taking W_0 sufficiently small) and the functions a_j are analytitic. Observe that we are here using that z_0 is a persistently non hyperbolic fixed point of f_{w_0} because $w_0 \in H(f)$. This grants that the fixed point z_0 has an analytic continuation as fixed point $z(w)$ of f_w and that $f'_w(z'(w)) = \lambda$ for every w near w_0. We claim that if $a_j(w_0) = 0$ for $1 \leqq j < m$ and $a_m(w_0) \neq 0$ then $a_j(w) = 0$ for all $1 \leqq j < m$ and every w near to w_0. To prove this we shall again use the hypothesis $w_0 \in H(f)$. Write:

$$f^q_w(z) = z + \sum_{j=1}^{\infty} \tilde{a}_j(w)\, z^{q+j}$$

and observe that $\tilde{a}_j(w_0) = 0$ for $1 \leqq j < m$ and $\tilde{a}_m(w_0) \neq 0$. Then:

$$f^q_{w_0}(z) - z = \sum_{j=1}^{\infty} \tilde{a}_j(w_0)\, z^{q+j} = z^{q+m} \left(\sum_{j=m}^{\infty} \tilde{a}_j(w_0)\, z^{j-m} \right).$$

If V_0 is a neighborhood of 0 where the second factor in the last term is $\neq 0$, then $z = 0$ is the unique fixed point of f_{w_0} in V_0. The J-stability of f_{w_0} implies that f_w has a unique fixed point in V_0 for w near to w_0. This implies that $a_j(w) = 0$ for every $1 \leqq j < m$ and every w in a neighborhood of w_0. Hence $a_j(w) = 0$ for every w in the same neighborhood and $1 \leqq j < m$ completing the Proof of the Claim. Now with a linear change of coordinates we can write:

$$f_w(z) = \lambda z + \sum_{j=m+1}^{\infty} a_j(w)\, z^{q+j}.$$

By [2], f_w is equivalent to the map $z \to \lambda z + z^{q+m}$ in a neighborhood of 0 and the homeomorphism h_w that conjugates both maps can be chosen depending analytically in w.

Now let us prove Lemma III.6. To simplify the notation suppose that $m = 1$. Let $U_1 \subset U$ be an invariant annulus whose boundary has two analytic closed curves. We shall need the following Lemma, to be proved later:

LEMMA III.7. — *There exists a neighborhood* W_0 *of* w_0 *such that for every* $w \in W_0$, $\overline{U}_1$ *doesn't contain eventually periodic points of* f_w *or points of* $J(f_w)$.

Now take some point $z_1 \in U_1$ and if $C = \{ e^{in\theta} \mid n \geqq 0 \}$ define $h_w : C \to \overline{\mathbb{C}}$, $w \in W_0$, by $h_w(e^{in\theta}) = f^n_w(z_1)$. This family of maps depends analytically on the parameter $w \in W_0$. To apply the λ-Lemma to this family we have only to check that every h_w is injective. If h_w is not injective, there exist n and m such that $f^n_w(z_1) = f^m_w(z_1)$. Then z_1 is an eventually periodic point of f_w, contradicting Lemma III.5. Now, applying the λ-Lemma we obtain a conjugacy h_w between the rotation $z \to e^{i\theta} z$ in the circle $\overline{C}$ and the restriction of f_w to the Jordan f_w-invariant curve $h_w(\overline{\mathbb{C}})$. By Lemma III.5 this Jordan f_w-invariant curve must belong to some fixed connected component of $J(f_w)^c$. Clearly it must be either a Herman ring or a Siegel's disk. But $h_0(\overline{C})$ is a Jordan f_{w_0}-invariant curve and has points of $J(f_{w_0})$ in its interior if and only if A is a Herman ring. Moreover $h_w(\overline{C})$ contains points of $J(f_w)$ in its interior if and only if $h_0(\overline{C})$ does. Therefore $h_w(\overline{C})$ belongs to a Siegel disk or Herman ring if and only if $h_0(\overline{C})$ belongs to the same type of component. Moreover it is an analytic curve and since it belongs to a Herman ring or Siegel disk, the restriction of f_w to $h_w(\overline{C})$ is real analytically equivalent to a rotation of the circle. Say that the circle is $\overline{C}$ and let $F : h_w(\overline{C}) \to \overline{C}$ be this real analytic conjugacy. Then $F^{-1} h_w$ is a conjugacy between two rotations. Therefore the rotations must coincide. In particular h_w is real analytic. Now let $B_w = \{ z \mid r_1(w) < |z| < r_2(w) \}$ be the maximal ring where an analytic extension $H_w : B_w \to \overline{\mathbb{C}}$ of $h_w : \overline{C} \to \overline{\mathbb{C}}$ exists. From the identity principle it is easy to conclude that $f_w h_w(z) = h_w(e^{i\theta} z)$ for every $z \in B_w$. We claim that for values of w near to w_0 the relation $h_w(B_w) \supset U$ holds. This will clearly prove Lemma III.6. In fact we can show that $h_w(B_w)$ is the Herman ring $A(w)$ containing $h_w(\overline{C})$. This follows from the fact that if $\overline{h_w(B_w)} \subset A(w)$ then one of the boundaries of $h_w(B_w)$ must be an f_w-invariant real analytic curve or reduce to a point. In the first case, the extension property of conformal representations would show that B_w is not maximal. We leave to the reader to verify that the second only arises when $r_1(w) = 0$ and then h_w extends to an analytic map of $B_w \cup \{0\}$.

To prove Lemma III.7, take W_0 so small that $J(f_w) \cap \overline{U} = \emptyset$ for all $w \in W_0$. This is granted by Theorem B. It remains to show that there are no eventually periodic points of f_w in U_1, for w in W_0.

Take a neighborhood V of the periodic orbits of f_w not contained in $J(f_{w_0})$ or in A if A is a Siegel disk. Take V so small that $V \cap A = \emptyset$. Since $w_0 \in H(f)$, all the f_w-periodic orbits not in $J(f_w)$ are contained in V for all $w \in W_0$, if W_0 is chosen small enough. Now suppose that for some $w \in W_0$ and $z \in U_1$ there exists $N > 0$ such that $f_w^N(z)$ is f_w-periodic. Since $J(f_w) \cap U = \emptyset$, $f_w^N(z) \notin J(f_w)$. Hence $f_w^N(z) \in V$.

Moreover if $h_w : J(f_{w_0}) \to J(f_w)$ is the homeomorphism given by Theorem B, the fact that $\overline{V}$ and $\overline{U}_1$ belong to different connected components of the complement of $J(f_{w_0})$, implies that they are also in different connected components of $h_w(J(f_{w_0})) = J(f_w)$ for all $w \in W_0$, if W_0 is small enough.

But we can take W_0 such that $f_w(U_1) \cap U_1 \neq \emptyset$ all $w \in W_0$. Then $f_w^{n+1}(U_1) \cap f_w^n(U_1) \neq \emptyset$ for all n, thus implying that $\bigcup_{n \geq 0} f_w^n(U_1)$ is connected. On the other hand the property $f_w^n(z) \in V$ implies that $\bigcup_{n \geq 0} f_w^n(U_1)) \cap V \neq \emptyset$ then U and V belong to the same connected component of the complement of $J(f_w)$.

Lastly, suppose A is Siegel disk with fixed point z_0; let $z_0(w)$ be its local analytic continuation. If U_1 still belongs to the Siegel disk corresponding to $z_0(w)$, clearly $\bigcup_{n \geq 0} f_w^n(U) \not\ni z_0(w)$ for w close to w_0.

IV. – Proof of Theorem D

Suppose that $w_0 \in C(f)$ and that Λ is a compact totally invariant set of f_{w_0} i. e. $f_{w_0}^{-1}(\Lambda)=\Lambda$ such that its intersection with any Siegel disk or Herman ring is compact. We shall construct a set $\Lambda_0 \subset \Lambda$, a simply connected open neighborhood W_0 of w_0 and a map $h: W_0 \times \Lambda_0 \to \overline{\mathbb{C}}$ with the following properties:

(*a*) $f_{w_0}(\Lambda_0)=\Lambda_0$;

(*b*) $\Lambda=\overline{\bigcup_{n \geq 0} f_{w_0}^{-n}(\Lambda_0)}$;

(*c*) The map $w \to h_w(z)$ is analytic for every $z \in \Lambda_0$, $h_{w_0}(z)=z$ and h_w is injective for all $w \in W_0$;

(*d*) $h_w(z)$ belongs to the orbit of a critical point of f_w if and only if z belongs to the orbit of a critical point of f_{w_0}.

(*e*) $f_w(h_w(z))=h_w(f_{w_0}(z))$ for every $w \in W_0$, $z \in \Lambda_0$.

Let us show how to prove the partial stability follows from the existence of h, W_0 and Λ_0. Let $\hat{\Lambda}_0$ be Λ_0 minus the intersection of forward f_{w_0}-orbits of critical points with Λ_0. Set $\hat{\Lambda}=\bigcup_{n \geq 0} f_{w_0}^{-n}(\hat{\Lambda}_0)$. Then $\hat{\Lambda}$ is dense in Λ. Define $h_w: \hat{\Lambda} \to \overline{\mathbb{C}}$ as follows: if $z \in \hat{\Lambda}$ choose n such that $f_{w_0}^n(z) \in \hat{\Lambda}_0$ and let $\varphi: W_0 \to \overline{\mathbb{C}}$ be an analytic function such that $\varphi(w_0)=z$ and:

$$f_w^n(\varphi(w))=h_w(f_{w_0}^n(z)),$$

for all $w \in W_0$. Then define $h_w(z)=\varphi(w)$. The existence of φ in a neighborhood od w_0 is granted by the implicit function theorem and the fact that $(f_{w_0}^n)'(z) \neq 0$ (because orbits of critical points don't intersect $\hat{\Lambda}_0$). Now let $W_1 \subset W_0$ be the maximal simply connected open set to which φ can be extended. Suppose that $W_1 \neq W_0$. Then there exists a sequence $w_n \in W_1$ converging to some $w \in W_0 \setminus W_1$. Suppose that $\varphi(w_n)$ converges to z_∞. Then $f_{w_\infty}^n(z_\infty)=h_{w_\infty}(f_{w_0}^n(z))$. We claim that $(f_{w_\infty}^n)'(z) \neq 0$. If not, $h_{w_\infty}(f_{w_0}^n(z))$ belongs to the orbit of a critical point of f_{w_∞}. By (*e*) this can happen only if $f_{w_0}^n(z)$ has same property with respect to f_{w_0}. But this is impossible because $f_{w_0}^n(z) \in \hat{\Lambda}_0$. Then $(f_{w_\infty}^n)'(z_\infty) \neq 0$ and the implicit function theorem gives an analytic function $\hat{\varphi}: W_\infty \to \overline{\mathbb{C}}$, where W_∞ is a neighborhood of w_∞, satisfying $\hat{\varphi}(w_\infty)=z_\infty$ and $(f_w^n)\hat{\varphi}(w))=h_w(f_{w_0}^n(z))$ for all $w \in W$. Moreover, the uniqueness property of the implicit function theorem grants $\hat{\varphi}(w)=\varphi(w)$ for all $w \in W_\infty \cap W_1$. Then this contradicts the maximality of W_1 thus proving that $W_1=W_0$.

Now that we have well defined maps $h_w: \hat{\Lambda} \to \overline{\mathbb{C}}$, that depend analytically in w, let us show that every h_w is injective. Suppose that $z_1 \neq z_2$ and $h_w(z_1)=h_w(z_2)$. Take $n \geq 0$ such that $f^n(z_i) \in \hat{\Lambda}_0$, $i=1, 2$. Then:

$$h_w(f_{w_0}^n(z_1))=f_w^n(h_w(z_1))=f_w^n(h_w(z_2))=h_w(f_{w_0}^n(z_2)).$$

But h_w is injective on $\hat{\Lambda}_0$. Then $f_{w_0}^n(z_1)=f_{w_0}^n(z_2)$. Take the minimum $n_0>0$ such that $f_{w_0}^{n_0}(z_1)=f_{w_0}^{n_0}(z_2)$. On the other hand:

$$h_w(f_{w_0}^{n_0-1}(z_1))\neq h_w(f_{w_0}^{n_0-1}(z_2))$$

for values of w near to w_0. Then we can assume that there exists a sequence $w_n\to w$ such that:

$$h_{w_n}(f_{w_0}^{n_0-1}(z_1))\neq h_{w_n}(f_{w_0}^{n_0-1}(z_2)).$$

But:

$$(\star)\qquad f_{w_n}(h_{w_n}(f_{w_0}^{n_0-1}(z_1)))=h_{w_n}(f_{w_0}^{n_0}(z_1))=h_{w_n}(f_{w_0}^{n_0}(z_2))=f_{w_n}(h_{w_n}(f_{w_0}^{n_0-1}(z_2)))$$

Hence $h_{w_n}(f_{w_0}^{n_0-1}(z_1))$ and $h_{w_n}(f_{w_0}^{n_0-1}(z_2))$ are different, converge to the same point:

$$h_w(f_{w_0}^{n_0-1}(z_1))=f_w^{n_0-1}(h_w(z_1))=f_w^{n_0-1}(h_w(z_2))=h_w(f_{w_0}^{n_0-1}((z_2))),$$

and its images under f_{w_n} are by $(\star)$ the same. This means that:

$h_w(f_{w_0}^{n_0-1}(z_1))=h_w(f_{w_0}^{n_0-1}(z_2))$ is a critical point of f_w. But:

$$f^{n-(n_0-1)}(h_w(f_{w_0}^{n_0-1}(z_1)))=h_w(f_{w_0}^n(z_1)).$$

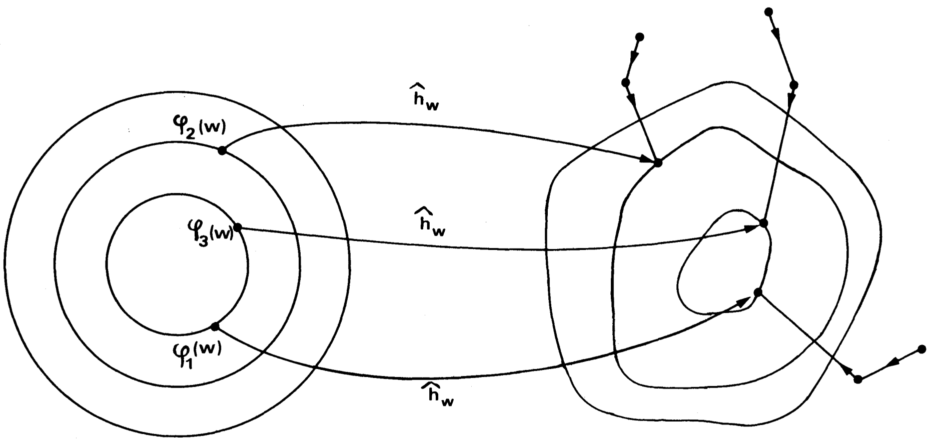
By (*d*) this means that $f_{w_0}^{n_0}(z_1)$ belongs to the orbit of a critical point, contradicting the definition of $\hat{\Lambda}_0$.

To finish the Proof of Theorem D we have only to show the existence of Λ_0, W_0 and $h: W_0\times\Lambda_0\to\overline{\mathbb{C}}$. To simplify the exposition suppose that every non hyperbolic periodic point or Herman ring is fixed. Without loss of generality we can suppose that Λ contains the attractive regions of attractive and superattractive fixed points. Let $S_1,\ldots,S_m$ be the Siegel disks of f_{w_0} and $H_1,\ldots,H_r$ its Herman rings. Let $z_1,\ldots,z_s$ be the attracting cycles and superattracting fixed points of f_{w_0} and let $V_1,\ldots,V_s$ be disjoint neighborhoods of $z_1,\ldots,z_s$ obtained by the application of Lemmas III.3 and III.4 to these attracting and superattracting fixed points. Finally, if $z_{s+1},\ldots,z_{s+l}$ are the parabolic periodic points, we take open sets $\Lambda_j\subset V_{s+j}$, $j=1,\ldots,l$ where V_{s+j} is obtained applying III.5 to these points, such that $f_{w_0}(\Lambda_j)\subset\Lambda_j$ and $\bigcup_{n=0} f^{-n}(\Lambda_j)$ is the union of all the parabolic domains with z_{s+j} in its boundary. The existence of these sets follows from the discussion in the introduction. First we shall show that if we take a neighborhood W_0 of w_0 contained in all the neighborhoods of w_0 given be Lemmas III.3 to III.6, and setting:

$$\Lambda_0=(\bigcup_i S_i\cap\Lambda)\cup(\bigcup_i H_i\cap\Lambda)\cup(\bigcup_i V_i)\cup(\bigcup_i\Lambda_i),$$

then, it is possible to define the map $h: W_0\times\Lambda_0\to\overline{\mathbb{C}}$ with the desired properties [that Λ_0 satisfies (*a*) and (*b*) is obvious]. If we forget about condition (*d*), the construction of h_w can be easily completed as follows. Suppose we want to define h_w on S_1. Then look at the preimage of z under the map $\hbar_{w_0}$ given by Lemma III.4 and compose it with $\hbar_w$

i.e. $h_w(z)=\hat{h}_w \hat{h}_{w_0}^{-1}(z)$. This definition satisfies (c) and (e) (on S_1) but (d) may fail to be true. To satisfy also (d) we define h_w as the composition $\hat{h}_w g_w \hat{h}_{w_0}^{-1}$, where g_w is a homeomorphism of the domain D of the map $\hat{h}_{w_0}$ that maps the $\hat{h}_{w_0}$-preimage of the set of points where f_{w_0}-forward orbits of critical points of f_{w_0} first hit S_1 onto the $\hat{h}_w$-preimage of the set of points where f_w-forward orbits of critical points of f_w first hit $\hat{h}_w(T)$. Require also that g_w depends analytically in w and that commutes with the rotation $z \to ze^{i\theta}$, where θ is the rotation number of f_w/S_1. Then $h_w=h_w\, g_w h_{w_0}^{-1}$ satisfies also property (d). To formalize the construction of the rearrangement map g_w we need the following Lemma:



LEMMA IV.1. — *Let* D *be the unit disk,* U *a complete manifold, and* $\varphi_i: U \to D$, $1 \leqq i \leqq k$ *analytic functions without zeroes such that for all* $1 \leqq i < j \leqq k$ *either* $|\varphi_i(w)| \neq |\varphi_j(w)|$ *for all* $w \in U$ *or there exists* $\alpha \in \mathbb{R}$ *such that* $\varphi_i(w)=e^{i\alpha}\varphi_j(w)$ *for all* $w \in U$. *Then given* $w_0 \in U$ *there exists* $g: U \times D \to D$ *such that:*

(a) *for all* $w \in U$, $g_w(.)=g(w, .)$ *is a homeomorphism of* D *and* $g_{w_0}=Identity$;

(b) *for every* $z \in D$ *the map* $w \to g_w(z)$ *is analytic;*

(c) $g_w(\varphi_i(w_0))=\varphi_i(w)$ *for all* $w \in U$, $1 \leqq i \leqq k$;

(d) $g_w(e^{i\theta}z)=e^{i\theta}g_w(z)$ *for all* $w \in U$, $z \in D$, $\theta \in \mathbb{R}$.

Proof. — Arrange the indexes of the family $\{\varphi_i\}$ in such a way that:

$$|\varphi_1(w_0)| < |\varphi_2(w_0)| < \ldots < |\varphi_n(w_0)|$$

and for all $n < j \leqq k$, there exists $1 \leqq i(j) \leqq n$ satisfying:

$$|\varphi_j(w_0)| = |\varphi_{i(j)}(w_0)|.$$

Then

$$(\star) \qquad 0 < |\varphi_1(w)| < |\varphi_2(w)| < \ldots < |\varphi_n(w)| < 1$$

and

$$|\varphi_j(w)| = |\varphi_{i(j)}(w)|,$$

for every $w \in U$, $n<j \leqq k$. Write the functions φ_i in polar coordinates as:

$$\varphi_i(w)=(r_i(w), \theta_i(w)).$$

Let F_w: [0, 1] be the unique monotone continuous function such that $\log F_w$ is piecewise linear and $\log F_w(r_i(w_0))=\log r_i(w)$ for $1 \leqq i \leqq n$, $F_w(0)=0$, $F_w(1)=1$.

The existence of F_w follows from ($\star$).

Let G_w: $[0, 1] \to \mathbb{R}$ be the continuous piecewise linear function satisfying:

$$G_w(1)=0,$$
$$G_w(0)=0,$$
$$G_w(r_i(w_0))=\theta_i(w)-\theta_i(w_0).$$

Now define (in polar coordinates) $g_w(r, \theta)=(F_w(r), G_w(r)+\theta)$. Clearly every g_w is a homeomorphisms satisfying (*a*), (*c*) and (*d*). To prove (*b*) observe that in each annulus $\{(r, \theta) \mid r_i(w_0)<r<r_{i+1}(w_0)\}$ we can write:

$$\log F_w(r)=[r_{i+1}(w_0)-r_i(w_0)]^{-1}[(r-r_i(w_0))\log r_{i+1}(w)-(r-r_{i+1}(w_0))\log r_i(w)]$$

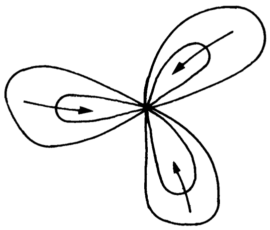
and:

$$\log G_w(r)=[r_{i+1}(w_0)-r_i(w_0)]^{-1}[(r-r_i(w_0))\,\theta_{i+1}(w)-(r-r_{i+1}(w_0))\,\theta_i(w)].$$

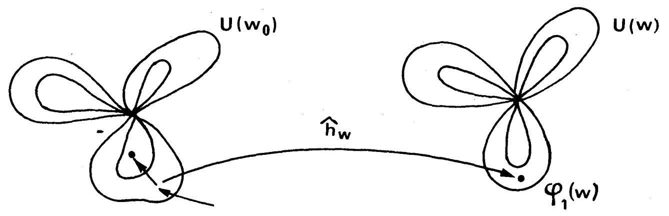
Clearly, for each fixed r, these are harmonic conjugate functions. Hence (*b*) is satisfied in all these annulus. A similar computation shows that it is also satisfied in the disk $r<r_i(w_0)$ and in the annulus $r_n(w_0)<r<1$, so the Lemma is proved.

Now take analytic functions $\varphi_i: W_0 \to D$, $i=1, \ldots, k$ such that $h_w(\varphi_i(w))$, $1 \leqq i \leqq k$, are the points where the f_w-forward orbits of critical points of f_w first hit S_1. If we show that these functions satisfy the hypothesis of Lemma IV.1 we are done. Restrict W_0 to grant that if $|\varphi_i(w_0)| \neq |\varphi_j(w_0)|$ for some $1 \leqq i<j \leqq n$, then $|\varphi_i(w)| \neq |\varphi_j(w_0)|$ for all $w \in W_0$. Now suppose that $|\varphi_i(w_0)|=|\varphi_j(w_0)|$ for some $1 \leqq i<j \leqq n$. Since $w_0 \in C(f)$ we must have $|\varphi_i(w)|=|\varphi_j(w)|$ for every $w \in W_0$. Hence φ_i/φ_j has constant modulus. Therefore it is constant i. e. $\varphi_i(w)=e^{i\alpha}\varphi_j(w)$ for all $w \in W_0$.

Now we shall show how to construct h_w on a set of type Λ_i. To simplify the notation set $i=1$ and let V a neighborhood of z_1 is defined a map $h: W_0 \times V \to \overline{\mathbb{C}}$ such that $\hat{h}_w$ is a local conjugacy between f_{w_0} and f_w. This map is given by Lemma III.5. We can take Λ_1



such that $\Lambda_1/f_{w_0}(\Lambda_1)$ is a union of sets whose boundaries are closed arcs: as in the Figure (*see* the discusion in the Introduction). Then every f_{w_0}-orbit intersects $U(w_0)=\Lambda_1 \setminus f_{w_0}(\Lambda_1)$ in at most one point, and, by the existence of a local conjugacy, every f_w-orbit intersects $U(w)=\hat{h}_w(U(w_0))$ in at most one point. Take the images by $\hat{h}_w^{-1}$ of the intersections of the forward f_w-orbits of critical points of f_w with $U(w)$.

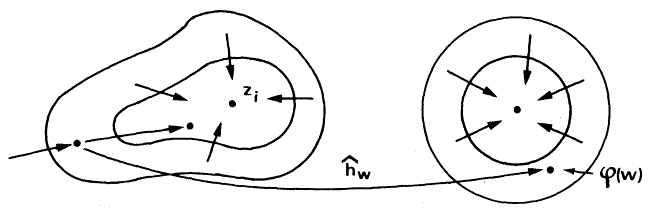


Describe these images by analytic functions $\varphi_1(w), \ldots, \varphi_m(w)$, $w \in W_0$. Restricting W_0 if necessary it is easy to construct a map $g_w: U(w_0) \hookleftarrow$, that is a homeomorphism for all $w \in W_0$, is the identity in the boundary of $U(w_0)$, depends analytically in w, is the identity for $w=w_0$ and maps $\varphi_i(w_0)$ in $\varphi_i(w)$, $1 \leqq i \leqq m$. Extend $g_w: U(w_0) \hookleftarrow$ to $g_n: \Lambda_1 \hookleftarrow$ using f_{w_0} i.e. :

$$g_w(f_{w_0}^n(z)) = f_{w_0}^n(g_w(z)) \quad \text{if} \quad n \geqq 0, \quad w \in W_0, \quad z \in U(w_0).$$

Now define $h_w: \Lambda_i \to \overline{\mathbb{C}}$ by $h_w = \hat{h}_w g_w \hat{h}_w^{-1}$. Clearly h_w satisfies conditions (*a*)-(*d*). As in the case of Siegel disks and Herman rings the role of the rearrangement map g_w is to make h_w satisfy condition (*d*).

To construct h_w on sets of type V_i, associated to an attractive fixed point, the construction is exactly the same replacing Lemma III.5 by Lemma III.3. Let $\hat{h}: W_0 \times V_i \to \overline{\mathbb{C}}$ be the parameterized linearization given by III.3. Set $\lambda = (f_{w_0})'(z_i)$. By composing $\hat{h}_w$ with a convenient map we can suppose that $\hat{h}_w$ is a conjugacy between $f_w: V_i \hookleftarrow$ and $D \ni z \to \lambda z \in D$, where D is a disk. Set $U = D \setminus \lambda D$. This set is a ring and we have analytic functions $\varphi_1(w), \ldots, \varphi_m(w)$ indicating the images by $\hat{h}_w$ of the points where forward f_w-orbits of critical points of f_w hit $V_i \setminus f_w(V_i)$. Take $g_w: U \hookleftarrow$ as before and extend it to $g_w: D \hookleftarrow$ by: $g_w(\lambda^n z) = \lambda^n g_w(z)$. Finally define: $h_w = \hat{h}_w^{-1} g_w h_{w_0}$.



This map satisfies the required conditions. Once more observe that $\hat{h}_w^{-1} \hat{h}_{w_0}$ would satisfy (*a*), (*b*), (*c*) and (*e*), and that the role of g_w is to make it satisfy also (*d*). It remains to consider the case of super attracttive fixed points.

By Lemma III.4 there is a map $\hat{h}: W_0 \times V_i \to D$, where D is a disk centered at 0, such that each h_w conjugates $f_w: V_i \circlearrowleft$ with $D: z \to z^n \in D$. Set $U = D \setminus \{z^n \mid z \in D\}$. Let $\varphi_1(w), \ldots, \varphi_m(w)$ be analytic functions describing the images by h_w of the points of f_w intersect U. Let $g_w: U \circlearrowleft$ be a family of homeomorphisms of U, that are the identity in the boundary of U, depending analytically in w, and satisfying $g_w(\varphi_i(w_0)) = \varphi_i(w)$. Require also that g_w acts as a linear map in circles $z = \text{Const}$. Such family is given by Lemma IV.1. Now extend g_w to D by setting $g_w(z) = g_w(z_0)^{n^m}$ if $z \in D$, $z_0 \in U$ and $z_0^{n^m} = z$. The fact that g_w acts as a linear map on circles grants that this definition is independent of the root z_0 of z used. Now define $h_w = \hat{h}_w^{-1} g_w \hat{h}_{w_0}$.

To prove the converse property is sufficient to observe that a map h_w satisfying (i) maps the set of critical points of f_{w_0} onto the set of critical points of f_w, preserving their orders, and maps the foliations of superattractive regions, Herman rings and Siegel disks of f_{w_0} onto the corresponding foliations for f_w. Those properties immediately yield that $w_0 \in C(f)$.

V. – Proof of Theorem E

We start proving the following property:

LEMMA V.1. – *Suppose that f and g are analytic endomorphisms of $\overline{\mathbb{C}}$ such that there exists a quasi-conformal homeomorphism $h: \overline{\mathbb{C}} \circlearrowleft$ satisfying $gh = hf$ and analytic in the complement of $J(f)$. Then, if f has no invariant line fields in $J(f)$, h is analytic.*

Proof. – Associate to a. e. $z \in \overline{\mathbb{C}}$ the ellipse $C(z) = h(z)^{-1} C_0(h(z))$ where $C_0(h(z))$ is the unit circle of $T_{h(z)}\overline{\mathbb{C}}$. The relation $gh = hf$ implies:

$$(\star) \qquad f'(z)\, C(z) = C(f(z)),$$

for a. e. $z \in \overline{\mathbb{C}}$. If $J(f)$ has measure zero, the Lemma is proved because h is in this case 1-quasi-conformal, hence conformal. Then we can suppose that $J(f)$ has non zero measure. Let Σ be the set of points of $z \in J(f)$ where $C(z)$ is defined and is not a circle. Then $(\star)$ shows that $f^{-1}(\Sigma) = \Sigma$. If $z \in \Sigma$ define $E(z)$ as the one dimensional subspace of $T_z\overline{\mathbb{C}}$ containing the major axis of $C(z)$. Again, $(\star)$ proves that $f'(z)\, E(z) = E(f(z))$ for $z \in \Sigma$. Since f has no invariant line fields in $J(f)$, it follows that the measure of Σ is zero. This means that $C(z)$ is a circle for a. e. $z \in J(f)$. Outside $J(f)$, $C(z)$ is always a circle by the analyticity of h in the complement of $J(f)$. Therefore $C(z)$ is a circle for a. e. $z \in \overline{\mathbb{C}}$. This means that h is 1-quasi-conformal, therefore conformal as we wanted to prove.

Now suppose that $w_0 \in C(f)$ satisfies the hypothesis of Theorem E. First of all observe that the hypothesis $w_0 \in C(f)$ plus the non existence of Herman rings and non hyperbolic periodic points imply that $J(f_{w_0}) \neq \overline{\mathbb{C}}$ because by Theorem D, every f_w with w near w_0, is topologically equivalent to f_{w_0} *via* a quasi-conformal map. By the previous Lemma this means that these maps are analytic, thus implying that every f_w, with w near w_0, is analytically equivalent to f_{w_0} contradicting the hypothesis that the family is reduced. Then

$J(f_{w_0})=\overline{\mathbb{C}}$, and since there are no Herman rings or Siegel disks, there exist attracting cycles and superattracting cycles whose attractive sets cover the complement of $J(f_{w_0})$. Recall $x \approxeq y$ if $f_w^n(x) \approx f_w^m(y)$ for some $n \geqq 0$, $m \geqq 0$. Let $r_1(w)$ be the number of ($\approxeq$)-equivalence classes of critical points not contained in $J(f_w)$ and not eventually periodic. The next step of the proof is the construction of an analytic function $\Psi: W_1 \to M$ of constant rank, where $W_1 \subset W$ is open and M is a complex manifold with dimension $r_1(w_0)$, such that for every value $c \in \Psi(W_1)$ and every connected component S of $\Psi^{-1}(c)$, all the endomorphisms f_w with $w \in S$ are analytically conjugate. If such a function exists, then the fact that the family of endomorphisms that we are considering is reduced implies that every S must consist of at most one point. Hence $\dim W \leqq r_1(w_0)$. But on the other hand $\dim W \geqq r(w_0)$ by hypothesis. Then $r_1(w_0)=r(w_0)$. This means that there are no critical points in $J(f_{w_0})$, hence, as observed in the introduction, f_{w_0} satisfies Axiom A.

Therefore the proof of Theorem E is now reduced to construct the function Ψ. In fact we shall construct this function satisfying the *a priori* weaker property that all the endomorphisms f_w with w in a set S as above, are topologically equivalent *via* a conjugacy that is quasi-conformal and analytic outside $J(f_w)$. But then Lemma V.1 shows that the conjugacy is in fact analytic. To simplify the construction of Ψ, we shall suppose that every attractive periodic point of f_{w_0} is fixed. Then the same property holds for every w in W_0, if W_0 is small enough. Let $\varphi_i: W_0 \to \overline{\mathbb{C}}$, $i=1, \ldots, k$ be analytic functions describing the position of the attractive and superattractive fixed points $\varphi_1(w), \ldots, \varphi_k(w)$ of f_w, $w \in W_0$. Suppose that $\varphi_i(w_0)$ is superattractive for $i=1, \ldots, k_1$ and that $(f_{w_0})'(\varphi_i(w_0)) \neq 0$ for $k_1 < i \leqq k$. Define $\Psi_i: W_0 \to \overline{\mathbb{C}}$, $k_1 < i \leqq k$, by $\Psi_i(w)=(f_w)'(\varphi_i(w))$. Let $V_1, \ldots, V_k$ be neighborhoods of $\varphi_1(w_0), \ldots, \varphi_k(w_0)$ given by Lemma III.4. Assume that W_0 is so small that the maps $\hat{h}_w$ given by this Lemma are defined for $w \in W_0$. If $1 \leqq i \leqq k_1$, let n_1 be the number of ($\approxeq$)-equivalence classes of critical points of f_{w_0} whose orbits intersect V_i and don't coincide eventually with $\varphi_i(w_0)$. For each $1 \leqq i \leqq k_1$ take an analytic function $\hat{\Psi}_i: W_0 \to \mathbb{C}^{n_i}$ such that each coordinate is the image under $\hat{h}_w$ of a point in the forward f_w-orbit of a critical point of f_w and moreover different coordinates correspond to different ($\approxeq$)-equivalence classes of critical points. For $k_1 < i \leqq k$, define n_i as before and let M_i be the quotient of $(\mathbb{C}-\{0\})^{n_i}$ by the action of $\mathbb{C}-\{0\}$ given by $(z_1, \ldots, z_{n_i})=(\lambda z_1, \ldots, \lambda z_{n_i})$. Define $\hat{\Psi}_i: W_0 \to M_i$ as an analytic function that to every $w \in W_0$ associates the element of M_i determined by the images under $\hat{h}_w$ of the forward f_w-orbits of critical points. Finally, define:

$$\Psi: \quad W_0 \to \mathbb{C}^{n_1} \times \ldots \times \mathbb{C}^{n_{k_1}} \times M_{k_1+1} \times \ldots \times M_k \times \mathbb{C} \times \ldots \times \mathbb{C},$$

by $\Psi(w)=(\hat{\Psi}_1(w), \ldots, \hat{\Psi}_k(w), \Psi_{n_k+1}(w), \ldots, \Psi_k(w))$. The manifold at right has dimension:

$$n_1 + \ldots + n_{k_1} + (n_{k_1+1}-1) + \ldots + (n_k - 1) + (k - k_1) = r_1(w_0),$$

as we wished. Now restrict Ψ to $W_1 \subset W_0$ where it has maximal rank. Let S be a connected component of $\Psi^{-1}(c)$ where $c \in \Psi(W_1)$. Fix some $w_1 \in S$. Let Λ be the union of the attractive sets of the attractive fixed points of f_w minus the orbits of the eventually fixed points. In a neighborhood W_2 of w_1 there exist, for every $w \in W_2$, an analytic conjugacy

$h_w : \Lambda \to h_w(\Lambda)$ between f_{w_0}/Λ and $f_w/h_w(\Lambda)$, depending analytically in w, constructed by the same method used in Section IV to prove Theorem D i. e., starting with local conjugacies nearby the attractive fixed points that map f_{w_0}-orbits of critical points of f_{w_0} onto f_w-orbits of critical points of f_w, and then pulling back these conjugacies by f_{w_0} and f_w, in order to fill Λ. The new fact we have now is that w varies in S. Then the property $\Psi(w)=c=\Psi(w_0)$ makes it possible to take these local conjugacies to be analytic because now that there are only attractive and superattractive fixed points to consider and the rearrangement functions g_w can be taken as linear maps depending in w. Then $h_w : \Lambda \to h_w(\Lambda)$ is analytic. The extension of h_w to $\overline{\mathbb{C}}$ (granted as before by the λ-Lemma) is (also by the λ-Lemma) quasi-conformal and analytic in Λ. Since Λ is $\overline{\mathbb{C}} - J(f_{w_0})$ minus a discrete set of points, it follows that h_w is analytic in $\Lambda - J(f_{w_0})$ as we wished.

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