ANNALES SCIENTIFIQUES DE L'É.N.S.

O. GABBER

A. JOSEPH

Towards the Kazhdan-Lusztig conjecture

Annales scientifiques de l'É.N.S. 4e série, tome 14, nº 3 (1981), p. 261-302

<http://www.numdam.org/item?id=ASENS_1981_4_14_3_261_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 14, 1981, p. 261 à 302

TOWARDS THE KAZHDAN-LUSZTIG CONJECTURE (*)

BY O. GABBER AND A. JOSEPH

ABSTRACT. — Let U ($\mathfrak{g}$) be the enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$. Many questions concerning U ($\mathfrak{g}$) (for example the ordering [14] in Prim U ($\mathfrak{g}$)) can be answered in terms of the multiplicities in the composition series of the Verma modules. The main result of this paper shows that a conjecture of Jantzen ([9], 5.18) concerning a certain filtration of the Verma modules implies precisely the formula for these multiplicities which was recently conjectured by Kazhdan and Lusztig ([10], Conj. 1.5). Its proof requires a study of the appropriate generalizations of the so-called Bernstein-Gelfand-Gelfand $\mathcal{O}$ category which arise when the base field is replaced by a commutative ring (defined over the integers). Several results on the corresponding extension groups are also obtained.

0. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A basic problem in the representation theory of $\mathfrak{g}$ is the determination of the composition series for the Verma modules. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^*$ the dual of $\mathfrak{h}$, $R \subset \mathfrak{h}^*$ the set of non-zero roots and $B \subset R$ a basis for R. Then for each $\lambda \in \mathfrak{h}^*$ the quadruplet $\mathfrak{g}$, $\mathfrak{h}$, B, λ determines a Verma module M (λ). Denote its unique simple quotient by L (λ). For each $\alpha \in R$, denote by $s_\alpha \in \operatorname{Aut} \mathfrak{h}^*$ the corresponding reflection, $S := \{s_\alpha : \alpha \in B\}$ and W the subgroup of Aut $\mathfrak{h}^*$ with generating set S. For the moment to simplify matters assume that $-2(\lambda, \alpha)/(\alpha, \alpha)$ is a positive integer for each $\alpha \in B$. Then after Verma each $M(w\lambda) : w \in W$ has finite length with composition factors amongst the $L(w'\lambda) : w' \in W$. Furthermore after Bernstein-Gelfand-Gelfand (in short, BGG) the simple factor $L(w'\lambda)$ occurs in $M(w\lambda)$ if and only if $w' \leqq w$ where $\leqq$ denotes the Bruhat ordering on W. (For further details *see* [6], Chapt. 7.) Thus it remains to determine the multiplicity of each factor — a problem on which important progress was made by Jantzen ([8], [9]).

(*) Shortly after the communication of this paper, we learnt that J. L. Brylinski and M. Kashiwara ([16], [17]) had just announced a proof of what we consider here to be a special (but important) case of the Kazhdan-Lusztig conjecture, namely for λ integral. (This generalization is so natural for representation theory that like Vogan [14] we gave it without comment.) At the same time a similar result was announced by A. A. Beilinson and I. N. Bernstein [15]. These authors establish an equivalence with a geometric problem solved for what corresponds to the integral case by D. A. Kazhdan and G. Lusztig [11] using the hard Lefschetz theorem developed by P. Deligne. This method does not at present resolve the case when B_λ (*see* text) cannot be conjugated into B. Again there has still to be a geometric interpretation of the Jantzen filtration in order to obtain the more refined and important (*cf.* [19], Sect. 4) information concerning the multiplicities in each step of the Jantzen filtration (*cf.* 4.9). More recently the possibility of this more refined data was conjectured by S. Gelfand and R. MacPherson [18] who refer to it as the generalized Kazhdan-Lusztig conjecture.

Recently, Kazhdan and Lusztig ([10], Sect. 1) have associated to each pair w, $w' \in W$ a polynomial $P_{w, w'}(q)$ in an indeterminate q. This is determined by an algorithm involving only the pair (W, S) viewed as a Coxeter group. Let w_0 denote the unique maximal element of W under the Bruhat order. They conjecture ([10], Conj. 1.5) that $L(w'\lambda)$ occurs precisely $P_{w_0 w, w_0 w'}(1)$ times in $M(w\lambda)$. This was motivated by the role of the Hecke algebra in "correcting" the failure of Poincaré duality on Schubert cells [11]. Again during the preparation of this work, Vogan remarked that their conjecture is equivalent to the formula:

$$P_{w, w'}(q) = \sum_{k=0}^{\infty} q^{(l(w')-l(w)-k)/2} \dim \operatorname{Ext}^k(M(w\lambda), L(w'\lambda)).$$

[Here Ext refers to the $\mathcal{O}$ category of BGG and $l(.)$ denotes the reduced length of $w \in W$.] Vogan further showed ([13], Sect. 3) that this was equivalent to a conjectured semisimplicity of a module $U_\alpha L(w\lambda)$ defined as follows. To each simple root α there is defined an exact functor θ_α on $\mathcal{O}$. Furthermore one has a complex $0 \to L(w\lambda) \to \theta_\alpha L(w\lambda) \to L(w\lambda) \to 0$ and $U_\alpha L(w\lambda)$ is defined to be its cohomology. (*See* 3.3, 3.11 for further details.)

The present work recalls an even earlier conjecture of Jantzen and shows that it implies the semisimplicity of $U_\alpha L(w\lambda)$. Actually a more precise result is obtained. First recall that in the work of Jantzen ([9], Chapt. 5) a certain contravariant form is used to define a filtration on Verma modules. With respect to the embedding of $M(w'\lambda)$ in $M(w\lambda)$ for $w' \leqq w$ and up to a shift determined by $l(w)-l(w')$, Jantzen's conjecture ([9], 5.18) asserts that this filtration ought to be hereditary. Now $\theta_\alpha M(w\lambda)$, which is an extension of $M(w\lambda)$ by $M(ws_\alpha \lambda)$, also admits a filtration *via* Jantzen's construction. This leads to two identities relating the multiplicities in $gr\, M(w\lambda)$ and in $gr\, M(ws_\alpha \lambda)$. The first [4.3(v)], which is independent of the conjecture, can be viewed as an analogue of a corresponding identity relating $\dim \operatorname{Ext}^*(M(w\lambda), L(w'\lambda))$ to $\dim \operatorname{Ext}^*(M(ws_\alpha\lambda), L(w'\lambda))$ given $w' s_\alpha < w'$ derived in [7], 2.2, and shown there to imply the BGG resolution for $L(w_0\lambda)$. Now assuming this Jantzen conjecture, a second deeper relation [4.8(iii)] (which has also a $\dim \operatorname{Ext}^*$ analogue) is obtained and taken together with the first determines the multiplicities in each filtration step. These multiplicities which can also be specified by the polynomials $P_{w, w'}(q)$ are found to be consistent with the Jantzen sum formula ([9], 5.3) obtained from the determinant of the contravariant form (4.10). At the same time the conjecture is shown to imply the semisimplicity of $gr\, M(w\lambda)$. It is also noteworthy that we are able to recognize a Hecke algebra in Jantzen's work (1.10.6 and 3.7).

Finally the Vogan method is used to partially compute $\dim \operatorname{Ext}^k(M(w\lambda), M(w'\lambda))$. One of the expected relations is obtained precisely (5.2.1), the second deeper relation, up to an inequality (5.2.3).

We should like to thank D. A. Kazhdan and G. Lusztig for advance knowledge of their conjecture. One of us (A.J.) benefited from the hospitality of the Sonderforchungsbereich, Bonn and many stimulating discussions with J.-C. Jantzen. The results of this paper were presented at a meeting on non-commutative harmonic analysis held in Marseille-Luminy during 16-20 June 1980. We should also like to thank the referee whose careful reading of the manuscript eliminated many ambiguities (due to one of us).

1. The BGG category over a commutative ring

1.1. SUMMARY. — In the light of the work of Jantzen ([9], Chapt. 5) the indeterminate present in the Kazhdan-Lusztig conjecture motivates the replacement of the base field by a commutative ring A. It is therefore natural to try to carry over the notion of the $\mathcal{O}$ category of BGG (*see* [1], [2], [5]) to this situation. This is not entirely straightforward and we felt the results may be of independent interest. Thus this rather long first section develops in a little more detail than we actually need most of the natural generalizations. The main results include: a comparison theory of Ext in various categories (1.5, 1.8.9), its relationship with $\mathfrak{n}^+$ cohomology (1.5.8), a change of ring formula (1.6), primary decomposition when A is a local ring (1.8.4), a comparison of Ext with Ext in specialization when A is a discrete valuation ring (1.9), and the definition of a symbol (1.10.5) for modules with a *p*-filtration.

1.2. THE CHEVALLEY BASIS.

1.2.1. Define $\mathfrak{g}$, $\mathfrak{h}$, R, B as in the introduction. One can choose an involutory antiautomorphism σ of $\mathfrak{g}$ satisfying $\sigma(H)=H$ for all $H\in\mathfrak{h}$. For each $\alpha\in R$ pick a basis vector X_α in the corresponding root space such that $\sigma(X_\alpha)=X_{-\alpha}$ and set $H_\alpha=[X_\alpha, X_{-\alpha}]$. Let $\mathfrak{g}_{\mathbb{Z}}$ be the additive subgroup of $\mathfrak{g}$ with basis $(X_\alpha)_{\alpha\in R}$, $(H_\alpha)_{\alpha\in B}$. Then $\mathfrak{g}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}\xrightarrow{\sim}\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{Z}}$ is a Lie algebra. Set $R^+=\mathbb{N}B\cap R$ and

$$\mathfrak{h}_{\mathbb{Z}}=\sum_{\alpha\in B}\mathbb{Z}H_\alpha,\qquad \mathfrak{n}_{\mathbb{Z}}^{\pm}=\sum_{\alpha\in R^+}\mathbb{Z}X_{\pm\alpha},\qquad \mathfrak{b}_{\mathbb{Z}}=\mathfrak{h}_{\mathbb{Z}}\oplus\mathfrak{h}_{\mathbb{Z}}^{+}$$

which are Lie subalgebras of $\mathfrak{g}_{\mathbb{Z}}$.

1.2.2. If $A\neq 0$ is any commutative ring and $\mathfrak{a}_{\mathbb{Z}}$ any Lie algebra, we define $\mathfrak{a}_A=A\otimes_{\mathbb{Z}}\mathfrak{a}_{\mathbb{Z}}$. We shall always assume that A is a $\mathbb{Q}$ algebra. Define $U(\mathfrak{a}_A)$ to be the enveloping algebra of $\mathfrak{a}_A$ and let $Z(\mathfrak{a}_A)$ denote its centre. Let $S(\mathfrak{a}_A)$ denote the symmetric algebra over $\mathfrak{a}_A$.

1.2.3. Let ρ be the half sum of the positive roots. For each $\lambda\in\mathfrak{h}_A^*$, let A_λ be the $U(\mathfrak{h}_A)$ module which is A as an A module and in which any $H\in\mathfrak{h}$ acts through multiplication by (λ, H). Extend A_λ to a $U(\mathfrak{b}_A)$ module by letting $X\in\mathfrak{n}_A^+$ act by zero. Define the $U(\mathfrak{g}_A)$ module $M(\lambda):=U(\mathfrak{g}_A)\otimes_{U(\mathfrak{b}_A)}A_{\lambda-\rho}$. It is a free rank one $U(\mathfrak{n}_A^-)$ module with canonical generator which we denote by $v_{\lambda-\rho}$ (or simply, v). It is called a Verma module over $U(\mathfrak{g}_A)$.

1.2.4. Set $Q(R):=\mathbb{Z}R$. Set $\alpha^\vee=2\alpha/(\alpha,\alpha)$ and define

$$P(R)=\{\lambda\in\mathfrak{h}_A^*:(\lambda,\alpha^\vee)\in\mathbb{Z}\}.$$

Define an ordering on P (R) through $\mu\leqq\nu$ given $\nu-\mu\in\mathbb{N}B$.

1.2.5. If M is a $U(\mathfrak{h}_A)$ module and $\mu\in\mathfrak{h}_A^*$, we define

$$M_\mu:=\{m\in M\mid Hm=(\mu,H)\,m,\forall\ H\in\mathfrak{h}_A\}$$

to be its μ weight submodule. In particular μ is called a weight of M if $M_\mu\neq 0$ and we let $\Omega(M)$ denote the set of all weights of M. For example, $\Omega(M(\lambda))=\lambda-\rho-\mathbb{N}B$.

Define:

$$\mathrm{M}(\lambda)^{-} := \oplus_{\mu<\lambda} \mathrm{M}(\lambda)_{\mu-\rho}.$$

1.3. THE HARISH-CHANDRA HOMOMORPHISM.

1.3.1. One has $\mathfrak{g}_A = \mathfrak{h}_A^{+} \oplus \mathfrak{h}_A \oplus \mathfrak{n}_A^{-}$ and all these algebras are free over A. Then by PBW the A homomorphism $u \otimes h \otimes v \mapsto uhv$ of $U(\mathfrak{n}_A^{+}) \otimes U(\mathfrak{h}_A) \otimes U(\mathfrak{n}_A^{-})$ into $U(\mathfrak{g}_A)$ is bijective and we have a direct sum decomposition $U(\mathfrak{g}_A) = U(\mathfrak{h}_A) \oplus (\mathfrak{n}_A^{-} U(\mathfrak{g}_A) + U(\mathfrak{g}_A)\mathfrak{n}_A^{+})$. Let P denote the projection of $U(\mathfrak{g}_A)$ onto $U(\mathfrak{h}_A)$ defined by this decomposition. Define an automorphism $\tau : U(\mathfrak{h}_A) \to U(\mathfrak{h}_A)$ of A algebras through $\tau(X) = X - \rho(X)$ for $x \in \mathfrak{h}_A$ and set $P' = \tau P$.

1.3.2. Consider $U(\mathfrak{g}_A)$ as an $\mathfrak{h}_A$ module through the adjoint action. Since $\mathfrak{h}_A$ is commutative $U(\mathfrak{h}_A) \cong S(\mathfrak{h}_A)$. By taking contragradient action, W acts on $\mathfrak{h}_{\mathbb{Z}}$ and hence on $\mathfrak{h}_A$, $\mathfrak{h}_A^*$, $S(\mathfrak{h}_A)$.

LEMMA. — (i) $P(U(\mathfrak{g}_A)_\mu) = 0$ *if* $\mu \neq 0$.

(ii) *The restriction of* P to $U(\mathfrak{g}_A)_0$ *is an* A-*algebra homomorphism.*

(iii) *The restriction of* P′ *to* $Z(\mathfrak{g}_A)$ *is injective with image* $S(\mathfrak{h}_A)^W$.

(i), (ii) follow as in case when A is a field ([6], 7.4.2, 7.4.6). For (iii) we apply $A \otimes_{\mathbb{Q}}$ to the case $A = \mathbb{Q}$, noting the isomorphisms $Z(\mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} A) \xrightarrow{\sim} Z(\mathfrak{g}_A)$ and

$$S(\mathfrak{h}_{\mathbb{Q}})^W \otimes_{\mathbb{Q}} A \xrightarrow{\sim} S(\mathfrak{h}_A)^W.$$

1.3.3. Extend σ to an antiautomorphism $a \mapsto \sigma(a)$ of $U(\mathfrak{g}_A)$. Let M be a $U(\mathfrak{g}_A)$ module. A symmetric bilinear form $m \times n \mapsto \mathscr{F}(m, n)$ on M with values in a commutative ring A′ is said to be contravariant if $\mathscr{F}(am, n) = \mathscr{F}(m, \sigma(a)n)$ for all $m, n \in M$, $a \in U(\mathfrak{g}_A)$.

1.3.4. Since σ interchanges $\mathfrak{n}_A^{+}$ and $\mathfrak{n}_A^{-}$ and operates by the identity on $U(\mathfrak{h}_A)$, it follows that $P(\sigma(z)) = P(z)$ for all $z \in U(\mathfrak{g}_A)$. Then $\mathscr{F}(a, b) := P(\sigma(a)b)$ is a symmetric bilinear contravariant form on $U(\mathfrak{g}_A)$ with values in $S(\mathfrak{h}_A)$.

1.3.5. If $\lambda \in \mathfrak{h}_A^*$ then λ defines an epimorphism $e_\lambda : S(\mathfrak{h}_A) \to A$ through $e_\lambda(H) = (\lambda, H)$ for all $H \in \mathfrak{h}_A$. Define an A module homomorphism $\chi'_\lambda : U(\mathfrak{g}_A) \to A$ by $\chi'_\lambda = e_\lambda P'$. Set $\chi_\lambda := \chi'_\lambda |_{Z(\mathfrak{g}_A)}$, which by 1.3.2 (ii) is an algebra homomorphism. As in ([9], 1.5) we obtain:

LEMMA. — *For all* $a \in U(\mathfrak{g}_A)$, $z \in Z(\mathfrak{g}_A)$, $m \in M(\lambda)$:

(i) $av - \chi'_\lambda(a)v \in M(\lambda)^{-}$.

(ii) $zm = \chi_\lambda(z)m$.

1.3.6. COROLLARY. — *Set* $\mathscr{F}'_\lambda(a, b) = \chi'_\lambda(\sigma(a)b)$. *Then* $\mathscr{F}'_\lambda$ *is a symmetric bilinear contravariant form on* $U(\mathfrak{g}_A)$ *with values in* A. *It defines by passage to the quotient a contravariant form* $\mathscr{F}_\lambda$ *on* $M(\lambda)$.

The first part follows from 1.3.4. Then if $a \in \operatorname{Ann} v$ we have by 1.3.5 (i) that $\chi'_\lambda(U(\mathfrak{g}_A)a) = 0$ and so $\mathscr{F}'_\lambda(U(\mathfrak{g}_A), a) = 0$ as required.

1.3.7. We call $\mathscr{F}_\lambda$ the canonical contravariant form defined on $M(\lambda)$. It is determined uniquely as a contravariant form by the property $\mathscr{F}_\lambda(v_{\lambda-\rho}, v_{\lambda-\rho}) = 1$.

1.4. CATEGORIES.

1.4.1. If $C \subset \mathfrak{h}_A^*$ is of the form $\lambda + P(R)$ for some $\lambda \in \mathfrak{h}_A^*$, define K_C'' to be the category of $U(\mathfrak{g}_A)$ modules M such that:

$$M = \sum_{\mu \in C - \rho} M_\mu.$$

Let K_C' be the full subcategory of all $M \in \mathrm{Ob}\, K_C''$ such that for each $m \in M$, $U(\mathfrak{n}_A^+)\, m$ is finitely generated as an A module. Let K_C be the full subcategory of K_C' consisting of modules which are finitely generated over $U(\mathfrak{g}_A)$. For example, if $C = \lambda + P(R)$, then $M(\lambda) \in \mathrm{Ob}\, K_C$.

1.4.2. Let A′ be any commutative ring and $\{J_i\}_{i \in I}$ a family of ideals of A′ such that $J_i + J_j = A'$, whenever $i \neq j$. Let K be the category of A′ modules $M = \sum_{i \in I} M_i$, where $M_i := \{m \in M \mid \forall x \in J_i, \exists n \in \mathbb{N}^+ \text{ s.t. } x^n m = 0\}$. Observe that each M_i is a A′ submodule.

LEMMA. — *$M \in \mathrm{Ob}\, K$ if and only if for each $m \in M$ there exists a finite set $F \subset I$ such that for all $x_i \in J_i : i \in F$, one has $n_i \in \mathbb{N}^+$ satisfying:*

$$\prod_{i \in F} x_i^{n_i} m = 0.$$

Necessity is clear. Sufficiency is by induction on card F.

1.4.3. Retain the hypotheses of 1.4.2.

LEMMA. — (i) *If $M \in \mathrm{Ob}\, K$, then $M \simeq \oplus_{i \in I} M_i$.*
(ii) *K is closed under subquotients.*
(iii) *$M \mapsto M_i$ is an exact functor on K.*

(i) Suppose $F \subset I$ is finite and $0 = \sum_{i \in F} m_i : m_i \in M_i$. We must show that $m_i = 0$. The proof is by induction on card F.

(ii) is an immediate consequence of 1.4.2 (iii) follows from (i), (ii).

Remark. — 1.4.2 and 1.4.3 still hold (and are easier to prove) when we define $M_i = \{m \in M \mid J_i m = 0\}$.

1.4.4. In 1.4.2 and 1.4.3 take $A' = U(\mathfrak{h}_A)$ and set $J_\mu = \mathrm{Ker}\, e_\mu$.

COROLLARY. — *If $M \in \mathrm{Ob}\, K_C''$, then:*
(i) $M \simeq \oplus_{\mu \in C - \rho} M_\mu$.
(ii) *$M \mapsto M_\mu$ is an exact functor on K_C''.*
(iii) *K_C'' is closed under subquotients.*
(iv) *K_C'' is closed under arbitrary direct sums.*

It is enough to check that $J_{\mu_1} + J_{\mu_2} = U(\mathfrak{h}_A)$ when μ_1, μ_2 are distinct elements of $C - \rho$. By By hypothesis $\mu_1 - \mu_2 \in P(R) - \{0\}$ and so we can find $H \in \mathfrak{h}_A$ such that $(\mu_1(H) - H) - (\mu_2(H) - H) = (\mu_1 - \mu_2)(H) \in \mathbb{Z} - \{0\}$ (that is an invertible element of A).

1.4.5. For each $\mu \in C$, we set $Q(\mu) := U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} A_{\mu-\rho} \in \operatorname{Ob} K_C''$. If M is any $U(\mathfrak{g}_A)$ module, the map $f \mapsto f(1 \otimes 1)$ of $\operatorname{Hom}(Q(\mu), M)$ into $M_{\mu-\rho}$ is a natural isomorphism of A modules. Then by 1.4.4 (ii), $Q(\mu)$ is projective in K_C'' and by 1.4.4 (iv) each $M \in \operatorname{Ob} K_C''$ is a quotient of a projective object. Again by 1.4.4 (iii), (iv) it follows that K_C'' is closed under direct limits and by 1.4.4 (ii) it follows that $\operatorname{Hom}(Q(\mu), -)$ commutes with direct limits.

1.4.6. Lemma. — *If* $M \in \operatorname{Ob} K_C''$, *then* $M \in \operatorname{Ob} K_C'$ *if and only if for all* $m \in M$, *there exists* $s \in \mathbb{N}$ *such that* $(\mathfrak{n}_A^+)^s m = 0$.

Sufficiency follows from the fact that $\mathfrak{n}_A^+$ and hence $(\mathfrak{n}_A^+)^s$ is finitely generated over A. Conversely take $m \in M$. By hypothesis $N := U(\mathfrak{n}_A^+) m$ is a finitely generated A module. Then there exists a finite subset $F \subset C$ such that $N \subset \sum_{\mu \in F-\rho} M_\mu$. Since $X_\alpha M_\mu \subset M_{\mu+\alpha}$ for all $\alpha \in R^+$, the assertion follows easily.

1.4.7. Corollary. — K_C' *is closed under subquotients and arbitrary direct sums.*

1.4.8. Lemma. — *If* $M \in \operatorname{Ob} K_C$, *then* M *is a finitely generated* $U(\mathfrak{n}_A^-)$ *module.*

This is immediate from $U(\mathfrak{g}_A) \rightleftarrows U(\mathfrak{n}_A^-) \otimes U(\mathfrak{h}_A) \otimes U(\mathfrak{n}_A^+)$.

1.4.9. Lemma. — *If* $M \in \operatorname{Ob} K_C$, *then* M *has a finite filtration with quotients isomorphic to quotients of the* $M(\mu) : \mu \in C$.

By 1.4.8 we can assume M to be of the form:

$$\sum_{i=1}^{s} U(\mathfrak{n}_A^-) v_{s-i}, \qquad v_i \in M_{\mu_i},$$

where we choose the labelling to satisfy $i \leqq j \Rightarrow \mu_i \ngeq \mu_j$. Set

$$F^{s-t} M = \sum_{j=1}^{t} U(\mathfrak{n}_A^-) v_{s-j}.$$

For each $\alpha \in R^+$, one has $X_\alpha v_j \in M_{\mu_j+\alpha}$. If $i \leqq j$, then $(U(\mathfrak{n}_A^-) v_i)_{\mu_j+\alpha} = 0$, for otherwise we should have $\mu_i > \mu_j$. It thus follows from 1.4.4 (i) that $X_\alpha v_j \in F^{j+1} M$. That is each $F^j M$ is a $U(\mathfrak{g}_A)$ module and if we let $\bar{v}_j$ denote the image of v_j in $F^j M / F^{j+1} M$, then $X_\alpha \bar{v}_j = 0$ for all $\alpha \in R^+$. Hence $F^j M / F^{j+1} M$ is isomorphic to a quotient of $M(\mu_j + \rho)$ and $\{F^j M\}_{j=0}^{s}$ is the required filtration.

1.4.10. Corollary. — *If* $M \in \operatorname{Ob} K_C$, *then each* $M_\mu : \mu \in \Omega(M)$ *is finitely generated as an* A *module.*

By 1.4.9 the assertion is reduced to the case $M = M(\mu)$ [or equivalently for $U(\mathfrak{n}_A^-)$] for which it holds easily.

1.5. Comparison of Ext in K_C'' and K_C'. — Throughout this section we fix a $P(R)$ coset $C \subset \mathfrak{h}_A^*$. By Ext we mean $\operatorname{Ext}_{K_C''}$.

1.5.1. Define categories $K_C''(\mathfrak{b})$, etc., of $U(\mathfrak{b}_A)$ modules by replacing $\mathfrak{g}_A$ by $\mathfrak{b}_A$. It is immediate that these satisfy assertions analogous to 1.4.4, 1.4.6, 1.4.7, 1.4.8. Again if $\mu \in C$, then $Q_{\mathfrak{b}}(\mu) := U(\mathfrak{b}_A) \otimes_{U(\mathfrak{h}_A)} A_{\mu-\rho}$ is projective in $\operatorname{Ob} K_C''(\mathfrak{b})$ and satisfies the assertions of 1.4.5.

1.5.2. LEMMA. — *Suppose* $N \in \mathrm{Ob}\, K_C''(\mathfrak{b})$ *and as a* $U(\mathfrak{h}_A)$ *module is a finite direct sum of some* $A_{\mu_i - \rho} : \mu_i \in C$. *Then* N *has a projective resolution* (X^*, ε_1) *in* $K_C''(\mathfrak{b})$ *such that each* X^j *is a finite direct sum of the* $Q_{\mathfrak{b}}(\nu) : \nu \in C$ *and satisfies* $\Omega(X^j) \subset \Omega(N) + \mathbb{N}B$.

For each $j \in \mathbb{N}$, the wedge product $\Lambda^j \mathfrak{n}_A^+$ considered as an $\mathfrak{h}_A$ module for adjoint action is a finite direct sum of the $A_\nu : \nu \in \mathbb{N} B$. Consider the standard resolution $(Y^*, \varepsilon) : Y^j = U(\mathfrak{n}_A^+) \otimes_A (\Lambda^j \mathfrak{n}_A^+)$ of A as a $U(\mathfrak{n}_A^+)$ module. Endow Y^j with a $\mathfrak{b}_A$ module structure by identification with $U(\mathfrak{b}_A) \otimes_{U(\mathfrak{h}_A)} \Lambda^j \mathfrak{n}_A^+$. Apply the functor $R \to R \otimes_A N$ on $U(\mathfrak{b}_A)$ modules which is exact because N is a free A module to get a resolution (X^*, ε_1) of N. For any $U(\mathfrak{h}_A)$ module M we have a bijection of $U(\mathfrak{b}_A)$ modules:

$$U(\mathfrak{b}_A) \otimes_{U(\mathfrak{h}_A)} (M \otimes_A N|_{U(\mathfrak{h}_A)}) \to (U(\mathfrak{b}_A) \otimes_{U(\mathfrak{h}_A)} M) \otimes_A N,$$

defined by the universality of the tensor product.

It follows that X^j is isomorphic to the direct sum of the $Q_{\mathfrak{b}}(\mu+\nu) : \mu \in \Omega(N), \nu \in \Omega(\Lambda^j \mathfrak{n}_A^+)$. Hence the assertion of the Lemma.

1.5.3. Fix $\mu \in C$. For each $s \in \mathbb{N}^+$ let $Q^s(\mu)$ be the quotient of $Q(\mu)$ by the submodule L^s with generator $(\mathfrak{n}_A^+)^s (1 \otimes 1)$.

Let M be a $U(\mathfrak{g}_A)$ module. From the surjection $Q(\mu) \to Q^s(\mu)$ we obtain a monomorphism:

$$\xi : \varinjlim_s \mathrm{Hom}_{U(\mathfrak{g}_A)}(Q^s(\mu), M) \to \mathrm{Hom}_{U(\mathfrak{g}_A)}(Q(\mu), M).$$

LEMMA. — *Suppose* $M \in \mathrm{Ob}\, K_C'$. *Then* ξ *is surjective.*

Given $\varphi \in \mathrm{Hom}_{U(\mathfrak{g}_A)}(Q(\mu), M)$, set $u = \varphi(1 \otimes 1)$. Since $M \in \mathrm{Ob}\, K_C'$ one has by 1.4.6, $s \in \mathbb{N}$ such that $0 = (\mathfrak{n}_A^+)^s u = \varphi(\mathfrak{n}_A^s (1 \otimes 1))$ and so φ is in the image of ξ.

1.5.4. LEMMA. — *Suppose* $M \in \mathrm{Ob}\, K_C$. *Then for all s sufficiently large, one has* $\mathrm{Ext}^i(Q^s(\mu), M) = 0 : i > 0$.

Since $Q(\mu)$ is projective in K_C'' we have a surjection

$$\mathrm{Ext}^{i-1}(L^s, M) \to \mathrm{Ext}^i(Q^s(\mu), M).$$

Let N^s be the $U(\mathfrak{b}_A)$ submodule of $Q_b(\mu)$ with generator $(\mathfrak{n}_A^+)^s (1 \otimes 1)$. Let (X^*, ε_1) be the projective resolution of N^s in $K_C''(\mathfrak{b})$ defined through the conclusion of 1.5.2. Since $U(\mathfrak{g}_A)$ is a free right $U(\mathfrak{b}_A)$ module we have $L^s \cong U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} N^s$ and $(U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} X^*, 1 \otimes \varepsilon_1)$ is a projective resolution of L^s in K_C''. Since $M \in \mathrm{Ob}\, K_C$ applying 1.4.9 gives $(\Omega(N^s) + \mathbb{N}B) \cap \Omega(M) = \emptyset$, for all s sufficiently large. For such a choice of s we have by 1.5.2 that:

$$\mathrm{Hom}_{U(\mathfrak{g}_A)}(U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} X^i, M) \cong \mathrm{Hom}_{U(\mathfrak{b}_A)}(X^i, M) = 0, \quad \text{for all } i \geqq 0.$$

Hence the assertion of the Lemma.

1.5.5. COROLLARY. — *Suppose* $M \in \mathrm{Ob}\, K_C'$. *Then:*

$$\varinjlim_s \mathrm{Ext}^i(Q^s(\mu), M) = 0, \qquad i > 0.$$

As in 1.5.4 it follows from 1.5.2 that $Q^s(\mu)$ has a projective resolution (Y^*, ε) in K''_C such that each Y^j is a finite direct sum of the $Q(\nu) : \nu \in C$.

Choose $\xi \in \mathrm{Ext}^i(Q^s(\mu), M) : i>0$. Since $\mathrm{Hom}(Q(\nu), -)$ commutes with direct limits we can find a finitely generated submodule M_1 of M and $\xi_1 \in \mathrm{Ext}^i(Q^s(\mu), M_1)$ such that ξ is the image of ξ_1 under the map $\mathrm{Ext}^i(Q^s(\mu), M_1) \to \mathrm{Ext}^i(Q^s(\mu), M)$. Now $M_1 \in \mathrm{Ob}\, K_C$, so for all t sufficiently large we have by 1.5.3 that $\mathrm{Ext}^i(Q^t(\mu), M_1)=0$. From the commuting square:

$$\begin{array}{ccc} \mathrm{Ext}^i(Q^s(\mu), M) & \to & \mathrm{Ext}^i(Q^t(\mu), M) \\ \uparrow & & \uparrow \\ \mathrm{Ext}^i(Q^s(\mu), M_1) & \to & \mathrm{Ext}^i(Q^t(\mu), M_1) \end{array}$$

it follows that the image of ξ in $\mathrm{Ext}^i(Q^t(\mu), M)$ is zero, as required.

1.5.5. Let M be any $U(\mathfrak{g}_A)$ module. We denote by τM the unique largest submodule of M such that $\tau M \in \mathrm{Ob}\, K''_C$. Given $N \in \mathrm{Ob}\, K''_C$, we denote by ηN the unique largest submodule of N such that $\eta N \in \mathrm{Ob}\, K'_C$. Fix an injective A module J and for each $M \in \mathrm{Ob}\, K''_C$ consider $\mathrm{Hom}_A(M, J)$ as a $U(\mathfrak{g}_A)$ module through the action $(\sigma(a).\varphi)\ (m)=\varphi(am), \forall\ a \in U(\mathfrak{g}_A),\ m \in M$. Given $\mu \in C$, set $I(\mu)=\tau(\mathrm{Hom}_A(Q(\mu), J))$. Through the natural isomorphism:

$$\mathrm{Hom}_{U(\mathfrak{g}_A)}(M, \mathrm{Hom}_A(Q(\mu), J)) \cong \mathrm{Hom}_{U(\mathfrak{g}_A)}(Q(\mu), \mathrm{Hom}_A(M, J)),$$

we obtain for each $M \in \mathrm{Ob}\, K''_C$ a natural isomorphism:

$$\mathrm{Hom}_{U(\mathfrak{g}_A)}(M, I(\mu)) \cong \mathrm{Hom}_A(M_{\mu-\rho}, J).$$

Hence $I(\mu)$ is injective in K''_C and each $M \in \mathrm{Ob}\, K''_C$ has an injective hull consisting of a possibly infinite direct sum of the $I(\mu) : \mu-\rho \in \Omega(M)$.

1.5.7. Proposition. — *Suppose* M, $N \in \mathrm{Ob}\, K'_C$. *Let* $0 \to N \to I_0 \to I_1 \to \ldots$, *be an injective resolution of* N in K''_C.

Then:

(i) $0 \to N \to \eta I_0 \to \eta I_1 \to \ldots$, *is an injective resolution of* N *in* K'_C.

(ii) $\mathrm{Ext}^*_{K''_C}(M, N) \cong \mathrm{Ext}^*_{K'_C}(M, N)$.

It is standard that the ηI_i are injective in K'_C. By definition of Ext and since $\eta M = M$ we have:

$$(\star) \qquad \mathrm{Ext}^i_{K''_C}(M, N) \cong H^i(\mathrm{Hom}(M, \eta I_*)).$$

Now for each $\mu \in C$ and all $i>0$, we have:

$$\begin{aligned}
H^i((\eta\, I_*)_{\mu-\rho}) &\cong H^i(\mathrm{Hom}(Q(\mu), \eta\, I_*)), \quad \text{by } 1.4.5,\\
&\cong H^i(\varinjlim_s \mathrm{Hom}(Q^s(\mu), \eta\, I_*)), \quad \text{by } 1.5.3,\\
&\cong \varinjlim_s H^i(\mathrm{Hom}(Q^s(\mu), \eta\, I_*)), \quad \text{by exactness of direct limit},\\
&\cong \varinjlim_s \mathrm{Ext}^i(Q^s(\mu), N), \quad \text{applying } (\star) \text{ to } M=Q^s(\mu),\\
&=0, \quad \text{by } 1.5.5.
\end{aligned}$$

Hence (i). Combined with $(\star)$ we obtain (ii).

1.5.8. Given $N \in \mathrm{Ob}\, K_C''$, then $H^*(\mathfrak{n}_A^+, N)$ is by the standard complex a semisimple $U(\mathfrak{h}_A)$ module with weights in $C-\rho$. Furthermore:

LEMMA. — *For all $\lambda \in C$, one has:*

$$\mathrm{Ext}^*(M(\lambda), N) \cong H^*(\mathfrak{n}_A^+, N)_{\lambda-\rho}.$$

Set $Y^j = (U(\mathfrak{b}_A) \otimes_{U(\mathfrak{h}_A)} \Lambda^j \mathfrak{n}_A^+) \otimes_A A_{\lambda-\rho}$. Then (Y^*, ε) is a projective resolution of $A_{\lambda-\rho}$ [considered as a $U(\mathfrak{b}_A)$ module by letting $X \in \mathfrak{n}_A^+$ act by scalars] in $K_C''(\mathfrak{b})$. It follows that $(U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} Y^*, 1 \otimes \varepsilon)$ is a projective resolution of $M(\lambda)$ in K_C'' and since:

$$\begin{aligned}
\mathrm{Hom}_{U(\mathfrak{g}_A)}(U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} Y^*, N) &\cong \mathrm{Hom}_{U(\mathfrak{h}_A)}(\Lambda^* \mathfrak{n}_A^+ \otimes_A A_{\lambda-\rho}, N),\\
&\cong H^*(\mathfrak{n}_A^+, N)_{\lambda-\rho},
\end{aligned}$$

the required assertion follows easily.

1.6. Change of rings

1.6.1. Let $\varphi : A \to \bar{A}$ be a homomorphism of (non-zero) $\mathbb{Q}$ algebras. Fix a $P(R)$ coset C in $\mathfrak{h}_A^*$ and let $\bar{C}$ be its image in $\mathfrak{h}_{\bar{A}}^*$. Since $\bar{A} \otimes_A U(\mathfrak{g}_A) \cong U(\mathfrak{g}_{\bar{A}})$, the functor

$$M \mapsto TM. = U(\mathfrak{g}_{\bar{A}}) \otimes_{U(\mathfrak{g}_A)} M$$

sends K_C'' to $K_{\bar{C}}''$ and the forgetful functor $M \mapsto T'M := M|_{U(\mathfrak{g}_A)}$, sends $K_{\bar{C}}''$ to K_C''. The map

$$t_0 : \mathrm{Hom}_{U(\mathfrak{g}_{\bar{A}})}(TM, N) \to \mathrm{Hom}_{U(\mathfrak{g}_A)}(M, T'N)$$

defined by $t_0(f) = (m \mapsto f(1 \otimes m))$ is an isomorphism of $\bar{A}$ modules. Since T' is right exact, it follows that if P is projective in K_C'', then TP is projective in $K_{\bar{C}}''$.

1.6.2. Let (X^*, ε) be a projective resolution of $M \in \mathrm{Ob}\, K_C''$. By 1.6.1, TX^* is a projective complex over TM. If $(\tilde{X}^*, \tilde{\varepsilon})$ is a resolution in $K_{\bar{C}}''$ of TM then [4], Prop. 11, p. 76, there exists a map $TX^* \overset{F^*}{\to} \tilde{X}^*$ of complexes such that:

$$\begin{tikzcd} TX^* \arrow[rr, "F^*"] \arrow[dr, "T\varepsilon"'] & & \tilde{X}^* \arrow[dl, "\tilde{\varepsilon}"] \\ & TM & \end{tikzcd}$$

commutes and furthermore any two such F^* are homotopic. We take (X^*, ε) to be a projective resolution of TM. Then for each $N \in \mathrm{Ob}\, K''_C$ the map $\mathrm{Hom}(F^*, N)$ of complexes $\mathrm{Hom}(\tilde{X}^*, N) \to \mathrm{Hom}(TX^*, N)$ gives on taking cohomology a map

$$t_i : \ \mathrm{Ext}^i_{K''_C}(TM, N) \to H^i(\mathrm{Hom}(TX^*, N)) \underset{t_0}{\cong} H^i(\mathrm{Hom}\,(X^*, T'N)) = \mathrm{Ext}^i_{K''_C}(M, T'N).$$

One checks that the map t_i is independent of the projective resolution taken and is natural.

LEMMA. — *If* M *or* $\overline{A}$ *is a flat* A *module, then* t_i *is an isomorphism of* $\overline{A}$ *modules.*

It is enough to show that either hypothesis implies that $(TX^*, T\varepsilon)$ is acyclic. Let Q be projective in K''_C. We show that Q is A projective. Since both $\mathfrak{g}_A$ and $\mathfrak{h}_A$ are free A modules it follows from PBW that $Q(\mu) : \mu \in C$ is a free A module. By 1.4.5 Q is an image (and hence a direct summand) of a suitable direct sum of the $Q(\mu) : \mu \in C$. Hence Q is A projective. It follows that (X^*, ε) is also a projective resolution of M in the category of A modules, so the i-th homology of TX^* is $\mathrm{Tor}^A_i(M, \overline{A})$. The latter vanishes if $i > 0$ and if either M or $\overline{A}$ is A flat.

1.7. SIMPLE MODULES. — Fix $\lambda \in \mathfrak{h}^*_A$ and set $C = \lambda + P(R)$.

1.7.1. LEMMA. — *Suppose* $\mu \in \mathfrak{h}^*_A$:

(i) *If* $N \subset M(\mu)$ *is a maximal submodule, then there exists a unique maximal ideal* $m \subset A$ *such that* $m\,M(\mu) \subset N$.

(ii) If $m \subset A$ *is a maximal ideal, there exists a unique maximal submodule* N *of* $M(\mu)$ *such that* $m\,M(\mu) \subset N$.

(i) Set $L = M(\mu)/N$. As $L \neq 0$, the image $\bar{v}$ of $v_{\mu-\rho}$ in L is non-zero, so $L_{\mu-\rho} \neq 0$. By the exactness of the functor $M \mapsto M_{\mu-\rho}$ in K''_C, it follows that $L_{\mu-\rho}$ is the image of $M_{\mu-\rho}$ and so equals $A\bar{v}$. Pick $m \supset \mathrm{Ann}_A \bar{v}$ maximal. Then $A\bar{v} \xleftarrow{\sim} A/\mathrm{Ann}_A \bar{v}$ and $m\bar{v} \xleftarrow{\sim} m/\mathrm{Ann}_A \bar{v}$, so $m\bar{v} \neq A\bar{v}$. Hence $m L \neq L$, so $m L = 0$ because L is simple. This proves existence. Uniqueness is obvious.

(ii) One has $(m\,M(\mu))_{\mu-\rho} = m v_{\mu-\rho} \subsetneq A\, v_{\mu-\rho}$.

Thus $m\,M(\mu)$ is a proper submodule. Because $M(\mu)$ is finitely generated, $m\,M(\mu)$ is contained in a maximal submodule N. By the maximality of $m v_{\mu-\rho}$ as an A submodule of $A v_{\mu-\rho}$ it follows that $N_{\mu-\rho} = m v_{\mu-\rho}$. From this property, the uniqueness of N follows easily.

1.7.2. If $\mu \in C$ and $m \subset A$ is a maximal ideal, we let $M(m, \mu)'$ denote the unique maximal submodule of $M(\mu)$ containing $m\,M(\mu)$ and set $L(m, \mu) = M(\mu)/M(m, \mu)'$. If A is a local ring, we simply write $M(\mu)'$ and $L(\mu)$.

1.7.3. LEMMA. — *In* K'_C, *every simple module* L *is isomorphic to some* $L(m, \mu)$ *with* $(m, \mu) \in \operatorname{Max} A \times C$, *and the pair* (m, μ) *is uniquely determined by* L.

Obviously $L \in \operatorname{Ob} K_C$ and so L is isomorphic by 1.4.9 to a simple quotient of some $M(\mu) : \mu \in C$. By 1.7.1 one further has $L \cong L(m, \mu)$ for some $m \in \operatorname{Max} A$. Finally note that μ is the unique maximal element of $\Omega(L)$ and $m = \operatorname{Ann}_A L(m, \mu)$.

1.7.4. If $D \subset C$ we let K'_D (resp. K_D) denote the full subcategory of K'_C (resp. K_C) consisting of all those modules whose simple subquotients are amongst the $L(m, \mu) : (m, \mu) \in \operatorname{Max} A \times D$.

1.7.5. LEMMA. — *Suppose* $D \subset C$. *Then:*

(i) K'_D *is closed under subquotients.*

(ii) *If* $M \in \operatorname{Ob} K'_C$ *and* $M = \sum_{i \in I} M_i : M_i \in \operatorname{Ob} K'_D$, *then* $M \in \operatorname{Ob} K'_D$.

(i) Follows from the definition and 1.4.7. If F is a subset of I, set $M_F = \sum_{i \in F} M_i$. Let $N_1 \subset N_2 \subset M$ be submodules of M with N_2/N_1 simple. Given $v \in N_2 \setminus N_1$, choose $F \subset I$ finite such that $v \in M_F$. Then $N_1 \cap M_F \neq N_2 \cap M_F$ and we assume F minimal with this property. Choose $i \in F$ and set $F' = F \setminus \{i\}$. Then $N_1 \cap M'_F = N_2 \cap M''_F$ and so by the Zassenhaus Lemma, N_1/N_2 is isomorphic to a simple subquotient of M_i. Hence (ii).

1.7.6. LEMMA. — *If* $0 \to M_1 \to M \to M_2 \to 0$ *is an exact sequence in* K''_C *and* M_1, $M_2 \in \operatorname{Ob} K'_D$, *for some* $D \subset C$, *then* $M \in \operatorname{Ob} K'_D$.

By 1.4.6 it is obvious that $M \in \operatorname{Ob} K'_C$. Now let $N_1 \subset N_2 \subset M$ be submodules of M with N_2/N_1 simple. Through the Zassenhaus Lemma N_2/N_1 is a subquotient of either M_1 or of M_2 and so $M \in \operatorname{Ob} K'_D$.

1.8. PRIMARY DECOMPOSITION.

1.8.1. In this subsection we assume that A is a local ring with m its unique maximal ideal and $k = A/m$ its residue field. For any A module M we denote by $m \mapsto \overline{m}$ the canonical projection (specialization) $M \to M/mM$. Identify $\mathfrak{h}^*_A/m\mathfrak{h}^*_A$ with $\mathfrak{h}^*_k$. Recall that the Weyl group W acts on $\mathfrak{h}^*_k$ and observe that $\overline{w\lambda} = w\overline{\lambda}$, for all $w \in W$, $\lambda \in \mathfrak{h}^*_A$. Given $\mu \in \mathfrak{h}^*_A$, set $R_{\overline{\mu}} = \{\alpha \in R : 2(\alpha, \overline{\mu})/(\alpha, \alpha) \in \mathbb{Z}\}$. This is itself a root system with Weyl group $W_{\overline{\mu}}$ generated by the $s_\alpha : \alpha \in R_{\overline{\mu}}$.

LEMMA [3]. — $W_{\overline{\mu}} = \{w \in W : w\overline{\mu} - \overline{\mu} \in Q(R)\}$.

1.8.2 Fix $\mu \in \mathfrak{h}^*_A$ and set $C = \mu + P(R)$. Call D a block if $\overline{D} := \{\overline{\lambda} : \lambda \in D\}$ is a W orbit. Since $\overline{\lambda} - \overline{\mu} \in P(R)$ for all $\lambda \in C$, it follows that $\overline{D}$ is also a $W_{\overline{\mu}}$ orbit.

LEMMA. — *Let* D *be a block. Then there exists* $\lambda \in \mathfrak{h}^*_A$, $\nu \in mh^*_A$ *such that:*

(i) $s_\alpha \lambda - \lambda \in \mathbb{Z}\alpha$, *for all* $\alpha \in R_{\overline{\lambda}}$.

(ii) $D = W_{\overline{\lambda}} \lambda + \nu$.

(i) Let AW denote the semidirect product $W \ltimes Q(R)$. Define an action of AW on $\mathfrak{h}^*_A$ through $(w, \xi)\, \eta = w\eta + \xi$. Pick $\lambda_1 \in D$ and let $AW(\overline{\lambda}_1)$ denote the stabilizer of $\overline{\lambda}_1$ in

AW. By 1.8.1, projection onto the first factor gives an isomorphism $AW(\bar{\lambda}_1) \simeq W_{\bar{\lambda}_1}$. Set:

$$\lambda = \frac{1}{\text{Card } AW(\bar{\lambda}_1)} \sum_{y \in AW(\bar{\lambda}_1)} y\lambda_1 \quad \text{and} \quad \nu = \lambda_1 - \lambda.$$

By construction $\bar{\lambda} = \bar{\lambda}_1$ and so $\nu \in m\mathfrak{h}_A^*$. Again for all $y \in AW(\bar{\lambda}_1)$ one has $y\lambda = \lambda$. By 1.8.1, this gives (i).

(ii) Take $\lambda_1 \in D$ and define λ as in (i). Pick $\lambda_2 \in D$. By definition of a block, $\lambda_1 - \lambda_2 \in P(R)$ and there exists $w \in W_{\bar{\lambda}}$ such that $w\bar{\lambda}_1 = \bar{\lambda}_2$. Set $\lambda_2' = w\lambda + \nu$.

Then $\bar{\lambda}_2' = w\bar{\lambda} = w\bar{\lambda}_1$. Again $\lambda_2' = w\lambda - \lambda + \lambda_1$ where we have $w\lambda - \lambda \in Q(R)$ [since it is modulo m and trivially $w\bar{\lambda} - \bar{\lambda} \in Q(R)$]. Now for fixed $w \in W_{\bar{\lambda}_1}$ there is at most one element of $\lambda_1 + P(R)$ equal to $w\bar{\lambda}$ modulo m. Hence $\lambda_2 = \lambda_2' \in W_{\bar{\lambda}}\lambda + \nu$, as required.

1.8.3. LEMMA (notation 1.3.5). — *Suppose* $\lambda, \mu \in C$ *are contained in distinct blocks. Then* $\text{Ker } \chi_\lambda + \text{Ker } \chi_\mu = Z(\mathfrak{g}_A)$.

Consider $\bar{\lambda}, \bar{\mu}$ as elements of $\mathfrak{h}_k^*$. The hypothesis implies that $\text{Ker } \chi_{\bar{\lambda}}$, $\text{Ker } \chi_{\bar{\mu}}$ are distinct maximal ideals of $Z(\mathfrak{g}_k)$. Hence $\text{Ker } \chi_\lambda + \text{Ker } \chi_\mu + mZ(\mathfrak{g}_A) = Z(\mathfrak{g}_A)$. It follows that we can choose $z \in \text{Ker } \chi_\mu$ such that $\overline{\chi_\lambda(z)} = 1$. Since A is local ring, $\chi_\lambda(z)$ is a unit in A. Hence $\chi_\lambda(\text{Ker } \chi_\mu) = A$ and so $\text{Ker } \chi_\lambda + \text{Ker } \chi_\mu = Z(\mathfrak{g}_A)$.

1.8.4. Let $D \subset C$ be a block and set $J_D = \bigcap_{\lambda \in D} \text{Ker } \chi_\lambda$. Given M a $U(\mathfrak{g}_A)$ module define M_D with respect to J_D as in 1.4.2. We call M_D the primary component of M with respect to the block D. Observe that $C = \coprod_{i \in I} D_i$, that is C is a countable disjoint union of its distinct blocks. When $D = D_i$ we simply write J_i, M_i for J_D, M_D.

PROPOSITION (primary decomposition). — *Given* $M \in \text{Ob } K_C'$ *then* $M = \oplus_{i \in I} M_i$. *All but finitely many* M_i *are zero if* $M \in \text{Ob } K_C$.

Take $A' = Z(\mathfrak{g}_A)$ in 1.4.2. By 1.8.3 the J_i satisfy $J_i + J_j = A'$ if $i \neq j$. Then by 1.4.2 and 1.4.3 it is enough to prove the second assertion, that is to show that $M \in \text{Ob } K_C$ satisfies the hypothesis of 1.4.2. This follows easily from 1.4.9 and 1.3.5 (ii).

1.8.5. COROLLARY (notation 1.7.4). — *Let* $D \subset C$ *be a block. Then:*

(i) $M \in \text{Ob } K_D' \Leftrightarrow M \in \text{Ob } K_C'$ *and* $M = M_D$.

(ii) $M \in \text{Ob } K_D \Leftrightarrow M \in \text{Ob } K_C$ *and* $J_D^s M = 0$, *for all s sufficiently large.*

For (i) it suffices to show that if $0 \neq M = M_D^s$ then it admits a simple factor $L \in \text{Ob } K_D$. It is enough to take M finitely generated and then the assertion follows from 1.4.9 and 1.7.1. Then (ii) follows from (i), 1.4.9 and 1.3.5 (ii).

1.8.6. Let $D \subset C$ be a block. By 1.8.4, 1.8.5 we have a functor $F_D : K_C' \to K_D'$ defined by primary decomposition. It is exact on K_C'. If $M \in \text{Ob } K_C$, then $F_D M \in \text{Ob } K_D$.

1.8.7. LEMMA. — *For each* $\lambda \in C$, $D \subset C$ *the module* $F_D Q^s(\lambda)$ *is independent of s for s sufficiently large.*

Since F_D is exact it is enough to show that $F_D(Q^{s+t}(\lambda)/Q^s(\lambda))=0$, for all $t\in\mathbb{N}$ and s sufficiently large. Now the $\mu_i\in C$, in the conclusion of 1.4.9 applied to $Q^{s+t}(\lambda)/Q^s(\lambda)$, clearly lie in the set $\lambda+s\,B+\mathbb{N}\,B$. If D is fixed, then $D\cap(\lambda+s\,B+\mathbb{N}\,B)=\varnothing$, for s sufficiently large. In fact it is enough to prove the corresponding statement in k and this is well-known. By 1.3.5 (ii) it establishes the required assertion.

1.8.8. Given $\lambda\in C$, $D\subset C$ we set $Q(\lambda, D)=\lim\limits_{s\to\infty} F_D Q^s(\lambda)$ (which is defined by 1.8.7). Then $Q(\lambda, D)\in \mathrm{Ob}\, K_D$.

1.8.9. Fix $\lambda\in C$, $D\subset C$ and recall once again that A is a local ring.

PROPOSITION. — (i) *If* $M\in\mathrm{Ob}\,K'_C$ *one has a natural isomorphism of* A *modules* $\mathrm{Hom}_{U(\mathfrak{g}_A)}(Q(\lambda, D), M)\cong(M_D)_\lambda$.

(ii) $Q(\lambda, D)$ *is projective in* K'_C, *and every* $M\in\mathrm{Ob}\,K'_C$ (*resp.* $\mathrm{Ob}\,K_C$) *is the image of a direct sum* (*resp. finite direct sum*) *of the* $Q(\lambda, D)$.

(iii) $\mathrm{Ext}^*_{K'_C}(M, N)\cong\mathrm{Ext}^*_{K_C}(M, N)$, *for all* $M, N\in\mathrm{Ob}\,K_C$.

$$
\begin{aligned}
\text{(i)}\quad \mathrm{Hom}(Q(\lambda, D), M) &\cong \mathrm{Hom}(\varinjlim_s F_D Q^s(\lambda), M), &&\text{by definition,}\\
&\cong \varinjlim_s \mathrm{Hom}(F_D Q^s(\lambda), M), &&\text{by exactness of direct limits,}\\
&\cong \varinjlim_s \mathrm{Hom}(Q^s(\lambda), F_D M), &&\text{by 1.8.6,}\\
&\cong \mathrm{Hom}(Q(\lambda), F_D M), &&\text{by 1.5.3,}\\
&\cong (M_D)_\lambda, &&\text{by 1.4.5.}
\end{aligned}
$$

The first part of (ii) follows from (i). The second part by 1.4.9 and 1.8.7. (iii) follows from (ii).

1.9. COMPARISON OF Ext WITH $\overline{\mathrm{Ext}}$ IN SPECIALIZATION.

1.9.1. In this subsection we assume that A is a discrete valuation ring over $\mathbb{Q}$. Let $\pi\in A$ be a generator of the maximal ideal, $k=A/\pi A$ and $\lambda\mapsto\bar\lambda$ denote specialization. Fix a P(R) coset $C\subset\mathfrak{h}_A^*$. By Ext* we shall mean $\mathrm{Ext}^*_{K'_C}$ [*cf.* 1.5.7 (ii), 1.8.8 (iii)]. By $\overline{\mathrm{Ext}}^*$ we shall mean $\mathrm{Ext}^*_{K''_{\bar C}}$ (where $\bar C$ denotes the image of C in $\mathfrak{h}_k^*$).

1.9.2. Call $\bar\lambda\in\mathfrak{h}_k^*$ dominant (resp. regular) if $(\bar\lambda, \alpha)\geqq 0$ [resp. $(\bar\lambda, \alpha)\neq 0$] for all $\alpha\in R^+_{\bar\lambda} := R_{\bar\lambda}\cap R^+$. Suppose $\xi\in\mathfrak{h}_A^*$, $l\in\mathbb{N}$, and define differential operators $\partial_\xi^{(l)}\in\mathrm{End}_A\, S(\mathfrak{h}_A)$ by identification of coefficients of t^l (t an indeterminate) in the expression (notation 1.3.5):

$$e_{y+t\xi}(x)=\sum_{l=0}^{\infty} t^l\, e_y(\partial_\xi^{(l)}(x)),\qquad y\in\mathfrak{h}_A^*,\quad x\in S(\mathfrak{h}_A).$$

1.9.3. LEMMA. — *If* $\bar\xi\in\mathfrak{h}_k^*$ *is regular and* $0\neq\bar z\in\mathfrak{h}_k^*$, *then there exists* $x\in S(\mathfrak{h}_A)^W$ *such that* $e_{\bar\xi}(\partial_{\bar z}(\bar x))\neq 0$.

We can take:

$$x = \sum_{w \in \mathrm{W}} wy,$$

for some $y \in \mathrm{S}(\mathfrak{h}_{\mathrm{A}})$ and then we require that:

$$\sum_{w \in \mathrm{W}} e_{w^{-1}\overline{\xi}}(\partial_{w^{-1}\overline{z}}(\overline{y})) \neq 0.$$

Set $\mathrm{J}_w = \operatorname{Ker} e_{w^{-1}\overline{\xi}}$, which is a maximal ideal of $\mathrm{S}(\mathfrak{h}_k)$. Since $\overline{\xi}$ is regular, the points $\{w^{-1}\overline{\xi}\}_{w \in \mathrm{W}}$ are distinct and so $\mathrm{J}_w + \mathrm{J}_{w'} = (1)$ *if* $w \neq w'$. Now the map $f_w : \varphi \mapsto e_{w^{-1}\overline{\xi}}(\partial_{w^{-1}\overline{z}}(\varphi))$ of $\mathrm{S}(\mathfrak{h}_k)$ to k is non-zero and vanishes on J_w^2. Then by say 1.4.3 we can choose $\overline{y} \in \mathrm{S}(\mathfrak{h}_k)$ so that all but one of the numbers $f_w(\overline{y}) : w \in \mathrm{W}$ vanish.

1.9.4. Let $\mathrm{D} \subset \mathrm{C}$ be a block. By 1.8.2 we can write $\mathrm{D} = \mathrm{W}_{\overline{\lambda}}\lambda + \nu\pi$ for some λ, $\nu \in \mathfrak{h}_{\mathrm{A}}^*$. A block is called semi-regular if the stabilizer of $\overline{\nu}$ in $\mathrm{W}_{\overline{\lambda}}$ is trivial and called regular if in addition $\overline{\lambda}$ is regular. Observe that if $\mathrm{D} \subset \mathrm{C}$ is semi-regular, then every sub-block of C is semi-regular.

Example. — Take $\mathrm{A} = \mathbb{C}[t]_{(t)}$ *and* $\overline{\lambda}$, $\overline{\delta} \in \mathfrak{h}^*$. Set $\lambda = \overline{\lambda} \otimes 1$, $\delta = \overline{\delta} \otimes 1$ and take $\mathrm{C} = \lambda + \delta t + \mathrm{P}(\mathrm{R})$. Then $\mathrm{D} = \mathrm{W}_{\overline{\lambda}} + \delta t \subset \mathrm{C}$ is semi-regular if $\overline{\delta}$ is regular and regular if both $\overline{\lambda}$, $\overline{\delta}$ are regular.

Lemma. — *Let* D *be a semi-regular block. If* $\lambda_1, \lambda_2 \in \mathrm{D}$ *are distinct, then there exists* $x \in \mathrm{S}(\mathfrak{h}_{\mathrm{A}})^{\mathrm{W}}$ *such that* $e_{\lambda_1}(x) \neq e_{\lambda_2}(x)$.

If $e_{\lambda_1}(x) = e_{\lambda_2}(x)$ for all $x \in \mathrm{S}(\mathfrak{h}_{\mathrm{A}})^{\mathrm{W}}$ then $\lambda_2 = w\lambda_1$ for some $w \in \mathrm{W}$. Write $\lambda_i = w_i\lambda + \pi\nu : w_i \in \mathrm{W}_{\overline{\lambda}}$, $i = 1, 2$. Then $w_2^{-1}\pi(w\nu - \nu) = w_2^{-1}ww_1\lambda - \lambda \in \mathrm{Q}(\mathrm{R})$ and so both sides must vanish. It follows that $w\nu - \nu = 0$ and $w \in \mathrm{W}_{\overline{\lambda}}$. By semi-regularity $w = 1$ and so $\lambda_1 = \lambda_2$.

1.9.5. Proposition. — *Let* $\mathrm{D} \subset \mathrm{C}$ *be a regular (resp. semi-regular) block. Suppose* λ_1, $\lambda_2 \in \mathrm{D}$ *are distinct and set* $\mathrm{J} = \operatorname{Ker}\chi_{\lambda_1} + \operatorname{Ker}\chi_{\lambda_2}$. *Then for all* $j \in \mathbb{N}$;

(i) $\pi \in \mathrm{J}$ *(resp.* $\pi^l \in \mathrm{J} : l$ *sufficiently large).*

(ii) $\operatorname{Ext}^j(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2))$ *is annihilated by* π *(resp.* $\pi^l : l$ *sufficiently large).*

(iii) $\operatorname{Hom}(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2)) = 0$.

(iv) *If* D *is regular, there is a short exact sequence of* k *modules:*

$$0 \to \operatorname{Ext}^j(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2)) \to \overline{\operatorname{Ext}^j}(\mathrm{M}(\overline{\lambda}_1), \mathrm{M}(\overline{\lambda}_2)) \to \operatorname{Ext}^{j+1}(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2)) \to 0.$$

(i) If $z \in \mathrm{Z}(\mathfrak{g}_{\mathrm{A}})$, then:

$$(z - \chi_{\lambda_1}(z)) - (z - \chi_{\lambda_2}(z)) = \chi_{\lambda_2}(z) - \chi_{\lambda_1}(z) = e_{\lambda_2}(\mathrm{P}'(z)) - e_{\lambda_1}(\mathrm{P}'(z)).$$

By 1.3.2 (ii) it is then enough to show that $\pi = (e_{\lambda_2}(x) - e_{\lambda_1}(x))u$ [resp. $\pi^l = (e_{\lambda_2}(x) - e_{\lambda_1}(x))u$: l sufficiently large] for some unit $u \in \mathrm{A}$ and some $x \in \mathrm{S}(\mathfrak{h}_{\mathrm{A}})^{\mathrm{W}}$. Thus the assertion for D semi-regular follows from 1.9.4. For D regular we write $\lambda_1 = w_1\lambda + \pi\nu$, $\lambda_2 = w_2\lambda + \pi\nu$. Then setting $\nu_i = w_i^{-1}\nu : i=1, 2$ and using the W invariance of x, we obtain:

$$e_{\lambda_2}(x) - e_{\lambda_1}(x) = e_{\lambda+\pi\nu_1}(x) - e_{\lambda+\pi\nu_2}(x) = \sum_{l=0}^{\infty} \pi^l e_\lambda (\partial_{\nu_1}^{(l)}(x) - \partial_{\nu_2}^{(l)}(x)).$$

It hence remains to show that $e_{\bar\lambda}(\partial_{\bar\nu_1 - \bar\nu_2}(x)) \neq 0$. Now $\bar\nu_1 - \bar\nu_2 = w_1^{-1}\bar\nu - w_2^{-1}\bar\nu \neq 0$, by the hypothesis of regularity. Then since $\bar\lambda$ is regular the required assertion follows from 1.9.3.

(ii) Let M, $N \in \mathrm{Ob}\, K_C''$. Take a projective resolution (X^*, ε) of M. The terms $\mathrm{Hom}_{U(\mathfrak{g}_A)}(X^j, N)$ admit a $Z(\mathfrak{g}_A)$ module structure coming either from the action of $Z(\mathfrak{g}_A)$ on X^j, or on N. Thus $\mathrm{Ext}^j(M, N)$ is a $Z(\mathfrak{g}_A)$ module, where the $Z(\mathfrak{g}_A)$ action is the one defined by functoriality of Ext^j using the $Z(\mathfrak{g}_A)$ action on M or on N. By 1.3.5 (ii), Ker χ_{λ_i} acts by zero on $M(\lambda_i) : i=1, 2$. Hence J annihilates $\mathrm{Ext}^j(M, N)$ and so (ii) follows from (i).

To prove (iii), (iv), consider the short exact sequence:

$$0 \to M(\lambda_2) \xrightarrow{\pi} M(\lambda_2) \to M(\bar\lambda_2) \to 0,$$

in K_C''. This given an injection:

$$0 \to \mathrm{Hom}(M(\lambda_1), M(\lambda_2)) \xrightarrow{\pi} \mathrm{Hom}(M(\lambda_1), M(\lambda_2))$$

and so (ii) implies (iii). Again if D is regular, then the long exact sequence for $\mathrm{Ext}^j(M(\lambda_1), -)$ decomposes by (ii) into short exact sequences:

$$0 \to \mathrm{Ext}^j(M(\lambda_1), M(\lambda_2)) \to \mathrm{Ext}^j(M(\lambda_1), M(\bar\lambda_2)) \to \mathrm{Ext}^{j+1}(M(\lambda_1), M(\lambda_2)) \to 0$$

of A modules and hence, by (ii) again, of k modules. Now $M(\lambda_1)$ is a free A module and so taking φ to be specialization in 1.6.2 we can replace the middle term by $\overline{\mathrm{Ext}^j}(M(\bar\lambda_1), M(\bar\lambda_2))$. Hence (iv).

1.9.6. Lemma. — *Let* D, $D' \subset C$ *be distinct blocks. Then:*

(i) $\mathrm{Ext}^*(M, N) = 0$ *for all* $M \in \mathrm{Ob}\, K_D'$, $N \in \mathrm{Ob}\, K_{D'}'$.

(ii) *For each* $j \in \mathbb{N}$, $\mathrm{Ext}^j(M, N) : M, N \in \mathrm{Ob}\, K_D$ *is a finitely generated* A *module.*

(i) As in 1.9.5 (ii) this follows from 1.8.3 on taking account of the action of $Z(\mathfrak{g}_A)$ on $\mathrm{Ext}^*(M, N)$. Since any discrete valuation ring is Noetherian, (ii) follows from 1.4.10 and 1.8.9.

1.9.7. Corollary. — *Suppose* $\overline{\mathrm{Ext}^j}(M(\bar\lambda_1), M(\bar\lambda_2)) = 0$, *for some* $\lambda_1, \lambda_2 \in C, j \in \mathbb{N}$. *Then* $\mathrm{Ext}^j(M(\lambda_1), M(\lambda_2)) = 0$.

The short exact sequence $0 \to M(\lambda_2) \xrightarrow{\pi} M(\lambda_2) \to M(\bar\lambda_2) \to 0$, gives the long exact sequence:

$$\to \mathrm{Ext}^j(M(\lambda_1), M(\lambda_2)) \xrightarrow{\pi} \mathrm{Ext}^j(M(\lambda_1), M(\lambda_2)) \to \mathrm{Ext}^j(M(\lambda_1), M(\bar\lambda_2)) \to.$$

Under the hypothesis and 1.5.2, it follows that π is surjective. The assertion then follows from 1.9.6 (ii) and Nakayama's Lemma.

1.9.8. Let D be a regular block. Then 1.9.5 and 1.9.7 reduce the determination of $\mathrm{Ext}^*(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2)) : \lambda_1, \lambda_2 \in \mathrm{D}$ as an A module to the determination of $\overline{\mathrm{Ext}^*}(\mathrm{M}(\bar{\lambda}_1), \mathrm{M}(\bar{\lambda}_2))$ as a k vector space. In 5.2 we shall show how to get quite precise information on the latter. For the moment we note that 1.9.5, 1.9.7 can be used with ([6], 7.6.23) to determine $\mathrm{Ext}^1(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2))$.

When $\mathrm{Hom}_{\mathrm{U}(\mathfrak{g}_k)}(\mathrm{M}(\bar{\lambda}_1), \mathrm{M}(\bar{\lambda}_2)) \neq 0$ we write $\bar{\lambda}_1 \prec \bar{\lambda}_2$. By ([6], 7.6.23) $\prec$ is a partial ordering on $\overline{\mathrm{C}}$.

Lemma. — *Let* $\mathrm{D} \subset \mathrm{C}$ *be a regular block. For each* $\lambda_1, \lambda_2 \in \mathrm{D}$ *one has:*

$$\dim \mathrm{Ext}^1(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2)) = \begin{cases} 1, & \lambda_1 \neq \lambda_2, \quad \bar{\lambda}_1 \prec \bar{\lambda}_2, \\ 0, & \textit{otherwise}. \end{cases}$$

Suppose $\lambda_1 \neq \lambda_2$. By 1.9.5 (iii) one has $\mathrm{Hom}(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2)) = 0$. Then by 1.9.5 (iv) with $j=0$, we obtain the required conclusions. If $\lambda_1 = \lambda_2$, apply 1.9.7 observing the obvious fact that $\overline{\mathrm{Ext}^1}(\mathrm{M}(\bar{\lambda}_1), \mathrm{M}(\bar{\lambda}_1)) = 0$.

1.9.9. The Lemma fails for non-regular blocks; but we still have vanishing. This follows from the corresponding result in specialization (*cf.* [5], Thm. 4) and 1.9.7.

Lemma. — *Let* $\mathrm{D} \subset \mathrm{C}$ *be a block. For each* $\lambda_1, \lambda_2 \in \mathrm{D}$ *and all* $j > 0$ *one has:*

$$\mathrm{Ext}^j(\mathrm{M}(\lambda_1), \mathrm{M}(\lambda_2)) = 0 \qquad \textit{unless} \quad \lambda_1 \neq \lambda_2 \quad \textit{and} \quad \bar{\lambda}_1 \prec \bar{\lambda}_2.$$

1.10. Modules with a p-filtration. — We assume from now on the hypotheses of 1.9 on A.

1.10.1. Fix a P(R) coset C and take $\mathrm{M} \in \mathrm{Ob}\, \mathrm{K}_{\mathrm{C}}$. A p-filtration of M (if it exists) is a finite decreasing filtration $\{ \mathrm{F}^j \mathrm{M} \}_{j=1}^{s}$ with factors isomorphic to Verma modules. Given an exact sequence $0 \to \mathrm{M}' \to \mathrm{M} \to \mathrm{M}(\mu) \to 0$ we obtain an exact sequence:

$$\mathrm{Tor}_1^{\mathrm{A}}(k, \mathrm{M}(\mu)) \to k \otimes_{\mathrm{A}} \mathrm{M}' \to k \otimes_{\mathrm{A}} \mathrm{M} \to k \otimes_{\mathrm{A}} \mathrm{M}(\mu) \to 0.$$

Since $\mathrm{M}(\mu)$ is A free, the term on the extreme left is zero. Again one has $k \otimes_{\mathrm{A}} \mathrm{M}(\mu) \cong \mathrm{M}(\bar{\mu})$. Thus if M admits a p-filtration, so does $k \otimes_{\mathrm{A}} \mathrm{M}$. Fix a p-filtration of M and let $[\mathrm{M} : \mathrm{M}(\mu)] : \mu \in \mathrm{C}$ denote the number of factors isomorphic to $\mathrm{M}(\mu)$.

From our previous observation $[k \otimes_{\mathrm{A}} \mathrm{M} : \mathrm{M}(\bar{\mu})] = [\mathrm{M} : \mathrm{M}(\mu)]$. Now given $\mu_1, \mu_2 \in \mathrm{C}$ one has $\mu_1 - \mu_2 \in \mathrm{P}(\mathrm{R})$ and so $\bar{\mu}_1 = \bar{\mu}_2$ if and only $\mu_1 = \mu_2$. Hence $[\mathrm{M} : \mathrm{M}(\mu)]$ is independent of the p-filtration chosen, since this result holds for a field ([1], Sect. 6).

1.10.2. Let E be a finite dimensional simple $\mathrm{U}(\mathfrak{g})$ module. Let $0 \neq v \in \mathrm{E}$ be a highest weight vector and set $\mathrm{E}_{\mathbb{Z}} = \mathrm{U}(\mathfrak{g}_{\mathbb{Z}})\, v$. One has $\mathrm{E}_{\mathbb{Z}} = \mathrm{U}(\mathfrak{n}_{\mathbb{Z}}^-)\, v$ and since $(\mathfrak{n}_{\mathbb{Z}}^-)^l\, v = 0$ for all l sufficiently large, it follows that $\mathrm{E}_{\mathbb{Z}}$ is a finitely generated torsion free and hence free $\mathbb{Z}$ module. Then $\mathrm{E}_{\mathrm{A}} := \mathrm{A} \otimes_{\mathbb{Z}} \mathrm{E}_{\mathbb{Z}}$ is a free A module of rank equal to dim E. Since $\mathrm{A} \cong \mathrm{A}^{**}$ as an A module, $\mathrm{E}_{\mathrm{A}} \cong \mathrm{E}_{\mathrm{A}}^{**}$.

1.10.3. Let K''' denote the category of all $U(\mathfrak{g}_A)$ modules. Given $M \in \operatorname{Ob} K'''$, we define $E_A \otimes_A M \in \operatorname{Ob} K'''$ through the diagonal action of $U(\mathfrak{g}_A)$ and let θ_E denote the functor $M \mapsto E_A \otimes_A M$. Since $\Omega(E_A) \subset P(R)$ one has $\theta_E M \in \operatorname{Ob} K_C$ whenever $M \in \operatorname{Ob} K_C$.

LEMMA. — *If* $M \in \operatorname{Ob} K_C$ *admits a p-filtration, then so does* $\theta_E M$.

Since θ_E is exact, it suffices to prove to the assertion when $M \cong M(\mu)$. This obtains exactly as in [6], 7.6.14.

1.10.4. Suppose $M \in \operatorname{Ob} K_C$ admits a p-filtration. If $M = M_1 \oplus M_2$, then $M_i \in \operatorname{Ob} K_C : i = 1, 2$ admit p-filtrations because any Verma module is indecomposable. Now suppose that M itself is indecomposable. By primary decomposition (1.8.4) it follows that $M \in \operatorname{Ob} K_D$ for some block $D \subset C$. Recall that the map $\lambda \mapsto \bar{\lambda}$ of D to $\bar{D}$ is injective and define a partial order $\prec$ on D through $\lambda_1 \prec \lambda_2$ given $\bar{\lambda}_1 \prec \bar{\lambda}_2$. By 1.9.9 we obtain the:

LEMMA. — *Suppose* $M \in \operatorname{Ob} K_D$ *admits a p-filtration. Then* M *admits a finite filtration* $\{F^i M\}_{i=1}^s$ *such that* $F^i M / F^{i+1} M \cong M(\lambda_i)^{n_i} : n_i \in \mathbb{N}^+$ *with the* λ_i *pairwise distinct and satisfying* $\lambda_i \succ \lambda_j \Rightarrow i > j$.

1.10.5. In the remainder of section 1.10 we assume that the blocks of C are all semi-regular (1.9.4). Let $D \subset C$ be a (semi-regular) block. We can write $D = W_{\bar{\lambda}} \lambda + \pi \nu$ for some $\lambda, \nu \in \mathfrak{h}_A^*$ and we let $W(\lambda)$ denote the stabilizer of λ in W (equivalently in $W_{\bar{\lambda}}$). To a module $M \in \operatorname{Ob} K_D$ with a p-filtration and equipped with a non-degenerate contravariant form $\mathscr{F}$ we associate a symbol $p(M, \mathscr{F}) \in \mathbb{N}[\pi, \pi^{-1}] W_{\bar{\lambda}} / W(\lambda)$ defined as follows. Let $\{F^i M\}_{i=1}^s$ be a p-filtration of M satisfying the conclusion of 1.10.4 and set $\lambda_i = w_i \lambda + \pi \nu$. By 1.9.9 each factor $F^i M / F^{i+1} M$ is a direct sum of n_i copies of $M(w_i \lambda + \pi \nu)$ and we denote their canonical generators by $\bar{v}_{i,r} : r = 1, 2, \ldots, n_i$. By 1.9.5 (ii) we can choose l sufficiently large so that $\pi^{-l} F^i M$ is a direct sum of the Verma modules $M(w_j \lambda + \pi \nu) : j \geqq i$. This allows us to choose $v'_{ir} \in F^i M$, $v''_{ir} \in F^{i+1} M$ such that $v_{ir} := v'_{ir} + \pi^{-l} v''_{ir}$ is a canonical generator for $M(w_i \lambda + \pi \nu)$ and a representative of $\bar{v}_{ir}$. Since the λ_i are pairwise distinct it follows from 1.9.5 (ii) that the A module V_i generated by the v_{ir} is uniquely determined by the following properties. One, V_i is a free A submodule of $\pi^{-l} F^i M$ of rank n_i. Two, each $v \in V_i$ is a highest weight vector of weight $\lambda_i - \rho$. Three, the image of V_i in $\pi^{-1} F^i M / \pi^{-l} F^{i+1} M$ is just the A submodule generated by the $\bar{v}_{ir} : r = 1, 2, \ldots, n_i$. The form $\mathscr{F}$ extends to $\pi^{-l} F^i M$ and we consider its restriction to V_i. Let us show that this restriction is non-degenerate. Set $\Pi = \{\pi^l : l \in \mathbb{N}\}$. It suffices to show that $\Pi^{-1} M = \oplus \Pi^{-1} M(\lambda_i)^{n_i}$ is an orthogonal direct sum for $\mathscr{F}$. By contravariance it is enough to consider the restriction of $\mathscr{F}$ to weight subspaces. Suppose u belongs to a highest weight subspace of $M(\lambda_i)^{n_i}$ and v to a weight space of $M(\lambda_j)^{n_j} : \lambda_j \neq \lambda_i$ having the same weight. The latter cannot be a highest weight space and so there exists $a \in U(\mathfrak{n}_A^-)_{\lambda_i - \lambda_j}$ such that $v = a v'$ for some highest weight vector $v' \in M(\lambda_j)^{n_j}$. Then $\sigma(a) u = 0$ and so $0 = \mathscr{F}(\sigma(a) u, v') = \mathscr{F}(u, a v') = \mathscr{F}(u, v)$. We conclude that $\mathscr{F}(u, M(\lambda_j)^{n_j}) = 0$ and so $\mathscr{F}(M(\lambda_i)^{n_i}, M(\lambda_j)^{n_j}) = 0$, as required.

We can now define the symbol $p(M, \mathscr{F})$. Since A is a principal ideal domain and in fact every ideal of A has the form $(\pi^l) : l \in \mathbb{N}$, we can choose bases $\{x_{ir}\}$, $\{y_{is}\}$ (*see* [9], 5.1 for example) for V_i such that:

$$\mathscr{F}(x_{ir}, y_{is}) = \begin{cases} \pi^{m_{ir}}, & r=s, \\ 0, & r \neq s, \end{cases}$$

with $m_{ir} \in \mathbb{Z}$. We define:

$$p\,\mathrm{M}, \mathscr{F}) = \sum_{i,\,r} \pi^{m_{ir}} w_i.$$

1.10.6. Let E be a finite dimensional simple $U(\mathfrak{g})$ module. Since E is a simple quotient of a Verma module it inherits a non-degenerate contravariant form from the canonical contravariant form defined (1.3.7) on the latter. This form restricts to $E_{\mathbb{Z}}$ and extends to a non-degenerate contravariant form $\mathscr{F}'$ on E_A. Now suppose $M \in \mathrm{Ob}\, K_C$ has a p-filtration and is equipped with some non-degenerate contravariant form $\mathscr{F}$. Then $\mathscr{F}' \otimes \mathscr{F}$ is defined as a non-degenerate contravariant form on $E_A \otimes_A M$. By contravariance, primary decomposition (1.8.6) is an orthogonal direct sum with respect to $\mathscr{F}' \otimes \mathscr{F}$ and so we obtain a non-degenerate contravariant $\mathscr{F}_i$ on each primary component M_i by restriction. Since each M_i admits a p-filtration, the symbol $p(M_i, \mathscr{F}_i)$ is defined. A basic problem is to compute the $p(M_i, \mathscr{F}_i)$ from $p(M, \mathscr{F})$. A formula of Jantzen ([8], Sect. 5, formula for a_μ) does just this in certain "multiplicity free" cases. It leads to operators defining a Hecke algebra—a fact which we believe to be the key to understanding the Kazhdan-Lusztig conjecture. Let θ be some product of the θ_E and F_D. When $\mathscr{F}$ defined on $\theta\, M(w\lambda)$ obtains from the canonical contravariant form on the Verma module $M(w\lambda)$ by applying the above procedure we simply write $p(\theta\, M(w\lambda))$ for $p(\theta\, M(w\lambda), \mathscr{F})$.

1.10.7. For any $\lambda \in C$ we consider (as in 1.2.3) $A_{\lambda-\rho}$ as a $U(\mathfrak{b}_A)$ module. Let N be a $U(\mathfrak{b}_A)$ module admitting a finite filtration with factors amongst the $A_{\lambda-\rho} : \lambda \in C$. Since $U(\mathfrak{g}_A)$ is a free right $U(\mathfrak{b}_A)$ module $M := U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} N$ admits a p-filtration with factors amongst the $M(\lambda) : \lambda \in C$. In particular $Q^s(\lambda)$ (notation 1.5.3) admits a p-filtration and by 1.8.7 so does $Q(\lambda, D)$ for any block $D \subset C$. Then by 1.8.9 every module projective in K'_C admits a p-filtration. Now choose $\lambda \in C$ so that $\bar{\lambda}$ is dominant and let Q be the projective cover of $M(\lambda)$ in K_C. By 1.9.9 and the above, one has $\mathrm{Ext}^1(M(\lambda), Q) = 0$ and so $Q \cong M(\lambda)$. Thus $M(\lambda)$ is projective and so is $\theta_E M(\lambda)$ for any finite dimension simple module E. By [6], 7.6.14 which extends easy to the present situation it follows that every indecomposable module projective in K'_C is a direct summand of the $\theta_E M(\lambda) : \lambda \in C$, $\bar{\lambda}$ dominant and E finite dimensional. Again by 1.10.4, $\theta_E M(\lambda)$ has a p-filtration and by [6], 7.6.14 $[\theta_E M(\lambda) : M(\mu)] = \dim E_{\lambda-\mu}$ for all $\lambda, \mu \in C$. Assuming $\bar{\mu}$ dominant implies $M(\mu)$ projective and hence the

LEMMA. — *For all* $\lambda, \mu \in C : \bar{\mu}$ *dominant, one has an isomorphism* $\mathrm{Hom}_{U(\mathfrak{g}_A)}(M(\mu), \theta_E M(\lambda)) \cong (E_A)_{\lambda-\mu}$ *of* A *modules.*

1.10.8. Take $\lambda \in C$ and set $Z(\lambda) := \{ z - \chi_\lambda(z) : z \in Z(\mathfrak{g}_A) \}$. By 1.3.5 we have $\mathrm{Ann}\, M(\lambda) \supset U(\mathfrak{g}_A) Z(\lambda)$ and we show that equality holds. Let $\mathscr{K}$ denote the image in $U(\mathfrak{g})$ of the harmonic elements of $S(\mathfrak{g})$. One has $U(\mathfrak{g}) = Z(\mathfrak{g}) \otimes_k \mathscr{K}$ ([6], 8.2.4). Let $\{ \mathscr{K}^i \}$

denote the filtration of $\mathscr{K}$ which derives from the canonical filtration $\{U^i(\mathfrak{g})\}$ of $U(\mathfrak{g})$. By ([6], 8.4.3) $\mathscr{K}$ acts faithfully on $M(\bar{\lambda})$. Choose $t \in \mathbb{N}$ such that $(\mathrm{ad}^t \mathfrak{n}^-)\mathfrak{g}=0$. An elementary argument shows that $a U^{ti}(\mathfrak{n}^-)\bar{v}=0 : a \in U^i(\mathfrak{g})$ implies $a U(\mathfrak{n}^-)\bar{v}=0$. Hence $\mathscr{K}^i$ acts faithfully on $\mathrm{Hom}_k(U^{ti}(\mathfrak{n}^-)\bar{v}, U^{t(i+1)}(\mathfrak{n}^-)\bar{v})$. It follows that the free finitely generated A module $\mathscr{K}^i \otimes_k A$ acts faithfully on the free finitely generated A module $\mathrm{Hom}_A(U^{ti}(\mathfrak{n}_A^-)v, U^{t(i+1)}(\mathfrak{n}_A^-)v)$ and so $\mathscr{K} \otimes_k A$ acts faithfully on $M(\lambda)$.

Since $U(\mathfrak{g}_A)=U(\mathfrak{g}_A)Z(\lambda)\oplus(\mathscr{K}\otimes_k A)$ this proves the required equality. Thus $U(\lambda) := U(\mathfrak{g}_A)/\mathrm{Ann}\, M(\lambda)$ is a free A module. Furthermore considered as a $U(\mathfrak{g}_A)$ module for adjoint action it is a direct sum of the E_A and by Kostant's Theorem ([6], 8.3.11) we obtain the

LEMMA. — *For each* $\lambda \in C$ *one has an isomorphism* $\mathrm{Hom}_{U(\mathfrak{g}_A)}(E_A, U(\lambda)) \cong (E_A^*)_0$ *of* A *modules.*

1.10.9. We extend to case of a ring a result of Bernstein and Gelfand [2], 3.5. Let K_λ''' denote the full subcategory of all $M \in \mathrm{Ob}\, K'''$ satisfying $Z(\lambda)M=0$, and $\theta_E(\lambda)$ the restriction of θ_E to K_λ'''. Call a functor $K_\lambda''' \to K'''$ a *projective* λ-*functor* if it is isomorphic to a direct summand of $\theta_E(\lambda)$ for some finite dimensional $U(\mathfrak{g})$ module E.

1.10.10. Let θ_1, θ_2 be projective λ-functors. Define a homomorphism:

$$i_\lambda : \quad \mathrm{Hom}(\theta_1, \theta_2) \to \mathrm{Hom}(\theta_1 M(\lambda), \theta_2 M(\lambda)),$$

via $i_\lambda(\varphi)=\varphi_{M(\lambda)}$ where $\varphi_{M(\lambda)} : \theta_1 M(\lambda) \mapsto \theta_2 M(\lambda)$ is the value of the functor morphism $\varphi : \theta_1 \mapsto \theta_2$ on $M(\lambda)$.

Consider $E_A \otimes_A U(\lambda)$ (notation 1.10.9) as a left $U(\mathfrak{g}_A)$ module and a right $U(\lambda)$ module through $(e\otimes u)v=e\otimes uv$ and $X(e\otimes u)=Xe\otimes u+e\otimes Xu$ for all $X \in \mathfrak{g}_A$, $e \in E_A$, $u, v \in U(\lambda)$. As in the case of a field ([2], 3.5) we obtain the:

PROPOSITION. — *If* $\bar{\lambda}$ *is dominant, then* i_λ *is an isomorphism.*

As in ([2], 3.5) it is enough to prove the assertion for $\theta_1=\theta_E(\lambda)$, $\theta_2=\theta_{E'}(\lambda)$ with E, E′ finite dimensional simple $U(\mathfrak{g})$ modules. Now:

$$\begin{aligned}
\mathrm{Hom}_{U(\mathfrak{g}_A)}(\theta_E M(\lambda), \theta_{E'} M(\lambda)) &\cong \mathrm{Hom}_{U(\mathfrak{g}_A)}(M(\lambda), (E_A^*\otimes_A E_A')\otimes_A M(\lambda)),\\
&\cong (E_A^*\otimes_A E_A')_0, \quad \text{by } 1.10.7,\\
&\cong \mathrm{Hom}_{U(\mathfrak{g}_A)}(E_A'^*\otimes_A E_A, U(\lambda)), \quad \text{by } 1.10.8,\\
&\cong \mathrm{Hom}_{U(\mathfrak{g}_A)}(E_A, E_A'\otimes_A U(\lambda)),\\
&\cong \mathrm{Hom}_{U(\mathfrak{g}_A)-U(\lambda)}(E_A\otimes_A U(\lambda), E_A'\otimes_A U(\lambda)), \quad \text{by ([2], 2.2)},\\
&\cong \mathrm{Hom}(\theta_E(\lambda), \theta_{E'}(\lambda)), \quad \text{by ([2], 1.3)},
\end{aligned}$$

since obviously $\theta_E M \cong E_A \otimes_A U(\lambda) \otimes_{U(\lambda)} M$, for all $M \in \mathrm{Ob}\, K_\lambda'''$.

2. The Kazhdan-Lusztig polynomials

2.1. Let W be a Weyl group with generating set S and length function $l(.)$. Following Kazhdan and Lusztig [10] we define for each $x, y \in W$ a polynomial $R_{x,y}$ in an indeterminate q

through the following relations valid for all $s \in S$:

(i) $$R_{x,1}(q) = \begin{cases} 1, & x=1, \\ 0, & \text{otherwise}, \end{cases}$$

(ii) $$R_{x,y}(q) = \begin{cases} R_{sx,sy}(q), & l(y)+l(sx)=l(x)+l(sy), \\ (q-1)\,R_{x,sy}(q)+q\,R_{sx,sy}(q), & \text{otherwise}. \end{cases}$$

Let $\mathscr{R}$ denote the matrix with entries $R_{x,y}$. Then $\mathscr{R}(1)$ is the identity matrix and $R_{x,y} \neq 0 \Leftrightarrow x \leqq y$, that is $\mathscr{R}$ is upper triangular (with respect to a basis $\{x_i\}$ of $\mathbb{C}$ W satisfying $x_i \leqq x_j \Rightarrow i<j$) with ones on the diagonal. We call such a matrix unipotent. Let prime denote differentiation. For the choice of the set of positive roots R^+ defined with respect to S one has for all $x \in W$, $\alpha \in R^+$ that $l(s_\alpha x) > x \Leftrightarrow x^{-1}\alpha \in R^-$.

2.2. Lemma. — *For each $x, y \in W$ one has:*

$$R'_{x,y}(1) = \begin{cases} 1, & y=s_\alpha x, \quad \alpha \in R^+, \ l(s_\alpha x) > x, \\ 0, & \textit{otherwise}. \end{cases}$$

The proof is by induction on $l(y)$. By (i) it holds if $l(y)=0$. Noting that $R_{sx,sy}(1)=0$ unless $x=y$, (ii) gives:

$(\star)$ $$R'_{x,y}(1) = \begin{cases} R'_{sx,sy}(1), & l(y)+l(sx)=l(sy)+l(x), \\ R'_{sx,sy}(1)+R_{x,sy}(1), & \text{otherwise}. \end{cases}$$

Assume $l(y) > l(sy)$. If $R_{x,sy}(1) \neq 0$, then $x=sy$ [which implies that $l(x) < l(sx)$] and so $R'_{sx,sy}(1)=0$. Then $(\star)$ gives the assertion for $R'_{sy,y}(1)$. Now assume $x \neq sy$. Then $R'_{x,y}(1) = R'_{sx,sy}(1)$ and by the induction hypothesis the right hand side equals 1 if and only if $sy = s_\alpha sx$ for some $\alpha \in R^+$ such that $(sx)^{-1}\alpha \in R^+$. Yet $y = ss_\alpha sx = s_{s\alpha}x$ and $s\alpha \in R^+$ (for otherwise $s=s_\alpha$ and $x=sy$) and $x^{-1}(s\alpha) \in R^+$ which proves the assertion in general.

2.3. For each $x, y \in W$, set $S_{x,y}(q) = q^{(l(x)-l(y))/2} R_{x,y}(q)$ and write $q=t^2$, $p=t-t^{-1}$. From the defining relations for $R_{x,y}$ it follows that each $S_{x,y}$ is polynomial in p and that the matrix $\mathscr{S}$ with entries $S_{x,y}$ is unipotent. Then $\mathscr{T} := \log \mathscr{S}$ is defined and has entries polynomial in p. From the definition ([6], 2.1 (i)) of $\mathscr{R}$ one has that $\mathscr{S}(p)\mathscr{S}(-p) = \mathrm{Id}$ and so $\mathscr{T}(p)+\mathscr{T}(-p)=0$. Hence we may write in a unique fashion $\mathscr{T}(p) = \mathscr{V}(t) - \mathscr{V}(t^{-1})$ where $\mathscr{V}$ is a strictly upper triangular matrix with entries polynomial in t. Define strictly upper triangular matrices $\mathscr{V}^{(i)} : i \in \mathbb{N}$ with entries polynomial in t inductively though $\mathscr{V}^{(0)} = \mathscr{V}$ and:

$$\exp(\mathscr{V}^{(i)}(t) - \mathscr{V}^{(i)}(t^{-1})) = \exp - \mathscr{V}^{(i-1)}(t) \exp(\mathscr{V}^{(i-1)}(t) - \mathscr{V}^{(i-1)}(t^{-1})) \exp \mathscr{V}^{(i-1)}(t^{-1}).$$

(Note that by Baker-Campbell-Hausdorff the right-hand side takes the form $\exp \mathscr{X}$ where by triangularity $\mathscr{X}$ has entries polynomial in t, t^{-1} and satisfies $\mathscr{X}(t) = -\mathscr{X}(t^{-1})$.) Eventually $\mathscr{V}^{(j)}=0$ and we set $\mathscr{Q} = \exp \mathscr{V}^{(1)} \exp \mathscr{V}^{(2)} \ldots \exp \mathscr{V}^{(j)}$ which is a unipotent matrix with entries $Q_{x,y}$ polynomial in t. By construction $\mathscr{S}(p) = \mathscr{Q}(t)(\mathscr{Q}(t^{-1}))^{-1}$ and $\mathscr{Q}(t)$ is uniquely determined by this relation and the requirement that it be polynomial in t with $Q_{x,x}(1)=1$,

$\forall x \in W$. Thus the expressions $P_{x,y}(t) := t^{l(y)-l(x)} Q_{x,y}(t^{-1})$ are polynomial in t, t^{-1} of degree $\leq l(y)-l(x)$, satisfy $P_{x,x}(1)=1$, $x \in W$ and the relation:

$$(\star) \qquad P_{x,y}(q^{-1}) q^{l(y)} = \sum_{z \in W} q^{l(x)} R_{x,z}(q) P_{z,y}(q).$$

By the construction these properties determine the matrix $\mathscr{P}$ with entries $P_{z,y}(q)$ completely. From the last relation $P_{z,y}$ is polynomial in q and coincides with the polynomial defined through ([10], 2.2 c).

LEMMA. — *For all $x, y \in W$, one has:*

$$2 P'_{x,y}(1) = (l(y)-l(x)) P_{x,y}(1) - \sum_{z \in W} R'_{x,z}(1) P_{z,y}(1).$$

Differentiate $(\star)$ and apply 2.2.

3. Operators of coherent continuation

3.1. We work from now on in the context of example 1.9.4. That is we take $A = \mathbb{C}[t]_{(t)}$ and $\lambda, \delta \in \mathfrak{h}^*$ regular with $C = \lambda + \delta t + P(R)$. (For convenience the bars have been omitted and we identify λ with $\lambda \otimes 1$.) Then the blocks of C are at least semi-regular. Assume further that $-\lambda$ is dominant.

3.2. For each $\mu \in \mathfrak{h}^*$ define R_μ, W_μ as in 1.8.1. Set $R^+_\mu = R^+ \cap R_\mu$ and let $B_\mu \subset R^+_\mu$ be a set of simple roots. Set $S_\mu = \{ s_\alpha : \alpha \in B_\mu \}$. We view the pair (W_μ, S_μ) as a Coxeter group and define the length function and Bruhat ordering accordingly. Let w_μ be the unique maximal element of W_μ.

3.3. Given $\nu \in P(R)$, let $E(\nu)$ denote the unique up to isomorphism simple $U(\mathfrak{g})$ module with extreme weight ν. Now recall that for each $\alpha \in B_\lambda$ we can choose $\nu_\alpha \in P(R)$ such that $-\lambda + \nu_\alpha$ is dominant and that $(\beta, -\lambda + \nu_\alpha) = 0 : \beta \in R^+$ is equivalent to $\beta = \alpha$. That is $(-\lambda + \nu_\alpha)$ "lies on the α-wall". Set $D = W_\lambda \lambda + \delta t$, $D_\alpha = W_\lambda (\lambda - \nu_\alpha) + \delta t$. Define an exact functor ψ_α on K_C through:

$$\psi_\alpha M = F_{D_\alpha}(E(-\nu_\alpha)_A \otimes_A F_D M).$$

It is called the translation functor to the α-wall. Define an exact function φ_α on K_C through:

$$\varphi_\alpha M = F_D(E(\nu_\alpha)_A \otimes_A F_{D_\alpha} M).$$

It is called the translation functor from the α-wall. Finally define an exact functor θ_α on K_C through $\theta_\alpha = \varphi_\alpha \psi_\alpha$. It is called the reflection functor (coherent continuation) across the α-wall.

3.4. Let M, N be $U(\mathfrak{g}_A)$ modules. We have a natural isomorphism $\mathrm{Hom}_{U(\mathfrak{g}_A)}(E_A \otimes_A M, N) \cong \mathrm{Hom}_{U(\mathfrak{g}_A)}(M, E^*_A \otimes_A N)$ of A modules. Since $E(\nu_\alpha)^*_A \cong E(-\nu_\alpha)_A$ it follows that φ_α is a left and a right adjoint to ψ_α. In particular on taking projective resolutions of any pair $M, N \in \mathrm{Ob}\, K_C$ we obtain the natural isomorphism $\mathrm{Ext}^*(\theta_\alpha M, N) \cong \mathrm{Ext}^*(M, \theta_\alpha N)$. Set $\lambda_\alpha = \lambda - \nu_\alpha$.

3.5. In the remainder of Section 3, we fix a pair $w \in W_\lambda$, $\alpha \in B_\lambda$ such that $ws_\alpha > w$. By 1.10.3 the module $\psi_\alpha M(w\lambda + \delta t)$ has a p-filtration and as in ([9], 2.3) it follows that for all $w' \in W_\lambda$, $[\psi_\alpha M(w\lambda+\delta t) : M(w'\lambda_\alpha + \delta t)] = \dim E(-\nu_\alpha)_{-\nu'}$ where $\nu' = w\lambda - w'\lambda_\alpha$. By ([9], 2.9) the latter equation has the unique solution namely $w'\lambda_\alpha = w\lambda_\alpha$ and $\nu' = w\nu_\alpha$. A similar calculation applies to $\psi_\alpha M(ws_\alpha \lambda + \delta t)$ and so we obtain:

$$\psi_\alpha M(w\lambda+\delta t) \cong M(w(\lambda - \nu_\alpha) + \delta t), \tag{1}$$

$$\psi_\alpha M(ws_\alpha \lambda + \delta t) \cong M(ws_\alpha(\lambda - \nu_\alpha) + \delta t). \tag{2}$$

Since $\lambda - \nu_\alpha = s_\alpha(\lambda - \nu_\alpha)$ the modules on the right hand sides coincide and we denote the common module by M. Since δ is regular, the canonical contravariant forms on $M(w\lambda + \delta t)$ and on $M(ws_\alpha \lambda + \delta t)$ are non-degenerate.

Now the procedure described in 1.10.6 defines *via* (i): $i = 1, 2$, a non-degenerate contravariant form $\mathscr{F}_i$ on M. Let $\overline{w}$ denote the image of w (or of ws_α) in $W_\lambda / W(\lambda - \nu_\alpha)$. Since the p filtration of M is multiplicity-free Jantzen's formula (for a_μ [8], Sect. 5) determines $p(M, \mathscr{F}_i) : i = 1, 2$, as below.

Take $i=1$. We show that $a_{\nu_\alpha}(w\lambda + \delta t - \rho)$ is a unit in A. Its numerator gains a factor of $t^n : n = \dim E(-\nu_\alpha)_{-(w\nu_\alpha - r\beta)}$ for each $r : = 2(\beta, w\lambda)/(\beta, \beta) \in \mathbb{N}^+$ and each $\beta \in R^+$.

Set $\nu' = s_\beta(w\nu_\alpha - r\beta)$. Then:

$$w\lambda - \nu' = w\lambda - s_\beta w \nu_\alpha - r\beta = s_\beta w \lambda - s_\beta w \nu_\alpha = s_\beta w \lambda_\alpha. \tag{$\star$}$$

For $n > 0$, ($\star$) has the unique solution $s_\beta w \lambda_\alpha = w \lambda_\alpha$ (as above) and so $s_\beta w = ws_\alpha$. Since $ws_\alpha > w$, this gives $w^{-1}\beta \in R^+$ and so $(\beta, w\lambda) < 0$ which contradicts the positivity of r. The denominator gains a factor of $t^n : n = \dim E(-\nu_\alpha)_{-(w\nu_\alpha - r\beta)}$ for each $r = -2(\beta, w(\lambda - \nu_\alpha))/(\beta, \beta) \in \mathbb{N}^+$ and each $\beta \in R^+$. Set $\nu' = w\nu_\alpha - r\beta$. Then $w\lambda - \nu' = w(\lambda - \nu_\alpha) + r\beta = s_\alpha w \lambda_\alpha$. For $n > 0$, this has (as above) the unique solution $\nu' = w\nu_\alpha$ and so $r = 0$. This contradicts the positivity of r, so the required assertion is proved.

Take $i=2$. We show that $a_{\nu_\alpha}(ws_\alpha \lambda + \delta t - \rho) = tu$, where u is a unit in A. Its numerator gains a factor of $t^n : n = \dim E(-\nu_\alpha)_{-(ws_\alpha \nu_\alpha - r\beta)}$ for each $r : = 2(\beta, ws_\alpha \lambda)/(\beta, \beta) \in \mathbb{N}^+$ and each $\beta \in R^+$. Set $\nu' = s_\beta(ws_\alpha \nu_\alpha - r\beta)$. Then:

$$ws_\alpha \lambda - \nu' = ws_\alpha \lambda - s_\beta ws_\alpha \nu_\alpha - r\beta = s_\beta ws_\alpha \lambda - s_\beta ws_\alpha \nu_\alpha = s_\beta ws_\alpha \lambda_\alpha.$$

For $n > 0$ we obtain the unique solution $s_\beta ws_\alpha \lambda_\alpha = w\lambda_\alpha$ and so $s_\beta w = ws_\alpha$. Since $ws_\alpha > w$, this gives $w^{-1}\beta = \alpha$ and so $2(\beta, ws_\alpha \lambda)/(\beta, \beta) \in \mathbb{N}^+$. As $\nu' \in W\nu_\alpha$ we have $n = 1$, so the numerator has a factor of t. A similar calculation to the above shows that the denominator has no factor of t and so we have proved the required assertion. (We remark that of course these computations are embedded in Jantzen's work.) We conclude that:

(1) $p(M, \mathscr{F}_1) = \overline{w}$;

(2) $p(M, \mathscr{F}_2) = t\overline{w}$.

3.6. Retain the above notation and set $X = M(ws_\alpha \lambda + \delta t)$, $Y = \theta_\alpha X$, $Z = M(w\lambda + \delta t)$.

LEMMA. — (i) $\theta_\alpha X \cong \theta_\alpha Z \cong Y$.

(ii) $\theta_\alpha L(ws_\alpha \lambda) = 0$.

(iii) *There is exact sequence* $0 \to X \to Y \to Z \to 0$.

(iv) Y *has a unique simple quotient. This is isomorphic to* $L(w\lambda)$.

(v) $\theta_\alpha L(w\lambda) \neq 0$. *In particular* φ_α *is faithful.*

(vi) $\theta_\alpha^2 = \theta_\alpha \oplus \theta_\alpha$.

(i) Obtains from 3.5. For (ii) observe that $L(ws_\alpha \lambda)$ is a quotient of $M(ws_\alpha \lambda)/M(w\lambda)$ and apply (i) (for $t=0$) using the exactness of θ_α. Set $\mu = \lambda - \nu_\alpha$. By 1.10.3 the module $\varphi_\alpha M(w\mu + \delta t)$ has a p-filtration and as in [9], 2.3, it follows that for all $w' \in W_\lambda$, $[\varphi_\alpha M(w\mu + \delta t) : M(w'\lambda + \delta t)] = \dim E(\nu_\alpha)_{\nu'}$ where $\nu' = w'\lambda - w\mu$. By [9], 2.9, the latter equation has just two solutions namely $w' = w$, $\nu' = w\nu_\alpha$ and $w' = ws_\alpha$, $\nu' = ws_\alpha \nu_\alpha$. In both cases ν' is an extreme weight of $E(\nu_\alpha)$ and hence occurs with multiplicity one. Thus Y has a two-step p-filtration with factors X, Z. Finally by 1.9.8 it follows from $ws_\alpha > w$ that Z is a quotient of Y. Hence (iii).

By (iii), $L(w\lambda)$ is a simple quotient of Y and by 1.7.1 any other simple quotient is isomorphic to $L(ws_\alpha \lambda)$. Yet by 3.4 and (ii), one has:

$$\operatorname{Hom}(\theta_\alpha X, L(ws_\alpha \lambda)) \cong \operatorname{Hom}(X, \theta_\alpha L(ws_\alpha \lambda)) = 0.$$

Hence (iv). Furthermore we also see that $\theta_\alpha L(w\lambda) \neq 0$. Since every simple object in K_D is isomorphic to some $\psi_\alpha L(w\lambda)$ with $ws_\alpha > w$ we obtain (v). By 1.10.10 it is enough to show for (vi) that $\theta_\alpha Y' = Y' \oplus Y'$ where $Y' = \theta_\alpha M(\mu + \delta t)$ and $\mu + \delta t \in C$ with μ dominant. Applying θ_α to (iii) and using (i) we obtain an exact sequence $0 \to Y' \to \theta_\alpha Y' \to Y' \to 0$. Yet $M(\mu + \delta t)$ and hence Y' is projective in K_C (1.10.7) and so this sequence splits.

3.7. Define M, Y as in 3.5, 3.6. Let $\mathscr{F}'_i : i = 1, 2$, be the form on Y which obtains from the form $\mathscr{F}_i$ on M defined (in 3.5) by applying the procedure of 1.10.6. Jantzen's formula for a_μ ([8], Sect. 5) gives as in 3.5:

(1) $p(Y, \mathscr{F}'_1) = ws_\alpha + t^{-1} w$.

(2) $p(Y, \mathscr{F}'_2) = tws_\alpha + w$.

3.8. We can interpret θ_α as a linear map $\tilde{\theta}_\alpha$ on $\mathbb{N}[t, t^{-1}] W_\lambda$ defined as follows:

$$(\star) \qquad \tilde{\theta}_\alpha v = \begin{cases} vs_\alpha + t^{-1} v : vs_\alpha > v, \\ tv + vs_\alpha : vs_\alpha < v. \end{cases}$$

Set $T_{s_\alpha} = t\tilde{\theta}_\alpha - 1$ and $q = t^2$. Then from $(\star)$ we obtain $(T_{s_\alpha} - q)(T_{s_\alpha} + 1) = 0$. Given $w \in W_\lambda$ with reduced decomposition $w = s_1 s_2 \ldots s_l$ where $s_i = s_{\alpha_i} : \alpha_i \in B_\lambda$ we set $T_w = T_{s_1} T_{s_2} \ldots T_{s_l}$. From $(\star)$ it is a simple exercise to show that T_w is independent of the reduced decomposition chosen. It follows that the $T_w : w \in W_\lambda$ generate over $\mathbb{Q}[q, q^{-1}]$ a Hecke algebra in the sense of [10], Sect. 1. Set $\theta_w = \theta_{\alpha_1} \theta_{\alpha_2} \ldots \theta_{\alpha_l}$; $\tilde{\theta}_w = \tilde{\theta}_{\alpha_1} \tilde{\theta}_{\alpha_2} \ldots \tilde{\theta}_{\alpha_l}$ and $M_w := \theta_w M(w_\lambda \lambda + \delta t)$ which is projective in K_C. An open question (Q1) is to show that $p(M_w) = \tilde{\theta}_w w_\lambda$. We have shown this for the case $l = 1$ and it also holds if the α_i are pairwise distinct. Indeed the "only" difficulty in the proof that can arise is that at some step M_w admits two non-isomorphic factors in its p-filtration which on translation to the appropriate wall become isomorphic. Unfortunately this difficulty is a very real one since by ($\star$) one has $\tilde{\theta}_\alpha^2 = t\tilde{\theta}_\alpha + t^{-1}\tilde{\theta}_\alpha$. Owing to the factor of t^{-1} one cannot have say:

$$p(\theta_\alpha^2 M(w\lambda)) = \tilde{\theta}_\alpha^2 w = t\tilde{\theta}_\alpha w + t^{-1}\tilde{\theta}_\alpha w = tp(\theta_\alpha M(w\lambda)) + t^{-1} p(\theta_\alpha M(w\lambda)).$$

Indeed this would contradict the splitting $\theta_\alpha^2 M(w\lambda) = \theta_\alpha M(w\lambda) \oplus \theta_\alpha M(w\lambda)$ implied by 3.6 (vi) and in fact $p(\theta_\alpha^2 M(w\lambda)) = 2p(\theta_\alpha M(w\lambda))$. Any proof of (Q1) must thus take account of the fact that $M(w_\lambda \lambda + \delta t)$ is projective and w is reduced.

3.9. For each $w \in W_\lambda$ let $P(w\lambda + \delta t)$ denote the projective cover of $L(w\lambda)$ in Ob K_C. It follows from 3.6 (iii) that M_w admits $P(w_\lambda w^{-1} \lambda + \delta t)$ as an indecomposable summand and furthermore the remaining summands are just the $P(w_\lambda w'^{-1} \lambda + \delta t) : w' < w$ (with appropriate multiplicities). A further open question (Q2) is to show that this sum is an orthogonal direct sum for the form on M_w. If we further assume the truth of the Kazhdan-Lusztig conjecture (which would determine the above multiplicities) positive answers to (Q1) and (Q2) would give the following result which we state as a conjecture.

CONJECTURE. — *For each $y \in W_\lambda$ one has:*

$$p(P(y\lambda + \delta t)) = \sum_{w \in W_\lambda} w t^{l(w) - l(y)} P_{w_\lambda w, w_\lambda y}(q^{-1}).$$

Observe that if $p(P(y\lambda + \delta t))$ is so given then it is polynomial in t and the coefficient of t^0 is just y. Conversely if (Q1) and (Q2) hold then this property determines the $p(P(y\lambda + \delta t))$ uniquely and implies the Kazhdan-Lusztig conjecture.

3.10. Let $M \in \text{Ob}\, K_{\bar{C}}$. Give M^* *a* $U(\mathfrak{g})$ module structure through $(am, n) = (m, \sigma(a) n)$ for all $m \in M$, $n \in M^*$, $a \in U(\mathfrak{g})$. Let $\delta(M)$ denote the submodule of M^* of all $\mathfrak{h}$ finite elements. (It is sometimes known as the $\mathcal{O}$ dual of M.) Then $\delta(M) \in \text{Ob}\, K_{\bar{C}}$. If M admits a non-degenerate contravariant form then $\delta(M) \cong M$. In particular $\delta(L(\mu)) \cong L(\mu)$, for $\mu \in \bar{C}$. By 1.4.9 and [6], 7.6.1, each $M \in \text{Ob}\, K_{\bar{C}}$ has finite length and we let $[M : L]$ denote the number of times the simple factor L occurs in M. Clearly $[\delta(M) : L(\mu)] = [M : L(\mu)]$ for each $\mu \in \bar{C}$.

3.11. Take λ, w, α as in 3.1, 3.5. Let $\mathcal{F}$ denote the non-degenerate contravariant form defined on $L(w\lambda)$ through the canonical form on $M(w\lambda + \delta t)$ and passage to quotient. Let $\mathcal{F}'$ be the non-degenerate contravariant form defined on $\theta_\alpha L(w\lambda)$ by the procedure of 1.10.6. By 3.6 (iii), (iv) the module $\theta_\alpha L(w\lambda)$ admits a unique simple quotient and this is isomorphic to $L(w\lambda)$. By 3.6 (vi), $L(w\lambda)$ cannot be all of $\theta_\alpha L(w\lambda)$. Let $(\theta_\alpha L(w\lambda))'$ be the unique maximal non-zero submodule of $\theta_\alpha L(w\lambda)$ which results. By 3.10 one has $\delta(\theta_\alpha L(w\lambda)) \cong \theta_\alpha L(w\lambda)$ and so $(\theta_\alpha L(w\lambda))'$ admits a unique simple submodule and this is isomorphic to $L(w\lambda)$. Set $U_\alpha L(w\lambda) = (\theta_\alpha L(w\lambda))'/L(w\lambda)$ which inherits a non-degenerate contravariant form from $\mathcal{F}'$.

LEMMA (*Vogan* [12], 3.7). — (i) $\theta_\alpha(U_\alpha L(w\lambda))=0$.

(ii) $[U_\alpha L(w\lambda) : L(y\lambda)]>0$, *implies* $l(y)-l(w)\leqq 1$ *and* $y=ws_\alpha$ *if equality holds, otherwise* $l(y)<l(w)$.

(iii) *Fix* $y\in W_\lambda$ *such that* $y>ys_\alpha$. *Then:*

$$\operatorname{Hom}(L(y\lambda), U_\alpha L(w\lambda))\cong \operatorname{Ext}^1(L(y\lambda), L(w\lambda)).$$

(iv) $[U_\alpha L(w\lambda) : L(ws_\alpha\lambda)]=1$.

We can take $t=0$ in 3.6. Then (i) follows from 3.6 (vi). (ii), (iv) follow from 3.6 (iv), the exactness of θ_α and [6], 7.6.23. (iii) follows from [12], 3.9 *c*, and 3.6 (ii).

Remarks. — Vogan has conjectured ([12], 3.15) that $U_\alpha L(w\lambda)$ is always semisimple and has shown ([13], 3.5) that this conjecture implies the truth of the Kazhdan-Lusztig conjecture. We shall show that both are also implied by the Jantzen conjecture (Sect. 4). At present it is not even known if $L(ws_\alpha\lambda)$ is a direct summand of $U_\alpha L(w\lambda)$. Vogan pointed out that the latter would give the implication $\operatorname{Ext}^j(M(y\lambda), L(w\lambda))\neq 0 \Rightarrow l(w)-l(y)-j$ even.

3.12. Fix $M\in \operatorname{Ob} K_C$. From the natural isomorphisms:

$$\operatorname{Hom}(\psi_\alpha M, \psi_\alpha M)\cong \operatorname{Hom}(\varphi_\alpha\psi_\alpha M, M)\cong \operatorname{Hom}(M, \varphi_\alpha\psi_\alpha M),$$

the identity map Id on $\psi_\alpha M$ induces maps $I'_M : M\to\theta_\alpha M$, $I''_M : \theta_\alpha M\to M$ (or simply I', I'').

LEMMA. — (i) $\operatorname{Ker} I'_M$ *is the largest submodule* N *of* M *satisfying* $\theta_\alpha N=0$.

(ii) $\operatorname{Coker} I''_M$ *is the largest quotient* Q *of* M *satisfying* $\theta_\alpha Q=0$.

Let N be a submodule of M. By functoriality we have the commuting square:

$$\begin{tikzcd}
\operatorname{Hom}(\psi_\alpha M, \psi_\alpha M) \arrow[r, "\sim"] \arrow[d, "\psi_\alpha N\to\psi_\alpha M"'] & \operatorname{Hom}(M, \theta_\alpha M) \arrow[d, "N\to M"] \\
\operatorname{Hom}(\psi_\alpha N, \psi_\alpha M) \arrow[r, "\sim"] & \operatorname{Hom}(N, \theta_\alpha M).
\end{tikzcd}$$

From this it easily follows that $I'_M N=0 \Leftrightarrow \psi_\alpha N=0$. Since φ_α is faithful [3.6(v)] we obtain (i). A similar argument gives (ii).

Remark. — A corresponding result holds in K_C^-.

3.13. Define X, Y, Z as in 3.6.

LEMMA. — *The sequence:*

$$0\to X\xrightarrow{I'_X} Y\xrightarrow{I''_Z} Z\to 0 \textit{ is exact.}$$

By 3.6 (iv), (v) and 3.12 (ii), I''_Z is surjective. Since Y is $\mathfrak{n}_A^-$ free either $I'_X=0$, or I'_X is injective. Now $\theta_\alpha X\neq 0$ by 3.6 (i), so I'_X is injective by 3.12 (i). In particular $\operatorname{Im} I'_X$ has a unique simple quotient and this is isomorphic to $L(ws_\alpha\lambda)$. Yet $L(ws_\alpha\lambda)$ is not a subquotient of Z and so $I_Z''I_X'=0$. By 3.6 (iii) it remains to show that $(\operatorname{Im} I_X')_{t=0}\neq 0$. Since φ_α, ψ_α commute with specialization we have $(\operatorname{Im} I_X')_{t=0}=\operatorname{Im} I'_{M(ws_\alpha\lambda)}\cong M(ws_\alpha\lambda)\neq 0$, as required.

3.14. Assume $-\lambda\in\mathfrak{h}^*$ dominant and regular. Take w, $w'\in W_\lambda$ with $w<w'$ and set $X=M(w'\lambda+\delta t)$, $Z=M(w\lambda+\delta t)$. A non-zero homomorphism $J_{w,w'}$ of $M(w\lambda)$ into $M(w'\lambda)$ defines by 1.9.5 (iv) a non-split extension Y of X by Z. By 1.9.5 (ii) we can identify Y with a submodule of $t^{-1}X\oplus Z$. The precise submodule is given by the following lemma in which $a\mapsto\bar{a}$ denotes specialization at $t=0$.

LEMMA:

$$Y=\{(t^{-1}a, b)\in X\times Z : J_{w,w'}(\bar{b})=\bar{a}\}.$$

Let P be a projective cover of Z. We have a commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & N \arrow[d] \arrow[ddll, "f_1"'] & \\
 & & & P \arrow[d, "\pi'"] \arrow[dl, "f"'] & \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r, "\pi"] & Z \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

with row and column exact and $f_1=f|_N$. By 1.9.5 (ii), $\operatorname{Ext}^1(Z, X)$ is annihilated by t, so tf_1 can be extended to a map f' of P into X. Given $y\in Y$ we can choose $p\in P$ such that $\pi'(p)=\pi(y)$. Then the map $g: y\mapsto y-f(p)-t^{-1}f'(p)$ of Y into $t^{-1}X$ is independent of the $p\in P$ chosen and coincides on X with the identity on X. Consequently the map $h: y\mapsto(g(y), \pi(y))$ of Y into $t^{-1}X\oplus Z$ is injective. Set $a'=-f'(p)$, $b=\pi'(p)$. If $b=0$, then $p\in N$ and $a'=-tf_1(p)$, so $\bar{a}'=0$. Thus we have a map $j: b\mapsto\bar{a}'$ of Z into X/tX. It is non-zero for otherwise $f'(p)\in tX$ and we should have $Y\cong X\oplus Z$. Obviously $\operatorname{Ker} j\supset tZ$ and so j defines by passage to the quotient an embedding $\bar{j}: M(w\lambda)\to M(w'\lambda)$. If $(t^{-1}a, b)\in\operatorname{Im} h$, then writing $b=\pi'(p)=\pi(y)$, we must have $a=t(y-f(p))+a$. Then $\bar{j}(\bar{b})=\bar{a}'=\bar{a}$. The converse obtains on noting that $y-f(p)\in X\subset\operatorname{Im} h$. Finally $\bar{j}$ identifies with $J_{w,w'}$ through the definition of the extension Y.

3.15. Take $w'=ws_\alpha$ in 3.14. Since [1.9.5 (ii), 1.9.8] $\operatorname{Ext}^1(Z, X)$ is a k vector space of dimension one we may regard Y to be the extension constructed in 3.6. Let $\mathscr{F}$ denote the canonical contravariant form defined on X or on Z and $\mathscr{F}_1'$ the form on Y defined in 3.7. Then with respect to the presentation of Y gives in 3.14 we may reformulate 3.7 (1) through the:

LEMMA. — *For all* $x=(a, b)$, $y=(a', b')\in Y$ *one has:*

$$\mathscr{F}_1'(x, y)=\mathscr{F}(a, a')+t^{-1}\mathscr{F}(b, b').$$

3.16. Assume $-\lambda \in \mathfrak{h}^*$ dominant and regular.

LEMMA. — *For all* $w, w' \in W_\lambda$;

(i) $\mathrm{Ext}^*(L(w\lambda), L(w'\lambda)) \cong \mathrm{Ext}^*(L(w'\lambda), L(w\lambda))$.

(ii) *For* $w \not> w'$ *the natural map:*

$$\mathrm{Ext}^1(L(w\lambda), L(w'\lambda)) \to \mathrm{Ext}^1(M(w\lambda), L(w'\lambda))$$

is injective.

(iii) $\mathrm{Ext}^1(L(w\lambda), L(w'\lambda)) = 0$, *unless* $w < w'$ *or* $w > w'$.

(iv) *If* $w \not< w'$, *then* (*notation* 1.7.2):

$$\mathrm{Ext}^1(L(w\lambda), L(w'\lambda)) \cong \mathrm{Hom}(M(w\lambda)', L(w'\lambda)).$$

(i) obtains by duality and the isomorphism $\delta(L(\mu)) = L(\mu)$: $\mu \in \overline{C}$. From the exact sequence:

$$0 \to M(w\lambda)' \to M(w\lambda) \to L(w\lambda) \to 0$$

we obtain the exact sequence:

$$(\star) \qquad \begin{aligned} &\to \mathrm{Hom}(M(w\lambda), L(w'\lambda)) \to \mathrm{Hom}(M(w\lambda)', L(w'\lambda)) \to \\ &\to \mathrm{Ext}^1(L(w\lambda), L(w'\lambda)) \to \mathrm{Ext}^1(M(w\lambda), L(w'\lambda)) \to . \end{aligned}$$

Under the hypothesis of (ii) the second term in ($\star$) vanishes and gives (ii). Then (iii) obtains from (i), (ii) and [5], Thm. 4. Under the hypothesis of (iv) the first and last terms of ($\star$) vanish ([6], 7.6.23; [5], Thm. 4). Hence (iv).

Remark. — Take $w' = ws_\alpha$; but reverse the roles of w, w' in (iv). By 3.6 one has $\dim \mathrm{Ext}^1(M(w\lambda), L(ws_\alpha\lambda)) \geqq 1$ and in fact equality holds. Then by 3.11 (iii) the map defined in (ii) is bijective $\Leftrightarrow$ $L(ws_\alpha\lambda)$ is a direct summand of $U_\alpha L(w\lambda) \Leftrightarrow L(w\lambda)$ is a quotient of $M(ws_\alpha\lambda)'$. The latter is an obvious consequence of assuming the Jantzen filtration (*see* Sect. 4) to be hereditary (*see* Sect. 4).

4. The Jantzen conjecture and main Theorem

As in 3.1 we take $A = \mathbb{C}[t]_{(t)}$ and $C = \lambda + \delta t + P(R)$ with both $\lambda, \delta \in \mathfrak{h}^*$ regular and $-\lambda$ dominant. Bar denotes specialization at $t = 0$.

4.1. Given $M \in K_C$ equipped with a contravariant form F we define $M^j := \{a \in M : \mathscr{F}(a, M) \in (t^j)\}$. Then $\{M^j\}_{j \in \mathbb{N}}$ is a filtration of M by $U(\mathfrak{g}_A)$ modules. Define a filtration of $\overline{M}: = M/tM$ through $\overline{M^j} = M^j/(tM \cap M^j)$. It is called the *Jantzen filtration* of $\overline{M}$ relative to the form $\mathscr{F}$. If M is a Verma module we shall always assume that $\mathscr{F}$ is its canonical form (1.3.7).

4.2. For each $w \in W_\lambda$, the filtration $\{M(w\lambda)^j\}_{j\in\mathbb{N}}$ of $M(w\lambda)$ is defined through 4.1 and the identification $M(w\lambda)=\overline{M(w\lambda+\delta t)}$. As suggested by Deodhar ([9], 5.3) one expects the $M(w\lambda)^j$ to be independent of the choice of δ regular; but this is still an open question.

Now fix $\alpha \in B_\lambda$, $w \in W_\lambda$ such that $ws_\alpha > w$. By [6], 7.6.23, there is an embedding $J_{w, ws_\alpha}: M(w\lambda) \to M(ws_\alpha\lambda)$ unique up to a scalar ([6], 7.1.8). Identify $M(w\lambda)$ with a submodule of $M(ws_\alpha\lambda)$.

CONJECTURE. — *For each* $j \in \mathbb{N}$, *one has:*

$$(\mathscr{C}) \qquad M(w\lambda)^j \subset M(ws_\alpha\lambda)^{j+1} \cap M(w\lambda).$$

We take J_{w, ws_α} (or simply, J) to define the extension Y of $M(ws_\alpha\lambda+\delta t)$ by $M(w\lambda+\delta t)$ in 3.14.

($\mathscr{C}$) is an old conjecture ([9], 5.18) of Jantzen who suggested in fact that equality should hold. In this stronger form the conjecture is equivalent to either:

$$M(w\lambda)^j = M(w_\lambda\lambda)^{j+l(w_\lambda)-l(w)} \cap M(w\lambda),$$

for all $j \in \mathbb{N}$, $w \in W_\lambda$, or:

$$M(w\lambda)^j = M(s_\alpha w\lambda)^{j+1} \cap M(w\lambda),$$

for all $j \in \mathbb{N}$, $w \in W_\lambda$ and $\alpha \in B_\lambda$ such that $s_\alpha w > w$. It is latter which should prove the easiest to establish. Here we shall show that ($\mathscr{C}$) implies the Kazhdan-Lusztig conjecture for the multiplicities of composition factors of $M(w\lambda)$ and Vogan's conjectural semisimplicity of $U_\alpha L(w\lambda)$.

We set $M(w\lambda)_j = M(w\lambda)^j/M(w\lambda)^{j+1}$ on which the induced form is non-degenerate ([9], 5.3).

4.3. (Notation 3.3, 4.1, 4.2). — With ν_α as in 3.3. Set $\mu=\lambda-\nu_\alpha$.

LEMMA. — *Suppose* $M \in \operatorname{Ob} K_C$ *admits a contravariant form, define a contravariant form on* $\psi_\alpha M$, $\varphi_\alpha M$ *through the procedure of* 1.10.6. *Then for all* $j \in \mathbb{N}$, $y \in W_\lambda : y < ys_\alpha$ *one has:*

(i) $(\psi_\alpha M)^j = \psi_\alpha M^j$.

(ii) $(\theta_\alpha M)^j = \theta_\alpha M^j$.

(iii) $\psi_\alpha M(w\lambda+\delta t)^j = M(w\mu+\delta t)^j$.

(iv) $\psi_\alpha M(ws_\alpha\lambda+\delta t)^{j+1} = M(w\mu+\delta t)^j$.

(v) $[M(ws_\alpha\lambda)^{j+1} : L(y\lambda)] = [M(w\lambda)^j : L(y\lambda)]$.

(i), (ii) are immediate from the definition of the forms on $\psi_\alpha M, \theta_\alpha M$. Then (iii), (iv) follow from (i) and 3.5 (1), (2). Since $\psi_\alpha L(y\lambda) = L(y\mu)$, when $y < ys_\alpha$ ([9], 2.11) we obtain (v) from (iii), (iv).

Remarks. — (v) expresses the fact that ($\mathscr{C}$) holds with respect to the simple factors which are not annihilated on passage to the α-wall. It easily follows that $M(w\lambda)^{l(w)} = \operatorname{Soc} M(w\lambda) \cong M(\lambda)$. This is an old result of Jantzen ([9], 5.3). One has $M(z\lambda)_0 \cong L(z\lambda)$, $z \in W_\lambda$ ([9], 5.3). Then by (iv) $\psi_\alpha L(ws_\alpha\lambda)=0$. The proof of the corresponding result in positive characteristic needed Jantzen's formula ([8], Sect. 5) for the behaviour of the contravariant form, whereas in zero characteristic we already have this result (*cf.* 3.6 (ii)); so Jantzen used to say that there was no need of his formula in characteristic zero!

4.4. Define X, Y, Z as in 3.6. Let $\mathscr{F}$ be the canonical form defined on X or on Z. Let $\mathscr{F}'_1$ the form on Y defined in 3.7 (1). In all three cases we shall simply denote the form by $\langle\ ,\ \rangle$. Both canonical surjections $Y \to Z$, $\overline{Y} \to \overline{Z}$ will be denoted by π which then commutes with specialization. We use $(\mathscr{C})$ to denote that conjecture 4.2 is assumed to hold.

LEMMA. — *For each* $j \in \mathbb{N}$ *one has:*

(i) $\pi(Y^j) \subset Z^j$.

(ii) $X^{j+1} \subset Y^j$.

$(\mathscr{C})$ (iii) $Z^{j+1} \subset \pi(Y^j)$.

$(\mathscr{C})$ (iv) $\overline{Y}^j \cap \overline{X} \subset \overline{X}^j$.

Recalling 3.14, consider Y as a submodule of $t^{-1}X \oplus Z$. If $y := (x, z) \in Y^j$, 3.15 gives $\langle x, x' \rangle + t^{-1}\langle z, z' \rangle \in (t^j)$ for all $(x', z') \in Y$. Now for each $z' \in Z$ one has by 3.14 that $(0, tz') \in Y$ and so $\langle z, z' \rangle \in (t^j)$, which gives $z \in Z^j$ and hence (i).

If $x \in X^{j+1}$ we have $(x, 0) \in Y$ by 3.14 and $\langle (x, 0), (t^{-1}x', z') \rangle = t^{-1}\langle x, x' \rangle \in (t^j)$, by 3.15. Hence (ii).

If $z \in Z^{j+1}$, then by $(\mathscr{C})$ we have $J(\overline{z}) \in J(\overline{Z}^{j+1}) \subset \overline{X}^{j+2}$, so there exists $x \in X^{j+2}$ such that $J(\overline{z}) = \overline{x}$. By 3.14 we have $(t^{-1}x, z) \in Y$ and so by 3.15 we obtain for all $x' \in X$, $z' \in Z$, that:

$$\langle (t^{-1}x, z), (t^{-1}x', z') \rangle = t^{-2}\langle x, x' \rangle + t^{-1}\langle z, z' \rangle \in (t^j).$$

Hence $(t^{-1}x, z) \in Y^j$ and so $z \in \pi(Y^j)$, which is (iii).

Now take $x \in X$ such that:

$$\overline{x} \in \overline{Y}^j = Y^j/(tY \cap Y^j).$$

By 3.14, $(x, 0) \in Y$ and there exists $(t^{-1}x_1, z_1) \in Y^j$ such that $(x - t^{-1}x_1, z_1) \in tY$. By 3.14 again, there exist $x_2 \in X$, $z_2 \in Z$ such that $x_1 = tx_2$, $z_1 = tz_2$ and $J(\overline{z}_2) = \overline{x} - \overline{x}_2$. By 3.15, we have:

$$\langle (t^{-1}x_1, z_1), (t^{-1}x', z') \rangle = t^{-1}\langle x_2, x' \rangle + \langle z_2, z' \rangle \in (t^j),$$

for all $x' \in X$, $z' \in Z$ such that $J(\overline{z}') = \overline{x}'$. Taking $x' = tx''$, $x'' \in X$, $z' = 0$ gives $x_2 \in X^j$. Taking $x' = 0$, $z' = tz''$; $z'' \in Z$ gives $z_2 \in Z^{j-1}$ and so by $(\mathscr{C})$ we have $J(\overline{z}_2) \in J(\overline{Z}^{j-1}) \subset \overline{X}^j$. Through our previous observations we may conclude that $\overline{x} \in \overline{X}^j$, as required.

4.5. Assume that $(\mathscr{C})$ holds. We set:

$$\overline{X}^a_{j+1} = \overline{X}^{j+1}/(\overline{Y}^{j+1} \cap \overline{X}), \qquad \overline{X}^b_{j+1} = (\overline{Y}^{j+1} \cap \overline{X})/\overline{X}^{j+2},$$

$$\overline{Y}^x_j = (\overline{Y}^j \cap \overline{X})/(\overline{Y}^{j+1} \cap \overline{X}),$$

which give the exact sequences:

(1) $$0 \to \overline{X}^b_{j+1} \to \overline{X}_{j+1} \to \overline{X}^a_{j+1} \to 0,$$

(2) $$0 \to \overline{X}^a_{j+1} \to \overline{Y}^x_j \to \overline{X}^b_j \to 0.$$

Again we set $\overline{Z}^a_j = \pi(\overline{Y}^j)/\overline{Z}^{j+1}$ and:

$$\overline{Z}^b_{j+1} = \overline{Z}^{j+1}/\pi(\overline{Y}^{j+1}), \qquad \overline{Y}^z_j = \pi(\overline{Y}^j)/\pi(\overline{Y}^{j+1}),$$

which give the exact sequences:

(3) $$0 \to \overline{Z}^a_{j+1} \to \overline{Z}_{j+1} \to \overline{Z}^b_{j+1} \to 0,$$

(4) $$0 \to \overline{Z}^b_{j+1} \to \overline{Y}^z_j \to \overline{Z}^a_j \to 0,$$

(5) $$0 \to \overline{Y}^x_j \to \overline{Y}_j \to \overline{Y}^z_j \to 0.$$

From (2), (4), (5) we get a four-step filtration (see p. 300) on $\overline{Y}_j$. We wish to relate this to the non-degenerate form on $\overline{Y}_j$ (denoted by $\langle\ ,\ \rangle$) defined by passage to quotient.

($\mathscr{C}$) LEMMA. — *For all* $j \in \mathbb{N}$:

(i) $\langle \overline{X}^a_{j+1}, \overline{Y}^x_j \rangle = 0$.

(ii) $\langle \overline{X}^a_{j+1}, \operatorname{Ker}(\overline{Y}_j \to \overline{Z}^a_j) \rangle = 0$.

Take $\bar{x} \in \overline{X}^a_{j+1}$, $\bar{y} \in \overline{Y}^x_j$. Let x (resp. y) be an element of X^{j+1} (resp. $Y^j \cap X$) whose image in $\overline{X}^a_{j+1}$ (resp. $\overline{Y}^x_j$) is $\bar{x}$ (resp. $\bar{y}$). By 3.14 we have $(x, 0)$, $(y, 0) \in Y$ and by 3.15 that $\langle (x, 0), (y, 0) \rangle = \langle x, y \rangle \in (t^{j+1})$, since $x \in X^{j+1}$. Hence $\langle x, y \rangle = 0$, which proves (i).

For (ii) we fix $\bar{x}$, x as in (i). By (i) it is enough to take $z \in \overline{Z}^b_{j+1}$ and to show that $\langle \bar{x}, \bar{z} \rangle = 0$. Choose $z \in Z^{j+1}$ whose image in $\overline{Z}^b_{j+1}$ is $\bar{z}$. By 4.2 (iii) there exists $x' \in X^{j+2}$ such that $(t^{-1}x', z) \in Y$. Then by 3.15 we have

$$\langle (x, 0), (t^{-1}x', z) \rangle = t^{-1} \langle x, x' \rangle \in (t^{j+1}).$$

That is $\langle \bar{x}, \bar{z} \rangle = 0$, which proves (ii).

4.6. Take $M \in \operatorname{Ob} K_C$. Since θ_α is exact we may define M^+ (resp. M^-) to be the smallest (resp. largest) submodule of M such that $\theta_\alpha(M/M^+) = 0$ (resp. $\theta_\alpha M^- = 0$). We remark that by 3.6 (ii), (v), $\theta_\alpha L = 0$ for a simple object $L \in \operatorname{Ob} \mathscr{K}_{\overline{D}}$ is equivalent to $L \cong L(y\lambda)$ with $y \in W_\lambda$ and $y > ys_\alpha$.

($\mathscr{C}$) LEMMA. — *For each* $j \in \mathbb{N}^+$:

(i) $\overline{Z}^+_j = \overline{Z}^a_j$.

(ii) $\overline{X}^-_j = \overline{X}^b_j$.

By 4.3 (ii), (iii) we may identify $\theta_\alpha(\overline{Z}^j)$ with $\overline{Y}^j$ and then the map $\pi_j : \overline{Y}^j \to \overline{Z}^j$ defined by restriction of π and 4.4 (i) identifies with the map $I^{\mu}_{\overline{Z}^j}$ defined in 3.12. By 3.12 (ii),

$\pi(\overline{Y}^j)=(\overline{Z}^j)^+$. Hence (i). Again by 4.3 (ii), (iv), we may identify $\theta_\alpha(\overline{X}^{j+1})$ with $\overline{Y}^j$ and then the map $\overline{X}^{j+1}\to\overline{Y}^j$ defined by 4.4 (ii) identifies with the map $I^L_{\overline{X}^{j+1}}$ defined in 3.12 (*cf.* 3.13). Thus (ii) follows from 3.12 (i).

4.6.1/2. Take $M\in \mathrm{Ob}\,K_{\overline{D}}$. Call M α-plus (resp. α-minus) decomposable if every simple factor L of M^+ (resp. of M/M^-) satisfies $\theta_\alpha L\neq 0$, and set $M_-=M/M^+$ (resp. $M_+=M/M^-$). Recall the definition of $\delta(M)$ given in 3.10.

LEMMA. — (i) M *is both* α*-plus and* α*-minus decomposable if and only if the exact sequence:*

$$0\to M^+\to M\to M_-\to 0$$

splits.

(ii) M *is* α*-plus decomposable if and only if* $\delta(M)$ *is* α*-minus decomposable.*

(iii) $M\cong\delta(M)$ *if* M *admits a non-degenerate contravariant form.*

These are immediate from the definitions and 3.10.

4.7. When 4.6.1/2 (i) holds we call M α-decomposable. Note that we can then identify M^- with M_-. Extend $U_\alpha(3.11)$ to any finite direct sum M of simple objects $L\in \mathrm{Ob}\,K_{\overline{D}}$ satisfying $\theta_\alpha L\neq 0$ through $U_\alpha(M\oplus N)=U_\alpha M\oplus U_\alpha N$.

($\mathscr{C}$) PROPOSITION. — *For each* $j\in\mathbb{N}$, *assume that* $\overline{Z}_j$ *is* α*-decomposable. Then:*

i) $\overline{Z}_j^b=\overline{Z}_j^-$.

(ii) $\overline{X}_j$ is α*-decomposable and* $\overline{X}_j^a=\overline{X}_j^+$.

If in addition $\overline{Z}_j^+$ *is semisimple, then* $U_\alpha\overline{Z}_j^+$ *is defined and:*

(iii) $\overline{X}_{j+1}^+\cong\overline{Z}_j^+$,

and:

(iv) *There is an exact sequence:*

$$0\to\overline{X}_j^-\to U_\alpha\overline{Z}_j^+\to\overline{Z}_{j+1}^-\to 0.$$

The hypothesis and 4.6 (i) gives (i). Through the non-degenerate form on $\overline{Y}_j$ and 4.5 (ii) it follows that $\overline{X}_{j+1}^a$ is isomorphic to a submodule of $\delta(\overline{Z}_j^+)$. Thus every simple factor L of $\overline{X}_{j+1}^a$ satisfies $\theta_\alpha L\neq 0$. Then by 4.6.1/2 (ii), it follows that $\overline{X}_{j+1}$ is α-minus decomposable, which through the non-degenerate form on $\overline{X}_{j+1}$ and 4.6 gives (ii). Consequently $\overline{X}_{j+1}^a=\overline{X}_{j+1}^+\quad\delta(\overline{Z}_j^+)$ through 4.3 (v). The semisimplicity of $\overline{Z}_j^+$ then implies (iii). By 3.5 (ii), 4.3 (ii) and α-decomposability one has $\theta_\alpha\overline{Z}_j=\theta_\alpha\overline{Z}_j^+=\overline{Y}_j$. Since $\overline{Z}_j^+$ is semisimple by hypothesis we obtain from 3.11 a complex $0\to Z_j^+\to\theta_\alpha Z_j^+\to Z_j^+\to 0$ with cohomology $U_\alpha\overline{Z}_j^+$ satisfying $\theta_\alpha(U_\alpha\overline{Z}_j^+)=0$. Thus (iv) results from (iii) and the middle two terms of the four-step filtration of $\overline{Y}_j$.

4.8. We may now give our main result. Recall that $-\lambda\in\mathfrak{h}^*$ is dominant, regular.

($\mathscr{C}$) THEOREM. — *Suppose* $-\lambda\in\mathfrak{h}^*$ *is dominant and regular. Then for each* $w\in W_\alpha$, $\alpha\in B_\lambda$ *satisfying* $ws_\alpha>w$ *one has:*

(i) $U_\alpha L(w\lambda)$ *is semisimple.*

For each $j \in \mathbb{N}$:

(ii) $M(w\lambda)_j$ *is semisimple.*

For each $y \in W_\lambda$ *such that* $y > ys_\alpha$:

(iii) $[U_\alpha M(w\lambda)_j : L(y\lambda)] = [M(ws_\alpha\lambda)_j : L(y\lambda)] + [M(w\lambda)_{j+1} : L(y\lambda)]$.

The proof is by induction on $l(w)$. Recalling that $M(\lambda)$ is a simple module, take $l \in \mathbb{N}$ and assume that (i) [resp. (ii)] holds for all $w \in W_\lambda$ such that $l(w) < l$ [resp. $l(w) \leqq l$]. Under this hypothesis each $\overline{Z}_j$ is semisimple and so the conditions of 4.7 are satisfied. By 4.7 (iv) we obtain the exact sequence:

$$0 \to L(ws_\alpha\lambda) \to U_\alpha L(w\lambda) \to \overline{Z}_1^- \to 0.$$

By 3.11 (iv) and the non-degenerate form on $U_\alpha L(w\lambda)$, it follows that the above sequence splits, from which (i) results. Again since the simple factors $L(z\lambda)$ of $M(w\lambda)$ satisfy $z \leqq w$, it follows by (i) and the hypothesis on $\overline{Z}_j$ that each $U_\alpha \overline{Z}_j^+$ is semisimple. By 4.6 (ii), 4.6.1/2 (i), 4.7 (ii), $M(ws_\alpha\lambda)_j = \overline{X}_j = \overline{X}_j^+ \oplus \overline{X}_j^-$. By 4.7 (iii) and the induction hypothesis $\overline{X}_j^+$ is semisimple. By 4.7 (iv) and the semisimplicity of $U_\alpha \overline{Z}_j^+$ it follows that $\overline{X}_j^-$ is semisimple. Hence $M(ws_\alpha\lambda)_j$ is semisimple. Finally (iii) obtains from 4.7 (iv).

4.9. It is clear that 4.3 (v) and 4.8 (iv) determine the composition factors of each $M(w\lambda)_j$. More precisely let q be an indeterminate and set:

$$P_{w_\lambda w,\, w_\lambda y}(q) = \sum_{j=0}^{\infty} q^{(l(w)-l(y)-j)/2} [M(w\lambda)_j : L(y\lambda)].$$

($\mathscr{C}$) COROLLARY. — *The* $P_{y,\, w}(q) : y,\ w \in W_\lambda$ *are the polynomials in* q *defined in* 2.3 *or equivalently by the recurrence relations* ([10], 2.2 *c*). *In particular* $P_{w_\lambda w,\, w_\lambda y}(1) = [M(w\lambda) : L(y\lambda)]$ *takes the form proposed by Kazhdan and Lusztig* ([10], *Conjecture* 1.5 *b*).

By 4.3 (v) we obtain:

$$(\star) \qquad P_{w_\lambda ws_\alpha,\, w_\lambda y}(q) = P_{w_\lambda w,\, w_\lambda y}(q) \qquad \text{if} \quad y < ys_\alpha.$$

Now suppose $y > ys_\alpha$. Given $z \in W_\lambda : zs_\alpha > z$ we define

$$\mu(w_{\lambda z}, w_\lambda y) = [M(z\lambda)_1 : L(y\lambda)].$$

By 4.8 (iii) with $j=0$ and recalling that $M(w\lambda)_0 \cong L(w\lambda)$ we obtain:

$$[U_\alpha L(z\lambda) : L(y\lambda)] = [L(zs_\alpha\lambda) : L(y\lambda)] + \mu(w_\lambda z, w_\lambda y).$$

Resubstitution in 4.8 (iii) gives:

$$P_{w_\lambda ws_\alpha,\, w_\lambda y}(q) + q\, P_{w_\lambda w,\, w_\lambda y}(q)$$
$$= \sum_{z \in W_\lambda \mid zs_\alpha > z} \mu(w_\lambda z, w_\lambda y)\, q^{(1/2)(l(z)-l(y)+1)} P_{w_\lambda w,\, w_\lambda z}(q) + P_{w_\lambda w,\, w_\lambda ys_\alpha}(q).$$

Now set $x=w_\lambda w$ and replace $w_\lambda y$ (resp. $w_\lambda z$) by y (resp. z). Then ($\star$) can be incorporated into the above equation to give for all $y \in W_\lambda : ys_\alpha > y$,

$$P_{x,\, ys_\alpha}(q) = q^u P_{xs_\alpha,\, y}(q) + q^{1-u} P_{x,\, y}(q) - \sum_{z \in W_\lambda \mid zs_\alpha < z} \mu(z,\, y)\, q^{(1/2)(l(y)-l(z)+1)} P_{x,\, z}(q),$$

where:

$$u = \begin{cases} 1, & xs_\alpha > x, \\ 0, & xs_\alpha < x. \end{cases}$$

This is just ([10], 2.2 c). Since $M(w_\lambda \lambda)_0 \cong L(w_\lambda \lambda)$ and $[M(w\lambda)_j : L(w_\lambda \lambda)] = 0$ unless $w = w_\lambda$ and $j=0$ we also obtain:

$$P_{x,\,1}(q) = \begin{cases} 1, & x=1, \\ 0, & \text{otherwise}. \end{cases}$$

These lead to the correct boundary conditions on $P_{x,\,y}(q)$, so the corollary is proved.

4.10. We may regard 4.9 as a conjecture for the multiplicities in each filtration step of $M(w\lambda)$. Let us show that this is consistent with the Jantzen sum formula ([9], 5.3). Using prime to denote the derivative we obtain:

$$P'_{w_\lambda w,\, w_\lambda y}(1) = \sum_{j=0}^{\infty} \frac{(l(w)-l(y)-j)}{2} [M(w\lambda)_j : L(y\lambda)],$$

$$= \left(\frac{l(w)-l(y)}{2}\right) P_{w_\lambda w,\, w_\lambda y}(1) - \frac{1}{2} \sum_{j=1}^{\infty} [M(w\lambda)^j : L(y\lambda)].$$

Substituting from 2.3:

$$\sum_{j=1}^{\infty} [M(w\lambda)^j : L(y\lambda)] = \sum_{z \in w_\lambda} R'_{w_\lambda w,\, w_\lambda z}(1)\, P_{w_\lambda z,\, w_\lambda y}(1),$$

$$= \sum_{\alpha \in R_\lambda^+ : w^{-1} w_\lambda \alpha \in R^-} P_{s_\alpha w_\lambda z,\, w_\lambda w}(1), \quad \text{by 2.2},$$

$$\sum_{\alpha \in R_\lambda^+ : w^{-1} \alpha \in R^-} [M(s_\alpha w \lambda) : L(y\lambda)], \quad \text{as required}.$$

4.11. The last result of this section would also be an immediate consequence of the truth of 4.9; but we show that it holds even without assuming the Jantzen conjecture.

LEMMA. — *Assume $-\lambda \in \mathfrak{h}^*$ dominant and regular. Then for all $w,\, y \in W_\lambda$, one has:*

(i) $j > l(w) - l(y) \Rightarrow [M(w\lambda)_j : L(y\lambda)] = 0$.

(ii) *If* $j = l(w) - l(y)$, *then* $[M(w\lambda)_j : L(y\lambda)] \leqq 1$.

The proof is by induction on $l(w)$. It is trivial if $l(w)=0$. Choose $\alpha \in B_\lambda$ such that $ws_\alpha > w$ and assume the assertion holds for $M(w\lambda)$ and establish it for $M(ws_\alpha \lambda)$. If $y < ys_\alpha$ then the assertion follows from 4.3 (v) and the induction hypothesis. Assume $y > ys_\alpha$.

Set $M = M(w\mu) : \mu = \lambda - \nu_\alpha$ (notation 3.3) and fix a composition series $M = M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_{t+1} = 0$. Each factor M_i / M_{i+1} is isomorphic to some

$\psi_\alpha L(x\lambda): x \in W_\lambda, x < xs_\alpha$. Then $\{\varphi_\alpha M_i\}_{i=0}^{t+1}$ is a normal series for $\theta_\alpha M(w\lambda)$ with factors isomorphic to the $\theta_\alpha L(x\lambda)$. Set $N_i := (\varphi_\alpha M_i) \cap M(ws_\alpha \lambda)$.

Then $\{N_i\}_{i=0}^{t+1}$ is a normal series for $M(ws_\alpha\lambda)$ with each quotient N_i/N_{i+1} isomorphic to a submodule $Q_{i,x}$ of some $\theta_\alpha L(x\lambda)$.

Set $N_i^j := ((\varphi_\alpha M_i) \cap M(ws_\alpha\lambda)^j)/((\varphi_\alpha M_i) \cap M(ws_\alpha\lambda)^{j+1})$. For each j, $\{N_i^j\}_{i=0}^{t+1}$ is a normal series for $M(ws_\alpha\lambda)_j$ with each quotient $Q_{i,x}^{j+1}$ being a subquotient of $Q_{i,x}$. Assume that $Q_{i,x}^{j+1}$ is a submodule of $Q_{i,x}$. We show that:

$$(\star) \qquad [Q_{i,x} : L(xs_\alpha\lambda)] = \begin{cases} 0, & j \geqq l(w) - l(x), \\ \leqq 1, & j = l(w) - l(x) - 1. \end{cases}$$

Set $X = M(ws_\alpha\lambda)$, $Q = Q_{i,x}^{j+1}$. By the hypothesis $\mathrm{Soc}\, Q \cong L(x\lambda)$. Since X admits a non-degenerate contravariant form we have $\delta X \cong X$ and this module admits δQ as a subquotient. Now δQ admits $L(x\lambda)$ as its unique quotient and so arises from the embedding $M(w\lambda) \hookrightarrow M(ws_\alpha\lambda)$. Furthermore this $L(x\lambda)$ arose from $M(ws_\alpha\lambda)^{j+1}$ and so as in 4.3 (v) it follows that δQ is isomorphic to a subquotient of $M(w\lambda)^j$. (Observe that $[M(ws_\alpha\lambda)^{j+1}/\iota(M(w\lambda)^j) : L(w\lambda)] = 0$.) Then

$$[Q : L(xs_\alpha\lambda)] = [\delta Q : L(xs_\alpha\lambda)] \leqq [M(w\lambda)^j : L(xs_\alpha\lambda)].$$

Since $l(xs_\alpha) = l(x) + 1$, we obtain $(\star)$ from the induction hypothesis.

Now consider (i). Suppose $Q_{i,x}^{j+1}$ is a subquotient of $M(ws_\alpha\lambda)_{j+1}$. If $Q_{i,x}^{j+1}$ is not a submodule of $Q_{i,x}$ we must have $[M(ws_\alpha\lambda)^{j+2} : L(x\lambda)] > 0$. Then by 4.3 (v) and the induction hypothesis $j+1 \leqq l(w) - l(x)$. By 3.11 (ii), $[Q_{i,x}^{j+1} : L(y\lambda)] > 0$ gives either, $l(y) < l(x)$ and so $j+1 < l(ws_\alpha) - l(y)$ or, $y = xs_\alpha$ and $j+1 \leqq l(ws_\alpha) - l(y)$. If $Q_{i,x}^{j+1}$ is a submodule of $Q_{i,x}$, then $j \leqq l(w) - l(x)$. If the inequality is strict we argue as above. Otherwise we use $(\star)$ to show that $[Q_{i,x}^{j+1} : L(y\lambda)] > 0$ implies $l(y) < l(x)$ and so $j+1 < l(ws_\alpha) - l(y)$. This gives (i). Finally we observe that $j+1 = l(ws_\alpha) - l(y)$ only if $j+1 = l(w) - l(x)$ and $y = xs_\alpha$. This gives (ii).

Remarks. — This result can be interpreted as saying that $P_{x,y}(q)$ defined through 4.9 is polynomial in $q^{1/2}$. For the corresponding expression defining $\mathrm{Ext}^k(M(w\lambda), L(y\lambda))$ (*see* introduction) the corresponding assertion is that the non-vanishing of this Ext group implies $k \leqq l(y) - l(w)$. The latter is a result of Casselman and Schmid (*cf.* [5], Thm. 4). When equality holds in (ii) it is clear that the unique smallest submodule of $M(w\lambda)_j$ admitting $L(y\lambda)$ as a factor is isomorphic to $M(y\lambda)$. The question of equality in (ii) is apparently quite deep for it would lead to a considerable simplification in the proof of ([9], 5.17).

5. Extensions of Verma modules

5.1. Adjoints. — In this first subsection we develop a property of θ_α used by Vogan ([13], Sect. 4) in his analysis of the Kazhdan-Lusztig conjecture.

5.1.1. Let K be an exact category. Given X, Y $\in$ Ob K, let Hom (X, Y) denote the set of all morphisms X $\to$ Y. Take Z $\in$ Ob K. Given $f \in$ Hom (X, Y) we denote by $S^Z f$ the covariant morphism functor $S^Z f: g \mapsto f_0 g$ of Hom(Z, X) to Hom(Z, Y) and by $S_Z f$ the contravariant morphism functor $S_Z f: g \mapsto g_0 f$ of Hom(Y, Z) to Hom(X, Z).

5.1.2. Let ψ be a functor on K with left adjoint φ. That is we have a functorial isomorphism:

$$J''(\ ,\ ):\quad \mathrm{Hom}(\ ,\psi(\ )) \simeq \mathrm{Hom}(\varphi(\ ),\ ).$$

LEMMA. — *Given morphisms* $X \xrightarrow{f} Y \xrightarrow{g} \psi Z$ *of objects of* K *one has:*

(i) $(J''(Y, Z)g) \circ \varphi(f) = J''(X, Z)(g \circ f)$.

(ii) $g \circ (J''(\psi X, Y)\psi(f)) = (J''(\psi(X), \psi(Z))\psi(g \circ f)$.

(i) By functoriality the diagram:

$$\begin{tikzcd} \mathrm{Hom}(Y, \psi Z) \arrow[r, "J''(Y,Z)"] \arrow[d, "S_{\psi Z} f"] & \mathrm{Hom}(\varphi Y, Z) \arrow[d, "S_{\psi Z}\varphi(f)"] \\ \mathrm{Hom}(X, \psi Z) \arrow[r, "J''(X,Z)"] & \mathrm{Hom}(\varphi X, Z) \end{tikzcd}$$

commutes. Then:

$$\begin{aligned}(J''(Y, Z)g) \circ \varphi(f) &= (S_{\psi Z}\varphi(f)) \circ (J''(Y, Z)g) = ((S_{\psi Z}\varphi(f)) \circ J''(Y, Z))g \\ &= (J''(X, Z) \circ (S_{\psi Z} f))g = J''(X, Z) \circ ((S_{\psi Z} f)g) = J''(X, Z)(g \circ f).\end{aligned}$$

(ii) Follows similarly.

5.1.3. Let ψ be a functor on K with right adjoint φ. That is we have a functorial isomorphism:

$$J'(\ ,\ ):\quad \mathrm{Hom}(\psi(\ ),\ ) \simeq \mathrm{Hom}(\ ,\varphi(\ )).$$

LEMMA. — *Given morphisms* $\psi X \xrightarrow{f} Y \xrightarrow{g} Z$ *of objects of* K *one has:*

(i) $\varphi(g) \circ (J'(X, Y)f) = J'(X, Z)(g_0 f)$.

(ii) $(J'(Y, \psi Z)\psi(g)) \circ f = J'(\psi X, \psi Z)\psi(g \circ f)$.

(i) By functoriality $(S^X \varphi(g)) \circ J'(X, Y) = J'(X, Z) \circ (S^{\psi X} g)$.

Applied to the element $f \in \mathrm{Hom}(\psi X, Y)$, this gives (i). (ii) follows similarly.

5.1.4. From now on we suppose that ψ is a functor on K with left and right adjoint φ. Set $\theta = \varphi\psi$.

LEMMA. — *Fix* X, Y, X', Y' $\in$ Ob K. *If the diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "l"'] & Y' \end{tikzcd}$$

commutes, then so does:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "J'(X,\psi X')\psi(k)"'] & Y \arrow[d, "J'(Y,\psi Y')\psi(g)"] \\ \theta X' \arrow[r, "\theta(l)"'] & \theta Y' \end{tikzcd}$$

By 5.1.3:

$$(J'(Y, \psi Y')\psi(g)) \circ f = J'(X, \psi Y')\psi(gf) = J'(X, \psi Y')\psi(lk)$$
$$= J'(X, \psi Y')(\psi(l) \circ \psi(k)) = \theta(l) \circ (J'(X, \psi X')\psi(k)).$$

5.1.5. Given $X \in \operatorname{Ob} K$, we define (as in 3.14) $I'_X : X \to \theta X$ by $J'(X, \psi X) \operatorname{Id}_{\psi X}$ and $I''_X : \theta X \to X$ by $I''_X = J''(\psi X, X) \operatorname{Id}_{\psi X}$.

LEMMA. — *For all* $X, Y \in \operatorname{Ob} K$, $f \in \operatorname{Hom}(\psi X, \psi Y)$ *the diagram:*

$$\begin{tikzcd} X \arrow[r, "J'(X,\psi Y)f"] \arrow[d, "I'_X"'] & \theta Y \arrow[d, "I''_Y"] \\ \theta X \arrow[r, "J''(\psi X, Y)f"'] & Y \end{tikzcd}$$

commutes.

By 5.1.2 (i), $(J''(\psi Y, Y) \operatorname{Id}_{\psi Y}) \circ \varphi(f) = J''(\psi X, Y) f$. By 5.1.3 (i),

$$\varphi(f) \circ (J'(\psi X, \psi X) \operatorname{Id}_{\psi X}) = J'(X, \psi Y) f.$$

Hence

$$(J''(\psi X, Y) f) \circ I'_X = I''_Y \circ \varphi(f) \circ I'_X = I''_Y \circ (J'(X, \psi Y) f).$$

5.1.6. COROLLARY. — *For all* $X, Y \in \operatorname{Ob} K$, *the diagram:*

$$\begin{tikzcd} \operatorname{Hom}(\psi X, \psi Y) \arrow[r, "J'(X,\psi Y)"] \arrow[d, "J''(\psi X, Y)"'] & \operatorname{Hom}(X, \theta Y) \arrow[d, "S^X I''_Y"] \\ \operatorname{Hom}(\theta X, Y) \arrow[r, "S_Y I'_X"'] & \operatorname{Hom}(X, Y) \end{tikzcd}$$

commutes.

For each $f \in \operatorname{Hom}(\psi X, \psi Y)$ one has:

$$((S^X I''_Y) \circ J'(X, \psi Y)) f = I''_Y \circ (J'(X, \psi Y) f) = (J''(\psi X, Y) f) \circ I'_X, \quad \text{by 5.1.5,}$$
$$= ((S_Y I'_X) \circ J''(\psi X, Y)) f.$$

5.1.7. Take $M \in \operatorname{Ob} K$. From now on $M \to \theta M$ (resp. $\theta M \to M$) denotes I'_M (resp. I''_M) and the functors S_M, S^M are to be understood. Composing J', J'' we have a functorial isomorphism:

$$J(\ ,\ ): \quad \operatorname{Hom}(\ , \theta(\ )) \xrightarrow{\sim} \operatorname{Hom}(\theta(\ ),\ ).$$

Now assume that K has enough projectives. Taking projective resolutions we obtain a functorial isomorphism $\mathrm{Ext}^*(\ ,\theta(\ )) \xrightarrow{\sim} \mathrm{Ext}^*(\theta(\ ),\ )$ which we also denote by $J(\ ,\ )$.

5.1.8. Take $M, N \in \mathrm{Ob}\,K$. Assume that ψ, φ (hence θ) are exact functors on K with φ a left and a right adjoint to ψ. Assume that K has enough projectives.

PROPOSITION. — *There is a commuting diagram of maps:*

$$\begin{tikzcd}
\to \mathrm{Ext}^j(N,\theta M) \arrow[r] \arrow[d, "J(N,M)"] & \mathrm{Ext}^j(N,\mathrm{Im}(\theta M \to M)) \arrow[r] \arrow[d, "\xi_1"] & \mathrm{Ext}^{j+1}(N,\mathrm{Ker}(\theta M \to M)) \to \arrow[d, "\xi_2"] \\
\to \mathrm{Ext}^j(\theta N, M) \arrow[r] & \mathrm{Ext}^j(\mathrm{Coim}(N \to \theta N), M) \arrow[r] & \mathrm{Ext}^{j+1}(\mathrm{Coker}(N \to \theta N), M) \to
\end{tikzcd}$$

with the rows exact. If $\mathrm{Ker}(N \to \theta N)=0$, *then* $\xi_1 = \mathrm{Im}(\theta M \to M) \to M$.

Let X^* be a projective resolution of N. Then θX^* is a projective resolution of θN. Set $C^* = \mathrm{Coker}\,(X^* \to \theta X^*)$, $D^* = \mathrm{Coim}(X^* \to \theta X^*)$. By 5.1.4 the diagram:

$$\begin{tikzcd}
X^* \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
\theta X^* \arrow[r] & \theta N \arrow[r] & 0
\end{tikzcd}$$

has exact rows and commutes, so C^* (resp. D^*) is a resolution of $\mathrm{Coker}\,(N \to \theta N)$ [resp. $\mathrm{Coim}(N \to \theta N)$]. Let:

$$0 \to E^* \to F^* \to G^* \to 0$$

be a projective resolution of the exact sequence:

$$0 \to \mathrm{Coim}(N \to \theta N) \to \theta N \to \mathrm{Coker}(N \to \theta N) \to 0.$$

By [4], Prop. 1.1, p. 76, we obtain for any $M \in \mathrm{Ob}\,K$ the commuting diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(C^*,M) \arrow[r] \arrow[d] & \mathrm{Hom}(\theta X^*,M) \arrow[r] \arrow[d] & \mathrm{Hom}(D^*,M) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(G^*,M) \arrow[r] & \mathrm{Hom}(F^*,M) \arrow[r] & \mathrm{Hom}(E^*,M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

By 5.1.6 and the projectivity of X* we obtain the commuting diagram:

$$(\star\star)\qquad \begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(X^*, \mathrm{Ker}(\theta M \to M)) & \to & \mathrm{Hom}(X^*, \theta M) & \to & \mathrm{Hom}(X^*, \mathrm{Im}(\theta M \to M)) & \to & 0 \\ & & & & \big\downarrow{\scriptstyle J(X^*, M)} & & \big\downarrow{\scriptstyle \mathrm{Im}(\theta M \to M) \to M} & & \\ & 0 & \to \mathrm{Hom}(C^*, M) & \to & \mathrm{Hom}(\theta X^*, M) & \to & \mathrm{Hom}(X^*, M) & & \end{array}$$

with exact rows. It follows that $J(X^*, M)$ defines by restriction an isomorphism $\mathrm{Hom}(X^*, \mathrm{Ker}(\theta M \to M)) \simeq \mathrm{Hom}(C^*, M)$. Composing the diagrams (*), (**) gives through ([4], Prop. 4.1, p. 85) the conclusion of the proposition.

5.2. With the conventions of Section 3, take $C = \lambda + \delta t + P(R) : \lambda, \delta \in \mathfrak{h}^*$ with $-\lambda$ dominant and regular. For the moment we do not assume δ regular. Fix $w, y \in W_\lambda, \alpha \in B_\lambda$ satisfying $ws_\alpha > w$, $ys_\alpha < y$. We apply 5.1 with $K = K_C$, $\theta = \theta_\alpha$ (notation 3.1). Set $s = s_\alpha$.

5.2.1. Lemma:

$$\mathrm{Ext}^j(M(y\lambda + \delta t), M(ws\lambda + \delta t)) \cong \mathrm{Ext}^j(M(ys\lambda + \delta t), M(w\lambda + \delta t)).$$

By 3.13 we have the exact sequences:

$$(\zeta) \qquad 0 \to M(y\lambda + \delta t) \xrightarrow{I'} \theta M(y\lambda + \delta t) \xrightarrow{I''} M(ys\lambda + \delta t) \to 0,$$

$$(\eta) \qquad 0 \to M(ws\lambda + \delta t) \xrightarrow{I'} \theta M(ws\lambda + \delta t) \xrightarrow{I''} M(w\lambda + \delta t) \to 0.$$

Take $M = M(w\lambda + \delta t)$, $N = M(y\lambda + \delta t)$ in 5.1.8. Since $J(N, M)$ is an isomorphism and ξ_1 is the identity map, it follows that ξ_2 is required isomorphism.

5.2.2. The exact sequences (ζ), (η) of 5.2.1 are of couse equally valid in specialization (equivalently setting $\delta = 0$). Recalling 3.6 (i) we let:

$$\zeta^j : \quad \mathrm{Ext}^j(\theta M(y\lambda), M(w\lambda)) \to \mathrm{Ext}^j(M(y\lambda), M(w\lambda)),$$

$$\eta^j : \quad \mathrm{Ext}^j(M(ys\lambda), \theta M(w\lambda)) \to \mathrm{Ext}^j(M(ys\lambda), M(w\lambda)),$$

be the resulting natural maps.

Lemma. — *For each j one has:*

$$\dim \mathrm{Ker}\, \zeta^j \leqq \dim \mathrm{Ker}\, \eta^j.$$

Recall that we have an embedding $J_{ys, y} : M(ys\lambda) \to M(y\lambda)$ and by 3.6 (i) that $\theta M(y\lambda) = \theta(J_{ys, y} M(ys\lambda))$. By functoriality the diagram:

$$\begin{array}{ccc} M(y\lambda) & \xrightarrow{I'_{M(y\lambda)}} & \theta M(y\lambda) \\ {\scriptstyle J_{ys, y}}\big\uparrow & & \big\uparrow{\scriptstyle \theta J_{ys, y}} \\ M(ys\lambda) & \xrightarrow[I'_{M(ys\lambda)}]{} & \theta M(ys\lambda) \end{array}$$

commutes and as in 3.13 one checks that both maps I′ are injective. Apply the functor $\mathrm{Ext}^j(-, \mathrm{M}(w\lambda))$. Again take $\mathrm{M}=\mathrm{M}(w\lambda)$, $\mathrm{N}=\mathrm{M}(ys\lambda)$ in 5.1.8. Together these give the commuting diagram:

$$\begin{tikzcd}
\mathrm{Ext}^j(\mathrm{M}(ys\lambda), \theta\,\mathrm{M}(w\lambda)) \arrow[r, "\eta_j"] \arrow[d, "\mathrm{J}(\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda))"] & \mathrm{Ext}^j(\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda)) \arrow[d, "\mathrm{Id}"] \\
\mathrm{Ext}^j(\theta\,\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda)) \arrow[r] & \mathrm{Ext}^j(\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda)) \\
\mathrm{Ext}^j(\theta\,\mathrm{M}(y\lambda), \mathrm{M}(w\lambda)) \arrow[r, "\zeta^j"] \arrow[u, "\theta\,\mathrm{J}_{ys,y}"] & \mathrm{Ext}^j(\mathrm{M}(y\lambda), \mathrm{M}(w\lambda)) \arrow[u, "\mathrm{J}_{ys,y}"]
\end{tikzcd}$$

Up to identifications defined by the isomorphisms $\mathrm{J}(\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda))$, $\theta\,\mathrm{J}_{ys,\,y}$, Id we have $\mathrm{J}_{ys,\,y}\zeta^j=\eta^j$. Hence the Lemma.

5.2.3. Corollary. — *For all* $j\in\mathbb{N}$ *one has:*

$$\begin{aligned}\dim \mathrm{Ext}^{j+1}(\mathrm{M}(ys\lambda), \mathrm{M}(ws\lambda)) - \dim \mathrm{Ext}^j(\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda))&\\ \geqq \dim \mathrm{Ext}^{j+1}(\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda)) - \dim \mathrm{Ext}^j(\mathrm{M}(y\lambda), \mathrm{M}(w\lambda)),&\end{aligned}$$

with equality if and only if $\operatorname{Ker}\zeta^j=\operatorname{Ker}\eta^j$, $\operatorname{Ker}\zeta^{j+1}=\operatorname{Ker}\eta^{j+1}$.

Applying the functor $\mathrm{Ext}^j(-, \mathrm{M}(w\lambda))$ to (ζ) (with $\delta=0$) gives:

$$\dim \mathrm{Ext}^{j+1}(\mathrm{M}(ys\lambda), \mathrm{M}(w\lambda)) - \dim \mathrm{Ext}^j(\mathrm{M}(y\lambda), \mathrm{M}(w\lambda)) = \dim \operatorname{Ker}\zeta^{j+1} - \dim \operatorname{Im}\zeta^j.$$

Applying the functor $\mathrm{Ext}^j(\mathrm{M}(ys\lambda), -)$ to (η) (with $\delta=0$) gives:

$$\begin{aligned}\dim \mathrm{Ext}^{j+1}(\mathrm{M}(ys\lambda), \mathrm{M}(ws\lambda)) - \dim \mathrm{Ext}^j(\mathrm{M}(ys\lambda), M(w\lambda))&\\ = \dim \operatorname{Ker}\eta^{j+1} - \dim \operatorname{Im}\eta^j.&\end{aligned}$$

Since ζ^j, η^j act on isomorphic modules we have:

$$\dim \operatorname{Im}\eta^j + \dim \operatorname{Ker}\eta^j = \dim \operatorname{Im}\zeta^j + \dim \operatorname{Ker}\zeta^j$$

and so by 5.2.2 the required assertion.

5.2.4. By the choice of w, y, s one has $ys\leqq w \Leftrightarrow ys\leqq ws$, so by [6], 7.6.23, 7.1.8, the validity of 5.2.3 extends to $j=-1$ if we set $\mathrm{Ext}^{-1}=0$. We define:

$$\mathrm{R}'_{x,\,y}(q)=\sum_{j=0}^{\infty} q^j(-1)^{l(y)-l(x)-j}\dim \mathrm{Ext}^j(\mathrm{M}(x\lambda), \mathrm{M}(y\lambda)).$$

Lemma. — *The expression* $\mathrm{R}'_{x,\,y}(q)$ *satisfies* 2.1 (i) *and the first relation in* 2.1 (ii). *The second relation is equivalent to equality in* 5.2.3.

By say 1.5.8 one has:

$$(\star)\qquad \operatorname{Ext}^j(\mathrm{M}(x\lambda),\ \mathrm{M}(\lambda)) \cong \mathrm{H}^j(\mathfrak{n}^+,\ \mathrm{M}(\lambda))_{x\lambda-\rho} \cong \mathrm{H}^j(\mathfrak{n}^-,\ \delta\mathrm{M}(\lambda))^*_{x\lambda-\rho}.$$

Yet $\mathrm{M}(\lambda) \cong \mathrm{L}(\lambda)$ and so $\delta(\mathrm{M}(\lambda)) \cong \mathrm{M}(\lambda)$ which is $\mathfrak{n}^-$ free. Hence 2.1 (i). The first relation in 2.1 (ii) obtains from 5.2.1 and the second from equality in 5.2.3.

5.2.5. Now take δ regular and set (recalling 1.9.5):

$$\widetilde{\mathrm{R}}_{x,\,y}(q) = \sum_{j=-1}^{\infty} q^j (-1)^{l(y)-l(x)-j} \dim \operatorname{Ext}^j(\mathrm{M}(x\lambda+\delta t),\ \mathrm{M}(y\lambda+\delta t)),$$

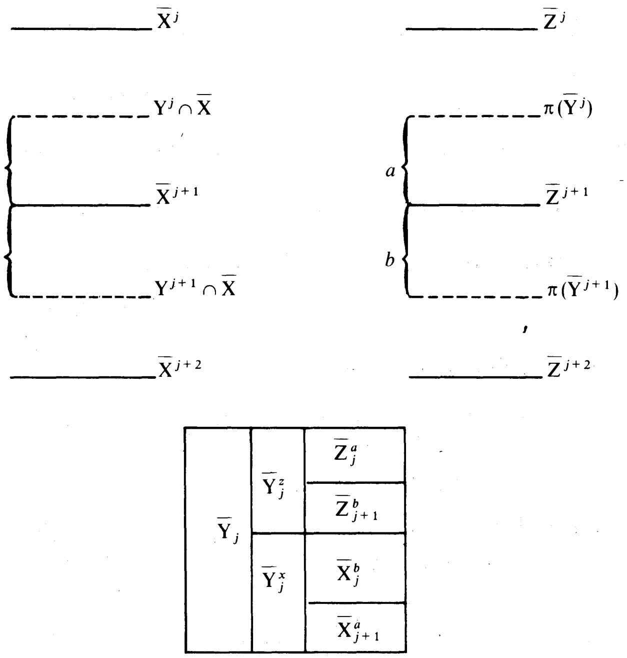
where we have formally defined:

$$\dim \operatorname{Ext}^{-1}(\mathrm{M}(x\lambda+\delta t),\ \mathrm{M}(y\lambda+\delta t)) = \begin{cases} 1, & x=y, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma. — *$\widetilde{\mathrm{R}}_{x,\,y}(q) = q^{-1}(q-1)\mathrm{R}'_{x,\,y}$, for all $x,\ y \in \mathrm{W}_\lambda$.*

Apply 1.9.5 (iv) with $\lambda_1 = x\lambda+\delta t$, $\lambda_2 = y\lambda+\delta t$.

5.2.6. We may regard 5.2.3 as a lower bound on $\dim \operatorname{Ext}^{j+1}(\mathrm{M}(ys\lambda),\ \mathrm{M}(ws\lambda))$ and ultimately a lower bound on $\operatorname{Ext}^{j+1}(\mathrm{M}(x\lambda),\ \mathrm{M}(w_\lambda\lambda))$. By (*) of 5.2.4, equality holds if and only if $\dim \mathrm{H}^j(\mathfrak{n}^+,\ \mathrm{M}(w_\lambda\lambda))$ equals precisely this lower bound.



Schematic presentation of the filtration of the modules $\mathrm{X} = \mathrm{M}(ws_\alpha\lambda+\delta t)$, $\mathrm{Z} = \mathrm{M}(w\lambda+\delta t)$ derived from the conclusion of Lemma 4.4 and of the four-step filtration of $\overline{\mathrm{Y}}_j$. The module Y is the extension of X by Z. Bar denotes specialization, superscripts (resp. subscripts) denote filtration (resp. gradation).

Index of notation

Symbols used frequently are given below in order of appearance. In addition, $\mathbb{N}^+$ denotes $\{1, 2, \ldots\}$ and $\mathbb{N}=\mathbb{N}^+ \cup \{0\}$.

0. $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{h}^*$, R, B, $M(\lambda)$, $L(\lambda)$, s_α, S, W, w_0.

1.2.1. σ, X_α, H_α, R^+, $\mathfrak{g}_{\mathbb{Z}}$.

1.2.2. A, $\mathfrak{a}_A$, $U(\mathfrak{a}_A)$, $Z(\mathfrak{a}_A)$, $S(\mathfrak{a}_A)$.

1.2.3. ρ, A_λ, $M(\lambda)$, $v_{\lambda-\rho}$.

1.2.4. $Q(R)$, $P(R)$.

1.2.5. M_μ, $\Omega(M)$.

1.3.1. P, P′.

1.3.5. e_λ, χ_λ.

1.3.6. $\mathscr{F}_\lambda$.

1.4.1. K''_C, K'_C, K_C.

1.4.5. $Q(\mu)$.

1.5.1. $K''_C(\mathfrak{b})$, $Q_{\mathfrak{b}}(\mu)$.

1.5.3. $Q^s(\mu)$.

1.7.4. K'_D, K_D.

1.8.1. k, $R_{\bar{\mu}}$, $W_{\bar{\mu}}$.

1.8.6. F_D.

1.8.8. $Q(\lambda, D)$.

1.10.1. $[M : M(\mu)]$.

1.10.2. E_A.

1.10.3 θ_E.

1.10.5 $p(M, \mathscr{F})$.

1.10.6. $p(M)$.

2.1. $R_{x, y}$.

2.3. $P_{x, y}$.

3.2. R^+_μ, B_μ, w_μ.

3.3. $E(\nu)$, ψ_α, φ_α, θ_α.

3.6. X, Y, Z.

3.9. $P(w\lambda+\delta t)$.

3.10. $\delta(M)$.

3.11. $(\theta_\alpha L(w\lambda))'$, $U_\alpha L(w\lambda)$.

3.12. I'_M, I''_M.

3.14. $J_{w, w'}$.

4.2. $M(w\lambda)^j$, $M(w\lambda)_j$.

REFERENCES

[1] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Category of $\mathfrak{g}$ Modules* (*Funct. Anal. Priloz.*, Vol.10, 1976, pp. 1-8).

[2] I. N. Bernstein and S. I. Gelfand, *Tensor Products of Finite and Infinite Dimensional Representations of Semisimple Lie Algebras* (*Compos. Math.*, Vol. 41, 1980, pp. 245-285).

[3] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 4-6, Éléments de mathématiques, XXXIV, Hermann, Paris, 1968.

[4] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton, New Jersey, 1976.

[5] P. Delorme, *Extensions dans la catégorie $\mathcal{O}$ de Bernstein-Gelfand-Gelfand. Applications*, preprint, Paris, 1978.

[6] J. Dixmier, *Algèbres enveloppantes*, cahiers scientifiques, XXXVII, Gauthier-Villars, Paris, 1974.

[7] O. Gabber and A. Joseph, *The Bernstein-Gelfand-Gelfand Resolution and the Duflo Sum Formula* (*Compos. Math.*, Vol. 43, 1981, pp. 107-131).

[8] J.-C. Jantzen, *Zur Charackterformel gewisser Darstellungen halbeinfacher Gruppen und Lie-Algebren* (*Math. Z.*, Vol. 140, 1974, pp. 127-149).

[9] J.-C. Jantzen, *Moduln mit einem höchsten Gewicht*, LN 750, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[10] D. A. Kazhdan and G. Lusztig, *Representations of Coxeter Groups and Hecke Algebras* (*Invent. Math.*, Vol. 53, 1979, pp. 165-184).

[11] D. A. Kazhdan and G. Lusztig, *Schubert Varieties and Poincaré Duality*, preprint, Harvard, 1979.

[12] D. Vogan, *Irreducible Characters of Semisimple Lie Groups I* (*Duke Math. J.*, Vol. 46, 1979, pp. 61-108).

[13] D. VOGAN, *Irreducible Characters of Semisimple Lie Groups II. The Kazhdan-Lusztig conjectures* (*Duke Math. J.*, Vol. 46, 1979, pp. 805-859).

[14] D. VOGAN, *Ordering in the Primitive Spectrum of a Semisimple Lie Algebra* (*Math. Ann.*, Vol. 248, 1980, pp. 195-203).

[15] A. A. BEILINSON and I. N. BERNSTEIN, *C. R. Acad. Sc.* (to appear).

[16] J.-L. BRYLINSKI and M. KASHIWARA, *Démonstration de la conjecture de Kazhdan-Lusztig sur les modules de Verma* [*C. R. Acad. Sc.* (to appear)].

[17] J.-L. BRYLINSKI and M. KASHIWARA, *Kazhdan-Lusztig Conjecture and Holonomic Systems* [*Invent. Math.* (to appear)].

[18] S. GELFAND and R. MACPHERSON, *Verma Modules and Schubert cells*, Preprint, I.H.E.S., November 1980.

[19] A. JOSEPH. *Goldie Rank in the Enveloping Algebra of a Semisimple Lie Algebra*, III [*J. Alg.* (in the press)].

(Manuscrit reçu le 28 septembre 1980,
révisé le 20 février 1981.)

O. GABBER,
Department of mathematical Sciences,
Tel-Aviv University,
Ramat-Aviv,
Tel-Aviv, Israël;

A. JOSEPH,
Université de Paris-VI,
U.E.R. de Mathématiques,
4, place Jussieu
75230 Paris Cedex 05.