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ON THE PICARD NUMBER OF A COMPLEX PROJECTIVE VARIETY

BY TETSUJI SHIODA

0. Introduction

Let X be a non-singular projective variety over $\mathbb{C}$. The Picard number $\rho(X)$ of X, i.e. the rank of Néron-Severi group of X, satisfies the well-known inequality:

$$(0.1) \qquad 1 \leqq \rho(X) \leqq h^{1,1}(X) = b_2(X) - 2h^{2,0}(X),$$

where $b_2(X)$ and $h^{i,j}(X)$ denote the 2nd Betti number and the Hodge numbers of X. In terms of the Lefschetz number $\lambda(X) = b_2(X) - \rho(X)$, (0.1) is equivalent to:

$$(0.2) \qquad 2h^{2,0}(X) \leqq \lambda(X) \leqq b_2(X) - 1.$$

In this paper, we study the Picard number of a non-singular projective variety over $\mathbb{C}$ having an automorphism of finite order. Given an automorphism g of finite order of such a variety X, we shall introduce two numerical invariants $L(X, g)$ and $\varphi(X, g)$ of the pair (X, g), which is defined in terms of the action of g on the space $H^{2,0}(X)$ of holomorphic 2-forms on X (Def. 1.2), and prove the inequality:

$$(0.3) \qquad 2h^{2,0}(X) \leqq L(X, g) \leqq \lambda(X),$$

and the congruence property:

$$(0.4) \qquad \lambda(X) \equiv 0 \bmod \varphi(X, g)$$

(*see* Theorem 1.3, § 1). These results improve the familiar estimate (0.1) or (0.2), reducing to the latter in case g is the identity. The proof will be given in paragraph 2 by considering the action of g on the group of transcendental cycles. As an application, we shall compute the Picard numbers of certain surfaces in $\mathbb{P}^3$ (§ 3-6). Among other things, we prove the following results:

(*a*) If X is a non-singular surface of a prime degree m in $\mathbb{P}^3$, defined by the equation:

$$w^m + F(x, y, z) = 0,$$

then we have:

$$\begin{cases} 1 \leqq \rho(X) \leqq h^{1,1}(X) - p_g(X), \\ \rho(X) \equiv 1 \bmod (m-1). \end{cases}$$

(*See* Proposition 3.2 for more general statements.)

(*b*) Moreover the following surface of degree m:

$$w^m + xy^{m-1} + yz^{m-1} + zx^{m-1} = 0,$$

has the Picard number 1 for any prime $m \geqq 5$ (Thm. 4.1).

Obviously the above (*b*) gives an elementary proof of the classical Theorem of M. Noether in the case of prime degree. Actually, when we started the present work, our first guess was that the Picard number of a surface with "many" automorphisms would be relatively big, and the above example, at first, was a surprise to us.

Thus our method, simple as it is, gives some new information on the study of Picard numbers. Still the problem of evaluating the Picard number is very difficult, and we mention in paragraph 6 some miscellaneous results for quintic surfaces in $\mathbb{P}^3$.

The last section, paragraph 7, deals with the extension of some of the above results to characteristic p. The extension to algebraic cycles of higher codimension is also possible, but it will be discussed elsewhere.

Finally we thank A. Furukawa for providing us with the proof of Lemma 4.3 which is given in the Appendix.

1. Invariants L (X, g) and φ (X, g)

First we recall some elementary algebraic facts, fixing the notation. Let G be a cyclic group of order d with a generator g, and let $\mathbb{Q}[G]$ or $\mathbb{C}[G]$ be the group ring of G over $\mathbb{Q}$ or $\mathbb{C}$. As is well-known, these rings are semi-simple and the decomposition into simple components is given as follows:

(1.1)
$$\begin{cases} \mathbb{Q}[G] \simeq \bigoplus_{n \mid d} W_n = \mathbb{Q}[t]/(\Phi_n(t)), \\ \mathbb{C}[G] \simeq \bigoplus_{\alpha^d = 1} U_\alpha, \qquad U_\alpha = \mathbb{C}[t]/(t-\alpha), \end{cases}$$

where $\Phi_n(t)$ is the n-th cyclotomic polynomial and where multiplication by g in the group rings corresponds to multiplication by t in the residue rings of $\mathbb{Q}[t]$ or $\mathbb{C}[t]$ on the right sides. Moreover we have:

(1.2)
$$W_n \otimes \mathbb{C} \simeq \bigoplus_{\alpha \in P_n} U_\alpha,$$

where P_n denotes the set of primitive n-th roots of unity. These facts easily follow from the relations:

(1.3)
$$t^d - 1 = \prod_{n \mid d} \Phi_n(t), \qquad \Phi_n(t) = \prod_{\alpha \in P_n} (t-\alpha),$$

and the irreducibility of $\Phi_n(t)$ in $\mathbb{Q}[t]$ (*see* e.g. [3]).

Now suppose that G acts on a vector space H of finite dimension over $\mathbb{C}$. By the semi-simplicity of $\mathbb{C}[G]$, we can write:

$$H = \bigoplus_{\alpha^d=1} V(\alpha), \qquad V(\alpha) \simeq U_\alpha^{\dim V(\alpha)}; \tag{1.4}$$

that is, $V(\alpha)$ is the eigenspace of g with eigenvalue α. In the following definition, $\varphi(.)$ is the Euler function, i. e. :

$$\varphi(n) = \# P_n = \deg \Phi_n(t).$$

DEFINITION 1.1. — With the above notation, we set:

$$\left\{\begin{array}{l} N(H, G) = \{n \mid V(\alpha) \neq 0 \text{ for some } \alpha \in P_n\}, \\ L(H, G) = \sum_{n \in N(H, G)} \max_{\alpha \in P_n} \{\dim V(\alpha) + \dim V(\bar{\alpha})\} \varphi(n), \\ \varphi(H, G) = \mathrm{GCD}\{\varphi(n) \mid n \in N(H, G)\}, \end{array}\right. \tag{1.5}$$

where $\bar{\alpha}$ denotes the complex conjugate of α.

We have the obvious inequality:

$$L(H, G) \geqq 2 \dim H, \tag{1.6}$$

because:

$$L(H, G) \geqq \sum_n \sum_{\alpha \in P_n} \{\dim V(\alpha) + \dim V(\bar{\alpha})\}.$$

Now let X be a non-singular projective variety over $\mathbb{C}$ and let g be an automorphism of X of order d. The group G generated by g acts on various cohomology groups of X or on their invariant subspaces. In particular, considering the action of G on the space $H^{2,0}(X)$ of holomorphic 2-forms on X, we make the following definition:

DEFINITION 1.2. — Using the notation of Definition 1.1, we set:

$$\left\{\begin{array}{l} N(X, g) = N(H^{2,0}(X), G), \\ L(X, g) = L(H^{2,0}(X), G), \\ \varphi(X, g) = \varphi(H^{2,0}(X), G). \end{array}\right. \tag{1.7}$$

By (1.6), we have:

$$L(X, g) \geqq 2 h^{2,0}(X). \tag{1.8}$$

We are ready to state the main Theorem of this paper.

THEOREM 1.3. — *The Lefschetz number* $\lambda(X)$ *of a non-singular projective variety* X *over* $\mathbb{C}$ *satisfies the following estimate and congruence:*

$$\left\{\begin{array}{l} \lambda(X) \geqq L(X, g), \\ \lambda(X) \equiv 0 \bmod \varphi(X, g), \end{array}\right. \tag{1.9}$$

for every automorphism g of finite order. Equivalently, the Picard number $\rho(X)$ *of* X *satisfies:*

(1.10) $$\begin{cases} \rho(X) \leqq b_2(X) - L(X, g), \\ \rho(X) \equiv b_2(X) \bmod \varphi(X, g). \end{cases}$$

The proof will be given in the next section.

We deduce some consequences:

COROLLARY 1.4. — *If a non-singular projective variety* X *has an automorphism g of finite order such that* $g^*(\omega) \neq \pm\omega$ *for any* $\omega \in H^{2,0}(X)$, $\omega \neq 0$, *then the Picard number of* X *has the same parity as the 2nd Betti number:*

$$\rho(X) \equiv b_2(X) \bmod 2.$$

Proof. — The assumption implies that $N(X, g)$ does not contain $n=1$ nor 2. Then $\varphi(n)$ is even for all $n \in N(X, g)$, and hence $\varphi(X, g)$ is also even. The assertion follows from (1.10).

Q.E.D.

COROLLARY 1.5. — *If a non-singular projective variety* X *has an automorphism g of finite order such that all the eigenvalues of* g^* *on* $H^{2,0}(X)$ *are primitive n-th roots of unity for some fixed integer n, then:*

$$\rho(X) \equiv b_2(X) \bmod \varphi(n).$$

Proof. — Under the assumption, we have $N(X, g) = \{n\}$ and hence $\varphi(X, g) = \varphi(n)$. Thus the assertion follows from (1.10).

Q.E.D.

The above Corollary applies, for instance, to varieties with $h^{2,0}(X)=1$. This has been observed, among other things, by Nikulin for the case of K3 surfaces (*cf.* [5], § 3).

2. The group of transcendental 2-cycles

Given a non-singular projective variety X over $\mathbb{C}$, we denote by $T(X)$ the group of transcendental 2-cycles on X, which is defined as the quotient of $H^2(X, \mathbb{Z})$ by the Néron-Severi group $NS(X)$:

(2.1) $$T(X) = H^2(X, \mathbb{Z})/NS(X).$$

If G is a cyclic group of order d generated by an automorphism g of X, then we can view

$$NS(X)_{\mathbb{Q}} = NS(X) \otimes \mathbb{Q} \quad \text{and} \quad T(X)_{\mathbb{Q}} = T(X) \otimes \mathbb{Q},$$

as $\mathbb{Q}[G]$-modules. By the semi-simplicity of $\mathbb{Q}[G]$, we can write:

(2.2) $$T(X)_{\mathbb{Q}} \simeq \bigoplus_{n \mid d} W_n^{r(n)},$$

for some $r(n) \geqq 0$, W_n being as in (1.1). Then it follows from (1.2) that:

$$T(X)_{\mathbb{C}} = T(X) \otimes \mathbb{C} \simeq \bigoplus_{n \mid d} \left(\bigoplus_{\alpha \in P_n} U_\alpha^{r(n)} \right). \tag{2.3}$$

This will be compared with the decomposition:

$$H^{2,0}(X) = \bigoplus_{n \in N(X, g)} \bigoplus_{\alpha \in P_n} V(\alpha), \qquad V(\alpha) \simeq U_\alpha^{\dim V(\alpha)}, \tag{2.4}$$

the notation being as in (1.4), (1.5) and (1.7).

LEMMA 2.1. — *With the above notation, we have:*
(i) $r(n) > 0 \iff n \in N(X, g)$,

and

(ii) $r(n) \geqq \max_{\alpha \in P_n} \{\dim V(\alpha) + \dim V(\bar{\alpha})\}$.

Proof. — By the Theorem of Lefschetz and Hodge, we have:

$$NS(X)_{\mathbb{Q}} = H^2(X, \mathbb{Q}) \cap H^{1,1}(X). \tag{2.5}$$

In particular, the natural map:

$$\begin{array}{ccc} H^2(X, \mathbb{C})/NS(X)_{\mathbb{C}} & \overset{\psi}{\rightarrow} & H^2(X, \mathbb{C})/H^{1,1}(X) \\ \| & & \wr\| \\ T(X)_{\mathbb{C}} & & H^{2,0}(X) \oplus H^{0,2}(X) \end{array} \tag{2.6}$$

is surjective. Obviously this map ψ is compatible with the actions of G on both spaces. Recall that $H^{0,2}$ is the complex conjugate of $H^{2,0}$. By looking at the eigenspaces with eigenvalue $\alpha \in P_n$, we have the induced surjective map:

$$U_\alpha^{r(n)} \rightarrow V(\alpha) \oplus \overline{V(\bar{\alpha})},$$

which proves the assertion (ii). It follows from this and the definition of $N(X, g)$ that $r(n) > 0$ if $n \in N(X, g)$. Now let:

$$T(X)_{\mathbb{Q}} = T_1 \oplus T_2,$$

where:

$$T_1 \simeq \bigoplus_{n \in N(X, g)} W_n^{r(n)}, \qquad T_2 \simeq \bigoplus_{n \notin N(X, g)} W_n^{r(n)}.$$

Then we see that ψ maps T_2 to 0 because of the compatibility of ψ with the G-actions. Therefore the inverse image of T_2 in $H^2(X, \mathbb{Q})$ under the natural map $H^2(X, \mathbb{Q}) \rightarrow T(X)_{\mathbb{Q}}$ lies in $H^2(X, \mathbb{Q}) \cap H^{1,1}$, hence in $NS(X)_{\mathbb{Q}}$ by (2.5). This proves that $T_2 = 0$, i.e. that $r(n) = 0$ if $n \notin N(X, g)$.

Q.E.D.

Summarizing the above, we have proved the following:

THEOREM 2.2. — *With the same notation as above, the* $\mathbb{Q}$ [G]*-module structure of* $T(X)_{\mathbb{Q}}$ *is given by:*

$$T(X)_{\mathbb{Q}} \simeq \bigoplus_{n \in N(X, g)} W_n^{r(n)}, \tag{2.7}$$

for some positive integers $r(n)$ *satisfying* (*ii*) *of Lemma* 2.1.

Proof of Theorem 1.3. — Comparing the dimensions of both sides of (2.7), we have:

$$\dim T(X)_{\mathbb{Q}} = \sum_{n \in N(X, g)} r(n) \dim W_n,$$

that is:

$$\lambda(X) = \sum_{n \in N(X, g)} r(n)\, \varphi(n). \tag{2.8}$$

Then, by Definitions 1.1, 1.2 and Lemma 2.1, we have:

$$\lambda(X) \geqq L(X, g)$$

and:

$$\lambda(X) \equiv 0 \bmod \varphi(X, g),$$

which proves Theorem 1.3.

Q.E.D.

REMARK 2.3. — We have defined the group of transcendental 2-cycles T (X) by (2.1) to consider varieties of arbitrary dimension. In dealing with surfaces, however, we may define the group of transcendental 2-cycles T′ (X) as the orthogonal complement of the Néron-Severi group in $H^2(X, \mathbb{Z})$ with respect to the cup product pairing. It should be noted that the structure of $\mathbb{Q}$ [G]-modules on $T(X)_{\mathbb{Q}}$ and $T'(X)_{\mathbb{Q}}$ is the same, and hence Theorem 1.3 is valid with T′ (X) in place of T (X).

3. Application to surfaces in $\mathbb{P}^3$

In subsequent sections, we evaluate Picard numbers of some surfaces in $\mathbb{P}^3$ by applying Theorem 1.3.

Fix $m \geqq 4$, and let X denote a non-singular surface of degree m in $\mathbb{P}^3$ or the minimal non-singular model of a surface of degree m in $\mathbb{P}^3$ having at most rational double points. By the theorem of simultaneous resolution, the diffeomorphism type of such a surface is uniquely determined by m (*cf.* Brieskorn [10]).

As is well known, the geometric genus $p_g(X)$ and the 2nd Betti number $b_2(X)$ are respectively given by:

$$p_g(m) = (m-1)(m-2)(m-3)/6, \tag{3.1}$$

$$b_2(m) = m(m^2 - 4m + 6) - 2 = (m-1)(m^2 - 3m + 3) + 1. \tag{3.2}$$

If X is defined by the homogeneous equation of degree m $F(x, y, z, w)=0$, then the space $H^{2,0}(X)$ of holomorphic 2-forms on X has the following basis $\{\omega_{ijk}\}$:

$$(3.3)\qquad \omega_{ijk}=\frac{x^i y^j w^k}{F_w}dx\wedge dy \qquad \left(\begin{matrix} i, j, k\geqq 0 \\ i+j+k\leqq m-4\end{matrix}\right),$$

in terms of the inhomogeneous coordinates (x, y, w) $(z=1)$, where F_w stands for $(\partial F/\partial w)(x, y, 1, w)$.

We shall mainly consider surfaces of the following types:

$$(3.4)\qquad F(x, y, z)+w^m=0 \quad (\S 3 \text{ and } 4),$$

$$(3.5)\qquad P(x, y)+Q(z, w)=0 \quad (\S 5),$$

where F, P, Q are homogeneous polynomials of degree m. In the following, we always assume that the equation in question defines a surface in $\mathbb{P}^3$, say X′, which is either non-singular or has at most rational double points, and that X is the minimal non-singular model of X′. For simplicity, we call such an X simply *a surface of degree m defined by* (3.4) or (3.5).

LEMMA 3.1. — *Let* X *be a surface defined by* (3.4), *and let g be the automorphism of order m of* X, *defined by*:

$$g:\ (x, y, z, w)\mapsto(x, y, z, \zeta_m w) \qquad [\zeta_m=\exp(2\pi i/m)].$$

Then, with the notation of Definition 1.2, *we have:*

$$(3.6)\qquad N(X, g)=\{n\,|\,n>1 \text{ and } n\,|\,m\},$$

and:

$$(3.7)\qquad L(X, g)=\sum_{\substack{n|m\\ n>1}} A_m(m/n)\,\varphi(n),$$

where:

$$(3.8)\qquad A_m(r)=\frac{1}{2}(m-r-1)(m-r-2)+\frac{1}{2}(r-1)(r-2) \qquad (1\leqq r\leqq m-1).$$

Proof. — The 2-forms in (3.3) are eigenforms of g^*, i. e.:

$$(3.9)\qquad g^*(\omega_{ijk})=\zeta_m^{k+1}\,\omega_{ijk} \qquad (0\leqq k\leqq m-4-i-j\leqq m-4).$$

Hence the subspace $V(\zeta_m^{k+1})$ of $H^{2,0}(X)$ corresponding to eigenvalue ζ_m^{k+1} has the dimension:

$$(3.10)\quad \dim V(\zeta_m^{k+1})=\#\{(i, j)\,|\,i, j\geqq 0,\ i+j\leqq m-4-k\}$$
$$=\begin{cases}\frac{1}{2}(m-k-2)(m-k-3) & (0\leqq k\leqq m-4),\\ 0 & (m-3\leqq k\leqq m-1).\end{cases}$$

Since ζ_{mi}^{k+1} is a primitive $m/(k+1, m)$-th root of unity, the set $N(X, g)$ consists of all divisors n of m different from 1. Let us compute the invariant $L(X, g)$. We have, by (3.10) and (3.8):

$$\dim V(\zeta_m^r) + \dim V(\bar{\zeta}_m^r) = A_m(r) \qquad (1 \leqq r \leqq m-1).$$

As is easily seen, $A_m(r) = A_m(m-r)$ and $A_m(r) > A_m(r+1)$ for $1 \leqq r < m/2$. Hence, for all $n \in N(X, g)$, we have:

$$\max_{(r, m) = m/n} A_m(r) = A_m(m/n).$$

It follows from the definition of $L(X, g)$ that:

$$L(X, g) = \sum_{\substack{n \mid m \\ n > 1}} A_m(m/n)\,\varphi(n),$$

proving (3.7).

Q.E.D.

PROPOSITION 3.2. — *If a surface X is defined by the equation*:

$$w^m + F(x, y, z) = 0,$$

then its Picard number has the following estimate:

$$\rho(X) \leqq b_2(m) - \sum_{\substack{n \mid m \\ n > 1}} A_m(m/n)\,\varphi(n), \tag{3.11}$$

where $A_m(r)$ *is defined by* (3.8). *In particular, if m is a prime, then:*

$$\rho(X) \leqq b_2(m) - 3p_g(m) = 1 + \frac{1}{2} m(m-1)^2, \tag{3.12}$$

and furthermore the following congruence holds:

$$\rho(X) \equiv 1 \bmod m-1. \tag{3.13}$$

More generally, if m is odd, then:

$$\rho(X) \equiv 1 \bmod 2. \tag{3.14}$$

Proof. — The first assertion is an immediate consequence of Theorem 1.3 and Lemma 3.1. If m is a prime, then $N(X, g)$, (3.6), consists of $\{m\}$ alone, and hence we have in this case:

$$L(X, g) = A_m(1)\,\varphi(m) = \frac{1}{2}(m-2)(m-3).(m-1) = 3p_g(m) \quad [\textit{see } (3.1)],$$

and:

$$\varphi(X, g)=\varphi(m)=m-1.$$

Thus the second assertion follows from Theorem 1.3 and (3.2). Finally, if m is odd, then $N(X, g)$ does not contain $n=1$ or 2, so that $\varphi(n)$ is even for all $n \in N(X, g)$. Hence $\varphi(X, g)$ is even, which proves the last assertion (*cf.* Cor. 1.4).

Q.E.D.

REMARK 3.3. — (i) Contrary to the last statement of Proposition 3.2, we do not know whether $\rho(X)$ is always even if m is even and $m>4$. For $m=4$, this is the case. In fact, the above proof shows that $N(X, g)=\{4\}$ and $\varphi(X, g)=\varphi(4)=2$ in this case.

(ii) For non-prime m, the estimate (3.12), which is stronger than (3.11), does not hold in general. A counter-example is given by the Fermat surface of degree $m=4$ or 6, for which $\rho=h^{1,1}$ holds (*cf.* [7]).

Now, as a supplement to Proposition 3.1, we prove:

PROPOSITION 3.4. — *If* X *is defined by the equation*:

$$w^m+\prod_{1 \leqq i \leqq m}(a_i x+b_i y+c_i z)=0, \tag{3.15}$$

where no three of m linear forms in the product have a non-trivial common zero, then:

$$\rho(X) \geqq 1+\frac{1}{2} m(m-1)^2=b_2(m)-3 p_g(m), \tag{3.16}$$

with the equality holding in case m is a prime.

Proof. — Denote by X′ the surface in $\mathbb{P}^3$ defined by (3.15) so that X is the minimal non-singular model of X′. Under the assumption, X′ has $m(m-1)/2$ singular points corresponding to the intersection points of m lines $a_i x+b_i y+c_i z=0$ in $\mathbb{P}^2$; each of them is a rational double point of type A_{m-1} (locally like $t^m=uv$). As is well known, such a singular point is resolved into $(m-1)$ rational curves on X, and the latter are numerically independent. Thus NS(X) contains $(m-1) . m(m-1)/2$ independent curves, in addition to the pull back to X of a hyperplane section of X′ in $\mathbb{P}^3$. This proves the inequality (3.16), and the equality statement for m prime follows from (3.12) of Proposition 3.2.

Q.E.D.

4. An explicit example of a non-singular surface in $\mathbb{P}^3$ with the Picard number one

As is well-known, the generic surface of degree $m \geqq 4$ in $\mathbb{P}^3$ has the Picard number 1 (Noether's Theorem, *cf.* Deligne [1]). It is, however, of independent interest to have an *explicit* example of a surface with this property. We shall give such an example below in case m is a prime.

For any m, let Y_m denote the non-singular surface of degree m with the equation:

$$w^m + xy^{m-1} + yz^{m-1} + zx^{m-1} = 0. \tag{4.1}$$

It has the following automorphism of order d:

$$\begin{cases} g: \ (x, y, z, w) \mapsto (\zeta_d^{-(m-1)^2} x, \zeta_d^{m-1} y, \zeta_d^{-1} z, \zeta_m w) \\ \quad [\zeta_d = \exp(2\pi i/d), \ \zeta_m = \exp(2\pi i/m)], \end{cases} \tag{4.2}$$

where:

$$d = (m-1)^3 + 1 = md_0, \quad d_0 = m^2 - 3m + 3. \tag{4.3}$$

THEOREM 4.1. — *Let m be a prime $\geqq 5$. Then the Picard number of the surface Y_m is equal to* 1.

Before proving the Theorem, we note that it implies (by standard specialization argument) the following result, which is stronger than Noether's Theorem in the case of prime degree:

COROLLARY 4.2. — *If Y is any surface of prime degree $m \geqq 5$ which specializes to* (4.1) (e. g. *generic surface of degree m in the sense of* [1]), *then the Picard number of* Y *is equal to* 1.

To prove the Theorem, let us first look at the action of g on the holomorphic 2-forms of (3.3). Noting that g takes the following form in terms of inhomogeneous coordinates (x, y, w):

$$g: \ (x, y, w) \to (\zeta_d^{m(2-m)} x, \zeta_d^m y, \zeta_d \zeta_m w), \tag{4.4}$$

we have:

$$g^*(\omega_{ijk}) = \zeta_d^{B(i,j,k)} \omega_{ijk}, \tag{4.5}$$

by setting:

$$\begin{cases} B(i, j, k) = m(2-m)(i+1) + mj + (d_0+1)(k+1) \\ \quad (0 \leqq i, j, k, \ i+j+k \leqq m-4). \end{cases} \tag{4.6}$$

LEMMA 4.3. — *Assume that m is a prime $\geqq 5$. Then, for any divisor n of $d_0 = m^2 - 3m + 3$, there exists some* $B(i, j, k)$ *such that* GCD $(d, B(i, j, k)) = n$.

Proof. — Here we prove this under the assumption that d_0 is also prime; the general case will be proven in the Appendix due to A. Furukawa. If d_0 is prime, only divisors of d_0 are 1 and d_0. Now we note:

$$\begin{cases} B(0, 1, 0) = m(2-m) + m + (d_0+1) = 4, \\ B(0, 0, m-4) = m(2-m) + (d_0+1)(m-3) = -4d_0 + d, \end{cases}$$

and hence, for m and d_0 both odd, we have:

$$\begin{cases} \mathrm{GCD}(d, B(0, 1, 0)) = (md_0, 4) = 1, \\ \mathrm{GCD}(d, B(0, 0, m-4)) = (d, -4d_0) = d_0, \end{cases}$$

proving the Lemma.

Q.E.D.

Proof of Theorem 4.1. — The above Lemma implies that, for the pair (Y_m, g) given by (4.1) and (4.2), the set $N(Y_m, g)$ contains all integers of the form $m\nu$ where ν is arbitrary divisor of d_0. Then:

$$(4.7)\quad L(Y_m, g) \geqq \sum_{\nu \mid d_0} \varphi(m\nu) = \varphi(m) \sum_{\nu \mid d_0} \varphi(\nu) = (m-1).d_0 = b_2(m) - 1 \quad (cf.\ (3.2)).$$

Since $\lambda(Y_m) \geqq L(Y_m, g)$ by Theorem 1.3, this proves:

$$\lambda(Y_m) = L(Y_m, g) = b_2(m) - 1,$$

and:

$$\rho(Y_m) = 1.$$

Q.E.D.

REMARK 4.3. — The Picard number of the surface (4.1) can be "large" if m is not a prime. For instance, we can show that, for $m=4$, $\rho(Y_4)=20$ (this is the maximum of Picard numbers of K3 surfaces) and for $m=6$, $\rho(Y_6)=32$.

5. Further application to surfaces in $\mathbb{P}^3$

Now we consider surfaces in $\mathbb{P}^3$ of the second type (3.5).

LEMMA 5.1. — *Let* X *be a surface of degree m defined by:*

$$(5.1)\qquad P(x, y) + Q(z, w) = 0,$$

where P *and* Q *are forms of degree m without multiple factors, and let g be the automorphism:*

$$(5.2)\qquad \left\{\begin{array}{l} g: \ (x, y, z, w) \mapsto (\zeta_m x, \zeta_m y, z, w) \\ \qquad [\zeta_m = \exp(2\pi i/m)]. \end{array}\right.$$

Assume that $m \geqq 7$ *or* $m=5$. *Then we have:*

$$(5.3)\qquad N(X, g) = \{ n \mid n > 1 \text{ and } n \mid m \},$$

and:

$$(5.4)\qquad L(X, g) = 2 \sum_{\substack{n \mid m \\ n > 1}} \ \max_{\substack{2 \leqq r \leqq m-2 \\ (r, m) = m/n}} \{ (r-1)(m-r-1) \} \varphi(n).$$

In particular, if m is a prime $\geqq 5$, *then* $N(X, g) = \{ m \}$ *and:*

$$(5.5)\qquad L(X, g) = \frac{1}{2}(m-1)^2 (m-3) = 3\, p_g(m) + \frac{1}{2}(m-1)(m-3).$$

Proof. — With respect to the 2-forms ω_{ijk} in (3.3), we have:

$$g^*(\omega_{ijk}) = \zeta_m^{i+j+2}\, \omega_{ijk} \qquad (i, j, k \geqq 0,\ i+j+k \leqq m-4).$$

and hence:

(5.6) $$\dim V(\zeta_m^r)=\begin{cases} 0, & r=0,\,1 \quad \text{or} \quad m-1,\\ (r-1)(m-r-1), & 2\leqq r\leqq m-2.\end{cases}$$

Then the Lemma is proven by a simple computation, as in Lemma 3.1.

Q.E.D.

PROPOSITION 5.2. — *The notation being as in Lemma* 5.1, *the Picard number of the surface* X *defined by* (5.1) *satisfies the inequality:*

(5.7) $$(m-1)^2+1\leqq\rho(X)\leqq b_2(m)-L(X,\,g).$$

In particular, if m is a prime $\geqq 5$, *then:*

(5.8) $$(m-1)^2+1\leqq\rho(X)\leqq\frac{1}{2}(m-1)(m^2-2m+3)+1,$$

and furthermore, the following congruence holds:

(5.9) $$\rho(X)\equiv 1 \bmod m-1.$$

More generally, if m is odd, then:

(5.10) $$\rho(X)\equiv 1 \bmod 2.$$

Proof. — Except for the lower extimate of $\rho(X)$, the assertions follow from Theorem 1.3 and Lemma 5.2 in the same way as in the proof of Proposition 3.2. This lower estimate is a consequence of the "inductive structure" of the equation (5.1). Let us briefly recall it (for the detail, *see* Sasakura [6], § 1, or Shioda-Katsura [8], Remark 1.10). Let C and C′ denote the following plane curves:

$$u^m=P(x,\,y) \qquad \text{and} \qquad v^m=Q(z,\,w).$$

Then the surface X is obtained from the product $C\times C'$ by the following three steps: (1) blow up m^2 points of $C\times C'$ defined by $u=v=0$, (2) form the quotient surface of the blown-up surface by the cyclic group μ_m of order m generated by:

$$(x,\,y,\,u;\,z,\,w,\,v)\mapsto(x,\,y,\,\zeta_m u;\,z,\,w,\,\zeta_m v),$$

and then, (3) blow down certain $2m$ non-singular rational curves in the quotient surface. This gives:

(5.11) $$\rho(X)=r+2+m^2-2m\geqq(m-1)^2+1,$$

where r is the rank of the group of classes of correspondences from C to C′ which are compatible with the μ_m-actions on C an C′.

Q.E.D.

REMARK 5.3. — (i) The relation (5.11) can be used to give another proof of the congruence (5.9), by showing that the said group of correspondence classes spans over $\mathbb{Q}$ a vector space over the cyclotomic field $\mathbb{Q}(\zeta_m)$ (*cf.* [6], § 5). In this approach, the connection of the Picard number and the periods of 2-forms is more explicit. Sasakura ([6], Thm. 4.2) proved, for example, that if m is a prime, then:

$$\rho(X)=\begin{cases}(m-1)^2+1 & \text{for generic P and Q,}\\ (m-1)^2+m & \text{for generic P, Q=P.}\end{cases}$$

(ii) In contrast to the last statement (5.10) for m odd, it is not true in general that $\rho(X)$ is even in case m is even. For example, if $P(x, y)=x^4+y^4+\lambda x^2 y^2$ and if X_λ is defined by $P(x, y)+P(z, w)=0$, then $\rho(X_\lambda)=19$ holds for all λ except for some countable values of λ (*cf.* Mizukami [4]).

(iii) The upper bound of (5.8) is attained, for example, by the Fermat surface of degree $m=5$, for which $\rho(X)=37$ (*cf.* [7]).

6. Remarks on Picard numbers of quintic surfaces

When a surface (or a variety) varies in a family, the Picard number takes various values in general, and it is usually not easy to determine which values are actually taken. In this section, we make some remarks on this problem for the family of surfaces in $\mathbb{P}^3$ of degree m, especially for $m=5$. (By the convention of paragraph 3, we mean by a surface in $\mathbb{P}^3$ either a non-singular surface in $\mathbb{P}^3$ or the minimal non-singular model of a surface in $\mathbb{P}^3$ having rational double points.)

For $m\leqq 4$, the solution to the above problem is well known. In case $m\leqq 3$, surfaces of degree m are rational surfaces and hence $\rho(X)=b_2(X)$ for all such X. For $m=4$, the local Torelli Theorem for K3 surfaces implies that the Picard number of a quartic surface takes all values in the allowable range (0.1):

$$1\leqq\rho(X)\leqq h^{1,1}=20.$$

For $m\geqq 5$, however, very little has been known. Let us consider the case $m=5$ in some detail, though analogous results can be obtained for any fixed prime m. For a quintic surface X, we have:

$$1\leqq\rho(X)\leqq h^{1,1}=45 \qquad (b_2=53,\ p_g=4).$$

Now the following Table shows some explicit examples of quintic surfaces for which the Picard number can be determined.

The verification of this table will be left to the reader. Roughly speaking, each surface X given below has a fairly large group of automorphisms ([1]), and the application of Theorem 1.3 gives sharp upper bound for $\rho(X)$. On the other hand, one has to find some

([1]) For a systematic study of hypersurfaces in $\mathbb{P}^n$ with an automorphism of large order, we refer to a paper of K. Ishii (in preparation).

independent algebraic cycles on X. This is done either directly as indicated in the Table for the case $\rho=5$, or by computing Hodge classes on X via the action of some automorphism group, as in the case of Fermat varieties (*cf.* [7]).

TABLE 6.1

	Equation of X	Remark
$\rho=1$	$(xy^4+yz^4+zx^4)+w^5=0$	Theorem 4.1
$\rho=5$	$(x^5+xy^4+yz^4)+w^5=0$ or $x^5+xy^4+yz^4+zw^4=0$	$NS(X)_{\mathbf{Q}}=\langle$ 5 lines on $y=0\rangle$; $NS(X)_{\mathbf{Q}}=\langle$ 5 lines on $z=0\rangle$
$\rho=9$	?	–
$\rho=13$	?	–
$\rho=17$	$P(x, y)+Q(z, w)=0$ (P, Q generic)	Remark 5.3 (i)
$\rho=21$	$P(x, y)+P(z, w)=0$ (P generic) or $xy^4+yz^4+zw^4+wx^4=0$	Remark 5.3 (i)
$\rho=25$	$(x^5+xy^4)+(z^5+w^5)=0$ or $(x^5+xy^4)+(z^4w+zw^4)=0$	–
$\rho=29$	$(x^5+xy^4)+(z^5+zw^4)=0$	
$\rho=33$	?	
$\rho=37$	$x^5+y^5+z^5+w^5=0$ or $(x^4y+xy^4)+(z^4w+zw^4)=0$	Fermat quintic (*cf.* [7])
$\rho=41$	$w^5+xyz(x+y+z)(ax+by+cz)=0$ ($a.b.c$: distinct, $\neq 0$)	Proposition 3.4
$\rho=45$	??	–

Among the missing values of ρ with $\rho\equiv 1 \pmod 4$ in the above table, it will be not too difficult to find some example of quintic surfaces with $\rho=9$, 13 or 33. For example, a non-singular quintic surface X, invariant by the automorphism $(x, y, z, w)\mapsto(x, \zeta y, \zeta^2 z, \zeta^3 w)$ $(\zeta=\zeta_5)$, is the universal covering of a Godeaux surface Y, and so $\rho(X)\geqq\rho(Y)=b_2(Y)=9$, and it is likely that a generic such X will have $\rho=9$. Also a quintic surface defined by $w^5+LQ_1Q_2=0$, where L (or Q_1, Q_2) is a linear (or quadratic) form in x, y, z, has $\rho\geqq 33$ (*cf.* the proof of Proposition 3.4), and a generic surface of this type seems to have $\rho=33$. We have no idea for the case $\rho=45$.

Concerning the above, we raise some questions:

QUESTION 6.2. – Let X be a surface of prime degree $m\geqq 5$.

(i) is there any X with $\rho(X)=h^{1,1}$?;

(ii) is there any X with $\rho(X)\not\equiv 1 \pmod{m-1}$? [2].

[2] According to P. Griffiths (oral communication), the answer to (ii) is YES in general. In fact, Picard numbers of surfaces of degree m can take any values 1, 2, 3... which are "not too close" to $h^{1,1}$, at least for sufficiently large m (m need not be prime). For instance, an example of a surface with $\rho=2$ can be given as follows. Fix a line L in $\mathbb{P}^3$ and look at the family of surfaces of degree m which contain L. Then the general member of this family will have $\rho=2$, at least if m is sufficiently large.

It will be an interesting problem to study the period mapping of these surfaces in connection with the above question.

7. Extension to characteristic p

As is well known, the familiar estimate (0.1) of the Picard number fails in general in characteristic $p>0$, but the weaker inequality $\rho(X) \leqq b_2(X)$ continues to hold, where $b_2(X)$ is the Betti number in l-adic cohomology (l : a prime $\neq p$). In this section, we note that, though the estimate of ρ in Theorem 1.3 also fails in characteristic p, the congruence property of ρ holds in a certain sense.

Let R be a discrete valuation ring with quotient field K of characteristic 0 and with residue field k of characteristic $p>0$. Let $\mathscr{X}$ be a scheme, smooth and projective over $S = \operatorname{Spec} R$, and let γ be an automorphism of finite order d of $\mathscr{X}$ over S. Assume that $d \not\equiv 0 \pmod p$. Let (X, g) and $(X'. g')$ be respectively the generic fibre over K and the special fibre over k, of the pair $(\mathscr{X}, \gamma)$; we also write $X' = X(p)$. We regard X defined over $\mathbb{C}$ by taking a suitable subfield of K, finitely generated over $\mathbb{Q}$, and then embedding it into $\mathbb{C}$. By the Picard number $\rho(X')$ of X' we understand the rank of Néron-Severi group of X' considered over the algebraic closure $\bar{k}$ of k.

PROPOSITION 7.1. – *Let (X', g') be a pair of a non-singular projective variety in characteristic p and its automorphism of order d, which lifts to a pair (X, g) in characteristic 0 as above. Then $\rho(X')$ and $\rho(X)$ are related by the following congruence:*

$$\rho(X') \equiv \rho(X) \bmod \varphi_l(X, g), \tag{7.1}$$

where $\varphi_l(X, g)$ for any prime l not dividing p.d is defined by:

$$\left\{\begin{array}{l} \varphi_l(X, g) = \operatorname{GCD}\{\varphi_l(n) \mid n \in N(X, g)\}, \\ \varphi_l(n) = \text{the least positive integer } f \text{ such that } l^f \equiv 1 \pmod n. \end{array}\right. \tag{7.2}$$

Proof (outline). – Let G be a cyclic group of order d. Via g and g', G acts on X and X', and hence on their l-adic cohomology groups for any prime $l \neq p$. By the general theory (*cf.* [2]), $H^i(\bar{X}, \mathbb{Q}_l)$ and $H^i(\bar{X}', \mathbb{Q}_l)$ ($\bar{X} = X \otimes \mathbb{C}$, $\bar{X}' = X' \otimes_k \bar{k}$) are isomorphic under the specialization map, and moreover this isomorphism is compatible with the actions of G by functoriality. Furthermore we have the commutative diagram of $\mathbb{Q}_l$ [G]-modules:

$$\left\{\begin{array}{ccc} H^2(\bar{X}, \mathbb{Q}_l) & \xrightarrow{\sim} & H^2(\bar{X}', \mathbb{Q}_l) \\ \cup & & \cup \\ NS(\bar{X}) \otimes \mathbb{Q}_l & \hookrightarrow & NS(\bar{X}') \otimes \mathbb{Q}_l. \end{array}\right. \tag{7.3}$$

By the comparison theorem, we have:

$$H^2(\bar{X}, \mathbb{Q}_l) \xrightarrow{\sim} H^2(\bar{X}, \mathbb{Q}) \otimes \mathbb{Q}_l, \tag{7.4}$$

the isomorphism being again compatible with the actions of G. By Theorem 2.2, the $\mathbb{Q}[G]$-module structure on $H^2(\overline{X}, \mathbb{Q})$ is given as follows:

$$(7.5)\qquad H^2(\overline{X}, \mathbb{Q}) \simeq NS(\overline{X})_{\mathbb{Q}} \oplus T(\overline{X})_{\mathbb{Q}} \simeq NS(\overline{X})_{\mathbb{Q}} \oplus \Big(\bigoplus_{n \in N(X,g)} W_n^{r(n)} \Big).$$

Now, for any prime $l \nmid n$, $W_n \otimes \mathbb{Q}_l$ decomposes into a direct sum of simple $\mathbb{Q}_l[G]$-modules, say $W_{n,i}$, each of which has rank $\varphi_l(n)$, the latter being defined by (7.2). This is immediate by considering the irreducible factors of the cyclotomic polynomial $\Phi_n(t)$ in $\mathbb{Q}_l[t]$ (say, by Hensel's lemma). Therefore it follows from (7.3) and (7.4) that the $\mathbb{Q}_l$ [G]-submodule $NS(\overline{X}')_{\mathbb{Q}_l}$ of $H^2(\overline{X}, \mathbb{Q}_l)$ has the following form:

$$NS(\overline{X}')_{\mathbb{Q}_l} \simeq NS(\overline{X})_{\mathbb{Q}_l} \oplus \Big(\bigoplus_{n \in N(X,g)} \bigoplus_i W_{n,i}^{s_i(n)} \Big)$$

for some $s_i(n) \leqq r(n)$. This proves:

$$(7.6)\qquad \rho(X') = \rho(X) + \sum_{n \in N(X,g)} \sum_i s_i(n)\, \varphi_l(n),$$

which implies (7.1).

Q.E.D.

Corollary 7.2. — *With the same notation as in Proposition* 7.1, *assume further that* $N(X, g) \not\ni 1, 2$. *Then:*

$$(7.7)\qquad \rho(X') \equiv b_2(X) \bmod 2.$$

Proof. — We choose a prime number $l \neq p$ such that $l \equiv -1 \pmod d$, d being the order of g. Then, for any $n \in N(X, g)$, $l \equiv -1 \pmod n$, but $l \not\equiv 1 \pmod n$ since $n \neq 1$ or 2 by assumption. Hence $\varphi_l(n)$ is even for all $n \in N(X, g)$, and $\varphi_l(X, g)$ is also even. The assertion now follows from Proposition 7.1 and Corollary 1.4.

Q.E.D.

Corollary 7.3. — *With the same notation as in Proposition* 7.1, *assume that* $N(X, g)$ *consists of a single element* n *which is of the form* $n = m^\nu$ *or* $2m^\nu$ *with* m *odd prome. Then:*

$$(7.8)\qquad \rho(X') \equiv b_2(X) \bmod \varphi(n).$$

Proof. — By assumption on n, the group $(\mathbb{Z}/n\mathbb{Z})^\times$ is cyclic. Hence (by Dirichlet's Theorem) we can find a prime number l such that $l \bmod n$ is a generator of this group and such that $l \nmid p.d$. For this choice of l, we have:

$$\varphi_l(X, g) = \varphi_l(n) = \varphi(n).$$

Then (7.8) is immediate from Proposition 7.1 and Corollary 1.5.

Q.E.D.

As an application of the above, let us consider the cose of surfaces in $\mathbb{P}^3$ as in paragraphs 3-6.

PROPOSITION 7.4. — *Assume* $m \not\equiv 0 \pmod p$. *Let* X *be a non-singular surface of degree m in* $\mathbb{P}^3$ *defined by the equation* (3.4) *or* (3.5) *over an algebraically closed field of characteristic p. Then:*

$$\rho(X) \equiv 1 \pmod 2 \quad \text{if} \quad m \text{ is odd} \tag{7.9}$$

and:

$$\rho(X) \equiv 1 \pmod{m-1} \quad \text{if} \quad m \text{ is prime}. \tag{7.10}$$

Proof. — This follows from Corollaries 7.2 and 7.3 (*cf.* Prop. 3.2 and 5.2).

Q.E.D.

REMARK 7.5. — It should be remarked that the congruence (7.7) or (7.9) is compatible with a consequence of the Tate conjecture, according to which the Picard number of a non-singular projective variety over a finite field should have the same parity as its 2nd Betti number (*cf.* [9]).

APPENDIX. — Proof of Lemma 4.3, by Akio Furukawa.

Let us recall the notation. Let m be a prime number $\geqq 5$ and set $d_0 = m^2 - 3m + 3$ and $d = md_0$. Put:

$$B(j, k) = m(2-m) + mj + (d_0+1)(k+1),$$

for $j \geqq 0, k \geqq 0, j+k \leqq m-4$. With the notation of (4.6), $B(j, k)$ equals $B(0, j, k)$. Now we shall prove:

LEMMA. — *Let n be arbitrary divisor of* d_0. *Then there exist some* (j, k) *such that* $j \geqq 0, k \geqq 0$, $j+k \leqq m-4$ *and* GCD $(B(j, k), d) = n$.

Proof. — First note that $B(j, k) \equiv 4(k+1) \bmod m$. Since m is a prime $\geqq 5$ and $1 \leqq k+1 \leqq m-3$, $B(j, k)$ and m are relatively prime. Hence we have GCD $(B(j, k), d) =$ GCD $(B(j, k), d_0)$, and we have only to consider $B(j, k) \bmod d_0$. We have then:

$$B(j, k) \equiv (j-1)m + (k+4) \bmod d_0.$$

We set:

$$N = \{ n \mid 1 \leqq n \leqq d_0,\ n \mid d_0 \},$$
$$M = \{ (j-1)m + (k+4) \mid 0 \leqq j \leqq m-4,\ 0 \leqq k \leqq (m-4) - j \}.$$

Moreover, for a subset S of $\mathbb{Z}$, we set:

$$D(S) = \{ (s, d_0) \mid s \in S \},$$

which is a subset of N. With this notation, we have to prove:

$$N \subset D(M). \tag{$\star$}$$

(1) First of all, we know that $D(M) \ni 1$ and d_0 (*see* the proof of Lemma 4.3 in the text). In particular, when d_0 is a prime (e.g. for $m = 5, 7, 17, 19, \ldots, d_0 = 13, 31, 241,$

307, ...), ($\bigstar$) is true. Thus we may assume that $m \geqq 11$. Under this assumption, we have $d_0/2 < (m-5)\,m+4 = B\,(m-4,\,0)$, and hence it suffices to show:

$$(\bigstar') \qquad D(\{1,\,2,\,\ldots,\,(m-5)\,m+4\}) \subset D(M).$$

(2) Considering the subset of elements of M with $j=0$ or 1, we see:

$$D(\{1,\,2,\,\ldots,\,m-1\}) \subset D(M).$$

(3) Now any element of $\{1,\,2,\,\ldots,\,(m-5)\,m+4\}$ which does not belong to M has one of the following forms:

$$\text{(i)} \qquad jm+1, \quad jm+2 \quad \text{or} \quad jm+3 \qquad (1 \leqq j \leqq m-5)$$

or:

$$\text{(ii)} \qquad jm-s \qquad (1 \leqq j \leqq m-5,\ 0 \leqq s \leqq j-1).$$

For an element of type (ii), if $s=0$, then $(jm,\,d_0)=(j,\,d_0) \in D(M)$ by (2). If $1 \leqq s \leqq j-1$, then we have:

$$d_0-(jm-s)=(m-j-3)\,m+(s+3)=(j'-1)\,m+(k'+4) \in M,$$

where $j'=m-j-2$ and $k'=s-1 \geqq 0$. This implies:

$$(jm-s,\,d_0)=(d_0-(jm-s),\,d_0) \in D(M).$$

(4) It remains to consider elements of type (i) and to show:

$$(\bigstar'') \qquad D(\{jm+k \mid 1 \leqq j \leqq m-5,\ k=1,\,2,\,3\}) \subset D(M).$$

Here we observe that $N \not\ni 2$, 3 or 5 since m is a prime $\geqq 5$. Hence, for any $n \in N$ with $n \neq d_0$, we have $n \leqq d_0/7$. Thus, instead of ($\bigstar''$), it suffices to prove:

$$(\bigstar''') \qquad D(\{jm+k \mid 1 \leqq j \leqq [(m-3)/7],\ k=1,\,2,\,3\}) \subset D(M).$$

(5) Let j be any integer such that $1 \leqq j \leqq (m-3)/7$. Since $m \geqq 11$, we have:

$$4\,j \leqq 4\,(m-3)/7 \leqq m-5 \qquad \text{and} \qquad 2\,j \leqq (m+2)/3 \leqq m-7.$$

Hence, recalling that d_0 is relatively prime to 2 or 3, we see:

$$(jm+1,\,d_0)=(4\,(jm+1),\,d_0)=((4\,j)\,m+4,\,d_0) \in D(M),$$
$$(jm+2,\,d_0)=(2\,(jm+2),\,d_0)=((2\,j)\,m+4,\,d_0) \in D(M),$$
$$(jm+3,\,d_0)=(2\,(jm+3),\,d_0)=((2\,j)\,m+6,\,d_0) \in D(M).$$

This proves ($\bigstar'''$), and hence ($\bigstar$).

Q.E.D.

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