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VARIATION OF MIXED HODGE STRUCTURES ARISING FROM FAMILY OF LOGARITHMIC DEFORMATIONS

BY SAMPEI USUI

CONTENTS

Introduction

The first fundamental results of Griffiths in [18] in the theory of period mappings are that, for a polarized, smooth, analytic family $f : \mathscr{X} \to S$ of projective manifolds:

(H.0) the Hodge filtration F on $R^n_{\mathscr{O}}(f) = R^n f_* \mathbb{Z} \otimes \mathscr{O}_S$ consists of holomorphic subbundles,

(H.1) the Gauss-Manin connection ∇ of $R^n_{\mathscr{O}}(f)$ satisfies:

$$\nabla F^p(R^n_{\mathscr{O}}(f)) \subset \Omega^1_S \otimes F^{p-1}(R^n_{\mathscr{O}}(f)),$$

(H.2) for any point $s \in S$,

$$(R^n_{\mathbb{Z}}(f),\ F)(s) \simeq (H^n(X_s, \mathbb{Z}),\ F),$$

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where the left-hand-side is the restriction to the fibre over s and the right is the Hodge structure on $H^n(X_s, \mathbb{Z})$, and

(H.P) there exists a locally constant bilinear form on the primitive part of $R^n_{\mathbb{Q}}(f)$ which satisfies the Hodge-Riemann bilinear relations.

In this article, we generalize these statements for a family $\mathcal{F}=(\mathcal{X}, \mathcal{Y}, \mathring{\mathcal{X}}, f, S, s_0, \varphi)$ of logarithmic deformations of a non-singular triple $(X, Y, \mathring{X})$ of Kawamata in [8], where we assume X to be projective and Y to be a divisor, by using the mixed Hodge structures of Deligne in [4].

That is, we can consider the relative logarithmic De Rham complex $\Omega^{\cdot}_f(\log \mathcal{Y})$, its weight filtration W and its Hodge filtration F. These define the spectral sequences

$$ {}_{W}E_1^{p,q}=\mathbb{R}^{p+q}f_*(\mathrm{Gr}^{W}_{-p}(\Omega^{\cdot}_f(\log \mathcal{Y}))) \Rightarrow E^{p+q}=\mathbb{R}^{p+q}f_*(\Omega^{\cdot}_f(\log \mathcal{Y})) $$

and

$$ {}_{F}E_1^{p,q}=\mathbb{R}^{p+q}f_*(\mathrm{Gr}^{p}_{F}(\Omega^{\cdot}_f(\log \mathcal{Y}))) \Rightarrow E^{p+q}=\mathbb{R}^{p+q}f_*(\Omega^{\cdot}_f(\log \mathcal{Y})) $$

of hypercohomology. For these, we prove Theorem (2.10):

(M.H. 0) $E^n=R^n\mathring{f}_*(\mathring{f}^{\cdot}\mathcal{O}_S)=R^n\mathring{f}_*\mathbb{Z}\otimes\mathcal{O}_S$, where $\mathring{\mathcal{X}}=\mathcal{X}-\mathcal{Y}$ and $\mathring{f}=f\,|\,\mathring{\mathcal{X}}:\mathring{\mathcal{X}}\to S$. Denote this by $R^n_{\mathcal{O}}(\mathring{f})$. The filtration W on $R^n_{\mathcal{O}}(\mathring{f})$ is induced from one on $R^n_{\mathbb{Q}}(\mathring{f})=R^n\mathring{f}_*\mathbb{Z}\otimes\mathbb{Q}$ consisting of its sub-local-systems. The filtration F on $R^n_{\mathcal{O}}(\mathring{f})$ consists of holomorphic subbundles of $R^n_{\mathcal{O}}(\mathring{f})$.

(M.H. 1) The Gauss-Manin connection ∇ of $R^n_{\mathcal{O}}(\mathring{f})$ satisfies:

$$ \nabla F^p(R^n_{\mathcal{O}}(\mathring{f}))\subset\Omega^1_S\otimes F^{p-1}(R^n_{\mathcal{O}}(\mathring{f})). $$

(M.H. 2) For any point $s\in S$,

$$ (R^n_{\mathbb{Z}}(\mathring{f}), W[n], F)(s)\simeq(H^n(\mathring{X}_s, \mathbb{Z}), W[n], F), $$

where the left-hand-side is the restriction to the fibre over s and the right is the mixed Hodge structure of Deligne in [4].

(M.H. 3):

$$ {}_{W}E_2={}_{W}E_{\infty}. \quad {}_{F}E_1={}_{F}E_{\infty}. $$

As well as in the classical case, in the proof of this theorem and the formulation of infinitesimal period mapping, an important role is played by an explicit calculation of the Gauss-Manin connection ∇ of $R^n_{\mathcal{O}}(\mathring{f})$, which is an immediate generalization of that in Katz-Oda [7].

We apply the above results to the local Torelli problem for the surfaces with $p_g=c_1^2=1$.

Historically, the first examples of such surfaces were constructed by Kinev [19] as counter examples to a conjecture of Griffiths, which stated that the local Torelli theorem holds for simply connected surfaces of general type with $p_g\geqq 1$. After this, the surfaces with $p_g=c_1^2=1$ have attracted our attention.

Caranese proved in [1] that the canonical models of these surfaces are represented as weighted complete intersections of type (6,6) in $\mathbb{P}(1, 2, 2, 3, 3)$, that the period mapping is generically finite, and that the differential of the period mapping has 2-dimensional kernel at those surfaces whose bicanonical mappings are Galois coverings of $\mathbb{P}^2$.

Todorov [10] and the author [11] showed independently and by different methods that there are positive dimensional fibres of the period mapping in question. The former restricted himself only to those surfaces whose bicanonical mappings are Galois coverings of $\mathbb{P}^2$. The latter pointed out other surfaces at which the period mapping has positive dimensional fibres, and also explained in [12] this curious phenomenon as an effect to the variation of Hodge structures caused by automorphisms of the surfaces in question.

Now, we can save the conjecture of Griffiths, at least for those surfaces with $p_g = c_1^2 = 1$ and with smooth, ample canonical divisors, if we pick up more data in the form of the mixed Hodge structures on the complements of the canonical divisors. (Note that the above conditions are satisfied for those surfaces whose bicanonical mappings are Galois coverings of $\mathbb{P}^2$.) This is our Theorem (5.2).

We believe that a generalization of the assertion (H. P) should be possible in our present context by virtue of Lemma (3.5) (*cf.* also (5.21) in [6]). Another problem which interests the author is a globalization of Theorem (2.10), i.e. we would like to find a nice relative divisor $\mathscr{Y}$ for a given smooth family $f : \mathscr{X} \to S$ of compact, Kähler manifolds so that we would be able to apply Theorem (2.10) for the induced family $(\mathscr{X}, \mathscr{Y}, \mathring{\mathscr{X}}, f, S)$. One of the candidates for such $\mathscr{Y}$ should be a general member of the m-ple relative canonical linear system $|K_f^{\otimes m}|$ for large enough m.

The present results were reported in the seminar of algebraic geometry at Kyoto University in September 1981 and in the seminar of geometry at Ecole Polytechnique in November 1981. The author would like to express his hearty gratitude to the professors at Kyoto and at Paris for their valuable discussions and encouragement, in particular, to Professor H. Esnault who kindly read and corrected the manuscript and also to the professors at Kochi University for the stimulating atmosphere.

1. Logarithmic deformation theory

In this section, we review the logarithmic deformation theory of Kawamata. We state only some definitions and results in [8] within our later use. For their proofs, *see* [8].

DEFINITIONS. — (1.1) A divisor Y of a complex manifold X is called a divisor with simple normal crossings if each irreducible component of Y is smooth and Y is of normal crossings.

(1.2) A non-singular triple $(X, Y, \mathring{X})$ is a triple consisting of a compact, complex manifold X, a divisor Y with simple normal crossings of X and $\mathring{X} = X - Y$.

(1.3) (Kawamata, Definition 3 in [8]). A family of logarithmic deformations of a non-singular triple $(X, Y, \mathring{X})$ is a 7-tuple $\mathscr{F} = (\mathscr{X}, \mathscr{Y}, \mathring{\mathscr{X}}, f, S, s_0, \varphi)$ satisfying the following conditions:

(i) $f : \mathscr{X} \to S$ is a proper, smooth morphism of complex spaces $\mathscr{X}$ and S.

(ii) $\mathscr{Y}$ is a closed analytic subspace of $\mathscr{X}$ and $\mathring{\mathscr{X}}=\mathscr{X}-\mathscr{Y}$.

(iii) $s_0 \in \mathrm{S}$ and $\varphi : \mathrm{X} \to f^{-1}(s_0)$ is an isomorphism such that $\varphi(\mathring{\mathrm{X}})=f^{-1}(s_0) \cap \mathring{\mathscr{X}}$.

(iv) f is locally a projection of a product space as well as its restriction to $\mathscr{Y}$, i. e. for each $x \in \mathscr{X}$ there exist an open neighborhood U of x and an isomorphism $\psi : \mathrm{U} \to \mathrm{V} \times \mathrm{W}$, where $\mathrm{V}=f(\mathrm{U})$ and $\mathrm{W}=\mathrm{U} \cap f^{-1}(f(x))$, such that the diagram

$$\begin{tikzcd} \mathrm{U} \arrow[r, "\psi"] \arrow[dr, "f"'] & \mathrm{V} \times \mathrm{W} \arrow[d, "\text{projection}"] \\ & \mathrm{V} \end{tikzcd}$$

is commutative and $\psi(\mathrm{U} \cap \mathscr{Y})=\mathrm{V} \times (\mathrm{W} \cap \mathscr{Y})$.

(1.4) (Saito, [9]). Let X be a complex manifold and Y a closed analytic subset of X. The logarithmic tangent sheaf $\mathrm{T}_{\mathrm{X}}(-\log \mathrm{Y})$ is the subsheaf of the tangent sheaf T_{X} of X consisting of derivations which send the $\mathscr{O}_{\mathrm{X}}$-ideal sheaf $\mathscr{I}_{\mathrm{Y}}$ of Y into itself.

(1.5) (Kawamata, Definition 5 in [8]). A family $\mathscr{F}=(\mathscr{X}, \mathscr{Y}, \mathring{\mathscr{X}}, f, \mathrm{S}, s_0, \varphi)$ of logarithmic deformations of a non-singular triple (X, Y, X) is semi-universal if for any family $\mathscr{F}'=(\mathscr{X}', \mathscr{Y}', \mathring{\mathscr{X}}', f', \mathrm{S}', s_0', \varphi')$ of logarithmic deformations of $(\mathrm{X}, \mathrm{Y}, \mathring{\mathrm{X}})$ there exist an open neighborhood S_1' of s_0' in S' and a morphism $\alpha_1 : \mathrm{S}_1' \to \mathrm{S}$ satisfying the following conditions:

(i) The restriction $\mathscr{F}'|_{\mathrm{S}_1'}$ of $\mathscr{F}'$ over S_1' is isomorphic to the induced family $\alpha_1^* \mathscr{F}$.

(ii) For any S_2' and α_2 which satisfy the condition (i), the differentials $d\alpha_1(s_0')$ and $d\alpha_2(s_0')$ from $\mathrm{T}_{\mathrm{S}'}(s_0')$ to $\mathrm{T}_{\mathrm{S}}(s_0)$ coincide.

Known results. — (1.6) Let $\mathscr{F}=(\mathscr{X}, \mathscr{Y}, \mathring{\mathscr{X}}, f, \mathrm{S}, s_0, \varphi)$ be a family of logarithmic deformations of a non-singular triple $(\mathrm{X}, \mathrm{Y}, \mathring{\mathrm{X}})$. The family $(\mathscr{X}, \mathscr{Y}, \mathring{\mathscr{X}}, f, \mathrm{S})$ is locally C^∞-trivial over S, i. e. there exist an open neighborhood S_1 of s_0 in S and a C^∞-isomorphism $\beta : f^{-1}(\mathrm{S}_1) \to \mathrm{S}_1 \times \mathrm{X}$ such that the diagram

$$\begin{tikzcd} f^{-1}(\mathrm{S}_1) \arrow[r, "\beta"] \arrow[dr, "f"'] & \mathrm{S}_1 \times \mathrm{X} \arrow[d, "\text{projection}"] \\ & \mathrm{S}_1 \end{tikzcd}$$

is commutative and $\beta(f^{-1}(\mathrm{S}_1) \cap \mathscr{Y})=\mathrm{S}_1 \times \mathrm{Y}$ (*cf.* Lemma 1 in [8]).

(1.7) For a non-singular triple $(\mathrm{X}, \mathrm{Y}, \mathring{\mathrm{X}})$, we get the following results similar to those in the classical case:

(i) $\mathrm{T}_{\mathrm{X}}(-\log \mathrm{Y})$ = the sheaf of infinitesimal automorphisms of X preserving Y.

(ii) $\mathrm{H}^1(\mathrm{X}, \mathrm{T}_{\mathrm{X}}(-\log \mathrm{Y}))$ = the set of infinitesimal logarithmic deformations.

(iii) $\mathrm{H}^2(\mathrm{X}, \mathrm{T}_{\mathrm{X}}(-\log \mathrm{Y}))$ = the set of obstructions.

(iv) The Kodaira-Spencer mapping

$$\rho_{s_0} : \mathrm{T}_{\mathrm{S}}(s_0) \to \mathrm{H}^1(\mathrm{X}, \mathrm{T}_{\mathrm{X}}(-\log \mathrm{Y}))$$

is defined in the usual way.

(1.8) The sequences

$$0 \to T_X(-Y) \to T_X(-\log Y) \to T_Y \to 0 \quad \text{and}$$

$$0 \to T_X(-\log Y) \to T_X \to N_{Y/X} \to 0$$

are exact, where $T_Y = \mathrm{Der}\, \mathcal{O}_Y$ and $N_{Y/X} = \mathrm{Coker}\,(T_Y \to T_X \otimes \mathcal{O}_Y)$.

(1.9) For any non-singular triple $(X, Y, \mathring{X})$, there exists a semi-universal family of logarithmic deformations of it. (A theorem of Kuranishi type, *cf.* Theorem 1 in [8].)

2. Variation of mixed Hodge structures

In this section, we give first some definitions and results without proofs, which are relative counterparts of those found in Deligne's work [4]. For their proofs, consult [3] and [4].

After these preparations, we state an existence theorem for the variation of mixed Hodge structures arising from a family of logarithmic deformations of a non-singular triple. The proof of this theorem will be found in the next section.

Let $\mathcal{F} = (\mathcal{X}, \mathcal{Y}, \mathring{\mathcal{X}}, f, S, s_0, \varphi)$ be a family of logarithmic deformations of a non-singular triple $(X, Y, \mathring{X})$.

We assume that X is projective. We assume also that S and hence $\mathcal{X}$ are smooth.

Let

$$\mathcal{Y} = \bigcup_{i \in I} \mathcal{Y}_i$$

be the decomposition into irreducible components. We fix once for all an order of these components.

We use the following notations:

$\mathcal{Y}^n$ = the union of the intersections of n of $\mathcal{Y}_i$'s.

$\tilde{\mathcal{Y}}^n$ = the disjoint union of the intersections of n of $\mathcal{Y}_i$'s, i. e. the normalization of $\mathcal{Y}^n$.

$i^n : \tilde{\mathcal{Y}}^n \to \mathcal{X}$ the projection.

$\tilde{g}^n = f \circ i^n : \tilde{\mathcal{Y}}^n \to S$.

$j : \mathring{\mathcal{X}} \hookrightarrow \mathcal{X}$ the inclusion.

$\mathring{f} = f \circ j : \mathring{\mathcal{X}} \to S$.

$R^n_{\mathbb{Z}}(\mathring{f}) = R^n \mathring{f}_* \mathbb{Z}$ modulo torsion.

$R^n_A(\mathring{f}) = R^n_{\mathbb{Z}}(\mathring{f}) \otimes A$ for an abelian sheaf A on S.

DEFINITIONS (2.1) (Deligne, (3.3.2) in [3]). — The relative logarithmic De Rham complex $\Omega^{\bullet}_f(\log \mathcal{Y})$ of a family $\mathcal{F}$ of logarithmic deformations is the smallest subcomplex of $j_* \Omega^{\bullet}_{\mathring{f}}$, where $\Omega^{\bullet}_{\mathring{f}}$ is the usual relative De Rham complex of the morphism $\mathring{f} : \mathring{\mathcal{X}} \to S$, satisfying the following conditions:

(i) Containing the relative De Rham complex $\Omega^{\bullet}_f$ of the morphism $f : \mathcal{X} \to S$.

(ii) $d\xi/\xi$ is a local section of $\Omega^1_f(\log \mathscr{Y})$ whenever ξ is a local section of $j_* \mathcal{O}_{\mathscr{X}}$ meromorphic along $\mathscr{Y}$.

(iii) Stable under exterior products.

(2.2) (Deligne, 3.5 in [3]). The weight filtration W of $\Omega^{\cdot}_f(\log \mathscr{Y})$ is defined by

$W_n(\Omega^{\cdot}_f(\log \mathscr{Y}))$ = the smallest $\mathcal{O}_{\mathscr{X}}$-submodule of $\Omega^{\cdot}_f(\log \mathscr{Y})$ satisfying the following conditions:

(i) Stable under the exterior products with local sections of $\Omega^{\cdot}_f$.

(ii) Containing the products $d\xi_1/\xi_1 \wedge \ldots \wedge d\xi_k/\xi_k$ whenever $k \leqq n$ and ξ_i are local sections of $j_* \Omega^{\cdot}_{f^*}$ meromorphic along $\mathscr{Y}$.

(2.3) The Hodge filtration F of $\Omega^{\cdot}_f(\log \mathscr{Y})$ is the stupid filtration, i. e.

$$F^p(\Omega^{\cdot}_f(\log \mathscr{Y})) = \sum_{p' \geqq p} \Omega^{p'}_f(\log \mathscr{Y}).$$

KNOWN RESULTS. — (2.4) For a local section ξ of $j_* \Omega^{\cdot}_{f^*}$, ξ belongs to $\Omega^{\cdot}_f(\log \mathscr{Y})$ if and only if both ξ and $d\xi$ have at most simple poles along $\mathscr{Y}$ (*cf.* (3.3.2) in [3]).

(2.5) $\Omega^1_f(\log \mathscr{Y})$ is locally free and dual to the relative logarithmic tangent sheaf $T_f(-\log \mathscr{Y})$, whose definition is similar to (1.4). And we have

$$\Omega^p_f(\log \mathscr{Y}) = \bigwedge^p \Omega^1_f(\log \mathscr{Y})$$

(*cf.* loc. cit.).

(2.6) If the family $\mathscr{F}$ is a product $\mathscr{F} = \mathscr{F}_1 \times_S \mathscr{F}_2$ over S, i. e. $S_1 = S_2 = S$, $\mathscr{X} = \mathscr{X}_1 \times_S \mathscr{X}_2$ and $\mathscr{Y} = \mathscr{X}_1 \times_S \mathscr{Y}_2 \cup \mathscr{Y}_1 \times_S \mathscr{X}_2$, then

$$\Omega^{\cdot}_f(\log \mathscr{Y}) \xleftarrow{\sim} pr_1^* \Omega^{\cdot}_{f_1}(\log \mathscr{Y}_1) \otimes pr_2^* \Omega^{\cdot}_{f_2}(\log \mathscr{Y}_2)$$

(*cf.* loc. cit.).

(2.7) There is an isomorphism

$$Gr^W_n(\Omega^{\cdot}_f(\log \mathscr{Y})) \xrightarrow{\sim} i^n{}_* \Omega^{\cdot}_{\tilde{g}^n}[-n]$$

called the Poincaré residue morphism, which is induced from

$$\alpha \wedge dt_{q(1)}/t_{q(1)} \wedge \ldots \wedge dt_{q(n)}/t_{q(n)} \mapsto \pm(\alpha \mid \mathscr{Y}_{q(1)} \cap \ldots \cap \mathscr{Y}_{q(x)}),$$

where

α is a local section of $\Omega^{\cdot}_f$,

t_i is a local equation of $\mathscr{Y}_i$,

q is a mapping from the set $\{1, \ldots, n\}$ to I, and the signature $\pm$ is determined by the orientation of q with respect to the fixed order on I (*cf.* 3.5 in [3]).

(2.8) The inclusion

$$\Omega^{\cdot}_f(\log \mathscr{Y}) \hookrightarrow j_* \Omega^{\cdot}_{f^*},$$

is a quasi-isomorphism (*cf.* 3.16 and 6.18 in [3]).

Now we consider the spectral sequences of hypercohomology

$$
(2.9.1)\qquad
\begin{aligned}
{}_{\mathrm{W}}\mathrm{E}_1^{p,q} &= \mathbb{R}^{p+q} f_*(\mathrm{Gr}^{\mathrm{W}}_{-p}(\Omega^{\bullet}_f(\log \mathscr{Y}))),\\
&= \mathbb{R}^{p+q} f_*(i^{-p}{}_*\Omega^{\bullet}_{\tilde{g}^{-p}}[p])\\
&= \mathbb{R}^{2p+q}\, \tilde{g}^{-p}{}_*\Omega^{\bullet}_{\tilde{g}^{-p}}\\
&= \mathrm{R}^{2p+q}\, \tilde{g}^{-p}{}_*(\mathbb{C})\otimes\mathscr{O}_{\mathrm{S}}
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\mathrm{E}^{p+q} &= \mathbb{R}^{p+q} f_*\,\Omega^{\bullet}_f(\log \mathscr{Y})\\
&= \mathbb{R}^{p+q} f_*(j_*\Omega^{\bullet}_{\mathring{f}}),\\
&= \mathbb{R}^{p+q} \mathring{f}_*\,\Omega^{\bullet}_{\mathring{f}},\\
&= \mathrm{R}^{p+q} \mathring{f}_*(\mathring{f}^{\bullet}\,\mathscr{O}_{\mathrm{S}})
\end{aligned}
$$

determined by the triple $(\Omega^{\bullet}_f(\log \mathscr{Y}), \mathrm{W}, f_*)$, and

$$
(2.9.2)\qquad
\begin{aligned}
{}_{\mathrm{F}}\mathrm{E}_1^{p,q} &= \mathbb{R}^{p+q} f_*(\mathrm{Gr}^p_{\mathrm{F}}(\Omega^{\bullet}_f(\log \mathscr{Y})))\\
&= \mathrm{R}^q f_*\,\Omega^p_f(\log \mathscr{Y}),
\end{aligned}
\quad\Rightarrow\quad
\mathrm{E}^{p+q} = \mathrm{R}^{p+q} \mathring{f}_*(\mathring{f}^{\bullet}\,\mathscr{O}_{\mathrm{S}}),
$$

determined by the triple $(\Omega^{\bullet}_f(\log \mathscr{Y}), \mathrm{F}, f_*)$.

For these spectral sequences, we can prove:

THEOREM (2.10). — (i) $\mathrm{E}^n = \mathrm{R}^n \mathring{f}_*(\mathring{f}^{\bullet}\,\mathscr{O}_{\mathrm{S}}) = \mathrm{R}^n_{\mathscr{O}_{\mathrm{S}}}(\mathring{f})$. *The filtration* W *on* $\mathrm{R}^n_{\mathscr{O}_{\mathrm{S}}}(\mathring{f})$, *which is the abutment of the spectral sequence* (2.9.1), *is induced from a filtration on* $\mathrm{R}^n_{\mathbb{Q}}(\mathring{f})$ *consisting of its sub-local-systems. The filtration* F *on* $\mathrm{R}^n_{\mathscr{O}_{\mathrm{S}}}(\mathring{f})$, *which is the abutment of the spectral sequence* (2.9.2), *consists of holomorphic sub-bundles of* $\mathrm{R}^n_{\mathscr{O}_{\mathrm{S}}}(\mathring{f})$.

(ii) *The triple* $(\mathrm{R}^n_{\mathbb{Z}}(\mathring{f}), \mathrm{W}[n], \mathrm{F})$ *is a variation of mixed Hodge structures on* S, *i. e. they have the following properties:*

(M.H.1) *The Gauss-Manin connection* ∇ *on* $\mathrm{R}^n_{\mathscr{O}_{\mathrm{S}}}(\mathring{f})$ *satisfies:*

$$\nabla \mathrm{F}^p(\mathrm{R}^n_{\mathscr{O}_{\mathrm{S}}}(\mathring{f})) \subset \Omega^1_{\mathrm{S}} \otimes \mathrm{F}^{p-1}(\mathrm{R}^n_{\mathscr{O}_{\mathrm{S}}}(\mathring{f})).$$

(M.H.2) *For any point* $s \in \mathrm{S}$, *the two filtrations* W[n] *and* F *induce the mixed Hodge structure of Deligne* [4] *on* $\mathrm{R}^n_{\mathbb{Z}}(\mathring{f})(s) = \mathrm{H}^n(\mathring{\mathrm{X}}_s, \mathbb{Z})$ *modulo torsion.*

(iii) *The spectral sequences* (2.9.1) *and* (2.9.2) *degenerate in*

$${}_{\mathrm{W}}\mathrm{E}_2 = {}_{\mathrm{W}}\mathrm{E}_{\infty}$$

and

$${}_{\mathrm{F}}\mathrm{E}_1 = {}_{\mathrm{F}}\mathrm{E}_{\infty}.$$

We will prove this theorem in the next section.

3. Proof of Theorem (2.10)

We use the notation in the previous section.

We prove Theorem (2.10) in sequence of lemmas.

LEMMA (3.1). — *Let*

$$\begin{tikzcd}
X \arrow[rr, "j"] \arrow[dr, "h"] \arrow[ddr, "f"'] & & Y \arrow[dr, "k"] \arrow[ddl, "g"] & \\
& X' \arrow[rr, "j'"] \arrow[d, "f'"] & & Y' \arrow[dll, "g'"] \\
& S & &
\end{tikzcd}$$

be a commutative diagram of holomorphic mappings f, g, j, f', g' and j' of complex manifolds and of homeomorphisms h and k. Then, for any sheaf A *of abelian groups on* S, *we have*

$$g'_*(\mathscr{C}^*(Y', j'_*(\mathscr{C}^*(X', f'^{\cdot} A)))) \simeq g_*(\mathscr{C}^*(Y, j_*(\mathscr{C}^*(X, f^{\cdot} A)))),$$

where $\mathscr{C}^*(\ ,\ )$ is the canonical resolution of Godement [14]. *In particular, the Leray spectral sequences*

$$R^p g_*(R^q j_*(f^{\cdot} A)) \Rightarrow R^{p+q} f_*(f^{\cdot} A)$$

and

$$R^p g'_*(R^q j'_*(f'^{\cdot} A)) \Rightarrow R^{p+q} f'_*(f'^{\cdot} A)$$

coincide.

We get immediately Lemma (3.1) by recalling the construction of the canonical resolution of Godement, and we omit its proof.

Let $\mathscr{F} = (\mathscr{X}, \mathscr{Y}, \mathring{\mathscr{X}}, f, S, s_0, \varphi)$ be a family of logarithmic deformations of a non-singular triple $(X, Y, \mathring{X})$ as in § 2. We identify $\varphi : X \xrightarrow{\sim} f^{-1}(s_0)$ and also $\varphi : Y \xrightarrow{\sim} f^{-1}(s_0) \cap \mathscr{Y}$.

LEMMA (3.2). — *The spectral sequences* $S.S.(\Omega^{\cdot}_f(\log \mathscr{Y}), W, f_*)$ *and* $S.S.(\Omega^{\cdot}_X(\log Y), W, \Gamma)$ *determined respectively by the triples* $(\Omega^{\cdot}_f(\log \mathscr{Y}), W, f_*)$ *and* $(\Omega^{\cdot}_X(\log Y), W, \Gamma)$ *are related by*

$$S.S.(\Omega^{\cdot}_f(\log \mathscr{Y}), W, f_*) = S.S.(\Omega^{\cdot}_X(\log Y), W, \Gamma) \otimes \mathscr{O}_S$$

when we consider both sides in the sense of germs at $s_0 \in S$.

Proof. — Recall that the spectral sequences in question are nothing but the Leray spectral sequences up to the change of indices $E_r^{p,q} \to E_{r+1}^{2p+q, -p}$. Hence by (1.6) and (3.1), we may assume S a small polydisc, $\mathscr{X} = X \times S$, $\mathscr{Y} = Y \times S$ and $f = pr_2$ the second projection.

Take a Stein open covering $\mathscr{W} = \{U_i\}$ of X, set $V_i = U_i \times S$ and denote $\mathscr{V} = \{V_i\}$. By (2.6), we have, in Čech compleces, that

$$\begin{aligned}
&\mathscr{C}^q(\mathscr{V}, \Omega^p_{pr_2}(\log(Y \times S))) \\
&= \mathscr{C}^q(\mathscr{V}, pr_1^{\cdot}(\Omega^p_X(\log Y) \otimes_{\mathbb{C}} pr_2^{\cdot} \mathscr{O}_S) \\
&= \mathscr{C}^q(\mathscr{W}, \Omega^p_X(\log Y)) \otimes_{\mathbb{C}} \Gamma(S, \mathscr{O}_S).
\end{aligned}$$

Since $\otimes_{\mathbb{C}} \Gamma(S, \mathscr{O}_S)$ is an exact functor, we get the assertion.

Q.E.D.

LEMMA (3.3). — $\nabla F^p(R^n_{\mathcal{O}_S}(\overset{\circ}{f})) \subset \Omega^1_S \otimes F^{p-1}(R^n_{\mathcal{O}_S}(\overset{\circ}{f}))$.

Proof. — Replacing the filtration of $\Omega^1_{\mathscr{X}}$ induced from the exact sequence

$$0 \to f^* \Omega^1_S \to \Omega^1_{\mathscr{X}} \to \Omega^1_f \to 0$$

by the filtration of $\Omega^1_{\mathscr{X}}(\log \mathscr{Y})$ induced from the exact sequence

$$0 \to f^* \Omega^1_S \to \Omega^1_{\mathscr{X}}(\log \mathscr{Y}) \to \Omega^1_f(\log \mathscr{Y}) \to 0,$$

we can calculate explicitly the Gauss-Manin connection ∇ of the logarithmic De Rham cohomology sheaf $\mathbb{R}^n f_* \Omega^{\cdot}_f(\log \mathscr{Y}) = R^n_{\mathcal{O}_S}(\overset{\circ}{f})$ exactly in the same way as Katz-Oda in [7] for that of the De Rham cohomology sheaf, and we get the assertion (*cf.* also [5]).

Q.E.D.

LEMMA (3.4). — *The spectral sequence* (2.9.1) *degenerates in*

$$_W E_2 = {}_W E_\infty.$$

Proof. — This is an immediate consequence of (3.2) and the absolute case of Deligne ((3.2.13) in [4]).

Q.E.D.

LEMMA (3.5). — *In the sense of germ at* $s_0 \in S$, $_W d_1$ *in the spectral sequence* (2.9.1) *is the alternating sum of Gysin mappings up to constant, i.e.*

$$_W d_1 = \frac{1}{2\pi\sqrt{-1}} \sum_{i_1 < \ldots < i_{-p}} \sum_j (-1)^j$$

$$(\textit{Gysin mapping: } R^{2p+q}\tilde{g}_{i_1 \ldots i_{-p}*}(\mathbb{C}) \otimes \mathcal{O}_S \to R^{2p+2+q}\tilde{g}_{i_1 \ldots \hat{i}_j \ldots i_{-p}*}(\mathbb{C}) \otimes \mathcal{O}_S),$$

where $\tilde{g}_{i_1 \ldots i_{-p}}$ *is the inclusion* $\mathscr{Y}_{i_1} \cap \ldots \cap \mathscr{Y}_{i_{-p}} \subsetneq \mathscr{X}$.

Proof. — This is also an immediate consequence of (3.2) and the absolute case of Griffiths-Schmid ((5.21) in [6]).

Q.E.D.

LEMMA (3.6). — *The spectral sequence* (2.9.2) *degenerates in*

$$_F E_1 = {}_F E_\infty.$$

Proof. — We can perform the procedure of Deligne ((3.2.13) in [4]) in our case by using (3.4) and (3.5). More precisely, consider the diagram:

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{l} {}_{\mathrm{W}}\mathrm{E}_1^{k,\,n-k} = \mathbb{R}^n f_*(\mathrm{Gr}^{\mathrm{W}}_{-k}(\Omega_f^{\cdot}(\log \mathscr{Y}))) \\ \quad = \mathrm{R}^{n+k}\tilde{g}^{-k}{}_*\mathbb{C}\otimes\mathcal{O}_{\mathrm{S}} \end{array} \arrow[dr, Rightarrow, "(2)"] & \\
\begin{array}{l} \mathrm{E}_1^{p,\,k,\,n-p-k} \\ \overset{\text{def}}{=} \mathbb{R}^n f_*(\mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}^{\mathrm{W}}_{-k}(\Omega_f^{\cdot}(\log \mathscr{Y}))) \\ = \mathrm{R}^{n-p} f_*(\mathrm{Gr}^{\mathrm{W}}_{-k}(\Omega_f^p(\log \mathscr{Y})) \\ = \mathrm{R}^{n-p}\tilde{g}^{-k}{}_*(\Omega^{p+k}_{\tilde{g}^{-k}}) \end{array} \arrow[ur, Rightarrow, "(1)", "(k\ :\ \text{fixed})"'] \arrow[dr, Rightarrow, "(p\ :\ \text{fixed})", "(3)"'] & & \begin{array}{l} \mathrm{E}^n = \mathbb{R}^n f_*(\Omega_f^{\cdot}(\log \mathscr{Y})) \\ \quad = \mathrm{R}^n f_*\,\mathbb{C}\otimes\mathcal{O}_{\mathrm{S}} \end{array} \\
 & \begin{array}{l} {}_{\mathrm{F}}\mathrm{E}_1^{p,\,n-p} = \mathbb{R}^n f_*(\mathrm{Gr}_{\mathrm{F}}^p(\Omega_f^{\cdot}(\log \mathscr{Y}))) \\ \quad = \mathrm{R}^{n-p} f_*(\Omega_f^p(\log \mathscr{Y})) \end{array} \arrow[ur, Rightarrow, "(4)"'] &
\end{tikzcd}$$

of four spectral sequences (1), (2), (3) and (4), where (2) and (4) are (2.9.1) and (2.9.2) respectively and (1) and (3) are defined respectively by

$$(\Omega^{\cdot}_{\tilde{g}^{-k}},\ \mathrm{F},\ \tilde{g}^{-k}_*) \qquad \text{and} \qquad (\Omega^p_f(\log \mathscr{Y}),\ \mathrm{W}, f_*).$$

By (3.5), ${}_{\mathrm{W}}d_1$ in (2) is strictly compatible with the Hodge filtration F, which is the abutment of the spectral sequence (1). With this and (3.4), we get

$$\mathrm{Gr}_{\mathrm{F}}^p({}_{\mathrm{W}}\mathrm{E}_2^{k,\,n-k}) \simeq \mathrm{H}^{\cdot}\,(\mathrm{E}_1^{p,\,k-1,\,n-k-p} \to \mathrm{E}_1^{p,\,k,\,n-k-p} \to \mathrm{E}_1^{p,\,k+1,\,n-k-p}) = \mathrm{E}_2^{p,\,k,\,n-k-p},$$

and hence

$$\sum_n \operatorname{rank} \mathrm{E}^n = \sum \operatorname{rank} \mathrm{E}_2^{\cdot\cdot}.$$

On the other hand, we see that

$$\sum_n \operatorname{rank} \mathrm{E}^n \leqq \sum \operatorname{rank} {}_{\mathrm{F}}\mathrm{E}_1^{\cdot} \leqq \sum \operatorname{rank} \mathrm{E}_2^{\cdot\cdot}.$$

Thus we can conclude that the spectral sequences (3) and (4) are degenerate at E_2 and in E_1 respectively.

Q.E.D.

LEMMA (3.7). — *For any point $s \in S$, the Hodge filtration* F *on* $R^n_{\mathcal{O}_S}(\mathring{f})$, *which is the abutment of the spectral sequence* (2.8.2), *induces on* $R^n_{\mathcal{O}_S}(\mathring{f})(s) = H^n(\mathring{X}_s, \mathbb{C})$ *the Hodge filtration of Deligne in* [4].

Proof. — Since $E^n = R^n_{\mathcal{O}_S}(\mathring{f}) = R^n \mathring{f}_* \mathbb{C} \otimes \mathcal{O}_S$ is locally free by (3.2) and

$$ {}_F E^{p,\, n-p}_\infty = {}_F E^{p,\, n-p}_1 = R^{n-p} f_* (\Omega^p_f (\log \mathscr{Y})) $$

by (3.6), the sheaf $\Omega^p_f(\log \mathscr{Y})$ on $\mathscr{X}$ is cohomologically flat with respect to the morphism f for each p (*cf.* [15]). Therefore we get our assertion by observing inductively on p the following commutative exact diagram :

$$
\begin{array}{ccccccccc}
0 \to & F^{p+1}(R^n_{\mathcal{O}_S}(\mathring{f}))(s) & \to & F^p(R^n_{\mathcal{O}_S}(\mathring{f}))(s) & \to & R^{n-p} f_*(\Omega^p_f(\log \mathscr{Y}))(s) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & F^{p+1}(H^n(\mathring{X}_s, \mathbb{C})) & \to & F^p(H^n(\mathring{X}_s, \mathbb{C})) & \to & H^{n-p}(X_s, \Omega^p_{X_s}(\log Y_s)) & \to 0,
\end{array}
$$

where F in the bottom line is the Hodge filtration of Deligne, i. e. the filtration on $H^n(\mathring{X}_s, \mathbb{C})$ of the abutment of the spectral sequence

$$ E^{p,\, n-p}_1 = H^{n-p}(X_s, \Omega^p_{X_s}(\log Y_s)) \quad \Rightarrow \quad E^n = H^n(\mathring{X}_s, \mathbb{C}). $$

Q.E.D.

At this stage, we have completely proved Theorem (2.10).

4. Infinitesimal period mapping

We can derive the formulation of the infinitesimal period mapping from (M.H.1) in Theorem (2.10), which is called transversality, horizontality or infinitesimal period relation, exactly in the same way as in case of the usual variation of Hodge structures (*cf.*, for example, [5]).

We continue to use the notation in the previous sections.

By virtue of (M.H.1) in Theorem (2.10), the Gauss-Manin connection ∇ induces an $\mathcal{O}_S$-linear homomorphism

$$
(4.1) \qquad \begin{array}{rccc}
\operatorname{Gr}^p_F(\nabla): & \operatorname{Gr}^p_F(R^n_{\mathcal{O}_S}(\mathring{f})) & \to & \Omega^1_S \otimes \operatorname{Gr}^{p-1}_F(R^n_{\mathcal{O}_S}(\mathring{f})), \\
& \wr\!\! \mid & & \wr\!\! \mid \\
& R^{n-p} f_* \Omega^p_f(\log \mathscr{Y}) & & \Omega^1_S \otimes R^{n-p+1} f_* \Omega^{p-1}_f(\log \mathscr{Y}).
\end{array}
$$

Hence we can associate to a local section θ of the tangent sheaf T_s of S an $\mathcal{O}_S$-linear homomorphism

$$
(4.2) \qquad \theta . \operatorname{Gr}^p_F(\nabla) : \quad R^q f_* \Omega^p_f(\log \mathscr{Y}) \to R^{q+1} f_* \Omega^{p-1}_f(\log \mathscr{Y})
$$

of the composite of $\operatorname{Gr}^p_F(\nabla)$ and the contraction with θ.

As we have mentioned in (iv) in (1.7), we can also associate to θ its Kodaira-Spencer class

$$(4.3) \qquad \rho(\theta)\in R^1 f_* T_f(-\log \mathscr{Y}).$$

PROPOSITION (4.4). — (4.2) *and* (4.3) *are related in the manner that the diagram*

$$\begin{tikzcd}
\theta \arrow[r, mapsto] & \theta\cdot \mathrm{Gr}^p(\nabla) \\[-20pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-20pt]
T_S \arrow[r] \arrow[d, "\substack{\text{Kodaira-}\\ \text{Spencer}\\ \text{mapping}\ \rho}"'] & \mathrm{Hom}_{\mathcal{O}_S}(R^q f_*(\Omega_f^p(\log \mathscr{Y}), R^{q+1} f_*\Omega_f^{p-1}(\log \mathscr{Y})) \\
R^1 f_* T_f(-\log \mathscr{Y}) \arrow[ur, "\text{contraction}"'] &
\end{tikzcd}$$

commutes up to $(-1)^p$.

The proof of this proposition is performed by observing the explicit calculations of Gauss-Manin connection ∇ and Kodaira-Spencer mapping ρ. With a brief change mentioned in the proof of Lemma (3.3), we can go on the same way as in [5], and we omit it (for explicit calculation of ∇, *cf.* [7]).

Let

$$\Phi: \quad S\to F1 = \text{flag manifold},$$

be the holomorphic mapping defined by the pair $(R^n_{\mathcal{O}_S}(\mathring{f}), F)$. Φ is called the period mapping associated to the variation of mixed Hodge structures $(R^n_{\mathcal{O}_S}(\mathring{f}), W[n], F)$ in Theorem (2.10).

Proposition (4.4) is paraphrased as:

COROLLARY (4.5). — *For a point* $s\in S$, *the following diagram is commutative up to* $\bigoplus_p(-1)^p$:

$$\begin{tikzcd}
T_S(s) \arrow[r, "d\Phi(s)"] \arrow[dddd, "\substack{\text{Kodaira-}\\ \text{Spencer}\\ \text{mapping}}"', "\rho\ \text{via contraction}"] & T_{F1}(\Phi(s)) \\
& \rotatebox{90}{$\simeq$} \\
& \bigoplus_p \mathrm{Hom}(F^p(s)/F^{p+1}(s), H^n(\mathring{X}_s, \mathbb{C})/F^p(s)) \\
& \bigcup \\
& \bigoplus_p \mathrm{Hom}(F^p(s)/F^{p+1}(s), F^{p-1}(s)/F^p(s)) \\
& \rotatebox{90}{$\simeq$} \\
H^1(T_{X_s}(-\log Y_s)) \arrow[r] & \bigoplus_p \mathrm{Hom}(H^{n-p}(\Omega^p_{X_s}(\log Y_s)), H^{n-p+1}(\Omega^{p-1}_{X_s}(\log Y_s))),
\end{tikzcd}$$

where $d\Phi(s)$ *is the differential of* Φ *at* s.

5. Application to surfaces with $p_g = c_1^2 = 1$.

In this section, we prove the local Torelli theorem for the surfaces with $p_g = c_1^2 = 1$ under the additional assumption that the canonical curve is ample and smooth by using the variation of mixed Hodge structures in Theorem (2.10).

Let X be a complete, smooth surface over $\mathbb{C}$ with $p_g = c_1^2 = 1$ and let C be its unique canonical curve, i.e. $C \in |K_X|$. Recall that X is simply connected (*cf.* [1]). We assume throughout this section that C is ample and smooth.

LEMMA (5.1). — *Let* $(\mathscr{X}, f, S, s_0, \varphi)$ *be the Kuranishi family of deformations of* X, *where* $f : \mathscr{X} \to S$, $s_0 \in S$ *and* $\varphi : X \simeq f^{-1}(s_0)$. *Then we have:*

(i) S *is smooth and* $\dim S = 18$.

(ii) C *can be extended uniquely to the relative canonical curve* $\mathscr{C}$ *of the family* $f : \mathscr{X} \to S$.

(iii) $\mathscr{C}$ *is smooth over* S.

(iv) $\mathscr{F} = (\mathscr{X}, \mathscr{C}, \mathring{\mathscr{X}}, f, S, s_0, \varphi)$ *form the semi-universal family of logarithmic deformations of the non-singular triple* $(X, C, \mathring{X})$, *where* $\mathring{\mathscr{X}} = \mathscr{X} - \mathscr{C}$ *and* $\mathring{X} = X - C$.

Proof. — (i) follows from $\dim H^1(X, T_X) = 18$ and $H^2(X, T_X) = 0$, and these are easily calculated (*cf.* for example, (1.4) in [11]). (ii) and (iii) follow immediately from the assumption that $p_g(X) = 1$ and C is smooth. (iv) is the straight forward consequence of the construction of $\mathscr{F}$ and the semi-universalities of $\mathscr{F}$ and the Kuranishi family.

Q.E.D.

THEOREM (5.2). — *If* C *is ample and smooth, the local Torelli theorem holds at* X *in the sense of variation of mixed Hodge structures, that is, the contraction mapping*

$$H^1(X, T_X(-\log C)) \to \operatorname{Hom}(H^0(X, \Omega_X^2(\log C)), H^1(X, \Omega_X^1(\log C)))$$

is injective.

We will prove this theorem in the sequence of lemmas.

LEMMA (5.3). — *The Poincaré residue morphism* (2.6) *induces exact sequences*

$$0 \to H^0(X, \Omega_X^2) \xrightarrow{\alpha} H^0(X, \Omega_X^2(\log C)) \to H^0(C, \Omega_C^1) \to 0$$

and

$$0 \to P^{1,1}(X) \to H^1(X, \Omega_X^1(\log C)) \xrightarrow{\beta} H^1(C, \mathcal{O}_C) \to 0,$$

where $P^{1,1}(X) =$ *the primitive part of* $H^1(X, \Omega_X^1)$ *with respect to* $\mathcal{O}_X(3C)$.

Proof. — Since X is simply connected, we have in particular

$$H^1(X, \Omega_X^2) = H^2(X, \Omega_X^1) = 0.$$

Hence our assertions are the special cases of the following facts:

(i) The spectral sequence defined by $(\Omega_X^p(\log C), W, \Gamma)$ is degenerated in $E_2 = E_\infty$ (Corollary (3.2, 13) (iii) in [4]).

(ii) The differential d_1 in the spectral sequence in (i) is coming from the Gysin mapping ((5.21) Lemma in [6]).

Q.E.D.

Set

$$T' = \{\theta \in \mathrm{Hom}(H^0(\Omega_X^2(\log C)),\ H^1(\Omega_X^1(\log C))) \mid \beta\theta\alpha = 0\}$$

and

$$T'_1 = \{\theta \in \mathrm{Hom}(H^0(\Omega_X^2(\log C)),\ H^1(\Omega_X^1(\log C))) \mid \beta\theta = 0\},$$

where α and β are the morphisms in Lemma (5.3).

LEMMA (5.4). — *We have a commutative exact diagram:*

$$\begin{tikzcd}
 & & & 0 \arrow[d, dash] \\
 & H^1(X, T_X(-C)) \arrow[r] \arrow[d, "\varphi_1"] & H^1(X, T_X(-\log C)) \arrow[r] \arrow[d, "\varphi"] & H^1(C, T_C) \arrow[d, "\varphi_2"] \\
0 \arrow[r] & T'_1 \arrow[r] & T' \arrow[r] & \mathrm{Hom}(H^0(\Omega_C^1), H^1(\mathcal{O}_C)),
\end{tikzcd}$$

where φ, φ_1 and φ_2 are the mappings defined by contraction.

The proof is easy, and we omit it.

Let $K^{\cdot} = K^{\cdot}(T_X(-C), K_X^{\otimes 2}, H^0(K_X^{\otimes 2}))$ be the Koszul complex of Liberman-Wilsker-Peters [16] which is defined in the following way:

$$K^p = (T_X(-C) \otimes_{\mathcal{O}_X} (K_X^{\otimes 2})^{\otimes p}) \otimes_{\mathbb{C}} \overset{p}{\wedge} H^0(K_X^{\otimes 2})^{\vee}.$$

$$d(x) = \sum_{i_1 < \dots < i_{p+1}} \Big(\sum_j (-1)^j f_{i_j} \otimes x_{i_1 \dots i_j \dots i_{p+1}}\Big) \otimes e_{i_1} \wedge \dots \wedge e_{i_{p+1}},$$

where $f_1, \dots, f_m$ $(m=3)$ form a basis of $H^0(K_X^{\otimes 2})$, $e_1, \dots, e_m$ its dual basis of $H^0(K_X^{\otimes 2})^{\vee}$

and

$$x = \sum_{i_1 < \dots < i_p} x_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_p \in K^p.$$

Let $\mathscr{C}^*(X, K^{\cdot})$ be the canonical resolution of Godement [14], and consider the spectral sequence of the double complex $\Gamma(\mathscr{C}^*(X, K^{\cdot}))$. Note that

$$(5.5) \qquad 0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to E^2,$$

is exact, and

$$(5.6) \qquad 'E_2^{0,1} = \mathrm{Ker}(H^1(T_X(-C)) \xrightarrow{d_1} H^1(\Omega_X^1) \otimes H^0(\Omega_X^2(\log C))^{\vee}).$$

LEMMA (5.7). — $E^1 = E^2 = 0$.

Proof. — Since $\mathrm{B}s|\mathrm{K}_{\mathrm{X}}^{\otimes 2}|=\varphi$ ([1]), the Koszul complex $\mathrm{K}^{\bullet}$ is exact. In particular, $''\mathrm{E}_1^{p,q}=0$ for all p and q, and we get the assertion.

Q.E.D.

LEMMA (5.8). — $'\mathrm{E}_2^{2,0}=0$.

Proof. — We use the fact that X is represented as a weighted complete intersection of type (6,6) in $\mathbb{P}=\mathbb{P}(1, 2, 2, 3, 3)$ 4 ([1]) and the exact diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & \check{\mathrm{N}}_{\mathrm{X/P}}\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2} \arrow[r] & \Omega_{\mathrm{P}}^1\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2} \arrow[r] \arrow[d] & \Omega_{\mathrm{X}}^1\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2} \arrow[r] & 0, \\ & & \bigoplus_i \mathcal{O}_{\mathrm{X}}(-e_i)\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2} \arrow[d] & & \\ & & \mathrm{K}_{\mathrm{X}}^{\otimes 2} \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

where $e_0=1$, $e_1=e_2=2$ and $e_3=e_4=3$.

Since $\mathrm{K}_{\mathrm{X}}^{\otimes 2}\simeq\mathcal{O}_{\mathrm{X}}(2)$, $\mathrm{H}^0(\bigoplus_i \mathcal{O}_{\mathrm{X}}(-e_i)\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2}) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{K}_{\mathrm{X}}^{\otimes 2})$ and $\mathrm{H}^1(\mathrm{K}_{\mathrm{X}}^{\otimes 2})=0$

and hence $\mathrm{H}^0(\Omega_{\mathbb{P}}^1\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2})=0$. On the other hand, $\mathrm{H}^1(\check{\mathrm{N}}_{\mathrm{X}/\mathbb{P}}\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2})=0$ because

$$\check{\mathrm{N}}_{\mathrm{X}/\mathbb{P}}\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2}\simeq\mathcal{O}_{\mathrm{X}}(-4)^{\oplus 2}.$$

Thus, we get $\mathrm{H}^0(\Omega_{\mathrm{X}}^1\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2})=0$, and hence

$$'\mathrm{E}_1^{2,0}=\mathrm{H}^0(\Omega_{\mathrm{X}}^1\otimes\mathrm{K}_{\mathrm{X}}^{\otimes 2})\otimes\overset{2}{\Lambda}\mathrm{H}^0(\mathrm{K}_{\mathrm{X}}^{\otimes 2})\check{}=0.$$

(*See* [17], for the calculation of the cohomology on weighted complete intersections.)

Q.E.D.

We can derive easily the injectivity of φ_1 in Lemma (5.4) from (5.5), (5.6), (5.7), (5.8) and the fact that the contraction mapping

$$\mathrm{H}^1(\mathrm{T}_{\mathrm{X}}(-\mathrm{C}))\otimes\mathrm{H}^0(\Omega_{\mathrm{X}}^2(\log \mathrm{C}))\to\mathrm{H}^1(\Omega_{\mathrm{X}}^1)$$

factors via $\mathrm{P}^{1,1}(\mathrm{X})\subsetneq\mathrm{H}^1(\Omega_{\mathrm{X}}^1)$. Hence φ is injective by Lemma (5.4). This concludes the proof of Theorem (5.2).

Remark (5.10). — Among the surfaces X with $p_g = c_1^2 = 1$ and K_X ample, those whose bicanonical mappings are Galois coverings of $\mathbb{P}^2$ were investigated by Catanese, Todorov and the author, and the second and the third proved that the period mapping, determined by the usual Hodge structures, has 2-dimensional fibres through the points, in the Kuranishi space, corresponding to these surfaces (*cf.* [10], [11] and also [12]).

However, since it is easy to see that the canonical curves C of those surfaces are smooth, our Theorem (5.2) asserts that even for those surfaces the local Torelli theorem holds if we associate to X the mixed Hodge structure on $\mathring{X} = X - C$ instead of the usual Hodge structure on X.

The geometrical interpretation of this result is the following. Recall first that the surface X in question is a ramified double covering of a K3 surface Y with nine rational double points of type A_1, which consists of the branch locus together with the image D of C (*cf.* [10]). Note that, by Lemma (3.5), we have the exact sequences

$$0 \to P^2 f_* \mathbb{Z} \to R^2 \mathring{f}_* \mathbb{Z} \to R^1 g_* \mathbb{Z} \to 0$$

and

$$0 \to f_* \Omega_f^2 \to f_* \Omega_f^2(\log \mathscr{Y}) \to g_* \Omega_g^1 \to 0,$$

where $P^2 f_* \mathbb{Z} = P^2 f_* (\mathbb{C}) \cap R^2 f_* \mathbb{Z}$. Take a flat frame $e_1, \ldots, e_{20}; e_{21}, \ldots, e_{24}$ of $R^2 \mathring{f}_* \mathbb{Z}$ such that $e_1, \ldots, e_{20}$ form a frame of $P^2 f_* \mathbb{Z}$. And let $\check{e}_i$ $(1 \leqq i \leqq 24)$ be its dual frame. Take also a frame $\omega_1; \omega_2, \omega_3$ of $f_* \Omega_f^2(\log \mathscr{Y})$ such that ω_1 is a frame of $f_* \Omega_f^2$. Then, in the period matrix

$$[(\omega_i, \check{e}_j)] = \begin{array}{c} \begin{matrix} \overbrace{}^{20} & \overbrace{}^{4} \end{matrix} \\ \left[\begin{array}{c|c} A_1 & 0 \\ \hline * & A_2 \end{array}\right] \end{array} \begin{array}{l} \}1 \\ \}2 \end{array}$$

A_1 distinguishes infinitesimally the K3 surface Y in their moduli space and A_2 distinguishes infinitesimally the branch locus D in the parameter space of the displacements in Y.

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