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Etale K-theory II:
CONNECTIONS WITH ALGEBRAIC K-THEORY

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We continue our study of etale K-theory begun in [11]. We introduce a natural transformation from algebraic K-theory to l-adic etale K-theory which extends our previously defined natural transformation in degree 0 and which admits associated natural transformations between K-theories with coefficients. Our expectation is that etale K-theory may soon become a successful tool for deciding geometric questions, especially those involving galois actions. Consequently, we investigate various relationships between algebraic and etale K-theory of varieties quasi-projective over an algebraically closed field. Work in progress indicates that etale K-theory of more general schemes should prove to be a useful tool for certain number theoretic problems.

The paper is divided into three sections, the first dedicated to constructing the natural transformations (Theorem 1.3) and verifying their multiplicative behavior. Section 1 also provides a particularly simple description of the l-adic natural transformation in degree 1, and verifies agreement with the construction of [11] in degree 0. Examples studied in Section 2 demonstrate the non-triviality of our natural transformations. These examples offer new computations of algebraic K-groups with finite coefficients (e. g., affine spaces and projective spaces minus unions of hyperplanes). One important technique we employ is the comparison of Mayer-Vietoris exact sequences presented in Theorem 3.5. The galois equivariance of our natural transformations (proved in Theorem 3.3) strongly restricts the image of algebraic K-theory in l-adic etale K-theory. Whereas the image in l-adic etale K-theory of algebraic K-theory in degree 0 is the subject of the Tate Conjecture, the image in positive degrees is even less well understood. Motivated by the example of an affine curve, we pose several questions concerning this image in positive degrees. We conclude with an analogue of C. Soulé's chern classes with denominators.

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1. Natural Transformations

We consider algebraic and etale K-theories of (simplicial) schemes defined over a fixed complete discrete valuation ring F with separably closed residue field. Although F is a (separably closed) field in examples of interest, the added generality enables "lifting to characteristic 0" arguments. We consider a prime l invertible in F, also fixed throughout this section.

Definition 1.1 presents a definition of etale K-theories more easily related to that of algebraic K-theory than the definition considered in [11] (which was based on generalized cohomology theories associated to complex K-theory); Proposition 1.2 demonstrates the equivalence of old and new definitions. Theorem 1.3 constructs the natural transformations $\hat{\rho}_*$ and $\bar{\rho}_*$ from integral and mod-l^v K-theory to l-adic and mod-l^v etale K-theory, shown to be multiplicative in Proposition 1.4 and Corollary 1.5. In Propositions 1.6, 1.7, and 1.8, we present explicit descriptions of $\hat{\rho}_0$, $\hat{\rho}_1$, $\bar{\rho}_0$, and $\bar{\rho}_1$, descriptions which are amenable to computations as seen in the next section.

We adopt the following notation: $\mathscr{S}$ (respectively, $\mathscr{S}_0$) denotes the category of simplicial sets (resp., pointed simplicial sets) and $\mathscr{H}$ (resp., $\mathscr{H}_0$) denotes the homotopy category of $\mathscr{S}$ (resp., $\mathscr{S}_0$) obtained by inverting weak equivalences. We employ the etale topological type functor:

$$(\ )_{\text{et}} : \ (\text{loc. noeth. s. schemes}) \to \text{pro-}\mathscr{S},$$

sending a somplicial scheme X. which is locally noetherian in each dimension to the pro-simplicial set $(X.)_{\text{et}}$ indexed by the category HRR(X.) of rigid hypercoverings of X. (*cf.* [12], 4.4); this functor $(\ )_{\text{et}}$ is extended to a functor on closed immersions of such simplicial schemes, sending $Y. \to X.$ to $(X., Y.)_{\text{et}} \in \text{pro-}\mathscr{S}^2$, an inverse system of inclusions of simplicial sets also indexed by HRR(X.) (*cf.* [12], 15.4.2). We let $(X./Y.)_{\text{et}} \in \text{pro-}\mathscr{S}_0$ be defined by collapsing the subsimplicial set of each pair of $(X., Y.)_{\text{et}} \in \text{pro-}\mathscr{S}^2$. In particular, if $Y. = \emptyset$, then $(X./Y.)_{\text{et}} = (X.)_{\text{et}} \coprod \text{pt}$.

We employ several constructions in homotopical algebra, including:

$$\#(\ ) : \mathscr{S} \to \text{pro-}\mathscr{S},$$
$$(\mathbf{Z}/l)_\infty(\ ) : \ \mathscr{S} \to \mathscr{S},$$
$$\underleftarrow{\text{holim}}(\ ) : \ \mathscr{S}^{\text{I}} \to \mathscr{S},$$

where $\#(S.) = \{\cos k_n S.;\ n \geqq 0\}$ is the canonical "Postnikov tower" of $S. \in \mathscr{S}$ (recall that $\cos k_n(\ )$ is right adjoint to $sk_n(\ )$ [3]), $(\mathbf{Z}/l)_\infty(\ )$ is the Bousfield-Kan $\mathbf{Z}/l$-completion functor, and $\underleftarrow{\text{holim}}(\ )$ is the Bousfield-Kan homotopy inverse limit functor [4]. Moreover, if S. and T. are pointed simplicial sets, then Hom. (S., T.) $\in \mathscr{S}_0$ denotes the function complex of pointed maps.

For any $n \geqq 0$, GL_n denotes the general linear group (scheme) over F (with $GL_0 = \text{Spec F}$) and BGL_n denotes the (geometrically) pointed simplicial scheme over F obtained by applying the bar construction to GL_n. We define $BGL_n^{\wedge} \in \text{pro-}\mathcal{S}_0$ by:

$$BGL_n^{\wedge} = \# \circ (\mathbf{Z}/l)_{\infty} \circ (BGL_n)_{et},$$

[where we have used the fact that if X. is pointed, then $(X.)_{et} \in \text{pro-}\mathcal{S}_0$].

The following definition incorporates a suggestion by the referee to consider $X. \times GL_1$ (fibre product over Spec F is implicit) when defining low dimensional etale K-groups.

DEFINITION 1.1. — Let $Y. \to X.$ be a closed immersion of simplicial schemes locally of finite type over F. For any $n \geqq 0$, we define:

$$BGL_n^{\wedge (X., Y.)} = \underleftarrow{\text{holim}}\ \underrightarrow{\text{colim}}\ \text{Hom.}((X./Y.)_{et}, BGL_n^{\wedge}),$$

where the homotopy inverse limit is indexed by the indexing category of $BGL_n^{\wedge}$ [namely, $HRR(BGL_n) \times N$] and the colimit is indexed by the indexing category for $(X./Y.)_{et}$ [namely, HRR (X.)]. We define the etale K-groups of (X., Y.) by:

$$\hat{K}_i^{et}(X., Y.) = \pi_i(BGL^{\wedge (X., Y.)}), \qquad i > 0,$$

$$\hat{K}_0^{et}(X., Y.) = \ker. \{ \hat{K}_1^{et}(X. \times GL_1, Y. \times GL_1) \to \hat{K}_1^{et}(X., Y.) \},$$

$$K_i^{et}(X., Y.; \mathbf{Z}/l^{\nu}) = \pi_i(BGL^{\wedge (X., Y.)}, \mathbf{Z}/l^{\nu}), \qquad i > 1,$$

$$K_{\varepsilon}^{et}(X., Y.; \mathbf{Z}/l^{\nu}) = \ker \{ K_{\varepsilon+1}^{et}(X. \times GL_1, Y. \times GL_1, \mathbf{Z}/l^{\nu}) \to K_{\varepsilon+1}^{et}(X., Y.; \mathbf{Z}/l^{\nu}) \}$$

$$\varepsilon = 0, 1$$

where $BGL^{\wedge (X., Y.)}$ is the colimit with respect to n of $BGL_n^{\wedge (X., Y.)}$, $(X., Y.) \to (X. \times GL_1, Y. \times GL_1)$ is induced by $e : X. \to \text{Spec F} \to GL_1$, and ν is any positive integer. ■

In [11], we defined etale K-groups using the classifying space $BU \times \mathbf{Z}$ of complex K-theory. Namely, for $\varepsilon = 0$ or 1, we defined:

$$\hat{K}^{\varepsilon}((X., Y.)_{et}) = \text{Hom}_{\text{pro-}\mathcal{H}_0}(\Sigma^{\varepsilon}(X./Y.)_{et}, \# (\text{Sin BU})^{\wedge} \times \mathbf{Z}^{\wedge}),$$

$$K^{\varepsilon}((X., Y.)_{et}, \mathbf{Z}/l^{\nu}) = \text{Hom}_{\text{pro-}\mathcal{H}_0}(\Sigma^{\varepsilon} C(l^{\nu}) \wedge (X./Y.)_{et}, \# (\text{Sin BU})),$$

where $(\ )^{\wedge} : \mathcal{H}_0 \to \text{pro-}\mathcal{H}_0$ denotes the Artin-Mazur l-adic completion functor [3], Sin( ) denotes the singular functor, Σ^0 is the identity and $\Sigma^1 = \Sigma$ is the simplicial suspension functor, and $C(l^{\nu})$ is the (simplicial) Moore space determined by the mapping cone of multiplication by l^{ν} on the circle.

PROPOSITION 1.2. — *Let* $Y. \to X.$ *be a closed immersion of simplicial schemes locally of finite type over* F *satisfying the condition that* $H^k(X., Y.; \mathbf{Z}/l)$ *is finite for all* $k \geqq 0$ *and is zero for all* k *sufficiently large. Then there are isomorphisms* (*natural with respect to maps over* F):

(1.2.1) $$\hat{K}_i^{et}(X., Y.) \simeq \hat{K}^{\langle i \rangle}((X., Y.)_{et}), \qquad i \geqq 0,$$

(1.2.2) $$K_i^{et}(X., Y., \mathbf{Z}/l^{\nu}) \simeq K^{\langle i \rangle}((X., Y.)_{et}, \mathbf{Z}/l^{\nu}), \qquad i \geqq 0, \nu > 0,$$

where $\langle i \rangle = 1/2\,(1 - (-1)^i)$ *is the parity of i.*

Proof. — For $i>0$, isomorphism (1.2.1) is the composition of the following chain of isomorphisms:

$$\hat{\mathrm{K}}_i^{\mathrm{et}}(\mathrm{X.}, \mathrm{Y.}) \simeq \underset{\overrightarrow{m}}{\mathrm{colim}}\, \pi_i(\underset{\longleftarrow}{\mathrm{holim}}\ \underset{\longrightarrow}{\mathrm{colim}}\ \mathrm{Hom.}\,((\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}},\ \#\circ\{(\mathbf{Z}/l)_n\}\circ \mathrm{Sin\,BU}_m))$$

$$\simeq \underset{\overrightarrow{m}}{\mathrm{colim}}\, \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}(\Sigma^i(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}},\ \#\circ\{(\mathbf{Z}/l)_n\}\circ \mathrm{Sin\,BU}_m)$$

$$\simeq \underset{\overrightarrow{m}}{\mathrm{colim}}\, \mathrm{Hom}_{\text{pro-}\mathscr{H}_0}(\Sigma^i(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}},\ \#\circ(\mathrm{Sin\,BU}_m)\hat{}\ ) \to \hat{\mathrm{K}}^{\langle i\rangle}((\mathrm{X.}, \mathrm{Y.})_{\mathrm{et}}),$$

where the first is given by [12], 13.10; the second by [12], 13.9; the third by the weak equivalence $(\mathrm{Sin\,BU}_m)\hat{}\to\{(\mathbf{Z}/l)_n(\mathrm{BU}_m); n>0\}$ of [12], 6.10 [where $(\ \ )\hat{}\ : \text{pro-}\mathscr{H}_0\to\text{pro-}\mathscr{H}_0$ is the Artin-Mazur l-adic completion functor], and the last by Bott periodicity and obstruction theory for the $2m$-equivalences $\#\circ(\mathrm{Sin\,BU}_m)\hat{}\to\#\circ(\mathrm{Sin\,BU})\hat{}$. To prove (1.2.1) for $i=0$, we employ the Kunneth Theorem (in this case, given by the smooth base change theorem for etale cohomology) to obtain the $\mathbf{Z}/l$-equivalences:

(1.2.3) $(\mathrm{X.}\times\mathrm{GL}_1/\mathrm{Y.}\times\mathrm{GL}_1)_{\mathrm{et}}\to(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}\times(\mathrm{GL}_1)_{\mathrm{et}}/(\mathrm{GL}_1)_{\mathrm{et}}\leftarrow(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}\times \mathrm{S}^1/\mathrm{S}^1$.

Consequently, (1.2.1) for $i=0$ follows from (1.2.1) for $i=1$ and the natural isomorphism:

$$\hat{\mathrm{K}}^1((\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}\times\mathrm{S}^1/\mathrm{S}^1)\simeq\hat{\mathrm{K}}^1((\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}})\oplus\hat{\mathrm{K}}^0((\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}),$$

implied by the cofibre triple:

$$(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}\to(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}\times\mathrm{S}^1/\mathrm{S}^1\to\Sigma(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}.$$

The isomorphism (1.2.2) for $i\geqq 2$ is obtained by modifying the above chain of isomorphisms used for (1.2.1) with $i\geqq 1$ by replacing $(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}$ by $\mathrm{C}(l^\nu)\wedge(\mathrm{X.}/\mathrm{Y.})_{\mathrm{et}}$. Using (1.2.3), we conclude the natural isomorphisms for $\varepsilon=0, 1$:

(1.2.4) $\mathrm{K}^\varepsilon((\mathrm{X.}\times\mathrm{GL}_1, \mathrm{Y.}\times\mathrm{GL}_1)_{\mathrm{et}}, \mathbf{Z}/l^\nu)\simeq\mathrm{K}^\varepsilon((\mathrm{X.}, \mathrm{Y.})_{\mathrm{et}}, \mathbf{Z}/l^\nu)\oplus\mathrm{K}^{\varepsilon\pm 1}((\mathrm{X.}, \mathrm{Y.})_{\mathrm{et}}, \mathbf{Z}/l^\nu)$.

Isomorphism (1.2.4) for $\varepsilon=0$ and isomorphism (1.2.2) for $i=2$ imply isomorphism (1.2.2) for $i=1$; similarly, isomorphism (1.2.4) for $\varepsilon=1$ and isomorphism (1.2.2) for $i=1$ imply isomorphism (1.2.2) for $i=0$. ■

The definitions of Definition 1.1 lead us to the following natural transformations from algebraic to etale K-theories. We recall that the finiteness theorem of [8] implies that any closed immersion $\mathrm{Y}\to\mathrm{X}$ of schemes quasi-projective over F satisfies the hypotheses of Proposition 1.2.

THEOREM 1.3. — *There are natural transformations of abelian group valued functors (from algebraic* K*-theory to "etale* K*-theory") for each* $i\geqq 0$, $\nu\geqq 1$:

$$\hat{\rho}_i:\ \mathrm{K}_i(\ \ )\to\hat{\mathrm{K}}_i^{\mathrm{et}}(\ \ ),\qquad \bar{\rho}_i:\ \mathrm{K}_i(\ \ ,\mathbf{Z}/l^\nu)\to\mathrm{K}_i^{\mathrm{et}}(\ \ ,\mathbf{Z}/l^\nu),$$

on the category of closed immersions of quasi-projective schemes over F. *In particular, if* X = Spec A *is an affine scheme of finite type over* F, *then:*

$$\hat{\rho}_i:\ \mathrm{K}_i(\mathrm{A})\to\hat{\mathrm{K}}_i^{\mathrm{et}}(\mathrm{X})\equiv\hat{\mathrm{K}}_i^{\mathrm{et}}(\mathrm{X},\emptyset)\qquad \textit{for}\quad i>0$$

and:

$$\bar{\rho}_i : \quad K_i(A, \mathbf{Z}/l^\nu) \to K_i^{et}(X, \mathbf{Z}/l^\nu) \equiv K_i^{et}(X, \emptyset; \mathbf{Z}/l^\nu) \qquad \textit{for} \quad i > 1,$$

are determined by an infinite loop space map:

$$\psi : \quad BGL(A)^+ \to BGL^{\hat{}X}.$$

Proof. — We first consider an affine scheme $X = \operatorname{Spec} A$. A t-simplex of the simplicial set $BGL_n(A)$ can be naturally identified with a map of simplicial schemes over F of the form $X \otimes \Delta[t] \to BGL_m$, where $X \otimes \Delta[t]$ is the naturally constructed simplicial scheme with $(X \otimes \Delta[t])_n$ equal to a disjoint union of copies of X indexed by $\Delta[t]_n$. Because the natural map $(X \otimes \Delta[t])_{et} \to X_{et} \otimes \Delta[t]$ is an isomorphism in pro-$\mathscr{S}$ by [12], 4.7, sending:

$$\alpha : \quad X \otimes \Delta[t] \to BGL_n \qquad \text{to} \quad \alpha_{et} : \quad X_{et} \otimes \Delta[t] \simeq (X \otimes \Delta[t])_{et} \to (BGL_n)_{et},$$

determines:

$$BGL_n(A) \to \underleftarrow{\lim}\ \underrightarrow{\operatorname{colim}} \operatorname{Hom.}((X/\varphi)_{et}, (BGL_n)_{et}).$$

Composing this with the maps induced by $(BGL_n)_{et} \to BGL_n^{\hat{}}$ and the canonical natural transformation $\underleftarrow{\lim}(\ ) \to \underleftarrow{\operatorname{holim}}(\ )$, we obtain the natural map:

$$\coprod_{n \geq 0} \psi_n : \quad \coprod_{n \geq 0} BGL_n(A) \to \coprod_{n \geq 0} BGL_n^{\hat{}X}.$$

To obtain a *natural* extension of $\coprod_{n \geq 0} \psi_n$ to a map of group completions which we identify with $\psi : BGL(A)^+ \times \mathbf{Z} \to BGL^{\hat{}X} \times \mathbf{Z}$, we introduce G. Segal's infinite loop space machinery [17] (this naturality is necessary only for the relative theory, otherwise naturality up to homotopy suffices). External direct sum determines a permutative category $\mathscr{G}l(A)$ whose object space is $\coprod_{n \geq 0}$ pt. and whose morphism space is $\coprod_{n \geq 0} GL_n(A)$ (both discrete); this determines a functor $\underset{\sim}{\mathscr{G}l}(A)$ from the category $\mathscr{F}$ of finite pointed sets to the category of permutative categories of simplicial sets (*cf.* [15]); applying the functor diag $\circ$ Nerve: (permutative categories) $\to \mathscr{S}_0$, we obtain:

$$\mathscr{B}GL(A) : \quad \mathscr{F} \to \mathscr{S}_0.$$

$\mathscr{B}GL(A)$ is a "Segal Γ-space" with $\mathscr{B}GL(A)(\underline{1}) = \coprod_{n \geq 0} BGL_n(A)$. Similarly, external direct sum determines a "permutative category of schemes" $\mathscr{G}l$ whose object scheme is $\coprod_{n \geq 0} \operatorname{Spec} F$ and whose morphism scheme is $\coprod_{n \geq 0} GL_n$; $\underset{\sim}{\mathscr{G}l}$ determines $\mathscr{G}l$ from $\mathscr{F}$ to the category of permutative categories of simplicial schemes; consequently, we obtain a "Segal Γ-simplicial scheme" as in [10], 9.1:

$$\mathscr{B}GL : \quad \mathscr{F} \to (\text{pointed simplicial schemes}).$$

The naturality of:

$$(\ )^{\hat{}X} : \quad (\text{pointed simplicial schemes}) \to \mathscr{S}_0,$$

together with the fact that $\# \circ (\mathbf{Z}/l)_\infty \circ (\ \ )_{et}$ commutes up to homotopy with products (i. e., fibre products over F) implies that $\mathscr{B}\mathrm{GL}$ determines a Segal Γ-space:

$$\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}}: \quad \mathscr{F} \to \mathscr{S}_0,$$

with $\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}}(\underline{1}) = \coprod_{n \geqq 0} \mathrm{BGL}_n^{\hat{}\mathrm{X}}$. As in [10], 5.2, we identify $\mathscr{B}\mathrm{GL}(\mathrm{A})(\underline{m})$ and $\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}}(\underline{m})$ with:

$$\coprod \Big(\prod_{i \in \mathrm{I}} \mathrm{BGL}_i(\mathrm{A}) \times \prod_{j \in \mathrm{J(I)}} \mathrm{EGL}_j(\mathrm{A})\Big), \qquad \coprod \Big(\prod_{i \in \mathrm{I}} \mathrm{BGL}_i \times \prod_{j \in \mathrm{J(I)}} \mathrm{EGL}_j\Big)^{\hat{}\mathrm{X}}$$

where the sum is indexed by ordered m-tuples of non-negative integers I, where the product $\prod_{j \in \mathrm{J(I)}}$ is indexed by the (appropriately ordered) set J (I) of all sums $j \in \mathrm{J(I)}$ of at least two of the entries of I, and where EGL_d is defined by applying the non-reduced bar construction to GL_d. Then we define a map of Segal Γ-spaces:

$$\Psi: \quad \mathscr{B}\mathrm{GL}(\mathrm{A}) \to \mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}},$$

by setting $\Psi(\underline{m}) = \coprod \psi_{\mathrm{I,J(I)}}$, where $\psi_{\mathrm{I,J(I)}}$ is defined by replacing BGL_n in the definition of ψ_n by:

$$\prod_{i \in \mathrm{I}} \mathrm{BGL}_i \times \prod_{j \in \mathrm{J(I)}} \mathrm{EGL}_j.$$

We may identify $\Psi(\mathrm{S}^1) = \mathrm{diag} \circ (k \mapsto \psi((\mathrm{S}^1)_k))$ (where S^1 is the minimal simplicial circle) with the group completion of the map $\Psi(\underline{1})$. Moreover, the natural H-map $\coprod_{n \geqq 0} \mathrm{BGL}_n^{\hat{}\mathrm{X}} \to \hat{\mathrm{BGL}}^{\hat{}\mathrm{X}} \times \mathbf{Z}$ is a group completion because the fact that the fibre of $\mathrm{BGL}_n^{\hat{}\mathrm{X}} \to \mathrm{BGL}^{\hat{}\mathrm{X}}$ becomes more highly connected as n increases implies that the homology of $\mathrm{BGL}^{\hat{}\mathrm{X}} \times \mathbf{Z}$ is the localization of the homology of $\coprod_{n \geqq 0} \mathrm{BGL}_n^{\hat{}\mathrm{X}}$ with respect to $\mathbf{N}^+ \subset \mathbf{N} \subset \pi_0 (\coprod_{n \geqq 0} \mathrm{BGL}_n^{\hat{}\mathrm{X}})$ (*cf.* [14]). Consequently, we have obtained the asserted infinite loop space map (given by $\Psi(\mathrm{S}^1)$ with appropriate identifications):

$$\psi: \quad \mathrm{BGL}(\mathrm{A})^+ \to \mathrm{BGL}^{\hat{}\mathrm{X}}.$$

We define $\hat{\rho}_i = \pi_i \circ \psi$ for $i > 0$ and $\bar{\rho}_i = \pi_i(\ \ , \mathbf{Z}/l^\nu) \circ \psi$ for $i > 1$. These definitions are extended to $\hat{\rho}_0$, $\bar{\rho}_1$, and $\bar{\rho}_0$ by employing the map $\otimes\, t: \mathrm{K}_0(\mathrm{A}) \to \mathrm{Ker}\{\mathrm{K}_1(\mathrm{A}[t, t^{-1}]) \to \mathrm{K}_1(\mathrm{A})\}$ and the following (homotopy) commutative square:

$$\begin{array}{ccc} \mathrm{BGL}(\mathrm{A}[t, t^{-1}])^+ & \xrightarrow{\psi} & \mathrm{BGL}^{\hat{}\mathrm{X} \times \mathrm{GL}_1} \\ \downarrow & & \downarrow \\ \mathrm{BGL}(\mathrm{A})^+ & \xrightarrow{\psi} & \mathrm{BGL}^{\hat{}\mathrm{X}} \end{array}$$

To define $\hat{\rho}_i : K_i(A, A/I) \to \hat{K}_i^{et}(X, Y)$ and $\bar{\rho}_i : K_i(A, A/I; \mathbf{Z}/l^\nu) \to K_i^{et}(X, Y; \mathbf{Z}/l^\nu)$, we use the naturality of the preceding construction to get a commutative square:

$$
\begin{array}{ccc}
\mathscr{B}\,GL(A)(S^1) & \xrightarrow{\Psi} & \mathscr{B}\,GL^{\wedge X}(S^1) \\
\downarrow & & \downarrow \\
\mathscr{B}\,GL(A/I)(S^1) & \xrightarrow{\Psi} & \mathscr{B}\,GL^{\wedge Y}(S^1)
\end{array} \tag{1.3.1}
$$

which determines a well defined homotopy class of maps on homotopy fibres. This determines $\hat{\rho}_i$ for $i>0$ and $\bar{\rho}_i$ for $i>1$. For $\hat{\rho}_0$, $\bar{\rho}_1$, and $\bar{\rho}_0$, we use the fact that (1.3.1) fits in a commutative cube with the analogous square fore $A[t, t^{-1}] \to A/I[t, t^{-1}]$ and $Y \times GL_1 \to X \times GL_1$.

Finally, to extend $\hat{\rho}_i$ and $\bar{\rho}_i$ to a closed immersion of (not necessarily affine) quasi-projective schemes $Y \to X$, we employ Jouanalou's construction [13]: a scheme X quasi-projective over F admits an "affine resolution" $\alpha : \tilde{X} \to X$ with X affine and α locally in the etale topology on X a product projection with affine spaces as fibres. Because $\alpha_{et} : \tilde{X}_{et} \to X_{et}$ and $\alpha|_{et} : \tilde{Y}_{et} \to Y_{et}$ are $\mathbf{Z}/l$-equivalences (where $\alpha|$ is the pull-back of α *via* $Y \to X$), we may define $\hat{\rho}_i$ for $i>0$ and $\bar{\rho}_i$ for $i>1$ by:

$$\hat{\rho}_i = (\alpha_{et}^*)^{-1} \circ \hat{\rho}_i \circ \alpha^* : \quad K_i(X, Y) \to K_i(\tilde{X}, \tilde{Y}) \to \hat{K}_i^{et}(\tilde{X}, \tilde{Y}) \to \hat{K}_i^{et}(X, Y),$$

$$\bar{\rho}_i = (\alpha_{et}^*)^{-1} \circ \bar{\rho}_i \circ \alpha^* : \quad K_i(X, Y, \mathbf{Z}/l^\nu) \to K_i(\tilde{X}, \tilde{Y}; \mathbf{Z}/l^\nu) \to K_i^{et}(\tilde{X}, \tilde{Y}; \mathbf{Z}/l^\nu) \to K_i^{et}(X, Y; \mathbf{Z}/l^\nu).$$

These definitions are extended to $\hat{\rho}_0, \bar{\rho}_1, \bar{\rho}_0$ by applying the argument used in the affine case to $\tilde{X} \times GL_1$. So defined, $\hat{\rho}_i$ and $\bar{\rho}_i$ are independent of the affine resolution $\alpha : \tilde{X} \to X$ because any two affine resolutions are dominated by a third (their fibre product over X). Naturality is verified for a map $f : X' \to X$ by using the fact that $\tilde{X}' \times_X \tilde{X}$ is $(\mathbf{Z}/l)$-equivalent to $\tilde{X}'$ and maps to $\tilde{X}$ over f whenever $\alpha' : \tilde{X}' \to X'$ and $\alpha : \tilde{X} \to X$ are affine resolutions. ∎

We recall that tensor product determines a ring space structure on $BU \times \mathbf{Z}$, thereby determining associative, graded commutative ring structures on $\hat{K}^*((\ )_{et})$ and $K^*((\ )_{et}, \mathbf{Z}/l^\nu)$ for $l^\nu \neq 2$ (a universal choice of co-product on mod-l^ν Moore spaces must be made for l^ν even; *cf.* [2]). As argued in [11], 3.2, these ring structures correspond under the isomorphisms (1.2.1) and (1.2.2) to ring structures on $\hat{K}_*^{et}(\ )$ and $K_*^{et}(\ ;\mathbf{Z}/l^\nu)$ induced by the external tensor product homomorphisms:

$$\otimes : \quad GL_m \times GL_n \to GL_{mn}, \qquad m, n \geqq 0.$$

For an algebra A over F, these homomorphisms determine ring structures on $K_*(A)$ and $K_*(A, \mathbf{Z}/l^\nu)$ for $l^\nu \neq 2$ (for l^ν even, we use our choice of co-product on mod-l Moore spaces), associative and commutative for $l^\nu \neq 2, 3, 4, 8$ (*cf.* [2]).

Proposition 1.4. — *The natural transformations $\hat{\rho}_i$ and $\bar{\rho}_i$ for $l^\nu \neq 2$ and $i \geqq 0$ determine contravariant functors:*

$$\hat{\rho}_*, \quad \bar{\rho}_* : \begin{pmatrix} \textit{closed immersions of} \\ \textit{schemes quasi-projective}/F \end{pmatrix} \to (\textit{graded rings with unit}).$$

Proof. — We recall that a "Segal Γ-space with multiplication" consists of a Segal Γ-space $\mathscr{B}:\mathscr{F}\to\mathscr{S}_0$ together with an associated functor $\tilde{\mathscr{B}}:\mathscr{F}\times\mathscr{F}\to\mathscr{S}_0$ provided with natural transformations $i_1:\tilde{\mathscr{B}}\to\mathscr{B}\circ\mathrm{pr}_1$, $i_2:\tilde{\mathscr{B}}\to\mathscr{B}\circ\mathrm{pr}_2$, and $\mu:\tilde{\mathscr{B}}\to\tilde{\mathscr{B}}\to\mathscr{B}\circ\otimes$ [where $\otimes:\mathscr{F}\times\mathscr{F}\to\mathscr{F}$ sends (m, n) to mn] such that:

$$i_1\times i_2:\quad \tilde{\mathscr{B}}(m, n)\to\mathscr{B}(m)\times\mathscr{B}(n),$$

is a weak equivalence for all $m, n\geqq 0$ [17]. The Segal Γ-space $\mathscr{B}\mathrm{GL}(\mathrm{A}):\mathscr{F}\to\mathscr{S}_0$ (respectively, $\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}}:\mathscr{F}\to\mathscr{S}_0$) admits this added structure determined by the bi-permutative category structure on $\mathscr{G}l(\mathrm{A})$ (resp., $\mathscr{G}l$) whose second multiplication is determined by tensor product. The natural transformation $\Psi:\mathscr{B}\mathrm{GL}(\mathrm{A})\to\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}}$ extends to a natural transformation $\tilde{\Psi}:\tilde{\mathscr{B}}\mathrm{GL}(\mathrm{A})\to\tilde{\mathscr{B}}\mathrm{GL}^{\hat{}\mathrm{X}}$ commuting with i_1, i_2, μ. Therefore, $\psi:\mathrm{BGL}(\mathrm{A})^+\times\mathbf{Z}\to\mathrm{BGL}^{\hat{}\mathrm{X}}\times\mathbf{Z}$ is a map of ringed spaces so that:

$$(1.4.1)\qquad \otimes\circ(\hat{\rho}_i\times\hat{\rho}_{i'})=\hat{\rho}_{i+i'}\circ\otimes:\quad \mathrm{K}_i(\mathrm{A})\times\mathrm{K}_{i'}(\mathrm{A})\to\hat{\mathrm{K}}^{\mathrm{et}}_{i+i'}(\mathrm{X}),$$

$$(1.4.2)\qquad \otimes\circ(\bar{\rho}_j\times\bar{\rho}_{j'})=\bar{\rho}_{j+j'}\circ\otimes:\quad \mathrm{K}_j(\mathrm{A}, \mathbf{Z}/l^\nu)\times\mathrm{K}_{j'}(\mathrm{A}', \mathbf{Z}/l^\nu)\to\mathrm{K}^{\mathrm{et}}_{j+j'}(\mathrm{X}, \mathbf{Z}/l^\nu),$$

for $i, i'>0$ and $j, j'>1$.

More generally, we conclude that the natural transformation Ψ determines a homotopy commutative square of ringed spaces:

$$(1.4.3)\qquad \begin{array}{ccc} (\mathrm{BGL}(\mathrm{A})^+\times\mathbf{Z})\wedge(\mathrm{BGL}(\mathrm{B})^+\times\mathbf{Z}) & \xrightarrow{\otimes} & (\mathrm{BGL}(\mathrm{A}\otimes\mathrm{B})^+\times\mathbf{Z}) \\ \downarrow{\scriptstyle \psi\wedge\psi} & & \downarrow{\scriptstyle \psi} \\ (\mathrm{BGL}^{\hat{}\mathrm{X}}\times\mathbf{Z})\wedge(\mathrm{BGL}^{\hat{}\mathrm{Z}}\times\mathbf{Z}) & \xrightarrow{\otimes} & (\mathrm{BGL}^{\hat{}\mathrm{X}\times\mathrm{Z}}\times\mathbf{Z}) \end{array}$$

where $\mathrm{X}=\mathrm{Spec}\,\mathrm{A}$ and $\mathrm{Z}=\mathrm{Spec}\,\mathrm{B}$. Namely, (1.4.3) follows from the fact that Ψ extends to a natural transformation of functors on $\mathscr{F}\times\mathscr{F}$ to $\mathscr{S}_0$:

$$\langle\mathscr{B}\mathrm{GL}(\mathrm{A}), \mathscr{B}\mathrm{GL}(\mathrm{B})\rangle\to\langle\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}}, \mathscr{B}\mathrm{GL}^{\hat{}\mathrm{Z}}\rangle,$$

mapping to $\Psi:\mathscr{B}\mathrm{GL}(\mathrm{A})\to\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}}$ *via* i_1, $\Psi:\mathscr{B}\mathrm{GL}(\mathrm{B})\to\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{Z}}$ *via* i_2, and $\Psi:\mathscr{B}\mathrm{GL}(\mathrm{A}\otimes\mathrm{B})\to\mathscr{B}\mathrm{GL}^{\hat{}\mathrm{X}\times\mathrm{Z}}$ *via* μ.

We employ (1.4.3) to prove (1.4.1) for $i>0$ and $i'=0$ as follows. Consider the map of complexes:

$$(1.4.4)\qquad \begin{array}{ccccc} \mathrm{K}_i(\mathrm{A})\otimes\mathrm{K}_0(\mathrm{A}) & \to & \mathrm{K}_i(\mathrm{A})\otimes\mathrm{K}_1(\mathrm{A}[t, t^{-1}]) & \to & \mathrm{K}_i(\mathrm{A})\otimes\mathrm{K}_1(\mathrm{A}) \\ \downarrow{\scriptstyle\otimes} & & \downarrow{\scriptstyle\otimes} & & \downarrow{\scriptstyle\otimes} \\ \mathrm{K}_i(\mathrm{A}) & \to & \mathrm{K}_{i+1}(\mathrm{A}[t, t^{-1}]) & \to & \mathrm{K}_{i+1}(\mathrm{A}) \end{array}$$

By the naturality of (1.4.3), ψ_* maps the right hand square of (1.4.4) to:

$$\begin{array}{ccc} \hat{\mathrm{K}}^{\mathrm{et}}_i(\mathrm{X})\otimes\hat{\mathrm{K}}^{\mathrm{et}}_1(\mathrm{X}\times\mathrm{GL}_1) & \to & \hat{\mathrm{K}}^{\mathrm{et}}_i(\mathrm{X})\otimes\hat{\mathrm{K}}^{\mathrm{et}}_1(\mathrm{X}) \\ \downarrow{\scriptstyle\otimes} & & \downarrow{\scriptstyle\otimes} \\ \hat{\mathrm{K}}^{\mathrm{et}}_{i+1}(\mathrm{X}\times\mathrm{GL}_1) & \to & \hat{\mathrm{K}}^{\mathrm{et}}_{i+1}(\mathrm{X}) \end{array}$$

Consequently, ψ_* determines a commutative square:

$$\text{(1.4.5)}\qquad \begin{array}{ccc} K_i(A)\otimes K_0(A) & \to & \hat{K}_i^{et}(X)\otimes \ker\{\hat{K}_1^{et}(X\times GL_1)\to \hat{K}_1^{et}(X)\} \\ \downarrow\otimes & & \downarrow\otimes \\ K_i(A) & \to & \ker\{\hat{K}_{i+1}^{et}(X\times GL_1)\to \hat{K}_{i+1}^{et}(X)\} \end{array}$$

whose right vertical arrow determines (by definition) the tensor product multiplication:

$$\hat{K}_i^{et}(X)\otimes\hat{K}_0^{et}(X)\to \hat{K}_i^{et}(X)\simeq \ker\{\hat{K}_{i+1}^{et}(X\times GL_1)\to \hat{K}_{i+1}^{et}(X)\}.$$

The cases $i=0$, $i'>0$ for (1.4.1); $j\geqq 2$, $j'<2$ and $j<2$, $j'\geqq 2$ for (1.4.2) are treated in exactly the same manner as $i>0$, $i'=0$ for (1.4.1). The case $i=0=i'$ for (1.4.1) is treated by considering the following analogue of (1.4.4):

$$\begin{array}{ccccc} K_0(A)\otimes K_0(A) & \to & K_1(A[s,s^{-1}])\otimes K_1(A[t,t^{-1}]) & \to & (K_1(A[s,s^{-1}])\otimes K_1(A))\oplus(K_1(A)\otimes K_1(A[t,t^{-1}])) \\ \downarrow\otimes & & \downarrow\otimes & & \downarrow\otimes+\otimes \\ K_0(A) & \longrightarrow & K_2(A[s,s^{-1},t,t^{-1}]) & \longrightarrow & K_2(A[s,s^{-1}])\oplus K_2(A[t,t^{-1}]) \end{array}$$

The cases in which both j and j' are less than 2 in (1.4.2) are treated similarly.

For a closed immersion $Y=\operatorname{Spec}A/I\to X=\operatorname{Spec}A$, we use the commutative square (1.3.1), whose maps we've seen are maps of ringed spaces.

For X not necessarily affine, we verify that $\hat{\rho}_*$ and $\bar{\rho}_*$ are ring homomorphisms by observing that:

$$\alpha^*:\ K_*(X)\to K_*(\tilde{X}),\qquad \alpha^*:\ K_*(X,\mathbf{Z}/l^\nu)\to K_*(\tilde{X},\mathbf{Z}/l^\nu)$$

and:

$$\alpha_{et}^*:\ \hat{K}_*^{et}(X)\to \hat{K}_*^{et}(\tilde{X}),\qquad \alpha_{et}^*:\ K_*^{et}(X,\mathbf{Z}/l^\nu)\to K_*^{et}(\tilde{X},\mathbf{Z}/l^\nu)$$

are ring homomorphisms for any affine resolution $\alpha:\tilde{X}\to X$.

Finally, to prove that $\hat{\rho}_*$ and $\bar{\rho}_*$ preserve units, it suffices by naturality to check for $X=\operatorname{Spec}F$, $Y=\emptyset$. This special case follows from the fact (verified by inspection) that:

$$\hat{\rho}_0\otimes\mathbf{Z}_l:\ K_0(F)\otimes\mathbf{Z}_l\to \hat{K}_0^{et}(X),\qquad \bar{\rho}_0:\ K_0(F,\mathbf{Z}/l^\nu)\to K_0^{et}(X,\mathbf{Z}/l^\nu),$$

are ring isomorphisms (where $\mathbf{Z}_l=\varprojlim \mathbf{Z}/l^\nu$). ■

Because tensor product is the coproduct in the category of (graded) commutative, associative rings with unit, the following is an immediate corollary of Proposition 1.4.

COROLLARY 1.5. — *Let* $X\to W$, $Z\to W$ *be maps of schemes quasi-projective over* F. *Then there are commutative squares* (*of commutative, associative rings*):

$$\text{(1.5.1)}\qquad \begin{array}{ccc} K_*(X)\underset{K_*(W)}{\otimes} K_*(Z) & \xrightarrow{\otimes} & K_*(X\underset{W}{\times} Z) \\ \downarrow \hat{\rho}_*\otimes\hat{\rho}_* & & \downarrow \hat{\rho}_* \\ \hat{K}_*^{et}(X)\underset{\hat{K}_*^{et}(W)}{\otimes}\hat{K}_*^{et}(Z) & \xrightarrow{\otimes} & \hat{K}_*^{et}(X\underset{W}{\times} Z) \end{array}$$

$$(1.5.2)\qquad \begin{array}{ccc} K_*(X, \mathbf{Z}/l^\nu) \underset{K_*(W,\mathbf{Z}/l^\nu)}{\otimes} K_*(Z; \mathbf{Z}/l^\nu) & \xrightarrow{\otimes} & K_*(X \underset{W}{\times} Z; \mathbf{Z}/l^\nu) \\ \downarrow{\scriptstyle \bar\rho_* \otimes \bar\rho_*} & & \downarrow{\scriptstyle \bar\rho_*} \\ K^{et}_*(X, \mathbf{Z}/l^\nu) \underset{K^{et}_*(W,\mathbf{Z}/l^\nu)}{\otimes} K^{et}_*(Z, \mathbf{Z}/l^\nu) & \xrightarrow{\otimes} & K^{et}_*(X \underset{W}{\times} Z; \mathbf{Z}/l^\nu) \end{array}$$

provided that $l^\nu \neq 2, 3, 4, 8$.

The following proposition gives an explicit description of $\hat\rho_1$, leading to a more concrete description of $\hat\rho_0$ in Proposition 1.7 and $\bar\rho_1$, $\bar\rho_0$ in Proposition 1.8.

PROPOSITION 1.6. — *Let* $X = \operatorname{Spec} A$ *be an affine scheme of finite type over* F. *There is a natural isomorphism:*

$$\theta : \hat K^{et}_1(X) \simeq \varinjlim_{n>0} \operatorname{Hom}_{\text{pro-}\mathscr{H}_0}((X/\varphi)_{et}, GL_n^\wedge)$$

(*where* $GL_n^\wedge = \# \circ (\mathbf{Z}/l)_\infty (GL_n)_{et}$) *such that* $\theta \circ \hat\rho_1$ *is determined by sending* $\alpha \in GL_n(A) = \operatorname{Hom}(X, GL_n)$ *to* $\alpha_{et}^\wedge : X_{et} \to GL_n^\wedge$.

Proof. — Let $\gamma_n : GL_n(A) \to GL_n^{\wedge X} = \varprojlim{}^{h} \varinjlim \operatorname{Hom}.((X/\varphi)_{et}, GL_n^\wedge)$ be defined by sending $\alpha \in GL_n(A)$ to $\alpha_{et}^\wedge : X_{et} \to GL_n^\wedge$, so that the following diagram commutes:

$$(1.5.1)_n\qquad \begin{array}{ccccc} GL_n(A) & \to & EGL_n(A) & \to & BGL_n(A) \\ \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle E\psi_n} & & \downarrow{\scriptstyle \psi_n} \\ GL_n^{\wedge X} & \to & EGL_n^{\wedge X} & \to & BGL_n^{\wedge X} \end{array}$$

Because $GL_n^\wedge \to EGL_n^\wedge \to BGL_n^\wedge$ is equivalent to an inverse system of fibre triples, the colimit of $(1.5.1)_n$ with respect to n determines a map of fibre triples, $\varinjlim (1.5.1)_n$, with contractible total spaces. We define θ as the composition:

$$\theta : \hat K^{et}_1(X) \simeq \pi_0(GL^{\wedge X}) \simeq \varinjlim \operatorname{Hom}_{\text{pro-}\mathscr{H}_0}((X/\emptyset)_{et}, GL_n^\wedge),$$

where the first isomorphism is the connecting homomorphism of the bottom row of $\varinjlim (1.5.1)_n$ and the second is given by [12], 13.10. Because $\pi_1(BGL(A)) \to \pi_1(BGL(A)^+)$ is the abelianization map, $\hat\rho_1$ can be viewed as the abelianization of $\pi_1(\ )$ applied to $\varinjlim \psi_n$: $BGL(A) \to BGL^{\wedge X}$ or equally as the abelianization of $\pi_0(\ )$ applied to $\varinjlim \gamma_n$. ■

In [11], we studied in detail a homomorphism:

$$\hat\rho : K_0(X, Y) \to \hat K^0_{et}(X, Y)$$
$$= H^0(X, Y; \mathbf{Z}_l) \times \varprojlim_r \varinjlim_{m,n} \operatorname{Hom}_{\text{pro-}\mathscr{H}_0}((X/Y)_{et}, \operatorname{cosk}_r(\mathrm{Grass}_{m+n,n})^\wedge),$$

defined for $X = \operatorname{Spec} A$, $Y = \emptyset$ by sending a rank n, projective A-module P to $(n, (\tau_p)_{et}^\wedge)$, where $\tau_p : X \to \mathrm{Grass}_{m+n,n}$ is a classifying map for P. The next proposition verifies that $\hat\rho$ may be identified with $\hat\rho_0$ of Theorem 1.3.

PROPOSITION 1.7. — *For any scheme* X *quasi-projective over* F, *there exists a natural isomorphism* $\Lambda : \hat{K}^0_{et}(X) \simeq \hat{K}^{et}_0(X)$ *with the property that:*

$$\hat{\rho}_0 = \Lambda \circ \hat{\rho} : \quad K_0(X) \to \hat{K}^{et}_0(X).$$

Proof. — The isomorphism $\Lambda^{-1} : \hat{K}^{et}_0(X) \simeq \hat{K}^0_{et}(X)$ is given by the composition of the isomorphism $\hat{K}^{et}_0(X) \simeq \hat{K}^0(X_{et})$ of Proposition 1.2 and the isomorphism $\hat{K}^0(X_{et}) \simeq \hat{K}^0_{et}(X)$ implied by the weak equivalence $\#(\mathrm{Sin\,BU})\hat{} \to \{\cos k_r \, \widehat{\mathrm{Grass}}_{2r,r}\}$.

Using the naturality of $\hat{\rho}_0$ and $\hat{\rho}$ together with an affine resolution $\alpha : \tilde{X} \to X$, we conclude that it suffices to prove the commutativity of the following square:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\Lambda \circ \hat{\rho}} & \hat{K}^{et}_0(X) \\ \downarrow{\scriptstyle \otimes t} & & \downarrow{\scriptstyle i} \\ K_1(A[t, t^{-1}]) & \xrightarrow{\hat{\rho}_1} & \hat{K}^{et}_1(X \times GL_1) \end{array}$$

where the right vertical arrow is the defining inclusion and X = Spec A.

By naturality, it suffices to verify that $\hat{\rho}_1 \circ (\otimes t) = i \circ \Lambda \circ \hat{\rho}$ on the universal projective A module over $GL_{m+n}/GL_m \times GL_n = \mathrm{Spec}\, A \ (= \tilde{\mathrm{G}}\mathrm{rass}_{m+n,n})$ for $m, n > 0$. Using standard "lifting to characteristic 0" arguments (*cf.* [11], 3.6), we conclude that it suffices to assume $F = \mathbf{C}$ because the universal projective module over $\tilde{\mathrm{G}}\mathrm{rass}_{m+n,n}$ is the reduction of a (universal) projective module over the lifting of $\tilde{\mathrm{G}}\mathrm{rass}_{m+n,n}$ to the Witt vectors of the residue field of F. Using the relationship between $\hat{\rho}$ and the forgetful functor ρ sending an algebraic vector bundle (i. e., projective module) to its associated topological vector bundle given by [11], 3.5, and employing Proposition 1.6, we conclude that it suffices to prove the commutativity of the following square for the complex affine variety X = Spec A with associated analytic space X^{top}:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\rho} & [X^{top}, BU \times \mathbf{Z}] \\ \downarrow{\scriptstyle \otimes t} & & \downarrow{\scriptstyle i} \\ K_1(A[t, t^{-1}]) & \xrightarrow{\gamma} & [X^{top} \times \mathbf{C}^*, U] \end{array} \tag{1.7.1}$$

In (1.7, 1), [,] indicates homotopy classes of maps, γ is determined by the map sending $\alpha : X \times GL_1 \to GL_N$ in $GL_N(A[t, t^{-1}])$ to $\alpha^{top} : X^{top} \times \mathbf{C}^* \to U_N$, and i sends $f : X^{top} \to BU \times \mathbf{Z}$ to $\otimes \circ (f, t) : X^{top} \times S^1 \to (BU \times \mathbf{Z}) \times U \to U$. We recall that $\otimes t : K_0(A) \to K_1(A[t, t^{-1}])$ sends the projective A-module P with $P \oplus Q \simeq A^N$ to $(\otimes t)(P)$ represented by $t \oplus 1$:

$$P[t, t^{-1}] \oplus Q[t, t^{-1}] \to P[t, t^{-1}] \oplus Q[t, t^{-1}] \quad \text{in } GL_N(A[t, t^{-1}]).$$

Thus, $\gamma \circ (\otimes t)(P)$ is represented by $X^{top} \times \mathbf{C}^* \to GL_N(\mathbf{C})$ defined to send (x, s) to $s \oplus 1 : P_x \oplus Q_x \simeq P_x \oplus Q_x$. This map is well known to represent $i \circ \rho(P)$. ■

In conjunction with Propositions 1.6 and 1.7, the following proposition (in the absolute case $Y = \emptyset$) determines $\bar{\rho}_1$ up to extension and $\bar{\rho}_0$ in terms of more explicit constructions than those of Theorem 1.3.

PROPOSITION 1.8. — *For any closed immersion* $Y \to X$ *of schemes quasi-projective over* F, $\hat{\rho}_i$ *and* $\bar{\rho}_i$ *with* $i \geqq 0$, $\nu \geqq 1$ *determine the following map* (i. e., "*commutative ladder*") *of long exact sequences* (1.8.1):

$$\begin{array}{ccccccccccccc} \ldots \to K_2(X, \mathbf{Z}/l^\nu) & \to & K_1(X) & \to & K_1(X) & \to & K_1(X, \mathbf{Z}/l^\nu) & \to & K_0(X) & \to & K_0(X) & \to & K_0(X, \mathbf{Z}/l^\nu) \\ \downarrow & (0) & \downarrow & & \downarrow & (I) & \downarrow & (II) & \downarrow & & \downarrow & (III) & \downarrow \\ \ldots \to K_2^{et}(X, \mathbf{Z}/l^\nu) & \to & \hat{K}_1^{et}(X) & \to & \hat{K}_1^{et}(X) & \to & K_1^{et}(X, \mathbf{Z}/l^\nu) & \to & \hat{K}_0^{et}(X) & \to & \hat{K}_0^{et}(X) & \to & K_0^{et}(X, \mathbf{Z}/l^\nu) \end{array}$$

Proof. — The upper long exact sequence (with $K_i(X) \to K_i(X)$ equal to multiplication by l^ν) is a formal consequence of the definition of $K_*(X, \mathbf{Z}/l^\nu)$; the lower exact sequence, in view of Proposition 1.2, is also a formality. The commutativity of all squares in the above ladder except (I), (II), and (III) is clear. The commutativity of (I) follows from the definition of $K_1^{et}(X, Y; \mathbf{Z}/l^\nu)$ as $\operatorname{Ker}\{K_2^{et}(X \times GL_1, Y \times GL_1; \mathbf{Z}/l^\nu) \to K_2^{et}(X, Y; \mathbf{Z}/l^\nu)\}$ and the commutativity of the following diagram:

$$\begin{array}{ccccccc} K_1(X) & \xrightarrow{\otimes l} & K_2(X \times GL_1) & \to & K_2(X \times GL_1, \mathbf{Z}/l^\nu) & \to & K_2(X, \mathbf{Z}/l^\nu) \\ \downarrow \hat{\rho}_1 & & \downarrow \hat{\rho}_2 & & \downarrow \bar{\rho}_2 & & \downarrow \bar{\rho}_2 \\ \hat{K}_1^{et}(X) & \xrightarrow{\otimes l} & \hat{K}_2^{et}(X \times GL_1) & \to & K_2^{et}(X \times GL_1, \mathbf{Z}/l^\nu) & \to & K_2^{et}(X, \mathbf{Z}/l^\nu) \end{array}$$

The commutativity of (II) and (III) follow directly from the naturality of the top and bottom rows of (1.8.1) and the commutativity of (0) and (I) for both X and $X \times GL_1$.

2. Examples and Mayer-Vietrois

We examine the natural transformations $\hat{\rho}_*$ and $\bar{\rho}_*$ in a few specific cases: a point (Example 2.1), a torus (Proposition 2.4), a multi-punctured affine line (Corollary 2.6), affine and projective space minus hyperplanes (Propositions 2.7 and 2.8), and a smooth curve (Proposition 2.9). These examples serve to indicate the non-triviality of our natural transformations as well as provide new computations of algebraic K-groups with coefficients.

Throughout this section, we consider a separably closed field F of characteristic p and a prime $l \neq p$. Of particular interest is the special case $F = \overline{\mathbf{F}}_p$, the algebraic closure of the prime field. Computations restricted to this special case apply to more general separably closed fields F provided that $\overline{\mathbf{F}}_p \to F$ induces an isomorphism $K_*(\overline{\mathbf{F}}_p, \mathbf{Z}/l) \simeq K_*(F, \mathbf{Z}/l)$ (or that we consider the natural transformation $\bar{\rho}_* : K_*(\ , \mathbf{Z}/l^\nu)[1/\beta_x] \to K_*^{et}(\ , \mathbf{Z}/l^\nu)$ of Proposition 2.2 and assume that $\overline{\mathbf{F}}_p \to F$ induces an isomorphism :

$$K_*(\overline{\mathbf{F}}_p, \mathbf{Z}/l^\nu)[1/\beta_x] \simeq K_*(F, \mathbf{Z}/l^\nu)[1/\beta_x]).$$

Example 2.1. — Let P denote $\operatorname{Spec} \overline{\mathbf{F}}_p$. Because P_{et} is contractible :

$$\hat{K}_{2i}^{et}(P) = \mathbf{Z}_l, \qquad \hat{K}_{2i}^{et}(P) = 0, \qquad K_{2i}^{et}(P, \mathbf{Z}/l^\nu) = \mathbf{Z}/l^\nu, \qquad K_{2i+1}^{et}(P, \mathbf{Z}/l^\nu) = 0,$$

for any $i \geqq 0$, $\nu > 0$, whereas $K_{2i}(\overline{\mathbf{F}}_p) = 0$ and $K_{2i+1}(\overline{\mathbf{F}}_p) = \varinjlim_{(n,p)=1} \mathbf{Z}/n$. Consequently,

$\hat{\rho}_i = 0$ for $i > 0$. On the other hand, $\psi_m : BGL_m(\overline{\mathbf{F}}_p) \to BGL_m^{\wedge}$ induces an isomorphism in $\mathbf{Z}/l$ cohomology for each $m \geqq 0$ (*cf.* [9]), so that $\psi : BGL(\overline{\mathbf{F}}_p)^+ \to BGL^{\wedge p}$ is a $\mathbf{Z}/l$-equivalence. Consequently, for any $l^\nu \neq 2$:

$$\bar{\rho}_* : \quad K_*(\overline{\mathbf{F}}_p, \mathbf{Z}/l^\nu) \to K_*^{et}(P, \mathbf{Z}/l^\nu) \simeq \mathbf{Z}/l^\nu[\beta], \qquad \deg(\beta) = 2,$$

is a ring isomorphism as observed by Browder in [5]. ■

Using Example 2.1, we next verify that $\bar{\rho}_*$ factors through the map from $K_*(X, \mathbf{Z}/l^\nu)$ to "$K_*(X, \mathbf{Z}/l^\nu)$ with the Bott element inverted." The latter theory is considered by R. Thomason in [19].

PROPOSITION 2.2. — *Let $\beta \in K_2(\overline{\mathbf{F}}_p, \mathbf{Z}/l^\nu)$ be a generator (corresponding to a primitive l^ν-th root of unity in $\overline{\mathbf{F}}_p$), for $l^\nu \neq 2$. For any quasi-projective variety X over F, $\bar{\rho}_*$: $K_*(X, \mathbf{Z}/l^\nu) \to K_*^{et}(X, \mathbf{Z}/l^\nu)$ factors through the localization:*

$$K_*(X, \mathbf{Z}/l^\nu) \to K_*(X, \mathbf{Z}/l^\nu)[1/\beta_x]^+,$$

where $\beta_x \in K_2(X, \mathbf{Z}/l^\nu)$ is the image of β and $K_*(X, \mathbf{Z}/l^\nu)[1/\beta_x]^+$ is the subring of $K_*(X, \mathbf{Z}/l^\nu)$ $[1/\beta_x]$ of elements of non-negative degree.

Proof. — Because $K_*^{et}(X, \mathbf{Z}/l^\nu) \simeq K_*(X_{et}, \mathbf{Z}/l^\nu)^+$, where $K_*(X_{et}, \mathbf{Z}/l^\nu)$ is the $\mathbf{Z}$-graded, $\mathbf{Z}/2$-periodic, mod-l^ν K-theory of X_{et}, it suffices to prove that $\bar{\rho}_2(\beta_x)$ is invertible in $K_*(X_{et}, \mathbf{Z}/l^\nu)$. By naturality, it suffices to observe that the image of $\beta \in K_2(F, \mathbf{Z}/l^\nu)$ under $\bar{\rho}_2$ is an invertible multiple of the Bott element in the mod-l^ν homotopy of $BGL^{\wedge \mathrm{Spec} F}$ (weakly equivalent to $(\mathbf{Z}_l)_\infty \circ (\mathrm{Sin}\ BU)$), because the composition $BGL(\overline{\mathbf{F}}_p)^+ \to BGL(F)^+ \to BGL^{\wedge \mathrm{Spec} F}$ is a $\mathbf{Z}/l$-equivalence. ■

We next employ Proposition 1.6 to study the example of GL_1.

Example 2.3. — Let $GL_1 = \mathrm{Spec}\ \overline{\mathbf{F}}_p[t, t^{-1}]$ and $l^\nu \neq 2$. Because:

$$K_i(\overline{\mathbf{F}}_p[t, t^{-1}]) = K_i(\overline{\mathbf{F}}_p) \oplus K_{i-1}(\overline{\mathbf{F}}_p),$$

we conclude that:

$$K_i(\overline{\mathbf{F}}_p[t, t^{-1}], \mathbf{Z}/l^\nu) = \mathbf{Z}/l^\nu \quad \text{for any } i \geqq 0.$$

Because $S^1 \to (GL_1)_{et}$ is a $\mathbf{Z}/l$-equivalence, we conclude that:

$$\hat{K}_*^{et}(GL_1) \simeq \mathbf{Z}_l[t, \beta]/t^2, \qquad K_*^{et}(GL_1, \mathbf{Z}/l^\nu) \simeq \mathbf{Z}/l^\nu[t, \beta]/t^2,$$

with $\deg(t) = 1$ and $\deg(\beta) = 2$. The unit $t \in \overline{\mathbf{F}}_p[t, t^{-1}]$ corresponds to $1 : GL_1 \to GL_1$, so that $t_{et} : (GL_1)_{et} \to GL_n^{\wedge}$ is a topological generator of $\pi_0(GL_n^{\wedge GL_1})$ for any $n \geqq 1$. Hence, Proposition 1.6 implies that $\hat{\rho}_1 : K_1(\overline{\mathbf{F}}_p[t, t^{-1}]) \to \hat{K}_1^{et}(GL_1)$ sends the class of t to a topological generator. We conclude using Proposition 1.8 that

$$\bar{\rho}_1 : \quad K_1(\overline{\mathbf{F}}_p[t, t^{-1}], \mathbf{Z}/l^\nu) \to K_1^{et}(GL_1, \mathbf{Z}/l^\nu),$$

is an isomorphism.

We employ the following commutative squares for $\varepsilon=0$ and 1.

$$(2.3.1)\qquad \begin{array}{ccc} K_{2i}(\bar{\mathbf{F}}_p, \mathbf{Z}/l^\nu)\otimes K_\varepsilon(\bar{\mathbf{F}}_p[t, t^{-1}], \mathbf{Z}/l^\nu) & \xrightarrow{\otimes} & K_{2i+\varepsilon}(\bar{\mathbf{F}}_p[t, t^{-1}], \mathbf{Z}/l^\nu) \\ \downarrow{\scriptstyle \bar{\rho}_{2i}\otimes\bar{\rho}_\varepsilon} & & \downarrow{\scriptstyle \bar{\rho}_{2i+\varepsilon}} \\ K^{et}_{2i}(P, \mathbf{Z}/l^\nu)\otimes K^{et}_\varepsilon(GL_1, \mathbf{Z}/l^\nu) & \xrightarrow{\otimes} & K^{et}_{2i+\varepsilon}(GL_1, \mathbf{Z}/l^\nu) \end{array}$$

implied by the proof of Proposition 1.4 (*cf.* (1.4.3)). By Example 2.1 and the above discussion, $\bar{\rho}_{2i}\otimes\bar{\rho}_\varepsilon$ is an isomorphism. Clearly, the horizontal maps of (2.3.1) are isomorphisms for $\varepsilon=0$; for $\varepsilon=1$, the top arrow is the mod-l^ν reduction of the isomorphism:

$$K_{2i}(\bar{\mathbf{F}}_p, \mathbf{Z}/l^\nu)\otimes K_1(\bar{\mathbf{F}}_p[t, t^{-1}])\to K_{2i+1}(\bar{\mathbf{F}}_p[t, t^{-1}], \mathbf{Z}/l^\nu)$$

(given by the "fundamental theorem of K-theory") and the bottom arrow is an isomorphism essentially by definition. Consequently, we conclude that:

$$(2.2.2)\qquad \bar{\rho}_*:\ K_*(\bar{\mathbf{F}}_p[t, t^{-1}], \mathbf{Z}/l^\nu)\to K^{et}_*(GL_1, \mathbf{Z}/l^\nu),$$

is an isomorphism of (commutative, associative) rings for $l^\nu\neq 2$. ■

The following proposition generalizes Example 2.3. In particular, we obtain elementary examples for which $\bar{\rho}_i$ is not an isomorphism for small i.

PROPOSITION 2.4. — *For $l^\nu\neq 2$, 3, 4, or 8, there is a ring isomorphism:*

$$K_*(\bar{\mathbf{F}}_p[x_1, \ldots, x_m, y_1, y_1^{-1}, \ldots, y_s, y_s^{-1}], \mathbf{Z}/l^\nu)\simeq \bigotimes_{i=1}^{s} K_*(\bar{\mathbf{F}}_p[t, t^{-1}], \mathbf{Z}/l^\nu),$$

where the tensor product is as algebras over $K_*(\bar{\mathbf{F}}_p, \mathbf{Z}/l^\nu)$. Furthermore:

$$\bar{\rho}_i:\ K_i(\bar{\mathbf{F}}_p[x_1, \ldots, x_m, y_1, y_1^{-1}, \ldots, y_s, y_s^{-1}], \mathbf{Z}/l^\nu)\to K^{et}_i(\mathbf{A}^m\times GL_1^{\times s}, \mathbf{Z}/l^\nu),$$

is injective for all $i\geqq 0$ and bijective for $i\geqq s-1$, where:

$$\operatorname{Spec}\bar{\mathbf{F}}_p[x_1, \ldots, x_m, y_1, y_1^{-1}, \ldots, y_s, y_s^{-1}]=\mathbf{A}^m\times GL_1^{\times s}.$$

Proof. — Quilen's fundamental theorem for regular rings [16], Corollary to Theorem 8, implies the isomorphism:

$$K_*(GL_1^{\times s}, \mathbf{Z}/l^\nu)\xrightarrow{\sim} K_*(\mathbf{A}^m\times GL_1^{\times s}, \mathbf{Z}/l^\nu).$$

Moreover, the fact that $\mathbf{A}^m\times GL_1^{\times s}\to GL_1^{\times s}$ is a $\mathbf{Z}/l$-equivalence implies the isomorphism $K^{et}_*(GL_1^{\times s}, \mathbf{Z}/l^\nu)\xrightarrow{\sim} K^{et}_*(\mathbf{A}^m\times GL_1^{\times s}, \mathbf{Z}/l^\nu)$. Thus, we may assume $m=0$. We proceed by induction on s, where the cases $s=0$ and $s=1$ are verified in Examples 2.1 and 2.3.

We consider the following special case of (1.5.2):

$$(2.4.1)\qquad \begin{array}{ccc} K_*(GL_1^{\times s-1}, \mathbf{Z}/l^\nu)\underset{K_*(P, \mathbf{Z}/l^\nu)}{\otimes} K_*(GL_1, \mathbf{Z}/l^\nu) & \to & K_*(GL_1^{\times s}, \mathbf{Z}/l^\nu) \\ \downarrow{\scriptstyle \bar{\rho}_*\otimes\bar{\rho}_*} & & \downarrow{\scriptstyle \bar{\rho}_*} \\ K^{et}_*(GL_1^{\times s-1}, \mathbf{Z}/l^\nu)\underset{K^{et}_*(P, \mathbf{Z}/l^\nu)}{\otimes} K^{et}_*(GL_1, \mathbf{Z}/l^\nu) & \to & K^{et}_*(GL_1^{\times s}, \mathbf{Z}/l^\nu) \end{array}$$

The Kunneth Theorem implies that the lower horizontal arrow of (2.4.1) is an isomorphism. Using Examples 2.1 and 2.3, we re-write (2.4.1) as follows:

$$\begin{array}{ccc} K_*(GL_1^{\times s-1}, \mathbf{Z}/l^\nu) \underset{\mathbf{Z}/l^\nu[\beta]}{\otimes} \mathbf{Z}/l^\nu[\beta, t]/t^2 & \to & K_*(GL_1^{\times s}, \mathbf{Z}/l^\nu) \\ \downarrow & & \downarrow \\ K_*^{et}(GL_1^{\times s-1}, \mathbf{Z}/l^\nu) \underset{\mathbf{Z}/l^\nu[\beta]}{\otimes} \mathbf{Z}/l^\nu[\beta, t]/t^2 & \to & K_*^{et}(GL_1^{\times s}, \mathbf{Z}/l^\nu) \end{array} \tag{2.4.2}$$

By induction, the left vertical arrow is an injection in all degrees and a bijection in degrees $i \geqq s-1$. Therefore, the upper horizontal arrow of (2.4.2) is an injection; by Quillen's fundamental theorem for regular rings, this injection must be a surjection. The proposition now follows, since we have shown that the two vertical arrows of (2.4.2) are isomorphic. ■

In order to obtain further examples, we employ the following comparison of "Mayer-Vietoris" long exact sequences.

THEOREM 2.5. — *Let* X *be a connected, smooth quasi-projective variety over* F, *and let* U, V *be Zariski opens of* X *with* $U \cup V = X$ *and* $U \cap V = W$. *Then* $\hat{\rho}_i$ *and* $\bar{\rho}_i$ *determine maps of long exact sequences for any* $\nu > 0$:

$$\begin{array}{ccccccccc} \ldots \to & K_i(X) & \to & K_i(U) \oplus K_i(V) & \to & K_i(W) & \to & K_{i-1}(X) & \to \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & \hat{K}_i^{et}(X) & \to & \hat{K}_i^{et}(U) \oplus \hat{K}_i^{et}(V) & \to & \hat{K}_i^{et}(W) & \to & \hat{K}_{i-1}^{et}(W) & \to \end{array} \tag{2.5.1}$$

$$\begin{array}{ccccccc} \ldots \to & K_i(X, \mathbf{Z}/l^\nu) & \to & K_i(U, \mathbf{Z}/l^\nu) \oplus K_i(V, \mathbf{Z}/l^\nu) & \to & K_i(W, \mathbf{Z}/l^\nu) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & K_i^{et}(X, \mathbf{Z}/l^\nu) & \to & K_i^{et}(U, \mathbf{Z}/l^\nu) \oplus K_i^{et}(V, \mathbf{Z}/l^\nu) & \to & K_i^{et}(W, \mathbf{Z}/l^\nu) & \to \ldots \end{array} \tag{2.5.2}$$

Proof. — An shown in [6], the following square is homotopy cartesian:

$$\begin{array}{ccc} BQ\underline{\underline{P}}(X) & \to & BQ\underline{\underline{P}}(V) \\ \downarrow & & \downarrow \\ BQ\underline{\underline{P}}(U) & \to & BQ\underline{\underline{P}}(W) \end{array} \tag{2.5.3}$$

where $\underline{\underline{P}}(X)$ is the exact category of locally free, finite rank 0_X modules and $Q\underline{\underline{P}}(X)$ is the Quillen Q-category associated to $\underline{\underline{P}}(X)$ [16]. Because $\pi_i(BQ\underline{\underline{P}}(X)) = K_{i-1}(X)$, (2.5.3) implies the long exact sequences constituting the top rows of (2.5.1) and (2.5.2). The fact that the natural map from the homotopy push-out of $U_{et} \leftarrow W_{et} \to V_{et}$ to X_{et} is a $\mathbf{Z}/l$-equivalence ([12], 14.10) implies that the following commutative square is homotopy cartesian [thus determining the long exact sequences in positive degrees of the bottom rows of (2.5.1) and (2.5.2)]:

$$\begin{array}{ccc} BGL^{\wedge X} \times \mathbf{Z} & \to & BGL^{\wedge V} \times \mathbf{Z} \\ \downarrow & & \downarrow \\ BGL^{\wedge U} \times \mathbf{Z} & \to & BGL^{\wedge W} \times \mathbf{Z} \end{array} \tag{2.5.4}$$

Let $\tilde{X}=\text{Spec } A \to X$, $\tilde{U}=\text{Spec } B \to U$, and $\tilde{V}=\text{Spec } C \to V$ be affine resolutions with $\tilde{U}$ and $\tilde{V}$ mapping to $\tilde{X}$ over X, and let $\tilde{W}=\text{Spec } O \to W$ be the affine resolution $\tilde{W}=\tilde{U} \times_{\tilde{X}} \tilde{V} \to W$.

The homotopy property for affine bundles ([16], 4.1) and the fact that (2.5.3) is homotopy cartesian imply that:

$$\begin{array}{ccc} \mathscr{B}\,GL(A)(S^1) & \to & \mathscr{B}\,GL(C)(S^1) \\ \downarrow & & \downarrow \\ \mathscr{B}\,GL(B)(S^1) & \to & \mathscr{B}\,GL(D)(S^1) \end{array} \tag{2.5.5}$$

is homotopy cartesian in dimensions greater than 1 [providing the top rows of (2.5.1) and (2.5.2) in positive degrees]. The natural transformation Ψ of Theorem 1.3 determines a map from (2.5.5) to the commutative square:

$$\begin{array}{ccc} \mathscr{B}\,GL^{\wedge\tilde{X}}(S^1) & \to & \mathscr{B}\,GL^{\wedge\tilde{V}}(S^1) \\ \downarrow & & \downarrow \\ \mathscr{B}\,GL^{\wedge\tilde{U}}(S^1) & \to & \mathscr{B}\,GL^{\wedge\tilde{W}}(S^1) \end{array} \tag{2.5.6}$$

Moreover, the $\mathbf{Z}/l$-equivalences $\tilde{X} \to X$, $\tilde{U} \to U$, $\tilde{V} \to V$, $\tilde{W} \to W$ fit in a map of commutative squares, thereby determining a weak equivalence from the following commutative square:

$$\begin{array}{ccc} \mathscr{B}\,GL^{\wedge X}(S^1) & \to & \mathscr{B}\,GL^{\wedge V}(S^1) \\ \downarrow & & \downarrow \\ \mathscr{B}\,GL^{\wedge U}(S^1) & \to & \mathscr{B}\,GL^{\wedge W}(S^1) \end{array} \tag{2.5.7}$$

to (2.5.6). Because $\Omega \circ (2.5.7)$ is equivalent to (2.5.4), we obtain the asserted "commutative ladders" (2.5.1) and (2.5.2) except for those squares involving $\hat{\rho}_0$, $\bar{\rho}_1$, and $\bar{\rho}_0$. The commutativity of these low dimensional squares (as well as the exactness of the bottom rows in dimension 0) is obtained by repeating the argument for $X \times GL_1 = (U \times GL_1) \cup (V \times GL_1)$ and then considering the map between ladders induced by $1 \times e : X \to X \times GL_1$. ■

COROLLARY 2.6. — *Let* $W = \text{Spec}\,\bar{\mathbf{F}}_p[x, (x-\alpha_1)^{-1}, \ldots, (x-\alpha_s)^{-1}]$, *where* $\alpha_1, \ldots, \alpha_s$ *are distinct elements of* $\bar{\mathbf{F}}_p$. *For* $l^\nu \neq 2$:

$$\bar{\rho}_* : \quad K_*(\bar{\mathbf{F}}_p[x, (x-\alpha_1)^{-1}, \ldots, (x-\alpha_s)^{-1}], \mathbf{Z}/l^\nu) \to K_*^{et}(W, \mathbf{Z}/l^\nu),$$

is an isomorphism.

Proof. — We proceed by induction on s, the case $s=1$ having been considered in Example 2.3 (and the case $s=0$ implied by Example 2.1). Let:

$$X = \text{Spec}\,\bar{\mathbf{F}}_p[x, (x-\alpha_1)^{-1}, \ldots, (x-\alpha_{s-2})^{-1}],$$

$$U = \text{Spec}\,\bar{\mathbf{F}}_p[x, (x-\alpha_1)^{-1}, \ldots, (x-\alpha_{s-1})^{-1}],$$

and:

$$V = \text{Spec}\,\bar{\mathbf{F}}_p[x, (x-\alpha_1)^{-1}, \ldots, (x-\alpha_{s-2})^{-1}, (x-\alpha_s)^{-1}],$$

so that $X = U \cup V$ and $W = U \cap V$. For positive degrees, the corollary follows from (2.5.2), induction, and the 5-Lemma. For degree 0, the corollary is implied by the isomorphism:

$$K_0(\overline{\mathbf{F}}_p[x, (x-\alpha_1)^{-1}, \ldots, (x-\alpha_s)^{-1}], \mathbf{Z}/l^\nu) \simeq \mathbf{Z}/l^\nu$$

and the weak $\mathbf{Z}/l$-equivalence $S^1 \vee \ldots \vee S^1 \to W_{et}$.

■

The next proposition demonstrates how Theorem 2.5 enables one to avoid the identification of maps in the Mayer-Vietoris sequences of algebraic K-theory.

PROPOSITION 2.7. — *Let* $\mathbf{A}^n = \operatorname{Spec} \overline{\mathbf{F}}_p[x_1, \ldots, x_n]$ *and let:*

$$U_i = \operatorname{Spec} \overline{\mathbf{F}}_p[x_1, \ldots, x_i, x_i^{-1}, \ldots, x_n] \subset \mathbf{A}^n.$$

For $l^\nu \neq 2, 3, 4,$ *or* 8:

$$\bar{\rho}_j : \; K_j(U_{1,\ldots,s} \cup U_{1,\ldots,s-1,s+1} \cup \ldots \cup U_{1,\ldots,s-1,m}; \mathbf{Z}/l^\nu) \\ \to K_j^{et}(U_{1,\ldots,s} \cup U_{1,\ldots,s-1,s+1} \cup \ldots \cup U_{1,\ldots,s-1,m}; \mathbf{Z}/l^\nu)$$

is injective for $j \geqq s-2$ *and bijective for* $j \geqq s-1$, *where m is any integer with* $s \leqq m \leqq n$ *and* $U_{1,\ldots,s-1,k} = U_1 \cap \ldots \cap U_{s-1} \cap U_k$. *In particular (taking* $s=1$ *and* $m=n$*):*

$$\bar{\rho}_* : \; K_*(\mathbf{A}^n - \{0\}, \mathbf{Z}/l^\nu) \to K_*^{et}(\mathbf{A}^n - \{0\}, \mathbf{Z}/l^\nu),$$

is an isomorphism.

Proof. — Let $Y_s^m = U_{1,\ldots,s} \cup U_{1,\ldots,s-1,s+1} \cup \ldots \cup U_{1,\ldots,s-1,m}$, so that:

$$Y_s^m = Y_s^{m-1} \cup U_{1,\ldots,s-1,m}, \quad Y_{s+1}^m \approx Y_s^{m-1} \cap U_{1,\ldots,s-1,m}.$$

By Proposition 2.4, $\bar{\rho}_j : K_j(U_{1,\ldots,s-1,m}, \mathbf{Z}/l) \to K_j^{et}(U_{1,\ldots,s-1,m}, \mathbf{Z}/l)$ is injective for $j \geqq 0$ and bijective for $j \geqq s-1$. Using descending induction on s and ascending induction on $m \geqq s$ (for a given s), we conclude the proposition by applying the 5-Lemma to (2.5.2). ■

The proof of Proposition 2.7 applies essentially verbatim to the following analogue concerning the standard opens of projective space.

PROPOSITION 2.8. — *Let* $\mathbf{P}^n = \operatorname{Proj} \overline{\mathbf{F}}_p[X_0, \ldots, X_n]$ *and let* $V_i \simeq \mathbf{A}^n \subset \mathbf{P}^n$ *be the complement of the hyperplane* $X_i = 0$. *For* $l^\nu \neq 2, 3, 4,$ *or* 8:

$$\bar{\rho}_j : \; K_j(V_{0,\ldots,s-1,s} \cup V_{0,\ldots,s-1,s+1} \cup \ldots \cup V_{0,\ldots,s-1,m}, \mathbf{Z}/l^\nu) \\ \to K_j^{et}(V_{0,\ldots,s-1,s} \cup V_{0,\ldots,s-1,s+1} \cup \ldots \cup V_{0,\ldots,s-1,m}, \mathbf{Z}/l^\nu),$$

is injective for $j \geqq s-2$ *and bijective for* $j \geqq s-1$, *where m is any integer with* $s \leqq m \leqq n$ *and* $V_{0,\ldots,s-1,m} = V_0 \cap \ldots \cap V_{s-1} \cap V_m$. *In particular (taking* $s=0$ *and* $m=n$*):*

$$\bar{\rho}_* : \; K_*(\mathbf{P}^n, \mathbf{Z}/l^\nu) \to K_*^{et}(\mathbf{P}^n, \mathbf{Z}/l^\nu)$$

is an isomorphism. ■

We apply the description of $\bar{\rho}_1$ implicit in Proposition 1.8 to provide a proof of the following example announced by R. Thomason. Thomason's example is particularly interesting because it uses K-theory with coefficients in an essential way: torsion classes in $K_0(X)$ determine classes in $K_1(X, \mathbf{Z}/l^\nu)$ which are mapped via $\bar{\rho}_1$ to the reduction of classes in $\hat{K}_1^{et}(X)$.

PROPOSITION 2.9. — *Let* X *be a smooth connected curve over* F. *If* $l^\nu \neq 2$, *then:*

$$\bar{\rho}_* : \ K_*(X, \mathbf{Z}/l^\nu) \to K_*^{et}(X, \mathbf{Z}/l^\nu),$$

is surjective.

Proof. — Because $\bar{\rho}_*$ is a ring homomorphism and $K_*^{et}(X, \mathbf{Z}/l^\nu)$ is periodic of period 2 via multiplication by $\bar{\rho}_2(\beta_x)$ (*cf.* Proposition 2.2), it suffices to prove the surjectivity of $\bar{\rho}_0$ and $\bar{\rho}_1$. Because X_{et} is $\mathbf{Z}/l$-equivalent to the homotopy type of a (not necessarily compact) Riemann surface, we conclude that $\hat{K}_*^{et}(X)$ is torsion free.

Proposition 1.8 implies that it suffices to prove that $\hat{\rho}_0 : K_0(X) \to \hat{K}_0^{et}(X)$ has dense image in order to prove that $\bar{\rho}_0$ is surjective. This is proved by observing that the following map has dense image:

$$(\deg \oplus c_1) \circ \hat{\rho} : K_0(X) \to \hat{K}_{et}^0(X) \simeq \varprojlim H^0(X_{et}, \mathbf{Z}/l^r) \oplus \varprojlim H^2(X_{et}, \mathbf{Z}/l^r),$$

where $c_1(\alpha : X_{et} \to \mathrm{Grass}\hat{}_{m+n,\,n}) = \alpha^*(c_1)$ for $c_1 \in \varprojlim_r H^2(\mathrm{Grass}\hat{}_{m+n,\,n}, \mathbf{Z}/l^r)$ the universal chern class, so that Proposition 1.7 implies that $\hat{\rho}_0$ has dense image.

To prove that $\bar{\rho}_1$ is surjective, we use Proposition 1.6 to define:

$$\bar{c}_{1,1} : \ \hat{K}_1^{et}(X) \to H^1(X_{et}, \mathbf{Z}/l^\nu),$$

by sending $\alpha : X_{et} \to GL\hat{}_n$ to $\alpha^*(\bar{c}_{1,1})$, where $\bar{c}_{1,1}$ is the pull-back *via* det: $GL\hat{}_n \to GL\hat{}_1$ of the canonical class in $H^1((GL_1)_{et}, \mathbf{Z}/l^\nu)$. Because $\hat{K}_0^{et}(X)$ is torsion free, $\bar{c}_{1,1}$ induces:

$$\bar{c}_{1,1} : \ K_1^{et}(X, \mathbf{Z}/l^\nu) \to H^1(X_{et}, \mathbf{Z}/l^\nu),$$

which is clearly an isomorphism for X a Riemann surface and is seen to be an isomorphism for $F \neq \mathbf{C}$ by "lifting to characteristic 0".

Applying the 5-Lemma to (2.5.2) and using the surjectivity of $\bar{\rho}_2$, we conclude that the surjectivity of $\bar{\rho}_1$ for general X follows from the bijectivity of $\bar{c}_{1,1} \circ \bar{\rho}_1$ for $X = \mathrm{Spec}\ A$. Using the exact sequences:

$$0 \to K_1(A) \otimes \mathbf{Z}/l^\nu \to K_1(A, \mathbf{Z}/l^\nu) \to {}_{l^\nu}K_0(A) \to 0,$$
$$0 \to A^* \otimes \mathbf{Z}/l^\nu \to H^1(X_{et}, \mathbf{Z}/l^\nu) \to {}_{l^\nu}\mathrm{Pic}(A) \to 0,$$

we conclude that the order of $K_1(A, \mathbf{Z}/l^\nu)$ is greater than or equal to the order of $H^1(X_{et}, \mathbf{Z}/l^\nu)$. Consequently, it suffices to prove that

$$\bar{c}_{1,1} \circ \bar{\rho}_1 : K_1(A, \mathbf{Z}/l^\nu) \to H^1(X_{et}, \mathbf{Z}/l^\nu)$$

is injective.

By construction, $\bar{c}_{1,1} \circ \hat{\rho}_1 : K_1(F[t, t^{-1}]) \to H^1((GL_1)_{et}, \mathbf{Z}/l^\nu)$ sends t to its image under the natural inclusion $F[t, t^{-1}]^* \otimes \mathbf{Z}/l^\nu \to H^1((GL_1)_{et}, \mathbf{Z}/l^\nu)$. By naturality, $\bar{c}_{1,1} \circ \bar{\rho}_1$ when restricted to $A^* \otimes \mathbf{Z}/l^\nu \subset K_1(A, \mathbf{Z}/l^\nu)$ is the natural inclusion

$$A^* \otimes \mathbf{Z}/l^\nu \to H^1(X_{et}, \mathbf{Z}/l^\nu).$$

Consider the following commutative square:

$$(2.9.1)\qquad \begin{array}{ccccc} K_1(A, \mathbf{Z}/l^\nu) & \to & \operatorname{colim} K_1(A_f, \mathbf{Z}/l^\nu) & \simeq & k(A)^* \otimes \mathbf{Z}/l^\nu \\ \downarrow{\scriptstyle \bar{c}_{1,1}\circ\bar{\rho}_1} & & \downarrow{\scriptstyle \bar{c}_{1,1}\circ\bar{\rho}_1} & & \downarrow{\scriptstyle 1} \\ H^1(X_{et}, \mathbf{Z}/l^\nu) & \to & \operatorname{colim} H^1((X_f)_{et}, \mathbf{Z}/l^\nu) & \simeq & k(A)^* \otimes \mathbf{Z}/l^\nu \end{array}$$

where the colimit is indexed by $f \in A^*$, $k(A)$ is the field of fractions of A, and the commutativity of the right hand square is implied by the preceding argument. Because:

$$K'_1(A/f, \mathbf{Z}/l^\nu) \simeq \Pi K_1(F, \mathbf{Z}/l^\nu) = 0,$$

by Quillen's devissage theorem, Quillen's localization theorem implies that each $K_1(A, \mathbf{Z}/l^\nu) \to K_1(A_f, \mathbf{Z}/l^\nu)$ is injective (*cf.* [16]). *Consequently*, (2.9.1) implies that $\bar{c}_{1,1} \circ \bar{\rho}_1 : K_1(A, \mathbf{Z}/l^\nu) \to H^1(X_{et}, \mathbf{Z}/l^\nu)$ is also injective as required. ■

3. Galois actions

We investigate galois actions on algebraic and etale K-theory, as described in Proposition 3.1 and Definition 3.2. Our basic result, Theorem 3.3, asserts that $\hat{\rho}_* : K_*(X) \to \hat{K}_*^{et}(X)$ is galois equivariant. Throughout this section, we consider a fixed prime l, a given field k finitely generated over the prime field with $1/l \in k$, and a chosen separable algebraic closure F of k. The galois actions we shall consider will be those of Gal (F, L) ⊂ Gal(F, k) for various finite extensions L/k. [If k were not finitely generated over the prime field—for example, $k = \overline{\mathbf{F}}_p$—then L might contain all l-primary roots of unity so that the action of Gal(F, L) would be less interesting.]

In Proposition 3.5, we use galois equivariance to prove that $\hat{\rho}_i \otimes \mathbf{Q} = 0$ for $i > 0$ whenever X is proper and smooth over $\overline{\mathbf{F}}_p$. This is to be contrasted with the Lichtenbaum-Quillen conjecture concerning $\bar{\rho}_*$ (Conjecture 3.9). Motivated by the example of an affine curve studied in Proposition 3.6, we pose several questions concerning the image of $\hat{\rho}_*$. We conclude by introducing chern character maps related to the work of C. Soulé [18].

We begin with the following proposition which describes the natural action of Gal (F, L) on $K_*(A)$ in a manner which is easily translated to etale K-theory.

PROPOSITION 3.1. — *Let* A *be an* F-*algebra of finite type and let* L *be a finite extension of k be chosen so that* A *is defined over* L (*i.e.*, $A = A_L \otimes_L F$ *for some algebra* A_L *of finite type over* L). *For* σ ∈ Gal (F, L), *let:*

$$\sigma : BGL_m(A) \to BGL_m(A), \qquad m \geqq 0,$$

be induced by $1 \otimes \sigma : A \simeq A_L \otimes_L F \xrightarrow{\sim} A$. *Equivalently*, σ *sends:*

$$\alpha : X \otimes \Delta[t] \to BGL_m \quad \text{to} \quad (1 \otimes \sigma^{-1}) \circ \alpha \circ (1 \otimes \sigma) : X \otimes \Delta[t] \to BGL_m,$$

where X *equals* Spec A, $GL_m = GL_{m,F}$ *denotes the general linear group scheme of* m *by* m *matrices over* F, *and:*

$$1\otimes\sigma : \quad GL_m \simeq GL_{m,L} \underset{\text{Spec L}}{\times} \text{Spec F} \rightrightarrows GL_m$$

equals $1\times\sigma$. *Then the natural action of* σ *on* $K_i(A)$ *for* $i>0$ *is that induced by the above* $\sigma : BGL_m(A) \to BGL_m(A)$ *on the i-th homotopy group of the group completion of* $\coprod_{m\geq 0} BGL_m(A)$.

Proof. — The assertion that σ on $K_i(A)$ is determined by $\sigma : BGL_m(A)\to BGL_m(A)$ is essentially immediate from the definition of the functor $K_i(\ )$. The automorphism σ : $BGL_m(A)\to BGL_m(A)$ is clearly that sending a t-simplex α of $BGL_m(A)$ corresponding to $\alpha_Z : X\otimes\Delta[t]\to BGL_{m,Z}$ to $\alpha_Z\circ(1\otimes\sigma)$. We verify by inspection that if $\alpha : X\otimes\Delta[t]\to BGL_m$ is the F-linear map determined by α_Z, then $(1\otimes\sigma^{-1})\circ\alpha\circ(1\otimes\sigma)$ is the F-linear map determined by $\alpha_Z\circ(1\otimes\sigma)$, thereby verifying the equivalent description of $\sigma : BGL_m\to BGL_m$. ■

With Proposition 3.1 as a guide, we proceed to define the action of Gal (F, L) on etale K-theory.

DEFINITION 3.2. — Let $Y_\cdot\to X_\cdot$ be a closed immersion of simplicial schemes locally of finite type over F, defined over some finite extension L of k (i. e., $Y_\cdot\to X_\cdot$ is the pull-back via Spec F → Spec L of some $Y^L_\cdot\to X^L_\cdot$ over L). For any $\sigma\in \text{Gal}(F, L)$ and any $m\geq 0$, define:

$$\sigma : \quad BGL_m^{\wedge(X_\cdot, Y_\cdot)}\to BGL_m^{\wedge(X_\cdot, Y_\cdot)},$$

to be the map induced by composition on the left with $(1\otimes\sigma^{-1})^\wedge : BGL_m^\wedge\to BGL_m^\wedge$ and on the right with $(1\otimes\sigma)_{et} : (X_\cdot/Y_\cdot)_{et}\to(X_\cdot/Y_\cdot)_{et}$. We define the natural actions:

$$\text{Gal}(F, L)\times\hat{K}_i^{et}(X_\cdot, Y_\cdot)\to\hat{K}_i^{et}(X_\cdot, Y_\cdot),$$
$$\text{Gal}(F, L)\times K_i^{et}(X_\cdot, Y_\cdot; \mathbf{Z}/l^\nu)\to K_i^{et}(X_\cdot, Y_\cdot; \mathbf{Z}/l^\nu), \qquad i\geq 0,$$

to be those determined by the above defined actions of $\sigma\in\text{Gal}(F, L)$ on $BGL_m^{\wedge(X_\cdot, Y_\cdot)}$ (and $BGL_m^{\wedge(X_\cdot\times GL_1, Y_\cdot\times GL_1)}$). ■

In verifying that $\hat{\rho}_*$ and $\bar{\rho}_*$ are Galois equivariant, we obtain a refinement of $\hat{\rho}_*$ (for $\hat{\rho}_0$, this refinement was obtained in [11], 5.3).

THEOREM 3.3. — *For any closed immersion* Y → X *of schemes quasi-projective over* F *and any finite extension* L *of* k *over which* Y → X *is defined, the natural maps:*

$$\hat{\rho}_* : \quad K_*(X, Y)\to\hat{K}_*^{et}(X, Y),$$
$$\bar{\rho}_* : \quad K_*(X, Y; \mathbf{Z}/l^\nu)\to K_*^{et}(X, Y; \mathbf{Z}/l^\nu),$$

are Gal (F, L)-*equivariant. If* $Y=\emptyset$, *then the image of* $\hat{\rho}_i$ *for any* $i\geq 0$ *is contained in the "discrete" submodule* $\hat{K}_i^{et}(X)^\delta$ *of* $\hat{K}_i^{et}(X)$ *consisting of those elements left fixed by some subgroup* *of* Gal(F, L) *of finite index. Consequently, the natural transformation* $\hat{\rho}_*$ *restricts to a natural transformation:*

$$\hat{\rho}_* : \quad K_*(\ ) \to \hat{K}^{et}_*(\ )^{\delta},$$

from schemes quasi-projective over F *to rings.*

Proof. — For X = Spec A, Y = ∅, the galois equivariance of $\hat{\rho}_*$ and $\bar{\rho}_*$ follows from the galois equivariance of:

$$\coprod \psi_m : \quad \coprod_{m \geq 0} BGL_m(A) \to \coprod_{m \geq 0} BGL_m^{\hat{}X}$$

and:

$$\coprod \psi_m : \quad \coprod_{m \geq 0} BGL_m(A[t, t^{-1}]) \to \coprod_{m \geq 0} BGL_m^{\hat{}X \times GL_1}$$

apparent from Proposition 3.1 and Definition 3.2 (and the fact that $\otimes t : K_*(A) \to K_{*+1}(A[t, t^{-1}])$ is Gal(F, L) equivariant). For X = Spec A and Y = Spec A/I, we employ the fact that the horizontal arrows Ψ of (1.3.1) are also galois equivariant (proved exactly as for ψ_m) to prove the galois equivariance of $\hat{\rho}_*$ and $\bar{\rho}_*$. For X not necessarily affine, we utilize an affine resolution $\tilde{X} \to X$.

To prove that $\hat{\rho}_i(K_i(X))$ is contained in $\hat{K}^{et}_i(X)^{\delta}$, it suffices to assume that X = Spec A with $A = A_L \underset{L}{\otimes} F$ and to prove that each element of $K_i(A)$ is invariant under some subgroup of Gal(F, L) of finite index. For $i = 0$, this follows from the fact that Gal(F, L′) leaves $[P] \in K_0(A)$ invariant whenever the projective A-module P is defined over L′. For $i > 0$, we consider $\mathscr{B}GL(A)(S^1)$. A t-simplex of $\mathscr{B}GL(A)(S^1)$ corresponds to a finite sequence of matrices with coefficients in A (namely, a t-simplex of $\mathscr{B}GL(A)(\underline{t})$ which has been explicitly described in the proof of Theorem 1.3); this t-simplex is invariant under Gal(F, L′) whenever each entry of each matrix of the sequence is defined over L′. Consequently, any compact subspace of the geometric realization of $\mathscr{B}GL(A)(S^1)$ is invariant under some subgroup of Gal(F, L) of finite index. This implies that any element of $\pi_{i+1}(\mathscr{B}GL(A)(S^1)) = K_i(A)$ (for $i > 0$) is invariant under some subgroup of Gal (F, L) of finite index.

To conclude that $\hat{\rho}_* : K_*(\ ) \to \hat{K}^{et}_*(\ )^{\delta}$ is a natural transformation of ring-valued functors, it suffices to prove that $\hat{K}^{et}_*(\ )^{\delta}$ is a functor from schemes quasi-projective over F to rings. To prove that $\hat{K}^{et}_*(X)^{\delta} \subset \hat{K}^{et}_*(X)$ is closed under multiplication, it suffices to observe that the maps (*cf.* proof of Proposition 1.4) $i_1 : \tilde{\mathscr{B}}GL^{\hat{}X} \to \mathscr{B}GL^{\hat{}X} \circ pr_1$, $i_2 : \tilde{\mathscr{B}}GL^{\hat{}X} \to \mathscr{B}GL^{\hat{}X} \circ pr_2$, and $\mu : \tilde{\mathscr{B}}GL^{\hat{}X} \to \mathscr{B}GL^{\hat{}X} \circ \otimes$ are Gal(F, L) equivariant whenever X is defined over L. If $f : X' \to X$ is a map of schemes quasi-projective over F, then f is defined over some finite extension L of k. Consequently, if $\alpha \in \hat{K}^{et}_*(X)$ is invariant under Gal(F, L′), then $f^*(\alpha) \in \hat{K}^{et}_*(X')$ is invariant under Gal(F, L″) where L″ is the join of L and L′. ∎

In the following proposition, we recall the definition of (etale) l-adic cohomology with explicit "Tate twist" and the existence of galois equivariant chern character components.

PROPOSITION 3.4. — *Let μ_{l^n} denote the subgroup of units of* F *consisting of l^n-th roots of unity. We define $\mathbf{Q}_l(s)$ as the following* Gal(F, *k*)*-module (a 1-dimensional $\mathbf{Q}_l$ vector space):*

$$\mathbf{Q}_l(s)=\mathbf{Q}\otimes\left(\varprojlim_n \mu_{l^n}\right)^{\otimes s},\qquad s\geqq 0;\qquad \mathbf{Q}_l(s)=\mathrm{Hom}(\mathbf{Q}_l(-s),\mathbf{Q}),\qquad s<0.$$

For any closed immersion Y → X *of schemes over* F *defined over* L, *we define the* Gal(F, L)*-module* $H^m(X, Y; \mathbf{Q}_l(s))$ *by:*

$$H^m(X, Y; \mathbf{Q}_l(s))=\left(\varprojlim_n H^m((X/Y)_{et}, \mathbf{Z}/l^n)\right)\underset{\mathbf{Z}_l}{\otimes}\mathbf{Q}_l(s),$$

For all r, s with $2s\geqq r\geqq 0$, there are naturally defined, galois equivariant chern character components:

$$\mathrm{ch}_{s,r}:\quad \hat{K}^{et}_{2s-r}(X, Y)\to H^r(X, Y; \mathbf{Q}_l(s)),$$

Furthermore, the galois equivariant maps:

$$\oplus(\mathrm{ch}_{s,r}\otimes\mathbf{Q}):\quad \hat{K}^{et}_i(X, Y)\otimes\mathbf{Q}\to\bigoplus_{2s-r=i} H^r(X, Y; \mathbf{Q}_l(s)),$$

for each $i\geqq 0$ determine an isomorphism of graded rings:

$$\mathrm{ch}_*:\quad \hat{K}^{et}_*(X, Y)\otimes\mathbf{Q}\simeq\bigoplus_{r,s} H^r(X, Y; \mathbf{Q}_l(s)).$$

Proof. — If we forget the restriction that ε be equal to only 0 or 1, then the proof of [11], 5.5 applies to establish the existence of $\mathrm{ch}_{s,r}$. [If $\alpha:\Sigma^{2s-r}(X/Y)_{et}\to B\hat{GL}_N$ represents a class in $\hat{K}^{et}_{2s-r}(X)$, then $\mathrm{ch}_{s,r}(\alpha)$ is the pull-back via α of the universal chern character component $\mathrm{ch}_r\in H^{2r}(B\hat{GL}_N, \mathbf{Q}_l(r))$.] The fact that ch_* is an isomorphism of graded rings is implied by [11], 2.4 and 5.5. ■

With the aid of the isomorphism:

$$\mathrm{ch}_0\otimes\mathbf{Q}:\quad \hat{K}^{et}_0(X)\otimes\mathbf{Q}\to\oplus H^{2r}(X, \mathbf{Q}_l(r)),$$

we studied the highly non-trivial map $\hat{\rho}_0\otimes\mathbf{Q}$ in [11]. On the other hand, we see below that the Riemann Hypothesis for finite fields implies that $\hat{\rho}_i\otimes\mathbf{Q}=0$ for X projective and smooth over $F=\overline{\mathbf{F}}_p$ whenever $i>0$.

PROPOSITION 3.5. — *If* X *is a projective and smooth algebraic variety over* $F=\overline{\mathbf{F}}_p$, *then* $\hat{K}^{et}_i(X)^\delta\otimes\mathbf{Q}=0$ *for* $i>0$. *Consequently for such* X:

$$\hat{\rho}_i\otimes\mathbf{Q}:\quad K_i(X)\otimes\mathbf{Q}\to\hat{K}^{et}_i(X)\otimes\mathbf{Q},$$

is trivial for any $i>0$.

Proof. — The Riemann Hypothesis for finite fields proved by Deligne [7] implies that the discrete submodule $H^r(X, \mathbf{Q}_l(s))^\delta$ of $H^r(X, \mathbf{Q}_l(s))$ is 0 for $2s-r>0$; namely, for X defined over $\mathbf{F}_q$, the absolute values of the eigenvalues of the arithmetic frobenius $\sigma \in \mathrm{Gal}(F, \mathbf{F}_q)$ acting on $H^r(X, \mathbf{Q}_l(s))$ are all equal to $q^{(2s-r)/2}$. By Proposition 3.4, $\hat{K}_i^{et}(X)^\delta \otimes \mathbf{Q} = 0$ for $i > 0$. The proposition now follows from Theorem 3.3. ■

We next investigate $\hat{\rho}_*$ for an affine curve.

PROPOSITION 3.6. — *Let* $X = \mathrm{Spec}\, A$ *be a smooth affine curve of genus* g *over* F. *Then* $\hat{\rho}_1 : K_1(X) \to \hat{K}_1^{et}(X)$ *has dense image in* $\hat{K}_1^{et}(X)^\delta$ *which is a free* $\mathbf{Z}_l$ *module of rank* t, *where* $t+1$ *is the number of closed points of the complement of* X *in a smooth, projective closure* $\overline{X}$. *Moreover, for any* $i > 1$, $\hat{K}_i^{et}(X)^\delta = 0$ *so that* $\hat{\rho}_i = 0$.

Proof. — The map $X_{et} \to \overline{X}_{et}$ is $\mathbf{Z}/l$ equivalent to the inclusion in a Riemann surface of genus g of the complement of $t+1$ points. Consequently, $\hat{K}_1^{et}(\overline{X}) \to \hat{K}_1^{et}(X)$ is split injective $\left(\text{isomorphic to } \varprojlim_n H^1(\overline{X}, \mathbf{Z}/l^n(1)) \to \varprojlim_n H^1(X, \mathbf{Z}/l^n(1))\right)$ with image a free $\mathbf{Z}_l$ module of rank $2g$. Because $\hat{K}_1^{et}(\overline{X})^\delta \subset \hat{K}_1^{et}(\overline{X})^\delta \otimes \mathbf{Q}$ equals 0 by Proposition 3.5, it suffices to prove that $\hat{\rho}_1(K_1(X))$ is dense in a submodule of rank at least t of $\hat{K}_1^{et}(X) \simeq \mathbf{Z}_l^{\oplus 2g+t}$ in order that we may conclude that $\hat{\rho}_1(K_1(X))$ is dense in $\hat{K}_1^{et}(X)^\delta$ and that $\hat{K}_1^{et}(X)^\delta$ is a free $\mathbf{Z}_l$ module of rank t.

Choose a rational function $f : \overline{X} \to \mathbf{P}^1$ with the property that f separates the (rational) points of $\overline{X} - X$ and is etale at each of these points. Let $W \subset \mathbf{P}^1$ denote the (Zariski) open complement of $f(\overline{X} - X)$. By excision, the composition (factoring through $H^1(W, \mathbf{Z}/l) \to H^1(X, \mathbf{Z}/l)$):

$$H^1(W, \mathbf{Z}/l) \to H^2(\mathbf{P}^1, W; \mathbf{Z}/l) \to H^2(\overline{X}, X; \mathbf{Z}/l),$$

is injective. Because $\hat{K}_1^{et}(W) \simeq \varprojlim H^1(W, \mathbf{Z}/l^n(1))$ and $\hat{K}_1^{et}(X) \simeq \varprojlim H^2(X, \mathbf{Z}/l^n(1))$ are torsion free, we conclude that $f^* : \hat{K}_1^{et}(W) \to \hat{K}_1^{et}(X)$ is injective with rank equal to t. Corollary 2.6 and the fact that $K_1(W, \mathbf{Z}/l^\nu) \simeq K_1(W) \otimes \mathbf{Z}/l^\nu$ imply that $\hat{\rho}_1(K_1(W))$ is dense in $\hat{K}_1^{et}(W)$. Therefore, the naturality of $\hat{\rho}_1$ implies that $\hat{\rho}_1(K_1(X))$ is dense in a submodule containing the rank t submodule $f^*(\hat{K}_1^{et}(W))$.

Because $\hat{K}_i^{et}(X)$ is torsion free, the vanishing of $\hat{K}_i^{et}(X)^\delta \otimes \mathbf{Q}$ implies the vanishing of $\hat{K}_i(X)^\delta$. Using Proposition 3.4, we conclude that it suffices to prove that $H^r(X, \mathbf{Q}_l(s))^\delta = 0$ for $2s - r \geqq 2$ in order to conclude that $\hat{K}_i^{et}(X)^\delta = 0$ for $i \geqq 2$. Because X has $\mathbf{Z}/l$ cohomological dimension 1, we need only observe that:

$$H^0(X, \mathbf{Q}_l(s))^\delta = \mathbf{Q}_l(s)^\delta = 0, \qquad s > 0,$$

$$H^1(X, \mathbf{Q}_l(s))^\delta = H^1(\overline{X}, \mathbf{Q}_l(s))^\delta \oplus H^1(W, \mathbf{Q}_l(s))^\delta = 0, \qquad s > 1,$$

where the second vanishing statement follows from Proposition 3.5 and the fact that $H^1(W, \mathbf{Q}_l(1)) = \mathbf{Q}_l^{\oplus t}$ as Gal(F, L)-modules (because $\hat{K}_1^{et}(W)^\delta = \hat{K}_1^{et}(W) \simeq H^1(W, \mathbf{Q}_l(1))$ as shown above). ■

We recall that the Tate Conjecture concerning the identification of algebraic cycles in terms of their behavior under Galois actions is equivalent to the conjecture that $\hat{\rho}_0 \otimes \mathbf{Q}_l : K_0(X) \otimes \mathbf{Q}_l \to \hat{K}_0^{et}(X)^\delta \otimes \mathbf{Q}$ is surjective [11]. Theorem 3.3, Propositions 3.5 and 3.6 suggest the following questions.

QUESTION 3.7. — For which smooth, quasi-projective varieties X over F is the natural map:

$$\hat{\rho}_* \otimes \mathbf{Q}_l : \quad K_*(X) \otimes \mathbf{Q}_l \to \hat{K}_*^{et}(X)^\delta \otimes \mathbf{Q},$$

surjective ■

QUESTION 3.8. — If X is a Zariski open of some smooth, projective variety $\bar{X}$ over F, what is the relationship between the vanishing range of $\hat{K}_*^{et}(X)^\delta$ and the codimension of $\bar{X}-X$ in $\bar{X}$? ■

Questions 3.7 and 3.8 should be contrasted with the following reformulation of a conjecture of Lichtenbaum and Quillen. The reader should observe that the Galois action is not necessarily discrete on $\varprojlim K_i(X, \mathbf{Z}/l^n)$. For example, $\varprojlim_n K_2(F, \mathbf{Z}/l^n) \simeq \mathbf{Z}_l(1)$ [no non-zero element of $\mathbf{Z}_l(1)$ is invariant under any subgroup of Gal(F, k) of finite index, because the field $k(\mu_{l^\infty})$ obtained by adjoining all l-primary roots of unity to k is not finite over k].

CONJECTURE 3.9 (Lichtenbaum-Quillen). — If X is a smooth, quasi-projective variety over F, then:

$$\varprojlim_n \bar{\rho}_i : \quad \varprojlim K_i(X, \mathbf{Z}/l^n) \to \hat{K}_i^{et}(X),$$

is an isomorphism for i greater than the l-cohomological dimension of X. ■

Our last proposition is inspired by the work of C. Soulé on the K-theory of rings of integers in number fields [18]. Because the maps $ch_{i,k}$ of Proposition 3.9 have the conjectured form (corresponding to Soulé's chern class components divided by $(i-1)!$), the method of proof of Proposition 3.10 suggests a possible means of extending Soulé's results.

PROPOSITION 3.10. — *Let* X *be a quasi-projective variety over* F *defined over some finite extension* L *of* k. *Then there exists* Gal(F, L)-*equivariant maps:*

$$ch_{s,r} : \quad K_{2s-r}(X, \mathbf{Z}/l^\nu) \to H^r(X_{et}, \mathbf{Z}/l^\nu(s)),$$

for $r=0$, 1, *or* 2 *and* s *with* $2s-r \geqq 0$.

Proof. — We assume X is connected (treating one component at a time) and affine (by replacing X by an affine resolution $\tilde{X} \to X$ otherwise). Let $B\hat{GL}_N^{(i)}$ denote the homotopy fibre of $B\hat{GL}_N \to \cos k_i B\hat{GL}_N$ (adjoint to the inclusion $sk_i B\hat{GL}_N \to B\hat{GL}_N$), so that $\pi_m(B\hat{GL}_N^{(i)})=0$ for $m<i$ and $\pi_m(B\hat{GL}_N^{(i)}) \simeq \pi_m(B\hat{GL}_N)$ for $m \geqq i$. Using obstruction theory as in the proof of Proposition 1.2, we conclude the isomorphisms:

$$(3.10.1) \qquad K_i^{et}(X, \mathbf{Z}/l^\nu) \simeq \mathrm{Hom}_{\text{pro}-\mathscr{H}_0}(\Sigma^{i-2} C(l^\nu) \wedge (X/\emptyset)_{et}, B\hat{GL}_N^{(i)})$$

for $N \gg 0$. We recall from the integrality theorem of Adams [1], that the restriction of $ch_s \in H^{2s}(BU, \mathbf{Q})$ in $H^{2s}(BU^{(i)}, \mathbf{Q})$ lies in $H^{2s}(BU^{(i)}, \mathbf{Z})$ if $i=2s$ or $2s-1$. Consequently, the restriction of $ch_s \in H^{2s}(B\hat{GL}_N, \mathbf{Q}_l(s))$ for $N \gg 0$ (*cf.* Proposition 3.4) lies in the image of:

$$\varprojlim_n H^{2s}(B\hat{GL}_N^{(i)}, \mathbf{Z}/l^n(s)) \to H^{2s}(B\hat{GL}_N^{(i)}, \mathbf{Q}_l(s)),$$

provided that $i=2s$ or $2s-1$. Thus, if $r=2s-i$ equals 0 or 1 and if $i \geqq 2$, we define $ch_{s,r}$ on $\alpha \in K_i(X, \mathbf{Z}/l^\nu)$ by:

$$ch_{s,r}(\alpha)=(\bar{\rho}_i(\alpha))^*(ch_s) \in H^r(X, \mathbf{Z}/l^\nu(s)),$$

where $\bar{\rho}_i(\alpha)$ is viewed as a map $\Sigma^{i-2} C(l^\nu) \wedge (X/\emptyset)_{et} \to BGL_N^{(i)}$ via (3.10.1).

To treat the case $r=2$, we choose a geometric point for X which determines a retract $\gamma : \Sigma^{i-2} C(l^\nu) \to \Sigma^{i-2} C(l^\nu) \wedge (X/\emptyset)_{et}$ with retraction τ. For any $\alpha \in K_i(X, \mathbf{Z}/l^\nu)$, with $i=2s-2 \geqq 2$, we observe that:

$$(3.10.2) \qquad \bar{\rho}_i(\alpha)-\bar{\rho}_i(\alpha)\circ\gamma\circ\tau : \quad \Sigma^{i-2} C(l^\nu) \wedge (X/\emptyset)_{et} \to B\hat{GL}_N^{(i)},$$

satisfies $(\bar{\rho}_i(\alpha)-\bar{\rho}_i(\alpha)\circ\gamma\circ\tau)^*(ch_i)=0$. Consequently, (3.10.2) lifts to:

$$\bar{\rho}_i(\alpha)^\sim : \quad \Sigma^{i-2} C(l^\nu) \wedge (X/\emptyset)_{et} \to B\hat{GL}_N^{(i+2)}.$$

We define $ch_{s,2}$ on $\alpha \in K_i(X, \mathbf{Z}/l^\nu)$ for $i=2s-2 \geqq 2$ by:

$$ch_{s,2}(\alpha)=(\bar{\rho}_i(\alpha)^\sim)^*(ch_s) \in H^2(X, \mathbf{Z}/l^\nu(s)).$$

For $i=1$, we define $ch_{1,1}$ to be that map determined by $ch_{2,2}$ on $K_2(X \times GL_1, \mathbf{Z}/l^\nu)$ and $K_2(X, \mathbf{Z}/l^\nu)$. More concretely,

$$ch_{1,1}(\alpha) \otimes \beta = ch_{2,1}(\alpha \otimes \beta) \in H^1(X, \mathbf{Z}/l^\nu(1)) \otimes \mathbf{Z}/l^\nu(1) \simeq H^1(X, \mathbf{Z}/l^\nu(2))$$

where $\beta \in \mathbf{Z}/l^\nu(1)$ is a generator. We define $ch_{0,0}$ to be the $\mathbf{Z}/l^\nu$ reduction of the rank function and $ch_{1,2}$ to be the (galois equivariant) first chern class.

The galois equivariance of the $ch_{s,r}(\ )$ so defined is proved as is Proposition 3.4 (*cf.* [11], 5.5). For example, if $\sigma \in Gal(F, L)$, then $\bar{\rho}(\sigma \circ \alpha)^*(ch_s)$ is the pull-back of ch_s via the composition:

$$(\sigma^{-1})^\sim \circ \bar{\rho}_i(\alpha) \circ (1 \wedge \sigma_{et}) : \quad \Sigma^{i-2} C(l^\nu) \wedge (X/\emptyset)_{et}$$
$$\to \Sigma^{i-2} C(l^\nu) \wedge (X/\emptyset)_{et} \to B\hat{GL}_N^{(i)} \to B\hat{GL}_N^{(i)},$$

by Theorem 3.3. On the other hand, the action of σ on $\bar{\rho}(\alpha)^*(ch_s) \in H^{2s-i}(X, \mathbf{Z}/l^\nu(s))$ is also given by this pull-back of ch_s, because the effect of $(\sigma^{-1})^{\wedge *}$ on ch_s when viewed in $H^{2s}(B\hat{GL}_N^{(i)}, \mathbf{Z}/l^\nu)$ is that of σ on $\mathbf{Z}/l^\nu(s)$. ■

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