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ON THE CHOW GROUPS OF CERTAIN RATIONAL SURFACES

BY SPENCER BLOCH (1)

Let X be a smooth projective surface defined over a field k. X is said to be *rational* if $X_{k'}$ is birational with $\mathbb{P}^2_{k'}$ for some k'/k. I will say X is *split* over k' if $X_{k'}$ can be obtained by blowing up a finite number of k'-points starting from $\mathbb{P}^2_{k'}$ and then blowing down a finite number of exceptional curves of the first kind defined and absolutely irreducible over k'. Any rational surface over k admits a splitting field k' finite over k. The following notation will be used throughout: X = rational surface/k, k' = splitting field for X, and:

$X' = X_{k'}$; N = Néron-Severi group of X'.

F = quotient field of X; $F' = F k'$.

$CH_0(X)$ = Chow group of zero cycles on X modulo rational equivalence.

$A_0(X) = \operatorname{Ker}(CH_0(X) \overset{\deg}{\to} \mathbb{Z}) = \operatorname{Ker}(CH_0(X) \to CH_0(X'))$.

X^i = points of codim i on X; $K_2(.)$ = K_2 group Milnor [11].

Br(.) = Brauer group; $k_*(.)$ = Mod 2 Milnor ring [10] [e.g. $k_1(k) = k^*/k^{*2}$].

Finally, to avoid technical problems associated with characteristic p, we will work only with rational surfaces X which are *separably split*, i.e. such that the splitting field k' can be taken to be separable over k.

The objective of this paper is to study a certain map Φ associating to a cycle of degree 0 in the Chow group $A_0(X)$ of a rational surface X a *torseur* for the Néron-Severi torus:

$$\Phi: \quad A_0(X) \to H^1(k'/k, N \otimes k'^*). \quad (2)$$

We show Image Φ is finite when k is local or global, and that Image Φ = 0 if k is non-archimedean local and X has good reduction over the closed fibre. When X is a *conic bundle surface*, i.e., when there exists $\pi : X \to \mathbb{P}^1_k$ a rational map with generic fibre a conic curve we

(1) Partially supported by National Science Foundation under NSF MCS77-01931.

(2) Colliot-Thélène has pointed out to me that the existence of such a map Φ is immediate from [5] and [15], where results like (0.2) and an even stronger version of the finiteness of $X(k)/B$ are also announced. The reader of the present paper should perhaps focus on the role of K-theory in evaluating Ker Φ and coker Φ as in (0.1.1), (0.3) and (0.4).

also show Ker Φ is finite when k is local or global and indeed Ker $\Phi=0$ if $k \not\subset \mathbb{R}$. Interesting examples of such surfaces are the *Châtelet surfaces* $X : T_0^2 - aT_1^2 = x(x-a_1)(x-a_2)$. It is possible to give examples of such surfaces with $A_0(X) \neq (0)$ [4]. In an appendix, we verify a compatibility between Φ and the notion of *Brauer equivalence* defined in [9]. Using this and the above results, we show that the set $X(k)/B$ of k-points of X modulo Brauer equivalence is finite for k global and X any rational surface.

Here, in more precise form are the main results.

THEOREM (0.1). — *Let X be a smooth projective surface over a field k. Assume X is rational, split by a galois extension k'/k. Then there is an exact sequence:*

$$(0.1.1)\quad \Gamma(k'/k, N \otimes k'^*) \to H^1(k'/k, K_2(F')/K_2(k')) \xrightarrow{\mu} A_0(X)$$

$$\xrightarrow{\Phi} H^1(k'/k, N \otimes k'^*) \xrightarrow{\theta} H^2(k'/k, K_2(F')/K_2(k')).$$

If the g.c.d. of the degrees of all closed points on X *is* 1, *then there is a map:*

$$H^2(k'/k, K_2(F')/K_2(k')) \to \coprod_{x \in X^1} \mathrm{Br}(k(x)),$$

extending the above exact sequence one place to the right.

THEOREM (0.2). — *With notation as above, if k a local field and k'/k is an arbitrary (not necessarily finite) galois extension, then* $H^1(k'/k, N \otimes k'^*)$ *is finite. If, moreover, k is non-archimedean and there exists a lifting of* X *to* $\tilde{X}$ *smooth and projective over the ring of integers of k, then* Im $\Phi=0$. *Finally, if k is global and k'/k is finite galois, then* Im Φ *is finite.*

THEOREM (0.3). — *Let k be a field of characteristic* $\neq 2$, *which is either local, global, or* C_i *for* $i \leqq 3$, *and let* $\bar{k}$ *be the separable closure of k. Let* X *be a separably split conic bundle surface over k. Then* $H^1(\bar{k}/k, K_2(F\bar{k})/K_2(\bar{k}))$ *is subquotient of* $k_3(k)$. *In particular, for k non-archimedean local,* C_1, C_2, *or totally imaginary global this group is zero. For k arbitrary local or global the group is finite.*

THEOREM (0.4). — *Let* X *be a conic bundle surface defined over a field k of characteristic* $\neq 2$:

(i) *If k is a* C_1 *field then* $A_0(X)=(0)$.

(ii) *If k is a local or global field, then* $A_0(X)$ *is finite.*

(iii) *If k* is non-archimedean local and X *has good reduction in the sense that there exists* $\tilde{X}$ *as in* (0.2), *then* $A_0(X)=(0)$.

The conic bundle surfaces given birationally by:

$$\mathrm{X}: \quad \mathrm{T}_1^2 - a\mathrm{T}_2^2 = \prod_{i=1}^{3}(x-a_i).\mathrm{T}_0^2 \quad \text{in } \mathbb{P}_k^2 \times \mathbb{A}_k^i, \qquad a,\ a_i \in k^*$$

with homogeneous coordinates T_0, T_1, T_2 and affine coordinate x were studied by Châtelet [3]. Finiteness of $\mathrm{A}_0(\mathrm{X})$ for such surfaces with k a number field was proven by Colliot-Thélène and Coray ([4], Remarques 6.7) using results of Châtelet [3] which can be found in Manin [9].

Manin ([8], [9]) constructed a pairing $\mathrm{A}_0(\mathrm{X}) \times \mathrm{H}^1(k'/k, \mathrm{N}) \to \mathrm{Br}(k)$ which he used to show (among other things) non-triviality of $\mathrm{A}_0(\mathrm{X})$. From our point of view, this pairing arises *via:*

$$\mathrm{A}_0(\mathrm{X}) \otimes \mathrm{H}^1(k'/k,\ \mathrm{N}) \to \mathrm{H}^1(k'/k,\ \mathrm{N} \otimes k'^*) \ \otimes \mathrm{H}^1(k'/k,\ \mathrm{N}) \xrightarrow[\substack{\text{intersection}\\ \text{product}}]{} \mathrm{H}^2(k'/k,\ k'^*).$$

In considering cycles on rational varieties of dimension >2 one wants to define the Néron-Severi torus to be the torus with character group N, i.e.:

$$\text{Néron-Severi torus } (k) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{N},\ k'^*)^{\mathrm{Gal}(k'/k)}.$$

This definition coincides with the one used in this paper because for X rational of dimension 2, N is self-dual. A very beautiful paper of Colliot-Thélène and Sansuc [5] contains among other results a proof that for X a smooth compactification of a torus T of arbitrary dimension, the group of torseurs for the Néron-Severi torus is isomorphic to the group $\mathrm{T}(k)/\mathrm{R}$. Here points on T are said to be R-equivalent if they can be connected by a chain of rational curves.

Among the open questions remaining in this area let me single out the question of finiteness of $\mathrm{A}_0(\mathrm{X})$ for any smooth rational surface over a local or global field, as well as the question of injectivity of $\mathrm{A}_0(\mathrm{X}) \to \mathrm{H}^1(k'/k, \mathrm{N} \otimes k'^*)$. I know of no counter-example to these assertions over *any* ground field. The relation between the proof of (0.3) and the Eichler norm Theorem suggests one should study the injectivity question when $k = \mathbb{R}$.

I should like to acknowledge many helpful conversations with J.-L. Colliot-Thélène, D. Coray and J.-J. Sansuc (3).

1. The fundamental exact sequence

The following complex is defined for any variety X [12]:

$$(1.1) \qquad \mathrm{K}_2(\mathrm{F}) \xrightarrow{\text{tame}} \coprod_{x \in \mathrm{X}^1} k(x)^* \to \coprod_{x \in \mathrm{X}^2} \mathbb{Z}.$$

(Here "tame" denotes the tame symbol [11].) Suppose for simplicity X is a smooth surface over k and let $\pi : \mathrm{Y} \to \mathrm{X}$ be obtained by blowing up a k-point on X. Let $\mathrm{E} \cong \mathbb{P}_k^1 \subset \mathrm{Y}$ be the

(3) *Added in proof:* in a recent paper, On the Chow groups of certain rational surfaces: complements to a paper of S. Bloch, Colliot-Thélène and Sansuc have strengthened a number of results found here. In particular they calculate the H^1 group in (3.4) for conic bundle surfaces and give examples where it is $\neq 0$.

exceptional fibre, and let $e \in \mathrm{E}$ be the generic point. The top row and middle and right-hand columns of the following diagram are exact:

$$(1.2)\qquad \begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & k^* \arrow[r] & k(e)^* \arrow[r] \arrow[d] & \overset{0}{\coprod\limits_{\mathrm{E}^1}} \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ & \mathrm{K}_2(\mathrm{F}) \arrow[r] \arrow[d, equal] & \coprod\limits_{y \in \mathrm{Y}^1} k(y)^* \arrow[r] \arrow[d] & \coprod\limits_{\mathrm{Y}^2} \mathbb{Z} \arrow[d] & \\ & \mathrm{K}_2(\mathrm{F}) \arrow[r] & \coprod\limits_{x \in \mathrm{X}^1} k(x)^* \arrow[r] \arrow[d] & \coprod\limits_{\mathrm{X}^2} \mathbb{Z} \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

[Here $\overset{0}{\coprod} \mathbb{Z} = \mathrm{Ker}\,(\coprod \mathbb{Z} \xrightarrow{\text{sum}} \mathbb{Z})$.]

PROPOSITION (1.3). — (i) *The complex* (1.1) *computes the cohomology of the Zariski sheaf* $\mathscr{K}_2$ *on* X, *i.e., it has cohomology groups reading from left to right* $\Gamma(\mathrm{X}, \mathscr{K}_2)$, $\mathrm{H}^1(\mathrm{X}, \mathscr{K}_2)$, *and* $\mathrm{H}^2(\mathrm{X}, \mathscr{K}_2) \simeq \mathrm{CH}_0(\mathrm{X})$.

(ii) *With* Y *as above*, $\Gamma(\mathrm{Y}, \mathscr{K}_2) \simeq \Gamma(\mathrm{X}, \mathscr{K}_2)$ *and* $\mathrm{H}^1(\mathrm{Y}, \mathscr{K}_2) \simeq \mathrm{H}^1(\mathrm{X}, \mathscr{K}_2) \oplus k^*$.

Proof. — (i) is a standard result in algebraic K-theory [12]. For (ii), using the contravariant functoriality of $\Gamma(\,.\,, \mathscr{K}_2)$ and the covariant functoriality of (1.1) for proper maps (4), one gets:

$$\begin{tikzcd} \Gamma(\mathrm{Y}, \mathscr{K}_2) \arrow[dr, hook] \arrow[d, shift right, "\pi_*"'] & \\ \Gamma(\mathrm{X}, \mathscr{K}_2) \arrow[r, hook] \arrow[u, shift right, "\pi^*"'] & \mathrm{K}_2(\mathrm{F}) \end{tikzcd}$$

where $\Gamma(\mathrm{X}, \mathscr{K}_2) \simeq \Gamma(\mathrm{Y}, \mathscr{K}_2)$. The isomorphism $\mathrm{H}^1(\mathrm{X}, \mathscr{K}_2) \oplus k^* \simeq \mathrm{H}^1(\mathrm{Y}, \mathscr{K}_2)$ now follows from a diagram chase using (1.2).

Q.E.D.

COROLLARY (1.4). — *Assume* X *is a smooth projective rational surface split by* k. *Then* $\Gamma(\mathrm{X}, \mathscr{K}_2) \simeq \mathrm{K}_2(k)$ *and* $\mathrm{H}^1(\mathrm{X}, \mathscr{K}_2) \simeq \mathrm{N} \otimes k^*$, *where* N = *Néron-Severi group of* X.

Proof. — Follows from (1.3) and the corresponding result for $\mathrm{X} = \mathbb{P}^2_k$ [14].

Remark (1.5). — With X as above, we have $\mathrm{N} \cong \mathrm{H}^1(\mathrm{X}, \mathscr{O}^*_{\mathrm{X}})$ and the isomorphism $\mathrm{H}^1(\mathrm{X}, \mathscr{O}^*_{\mathrm{X}}) \otimes k^* \to \mathrm{H}^1(\mathrm{X}, \mathrm{K}_2)$ is given by the symbol map.

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(4) This functionality results from covariant functoriality of the localization sequence. The reader who wishes a detailed discussion in an analogous (cohomological) context is referred to S. BLOCH and A. OGUS, *Gersten's Conjecture and the Homology of Schemas* (*Ann. scient. Éc. Norm. Sup.*, T. 7, fasc. 2, 1974).

THEOREM (1.6). — *Let* X *be a smooth projective rational surface over a field k, and let k′ be galois over k splitting* X. *Then there is an exact sequence:*

$$\Gamma(k'/k, \mathrm{N} \underset{\mathbb{Z}}{\otimes} k'^*) \to \mathrm{H}^1(k'/k, \mathrm{K}_2(\mathrm{F}')/\mathrm{K}_2(k'))$$
$$\xrightarrow{\mu} \mathrm{A}_0(\mathrm{X}) \xrightarrow{\Phi} \mathrm{H}^1(k'/k, \mathrm{N}\otimes k'^*) \xrightarrow{\theta} \mathrm{H}^2(k'/k, \mathrm{K}_2(\mathrm{F}')/\mathrm{K}_2(k')).$$

Proof. — One knows that $\mathrm{CH}_0(\mathrm{X}') \cong \mathbb{Z}$, so there are exact sequences (defining $\mathbb{Z}$):

$$(1.7) \qquad \begin{cases} 0 \to \mathbb{Z} \to \coprod_{x' \in \mathrm{X}'^1} k'(x')^* \to \overset{0}{\coprod_{\mathrm{X}'^2}} \mathbb{Z} \to 0, \\ 0 \to \mathrm{K}_2(\mathrm{F}')/\mathrm{K}_2(k') \to \mathrm{Z} \to \mathrm{N}\otimes k'^* \to 0. \end{cases}$$

One has:

$$\mathrm{H}^*(k'/k, \coprod_{\mathrm{X}'^1} k'(x')^*) \cong \coprod_{x \in \mathrm{X}^1} \mathrm{H}^*(k'/k_{x'}, k'(x')^*),$$

where $k_x = k' \cap k(x)$, and on the right $x' \in \mathrm{X}'^1$ is chosen lying above $x \in \mathrm{X}$. Also:

$$\Gamma\left(k'/k, \overset{0}{\coprod_{\mathrm{X}'^2}} \mathbb{Z}\right) \cong \overset{0}{\coprod_{\mathrm{X}^2}} \mathbb{Z}.$$

It follows from the first sequence in (1.7) that $\mathrm{A}_0(\mathrm{X}) \cong \mathrm{H}^1(k'/k, \mathrm{Z})$, and substituting this value in the cohomology sequence from the second exact sequence yields the Theorem.

Remark (1.7). — If the g. c. d. of the degrees of all closed points of X is 1 [e. g., if $\mathrm{X}(k) \neq \emptyset$], then $\mathrm{H}^1\left(k'/k, \overset{0}{\coprod_{\mathrm{X}'^2}} \mathbb{Z}\right) = 0$ and the above sequence can be extended one more term to the right *via* a map:

$$\mathrm{H}^2(k'/k, \mathrm{K}_2(\mathrm{F}')/\mathrm{K}_2(k')) \to \coprod_{x \in \mathrm{X}^1} \mathrm{Br}(k(x)).$$

2. Finiteness of Im Φ

THEOREM (2.1). — *Let k be a local field, k′/k a (not necessarily finite) galois extension. Let* N *be a free f.g.* $\mathbb{Z}$*-module with a continuous* $\mathrm{Gal}(k'/k)$*-action. Let* $\mathrm{N}^v = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{N}, \mathbb{Z})$. *Then* $\mathrm{H}^1(k'/k, \mathrm{N} \underset{\mathbb{Z}}{\otimes} k'^*)$ *is a finite group, dual to* $\mathrm{H}^1(k'/k, \mathrm{N}^v)$.

Proof. — This is a standard result in local duality [13].

THEOREM (2.2). — *Let k be a global field, k′/k a finite galois extension. Let* X *be a smooth projective rational surface defined over k split over k′. Then the kernel of the map:*

$$\theta: \quad \mathrm{H}^1(k'/k, \mathrm{N}\otimes k'^*) \to \mathrm{H}^2(k'/k, \mathrm{K}_2(\mathrm{F}')/\mathrm{K}_2(k')),$$

defined in (1.6) *is finite.*

Proof. — The geometric situation, viz. X together with an isomorphism $X_{k'} \cong (\mathbb{P}^2_{k'}$ with a finite number of k' points blown up and down), can be spread out over the ring of S-integers $\mathcal{O}_{k,S}$ for some finite set S of primes of k. Thus for each $\mathfrak{p} \notin S$ we may assume the reduction $X(\mathfrak{p})$ is a smooth projective rational surface over $k(\mathfrak{p})$, and that we are given $X(\mathfrak{p}) \underset{k(\mathfrak{p})}{\times} k'(\mathfrak{p}') \simeq [\mathbb{P}^2_{k'(\mathfrak{p})}$ with finite number of $k'(\mathfrak{p}')$-points blown up and down]. Enlarging S, we may assume it contains all primes which ramify in k', and that the class group of $\mathcal{O}_{k',S'}$ is trivial, where $S' = \{$primes lying over S$\}$.

Tensoring the exact sequence:

$$0 \to \mathcal{O}^*_{k',S'} \to k'^* \to \coprod_{\mathfrak{p}' \notin S'} \mathbb{Z} \to 0,$$

with N, we obtain a map $\psi : H^1(k'/k, N \otimes k'^*) \to \coprod_{\mathfrak{p} \notin S} H^1(D_\mathfrak{p}, N)$, where $D_\mathfrak{p} \subset \mathrm{Gal}(k'/k)$ is a decomposition group at $\mathfrak{p}$. Note Ker ψ is a quotient of $H^1(k'/k, N \otimes \mathcal{O}^*_{k',S'})$ which is finite since $\mathcal{O}^*_{k',S'}$ is *f.g.* Thus it will suffice to show Ker $\theta \subset$ Ker ψ.

We may think now of X' as smooth and projective over Sp $\mathcal{O}_{k',S'}$, so $\mathfrak{p}' \notin S'$ defines a discrete valuation on F', and hence a tame symbol $K_2(F') \to F'(\mathfrak{p}')^*$, where $F'(\mathfrak{p}')$ = function field of $X'(\mathfrak{p}')$. Our objective is to define injective maps:

$$\tau_\mathfrak{p} : \quad H^1(D_\mathfrak{p}, N) \hookrightarrow H^2(D_\mathfrak{p}, F'(\mathfrak{p}')^*/k'(\mathfrak{p}')^*)$$

(where $\mathfrak{p}'$ denotes a fixed prime above $\mathfrak{p}$) fitting into a commutative square:

$$\text{(2.2.1)} \qquad \begin{tikzcd} H^1(k'/k, N\otimes k'^*) \arrow[r, "\theta"] \arrow[d, "\psi"] & H^2(k'/k, K_2(F')/K_2(k')) \arrow[d, "\text{tame}"] \\ \coprod_{\mathfrak{p}\notin S} H^1(D_\mathfrak{p}, N) \arrow[r, hook, "\amalg \tau_\mathfrak{p}"] & \coprod_{\mathfrak{p}\notin S} H^2(D_\mathfrak{p}, F'(\mathfrak{p}')^*/k'(\mathfrak{p}')^*) \end{tikzcd}$$

The existence of such a commutative square (2.2.1) will prove Ker $\theta \subset$ Ker ψ as claimed.

LEMMA (2.2.2). — *Define $\tau_\mathfrak{p}$ to be the boundary map associated to the exact sequence:*

$$0 \to F'(\mathfrak{p})^*/k'(\mathfrak{p}')^* \to \mathrm{Div}(X'(\mathfrak{p}')) \to N \to 0.$$

Then $\tau_\mathfrak{p}$ is injective and the diagram (2.2.1) *commutes.*

Proof. — Notice first that

$$N = \text{Néron-Severi } (X') \cong \text{Néron-Severi } (X'(\mathfrak{p}'))$$

because both surfaces are split over their fields of definition, so the above exact sequence is defined. Moreover, if $C' \subset X'$ is an irreducible curve it can be "closed-up" and normalized to a normal family of curves $\tilde{C}$ flat over $\mathcal{O}_{k',S'}$, so $\mathfrak{p}'$ defines a divisor $C'(\mathfrak{p}')$ on $\tilde{C}$, hence a map:

$$k'(C')^* \xrightarrow{\alpha_0} \coprod \mathbb{Z},$$

where the sum runs over irreducible components of $C'(\mathfrak{p}')$.

Now let C′ run through all irreducible curves on X′ and identify components of C′ (𝔭′) with divisors on X′ (𝔭′) *via* cycle-theoretic direct image to get a square:

$$\begin{array}{ccc} K_2(F') & \xrightarrow{\text{tame}} & \coprod_{x' \in X'^1} k'(x')^* \\ {\scriptstyle \text{tame}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ F'(\mathfrak{p}')^* & \longrightarrow & \mathrm{Div}(X'(\mathfrak{p}')) \end{array}$$

which commutes up to sign (5). We can now consider a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \rightarrow & K_2(F')/K_2(k') & \longrightarrow & Z & \longrightarrow & N \otimes k'^* & \rightarrow & 0 \\ & & {\scriptstyle \text{tame}}\downarrow & & \downarrow & & \downarrow{\scriptstyle V_{\mathfrak{p}'}} & & \\ 0 & \rightarrow & F'(\mathfrak{p}')^*/k'(\mathfrak{p}')^* & \longrightarrow & \mathrm{Div}(X'(\mathfrak{p}')) & \rightarrow & N & \longrightarrow & 0 \end{array} \qquad (6)$$

where the middle vertical arrow is defined *via* α in the square above. This suffices to show that (2.2.1) commutes. Finally, note Div (X′ (𝔭)) is a direct limit of permutation modules so $H^1(D_\mathfrak{p}, \mathrm{Div}(X'(\mathfrak{p}'))=0$ and $\tau_\mathfrak{p}$ is injective.

Q.E.D.

On reflection, it seems that the same argument should yield something in the local case as well:

THEOREM (2.3). — *Let* X *be a smooth, projective rational surface defined over a non-archimedean local field k with residue field* $\mathbb{F}_q$ *and ring of integers* $\mathcal{O}$. *Assume* X *has good reduction in the sense that there exists an* $\tilde{X}$ *smooth and projective over* $\mathrm{Sp}\,\mathcal{O}$ *with generic fibre* X. *Then there exists k′/k finite unramified splitting* X, *and*:

$$\mathrm{Im}\,\Phi = \mathrm{Ker}(\theta : H^1(k'/k, N \otimes k'^*) \to H^2(k'/k, K_2(F')/K_2(k'))).$$

is zero.

Proof. — Let $X_0 \to \mathrm{Sp}\,\mathbb{F}_q$ be the closed fibre of $\tilde{X}$. Then X_0 is rational [the canonical bundle has negative degree and $q=0$, since both these properties specialize from X (7).]. The fact that X has an unramified splitting field follows from:

LEMMA (2.3.1). — *Let* R *be a complete discrete valuation ring with residue field* $\mathbb{F}$ *and quotient field k. Let* $f : \tilde{X} \to \mathrm{Sp}\,R$ *be smooth, projective, geometrically irreducible, and*

(5) The referee is dubious. Quite generally, for a noetherian scheme Y, there are complexes

$$\coprod_{Y^0} K_n(k(y)) \to \coprod_{Y^1} K_{n-1}(k(y)) \to \coprod_{Y^2} K_{n-2}(k(y)) \to \dots \quad [12].$$

Apply this with $n=2$ and Y = "spreading out" of X′ over the local ring at 𝔭′ on $\mathcal{O}_{k'}$.

(6) If the curve $C' \subset X'$ discussed above has good reduction at 𝔭′, and if $a \in k'^*$, then clearly $\alpha_0(a) = V_{\mathfrak{p}'}(a)$. Since N is generated by such curves, we get $V_{\mathfrak{p}'}$ here.

(7) Let ω = canonical bundle of $\tilde{X}/\mathrm{Sp}\,\mathcal{O}$, $\omega_k = \omega|_X$, $\omega_0 = \omega|_{X_0}$. Then $0 > (\omega_k . L_k) = (\omega_0 . L_0)$. The assertion about q follows because q is one half the first Betti number in etale cohomology, and the etale Betti numbers are constant in somooth families.

assume the closed fibre X_0 is rational and split over $\mathbb{F}$. Then the generic fibre X is rational and split over k.

Proof. — An $\mathbb{F}$-valued point of X_0 lifts by smoothness to an R-valued point of $\tilde{X}$, so a morphism $X_0(1) \to X_0$ obtained by blowing up an $\mathbb{F}$-point lifts to a diagram:

$$\begin{tikzcd} X_0(1) \arrow[d] & \subset & \tilde{X}(1) \arrow[d, "\text{blow up an R-point.}"] \\ X_0 & \subset & \tilde{X} \end{tikzcd}$$

We may thus build a figure:

$$\begin{tikzcd} & \tilde{Y} \subset Y_0 \arrow[dl, "\text{blowings up}"'] \arrow[dr, "\text{blowings down}"] & \\ \tilde{X} \subset X_0 & & \mathbb{P}^2_{\mathbb{F}} \subset \mathbb{P}^2_{R} \end{tikzcd}$$

Let $E_0 \subset Y_0$ be an exceptional curve of the first kind. There exists a unique family of exceptional curves of the first kind $\tilde{E} \subset \tilde{Y}$ lifting E_0. Indeed, the normal bundle of $E_0 \cong \mathbb{P}^1$ in Y_0 is $\mathcal{O}(-1)$, so both the zeroeth and first cohomology of the normal bundle vanish, implying existence and uniqueness of lifting ([8]). We may then blow down $\tilde{E}$ on $\tilde{Y}$ in much the same way as E_0 on Y_0 ([9]). Iterating we get a diagram:

$$\begin{tikzcd} Y_0 \arrow[d] & \subset & \tilde{Y} \arrow[d] \\ \mathbb{P}^2_{\mathbb{F}} & \subset & \tilde{Z} \end{tikzcd}$$

with $\tilde{Z}$ smooth and projective over Spec R with closed fibre $\mathbb{P}^2_{\mathbb{F}}$. Since $\mathbb{P}^2_{\mathbb{F}}$ is rigid also, it follows that $\tilde{Z} \cong \mathbb{P}^2_{R}$. Passing now to the generic fibres we get $X_k \leftarrow Y_k \to Z_k \cong \mathbb{P}^2_k$, both morphisms being obtained by successive blowings up of k-points.

Q.E.D.

Returning to the proof of (2.3), let k'/k be unramified with residue field splitting X_0. The units $\mathcal{O}'^*$ in the ring of integers $\mathcal{O}' \subset k'$ are cohomologically trivial so $N \otimes \mathcal{O}'^*$ is cohomologically trivial ([13], Thm. 9, p. 152) and:

$$H^1(k'/k,\ N \otimes k'^*) \cong H^1(k'/k,\ N).$$

([8]) For this sort of argument, *cf.* A. Grothendieck, *SGA I*, fasc. I, exp. III, Paris, 1960-1961.

([9]) Let $\eta \to \operatorname{Sp} R$ be the generic point. Let A_η be the ring of germs of regular functions on Zariski neighborhoods of $E_\eta \subset Y_\eta$. Since E_η is rational with self-intersection -1 it blows down so A_η is smooth of dim. 2 over $k(\eta)$. Let $A \subset A_\eta$ be the R-algebra of functions along $\tilde{E} \subset \tilde{Y}$, $I \subset A$ the ideal of functions vanishing along $\tilde{E}$. Then the germ of $\tilde{E} \in \tilde{Y}$ comes by blowing up $I \subset A$.

Writing $\mathbb{F}'$ and F'_0 for the residue field of k' and the function field of $X_0 \times_{\mathbb{F}} \mathbb{F}'$ respectively we obtain as in (2.2.1):

$$
\begin{array}{ccccc}
 & H^1(k'/k, N \otimes k'^*) & \xrightarrow{\nu} & H^2(k'/k, K_2(F')/K_2(k')) \\
 & \downarrow \wr & & \downarrow \text{tame} \\
0 \longrightarrow & H^1(k'/k, N) & \xrightarrow{\tau} & H^2(k'/k, F_0'^*/\mathbb{F}'^*)
\end{array}
$$

It follows that Ker $\theta = 0$. Q.E.D.

3. Applications to conic bundle surfaces

Proposition (3.1). — *Let* X *be as in paragraph* 2 *and let* $k' \subset k''$ *be galois extensions of* k *splitting* X. *Then the triangle:*

$$
\begin{tikzcd}
H^1(k'/k, K_2(F')/K_2(k')) \arrow[rr] \arrow[dr, "\mu'"'] & & H^1(k''/k, K_2(F'')/K_2(k'')) \arrow[dl, "\mu''"] \\
 & A_0(X) &
\end{tikzcd}
$$

commutes. In particular, Image $\mu'' \supset$ Image μ'.

The proof is immediate.

We assume henceforth when discussing conic bundle surfaces that char $k \neq 2$.

Theorem (3.2). — *Let* X *be a conic bundle surface, i. e., suppose given a rational map* $\pi : X \to \mathbb{P}^1_k$ *with generic fibre a conic curve (Severi-Brauer variety of dimension* 1). *Assume* k *is a local field, a global field or a* C_i *field for* $i \leqq 3$. *Then* $H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k}))$ *is a subquotient of* $k_3(k)$, *the third graded piece of the Milnor ring of* k. *(Here* $\bar{k}$ *is the separable closure of* k, $\bar{F} = F\bar{k}$.)

Corollary (3.3). — *With hypotheses as above, if* k *is a* C_2*-field, a local field* $\neq \mathbb{R}$, *or a totally imaginary global field, then* $H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k})) = 0$. *If* k *is any number field, or* $k = \mathbb{R}$, *then* $H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k})) \cong \bigoplus_s \mathbb{Z}/2\mathbb{Z}$, *where* $s \leqq$ *the number of real places of* k.

Proof. — With the exception of the C_2 case, all assertions follow from (3.2) together with results about $k_3(k)$ proved in [10]. For k C_2 (or non-archimedean local) any quadratic form in five variables represents zero. Let

$$l : k^* \to k_1(k) = k^*/k^{*2}$$

and consider an element

$$l(a)\, l(b)\, l(c) \in k_3(k).$$

Let A be the quaternion algebra $T^2 = a$, $U^2 = b$, $TU = -UT$. The reduced norm $N : A \to k$ is a quadratic form in four variables, so it is surjective and there exists $\alpha \in A$, $N(\alpha) = c$. If $\alpha \in k$, then $c \in k^{*2}$ so $l(a)\, l(b)\, l(c) = 0$. If $\alpha \notin k$, then $k(\alpha) \subset A$ is a commutative subfield splitting A. If k [and hence also $k(\alpha)$] is C_2 or local, the map $k_2(k) \to {}_2\operatorname{Br}(k)$, $l(a)\, l(b) \to [A]$

is known to be injective [6], so the symbol $l(a)\,l(b) \to 0$ in $k_2(k(\alpha))$. Since the reduced norm on A coincides with the field norm from $k(\alpha)$ to k, we may use the projection formula [2]:

$$l(a)\,l(b)\,l(c)=l(a)\,l(b)\,\mathrm{N}(l(\alpha))=\mathrm{N}_{k_3(k(\alpha))/k_3(k)}(l(a)\,l(b)\,l(\alpha))=0.$$

Q.E.D.

Remark (3.4). — I don't know whether $\mathrm{H}^1(\bar{k}/k, \mathrm{K}_2(\bar{\mathrm{F}})/\mathrm{K}_2(\bar{k}))$ is ever non-zero.

COROLLARY (3.5). — *Let* X *be a conic bundle surface over a local or global field* k. *Then* $\mathrm{A}_0(\mathrm{X})$ *is finite.*

Proof. — Combine (3.1), (3.3), the results in paragraph 2 on finiteness of Im Φ, and (1.6).

COROLLARY (3.6). — *Let* X *be a conic bundle surface over a non-archimedean local field, and assume* X *has good reduction over the residue field in the sense of* (2.3). *Then* $\mathrm{A}_0(\mathrm{X})=(0)$.

Proof. — Combine (2.3), (3.1), (3.3), and (1.6).

For the proof of (3.2), fix a conic bundle structure $\pi : \mathrm{X} \to \mathbb{P}^1_k$ on X. Let $\mathrm{K}=k(\mathbb{P}^1)$, $\bar{\mathrm{K}}=\bar{k}(\mathbb{P}^1)$. Note that $\mathrm{X}_{\bar{\mathrm{K}}} \cong \mathbb{P}^1_{\bar{\mathrm{K}}}$ so there exists an exact sequence of Gal $(\bar{k}/k)$-modules:

$$0 \to \mathrm{K}_2(\bar{\mathrm{K}}) \to \mathrm{K}_2(\bar{\mathrm{F}}) \to \coprod_{\bar{x}\in \mathrm{X}_{\bar{\mathrm{K}}}} \bar{\mathrm{K}}(\bar{x})^* \xrightarrow{\mathrm{N}} \bar{\mathrm{K}}^* \to 0. \tag{3.7}$$

Note also:

$$\mathrm{K}_2(\bar{\mathrm{K}}) \cong \mathrm{K}_2(\bar{k}) \oplus \coprod^{0}_{\bar{y}\in \mathbb{P}^1_{\bar{k}}} \bar{k}(\bar{y})^*. \tag{3.8}$$

Let $\mathcal{N} \subset \mathrm{K}^*$ be the image under the norm map of $\coprod_{x\in \mathrm{X}_{\mathrm{K}}} \mathrm{K}(x)^*$. It follows from (3.7) and (3.8) that there is an exact sequence:

$$0 \to \mathrm{H}^1(\bar{k}/k, \mathrm{K}_2(\bar{\mathrm{F}})/\mathrm{K}_2(\bar{k})) \to \mathrm{K}^*/\mathcal{N} \xrightarrow{\psi} \coprod_{y\in \mathbb{P}^1_k} \mathrm{Br}(k(y)). \tag{3.9}$$

Fix a, $b\in \mathrm{K}^*$ such that X_{K} is isomorphic to the conic curve defined by $\mathrm{T}_0^2-a\mathrm{T}_1^2-b\mathrm{T}_2^2=0$. We will work with $l(a)\,l(b)\in k_2(\mathrm{K})$ as well as with the quaternion algebra A defined by $\mathrm{T}^2=a$, $\mathrm{U}^2=b$, $\mathrm{TU}=-\mathrm{UT}$.

LEMMA (3.10). — *Let* $\mathrm{N} : \mathrm{A}^* \to \mathrm{K}^*$ *be the reduced norm. Then* $\mathrm{NA}^* \subset \mathcal{N}$.

Proof. — If $\alpha\in \mathrm{A}$, $\alpha \notin \mathrm{K}$ then $\mathrm{K}(\alpha)$ splits A so X_{K} has a $\mathrm{K}(\alpha)$-rational point. Since the norm from A coincides on $\mathrm{K}(\alpha)$ with the field norm we get $\mathrm{N}(\alpha)\in\mathcal{N}$. If $\alpha\in \mathrm{K}$ then $\mathrm{N}(\alpha)=\alpha^2\in\mathcal{N}$.

Q.E.D.

Returning to the proof of (3.2) we now have:

$$
(3.11)\qquad
\begin{tikzcd}
0 \arrow[r] & H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k})) \arrow[r] & K^*/\mathcal{N} \arrow[r, "\psi"] & \coprod_{y\in\mathbb{P}^1_k} Br(k(y)) \\
& & K^*/NA^* \arrow[u] \arrow[ur, "\psi'"'] &
\end{tikzcd}
$$

and it will suffice for the proof to show ψ' injective.

LEMMA (3.12). — *Let k be as in* (3.2). *The diagram below is commutative:*

$$
\begin{tikzcd}
K^*/NA^* \arrow[r, "\psi'"] \arrow[d, ".l(a).l(b)"'] & \coprod Br(k(y)) \\
k_3(K) \arrow[r, "\text{tame symbol}"] & \coprod k_2(k(y)). \arrow[u, "\text{galois symbol}"']
\end{tikzcd}
$$

Note that $NA^*.l(a).l(b)=(0)$ by the sort of projection formula argument already used in the proof of (3.3). I will postpone the proof of (3.12) to go directly to the main point.

LEMMA (3.13). — *The maps in the above square have the following properties:*

(i) "*galois symbol*" *is injective.*

(ii) "*tame symbol*" *has kernel* $k_3(k)\subset k_3(K)$.

(iii) $.l(a).l(b)$ *is injective.*

Proof. — Injectivity of the galois symbol follows from results of [6] together with our assumptions on k. The fact that Ker (tame) $=k_3(k)$ is proved in [10]. To show injectivity of the left-hand arrow, we consider the map defined in *op. cit.* :

$$k_3(K)\to I^3/I^4,$$
$$l(a)\,l(b)\,l(c)\to(\langle c\rangle+\langle -1\rangle)(\langle b\rangle+\langle -1\rangle)(\langle a\rangle+\langle -1\rangle),$$

where I is the augmentation ideal in the Witt ring of K. The above quadratic form is a *Pfister form* in the terminology of [7]. In particular, this form lies in I^4 if and only if it is *hyperbolic* (Theorem of Arason-Pfister, *op. cit.*, Cor. 3.4, p. 290). In particular, $l(a)\,l(b)\,l(c)=0$ in $k_3(K)$ implies there exists $x_1,\ldots,x_8\in K$ not all zero such that:

$$abcx_1^2-abx_2^2-acx_3^2-bcx_4^2+ax_5^2+bx_6^2+cx_7^2-x_8^2=0.$$

Formally, then we may factor and write:

$$c=\frac{abx_2^2-ax_5^2-bx_6^2+x_8^2}{abx_1^2-ax_3^2-bx_4^2+x_7^2}.$$

Notice that both numerator and denominator are norms from A^*. We may assume $A\not\cong M_2(K)$ else $X_K\cong\mathbb{P}^1_K$ and the whole discussion is silly. Thus the denominator in the above equation vanishes if and only if $x_1=x_3=x_4=x_7=0$. But vanishing of the denominator implies vanishing of the numerator, and hence of all the x_i, a contradiction. Hence, neither numerator nor denominator vanishes and we have written c as a norm from A^*.

Q.E.D.

The proof of (3.2) is now immediate. Indeed, $H^1(\bar{k}/k, K_2(\bar{F})/K_2(\bar{k}))$ is a quotient of $\operatorname{Ker}\psi'$ by (3.11), and $\operatorname{Ker}\psi' \subset k_3(k)$ by (3.12).

Q.E.D.

It remains to prove (3.12).

LEMMA (3.14). — *Let* X_K *be a conic curve over* $\operatorname{Sp} K$ *with* $\bar{K}/K$ *galois splitting* X_K. *Let* $[X_K] \in H^2(\bar{K}/K, \bar{K}^*)$ *be the class of* X_K *as a Severi-Brauer variety, and assume* $[X_K] \neq 0$. *Let* $\bar{F}$ = *quotient field of* $X_{\bar{K}}$ *and consider the exact sequence of* $\operatorname{Gal}(\bar{K}/K)$*-modules:*

$$0 \to \bar{K}^* \to \bar{F}^* \to \coprod_{\bar{x} \in X_{\bar{K}}} \mathbb{Z} \to \mathbb{Z} \to 0.$$

Then $[X_K] = \partial_2 \partial_1(1)$ *where* ∂_i *are the boundary maps associated to this exact sequence.*

Proof. — $\left(\coprod_{x \in X_{\bar{K}}} \mathbb{Z}\right)^{\operatorname{Gal}(\bar{K}/K)} = \coprod_{x \in X_K} \mathbb{Z}$. Since any $x \in X_K$ has even degree over $\operatorname{Sp} K$ [this follows from a norm argument using the fact that X_K splits over $K(x)$], we get an exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to H^2(\bar{K}/K, \bar{K}^*) \to H^2(\bar{F}/F, \bar{F}^*),$$
$$1 \mapsto \partial_2 \partial_1(1).$$

Since $[X_K] \to 0$ in $H^2(\bar{F}/F, \bar{F}^*)$, we are done.

Q.E.D.

Tensoring the sequence in (3.14) with K^* and using the symbol map, we obtain a commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & K^* \otimes_{\mathbb{Z}} K^* & \to & \bar{F}^* \otimes K^* & \to & \coprod_{\bar{x} \in X^1_{\bar{K}}} K^* & \to & K^* & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & K_2(\bar{K}) & \to & K_2(\bar{F}) & \to & \coprod \bar{K}(\bar{x})^* & \to & \bar{K}^* & \to 0.
\end{array}$$

Writing ∂'_i for the boundary maps on cohomology associated to the bottom row, we find a commutative triangle:

$$\begin{tikzcd}
K^* \arrow[rr, "{[X_K] \otimes \cdot}"] \arrow[dr, "\partial'_0 \circ \partial'_1"'] & & H^2(\bar{K}/K, \bar{K}^*) \otimes_{\mathbb{Z}} K^* \arrow[dl, "\text{symbol}"] \\
& H^2(\bar{K}/K, K_2(\bar{K})) &
\end{tikzcd}$$

Now take $\mathrm{K}=k(\mathbb{P}^1)$, $\overline{\mathrm{K}}=\bar{k}(\mathbb{P}^1)$ as in (3.12) and compose with the tame symbol to get a commutative square:

$$\begin{tikzcd}
\mathrm{K}^* \arrow[r, "\Psi"] \arrow[d, "{[\chi_\mathrm{K}]\otimes\cdot}"'] & \coprod_{y\in\mathbb{P}^1_k} \mathrm{Br}(k(y)) \\
\mathrm{H}^2(\bar{k}/k, \overline{\mathrm{K}}^*)\otimes\mathrm{K}^* \arrow[r] & \mathrm{H}^2(\bar{k}/k, \mathrm{K}_2(\overline{\mathrm{K}})) \arrow[u, "\text{tame}"']
\end{tikzcd}$$

The proof of (3.12) now follows from:

LEMMA (3.15). — *Let k be a field of characteristic $\neq 2$, $\bar{k}$ = separable closure of k. Let* K *be an extension field of transcendence degree* 1 *over k, and write $\overline{\mathrm{K}}=\mathrm{K}\bar{k}$. Let $a, b\in\mathrm{K}^*$ and write $l(a)\,l(b)\in k_2(\mathrm{K})$, $(a, b)\in {}_2\mathrm{Br}(\mathrm{K})$. Then for torsion prime to the characteristic:*

$$\mathrm{Br}(\mathrm{K})\cong\mathrm{H}^2(\bar{k}/k, \overline{\mathrm{K}}^*).$$

Moreover, if y is a place of K *over k with residue field $k(y)$, the diagram:*

(3.15.1)
$$\begin{tikzcd}[column sep=small]
 & \mathrm{H}^2(\bar{k}/k, \overline{\mathrm{K}}^*)\otimes\mathrm{K}^* \arrow[dr] & & \\
\mathrm{K}^* \arrow[ur, "{(a,b)\otimes\cdot}"] \arrow[dr, "{\cdot l(a)\cdot l(b)}"'] & & & \mathrm{H}^2(\bar{k}/k, \mathrm{K}_2(\overline{\mathrm{K}})) \arrow[dl, "\mathrm{T}_1"] \\
 & k_3(\mathrm{K}) \arrow[r, "\mathrm{Gal}\circ\mathrm{T}_2"] & \mathrm{Br}(k(y)) &
\end{tikzcd}$$

(*where* T_1 *and* T_2 *are tame symbols*) *commutes.*

Proof. — Replacing k by its perfect closure, we may assume $k(y)$ separable over k. Next replacing K by its completion at y we may suppose $k(y)\subset\mathrm{K}$ and $\overline{\mathrm{K}}=\coprod\overline{\mathrm{K}}_i$, one copy for each place lying over y. Now replacing the galois group by the decomposition group for one of the $\overline{\mathrm{K}}_i$ we may assume $k=k(y)$, $\mathrm{K}=k((\pi))$.

We now have split exact sequences:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{O}^* \arrow[r] & \mathrm{K}^* \arrow[r] & \mathbb{Z} \arrow[r] \arrow[l, bend left, "\pi \quad 1"] & 0,
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
0 \arrow[r] & \overline{\mathcal{O}}^* \arrow[r] & \overline{\mathrm{K}}^* \arrow[r] & \mathbb{Z} \arrow[r] \arrow[l, bend left] & 0.
\end{tikzcd}$$

Note that $1+\pi\overline{\mathcal{O}}$ is cohomologically trivial, so:

$$\mathrm{H}^2(\bar{k}/k, \overline{\mathrm{K}}^*)\cong\mathrm{H}^2(\bar{k}/k, \bar{k}^*)\oplus\mathrm{H}^2(\bar{k}/k, \mathbb{Z}).$$

Case 1. — $a, b\in\mathcal{O}^*$. Let $a_0, b_0\in k^*$ denote the mod π reductions of a, b. In this case $(a, b)=(a_0, b_0)\in\mathrm{H}^2(\bar{k}/k, \bar{k}^*)$. It is easy enough to see that going either way around (3.15.1), $f\in\mathrm{K}^*$ gets taken to $\mathrm{ord}(f).(a_0, b_0)\in\mathrm{Br}(k)$.

Using linearity, it remains only to consider.

Case 2. — $b=\pi$, $a\in\mathcal{O}^*$. In this case let $\mathrm{G}=\mathrm{Gal}(\bar{k}/k)$ and let $\rho:\mathrm{G}\times\mathrm{G}\to\mathbb{Z}$ be a 2-cocycle representing the image of a under the composition:

(3.15.2)
$$\begin{cases} \mathcal{O}^*\to\mathrm{H}^1(\overline{\mathrm{K}}/\mathrm{K}, \mu_2)\xrightarrow{\delta}\mathrm{H}^2(\overline{\mathrm{K}}/\mathrm{K}, \mathbb{Z}), \\ \qquad a\mapsto\chi_a. \end{cases}$$

Here δ is the coboundary from the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. Then (a, b) is represented by the cocycle:

$$G \times G \to \overline{K}^*, \qquad (g_1, g_2) \mapsto \pi^{\rho(g_1, g_2)}.$$

Indeed, one knows (*cf.* [13], p. 214) that:

$$(a, \pi) = \pi . \delta\chi_a. \quad \text{(cup product)}.$$

With reference to (3.15.1) we must show:

$$T_1 \{ \pi^{\rho(g_1, g_2)}, f \} = T_2 (l(a)\ l(\pi)\ l(f)).$$

If $f = -\pi$ this is clear as both sides are trivial. If, on the other hand, f is a unit with residue class f_0, we reduce to showing that $l(a_0)\ l(f_0) \in k_2(k)$ maps to the element in $\mathrm{Br}(k)$ represented by the cocyle $f_0^{\rho(g_1, g_2)}$. This follows as above with K in (3.15.2) replaced by k and $\overline{K}$ replaced by $\bar{k}$. This completes the proof of (3.15) and (3.12).

Brauer equivalence. Appendix

Let X be a rational surface over k, and let $\coprod_X^0 \mathbb{Z}$ be the group of 0-cycles of degree 0 on X. Following Manin ([8], [9]), we define a pairing:

$$(\ \ ):\ \coprod_X^0 \mathbb{Z} \times (\mathrm{Br}(X)/\mathrm{Br}(k)) \to \mathrm{Br}(k),$$

as follows. Given a cycle:

$$\sum n_i (x_i) \quad \text{and} \quad a \in \mathrm{Br}(X), \qquad (\sum n_i (x_i), a) = \prod_i \mathrm{cor}_{k(x_i)/k} (a(x_i))^{n_i}.$$

Two k-points x_1, x_2 are said to be Brauer equivalent (written $x_1 \underset{B}{\sim} x_2$) if $(x_1 - x_2, a) = 1$ for all $a \in \mathrm{Br}(X)$. Manin shows (*op. cit.*, Thm. 44.2) that $X(k)/B$ is finite for X a smooth cubic surface over a global field. In this appendix, I want to establish a compatibility between the Manin pairing and the map $\Phi : A_0(X) \to H^1(\bar{k}/k, N \otimes \bar{k}^*)$. It will follow that $X(k)/B$ is finite for *any* smooth rational surface over a global field, and that all points are Brauer equivalent at good reduction places. This explains and generalizes a number of Manin's calculations (*op. cit.*, 45.5, 45.6, 45.11, 45.12).

The key fact is:

PROPOSITION (A.1). — *The following diagram commutes:*

$$\begin{array}{ccc} \coprod\limits_{p\in X}^{0} \mathbb{Z}\times(\mathrm{Br}(X)/\mathrm{Br}(k)) & \xrightarrow{\ (\ ,\ )\ } & \mathrm{Br}(k)=H^2(\bar{k}/k,\bar{k}^*) \\ \downarrow & & \uparrow \scriptstyle\text{Intersection pairing} \\ A_0(X)\times H^1(\bar{k}/k,N) & \xrightarrow{\ \Phi\times 1\ } & H^1(\bar{k}/k,N\otimes\bar{k}^*)\times H^1(\bar{k}/k,N) \end{array}$$

[*Here the map* Br (X)/Br (k) $\to$ H^1 ($\bar{k}/k$, N) *arises from the spectral sequence* $H^p(\bar{k}/k, H^q(\bar{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)$.]

COROLLARY (A.2). — *Let k be a local or global field. Then* $X(k)/B$ *is finite.*

Proof. — Fix $x_0\in X(k)$ [if $X(k)=\emptyset$, of course, there is not much to prove] and define $\Psi: X(k)\to H^1(\bar{k}/k, N\otimes\bar{k}^*)$ by $\Psi(x)=\Phi((x)-(x_0))$. Of course Image $\Psi\subset$ Image Φ is finite. On the other hand, if $x, y\in X(k)$ and $x \underset{B}{\not\sim} y$ then $\Psi(x)\neq\Psi(y)$. Hence $\#(X(k)/B)\leqq\#\text{Image }\Phi$. Q.E.D.

COROLLARY (A.3). — *If k is non-archimedean local and* X *has good reduction over the residue field of k, then* $\#(X(k)/B)\leqq 1$.

Proof. — As above, using $\#(\text{Image }\Phi)=1$. Q.E.D.

We turn now to the proof of (A.1). Let $S\subset X$ be a union of divisors, and let $\bar{S}=S_{\bar{k}}\subset\bar{X}$. Let $\mathrm{Div}_{\bar{S}}(\bar{X})\subset\mathrm{Div}(\bar{X})=\coprod\limits_{D\subset\bar{X}}\mathbb{Z}$ be the group generated by divisors supported on $\bar{S}$, and assume S sufficiently large so $\mathrm{Div}_{\bar{S}}(\bar{X})\twoheadrightarrow N$. There is then an exact sequence:

$$0\to\Gamma(\bar{X}-\bar{S},\mathcal{O}_{\bar{X}}^*)/\bar{k}^*\to\mathrm{Div}_{\bar{S}}(\bar{X})\to N\to 0. \tag{A.4}$$

Given $\bar{D}\subset\bar{X}$ an irreducible divisor with $\bar{D}\not\subset\bar{S}$, let $\pi:\tilde{D}\to\bar{X}$ be the normalization, and let $\bar{k}(\bar{D})_{\bar{S}}^*$ denote those meromorphic functions on $\tilde{D}$ which are invertible at points of $\pi^{-1}(\bar{S})$. Define $Z_{\bar{S}}\subset Z$ and $(\coprod\limits_{X}^{0}\mathbb{Z})_{\bar{S}}\subset\coprod\limits_{\bar{X}}^{0}\mathbb{Z}$ so the sequence:

$$0\to Z_{\bar{S}}\to\coprod_{\bar{D}\not\subset\bar{S}}\bar{k}(\bar{D})_{\bar{S}}^*\to\left(\coprod_{\bar{X}}^{0}\mathbb{Z}\right)_{\bar{S}}\to 0, \tag{A.5}$$

is exact. We define a pairing:

$$\langle\ \rangle:\ \coprod_{\bar{D}\not\subset\bar{S}}\bar{k}(\bar{D})_{\bar{S}}^*\times\mathrm{Div}_{\bar{S}}(\bar{X})\to\bar{k}^*, \tag{A.6}$$

as follows. Let f be a function on $\tilde{D}$ with divisor (f) supported off $\pi^{-1}(\bar{S})$. Let $\Delta\subset\bar{S}$ be a divisor and write:

$$\pi^{-1}\Delta=\sum n_i(d_i)\quad\text{(as a cycle)}.$$

Define:

(A.7)
$$\langle f|_{\bar{\mathrm{D}}},\, \Delta\rangle = \prod f(d_i)^{n_i}.$$

Lemma (A.8). — *The following diagram is a commutative diagram of pairings:*

$$\begin{tikzcd}
\coprod\limits_{\bar{\mathrm{D}}\not\subset\bar{\mathrm{S}}} \bar{k}(\bar{\mathrm{D}})^*_{\bar{\mathrm{S}}} \arrow[d] & \times\, \mathrm{Div}_{\bar{\mathrm{S}}}(\bar{\mathrm{X}}) \arrow[r] & \bar{k}^* \arrow[d, equal] \\
\Big(\coprod\limits_{\bar{\mathrm{X}}}^{0} \mathbb{Z}\Big)_{\bar{\mathrm{S}}} & \times\, \Gamma(\bar{\mathrm{X}}-\bar{\mathrm{S}}, \mathcal{O}^*_{\mathrm{X}})/\bar{k}^* \arrow[u, "\mathrm{div}"'] \arrow[r] & \bar{k}^*.
\end{tikzcd}$$

Here the top pairing is defined in (A.6) *and the bottom is evaluation. The left hand vertical arrow is from* (A.5).

Proof. — Given $g\in\Gamma(\bar{\mathrm{X}}-\bar{\mathrm{S}}, \mathcal{O}^*_{\mathrm{X}})$ and given

$$\beta=\sum n_j(b_j)\in\Big(\coprod^{0}\mathbb{Z}\Big)_{\bar{\mathrm{S}}},$$

there exists $\sum f_i|_{\bar{\mathrm{D}}_i}\mapsto\beta$ with $f_i\in\bar{k}(\bar{\mathrm{D}}_i)^*_{\bar{\mathrm{S}}}$. We have:

$$\langle g,\, \beta\rangle = h(0)\,h(\infty)^{-1}, \qquad h=\prod \mathrm{N}_{g,\,\bar{\mathrm{D}}_i}(f_i).$$

Note the equality of divisors $(h)=g_*(\beta)=\sum n_j g(b_j)$. Thus $h=\mathrm{C}.\prod(t-g(b_j))^{n_j}$, where t is the standard parameter on $\mathbb{P}^1$ and C is some constant. It follows that:

$$h(0)\,h(\infty)^{-1}=\prod g(b_j)^{n_j}.$$

Q.E.D.

The following abstract result in group cohomology comes from setting $\mathrm{C}=\mathrm{C}'$, $\mathrm{C}''=0$ in ([16], Thm. 6, p. 112).

Lemma (A.9). — *Let* G *be a group,* A, A′, A″, B, B′, B″, *and* C *all* G-*modules. Suppose given exact sequences:*

$$0\to\mathrm{A}'\to\mathrm{A}\to\mathrm{A}''\to 0,$$
$$0\to\mathrm{B}'\to\mathrm{B}\to\mathrm{B}''\to 0$$

and a G-*pairing* $\mathrm{A}\times\mathrm{B}\xrightarrow{\langle\ \rangle}\mathrm{C}$. *Assume* $\langle\mathrm{A}',\mathrm{B}'\rangle=0$ *so there are induced pairings:*

$$\mathrm{A}'\times\mathrm{B}''\to\mathrm{C}, \qquad \mathrm{A}''\times\mathrm{B}'\to\mathrm{C}.$$

Then given $a \in \mathrm{H}^p(\mathrm{G}, \mathrm{A}'')$, $b \in \mathrm{H}^q(\mathrm{G}, \mathrm{B}'')$ *we have:*

$$\langle \partial_{\mathrm{A}} a, b \rangle = (-1)^{p+1} \langle a, \partial_{\mathrm{B}} b \rangle \in \mathrm{H}^{p+q+1}(\mathrm{G}, \mathrm{C}).$$

(*Here* ∂_{A} *and* ∂_{B} *are boundary maps for the corresponding long exact sequence of cohomology.*)

Applying this result to the sequences (A.4), (A.5) and the pairing into $\bar{k}^*$ described above, we get a commutative diagram of pairings:

$$\text{(A.10)} \qquad \begin{tikzcd} \mathrm{H}^0(\bar{k}/k, (\overset{\circ}{\coprod} \mathbb{Z})_{\bar{\mathrm{S}}}) \arrow[d, "\partial"] \times \mathrm{H}^2(\bar{k}/k, \Gamma(\bar{\mathrm{X}}-\bar{\mathrm{S}}, \mathcal{O}^*)/\bar{k}^*) \arrow[drr, "\text{evaluate}"] & & \\ \mathrm{H}^1(\bar{k}/k, \mathrm{Z}_{\bar{\mathrm{S}}}) \times \mathrm{H}^1(\bar{k}/k, \mathrm{N}) \arrow[u, "\partial"'] \arrow[rr, "\langle\ \rangle"] & & \mathrm{Br}(k) = \mathrm{H}^2(\bar{k}/k, \bar{k}^*) \end{tikzcd}$$

[Commutativity means $\langle \partial z, n \rangle =$ evaluation $(z, \partial n)$.]

The next step is to show that the pairing:

$$\mathrm{Z}_{\bar{\mathrm{S}}} \times \mathrm{N} \to \bar{k}^*,$$

factors through the intersection pairing $\mathrm{N} \otimes \bar{k}^* \otimes \mathrm{N} \to \bar{k}^*$ by means of $\mathrm{Z}_{\bar{\mathrm{S}}} \to \mathrm{Z} \to \mathrm{N} \otimes \bar{k}^*$. This is essentially a consequence of Weil reciprocity as follows. Think of $\mathrm{N} \otimes \bar{k}^* \simeq \mathrm{H}^1(\bar{\mathrm{X}}, \mathscr{K}_2)$. Given $\Delta \subset \bar{\mathrm{S}}$ an irreducible divisor, let $\pi : \tilde{\Delta} \to \bar{\mathrm{X}}$ be the normalization. There is a diagram:

$$\begin{tikzcd} \mathrm{K}_2(\bar{k}(\tilde{\Delta})) \arrow[r] \arrow[dr, "0"'] & \coprod_{x \in \tilde{\Delta}} \bar{k}(x)^* \arrow[r] \arrow[d, "\text{norm}"] & \mathrm{H}^1(\tilde{\Delta}, \mathscr{K}_2) \arrow[r] \arrow[dl, dashed] & 0 \\ & \bar{k}^* & & \end{tikzcd}$$

with exact top row. The diagonal arrow is zero by the reciprocity [*cf.* Lemma (A.8)], whence an induced map $\mathrm{H}^1(\tilde{\Delta}, \mathscr{K}_2) \to \bar{k}^*$.

If we associate to Δ the composed map:

$$\varphi_\Delta : \ \mathrm{H}^1(\bar{\mathrm{X}}, \mathscr{K}_2) \to \mathrm{H}^1(\tilde{\Delta}, \mathscr{K}_2) \to \bar{k}^*,$$

we obtain a pairing:

$$\mathrm{H}^1(\bar{\mathrm{X}}, \mathscr{K}_2) \times \mathrm{Div}_{\bar{\mathrm{S}}}(\bar{\mathrm{X}}) \to \bar{k}^*.$$

One verifies quite easily ([10]) that the diagram of pairings:

$$\text{(A.11)} \qquad \begin{tikzcd}
N\otimes\bar{k}^* \times N \arrow[drr, "\text{intersection pairing}"] & & \\
H^1(\bar{X},\mathcal{K}_2)\times \mathrm{Div}_{\bar{S}}(\bar{X}) \arrow[u, "\wr\|"'] \arrow[rr] & & \bar{k}^* \\
Z_{\bar{S}} \times \mathrm{Div}_{\bar{S}}(\bar{X}) \arrow[u, "\uparrow \quad \|"'] \arrow[urr, "\langle\ \rangle"'] & &
\end{tikzcd}$$

commutes.

Finally, noting that Pic $(\bar{X}-\bar{S})=(0)$, one obtains a commutative diagram up to sign ([5], appendice):

$$\text{(A.12)} \qquad \begin{tikzcd}
\mathrm{Br}(X)/\mathrm{Br}(k) \arrow[r] \arrow[d] & \mathrm{Br}(X-S)/\mathrm{Br}(k) \\
H^1(\bar{k}/k, N) \arrow[r, "\partial"] & H^2(\bar{k}/k, \Gamma(\bar{X}-\bar{S}, \mathcal{O}^*)/\bar{k}^*). \arrow[u]
\end{tikzcd}$$

Combining (A.10), (A.11), (A.12) one obtains the assertion of (A.1).

Q.E.D.

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([10]) Use functoriality of the map pic $(\bar{X})\otimes\bar{k}^* \to H^1(\bar{X}, K_2)$ for the morphism $\tilde{\Delta} \to X$, the fact that Pic $\bar{X} \to$ Pic $\bar{\Delta} \xrightarrow{\deg} \mathbb{Z}$ is the intersection number with $\bar{\Delta}$, and the above diagram.

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