ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 16, nº 1 (1983), p. 151-172

<http://www.numdam.org/item?id=ASENS_1983_4_16_1_151_0>

Ann. scient. Éc. Norm. Sup.
4e série, t. 16, 1983, p. 151 à 172.

SIMPLE QUOTIENTS OF GROUP C*-ALGEBRAS FOR TWO STEP NILPOTENT GROUPS AND CONNECTED LIE GROUPS

by Detlev POGUNTKE

Introduction

Let G be a locally compact group, and let $C^*(G)$ be its group C^*-algebra. Two classical tasks of representation theory are the determination of the sets $\hat{G}(\hat{\hat{G}})$ of all (quasi) equivalence classes of unitary strongly continuous irreducible (factor) representations of G in Hilbert spaces. To every such representation π there corresponds a representation π' of $C^*(G)$, and the kernel of π' is a primitive ideal in $C^*(G)$—at least if G is σ-compact. [For non σ-compact G, it is not known if there can be factor representations of $C^*(G)$ having kernels which are prime but not primitive.] In this way, one gets a map from $\hat{G}(\hat{\hat{G}})$ onto Prim(G), where Prim(G) denotes the set (or space with the Jacobson topology) of primitive ideals in $C^*(G)$. Thus the determination of $\hat{G}(\hat{\hat{G}})$ can be divided into two tasks, namely into the determination of Prim(G) and into the determination of the fibers of the maps $\hat{G} \to \text{Prim}(G)$ and $\hat{\hat{G}} \to \text{Prim}(G)$, respectively, i. e. the (quasi) equivalence classes of representations with a given kernel. Concerning Prim(G), a lot of work is done, especially for connected Lie groups. In the semisimple case [there $\hat{G} \to \text{Prim}(G)$ is a bijection, because G is of type I] a complete description is not yet available, but in the solvable case the set (not the space) Prim(G) is "known", *see* [16]. For the second task, a first step will be the determination of the structure of $C^*(G)/\mathscr{I}$, where $\mathscr{I}$ is a primitive ideal in $C^*(G)$—and this is the theme of the present article.

As indicated in the title, we want to study this question in two cases. In part I of the paper we consider compactly generated two step nilpotent groups G [then, by the way, Prim(G) can be parametrized very easily]. It turns out that the primitive quotients of $C^*(G)$ are simple and isomorphic to tensor products of the algebra of compact operators on a (finite- or infinite-dimensional) Hilbert space and twisted convolution algebras on free abelian groups of finite rank.

In part II we treat connected Lie groups G. By a result of Moore and Rosenberg [15], every primitive quotient $C^*(G)/\mathscr{I}$ contains a (unique) minimal non-zero closed ideal $\mathscr{M}=\mathscr{M}_{\mathscr{I}}$; $\mathscr{M}$ is a simple C*-algebra. What we actually do is the determination of the structure of $\mathscr{M}$. Note that in order to describe all irreducible (factor) representations of G with the given kernel $\mathscr{I}$, it suffices to know any non-zero ideal in $C^*(G)/\mathscr{I}$ (and its representations!). It turns out that $\mathscr{M}$ is isomorphic either to a finite-dimensional matrix algebra or to the tensor product of the algebra of compact operators on an infinite-dimensional Hilbert space and the twisted convolution algebra on a free abelian group of finite rank. Green has already obtained parts of this theorem [8]; he also posed the problem to determine the precise structure of $\mathscr{M}$. This theorem reduces (to some extent) the problem of the classification of all irreducible representations of a Lie group to the problem of the classification of all irreducible projective representations of free abelian groups (which is not solved yet; see the remarks at the end of the paper).

Part I and II are preceded by a section on twisted covariance algebras, in the sense of Green [7], and in the sense of Leinert [13]. This section will contain no single new result. It is taken up to make the paper better readable, and to introduce some notations which will differ slightly from those of Green's.

Twisted covariance algebras

Mackey analyzed systematically the representations of a locally compact group G *via* the representations of a given closed normal subgroup N [14]. The C*-algebraic equivalent will be to analyze $C^*(G)$ by using $C^*(N)$ and subquotients of it. This leads to the concept (or to concepts) of twisted covariance algebras. These concepts are useful especially in the case that one wants to study subquotients of $C^*(G)$ which are "given" by (G-invariant) subquotients of $C^*(N)$.

First, I will present Leinert's definition. Let $H=G/N$ be the quotient group. Suppose that there exists a measurable cross section $\sigma: H \to G$ with $\sigma(e)=e$. Using σ one can identify the Banach space $L^1(G)$ with $L^1(H, L^1(N))$. In order to transform the convolution and the involution on $L^1(G)$ into the new picture one introduces a "unitary factor system" P on H and an (in general non homomorphic) "action" T of H on $L^1(N)$. Let $\gamma: H \times H \to N$ be defined by $\gamma(x, y)=\sigma(y)^{-1}\sigma(x)^{-1}\sigma(xy)$, and let δ be the modular function of the action of G on N, i. e.

$$\int_N f(n)\,dn=\delta(x)\int_N f(x^{-1}nx)\,dn \qquad \text{for} \quad x\in G, \quad f\in L^1(N).$$

Put $\mathscr{A}=L^1(N)$, and denote by $\mathscr{A}^b$ the adjoint algebra of $\mathscr{A}$; $\mathscr{A}^b$ is also often called the multiplier algebra. For $x, y\in H$ the element $P_{x,y}$ in the unitary group of $\mathscr{A}^b$ is defined by

$$(P_{x,y}f)(n)=f(\gamma(x, y)n),$$

and the $\star$-automorphism T_x of $\mathscr{A}$ is defined by

$$(T_x f)(n)=\delta(\sigma(x^{-1})^{-1})f(\sigma(x^{-1})n\sigma(x^{-1})^{-1}).$$

One easily proves the following formulas:

(L) $$P_{e,x}=P_{x,e}=T_e=\text{identity} \quad \text{for all } x\in H,$$
$$f\star(P_{x,y}g)=(P_{x,y}T_{y^{-1}}T_{x^{-1}}T_{y^{-1}x^{-1}}^{-1}f)\star g \quad \text{for } x, y\in H \text{ and } f, g\in\mathscr{A},$$
$$P_{xy,z}T_{z^{-1}}P_{x,y}T_{z^{-1}}^{-1}=P_{x,yz}P_{y,z} \quad \text{for } x, y, z\in H.$$

It turns out that $L^1(G)$ is $\star$-isomorphic to $L^1(H, \mathscr{A}, P, T)$ if one defines the convolution and the involution in the latter algebra by:

$$(f\star g)(x)=\int_H [P_{xy,y^{-1}}T_y f(xy)]\star g(y^{-1})\,dy$$
$$f^*(x)=\Delta_H(x)^{-1}P_{x^{-1},x}^{-1}T_x^{-1}[f(x^{-1})^*],$$

where Δ_H denotes the modular function of the group H.

And whenever there are given a locally compact group H, an involutive Banach algebra $\mathscr{A}$, a (measurable) map T from H into the group of $\star$-automorphisms of $\mathscr{A}$ and a (measurable) map P (a so-called unitary factor system) from $H\times H$ in the unitary group of $\mathscr{A}^b$ satisfying (L) one may form the involutive algebra $L^1(H, \mathscr{A}, P, T)$ by the above formulas. This is Leinert's version of twisted covariance algebras.

In the following parts of the paper, we will use this construction to study $C^*(G)$. $C^*(G)$ is the C^*-hull of $L^1(G)\cong L^1(H, L^1(N), P, T)$ and also of $L^1(H, C^*(N), P, T)$; note that the factor system P and the action T can be extended to $C^*(N)$.

An important special case of twisted covariance algebras will be the case that T is trivial and P is a scalar multiple (of modulus one) of the identity. Then the conditions (L) simply mean that P is a (measurable) cocycle. And $L^1(H, \mathscr{A}, P)$ is isomorphic to the projective tensor product of $L^1(H, \mathbb{C}, P)$ and $\mathscr{A}$. Instead of $L^1(H, \mathbb{C}, P)$ we will simply write $L^1(H, P)$, its C^*-hull is denoted by $C^*(H, P)$. These algebras are called *twisted convolution algebras*. One should notice that equivalent cocycles give rise to isomorphic algebras: if the equivalence is established by the Borel function b then multiplication by b yields the desired isomorphism.

For abelian groups, we will use the notion of a non-degenerate cocycle. Recall [12], that to a cocycle P on a locally compact abelian group H one may associate an antisymmetric bicharacter $\varphi=\varphi_P$ by the formula $\varphi(x, y)=P(x, y)\,P(y, x)^{-1}$ (φ corresponds to the commutator in the central group extension defined by P). P is called non-degenerate if $\varphi(x, H)=1$ holds only for $x=0$ which means that the center of the group extension defined by P is equal to $\mathbb{T}$. Especially, if P is degenerate then the adjoint algebra of $C^*(H, P)$ has a non-trivial center, and $C^*(H, P)$ can't be a primitive algebra.

In the rest of this section I will discuss Green's notion of twisted covariance algebras. Let again G be a locally compact group, and let N be a closed normal subgroup of G. The basis of the construction is the observation that G can be considered (in a non-trivial way) as a quotient of the semidirect product $G\ltimes N$. This quotient map gives (by integration) quotient maps $L^1(G\ltimes N)\to L^1(G)$ and $C^*(G\ltimes N)\to C^*(G)$. $C^*(G\ltimes N)$ is isomorphic to an "ordinary covariance algebra" $C^*(G, C^*(N))$, and the kernel of $C^*(G, C^*(N))\to C^*(G)$

can be described in terms of a "twist τ" which will simply be left translation on $C^*(N)$. Let's make this procedure more precise. The semidirect product $G \ltimes N$ is, by definition, the set $G \times N$ with the multiplication law $(x, m)(y, n) = (xy, \alpha(y)^{-1}(m)n)$ where the homomorphism $\alpha : G \to \mathrm{Aut}(N)$ is given by $\alpha(x)(n) = xnx^{-1}$. Define the homomorphism $\varphi : G \ltimes N \to G$ by $\varphi(x, n) = xn$; φ is surjective and its kernel K is equal to $\{(n^{-1}, n); n \in N\}$. Integration along K gives a surjective $\star$-morphism Φ:

$$L^1(G \ltimes N) \to L^1(G), \qquad (\Phi f)(x) = \int_N \Delta(n)^{-1} f(xn^{-1}, n)\, dn$$

where Δ denotes the modular function of G or N. Put $\mathscr{A} = L^1(N)$ and define the (homomorphic) action T of G on $\mathscr{A}$ by $(T_x f)(n) = \delta(x) f(x^{-1}nx)$, δ as above. Then $L^1(G \ltimes N)$ is $\star$-isomorphic to $L^1(G, \mathscr{A}, T)$ (with trivial factor system). In order to compute the kernel of Φ we define the homomorphism τ (the "twist") from N into the unitary group of $\mathscr{A}^b$ by $\{\tau(n)a\}(m) = a(n^{-1}m)$.

T and τ are related by the formulas

$$\text{(G)} \qquad \begin{aligned} \tau(n)\, a\, \tau(n)^{-1} &= T_n a && \text{for } a \in \mathscr{A},\ n \in N, \\ \tau(xnx^{-1}) &= T_x(\tau(n)) && \text{for } x \in G,\ n \in N; \end{aligned}$$

in the second equation T_x denotes the extension to $\mathscr{A}^b$ of the automorphism T_x on $\mathscr{A}$. In a canonical way, $\mathscr{A}^b$ and $\mathscr{M}(G)$ (the space of bounded measures on G) can be identified with parts of $L^1(G, \mathscr{A}, T)^b$. Especially, $\tau(n) \in \mathscr{A}^b$ and ε_n (= point measure at the point $n \in N$) can be considered as elements of $L^1(G, \mathscr{A}, T)^b$. One can show that ker Φ is the closed two-sided ideal generated by $\varepsilon_n - \tau(n)$, $n \in N$, i. e. ker Φ is the smallest closed two-sided ideal in $L^1(G, \mathscr{A}, T)$ containing all the $f(\varepsilon_n - \tau(n))g$, $f, g \in L^1(G, \mathscr{A}, T)$, $n \in N$. Of course, the action T and the twist τ can be extended to the C*-hull of $\mathscr{A}$ ($= C^*(N)$). The $\star$-morphism $L^1(G, \mathscr{A}, T) \to L^1(G)$ extends to a surjective $\star$-morphism $C^*(G, C^*(N), T) \to C^*(G)$, also denoted by Φ, and its kernel can be described as above in terms of τ. This example was generalized by Green. Whenever there is given a locally compact group G, a closed normal subgroup N of G, a C*-algebra $\mathscr{A}$, a (strongly continuous) homomorphic action $T : G \to \mathrm{Aut}(\mathscr{A})$ and a continuous homomorphism τ from N into the unitary group of $\mathscr{A}^b$ such that (G) holds one calls $(G, N, \mathscr{A}, T, \tau)$ a *twisted covariance system* and defines the *twisted covariance algebra* $C^*(G, \mathscr{A}, T, \tau)$ as the quotient of $C^*(G, \mathscr{A}, T)$ modulo the closed ideal generated by $f(\varepsilon_n - \tau(n))g$, $f, g \in C^*(G, \mathscr{A}, T)$, $n \in N$. From Green's results on these algebras we will especially use the following.

THEOREM (Green [8]). — *Let* $(G, N, \mathscr{A}, T, \tau)$ *be a twisted covariance system. Let* H *be a closed subgroup of* G *containing* N *such that there exists a measurable cross section* $G/H \to G$ *taking relatively compact sets into relatively compact sets. Let* $p : \mathrm{Prim}(\mathscr{A}) \to G/H$ *be a continuous* G-*equivariant map, and let* $\mathscr{I}$ *be the kernel (in the "hull-kernel sense") of* $p^{-1}(eH)$. *Then*

(i) $C^*(G, \mathscr{A}, T, \tau)$ *is isomorphic to* $\mathscr{K}(L^2(G/H)) \otimes C^*(H, \mathscr{A}/\mathscr{I}, T, \tau)$

where the action of H *on* $\mathscr{A}/\mathscr{I}$ *and the twist of* N *on* $\mathscr{A}/\mathscr{I}$ *are the obvious ones.*

(ii) *If in addition p is injective and* $\mathscr{A}$ *is a type* I *algebra* (*hence* $\mathscr{A}/\mathscr{I}$ *is isomorphic to the algebra of compact operators*) *then* $C^*(G, \mathscr{A}, T, \tau)$ *is isomorphic to* $C^*(H/N, m)\otimes\mathscr{A}/\mathscr{I}\otimes\mathscr{K}(L^2(G/H))$ *where m is a cocycle on* H/N (*all the* H-*conjugates of one of the irreducible representations of* $\mathscr{A}/\mathscr{I}$ *are unitarily equivalent and give rise to an m-projective representation of* H).

This section is finished by some agreements. A cocycle on a (locally compact) group is always understood to be a 2-cocycle with values in $\mathbb{T}$, the group of complex numbers of modulus one; in most cases the cocycles will be continuous. By a (projective) representation of a locally compact group we always mean a strongly continuous (projective) unitary representation in a Hilbert space. If $\mathscr{H}$ is a Hilbert space then $\mathscr{K}(\mathscr{H})$ denotes its algebra of compact operators.

Part I:
Representation theory of compactly generated two step nilpotent groups

Let G be a locally compact two step nilpotent group, and let Z be any closed central subgroup of G containing the commutator subgroup. First we want to describe the primitive ideals in $C^*(G)$. So, let $\mathscr{I}$ be a primitive ideal in $C^*(G)$, and let π be an irreducible representation of $C^*(G)$ (or of G) with $\ker\pi=\mathscr{I}$. Then $\pi|_Z$ is a multiple of a unitary character of Z, say λ. Let $Z_\lambda/\ker\lambda$ be the center of $G/\ker\lambda$ (Z_λ might be strictly bigger than the center of G). The representation π can be considered as a representation of $G/\ker\lambda$. From the irreducibility of π it follows that $\pi|_{Z_\lambda}$ is a multiple of a unitary character of Z_λ, say μ, with $\mu|_Z=\lambda$. Now the procedure stops, i. e. the center of $G/\ker\mu$ is precisely $Z_\lambda/\ker\mu$. And Kaniuth has shown in [11] (following ideas of Howe in [10]) that the kernel [in $C^*(G)$] of the induced representation $\mathrm{ind}_{Z_\lambda}^G\mu$ coincides with $\ker\pi=\mathscr{I}$. Obviously, for different parameters λ, μ the kernels of the corresponding induced representations are different; and there are no inclusions between them which shows that all primitive ideals are maximal. Thus, we have obtained the following.

PROPOSITION (Howe, Kaniuth). — *Let* G *be a locally compact two step nilpotent group, let* Z *be a closed central subgroup of* G *containing the commutator subgroup, and let* Z_λ (*for* $\lambda\in\hat{Z}$) *be defined by* $Z_\lambda/\ker\lambda=$ *center of* $G/\ker\lambda$. *Then the map* $(\lambda, \mu)\to\ker\mathrm{ind}_{Z_\lambda}^G\mu$ *is a bijection from* $\{(\lambda, \mu);\ \lambda\in\hat{Z},\ \mu\in\hat{Z}_\lambda,\ \mu_{|Z}=\lambda\}$ *onto the set of primitive* (*maximal*) *ideals in* $C^*(G)$.

Under the additional hypothesis that G is compactly generated (then the structure of G/Z is known) we want to determine explicitly the structure of the simple quotients of $C^*(G)$, i. e. the quotients $C^*(G)/\ker\mathrm{ind}_{Z_\lambda}^G\mu$. One of the main tools in doing this will be part (ii) in Green's isomorphism theorem — see the first section.

THEOREM 1. — *Let* G *be a compactly generated locally compact two step nilpotent group. Then every primitive quotient of* $C^*(G)$ *is isomorphic to* $C^*(F, \nu)\otimes\mathscr{K}(\mathscr{H})$ *where* F *is a free abelian group of finite rank* (*including zero*), ν *is a non-degenerate cocycle on* F *and* $\mathscr{K}(\mathscr{H})$ *is the algebra of compact operators on the finite- or infinite-dimensional Hilbert space* $\mathscr{H}$.

Proof. — Let $\lambda \in \hat{Z}$ and $\mu \in \hat{Z}_\lambda$ with $\mu|_Z = \lambda$ be given.

Denote by $p : G \to G/Z_\lambda$ the quotient morphism, and let K be the maximal compact subgroup of G/Z_λ. Every irreducible unitary representation ρ of $p^{-1}(K)$ with the property that $\rho|_{Z_\lambda}$ is a multiple of μ is finite-dimensional [because $p^{-1}(K)/\ker \lambda$ is a "central topological group", i. e. compact modulo the center]. And the set X of (equivalence classes of) those representations is a discrete subset of the unitary dual of $p^{-1}(K)$. Moreover, X is the hull of the kernel of $\mathrm{ind}_{Z_\lambda}^G \mu |_{p^{-1}(K)}$. Since $\ker \mathrm{ind}_{Z_\lambda}^G \mu$ is primitive, X is an G-orbit. Let $\mathscr{A} := C^*(p^{-1}(K))/k(X)$. Write $C^*(G)$ as $C^*(G, C^*(p^{-1}(K)), T, \tau)$ where the (canonical) twist τ is defined on the normal subgroup $N := p^{-1}(K)$. Obviously, the kernel of the quotient map $C^*(G, C^*(N), T, \tau) \to C^*(G, \mathscr{A}, T, \tau)$ coincides with $\ker \mathrm{ind}_{Z_\lambda}^G \mu$. Hence $C^*(G)/\ker \mathrm{ind}_{Z_\lambda}^G \mu$ is isomorphic to $C^*(G, \mathscr{A}, T, \tau)$. Let ρ be a point in X, let G_ρ be its stabilizer, and let m be the corresponding cocycle on G_ρ (or better: on G_ρ/N). Since $\mathrm{Prim}(\mathscr{A})(=X)$ is G-homeomorphic to G/G_ρ it follows from part (ii) of Green's isomorphism theorem that $C^*(G, \mathscr{A}, T, \tau)$ is isomorphic to

$$C^*(G_\rho/p^{-1}(K), m) \otimes \mathscr{K}(L^2(G/G_\rho)) \otimes M_n(\mathbb{C})$$

with $n = \dim \rho$.

The group $G_\rho/p^{-1}(K)$ is isomorphic to the direct sum of a free abelian group F of finite rank and a vector group W [which is the connected component of the identity in $G_\rho/p^{-1}(K)$]. Hence we may assume that m is continuous (we may even assume that m is a bicharacter) — *see* [12]. Let $\varphi : (G_\rho/p^{-1}(K))^2 \to \mathbb{T}$ be defined by $\varphi(x, y) = m(x, y)\,\overline{m}(y, x)$; φ is an antisymmetric bicharacter. Denote by V the kernel of $\varphi|_{W^2}$, i. e. $V := \{x \in W;\ \varphi(x, W) = 1 (= \varphi(W, x))\}$. Now, we choose any vector space complement Y to V in W. Then we may represent the group $G_\rho/p^{-1}(K)$ as $H \oplus Y$ with $H = F \oplus V$. For twisted convolution algebras on direct sums of locally compact abelian groups one has the following structure.

LEMMA 1. — *Let* H *and* Y *be locally compact abelian groups and let m be a continuous cocycle on the direct sum* $H \oplus Y$. *Define* $\varphi : (H \oplus Y)^2 \to \mathbb{T}$ *by* $\varphi(a, b) = m(a, b)\,\overline{m}(b, a)$. *Then* $L^1(H \oplus Y, m)$ *is isomorphic as an involutive Banach algebra to* $L^1(H, L^1(Y, m_2), m_1, T)$ — *in the sense of Leinert — where* m_1 (m_2) *denotes the restriction of m to* H (Y) *and the action* T *of* H *on* $\mathscr{A} := L^1(Y, m_2)$ *is given by*

$$(T_x a)(y) = \varphi(x, y)\, a(y), \qquad x \in H, \quad y \in Y.$$

Proof. — For $f \in L^1(H \oplus Y, m)$ we define f'

$$H \to \mathscr{A} \text{ by } f'(x)(y) = \overline{m}(x, y) f(x, y).$$

Obviously, $f \to f'$ is an isometric isomorphism of Banach spaces. Some easy computations show that this map is in fact an $\star$-morphism.

This lemma shows that $C^*(G_\rho/p^{-1}(K), m)$ is isomorphic to the C^*-hull of $L^1(F \oplus V, L^1(Y, m_2), m_1, T)$ or to the C^*-hull of $L^1(F \oplus V, C^*(Y, m_2), m_1, T)$.

But since $\varphi|_{Y^2}$ is a non-degenerate antisymmetric bicharacter the Stone-von Neumann theorem tells us that the group Y has precisely one (up to equivalence) irreducible m_2-projective representation σ; m_2-projective means that $\sigma(y_1)\sigma(y_2)=m_2(y_1, y_2)\sigma(y_1+y_2)$. Using σ we may identify $C^*(Y, m_2)$ with the algebra $\mathcal{K}(\mathcal{H})$ of compact operators in the representation space $\mathcal{H}$ of σ. We have to compute how the action T of $H=F\oplus V$ on $L^1(Y, m_2)$ transforms under this identification, i.e. we are looking for an homomorphism T′ from H into the automorphism group of $\mathcal{K}(\mathcal{H})$ such that the diagram

$$\begin{tikzcd} L^1(Y, m_2) \arrow[r, "\sigma"] \arrow[d, "T_a"] & \mathcal{K}(\mathcal{H}) \arrow[d, "T'_a"] \\ L^1(Y, m_2) \arrow[r, "\sigma"] & \mathcal{K}(\mathcal{H}) \end{tikzcd}$$

commutes for all $a\in H$, where, of course, σ denotes the integrated form of the projective group representation. This is not hard. Since $\varphi|_{Y^2}$ is non-degenerate there exists a unique continuous homomorphism $R: H\to Y$ such that for given $a\in H$ the equation $\varphi(a, y)=\varphi(R(a), y)$ holds for all $y\in Y$. By the way, on the connected component V of H the homomorphism R is trivial because V is the kernel of $\varphi|_{W^2}$. For $a\in H$ we put $U_a:=\sigma(R(a))$ and define $T'_a:\mathcal{K}(\mathcal{H})\to\mathcal{K}(\mathcal{H})$ by $T'_a B=U_a B U_a^{-1}$. Let $f\in L^1(Y, m_2)$. We have to show that

$$U_a\sigma(f)U_a^{-1}=\sigma(T_a f).$$

From

$$\sigma(x)\sigma(y)=m_2(x, y)\sigma(x+y)=m_2(x, y)\sigma(y+x)=m_2(x, y)\overline{m}_2(y, x)\sigma(y)\sigma(x)$$

it follows that

$$\sigma(x)\sigma(y)\sigma(x)^{-1}=\varphi(x, y)\sigma(y) \quad \text{for all } x, y\in Y.$$

Using this fact one gets

$$\sigma(T_a f)=\int_Y f(y)\varphi(a, y)\sigma(y)\,dy=\int_Y f(y)\varphi(R(a), y)\sigma(y)\,dy$$
$$=\int_Y f(y)U_a\sigma(y)U_a^{-1}\,dy=U_a\sigma(f)U_a^{-1}.$$

For later use, we note that $U_a U_b=r(a, b)U_{a+b}$ for $a, b\in H$ if $r: H^2\to\mathbb{T}$ is defined by $r(a, b)=m_2(R(a), R(b))$; this follows immediately from the definition of U. From these considerations it follows that $C^*(G_\rho/p^{-1}(K), m)$ is isomorphic to the C*-hull of $L^1(H, \mathcal{K}(\mathcal{H}), m_1, T')$ where the action T′ of $H=F\oplus V$ is given by conjugation with the unitary operators U_a. Let the cocycle $w: H\times H\to\mathbb{T}$ be defined by $w(a,b)=m_1(a,b)\overline{r}(a,b)=m_1(a,b)\overline{m}_2(R(a),R(b))$, and form the algebra $L^1(H,\mathcal{K}(\mathcal{H}),w)$ where H acts trivially on $\mathcal{K}(\mathcal{H})$. For a continuous, compactly supported function $f: H\to\mathcal{K}(\mathcal{H})$ let $f': H\to\mathcal{K}(\mathcal{H})$ be defined by $f'(a)=U_a f(a)$; f' is also continuous and, of course, compactly supported. The map $f\to f'$ extends to an isometry from the

Banach space $L^1(H, \mathscr{K}(\mathscr{H}))$ onto itself and gives an $\star$-isomorphism from the Banach algebra $L^1(H, \mathscr{K}(\mathscr{H}), m_1, T')$ onto $L^1(H, \mathscr{K}(\mathscr{H}), w)$. Hence $C^*(G_\rho/p^{-1}(K), m)$ is isomorphic to $\mathscr{K}(\mathscr{H})\otimes C^*(H, w)$. The latter algebra is "better" than the former in so far as $w(a, b)=w(b, a)$ for a, b in the connected component V of H, i. e. the connected component in the central group extension corresponding to w is commutative.

Remark 1. — In the preceding part (in contrary to the following) of the proof we have not used that $C^*(G_\rho/p^{-1}(K), m)$ is primitive. In fact, what we have shown is the following: Let m be any (continuous) cocycle on $X\oplus W$, where X is any locally compact abelian group and W is vector group. Define $\varphi : X\oplus W \times X\oplus W \to \mathbb{T}$ by $\varphi(x, y)=m(x, y)\overline{m}(y, x)$, let V be the kernel of $\varphi|_{W^2}$, and let Y be any vector space complement to V in W. The homomorphism $R : X\oplus V \to Y$ is defined by requiring that $\varphi(a, y)=\varphi(R(a), y)$ holds for all $a\in X\oplus V$ and all $y\in Y$; R is trivial on V. Then $C^*(X\oplus W, m)$ is isomorphic to the tensor product of $C^*(X\oplus V, w)$ and $\mathscr{K}(\mathscr{H})$ where $w : X\oplus V\times X\oplus V \to \mathbb{T}$ is given by $w(a, b)=m(a, b)\overline{m}(R(a), R(b))$ and where $\mathscr{K}$ is equal to $\mathbb{C}$ (in case that $Y=0$) or to an infinite-dimensional separable Hilbert space.

In the next (and last) step we show that it is possible to reduce the dimension of V by splitting off an algebra of compact operators. Of course, this step is superfluous if $V=0$. So, let's assume that $\dim V>0$. The antisymmetric bicharacter $\Psi : H\times H\to\mathbb{T}$ is defined by $\Psi(a, b)=w(a, b)\overline{w}(b, a)$. The group H can be identified with $F\oplus V$. We choose a subgroup F_0 of rank one in F which admits a complement F' in F, i. e. $F=F'\oplus F_0$, and with the property that the restriction of Ψ to $(F_0\oplus V)^2$ is not trivial. Of course, such a subgroup F_0 exists: if not, $L^1(V, w)$ would be central in $L^1(H, w)$ and in $C^*(H, w)$, in contradiction to the facts that $C^*(H, w)$ is primitive and $\dim V>0$. Since Ψ is not trivial on $(F_0\oplus V)^2$ and since F' is a free group there exists a homomorphism $t : F'\to V$ with

$$\Psi((x, 0), (y, 0))=\overline{\Psi}((0, t(x)), (y, 0)) \quad \text{for all } x\in F', \quad y\in F_0.$$

Let $F_1 := \{(x, t(x))\,|\,x\in F'\}\leqq F\oplus V=H$. Then H can be identified with $F_1\oplus F_0\oplus V$ such that Ψ is trivial on $(\{0\}\oplus\{0\}\oplus V)^2$ and on $(F_1\oplus\{0\}\oplus\{0\})\times(\{0\}\oplus F_0\oplus\{0\})$, and Ψ is not trivial on $(\{0\}\oplus F_0\oplus V)^2$. Let $X := F_0\oplus V\leqq H$. By Lemma 1, $L^1(H, w)$ is isomorphic to $L^1(F_1, L^1(X, w_0), w_1, T)$ where w_0, w_1 are the restrictions of w to X and F_1, respectively, and the action T of F_1 on $L^1(X, w_0)$ is given by

$$(T_a f)(x)=\chi_a(x)f(x) \qquad \text{for} \quad a\in F_1, \quad f\in L^1(X, w_0)$$

where $\chi_a\in\hat{X}=(F_0\oplus V)\hat{}$ is defined by $\chi_a(y, v)=\Psi((a, 0, 0), (0, y, v))$. From the normalizations above it follows that χ_a is trivial on $F_0\oplus\{0\}$. From the fact that Ψ is trivial on V^2 one deduces very easily that there exists a bicharacter $\mu : F_0\times V\to\mathbb{T}$ such that w_0 is equivalent to the cocycle $Q : X\times X\to\mathbb{T}$ given by $Q(y\oplus v, z\oplus w)=\mu(y, w)$. Since Ψ in non-trivial on X^2, μ has to be non-trivial. Hence we may substitute $L^1(X, w_1)$ by the isomorphic algebra $L^1(X, Q)$, and we find that $L^1(H, w)$ is isomorphic to $L^1(F_1, L^1(X, Q), w_1, T)$ where the action T of F_1 on $L^1(X, Q)$ is given by the same formula as above because the canonical isomorphism from $L^1(X, w_1)$ onto $L^1(X, Q)$ commutes with multiplication by characters. Let $M := \{x\in X;\ Q(y, x)=Q(y,x) \text{ for all } y\in X\}$ or $M=\{0\oplus v;\ v\in V,$

$\mu(b, v)=1$ for all $b \in F_0\}$; so we may consider M as a part of V. It is easy to see that the primitive ideal space of $L^1(X, Q)$ [or of $C^*(X, Q)$] is homeomorphic to $\hat{M}$, and the general theory (Dixmier-Douady) predicts (the third cohomology group of $\hat{M}$ is zero) that $C^*(X, Q)$ is isomorphic to $C_\infty(\hat{M}, \mathcal{K}(\mathcal{H}))$. But since we need the action T on $C_\infty(\hat{M}, \mathcal{K}(\mathcal{H}))$ and since it is not very difficult we prefer to construct such an isomorphism. Denote by $p: V \to V/M (\cong \mathbb{T})$ the quotient morphism and choose a measurable cross section $s: V/M \to V$ to p. For a character $\eta \in \hat{M}$ we define the (Q-projective) representation π_η of X in $\mathcal{H} := L^2(V/M)$ by

$$\{\pi_\eta(b \oplus w)\xi\}(x)=\mu(b, s(x)-w)\,\eta(-s(x)+w+s(x-p(w)))\,\xi(x-p(w))$$

for $b \oplus w \in X$, $\xi \in \mathcal{H}$, $x \in V/M$. π_η is (up to equivalence) the unique Q-projective representation of X which is, restricted to M, a multiple of η. From the definition of π_η it is not completely obvious that π_η is a continuous representation. We postpone this question. First we construct some intertwining operators which will be needed later and which also help to prove the continuity of π_η. Let χ be a character of V; χ is also considered as a character of $X=F_0 \oplus V$ by putting $\chi(b \oplus v)=\chi(v)$. Let the unitary operator U_χ in $\mathcal{H}$ be defined by $(U_\chi \xi)(x)=\chi(s(x))\,\xi(x)$. Then U_χ is an intertwining operator between $\chi\pi_\eta$ and $\pi_{\eta \cdot \chi|_M}$, i. e. if we put $\eta' := \eta\chi|_M$ we have:

$$\chi(v)\,\pi_\eta(b \oplus v)=U_\chi\,\pi_{\eta'}(b \oplus v)\,U_\chi^{-1}$$

for all $b \in F_0$, $v \in V$, which can be easily verified. This formula shows that it suffices to prove the continuity of π_η only for the trivial character η, but in this case it is obvious. Moreover, for $f \in L^1(X, Q)$ the operator $\pi_\eta(f)$ is compact and the map $\eta \to \pi_\eta(f) =: (\mathcal{G} f)(\eta)$ from $\hat{M}$ into $\mathcal{K}(\mathcal{H})$ is continuous and vanishes at infinity. Hence we have obtained an injective $\star$-morphism $\mathcal{G}$ from $L^1(X, Q)$ into $C_\infty(\hat{M}, \mathcal{K}(\mathcal{H}))$ which can be extended to an isomorphism from $C^*(X, Q)$ onto $C_\infty(\hat{M}, \mathcal{K}(\mathcal{H}))$. Again we have to compute how the action T of F_1 on $L^1(X, Q)$ transforms under $\mathcal{G}$, i. e. we are looking for an automorphism T'_a, $a \in F_1$, on $C_\infty(\hat{M}, \mathcal{K}(\mathcal{H}))$ making the diagram

$$\begin{array}{ccc} L^1(X, Q) & \xrightarrow{\mathcal{G}} & C_\infty(\hat{M}, \mathcal{K}(\mathcal{H})) \\ \Big\downarrow {\scriptstyle T_a} & & \Big\downarrow {\scriptstyle T'_a} \\ L^1(X, Q) & \xrightarrow{\mathcal{G}} & C_\infty(\hat{M}, \mathcal{K}(\mathcal{H})) \end{array}$$

commute. For $f \in L^1(X, Q)$ we have

$$(\mathcal{G} T_a f)(\eta)=\int_X (T_a f)(x)\,\pi_\eta(x)\,dx=\int_X \chi_a(x)\,f(x)\,\pi_\eta(x)\,dx=\int_X f(x)\,U_a\,\pi_{\eta \cdot \chi_a|_M}(x)\,U_a^{-1}\,dx$$

where we have put (by a slight abuse of notation) $U_a := U_{\chi_a}$. Hence $(\mathcal{G} T_a f)(\eta)=U_a(\mathcal{G} f)(\eta\chi_{a|_M})U_a^{-1}$. Therefore, the above diagram commutes if $T'_a : C_\infty(\hat{M}, \mathcal{K}(\mathcal{H})) \to C_\infty(\hat{M}, \mathcal{K}(\mathcal{H}))$ is given by $(T'_a g)(\eta)=U_a g(\eta\chi_{a|_M})U_a^{-1}$. Define the action S of F_1 on $C_\infty(M)=C_\infty(\hat{M}, \mathbb{C})$ by $(S_a g)(\eta)=g(\eta\chi_{a|_M})$. $C^*(H, w)$ is the

C*-hull of $L^1(F_1, C_\infty(\hat{M}, \mathcal{K}(\mathcal{H})), w_1, T')$. From the fact that $a \to U_a$ is an ordinary representation, i.e. $U_a U_b = U_{a+b}$ for $a, b \in F_1$, one deduces very easily that $C^*(H, w)$ is isomorphic to the tensor product of $\mathcal{K}(\mathcal{H})$ and the C*-hull of $L^1(F_1, C_\infty(\hat{M}), w_1, S)$. Using the Fourier transform $\mathcal{F} : L^1(M) \to C_\infty(\hat{M})$ of the abelian group M, i.e. $(\mathcal{F} f)(\eta) = \int_M f(m)\eta(m)\,dm$, we consider $C_\infty(\hat{M})$ as the C*-hull of $L^1(M)$. If the action S′ of F_1 on $L^1(M)$ is defined by $(S'_a f)(m) = \chi_a(m) f(m)$ the diagram

$$\begin{tikzcd} L^1(M) \arrow[r, "\mathcal{F}"] \arrow[d, "S'_a"] & C_\infty(\hat{M}) \arrow[d, "S_a"] \\ L^1(M) \arrow[r, "\mathcal{F}"] & C_\infty(\hat{M}) \end{tikzcd}$$

commutes for all $a \in F_1$.

Hence the C*-hull of $L^1(F_1, C_\infty(\hat{M}), w_1, S)$ is isomorphic to the C*-hull of $L^1(F_1, L^1(M), w_1, S')$. Let $P : (F_1 \oplus M)^2 \to \mathbb{T}$ be defined by

$$P(a \oplus v, b \oplus w) = w_1(a, b)\chi_a(w) = w_1(a, b)\psi(a, w).$$

Then P is a cocycle on $F_1 \oplus M$, and by Lemma 1, $L^1(F_1 \oplus M, P)$ is isomorphic to $L^1(F_1, L^1(M), w_1, S')$. Hence $C^*(H, w)$ is isomorphic to $\mathcal{K}(\mathcal{H}) \otimes C^*(F_1 \oplus M, P)$. But the dimension of the connected component M_0 of the identity in M is equal to $\dim V - 1$ while the « discrete rank » of $F_1 \oplus M$, i.e. the rank of the free abelian group $F_1 \oplus M/M_0$, is equal to the rank of H/V. Repeating this procedure finitely often we obtain that $C^*(H, w)$ is isomorphic to $\mathcal{K}(\mathcal{H}) \otimes C^*(N, q)$ where $\mathcal{H}$ is $\mathbb{C}$ (in case that $V=0$) or a separable infinite-dimensional Hilbert space, N is a free abelian group of the same rank as H/V and q is a cocycle on N. Since $C^*(H, w)$ is primitive the cocycle q has to be nondegenerate.

The proof of the theorem gives a little bit more information than stated in the theorem. Especially, the proof shows when the factor $\mathcal{K}(\mathcal{H})$ is finite-dimensional and when the primitive quotient is of type I.

COROLLARY 1. — $C^*(G)/\ker \operatorname{ind}_{Z_\lambda}^G \mu$ *is isomorphic to the tensor product of a finite-dimensional matrix algebra and an algebra of the type* $C^*(N, q)$ *iff* G_ρ *is of finite index in* G *and* $G/p^{-1}(K)$ *is discrete. In this case the matrix algebra is the algebra of matrices in a space of dimension* $|G/G_\rho| \dim \rho$.

COROLLARY 2. — $C^*(G)/\operatorname{Ker} \operatorname{ind}_{Z_\lambda}^G \mu$ *is of type I iff* $G_\rho/p^{-1}(K)$ *is a vector group.*

Remark 2. — After establishing Theorem 1 it is very natural to put the following question: to what extent are the data F, ν, $\mathcal{H}$ determined by a given primitive ideal (quotient)? In other words: when are $C^*(F, \nu) \otimes \mathcal{K}(\mathcal{H})$ and $C^*(F', \nu') \otimes \mathcal{K}(\mathcal{H}')$ isomorphic? The best general result in this direction is due to Elliott [5], who computed — using the exact sequence of Pimsner-Voiculescu — the K-groups of these algebras. It turned out that $K_0(C^*(F, \nu)) \oplus K_1(C^*(F, \nu))$ is — as a group — isomorphic to the exterior algebra of F, which shows that the rank of F (and hence F) is an invariant of $C^*(F, \nu) \otimes \mathcal{K}(\mathcal{H})$

and, therefore, of a primitive ideal in $C^*(G)$. The cocycle ν (or even its cohomology class) is not uniquely determined, although perhaps its conjugacy class under $GL(F)$ is. (Rieffel proves this [20, Thm. 4] when $rk\, F=2$.) It is not clear if dim $\mathcal{H}$ is an invariant when it is finite, although the finite – dimensional and the infinite-dimensional cases can clearly be distinguished on the basis of whether or not the C*-algebra has a unit.

Part II: Simple subquotients of $C^*(G)$, G a connected Lie group

Let G be a connected Lie group, and let $\mathcal{I}$ be a primitive ideal in $C^*(G)$. By a result of Moore and Rosenberg [15], the primitive quotient $C^*(G)/\mathcal{I}$ contains a unique non-zero simple closed ideal $\mathcal{M}=\mathcal{M}_{\mathcal{I}}$. Green has shown in [8] that $\mathcal{M}$ is either finite-dimensional [and then, of course, $\mathcal{M}=C^*(G)/\mathcal{I}$] or a stable algebra. In the same paper, Green has raised the problem to determine the structure of $\mathcal{M}$ more explicitly. It is the purpose of this part to solve this problem, see the theorem below. By the way, the proof of this theorem gives also the results of Moore-Rosenberg and Green mentioned above. Besides Green's isomorphism theorem for twisted covariance algebras, the main tools will be Dixmier's theorem that locally algebraic groups are of type I and Pukanszky's theorem on the orbits of an algebraic group in the dual of a locally algebraic group. That is not surprising because these deep theorems are definitely the most powerful tools in dealing with non type I Lie groups as can be seen in many papers of Pukanszky.

THEOREM 2. – *Let* G *be a connected Lie group. Then every primitive quotient of* $C^*(G)$ *contains a unique simple closed ideal* $\mathcal{M}$ *which is either isomorphic to a finite-dimensional matrix algebra or to* $\mathcal{K}(\mathcal{H})\otimes C^*(F,\nu)$ *where* $\mathcal{H}$ *is an infinite-dimensional (separable) Hilbert space, where* F *is a free abelian group of finite rank (including zero), and* ν *is a non-degenerate cocycle on* F.

COROLLARY (Pukanszky, Green). – *Every primitive ideal of* $C^*(G)$ *is the kernel of a unique (up to quasi-equivalence) normal representation.*

Proof. – The proof of this theorem will be lengthy. To make it better readable I have divided it into sections with descriptive titles; these titles are only understandable if one knows the notations introduced during the proof.

First we note that the uniqueness of $\mathcal{M}$ is trivial: a primitive C*-algebra never contains two different simple closed ideals.

We suppose that G is simply connected. Let $\mathfrak{g}$ be the Lie algebra of G. Following Pukanszky we choose a faithful representation of $\mathfrak{g}$ in $\mathfrak{gl}(W)$ for some finite-dimensional real vector space W (theorem of Ado) and identify $\mathfrak{g}$ with its image. Let $\tilde{\mathfrak{g}}$ be the smallest algebraic Lie subalgebra of $\mathfrak{gl}(W)$ containing $\mathfrak{g}$. It is known that $[\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}]=[\mathfrak{g},\mathfrak{g}]$ is an algebraic Lie algebra. Let $\mathfrak{n}$ be any algebraic Lie algebra between $[\mathfrak{g},\mathfrak{g}]$ and $\mathfrak{g}$, and let N be the corresponding analytic (normal) subgroup of G. Moreover, let L be the analytic subgroup of $GL(W)$ corresponding to $\tilde{\mathfrak{g}}$, L is the connected component of a real algebraic group. There is a canonical homomorphism $G\to L$, and L acts by automorphisms on $\mathfrak{g}$, $\mathfrak{n}$, G and N and then also on Prim(N). Denote the homomorphism $L\to \mathrm{Aut}(G)$ by α; usually the argument will be written as a subscript. By a theorem of Dixmier [4], N is a group of type I. Hence $\hat{N}$ can be identified with Prim (N).

Let $\mathscr{I}$ be a primitive ideal in $C^*(G)$. Let $\mathscr{J}$ be the "restriction" of $\mathscr{I}$ to $C^*(N)$, i. e. $\mathscr{J} := \{ x \in C^*(N);\ x\, C^*(G) \subseteq \mathscr{I} \}$ where $C^*(G)$ is considered as a module over $C^*(N)$, and let X be the hull of $\mathscr{J}$ in Prim (N) (or in $\hat{N}$). By the famous theorem of Pukanszky [17], there exists an ρ in X such that X is contained in the closure of $L\rho$ in $\hat{N}$ and $L\rho$ is locally closed. Let $\mathscr{J}_1$ and $\mathscr{J}_2$ be the kernels of the closed sets $(L\rho)^- \setminus L\rho$ and $(L\rho)^-$, respectively.

Reduction to compact orbits

$C^*(G)$ is considered as $C^*(G, C^*(N), \tau)$ in the sense of Green's papers. The quotient map $C^*(G, C^*(N), \tau) \to C^*(G)/\mathscr{I}$ factorizes through $C^*(G, C^*(N)/\mathscr{J}_2, \tau)$ and gives a quotient map $\pi : C^*(G, C^*(N)/\mathscr{J}_2, \tau) \to C^*(G)/\mathscr{I}$ which is non zero on the ideal $C^*(G, \mathscr{J}_1/\mathscr{J}_2, \tau)$. The dual space $(\mathscr{J}_1/\mathscr{J}_2)\hat{}$ of the type I algebra $\mathscr{J}_1/\mathscr{J}_2$ is canonically homeomorphic to the connected abelian Lie group L/L_ρ where L_ρ denotes the stabilizer of ρ in L. And the G-action on $(\mathscr{J}_1/\mathscr{J}_2)\hat{}$ corresponds to translation via the homomorphism $q : G \to L/L_\rho$ which is the composition of $G \to L$ and $L \to L/L_\rho$. Denote by T the maximal compact subgroup in L/L_ρ, and let $G_1 := q^{-1}(T)$. The homomorphism q induces an injective homomorphism $\dot{q}$ from G/G_1 into $(L/L_\rho)/T$ which is a vector group. Hence G/G_1 is a vector group, too. Consequently, G_1 is connected. Moreover, $\dot{q}$ admits an inverse, i. e. there is a continuous homomorphism $s : (L/L_\rho)/T \to G/G_1$ with $s \circ \dot{q} = \mathrm{id}_{G/G_1}$. The composition of the quotient map from $L/L_\rho (= (\mathscr{J}_1/\mathscr{J}_2)\hat{}\,)$ onto $(L/L_\rho)/T$ and s gives a continuous G-equivariant map p from $(\mathscr{J}_1/\mathscr{J}_2)\hat{}$ onto G/G_1. There are two cases, namely that G_1 is a proper subgroup of G and that $G = G_1$.

Let's first assume that G_1 is a proper subgroup of G. Denote by $\mathscr{J}_0/\mathscr{J}_1$ the kernel of $p^{-1}(e\,G_1)$ in $\mathscr{J}_1/\mathscr{J}_2$. By part (i) of Green's isomorphism theorem, $C^*(G, \mathscr{J}_1/\mathscr{J}_2, \tau)$ is isomorphic to $\mathscr{K}(L^2(G/G_1)) \otimes C^*(G_1, \mathscr{J}_1/\mathscr{J}_0, \tau)$. Hence the primitive algebra $\pi[C^*(G, \mathscr{J}_1/\mathscr{J}_2, \tau)]$ [recall that this algebra is a non-zero ideal in $C^*(G)/\mathscr{I}$] is a quotient of $\mathscr{K}(L^2(G/G_1)) \otimes C^*(G_1, \mathscr{J}_1/\mathscr{J}_0, \tau)$. But $C^*(G_1, \mathscr{J}_1/\mathscr{J}_0, \tau)$ is an ideal in $C^*(G_1, C^*(N)/\mathscr{J}_0, \tau)$ which is a quotient of $C^*(G_1, C^*(N), \tau) \cong C^*(G_1)$. Since we may (inductively) assume that the theorem is true for the group G_1, the assertion of the theorem follows for the primitive quotient $C^*(G)/\mathscr{I}$. In this case, $C^*(G)/\mathscr{I}$ is infinite-dimensional, i. e. the second possibility occurs.

Construction of the compact group K

So, let's assume that $G = G_1$, i. e. that the closure of $q(G)$ in L/L_ρ is compact. First, we show that there exists a compact connected abelian subgroup K of L (i. e. a torus) which is mapped under $L \to L/L_\rho$ onto $q(G)^-$ such that $K \to q(G)^-$ has a finite kernel. This will be a consequence of the fact that L_ρ has only finitely many connected components (*see* Lemma 22 in [19]) and the following lemma which is certainly known but in Hochschild's book I found only a slightly weaker version which will be used in the proof.

LEMMA 2. — *Let* Y *be a connected Lie group, and let* Z *be a closed normal subgroup of* Y *with finitely many connected components. Then any maximal compact subgroup of* Y *is mapped under the quotient morphism* $Y \to Y/Z$ *onto one of the maximal compact subgroups of* Y/Z.

Proof. — Let Z_0 be the identity component of Z, let $p : Y \to Y/Z$ be the quotient morphism, let M be a maximal compact subgroup of Y, let T be a maximal compact subgroup of Y/Z with $p(M) \subseteq T$, and let $H := p^{-1}(T)$. Since Y/H is homeomorphic to $(Y/Z)/T$ which is homeomorphic to a euclidean space and in particular simply connected, it follows from the exact homotopy sequence for the fibration $H \to Y \to Y/H$ that H is connected. Moreover, as Z/Z_0 is finite, H/Z_0 is a finite covering of T and so is compact. From Theorem 3.7 in Chapt. XV of [9], p. 186, we get that $H = MZ_0$ and consequently $p(M) = T$.

Now, let T be the maximal compact subgroup of the connected abelian group L/L_ρ. Lemma 2 implies that there exists a compact (connected) subgroup M of L which is mapped onto T. From the known structure of compact connected Lie groups (*see* for instance [9], chapt XIII, Thm. 1.3, p. 144) it follows that the connected component $Z_0(M)$ in the center of M is mapped onto T. It is easy to see that there exists a subtorus K in $Z_0(M)$ which is mapped onto $q(G)^- \subseteq T$ such that the kernel of $K \to q(G)^-$ is finite. Since $K \subseteq L$ acts by automorphisms on $\mathfrak{g}$ and on G we may form the semi-direct product $H := K \ltimes G$ which will be used later.

Construction of $\mathscr{J}_3$ and realization of $C^*(G, \mathscr{J}_1/\mathscr{J}_3, \tau)$ in the "Leinert picture"

Recall that we want to compute the image of $\pi : C^*(G, \mathscr{J}_1/\mathscr{J}_2, \tau) \to C^*(G)/\mathscr{J}$. By a theorem of Dixmier, *see* [3], the algebra $\mathscr{C}$ of bounded continuous functions on $(\mathscr{J}_1/\mathscr{J}_2)\hat{} \cong L/L_\rho$ acts canonically on $\mathscr{J}_1/\mathscr{J}_2$; in fact the center of the adjoint algebra $(\mathscr{J}_1/\mathscr{J}_2)^b$ may be identified with $\mathscr{C}$. $\mathscr{C}$ may also be considered as part of the adjoint algebra of $C^*(G, \mathscr{J}_1/\mathscr{J}_2, \tau)$, and the algebra of G-fixpoints $\mathscr{C}^G$ [corresponding to the bounded continuous functions on $(L/L_\rho)/q(G)^-$] is central in this adjoint algebra. From this and the fact that ker π is a primitive ideal it follows that $\pi : C^*(G, \mathscr{J}_1/\mathscr{J}_2, \tau) \to C^*(G)/\mathscr{J}$ factorizes through

$$C^*(G, \mathscr{J}_1/\mathscr{J}_2, \tau) \to C^*(G, \mathscr{J}_1/\mathscr{J}_3, \tau)$$

where $\mathscr{J}_3/\mathscr{J}_2$ is the kernel in $\mathscr{J}_1/\mathscr{J}_2$ of the closure of $G\rho$ in $(\mathscr{J}_1/\mathscr{J}_2)\hat{}$. The dual space of $\mathscr{J}_1/\mathscr{J}_3$ can be identified with the closure of $G\rho$ in $(\mathscr{J}_1/\mathscr{J}_2)\hat{}$ which is equal to $K\rho$. Now, we will study the algebra $C^*(G, \mathscr{J}_1/\mathscr{J}_3, \tau)$ in more detail. In fact, we will show that all its primitive quotients are simple and either finite-dimensional or stable and stably isomorphic to an algebra of the form $C^*(F, \nu)$ (as stated in the theorem).

To this end, we realize $C^*(G, \mathscr{J}_1/\mathscr{J}_3, \tau)$ in the "Leinert picture", i.e. as the C*-hull of a convolution algebra of functions on $V := G/N$ with values in $\mathscr{J}_1/\mathscr{J}_3$. Let $\delta : G \to \mathbb{R}$ be the modular function of the action of G on N, i.e.

$$\int_N f(n)\,dn = \delta(x) \int_N f(x^{-1} n x)\,dn \qquad \text{for} \quad x \in G, \quad f \in L^1(N).$$

We fix a continuous cross section σ from $V=G/N$ into G with $\sigma(e)=e$ and $\sigma(x^{-1})=\sigma(x)^{-1}$ (this makes the formulas a little bit simpler). Then $\gamma: V\times V\to N$ is defined by $\gamma(x, y)=\sigma(y)^{-1}\sigma(x)^{-1}\sigma(xy)$. For $x, y\in V$ we define $P_{x, y}: L^1(N)\to L^1(N)$ by $(P_{x, y}f)(n)=f(\gamma(x, y)n)$ and $T_x: L^1(N)\to L^1(N)$ by:

$$(T_x f)(n)=\delta(\sigma(x))\, f(\sigma(x)^{-1}\, n\, \sigma(x)).$$

The operators $P_{x, y}$ and T_x can be extended to $C^*(N)$ and give operators on the subquotient $\mathscr{J}_1/\mathscr{J}_3$, denoted by the same letters. The C^*-hull of $L^1(V, \mathscr{J}_1/\mathscr{J}_3, P, T)$ is isomorphic to $C^*(G, \mathscr{J}_1/\mathscr{J}_3, \tau)$.

Realization of $\mathscr{J}_1/\mathscr{J}_3$ as an algebra of functions

In the next step, $\mathscr{J}_1/\mathscr{J}_3$ is realized as an algebra of continuous functions from K into the algebra of compact operators. To this end, we fix a representative in the equivalence class $\rho\in C^*(N)\hat{}$. Denote this representative also by the letter ρ, and let $\mathscr{H}$ be the representation space of ρ. ρ is also considered as a representation of $\mathscr{J}_1$ and of $\mathscr{J}_1/\mathscr{J}_3$, and we note that $\rho(\mathscr{J}_1)$ is equal to the algebra $\mathscr{K}(\mathscr{H})$ of compact operators on $\mathscr{H}$. For $k\in K$, let the representation ρ_k of N [and then of $C^*(N)$, $\mathscr{J}_1$, $\mathscr{J}_1/\mathscr{J}_3$] be defined by $\rho_k(n)=\rho(\alpha_k(n))$, i. e. $\rho_k=k^{-1}\rho$ [recall that $\alpha: L\to \mathrm{Aut}\ (G)$, and each α_x transforms N into itself]. For $f\in\mathscr{J}_1/\mathscr{J}_3$ the function $\mathscr{G}f: K\to\mathscr{K}(\mathscr{H})$ is defined by $(\mathscr{G}f)(k)=\rho_k(f)$. Since K acts strongly continuously on $C^*(N)$, $\mathscr{G}f$ is a continuous function. Moreover $\mathscr{G}: \mathscr{J}_1/\mathscr{J}_3\to C(K, \mathscr{K})$ is an injective morphism of C^*-algebras, where the latter space is equipped with the pointwise operations and the sup-norm. We have to determine the image of $\mathscr{G}$ and the transformed operations P and T. At this point, the group $H=K\ltimes G$ will be useful. Recall that the multiplication in H is given by $(k_1, g_1)(k_2, g_2)=(k_1 k_2, \alpha_{k_2^{-1}}(g_1)g_2)$. First we compute the stabilizer H_ρ in H of (the equivalence class of) ρ in $\hat{N}$. To the finite covering $K\to q(G)^-\subseteq L/L_\rho$ there exists a unique homomorphism $w: G\to K$ such that

$$\begin{tikzcd} & K \arrow[d] \\ G \arrow[ur, "w"] \arrow[r, "q"'] & q(G)^- \end{tikzcd}$$

commutes. The image of w is dense in K, and N is, of course, contained in the kernel of w. Hence we will also consider w as an homomorphism from $V=G/N$ into K. Moreover, for $x\in G$ the representation $x\rho$, given by $(x\rho)(n)=\rho(x^{-1}nx)$ is equivalent to $w(x)\rho$, given by $(w(x)\rho)(n)=\rho(\alpha_{w(x)^{-1}}(n))$. One computes that $H_\rho=\{(k, x)\in H;\ kw(x)\in K_\rho\}$ with $K_\rho := K\cap L_\rho$ which is a finite group. Since H_ρ/N is isomorphic to the direct product of G/N and K_ρ, every cocycle on H_ρ/N is equivalent to a continuous cocycle. Therefore, there exists a continuous cocycle m on H_ρ (living on H_ρ/N)

and a continuous m-projective representation U of H_ρ in $\mathscr{H}$ extending ρ; especially $U_{h_1} U_{h_2} = m(h_1, h_2) U_{h_1 h_2}$, $\rho(n) = U_n$, and $\rho(hnh^{-1}) = U_h \rho(n) U_h^*$ for h, h_1, $h_2 \in H_\rho$ and $n \in N$. Later, we will use the continuous function $r: K \times G \to H$ defined by $r(k, x) = (w(x)^{-1}, \alpha_{kw(x)}(x))$. Since $\alpha_{kw(x)}(x)$ is congruent to x modulo N, $w(x)$ is equal to $w(\alpha_{kw(x)}(x))$ and, therefore, the values of r are in H_ρ. Moreover, every element $(k, x) \in H$ is equal to $r(k, x)(kw(x), e)$, i. e. (k, x) can be decomposed into a product of an element of H_ρ and an element of K.

From the definitions it follows very easily that $(\mathscr{G} f)(kd) = U_d \, \mathscr{G} f(k) U_d^*$ for $k \in K$, $d \in K_\rho$ and $f \in \mathscr{J}_1/\mathscr{J}_3$. Therefore, the image of $\mathscr{G}$ is contained in

$$\mathscr{E} := \{ \varphi \in C(K, \mathscr{K}(\mathscr{H})) ; \varphi(kd) = U_d \varphi(k) U_d^* \text{ for } k \in K, d \in K_\rho \}.$$

From 11.1.6 (or even 11.1.4) in [2] it follows that the image of $\mathscr{G}$ coincides with $\mathscr{E}$; notice that there is a canonical bijection between $\mathscr{E}^\wedge$ and $(\mathscr{J}_1/\mathscr{J}_3)^\wedge$.

If for $x \in G/N = V$, the operator $T'_x : \mathscr{E} \to \mathscr{E}$ is defined by

$$(T'_x \varphi)(k) = U_{r(k, \sigma(x))} \varphi(kw(x)) U^*_{r(k, \sigma(x))};$$

an easy calculation shows that $T'_x \circ \mathscr{G} = \mathscr{G} \circ T_x$.

For x, $y \in V$ we define $P'_{x, y} : \mathscr{E} \to \mathscr{E}$ by

$$(P'_{x, y} \varphi)(k) = \rho(k \gamma(x, y)^{-1} k^{-1}) \varphi(k).$$

Then one proves the equation $P'_{x, y} \circ \mathscr{G} = \mathscr{G} \circ P_{x, y}$. Altogether, this shows that the C*-hull of $L^1(V, \mathscr{J}_1/\mathscr{J}_3, P, T)$ is isomorphic to the C*-hull of $L^1(V, \mathscr{E}, P', T')$.

Resolution of P′ and T′

For $f \in L^1(V, \mathscr{E})$, considered as a function from $V \times K$ in $\mathscr{K}(\mathscr{H})$, we define $\tilde{f} : V \times K \to \mathscr{K}(\mathscr{H})$ by

$$\tilde{f}(x, k) = U^*_{r(k, \sigma(x)^{-1})} f(x, k).$$

To prove continuity properties of $\tilde{f}$ we use the following lemma; its simple proof is omitted.

LEMMA 3. — *Let $\mathscr{K}(\mathscr{H})$ be the normed space of compact operators on the Hilbert space $\mathscr{H}$, let Y be a topological space, and let K be a compact space.*

(i) *If U is a strongly continuous map from Y into the group of unitaries on $\mathscr{H}$ and $\varphi : Y \to \mathscr{K}(\mathscr{H})$ is continuous then $y \to U(y) \varphi(y)$ is a continuous function from Y into $\mathscr{K}(\mathscr{H})$.*

(ii) *If U is a strongly continuous map from $Y \times K$ into the group of unitaries on $\mathscr{H}$ and h is a continuous map from Y into the normed (with sup-norm) space $\mathscr{C}(K, \mathscr{K}(\mathscr{H}))$ of continuous functions from K into $\mathscr{K}(\mathscr{H})$ then $g : Y \to \mathscr{C}(K, \mathscr{K}(\mathscr{H}))$, defined by $g(y)(k) = U(y, k) h(y)(k)$, is a continuous map.*

Part (i) of the lemma shows that, for fixed x, the function $k \to \tilde{f}(x, k)$ from K into $\mathcal{K}(\mathcal{H})$ is continuous. We claim that this function is even contained in $\mathcal{E}$. Since

$$\tilde{f}(x, kd) = U^*_{r(kd, \sigma(x)^{-1})} f(x, kd) = U^*_{r(kd, \sigma(x)^{-1})} U_d f(x, k) U^*_d$$

and

$$U_d \tilde{f}(x, k) U^*_d = U_d U^*_{r(k, \sigma(x)^{-1})} f(x, k) U^*_d$$

it suffices to show that

$$U_d U_{r(k, x)} = U_{r(kd, x)} U_d$$

holds for $k \in K$, $d \in K_\rho$, $x \in G$. One verifies that $dr(k, x) = r(kd, x)d$. Hence it remains to show that $m(d, r(k, x)) = m(r(kd, x), d)$. But $m(r(kd, x), d) = m(dr(k, x)d^{-1}, d) = m(r(k, x), d)$ as $dr(k, x)d^{-1} \equiv r(k, x)$ modulo N. The values of r are contained in the connected component $(H_\rho)_0$ of the identity in H_ρ. Since the bicharacter $\beta : H_\rho \times H_\rho \to \mathbb{T}$ associated to m, i.e. $\beta(a, b) = m(a, b)\overline{m}(b, a)$, vanishes on $K_\rho \times (H_\rho)_0$ [as K_ρ is finite and $(H_\rho)_0/N$ is divisible] we get $m(r(k, x), d) = m(d, r(k, x))$ and $m(r(kd, x), d) = m(d, r(k, x))$.

We have seen that $\tilde{f}$ may be considered as a function from V into $\mathcal{E}$. If $f : V \to \mathcal{E}$ is a continuous function with compact support then part (ii) of Lemma 3 tells us that $\tilde{f} : V \to \mathcal{E}$ is a continuous function, too (with compact support); moreover $\| f \|_1 = \| \tilde{f} \|_1$. Altogether we get that $f \to \tilde{f}$ can be extended to an isometry of the Banach space $L^1(V, \mathcal{E})$ onto itself.

Next, we introduce an action S of V on $\mathcal{E}$ and a factor system Q such that $f \to \tilde{f}$ will be an $\star$-isomorphism from $L^1(V, \mathcal{E}, P', T')$ onto $L^1(V, \mathcal{E}, Q, S)$. Q is simply a cocycle on V, namely $Q(a, b) = m((w(b), \sigma(b)^{-1}), (w(a), \sigma(a)^{-1}))$. Note that Q is independent on the choice of σ because m lives on H_ρ/N. S is defined by $(S_x \varphi)(k) = \varphi(w(x)k)$ for $k \in K$, $x \in V$, $\varphi \in \mathcal{E}$. First we show that $f \to \tilde{f}$ commutes with the involutions. The involution in $L^1(V, \mathcal{E}, P', T')$ is given by $f^*(x) = T'_{x^{-1}} f(x^{-1})^*$ or, considered as a function in two variables

$$f^*(x, k) = U_{r(k, \sigma(x)^{-1})} f(x^{-1}, kw(x)^{-1})^* U^*_{r(k, \sigma(x)^{-1})},$$

see the first section (notice that $P'_{x^{-1}, x} = 1$).

Hence $(f^*)^\sim(x, k) = f(x^{-1}, kw(x)^{-1})^* U^*_{r(k, \sigma(x)^{-1}}$,

The involution of $\tilde{f}$ is defined by

$$(\tilde{f})^*(x) = Q(x^{-1}, x)^{-1} S_{x^{-1}} \tilde{f}(x^{-1})^*.$$

Hence

$$\begin{aligned}(\tilde{f})^*(x, k) &= Q(x^{-1}, x)^{-1} \tilde{f}(x^{-1}, kw(x)^{-1})^* \\ &= Q(x^{-1}, x)^{-1} f(x^{-1}, kw(x)^{-1})^* U_{r(kw(x)^{-1}, \sigma(x))}.\end{aligned}$$

It remains to show that

$$Q(x^{-1}, x) = U_{r(kw(x)^{-1}, \sigma(x))} U_{r(k, \sigma(x)^{-1})}.$$

But $r(kw(x)^{-1}, \sigma(x)) r(k, \sigma(x)^{-1})$ is equal to the identity in H.

Hence

$$U_{r(kw(x)^{-1},\,\sigma(x))}\,U_{r(k,\,\sigma(x)^{-1})}=m(r(kw(x)^{-1},\,\sigma(x)),\,r(k,\,\sigma(x)^{-1}))$$
$$=m((w(x)^{-1},\,\alpha_k(\sigma(x))),\,(w(x),\,\alpha_{kw(x)^{-1}}(\sigma(x)^{-1})))=Q(x,\,x^{-1})$$

as m lives on H_ρ/N. Since $Q(x, x^{-1})=Q(x^{-1}, x)$ holds for every cocycle, this part is finished.

Next, we prove that $(f\star g)^\sim=\tilde{f}\star\tilde{g}$. By definition:

$$(\tilde{f}\star\tilde{g})\;(x,k)=\int_V dy\,Q(xy,\,y^{-1})\,\tilde{f}(xy,\,kw(y))\,\tilde{g}(y^{-1},\,k)$$
$$=\int_V dy\,Q(xy,\,y^{-1})\,U^*_{r(kw(y),\,\sigma(xy)^{-1})}\,f(xy,\,kw(y))\,U^*_{r(k,\,\sigma(y))}\,g(y^{-1},\,k).$$

And

$$(f\star g)^\sim\;(x,k)=U^*_{r(k,\,\sigma(x)^{-1})}\int_V dy\,[P'_{xy,\,y^{-1}}\,T'_y\,f(xy)](k)\,g(y^{-1},\,k)$$
$$=U^*_{r(k,\,\sigma(x)^{-1})}\int_V dy\,\rho(\alpha_k(\gamma(xy,\,y^{-1})^{-1})\,U_{r(k,\,\sigma(y))}$$
$$\times f(xy,\,kw(y))\,U^*_{r(k,\,\sigma(y))}\,g(y^{-1},\,k).$$

Hence, it suffices to show that:

$$U^*_{r(k,\,\sigma(x)^{-1})}\,\rho(\alpha_k(\gamma(xy,\,y^{-1})))^*\,U_{r(k,\,\sigma(y))}\,U_{r(kw(y),\,\sigma(xy)^{-1})}=Q(xy,\,y^{-1}).$$

The product of the two first factors gives U^*_a with $a=\alpha_k(\gamma(xy, y^{-1}))\,r(k, \sigma(x)^{-1})$. Let $b=r(k, \sigma(y))$ and $c=r(kw(y), \sigma(xy)^{-1})$. We have to compute $U^*_a\,U_b\,U_c$, which is equal to $\overline{m}(a, a^{-1}bc)\,m(b, c)\,U_{a^{-1}bc}$ (this follows immediately from the definition of an m-projective representation). A straightforward calculation gives that $a^{-1}bc$ is the identity element in H. Hence:

$$U^*_a\,U_b\,U_c=m(b,c)=m(r(k,\,\sigma(y)),\,r(kw(y),\,\sigma(xy)^{-1}))$$
$$=m((w(y)^{-1},\,\alpha_{kw(y)}(\sigma(y))),\,(w(xy),\,\alpha_{kw(x)^{-1}}(\sigma(xy)^{-1})))=Q(xy,\,y^{-1}),$$

because $\alpha l(z)$ is congruent z mod N for all $z\in G$ and all $l\in L$.

Stability

Recall that we want to determine the primitive quotients of $C^*(G, \mathcal{J}_1/\mathcal{J}_3, \tau)$. Until now, we have obtained a better realization of this algebra, namely that $C^*(G, \mathcal{J}_1/\mathcal{J}_3, \tau)$ is isomorphic to the C*-hull of $L^1(V, \mathcal{E}, Q, S)$. In the next step, it is shown that the primitive quotients of this algebra are stable or finite-dimensional. Since this is a known result [8], the presentation in this section of the paper will be less detailed than in others. I will use the following criterion for stability: a C*-algebra $\mathcal{A}$ is stable if (and only if) its adjoint algebra $\mathcal{A}^b$ contains a subalgebra $\mathcal{L}$, isomorphic to the algebra of compact operators on a separable infinite-dimensional Hilbert space, such that $\mathscr{L}\mathscr{A}$ is dense in $\mathscr{A}$. Moreover, if a C*-algebra is stable then the same holds true for all its primitive quotients.

Now, denote by $\mathscr{A}$ the C*-hull of $L^1(V, \mathscr{E}, Q, S)$.

We distinguish three cases.

Case 1. — Q is not equivalent to the trivial cocycle.

Let Z be the kernel of the associated antisymmetric bicharacter, i. e.:

$$Z=\{x\in V;\ Q(x, y)=Q(y, x) \text{ for all } y\in V\},$$

and let Y be any complementary vector space. Then $L^1(Y, \mathbb{C}, Q|_{Y\times Y})$ is contained in the adjoint algebra of $L^1(V, \mathscr{E}, Q, S)$, and $\mathscr{L}=C^*(Y, Q)$ is contained in $\mathscr{A}^b$. $\mathscr{L}$ satisfies the requirements of the criterion.

Case 2. — Q is equivalent to the trivial cocycle, K is not trivial.

Of course, we assume that $Q(x, y)=1$ for all $x, y\in V$. Let Z be the kernel of $w: V\to K$. $C^*(Z)$ is contained in the center of $\mathscr{A}^b$. Let π be an irreducible representation of $\mathscr{A}$. The restriction of π to $C^*(Z)$ corresponds to a character χ of Z. By tensoring π with a suitable character (which yields isomorphic primitive quotients) we may assume that χ is the trivial character. Hence π factorizes through $\mathscr{A}\to\dot{\mathscr{A}}$ where $\dot{\mathscr{A}}$ denotes the C*-hull of $L^1(V/Z, \mathscr{E}, S)$ with the obvious action S. We will show that $\dot{\mathscr{A}}$ is stable. The algebra $\mathscr{C}(K/K_\rho)$ of continuous functions on K/K_ρ is contained in $\mathscr{E}^b$ (in a canonical way), and V/Z acts by translations, *via* w, on $\mathscr{C}(K/K_\rho)$. The C*-hull $\mathscr{B}$ of $L^1(V/Z, \mathscr{C}(K/K_\rho), S)$ is considered as a subalgebra of $\dot{\mathscr{A}}^b$. It follows from the results of part I (one has to use that $w: V\to K$ has a dense image) that $\mathscr{B}$ has only finitely many primitive (simple) quotients [parametrized by the characters of $w^{-1}(K_\rho)/Z$], and all those quotients are stable. From the known structure of these quotients and from the Chinese remainder theorem it follows that $\mathscr{B}$ is isomorphic to $\mathscr{K}\otimes\mathscr{B}_0$ where $\mathscr{K}$ is the algebra of compact operators and $\mathscr{B}_0$ is a unital C*-algebra. $\mathscr{L}:=\mathscr{K}\otimes 1$ satisfies the requirements of the criterion for stability.

Case 3. — Q is equivalent to the trivial cocycle, K is trivial.

Then $\mathscr{A}$ is isomorphic to $C^*(V)\otimes\mathscr{K}(\mathscr{H})$, and the assertion is obvious. Let's interpret the conditions of case 3 in the original data: $K=\{e\}$ means that $G=H=H_\rho$, i. e. ρ can be extended to a projective representation $\tilde{\rho}$ of G. If, in addition, Q (and hence m) is trivial, one can even find an ordinary representation $\tilde{\rho}$ of G extending ρ.

Reduction to: $U|_{K_\rho}$ is irreducible

Knowing that the primitive quotients of $\mathscr{A}$ are stable (except trivial cases) we now reduce to the case that $U|_{K_\rho}$ is irreducible (and, consequently, $\mathscr{H}$ can be substituted by a finite-dimensional space). This is done by computing a "full corner" in $\mathscr{A}$, i. e. a subalgebra of the form $p\mathscr{A}p$ where p is a projection in $\mathscr{A}^b$ such that $\mathscr{A}p\mathscr{A}$ is dense in $\mathscr{A}$, *see* e. g. [1]. From the "theory of corners" it follows that the primitive quotients of $\mathscr{A}$ are stably isomorphic to the primitive quotients of $p\mathscr{A}p$. Hence it suffices to compute the latter.

In fact, we choose as $p \in \mathcal{K}(\mathcal{H})$ the projection onto an $U|_{K_\rho}$-irreducible subspace of $\mathcal{H}$. p is identified with the associated constant function $K \to \mathcal{K}(\mathcal{H})$, which is contained in $\mathcal{E}$. $\mathcal{E}$ is canonically embedded into $\mathcal{A}^b$; in this way p is considered as an element of $\mathcal{A}^b$. It is easy to see that $\mathcal{A} p \mathcal{A}$ is dense in $\mathcal{A}$, and $p \mathcal{A} p$ is the C*-hull of $p \star L^1(V, \mathcal{E}, Q, S) \star p$.

An $f \in L^1(V, \mathcal{E}, Q, S)$ is contained in $p \star L^1(V, \mathcal{E}, Q, S) \star p$ iff $f(x) \in p\mathcal{E}p =: \mathcal{E}_p$ for (almost) all $x \in V$. And $\mathcal{E}_p$ can be identified with the space of all continuous functions $\varphi : K \to \mathcal{B}(p\mathcal{H})$ such that $\varphi(kd) = U_d \varphi(k) U_d^*$ for $k \in K$, $d \in K_\rho$. Hence $p \star L^1(V, \mathcal{E}, Q, S) \star p$ is isomorphic to $L^1(V, \mathcal{E}_p, Q, S)$ where the new Q and S are formally the same as before and, therefore, denoted by the same letters.

Substitution of $\mathcal{E}_p$, end of the proof

It is unpleasant that $\mathcal{E}_p$ does not consist of all continuous functions from K into $\mathcal{B}(p\mathcal{H})$. We would like to substitute $\mathcal{E}_p$ by a " better algebra". This is done in the next step. In fact, the obstruction caused by the finite group K_ρ (and the representation $U|_{K_\rho}$) will "survive in the form of finite cocycle on a subgroup of the dual group $\hat{K}$ of K". We will use the following lemma which was certainly known already to Schur [21]. Its proof is simple and omitted.

LEMMA 4. — *Let* R *be a finite abelian group, let* m *be a cocycle on* R, *and let* U *be an irreducible* m*-projective representation of* R *in a finite-dimensional* (*complex*) *Hilbert space. Let* $Z := \{x \in R;\ m(x, y) = m(y, x) \text{ for all } y \in R\}$, *and let* D *be the annihilator of* Z *in the dual group* $\hat{R}$ *of* R. D *coincides with the group of characters* $\chi \in \hat{R}$ *such that* U *is equivalent to* $\chi \otimes U$. *For* $\chi \in D$, *Let* Y_χ *be a unitary intertwining operator between* U *and* $\chi \otimes U$, i. e. $U(x) Y_\chi = Y_\chi \chi(x) U(x)$ *for all* $x \in R$; Y_1 *is chosen as the identity. Then* $\chi \to Y_\chi$ *is an irreducible* λ*-projective representation of* D *for a certain cocycle* λ *on* D.

Of course, the lemma is applied to the representation $U|_{K_\rho}$ of $R = K_\rho$ in $p\mathcal{H}$. Let Z be as in the lemma, and let $\dot{D}$ be the annihilator of Z in $\hat{K}_\rho$. The inclusion $K_\rho \to K$ induces a surjection $\hat{K} \to \hat{K}_\rho$, denoted by $\chi \to \dot{\chi}$. Let $D \subset \hat{K}$ be the preimage of $\dot{D}$ under this map. For $\dot{\chi} \in \dot{D}$ let the unitary operator $Y_{\dot{\chi}}$ in $p\mathcal{H}$ be chosen as in the lemma. The corresponding cocycle λ on $\dot{D}$ is also considered as a cocycle on D. Then we form the twisted convolution algebra $L^1(D, \lambda)$ and define the "Fourier transform" $\mathcal{F} : L^1(D, \lambda) \to \mathcal{E}_p$ by:

$$(\mathcal{F} f)(k) = \int_D \chi(k) f(\chi) Y_{\dot{\chi}}\, d\chi.$$

One easily checks that the values of $\mathcal{F}$ are in $\mathcal{E}_p$, that $\mathcal{F}$ is a $\star$-morphism, and that $\mathcal{F}$ extends to an isomorphism from $C^*(D, \lambda)$ onto $\mathcal{E}_p$. Moreover, if the action T of V on $L^1(D, \lambda)$ is defined by $(T_x f)(\chi) = \chi(w(x)) f(\chi)$ then $S_x \circ \mathcal{F} = \mathcal{F} \circ T_x$ for all $x \in V$. Hence $p \mathcal{A} p$, i. e. the C*-hull of $L^1(V, \mathcal{E}_p, Q, S)$, is isomorphic to the C*-hull of $L^1(V, L^1(D, \lambda), Q, T)$. From Lemma 1 it follows that $L^1(V, L^1(D, \lambda), Q, T)$ is isomorphic to $L^1(V \oplus D, \mu)$ where the cocycle μ on $V \oplus D$ is (for instance) defined as:

$$\mu(x \oplus \chi, y \oplus \eta) = Q(x, 'y) \eta(w(x)) \lambda(\chi, \eta).$$

Since the primitive quotients of $C^*(V \oplus D, \mu)$ are known by the results of part I, the theorem is proved.

The corollary follows easily from the fact that $C^*(F, \nu)$ has a unique traceable factor representation, the "regular" representation in $L^2(F)$.

Remark 3. — Actually, all the algebras $C^*(\mathbb{Z}^n, \alpha) \otimes \mathcal{K}(\mathcal{H})$ (α a non-degenerate cocycle, $\mathcal{H}$ an infinite-dimensional Hilbert space) can be realized as quotients of $C^*(G)$. One may even choose G to be a metabelian connected Lie group. This can be seen as follows. Of course, we may assume that α is an antisymmetric bicharacter. There exists an antisymmetric continuous bicharacter B on $\mathbb{R}^n$ with $B|_{\mathbb{Z}^n \times \mathbb{Z}^n} = \alpha$. Let K be the kernel of this bicharacter, and let $V := \mathbb{R}^n/K$. B induces a non-degenerate bicharacter β on V. The composition γ of $\mathbb{Z}^n \to \mathbb{R}^n$ and $\mathbb{R}^n \to V$ is injective, and α is given in termes of γ and β by:

$$\alpha(r, s) = \beta(\gamma(r), \gamma(s)).$$

Define $\delta : V \to \mathbb{T}^n$ by $\langle r, \delta(y) \rangle = \beta(\gamma(r), y)^2$, $y \in V$, $y \in \mathbb{Z}^n$ where $\langle , \rangle$ denotes the duality between $\mathbb{Z}^n$ and $\mathbb{T}^n$. Then form the group $G = V \times \mathbb{T} \times \mathbb{C}^n$ with the multiplication:

$$(x, a, z)(y, b, w) = (x+y, ab\,\beta(x, y), \delta(y)^{-1}(z)+w)$$

where $\delta(y)^{-1}(z)$ means coordinatewise multiplication. If the character ρ on the abelian normal subgroup $N = \mathbb{T} \times \mathbb{C}^n$ is defined by $\rho(a, z_1, \ldots, z_n) = a e^{2\pi i \operatorname{Re}(z_1 + \ldots + z_n)}$ it is pretty easy to see that the primitive (simple) quotient of $C^*(G)$ corresponding to the orbit of ρ is isomorphic to $C^*(\mathbb{Z}^n, \bar{\alpha}) \otimes \mathcal{K}(\mathcal{H})$.

I conclude this article with some comments and some open problems; the first two comments were indicated by the referee.

(1) One may view Theorem 2 as a representation-theoretic version of a theorem in ergodic theory [J. Feldman, P. Hahn and C. Moore, *Orbit Structure and Countable Sections for Actions of Continuous Groups* (*Adv. in Math.*, Vol. 28, 1978, pp. 186-230)], which may be phrased loosely as saying that ergodic actions of continuous groups are stably isomorphic to actions of discrete groups. In fact, the analogy can be traced to the fact that each non-type I simple subquotient of $C^*(G)$ (for G connected Lie) arises from the ergodic action of G on the closure of $q(G)$ in L/L_ρ.

(2) Theorem 2, together with the results of [5] and recent work of G. Kasparov [*K-theory, group C*-algebras, and higher signatures*, preprint, Chernogolovka, 1981], helps complete the argument of P. Green (unpublished) sketched in [J. Rosenberg, *Group C*-algebras and Topological Invariants*, to appear in *Proc. International Conf. on Operator Algebras and Group Representations*, Neptun, Romania, 1980, Pitman Publ.]. One arrives at the following theorem: every square-integrable factor representation of a connected amenable unimodular Lie group is type I (and is thus quasi-equivalent to one of the square-integrable irreducible representations in the usual sense, which were classified by Nguyen Anh [*Classification of Connected Unimodular Lie Groups with Discrete Series* (*Ann. Inst. Fourier*, Grenoble, Vol. 30, 1980, pp. 159-192)]. To summarize the argument, Green has shown that for a unimodular Lie group G, each square-integrable factor representation contributes a simple direct summand $\mathcal{A}$ to $C_r^*(G)$. By Theorem 2, either $\mathcal{A} \cong \mathcal{K}(\mathcal{H})$,

or else $\mathscr{A} \cong C^*(F, \nu) \otimes \mathscr{K}$ for some nontrivial F. But in the latter case, Elliott [5] shows that $K_0(\mathscr{A})$ and $K_1(\mathscr{A})$ are both non-zero. This would contradict the conjecture of Rosenberg and Kasparov that $K_*(C_r^*(G))$ is concentrated in degree given by the parity of the dimension of G/M, where M is maximal compact in G. Kasparov's paper actually proves the conjecture if G is amenable, and gives reason to believe it holds in general.

(3) In case that a parametrization of Prim (G) is known, e. g. if G is a connected solvable Lie group, it would be desirable to have a description of the pair (F, ν) associated to a primitive ideal in terms of the parameters.

(4) Let G be a finite extension of a connected Lie group. Also in this case, every primitive quotient of C* (G) contains a simple ideal as was shown in my paper *Der Raum der primitiven Ideale von endlichen Erweiterungen lokalkompakter Gruppen* (*Arch. Math.*, Vol. 28, 1977, pp. 133-138). It should be possible to determine these simple algebras similar to Theorem 2. After that one may generalize to almost connected groups by the usual procedure.

(5) Theorem 2 tells that for the classification of all irreducible representations of a connected Lie group one needs a classification of all ν-projective irreducible representations of a free abelian group F (ν non-degenerate). This is generally believed to be impossible because by Glimm's theorem, $C^*(F, \nu)^\wedge$ is not standard. But anyway, $C^*(F, \nu)^\wedge$ is a well-defined set, and there are several groups acting on it. Perhaps it will be possible to determine the set of orbits for one of these groups. This remark is purely speculative; I must confess that I have no theorem in that direction.

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(Manuscrit reçu le 26 avril 1982, révisé le 18 octobre 1982).

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