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Representations of algebraic groups in prime characteristics

REPRESENTATIONS OF ALGEBRAIC GROUPS
IN PRIME CHARACTERISTICS (¹)

By George R. KEMPF

Let G be a connected reductive group over an algebraically closed field k of prime characteristic p. Building on the work of Curtis [6], Steinberg ([4], [12]) developed the initial theory of the (rational) irreducible representations of G. The special representations [11], which bear his name, have played a critical role in later developments. The Steinberg representations were used by Haboush [7] to prove Mumford's conjecture. More recently Haboush [8] and Andersen [2] have discovered that a very important role in the study of the cohomology of sheaves on homogeneous space is played by them.

In this paper I will rederive (with all deliberate hindsight) the basic Theorems of Steinberg's theory together with the above beautiful recent results, which motivated this work. The central Theorem 3.1 is proven by a global to infinitesimal argument. Grothendieck's group schemes are systematically exploited in connection with purely inseparable morphisms. I hope this paper will make the current frontier of research more accessible to fresh recruits.

I will use the notations and definitions contained in [10]. Groups schemes are assumed to be affine.

1. The main Lemma

We will work entirely in the category of k-schemes of finite type. Points are assumed to be k-rational. If X is scheme, $k[X]$ denotes the k-algebra of global sections of the structure sheaf $\mathcal{O}_X$. Varieties will be irreducible.

Given any variety X and positive integer n, we may define another variety X_n together with a purely inseparable morphism $F_n : X \to X_n$. As topological spaces, X and X_n are the same, but locally the regular functions on X_n are the p^n-th powers of regular functions on X. For any point x of X, the inverse image $X_{qx} \equiv F_n^{-1}(F_n(x))$ is a closed subscheme of X, which is concentrated at x, where $q = p^n$. So X_{qx} is the q-th order thickening of X along x as the ideal of X_{qx} in $\mathcal{O}_X$ is the ideal (ϑ^q) generated by q-th powers of elements of the ideal ϑ of x.

If H is a group variety, then $F_n : H \to H_n$ is a homomorphism of algebraic groups. In particular, its kernel $_nH \equiv F_n^{-1}(e) = H_{qe}$ is a group scheme with only one point, where $_nH$ is a characteristic subgroup scheme of H. Furthermore we have an induced isomorphism $H/_nH \approx H_n$, where $H/_nH$ is the quotient group scheme. Thus the regular functions on H_n are exactly the regular functions on H, which are invariant under the translation action of $_nH$. Explicitly:

$$k[H_n] = \{ f \in k[H] \mid \alpha^* f = \pi^* f \text{ in } k[_nH \times H] \},$$

where $\alpha : {}_nH \times H \to H$ is the translation action and $\pi : {}_nH \times H \to H$ is the projection.

More generally let M be any (algebraic) representation of a group scheme K. Thus M is a left-comodule over the Hopf algebra $k[K]$, where the coaction $M \to k[K] \otimes M$ will be denoted by α_M^*. An element m of M is called invariant under K if $\alpha_M^*(m) = 1 \otimes m$. A representation is called trivial if it consists entirely of invariant elements.

A group scheme U is unipotent if it is isomorphic to a closed subgroup scheme of the group U_m of strictly upper-triangular $m \times m$ matrices by some m. Trivially a closed subgroup scheme of a unipotent group is unipotent. The main tool in this paper is a direct generalization of the Engel-Kolchin theorem. I will give its proof although it should be well-known.

THEOREM 1.1. — *Let* U *be a group scheme. The following statements are equivalent:*

(a) U *is unipotent;*

(b) *any irreducible representation of* U *is trivial;*

(c) *any non-zero representation of* U *possesses a non-zero invariant element;*

(d) *there is an increasing sequence of* U-*subspaces* $0 \equiv V_0 \subset V_1 \subset \ldots \subset k[U]$ *of the left regular representation such that* $k[U] = \bigcup V_i$ *and the quotient representations of* U *on* V_{i+1}/V_i *are trivial.*

Proof. — $(b) \Rightarrow (c)$. By Cartier's Lemma, any representation M is the union of finite dimensional subrepresentations. Thus to prove (c) we may assume that the given representation is finite dimensional and, hence, possesses an irreducible subrepresentation. From here the implication is evident.

$(c) \Rightarrow (d)$. We will define the V_i's successively. Let V_{i+1} be the inverse image of the maximal trivial submodule of $k[U]/V_i$. To show that the V_i's exhaust $k[U]$, by Cartier's Lemma we need to show that any finite dimensional subrepresentation $N \subset k[U]$ is contained in some V_i. By applying (c) to the image of N in $k[U]/V_i$, we see that $N \subset V_{\dim N}$.

$(d) \Rightarrow (b)$. Let M be an irreducible representation. Let L be a non-zero linear functional on M. Then the composition:

$$\psi : \quad M \xrightarrow{\alpha_M} k[U] \otimes M \xrightarrow{1 \times L} k[U] \otimes k \approx k[U]$$

is a non-zero homomorphism of U-modules from M to the left regular representation. As M is irreducible, ψ is injective. Better yet, if (d) is true, M is isomorphic to a submodule of the quotient V_i/V_{i-1} for some i. Thus M is trivial as the quotient is a trivial U-module.

$(a) \Rightarrow (d)$. Assume that U is a closed subgroup scheme of the triangular group $\mathbf{U}_m$. An explicit calculation involving polynomials in matrix coefficients shows that the statement (d) is true for $\mathbf{U}_m$. As the restriction $k[\mathbf{U}_m] \to k[\mathbf{U}]$ is a surjective U-module homomorphism, the subgroup U inherits the property (d) from $\mathbf{U}_m$.

$(c) \Rightarrow (a)$. Let M be a faithful finite dimensional representation of U. By (c) and an evident induction, we may find a complete flag $M_1 \subset M_2 \subset \ldots \subset M_m = M$ of U-subspaces such that the action of U on the quotients M_i/M_{i-1} are trivial. Choose a basis $m_1, \ldots, m_m$ of M such that each M_i is spanned by $m_1, \ldots, m_i$. Let $\alpha_M(m_i) = \sum \alpha_i^j \otimes m_j$. Thus the matrix (α_i^j) of functions on U defines a closed embedding $U \subset \mathbf{U}_m$.

Q.E.D.

Now we are in a position to understand the main result of this section.

LEMMA 1.2. − *Let* U *be a unipotent group variety. Let* W *be a subspace of* $k[U]$ *which is a sub-representation as a* $_nU$-*module. We have two mutually exclusive possibilities;*

(a) *the restriction* $W \to k[_nU]$ *is injective, and*

(b) W *contains an element of the form* f^q *for some non-zero regular function* f *on* G, *which vanishes at the identity, where* $q = p^n$.

Proof. − Assume that (b) is true. As f vanishes at the identify, f^q vanishes at the identify, f^q vanishes when it is restricted to the subscheme $_nU$. As $0 \neq f^q \in W$, (a) is false.

Conversely, assume that (a) is false. Let K be the kernel of the whole restriction $r : k[U] \to k[_nU]$. By assumption, the intersection $W \cap K$ is not zero. Furthermore, because r is a homomorphism of $_nU$-modules, K and, hence, $W \cap K$ are $_nU$-submodules. As U is unipotent, its subgroup scheme $_nU$ is also unipotent. Therefore, by (c) of Theorem 1.1, we may find a non-zero regular function g in $W \cap K$, which is invariant under $_nU$. As g is contained in K, g has to vanish at the identity.

As g is invariant under $_nU$, g is a regular function on the quotient $U_n = U/_nU$. By the definition of U_n, g has the form f^q for some regular function f on U. Now f vanishes at the identity and is non-zero as with g. By construction, $f^q = g \in W$. Hence (b) is true.

Q.E.D.

Remark. − The kernel K of the restriction consists of the regular functions on U, whose terms of degree $< q$ in their Taylor series at the identity vanishes. The restriction $W \to k[_nU]$ being injective means that an element of W is determined by certain terms of low degree in its Taylor series expansion at the identity.

2. The injectivity Theorem

Let U be a unipotent group variety. First we will consider one procedure for producing $_nU$-subrepresentations W of $k[U]$ geometrically. The basic idea is to look for all regular functions on U that satisfy some type of linear $_nU$-invariant conditions on their behaviour at infinity. Although we will need only a very concrete form of this idea, I will present it in a general form at first.

Let $f : U \to X$ be an arbitrary continuous mapping to a topological space X. As the structure sheaf $\mathcal{O}_U$ is a sheaf of $_nU$-modules, its direct image $f_* \mathcal{O}_U$ is naturally a sheaf of $_nU$-modules on X. Let $\mathscr{W}$ be a sheaf of $_nU$-modules which is contained in $f_* \mathcal{O}_U$. Then $W = \Gamma(X, \mathscr{W})$ is a $_nU$-submodule of $\Gamma(X, f_* \mathcal{O}_U) \approx \Gamma(U, \mathcal{O}_U) = k[U]$.

Specializing further consider the case where $f : U \to X$ is a U-equivariant open embedding. This means that X is a variety with a morphic U-action and f is an open immersion such that $f(u_1 u_2) = u_1 f(u_2)$. In this case $f_* \mathcal{O}_U$ is just the sheaf of rational functions on X, which are locally regular on U. In this algebro-geometric situation, $f_* \mathcal{O}_U$ possesses a natural U-linearization. One naturally examines at first the coherent U-linearized subsheaves $\mathscr{W}$ of $f_* \mathcal{O}_U$. Next we will discuss the most obvious type of such sheaves for simplicity.

Consider the case where X is a smooth variety and the complement $X - U$ consists of a finite number of irreducible divisors D_i for i in an index set I. As U is irreducible, each of the divisors D_i are invariant under U. For any integral-valued function m on I, we have an invertible sheaf $\mathcal{O}_X(\sum m_i D_i)$ on X. As the divisors are invariant, this sheaf possesses a natural U-linearization extending the U-linearization of its restriction $\mathcal{O}_U$ to U. Thus the space $W(m) \equiv \Gamma(X, \mathcal{O}_X(\sum m_i D_i))$ is a U-subrepresentation of $k[U]$.

With these notations we may proceed to:

PROPOSITION 2.1. — *We have two mutually exclusive possibilities;*

(a) *the evaluation mapping* $W(m) \to k[_nU]$ *is injective, and*

(b) *there is a non-zero regular function f on U, which vanishes at the identity, such that* $q \cdot \mathrm{ord}_{D_i}(f) + m_i \geqq 0$ *for all i.*

Proof. — This is straightforward translation of Lemma 1.2 as the condition on the order of f along the divisors D_i is equivalent to f^q being contained in $W(m)$.

                                                                                        Q.E.D.

Before I continue, I want to make a clarifying remark about the space $W(m)$.

LEMMA 2.2. — (a) *the space $W(m)$ is not zero $\Leftrightarrow W(m)$ contains the constant functions $\Leftrightarrow$ each m_i is not negative;*

(b) *the space $W(m)$ contains a non-zero function which vanishes at the identity $\Leftrightarrow k \subsetneq W(m)$.*

Proof. — For (a), the second equivalence follows directly from the definition of $W(m)$. In the first equivalence, the implication $\Rightarrow$ is evident. Conversely, if $W(m)$ is not zero, then $W(m)$ contains a non-zero function which is invariant under U by Theorem 1.1. As such a function must be a non-zero constant, $W(m)$ contains the constants. So (a) is true.

For (b), clearly we have the implication $\Rightarrow$. Conversely, if f is a non-constant function in $W(m)$, $f - f$ (identity) is a non-zero function in $W(m)$, which vanishes at the identity, as $k \subset W(m)$. So (b) is true.

                                                                                        Q.E.D.

With this result, we may understand a less ambiguous consequence of the last Theorem.

THEOREM 2.3. — *Assume that $\Gamma(X, \mathcal{O}_X) = k$ and $m_i < q$ for all i. Then the restriction $W(m) \to k[_nU]$ is injective.*

Proof. — By Theorem 2.1, we need to show that there is no non-zero regular function f on U, which vanishes at the identity and satisfies $q \cdot \mathrm{ord}_{D_i}(f) + m_i \geq 0$ for each i. Assume that f were such a function. Let n be the function on the index set I given by $n_i = -\mathrm{ord}_{D_i}(f)$. By definition, $f \in W(n)$. As $f \neq 0$, by Lemma 2.2 (a), each $n_i \geq 0$. Thus the above system of inequalities becomes $m_i \geq q n_i$ for all i. By the assumption on m, we must have $0 = n_i$ for all i; i.e., $n = 0$. As $\Gamma(X, \mathcal{O}_X) = W(0) = W(n)$, f must be constant by the other assumption. As a non-zero constant function cannot vanish at the identity, we may conclude that no such f exists.

Q.E.D.

Let $\mathcal{L}$ be an arbitrary invertible sheaf on X. As $k[U]$ is a unique factorization domain, $\mathcal{L}|_U \approx \mathcal{O}_U$. As the units in $k[U]$ are constants, this isomorphism is unique up to constant multiple. Thus we have an isomorphism $\mathcal{L} \approx \mathcal{O}_X(\sum m_i D_i)$ for some uniquely determined function m. The integer m_i will be called the moment, $\mathrm{mo}(\mathcal{L}, D_i)$, of $\mathcal{L}$ along the divisor D_i. With these definitions, we can obviously reformulate the last theorem as:

THEOREM 2.4. — *Assume that* $\Gamma(X, \mathcal{O}_X) = k$ *and* $\mathrm{mo}(\mathcal{L}, D_i) < q$ *for all* i. *Then the restriction:*

$$\Gamma(X, \mathcal{L}) \to \Gamma(X_{qx}, \mathcal{L}|_{X_{qx}}) \quad \textit{is injective,}$$

where x is any point of X which is contained in the dense U-orbit.

3. Applications to reductive groups

Let G be a reductive group variety. Let f be a point of a complete homogeneous space X of the form G/P, where P is a parabolic subgroup of G. Let T be a maximal torus contained in the stabilizer P_f of f. Recall (in the notation of [9]) that there is a unipotent group variety U_f^- normalized by T such that the morphism $i : U_f^- \to X$ given by $i(u) = u.f$ is a U_f^--equivariant open embedding. Furthermore, as X is complete, an everywhere regular function on X must be constant. Hence the assumption $k = \Gamma(X, \mathcal{O}_X)$ of Theorem 2.4 is verified in this situation.

Given an invertible sheaf $\mathcal{L}$ on X, one defines a weight $p_f(\mathcal{L})$ of the root system of G with respect to T. This process gives an isomorphism between the group of $\mathcal{L}$'s upto isomorphism and the group of weights which are invariant under the subgroup W_f of the Weyl group W that fixes f. The irreducible divisors D_i correspond to fundamental weights $\bar{\omega}_i$. In terms of weights, the moments are determined by the equation $p_f(\mathcal{L}) = \sum \mathrm{mo}(\mathcal{L}, D_i) \bar{\omega}_i$. A weight $\sum m_i \bar{\omega}_i$ is said to be dominant if all of its moments m_i are non-negative. By Lemma 2.2, $p_f(\mathcal{L})$ is a dominant weight if and only if $\mathcal{L}$ possesses a non-zero global section.

We will work with G-linearized invertible sheaves to avoid the minor difficulties of projective linearizations and representations. Let χ be a character of the parabolic group P_f. We have a G-linearized invertible sheaf $\mathcal{L}_f(\chi)$ whose value $\mathcal{L}_f(\chi)|_f$ at f is the ground field k on which P_f acts by the character χ. The weight $p(\chi)$ of the sheaf $\mathcal{L}_f(\chi)$ is easily

computed in terms of the action of the Weyl group on the restriction of χ to T. The space $M_f(\chi)$ of global sections of $\mathscr{L}_f(\chi)$ is naturally a G-module which is called an induced representation.

Similarly we will denote the space $\Gamma(X_{q.f}, \mathscr{L}_f(\chi)|_{X_{q.f}})$ by $M_f^q(\chi)$. We want to study the group-theoretic properties of the restriction homomorphism $\rho : M_f(\chi) \to M_f^q(\chi)$. Let $P_{q.f}$ be the subgroup scheme of G, which is the stabilizer of the subscheme $X_{q.f}$ of X. Before I discuss $P_{q.f}$ in detail, at least $P_{q.f}$ acts on $X_{q.f}$ morphicly. Furthermore, the G-linearization of $\mathscr{L}_f(\chi)$ induces a natural $P_{q.f}$-linearization of $\mathscr{L}_f(\chi)|_{X_{q.f}}$. Taking global sections, we see that the restriction ρ is a homomorphism of $P_{q.f}$-modules.

Summarizing what we have proven so far about induced representations, we now know the truth of:

THEOREM 3.1. — *If the moments of the weight $p(\chi)$ are all strictly less than q, then the restriction $\rho : M_f(\chi) \to M_f^q(\chi)$ is an injective homomorphism of $P_{q.f}$-modules.*

To put our picture of the group scheme $P_{q.f}$ in proper perspective, we need to examine the inseparable morphism $F_n : X \to X_n$. As X is a homogeneous space under G of the form G/P for some parabolic subgroup P of G, X_n is a homogeneous space under G_n of the form G_n/P_n for the parabolic subgroup P_n of G_n. Furthermore the morphism $F_n : X \to X_n$ is equivariant with respect to the homomorphism $F_n : G \to G_n$ of groups [i.e., $F_n(g.x) = F_n(g).F_n(x)$]. If we regard X_n as a G-variety *via* the action of G_n on X_n composed with the homomorphism, the morphism is G-equivariant and X_n is a homogeneous space under G of the form $G/F_n^{-1} P_n$, where $F_n^{-1} P_n$ is the scheme-theoretic inverse image of P_n. Thus the fiber of the morphism is a homogeneous space under $F_n^{-1} P_n$ of the form $(F_n^{-1} P_n)/P_n$.

With these facts in mind, we see that the stabilizer $P_{q.f}$ of $X_{q.f}$ should be regarded as the closed subgroup scheme $F_n^{-1}(F_n(P_f)) = F_n^{-1}((P_f)_n)$ of G. In other words $P_{q.f}$ is the q-th order infinitesimal thickening of P_f in G. Furthermore $X_{q.f}$ should be regarded as the homogeneous space $P_{q.f}/P_f$ using the action of $P_{q.f}$ on the point f. This quotient representation actually gives much information.

By the big cell Theorem, multiplication in G gives an open immersion $U_f^- \times P_f \to G$. Consequently, multiplication induces two compatible isomorphisms of schemes $_nU_f^- \times _nP_f \approx _nG$ and $_nU_f^- \times P_f \approx P_{q.f}$. Hence we have natural isomorphisms $P_{q.f}/P_f \approx _nG/_nP_f \approx _nU_f^-$. Therefore we may regard $X_{q.f}$ as the homogeneous space $_nG/_nP_f$ under the action of $_nG$ or as the principal homogeneous space $_nU_f^-$ under the action of $_nU_f^-$. The first representation is of course equivariant under the action of P_f, which acts on $X_{q.x}$ by translation and on $_nG/_nP_f$ *via* the conjugation action by P_f on $_nG$. The isomorphism $_nU_f^- \approx X_{q.f}$ is equivariant under the Levi factor L_f (and, hence, T) of P_f with respect to T, which operates on $_nU_f^-$ by conjugation.

Thus we may regard the space $M_f^q(\chi)$ as the representation of $P_{q.f}$ (or $_nG$) induced by the restriction of χ to its closed subgroup scheme P_f (or $_nP_f$). In the alternative case, we may simply call $M_f^q(\chi)$ the infinitesimally induced representation of $_nG$.

4. More applications

The infinitesimally induced representations $M_f^q(\chi)$ are very concrete objects and they have some properties analogous to those of the globally induced representations $M_f(\chi)$. First I will mention some of these properties. Then I will show how the injectivity Theorem is related to Steinberg's Theorem concerning the infinitesimal irreducibility of certain irreducible representations of G.

Recall that, if the weight $p(\chi)$ of χ is dominant, the induced representation $M_f(\chi)$ has a canonical element $\psi_f(\chi)$, which is invariant under U_f^- and has value 1 in $\mathscr{L}(\chi)|_f = k(\chi)$. Furthermore $\psi_f(\chi)$ spans the space of all U_f^--invariants in $M_f(\chi)$ and $\psi_f(\chi)$ is a T (or even L_f) eigenvector of weight equal to the restriction χ' of χ to T. Even if we drop the dominant assumption, we have a section, say $\psi_f(\chi)$ again, in $\Gamma(U_f^- \cdot f, \mathscr{L}_f(\chi))$ with similar properties.

Let $\psi_f^q(\chi)$ denote the image of $\psi_f(\chi)$ in $\Gamma(X_{q.f}, \mathscr{L}_f(\chi)|_{X_{q.f}}) = M_f^q(\chi)$. Then ψ_f^q is $_n U_f^-$-invariant and we may write any element of $M_f^q(\chi)$ uniquely in the form $g \cdot \psi_f^q(\chi)$, where g is an element of $k[X_{q.f}] = M_f^q(1)$. As $\psi_f^q(\chi)$ is $_n U_f^-$-invariant and $X_{q.f} \approx {}_n U_f^-$, we have a natural isomorphism of $_n U_f^-$-modules between $M_f^q(\chi)$ and the left-regular representation on $k[{}_n U_f^-]$. Formally we have:

LEMMA 4.1. — (a) the space of $_n U_f^-$-invariants in $M_f^q(\chi)$ is the line $k \psi_f^q(\chi)$;

(b) $k \psi_f^q(\chi)$ is also the χ'-weight space for the action of T on $M_f^q(\chi)$;

(c) the characteristic of $M_f^q(\chi)$ as a T-module is given by:

$$[\chi'] \cdot \Pi \left(\sum_{0 \leq i < q} [\alpha^i] \right) = [\chi'] \cdot \Pi \left(\frac{[1] - [\alpha]^q}{[1] - [\alpha]} \right),$$

where the products are both taken for all roots α lying in U_f^-.

Proof. — For (a), the only $_n U_f^-$-invariants in its left regular representation are the constants. In terms of the isomorphism $M_f^q(\chi) \approx k[{}_n U_f^-]$, this means that $k \psi_f^q(\chi)$ is the space of invariants in $M_f^q(\chi)$.

For (b), as U_f^- is the unipotent radical of the parabolic subgroup $L_f \cdot U_f^-$ of G, the set $R(U_f^-)$ of roots in U_f^- all lie in an open half space in the usual additive picture of the root system. Thus the trivial character 1 of T is not a product $\Pi \, \alpha^{m_\alpha}$ with α in $R(U_f^-)$ and positive exponents m_α. Therefore (b) follows directly from the formula in (c).

For (c), let $k(\chi)$ denote the submodule of $k[T]$ spanned by a character χ of T. As $\psi_f(\chi)$ is a T-eigenvector of weight χ', the above isomorphism actually gives a T-isomorphism $M_f^q(\chi) \approx k[{}_n U_f^-] \otimes_k k(\chi')$ where T acts on $k[{}_n U_f^-]$ via conjugation. Therefore to prove (c) it will suffice to determine the characteristic of $k[{}_n U_f^-]$. Recalling that $k[U_f^-]$ is T-isomorphic to the symmetric algebra on the T-module $K \equiv \oplus k(\alpha)$ for α in $R(U_f^-)$, where K generates the ideal of the identity. Thus from the definition of $_n U_f^-$, $k[{}_n U_k^-]$ is T-isomorphic to Sym $K/(K^q) \approx \underset{0 \leq i < q}{\oplus} \text{Sym}^i K \approx \otimes (\underset{1 \leq i < q}{\oplus} k(\alpha^i))$. Hence (c) is evident.

Q.E.D.

For a representation $M_f(\chi)$ globally induced from character χ of dominant weight, the fact that it has a unique U_f^--invariant line together with Theorem 1.1 implies the following two facts; $M_f(\chi)$ is an indecomposable U_f^--module and $M_f(\chi)$ possesses a unique irreducible G-submodule $N_f(\chi)$. The same arguments apply [cf. Lemma 4.1 (a)] to the infinitesimal case for any χ. Thus $M_f^q(\chi)$ is an indecomposable $_nU_f^-$-module and $M_f^q(\chi)$ possesses a unique irreducible $_nG$-submodule $N_f^q(\chi)$. { Actually $N_f(\chi)$ [resp. $N_f^q(\chi)$] is the smallest U_f (resp. $_nU_f$)-module containing $\psi_f(\chi)$ [resp. $\psi_f^q(\chi)$], where U_f denotes the unipotent radical of P_f. }

We will discuss Steinberg's Theorem which relates these two kinds of induced representations.

THEOREM 4.2. — *Let χ be a character with a dominant weight which has moments strictly less than q. Then the restriction mapping induces an isomorphism $N_f(\chi) \to N_f^q(\chi)$. In other words, $N_f(\chi)$ remains irreducible when it is regarded as a $_nG$-module.*

Proof. — For any group scheme H and any element m of an H-module M, we may compute the minimal H-submodule $k\,H\,m$ of M containing m in a simple way. Write $\alpha_M^* m = \sum f_i \otimes m_i$ where the f_i's are linearly independent elements of $k[H]$ and the m_i's are linearly independent elements of M. Then the m_i's are a basis for the vector space $k\,H\,m$. In particular, if M is the left regular representation of H on $k[H]$, the m_i's (resp. f_i's) are a basis of the left (resp. right) H-submodule spanned by m in $k[H]$ (see [3]).

As usual we may identify the induced representation $N_f(\chi)$ with the G-subspace

$$\{ f \in k[G] \mid f(gp) = f(g)\chi(p) \text{ for all } g \text{ in } G \text{ and } p \text{ in } P_f \} = \{ f \in k[G] \mid \varkappa^* f = f . \chi \},$$

where $\varkappa^*$ is the cohomomorphism of the multiplication $\varkappa : G \times P_f \to G$, of the left regular representation. The analogous statement is true for the infinitesimally induced representation.

The element $\psi_f(\chi)$ in $N_f(\chi)$ is identified with a regular function $\chi \uparrow P_f$ on G, which satisfied a two-sided functional equation:

$$(\varkappa \uparrow P_f)(p'\,gp) = \chi_-(p') . \chi \uparrow P_f(g) . \chi(p),$$

for p' in $P_f^- = L_f . U_f^-$, g in G and p in P_f, where χ_- is the character of the opposite parabolic subgroup P_f^- with the same restriction as χ to the Levi subgroup $L_f = P_f \cap P_f^-$. Now write $\mu^*(\chi \uparrow P_f) = \sum f_i \otimes m_i$ where the f_i's (resp. m_i's) are linearly independent regular functions on G. By the above remarks, the m_i's are identified with a basis of the irreducible G-submodule $N_f(\chi)$ of $M_f(\chi)$. If we can prove that (†) the images $\bar{f}_i$'s (resp. $\bar{m}_i$'s) in $k[_nG]$ are still linearly independent, the $\bar{m}_i$'s are identified with a basis of the irreducible $_nG$-submodule $N_f^q(\chi)$ of $M_f^q(\chi)$. Therefore the statement (†) implies that the restriction gives an isomorphism $N_f(\chi) \approx N_f^q(\chi)$.

Thus it will suffice to prove the statement (†). By Theorem 3.1, the restriction $M_f(\chi) \to M_f^q(\chi)$ is injective. Hence the $\bar{m}_i$'s are linearly independent as the m_i's are. The

linear independence of the $\bar{f_i}$'s follows by the same reasoning using left-right symmetry. Explicitly we may identify the right G-subspace

$$\{ f \in k[G] \mid f(p'.g) = \chi_-(p') f(g) \}$$

with the space of sections of a right G-linearized invertible sheaf $\mathscr{L}$ on the right homogeneous space P_f^-/G. As the moments of $\mathscr{L}$ are the orders of $\chi \mid P_f(g)$ along the divisors in $G - U_f^- L_f U_f$, they are the same as the moments of $\mathscr{L}_f(\chi)$. Thus the assumptions of Theorem 3.1 are verified for $\mathscr{L}$ on $P_f^-\backslash G$. Hence the same reasoning shows that the $\bar{f_i}$'s are also linearly independent. Therefore (†) is true.

<div align="right">Q.E.D.</div>

Remark. — This Theorem shows that the original injectivity Theorem 2.4 has significant content. The Theorem 2.4 could be generalized to a statement about the sections of a G-linearized sheaf on G/P restricted to a Schubert variety (or modification of one), but at present the strength of such a generalization has not been fully tested.

5. Steinberg representations

We will first recall a general fact about finite group schemes. By definition a group scheme H is finite if $k[H]$ is a finite dimensional k-vector space (equivalently H has a finite number of points). A basic fact about finite group schemes is:

LEMMA 5.1. — *There is a unique hyperplane $L \subset k[H]$ such that L is a left-H-submodule and the induced representation on $k[H]/L$ is trivial.*

Proof. — This result can be found in Sweedler's book [13]. (When H is reduced, $L = \{ f \in k[H] \mid \sum_{h \in H} f(h) = 0 \}$.) I will give a sketch of Sweedler's proof in geometric language although in more classical language it involves convolution of linear functionals on $k[H]$. Let ω_H be the dualizing sheaf on H. By functoriality, ω_H is a H-linearized sheaf for the left action of H on itself. By Grothendieck's Theorem 90, we have a H-equivariant isomorphism $\omega_H \approx \mathcal{O}_H \otimes_k V$ for some k-vector space V. Thus $\Gamma(H, \omega_H) \approx k[H] \otimes_k V$ as H-linearized $k[H]$-modules and, hence, the space of H-invariants in $\Gamma(H, \omega_H) \approx k \otimes_k V$. By duality, $\Gamma(H, \omega_H) \approx \text{Hom}_k(k[H], k)$ and, hence, V must be one dimensional and the invariants $k \otimes V$ correspond to the linear functionals (i.e. integrals) defining such subspaces L in $k[H]$.

<div align="right">Q.E.D.</div>

Remark. — The hyperplane L is invariant under the right action of H (basically because the left and right actions commute). Thus an integral defining L must be a right eigenvector for H. Its weight corresponds to the multipliers in the functional equation for right translation of a left-invariant Haar measure. Integrals give a self-duality for $k[H]$. Perhaps this duality should be studied further when H is the Frobenius kernel $_nG$ of a reductive group G. We can now specialize the lemma to the unipotent case.

COROLLARY 5.2. — *If* H *is a finite unipotent group scheme:*

(a) *there is a unique hyperplane* L *in the (left) regular representation* $k[H]$;

(b) L *is the maximal proper* H-*submodule of* $k[H]$.

Proof. — By Theorem 1.1, any proper H-submodule of $k[H]$ is contained in some hyperplane L' of $k[H]$, which is an H-submodule. By Theorem 1.1 again, the line $k[H]/L'$ must be a trivial representation of H. By Lemma 5.1, the hyperplane L' is unique.

<div align="right">Q.E.D.</div>

We want to apply this Corollary when H is the previous unipotent group scheme $_nU_f^-$. In this case we can use the action of the torus T to give an explicit description of the situation. What we need to know is contained in:

LEMMA 5.3. — *The maximal proper* $_nU_f^-$-*submodule* L *of* $k[_nU_f^-]$ *is the* T-*subspace which is spanned by the eigenspaces of weight* $\neq \Pi \alpha^{q-1}$, *where* α *runs through the roots of* U_f^-.

Proof. — By Corollary 5.2 we need only see that the T-subspace in question is a $_nU_f^-$-submodule and has codimension one in $k[_nU_f^-]$. In Lemma 4.1, we have determined the weights occurring in $k[_nU_f^-]$ together with their multiplicity. In terms of the partial ordering of characters given by the roots $R[U_f^-]$ in U_f^-, the weight $\Pi \alpha^{q-1}$ for α in $R[U_f^-]$ is strictly higher than the other weight in $k[_nU_f^-]$. Therefore the T-subspace in question is a $_nU_f^-$-submodule of codimension equal to the multiplicity of $\Pi \alpha^{q-1}$, which we have determined to be one.

We can restate the last result in terms of the infinitesimally induced representations.

LEMMA 5.4. — *The infinitesimally induced representation* $M_f^q(\chi)$ *has a maximal* $_nU_f^-$-*subrepresentation which is the* T-*subspace complementary to the line of* T-*eigenvectors of weight* $\chi' \pi \alpha^{q-1}$ *where* α *runs through the roots in* U_f^-.

Proof. — Just use the isomorphism in the proof of Lemma 4.1 to deduce this result from the Lemma 5.3.

<div align="right">Q.E.D.</div>

Remark. — For the globally induced representations $M_f(\chi)$, the representation has a unique U_f^--submodule of codimension one if and only if $M_f(\chi)$ is irreducible. Thus the situation is not strictly analogous to $M_f^q(\chi)$. The correct analogy seems to be with U_f invariants in $M_f^q(\chi)$.

We are ready to discuss Steinberg representations. For the rest of this section, the homogeneous space will be assumed to have the form G/B where B is a Borel subgroup of G. The Steinberg character σ_q is by definition a character of T such that $\sigma_q^{-2} = \Pi \alpha^{q-1}$ where the product is taken over all roots α in U_f^-. Clearly the Steinberg character exists unless $p=2$ and in any case it is weight for the root system of G. It is well-known that σ_q is dominant and has moments equal to $q-1$. Furthermore, if w_0 is the longest symmetry in the Weyl group, then $w_0 . \sigma_q = \sigma_q^{-1}$. The Steinberg representation St_q is the globally

induced G-module $M_f(\sigma_q)$. The principal properties of Steinberg representations are listed in:

THEOREM 5.5. — (a) *the Steinberg representation* St_q *is an irreducible $_nG$-module and, hence, irreducible as a G-module;*

(b) *also* St_q *is naturally isomorphic to the infinitesimally induced P_f^q-module $M_f^q(\sigma_q)$;*

(c) *the characteristic of* St_q *as a T-module is* $[\sigma_q] \Pi ([1]-[\alpha^q])/([1]-[\alpha])$, *where* α *runs through the roots of* U_f^-.

Proof. — By the above remarks, the Theorem 3.1 applies to the Steinberg character σ_q. Thus we have two natural injections $N_f(\sigma_q) \subset M_f(\sigma_q) \equiv St_q \to M_f^q(\sigma_q)$. In particular we may regard $N_f(\sigma_q)$ as a $_nU_f^-$-subspace of $M_f^q(\sigma_q)$. To show that $N_f(\sigma_q) = M_f^q(\sigma_q)$ by Lemma 5.4, it will suffice to find a non-zero eigenvector in $N_f(\sigma_q)$ of weight $\sigma_q \Pi \alpha^{q-1}$ with α in $R(U_f^-)$. Now $N_f(\sigma_q)$ is a G-module with a non-zero eigenvector $\psi_f(\sigma_q)$ of weight σ_q. Consider the following element $\bar{w}_0^* \cdot \psi_f(\sigma_q)$ of $N_f(\sigma_q)$ where $\bar{w}_0$ is an element of $N(T)$ representing w_0. Clearly $\bar{w}_0^* \cdot \psi_f(\sigma_q)$ is non-zero and its weight is $w_0 \cdot \sigma_q$ which equals $\sigma_q^{-1} = \sigma_q \cdot (\Pi \alpha^{q-1})$. Therefore we have found the desired element and, hence, $N_f(\sigma_q) = M_f^q(\sigma_q) = St_q$.

By Theorem 4.2, $N_f(\sigma_q) = N_f^q(\sigma_q)$. Therefore, as $N_f^q(\sigma_q)$ is an irreducible $_nG$-module, St_q is $_nG$-irreducible. So (a) is true. Also $St_q = M_f^q(\chi)$ and, hence, (b) is true. Part (c) follows directly from (b) by Lemma 4.1 (c).

<div align="right">Q.E.D.</div>

Steinberg has given a partial converse. We will sketch its proof for completeness.

PROPOSITION 5.6. — *Let* χ *be a dominant character which has all its moments strictly less than* q. *If the infinitesimally induced representation* $M_f^q(\chi)$ *is an irreducible $_nG$-module, then* χ *is the Steinberg character* σ_q.

Proof. — In other words, $N_f^q(\chi) = M_f^q(\chi)$. By Theorem 4.2, $N_f(\chi) \approx N_f^q(\chi)$. Thus $N_f(\chi) \approx M_f^q(\chi)$ and we know the characteristic of $N_f(\chi)$ as a T-module by Lemma 4.1 (c). From this and highest weight considerations, we may conclude that $N_f(\chi)$ contains a non-zero U_f-invariant of weight $\chi' \Pi \alpha^{q-1}$. As such an invariant is a constant times $\bar{w}_0^* \psi_f(\chi)$, we have an equality of weights $w_0(\chi) = \chi \Pi \alpha^{q-1}$. By an elementary fact about root systems, this equation has only one solution $\chi = \sigma_q$ with the given restriction on the moments of χ.

<div align="right">Q.E.D.</div>

6. Applications to sheaves

First we will consider the meaning of the conclusion of the general Theorem 2.4. We have a quasi-coherent sheaf $\mathcal{M}$ on a variety Y such that:

$(\star)$ the restriction $\Gamma(Y, \mathcal{M}) \to \Gamma(Y_{qy}, \mathcal{M}|_{Y_{qy}})$,

is injective for y in an open dense subset of Y.

Let examine the quasi-coherent sheaf $(F_n)_* \mathcal{M}$ on Y_n which is the direct image of $\mathcal{M}$ under the purely inseparable morphism $F_n : Y \to Y_n$. For any arbitrary point $z = F_n(y)$ of Y_n, we have a natural isomorphism $[(F_n)_* \mathcal{M}]_z \overset{\approx}{\to} \Gamma(Y_{qy}, \mathcal{M}|_{Y_{qy}})$ as F_n is affine and $F_n^{-1}(z) = Y_{qy}$. As we have a natural isomorphism $\Gamma(Y, \mathcal{M}) \approx \Gamma(Y_n, (F_n)_* \mathcal{M})$, we have a canonical $\mathcal{O}_{Y_n}$-homomorphism:

$$\alpha: \quad \Gamma(Y, \mathcal{M}) \otimes_k \mathcal{O}_{Y_n} \to F_{n*} \mathcal{M}.$$

Thus our hypothesis $(\star)$ means that $\alpha|_z$ is injective for z in an open dense subset of Y_n. As a section of the kernel of α is determined by its values on such a subset, we may conclude the truth of the first statement in:

LEMMA 6.1. — *The condition* $(\star)$ *implies the following conclusions:*

(i) *the canonical mapping* α *is injective;*

(ii) *if* $\mathcal{N}$ *is a quasi-coherent sheaf on* Y_n *which is locally free along the support of the cokernel of* α*, then:*

(a) $\alpha \otimes \mathcal{N} : \Gamma(Y, \mathcal{M}) \otimes_k \mathcal{N} \to F_{n*}(\mathcal{M}) \otimes_{\mathcal{O}_{Y_n}} \mathcal{N}$ *is injective;*

(b) *the induced mapping* $\Gamma(Y, \mathcal{M}) \otimes_k \Gamma(Y_n, \mathcal{N}) \to \Gamma(Y_n, F_{n*}(\mathcal{M}) \otimes \mathcal{N})$ *on global sections is injective;*

(iii) *if* $\mathcal{N}$ *or* $\mathrm{cok}(\alpha)$ *is locally free everywhere on* Y_n*, then we have a natural injection,*

$$\Gamma(Y, \mathcal{M}) \otimes_k \Gamma(Y_n, \mathcal{N}) \hookrightarrow \Gamma(Y, \mathcal{M} \otimes_{\mathcal{O}_Y} (F_n)^* \mathcal{N});$$

(iv) *if* α *is an isomorphism, then:*

(c) *the injections in part* (iii) *are isomorphisms, and*

(d) *the restrictions in* $(\star)$ *are isomorphisms for all* y *in* Y. *Conversely, if they always are,* α *is an isomorphism.*

Proof. — For (ii) the assumption implies that $\mathcal{T}or^1_{\mathcal{O}_Y}(\mathrm{cok}(\alpha), \mathcal{N}) = 0$, which is stronger than the injectivity assertion (a). The global effect of (a) is (b). Part (iii) follows from (b) by the projection formula. In (iv), (c) and the first part of (d) are evident. The rest of (d) is a routine application of Nakayama's lemma.

Q.E.D.

Next I want to record the group-theoretic properties of the above homomorphisms. Given a group variety H, we have the homomorphism $\psi \equiv F_n : H \to H_n$. If H acts on the variety Y, then H_n acts on Y_n so that $F_n : Y \to Y_n$ is ψ-equivariant. Given a H_n-linearized sheaf $\mathcal{N}$ on Y_n or any H_n-module N, let $\mathcal{N}^\psi$ and M^ψ denote the H-structures induced by ψ. With the above notations, we have:

LEMMA 6.2. — *Given an H-linearization of* $\mathcal{M}$*, then:*

(a) $F_{n*} \mathcal{M}$ *has a naturally H-linearization and* $\alpha: \Gamma(Y, \mathcal{M}) \otimes_k \mathcal{O}_{Y_n}^\psi \to F_{n*} \mathcal{M}$ *is H-equivariant for the diagonal linearization of the first term;*

(b) *similarly, the natural homomorphisms:*

$$\Gamma(Y, \mathcal{M}) \otimes_k \Gamma(Y_n, \mathcal{N})^\psi \to \Gamma(Y_n, F_{n*}(\mathcal{M}) \otimes_{\mathcal{O}_{Y_n}} \mathcal{N}^\psi) \to \Gamma(Y, \mathcal{M} \otimes_{\mathcal{O}_Y} (F_n)^* \mathcal{N}^\psi)$$

are H-equivariant.

Q.E.D.

Proof. — Why not?

My last duty before I can state definitively the consequences of the previous theory is the change of scalars. Temporarily we will move outside of the category of k-schemes to do this. Let $\varkappa$: Spec $k \to$ Spec k be the morphism corresponding to the q-power homomorphism on the rings. Then $\varkappa$ is an isomorphism and the analogous $\varkappa$-morphism $Y \to Y$ factors naturally as:

$$Y \overset{F_n}{\to} Y_n = \text{Spec } k \times_k Y \overset{\varkappa \times 1_Y}{\longrightarrow} Y.$$

Further $\varkappa \times 1_Y$ is $\varkappa \times 1_H$-equivariant for any group variety H acting on Y. Given any H-linearization of a sheaf $\mathscr{R}$ on Y, we have a H_n-linearization of $(\varkappa \times 1_Y)^* \mathscr{R}$ and, hence, an H-linearization $(\varkappa \times 1_Y)^* \mathscr{R}^\psi \equiv \mathscr{R}^{[q]}$. Specializing to the case where Y is a point, a given representation M of H gives an H_n-representation on $k \otimes_\varkappa M$, which induces a representation $M^{[q]}$ of H. Clearly we may identify $\Gamma(Y_n, \mathscr{R}^{[q]})$ with $\Gamma(Y, \mathscr{R})^{[q]}$ in a G-equivariant manner.

Returning to the category of k-schemes and the notations of sections four and five, we have:

THEOREM 6.3. — *If the moments of the weight $p(\chi)$ of $\varkappa$ are all strictly less than q and ψ is any other character of the parabolic subgroup P_f:*

(a) *we have natural injective G-equivariant homomorphisms:*

$$\alpha: \quad M_f(\chi) \otimes_k \mathcal{O}_{X_n} \hookrightarrow (F_n)_* \mathscr{L}_f(\chi)$$

and, more generally:

$$\alpha_\psi: \quad M_f(\chi) \otimes_{\mathcal{O}_{X_n}} \mathscr{L}_f(\psi)^{[q]} \hookrightarrow (F_n)_* \mathscr{L}_f(\chi) \otimes_{\mathcal{O}_{X_n}} \mathscr{L}_f(\psi)^{[q]} \approx (F_n)_* \mathscr{L}_f(\chi \cdot \psi^q),$$

and

(b) *globally we have a natural injective G-homomorphism:*

$$A_\psi: \quad M_f(\chi) \otimes M_f(\psi)^{[q]} \hookrightarrow M_f(\chi \cdot \psi^q).$$

Proof. — By Theorem 3.1, the condition $(\star)$ at the beginning of this section is verified for $\mathscr{M} = \mathscr{L}_f(\chi)$ at the point f. By homogenity, the condition $(\star)$ is verified for all points of χ. Thus injectivity follows directly from Lemma 6.1 (i), (ii) and (iii). The Lemma 6.2 should explain the G-equivariance as $(F_n)^* (\mathscr{L}_f(\psi)^{[q]}) \approx \mathscr{L}_f(\psi^q)$.

Q.E.D.

In case where X is a total flag space and χ is the Steinberg character σ_q, we have much stronger conclusions (*see* [8] and [2]).

THEOREM 6.4. — *For an arbitrary character ψ of the Borel subgroup P_f:*

(a) *we have natural G-equivariant isomorphisms:*

$$\alpha: \quad St_q \otimes_k \mathcal{O}_{X_n} \overset{\approx}{\to} (F_n)_* \mathscr{L}_f(\sigma_q),$$

and, more generally,

$$\alpha_\psi: \quad St_q \otimes_{\mathcal{O}_{X_n}} \mathscr{L}_f(\psi)^{[q]} \approx (F_n)_* \mathscr{L}_f(\sigma_q \cdot \psi^q),$$

and

(b) *globally we have natural isomorphisms:*

$$\mathrm{St}_n \otimes_k \mathrm{M}_f(\psi)^{[q]} \overset{\sim}{\to} \mathrm{M}_f(\sigma_q \psi^q)$$

and in particular, for any non-negative integer r:

$$\mathrm{St}_q \otimes \mathrm{St}_{q^r}^{[q]} \overset{\sim}{\to} \mathrm{St}_{q^{r+1}}.$$

Proof. — One uses Theorem 5.5 b) together with conserve in Lemma 6.1 (iv) to verify that α is an isomorphism. Using the rest of that part (iv), the proof is similar to the last one. The last remark is a special case because $\sigma_q \sigma_{q^r}^q = \sigma_{q^{r+1}}$ [i. e. in terms of the exponents in the definition of σ_*, $q(q^r - 1) + (q - 1) = q^{r+1} - 1$].

<div align="right">Q.E.D.</div>

Remark. — The last statement implies a very special property for the graded k-algebra $\oplus_{k \geq 0} \mathrm{M}_f(\sigma_q^k)$ with respect to the n-th Frobenius homomorphism. One may ask whether there is any useful theory about such graded rings.

Next I will show how the global injectivity theorem implies the Steinberg tensor product formula.

THEOREM 6.5. — *In the situation of Theorem 6.3, the homomorphism* A_ψ *induces a* G-*isomorphism:*

$$\mathrm{N}_f(\chi) \otimes \mathrm{N}_f(\psi)^{[q]} \overset{\sim}{\to} \mathrm{N}_f(\chi \cdot \psi^q).$$

Proof. — We will see the general strategy from the proof of Theorem 4.2. Let $\mu^*(\chi \uparrow \mathrm{P}_f) = \sum_i f_i \otimes m_i$ and $\mu^*(\psi \uparrow \mathrm{P}_f) = \sum_j g_j \otimes n_j$ where $\{f_i\}$, $\{m_i\}$, $\{g_j\}$ and $\{n_j\}$ are all linearly independent. Now $\mu^*(\chi \cdot \psi^q \uparrow \mathrm{P}_f) = \sum_{i, j} f_i g_j^q \otimes m_i n_j^q$. Thus the content of the Theorem is that $\{f_i g_j^q\}$ and $\{m_i n_j^q\}$ are linearly independent. Using the left-right trick, we need only prove the statement for $\{f_i g_j^q\}$, but this just means that $\mathrm{N}_f(\chi) \otimes \mathrm{N}_f(\chi)^{[q]} \to \mathrm{M}_f(\chi \cdot \psi^q)$ is injective. As this is a weaker statement than Theorem 6.3, we are done.

<div align="right">Q.E.D.</div>

7. Application to cohomology

In the last section we have actually found that there is an interesting example of a quasi-coherent sheaf $\mathcal{M}$ on a variety Y such that $(\star\star)$ the natural homomorphism α: $\Gamma(\mathrm{Y}, \mathcal{M}) \otimes_k \mathcal{O}_{\mathrm{Y}_n} \to (\mathrm{F}_n)_* \mathcal{M}$ is an isomorphism.

This is a remarkable phenomenon. In fact $(\mathrm{F}_n)_* \mathcal{M}$ is a free $\mathcal{O}_{\mathrm{Y}_n}$-module. Hence as the morphism F_n is affine, $\mathcal{M}$ is flat over Y_n. The cohomological consequence of these facts are quite extreme.

LEMMA 7.1. — *Assume that the condition ($\star\star$) is verified and that $\mathscr{N}$ is an arbitrary quasi-coherent sheaf on Y_n. Then the natural multiplication:*

$$\Gamma(Y, \mathscr{M}) \otimes_k H^i(Y_n, \mathscr{N}) \to H^i(Y, \mathscr{M} \otimes_{\mathscr{O}_Y} (F_n)^* \mathscr{N}),$$

is an isomorphism for all integers i.

Proof. — Obviously we have an isomorphism:

$$\Gamma(Y, \mathscr{M}) \otimes_k H^i(Y_n, \mathscr{N}) \approx H^i(Y_n, \Gamma(Y, \mathscr{M}) \otimes_k \mathscr{N}).$$

Using the global effect of the isomorphism $\alpha_{\mathscr{N}}$ of Lemma 6.1 (iv), we have a natural isomorphism of the last cohomology group with $H^i(Y_n, (F_n)_* \mathscr{M} \otimes_{\mathscr{O}_{Y_n}} \mathscr{N})$. As $\mathscr{M}$ is flat and affine over Y_n, we have a projection formula isomorphism of the last group with $H^i(Y, \mathscr{M} \otimes_{\mathscr{O}_Y} (F_n)^* \mathscr{N})$. Therefore the composition of these isomorphisms is an isomorphism.

<div align="right">Q.E.D.</div>

In the presence of G-linearizations of the above sheaves, these isomorphisms are G-equivariant in terms of the induced G-module structure on the cohomology groups. The main application to cohomology of the preceeding theory is due to Haboush [8] and Andersen [2]. It is:

THEOREM 7.2. — *In the situation of Theorem 6.5, we have natural G-equivariant isomorphisms:*

$$B_\chi: \quad St_q \otimes_k H^i(Y, \mathscr{L}_f(\chi))^{[q]} \overset{\sim}{\to} H^i(Y, \mathscr{L}_f(\sigma_q \cdot \chi^q)).$$

Proof. — By Theorem 6.4 (a), we may apply our Lemma 7.1 with $\mathscr{M} = \mathscr{L}_f(\sigma_q)$ and $\mathscr{N} = \mathscr{L}_f(\chi)^{[q]}$. The result follows directly.

<div align="right">Q.E.D.</div>

The last Theorem provides an excellent solution to problem of finding a G-equivariant proof of the Borel-Weil type vanishing Theorem [9]. This proof is too easy to resist writting.

THEOREM 7.3. — *If the weight of the character χ is dominant, then the cohomology group $H^i(X, \mathscr{L}_f(\chi))$ are zero for positive i.*

Proof. — As $\sigma_q \chi^q$ has positive moments, by Theorem 7.2, it will be enough to consider the case where χ has positive moments; i.e. $\mathscr{L}_f(\chi)$ is ample. By Serre vanishing Theorem, $H^i(X, \mathscr{L}_f(\chi^m)) = 0$ for all $i > 0$ when $m \gg 0$. By the theorem again, the natural homomorphism $\gamma: H^i(X, \mathscr{L}_f(\chi))^{[q]} \to H^i(X, \mathscr{L}_f(\chi^q))$ is injective because $B_\chi = 1 \cup \gamma$ and $St_q \neq 0$. Hence $H^i(X, \mathscr{L}_f(\chi)) = 0$ for all $i > 0$.

<div align="right">Q.E.D.</div>

The reader may find other interesting applications of the Theorems in this paper in [1], [2], [5], [8], [10]. A significant fact about the Steinberg character σ_q can be seen in duality theory. Recall that, on the homogeneous space X, we may G-equivariantly identify the

dualizing sheaf ω_X with $\mathscr{L}_f(\psi)$ where $\psi = \Pi\alpha$ where α runs through the roots in U_f^-. Thus $\omega_{X_{qx}} \approx \mathscr{E}pt_{\mathcal{O}_X}^{\dim X}(\mathcal{O}_{X_{qx}}, \mathscr{L}_f(\psi))$. Furthermore the relative dualizing sheaf ω_F for the morphism: $F_n \colon X \to X_n$ is:

$$\mathscr{L}_f(\psi) \otimes (F_n)^*(\mathscr{L}_f(\psi)^{[q]})^{\otimes -1} \approx \mathscr{L}_f(\psi^{1-q}).$$

For a total flag space by the definition we have $\mathscr{L}_f(\sigma_q)^{\otimes 2} \approx \omega_F$.

Thus, *a priori*, we expect the direct image $(F_n)_* \mathscr{L}_f(\sigma_q)$ to be self-dual. Under the above isomorphisms with $St_q \otimes_k \mathcal{O}_{X_n}$, this self-duality corresponds to the irreducibility of St_q. Further reflexions on duality should convince the reader of the naturalness of the duality ideas expressed (or implicit everywhere) in Section 5.

Remark. — For split groups over a finite field F, Steinberg has developed an analogous representation theory. The methods of this paper can be modified to treat that case also. The Frobenius morphism $U \to {}_nU \setminus U$ must be replaced by the Artin-Schreier-Lang morphism $U \to U_F \setminus U$ and the role of the torus T is limited to the action of its group T_F of rational points.

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