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ON EXTREMAL SOLUTIONS OF MARTINGALE PROBLEMS

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Section (0). Introduction

This paper consists of two parts which, even in the eyes of the authors, are only loosely related to one another. Thus we deem it to be appropriate to split the introduction into two parts: the first part being for sections (1) through (4), and the second part for sections (5) through (9).

To explain what we are doing in sections (1)-(4), recall the following facts. If $\Omega=C([0,\infty),R^1)$ and $x(t)$ is the usual coordinate mapping on Ω, then Lévy showed that Wiener measure $\mathscr{W}$ is the one and only probability measure P on Ω such that $P(x(0)=0)=1$, $x(t)$ is a P-martingale, and $x^2(t)-t$ is a P-martingale. In other words, $\mathscr{W}$ is uniquely characterized by the equations

$$\mathscr{W}(x(0)=0)=1$$

and

$$E^{\mathscr{W}}[(x(t_2)-x(t_1)), A]=E^{\mathscr{W}}[(x^2(t_2)-x^2(t_1)-(t_2-t_1)), A]=0$$

for all $0\leqq t_1<t_2$ and $A\in\mathscr{M}_{t_1}\equiv\sigma(x(t):0\leqq t\leqq t_1)$. Lévy's beautiful characterization shows that the functions $(x(t_2)-x(t_1))\chi_A$ and $(x^2(t_2)-x^2(t_1)-(t_2-t_1))\chi_A$, $0\leqq t_1<t_2$ and $A\in\mathscr{M}_{t_1}$, must play a central role in the structure of $\mathscr{W}$. In fact, since $x^2(t)-t=2\int_0^t x(u)\,dx(u)$ (a.s., $\mathscr{W}$) and $\int_0^t x(u)\,dx(u)$ is in the $L^2(\mathscr{W})$-closure of span $\{(x(t_2)-x(t_1))\chi_A:0\leqq t_1<t_2$ and $A\in\mathscr{M}_{t_1}\}$), one is lead to suspect that the functions in the class $\{(x(t_2)-x(t_1))\chi_A:0\leqq t_1<t_2$ and $A\in\mathscr{M}_{t_1}\}$ are by themselves the building blocks out of which all of $L^2(\mathscr{W})$ can be constructed. This suspicion is most dramatically confirmed by the Itô-Wiener theory of "homogeneous chaos" (*cf.* exercise 4.6.14 in [*S. and V.*]) from which one easily derives Itô's representation of every $\Phi\in L^2(\mathscr{W})$ in the form $E^{\mathscr{W}}[\Phi]+\int_0^\infty\theta(u)\,dx(u)$, where θ is a progressively

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measurable function with $E^{\mathscr{W}}\left[\int_0^\infty (\theta(u))^2\,du\right] < \infty$. What was not immediately apparent for many years is that Itô's representation theorem is, in fact, not only motivated by Lévy's characterization of $\mathscr{W}$ but is also a quite easy direct consequence of it [*cf.* remark (1.2) below]. The line of reasoning which connects these two properties of $\mathscr{W}$ was first investigated by Jacod [13] and later by Jacod and Yor [14] and Yor [16]. The essential ingredient which makes it easy to establish the connection between Lévy's and Itô's results is the theorem of R. Douglas: Theorem (1.1) below.

In section (1), we present Douglas's theorem and show how (in conjunction with a theorem about the convergence of sequences of martingales) it provides a simple proof of Itô's representation theorem.

Having introduced Douglas's theorem and shown how it can be applied in a special situation, we turn in section (2) to applications of Douglas's theorem to the study of more general martingale problems (*cf.* Chapt. VI of [*S. and V.*]). The point here is that even when a given martingale problem admits more than one solution, Douglas's theorem enables one to characterize the extreme solutions of this problem. The reason why one is interested in obtaining such a characterization is that all solutions to a given martingale problem can be reconstructed from a certain subset of the extreme ones: namely, those which are members of time-homogeneous strong Markov selections [*cf.* Thm. (2.9) below]. (A word of warning must be given at this point. For us, a *stopping* time τ is a function on the sample space such that $\{\tau \leqq t\}$ is measurable with respect to the path up until time t.

This is to be distinguished from what we call *extended stopping times* τ for which the condition is on $\{\tau < t\}$ instead of $\{\tau \leqq t\}$. In keeping with this convention, we say that a Markov process is *strong Markov* if it enjoys the Markov property at stopping times even if it does not at extended stopping times. That this distinction is real can be seen from example (2.13) below). In this connection, we show that under mild conditions, there always exists a selection of solutions which is strong Markov with respect to extended stopping times and which consists of extreme solutions; however, we have been unable to show that every such selection is necessarily made up of extreme solutions. If one drops the strong Markov condition for extended stopping times, then example (2.13) shows that the selection need not consist of extreme solutions.

Section (3) has as its basis the simple observation that every Markov process can be viewed as the unique solution of a martingale problem [*cf.* Lemma (3.1)]. Combining this observation with Douglas's theorem, we obtain a very simple proof of an important theorem due to Kunita and Watanabe [*cf.* Thm. (3.2) and remark (3.3)]. The machinery thereby obtained enables us to develop some criteria for determining when a given Markov process is made up of extreme solutions to the martingale problem determined by the operator of which its true generator is an extension [*cf.* Theorems (3.17) and (3.19) as well as Corollary (3.20)].

Section (4) is a collection of examples. Unfortunately, these examples serve best to demonstrate just how weak is our present understanding of the structure of the solutions to a martingale problem for which there is more than one solution. Nonetheless, it is our impression that, with the exception of Girsanov's now classic example [9], the literature contains very few instances of ill-posed martingale problems for which the structure of the solutions has been completely worked out. Section (4) can be viewed as a feeble attempt to remedy this situation.

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The second part of the paper is wholly devoted to the study of the extreme points of the (convex) set of all continuous one-dimensional (local) martingale distributions. Of course, the general results of the first two sections apply here and again allow us to relate extremality to a martingale representation property.

The starting point of our study, expounded in section (5) of this paper, is the Dubins-Schwarz theorem [20] which gives a sufficient condition for extremality in terms of the Brownian motion of which the martingale is a random time change (*cf.* Dambis [18] and Dubins-Schwarz [19]).

As discovered by Dubins and Schwarz [20], and again by Yor [32], this sufficient condition of Dubins and Schwarz is not necessary. We hope to clarify this situation with a result further relating extremality and the Dubins-Schwarz condition [*cf.* Thm. (7.3)] as well as a procedure – developed in section (6) – for constructing new examples of extreme martingales which do not satisfy the Dubins-Schwarz condition: in particular, Tsirel'son's example [29] of a stochastic differential equation having no strong non-anticipating solution allows to construct such a martingale [*cf.* Thm. (6.4)].

Finally, we conclude the second part with a list of problems whose resolution we believe to be the major questions left in this area.

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Section (1)

Much of what follows in this paper turns on the following simple observation due to Douglas [1]. We include the proof of Douglas's theorem only because it is so short.

THEOREM (1.1) (Douglas). – *Let* $(\Omega, \mathscr{M})$ *be a measurable space and let* $\mathscr{F}$ *be a set of* $\mathscr{M}$*-measurable* $f: \Omega \to \mathbf{R}^1$. *Define* $\mathscr{N}(\mathscr{F})$ *to be the set of all probability measures* μ *on* $(\Omega, \mathscr{M})$ *such that* $\mathscr{F} \subseteqq \mathrm{L}^1(\mu)$ *and* $\int f\,d\mu = 0$, $f \in \mathscr{F}$. *Then* $\mathscr{N}(\mathscr{F})$ *is convex, and* $\mu \in \mathscr{N}(\mathscr{F})$ *is an extreme point if and only if* $1 \oplus \operatorname{span}(\mathscr{F})$ *is dense in* $\mathrm{L}^1(\mu)$.

Proof. – Certainly $\mathscr{N}(\mathscr{F})$ is convex. Suppose that $\mu \in \mathscr{N}(\mathscr{F})$ is extreme and that $1 \oplus \operatorname{span}(\mathscr{F})$ is not dense in $\mathrm{L}^1(\mu)$. Then, by the Hahn-Banach theorem, we can find a non-zero $h \in \mathrm{L}^\infty(\mu)$ such that $\int h\,d\mu = 0$ and $\int hf\,d\mu = 0$ for each $f \in \mathscr{F}$. Furthermore, we may assume that $-1/2 \leqq h \leqq 1/2$ everywhere.

Now define ν_+ and ν_- so that $d\nu_\pm = (1 \pm h)\, d\mu$. Clearly $\nu_+, \nu_- \in \mathcal{N}(\mathcal{F})$ and $\nu_+ \neq \nu_-$. But $\mu = 1/2\, \nu_+ + 1/2\, \nu_-$, and so we have a contradiction.

Next suppose that $\mu \in \mathcal{N}(\mathcal{F})$, $1 \oplus \mathrm{span}(\mathcal{F})$ is dense in $L^1(\mu)$, and $\mu = \theta \nu_1 + (1-\theta)\nu_2$, where $0 < \theta < 1$ and $\nu_1, \nu_2 \in \mathcal{N}(\mathcal{F})$. Clearly, $\nu_i \ll \mu$ and $d\nu_i/d\mu \in L^\infty(\mu)$ for $i = 1, 2$. Hence $1 \oplus \mathrm{span}(\mathcal{F})$ is dense in $L^1(\nu_i)$ for $i = 1, 2$. At the same time, it is obvious that $\int g\, d\nu_1 = \int g\, d\nu_2$ for all $g \in 1 \oplus \mathrm{span}(\mathcal{F})$. Thus, $\nu_1 = \nu_2$.

Q.E.D.

Remark (1.2). — Before proceeding, we want to show how Douglas's theorem relates to the problem of representing martingales. For this purpose, consider the following example: $\Omega = C([0, \infty), R^1)$; for $\omega \in \Omega$, $x(t, \omega)$ is the position of ω at time $t \geqq 0$; $\mathcal{M}_t = \sigma(x(s) : 0 \leqq s \leqq t)$; $\mathcal{M} = \bigvee_{(t \geqq 0)} \mathcal{M}_t$ and $\mathcal{F}$ consists of functions of the form

$$f = (\varphi(x(t_2)) - \varphi(x(t_1)) - \int_{t_1}^{t_2} 1/2\, \varphi''(x(s))\, ds)\, \chi_A$$

where $0 \leqq t_1 < t_2$, $A \in \mathcal{M}_{t_1}$, and $\varphi \in C_0^\infty(R^1)$. Then one can easily show that Wiener measure $\mathcal{W}$ is the one and only element P of $\mathcal{N}(\mathcal{F})$ such that $P(x(0) = 0) = 1$ (*cf.* [*S. and V.*], Thm. 4.1.1). In particular, $\mathcal{W}$ is an extreme element of $\mathcal{N}(\mathcal{F})$ and so $1 \oplus \mathrm{span}(\mathcal{F})$ is dense in $L^1(\mathcal{W})$. At the same time, by Itô's formula

$$(\varphi(x(t_2)) - \varphi(x(t_1)) - \int_{t_1}^{t_2} 1/2\, \varphi''(x(s))\, ds)\, \chi_A = \int_0^\infty \theta(s)\, dx(s)$$

where

$$\theta(s, \omega) = \chi_A(\omega)\, \chi_{]t_1, t_2]}(s)\, \varphi'(x(s, \omega)).$$

Hence every $g \in 1 \oplus \mathrm{span}(\mathcal{F})$ is of the form

$$X = c + \int_0^\infty \theta(s)\, dx(s) \tag{1.3}$$

where $c \in R^1$ and $\theta : [0, \infty) \times \Omega \to R^1$ is previsible.

In other words, Douglas's theorem by itself shows that every element of $L^1(\mathcal{W})$ is the $L^1(\mathcal{W})$ limit of X's having the form given in (1.3). This is very close to Itô's representation theorem but is a slightly weaker statement. In order to complete the proof of Itô's theorem, we use the following fact about L^1-convergent martingales (*cf.* M. Yor [15]).

THEOREM (1.4). — *Let $(\Omega, \mathcal{M}, P)$ be a probability space and $\{\mathcal{M}_t : t \geqq 0\}$ a non-decreasing family of sub σ-algebras of $\mathcal{M}$ such that $\mathcal{M} = \sigma(\bigcup_{t \geqq 0} \mathcal{M}_t)$. Given $X \in L^1(P)$, let $X(t)$ be a right-continuous version of $E^P[X \mid \mathcal{M}_{t+0}]$ ($\mathcal{M}_{t+0} = \bigcap_{\varepsilon > 0} \mathcal{M}_{t+\varepsilon}$). Suppose that $X_n \to X$ in $L^1(P)$. Then there exist a sequence of stopping times $\{\tau_j\}_{j=1}^\infty$ and a subsequence $\{n_k\}_{k=1}^\infty$ of*

Z^+ *such that* $\tau_j \uparrow \infty$ *(a. e.,* P) *and for each* $j \geqq 1$:

$$E^P[\sup_{0\leqq t\leqq \tau_j} |X(t)|] < \infty \quad \textit{and} \quad E^P[\sup_{0\leqq t\leqq \tau_j} |X_{n_k}(t)-X(t)|] \to 0 \quad \textit{as } k\uparrow\infty.$$

Furthermore, if in addition for each n, $X_n(t)$ *is (a. s.,* P)*-continuous, then* $\{n_k\}_{k=1}^{\infty}$ *and* $\{\tau_j\}_{j=1}^{\infty}$ *can be chosen so that*

$$\sup_k \sup_{0\leqq t\leqq \tau_j} |X_{n_k}(t)-X(t)| \vee |X(t)| \leqq j \qquad \text{P } a.s. \quad on\ (\tau_j>0),$$

and therefore $E^P[\sup_{0\leqq t\leqq \tau_j} |X_{n_k}(t)-X(t)|^r;\ 1_{(\tau_j>0)}] \to 0$ *for each* $1 \leqq r < \infty$.

Proof. — Choose $\{n_k\}_{k=1}^{\infty}$ so that $\sum_k E^P[|X_{n_k}-X|]<\infty$ and define

$$\tau_j = \inf\{t\geqq 0 : (\exists k)|X_{n_k}(t)-X(t)| \vee |X(t)| \geqq j\}.$$

Clearly

$$E^P[\sup_{0\leqq t\leqq \tau_j} |X(t)|] \leqq j + E^P[|X(\tau_j)|] \leqq j + E[|X|] < \infty.$$

Also

$$E^P[\sup_{0\leqq t\leqq \tau_j} |X_{n_k}(t)-X(t)|] \leqq E^P[\sup_{0\leqq t< \tau_j} |X_{n_k}(t)-X(t)|] + E^P[|X_{n_k}(\tau_j)-X(\tau_j)|].$$

Since $E^P[|X_{n_k}(\tau_j)-X(\tau_j)|] \leqq E^P[|X_{n_k}-X|]$, the second term tends to 0. As for the first term, note that $\sup_{0\leqq t<\tau_j} |X_{n_k}(t)-X(t)| \leqq j$ for all k and that, by Doob's inequality, for each $\varepsilon>0$:

$$P(\sup_{t\geqq 0}|X_n(t)-X(t)| \geqq \varepsilon) \leqq \frac{1}{\varepsilon} E^P[|X_n - X|] \to 0.$$

Hence, by Lebesgue's dominated convergence theorem, the first term also tends to zero. Thus, we need only check that $\tau_j \uparrow \infty$ (a. s., P). But by Doob's inequality

$$P(\tau_j \leqq t) \leqq P(\sup_{0\leqq s\leqq t} |X(s)| \geqq j) + \sum_k P(\sup_{0\leqq s\leqq t} |X_{n_k}(s)-X(s)| \geqq j)$$
$$\leqq \frac{1}{j}(E^P[|X|] + \sum_k E^P[|X_{n_k}-X|]) \to 0 \qquad \text{as } j\uparrow\infty.$$

Finally, assume that $X_n(t)$ is (a. s., P)-continuous for each n. Then, by Doob's inequality, $X(t)$ is (a. s., P)-continuous. Hence

$$\sup_{0\leqq t\leqq \tau_j} |X_{n_k}(t)-X(t)| \vee |X(t)| \leqq j, \text{ P a.s.} \quad \text{on } (\tau_j>0).$$

Q.E.D.

Completion of Remark (1.2). — We can now finish the proof of Itô's representation theorem. Indeed, suppose that $X \in L^2(\mathscr{W})$. Choose $\{X_n\}_1^\infty$ of the form given in (1.3) so that $X_n \to X$ in $L^1(\mathscr{W})$. Clearly we can suppose that $E(X)=E(X_n)=0$. Furthermore, $X_n(t)$ is (a. s., $\mathscr{W}$) continuous for each $n \geqq 1$. Hence we can choose $\{\tau_j\}_{j=1}^\infty$ and $\{n_k\}_{k=1}^\infty$ as in Theorem (1.4) so that

$$E^{\mathscr{W}}[\sup_{0 \leqq t \leqq \tau_j} |X_{n_k}(t)-X(t)|^2] \to 0.$$

In particular, this means that

$$\lim_{k\to\infty} \sup_{l \geqq k} E^{\mathscr{W}}\left[\int_0^{\tau_j} |\theta_{n_l}(s)-\theta_{n_k}(s)|^2\,ds\right]=0.$$

Thus we can find a previsible θ such that

$$\int_0^{\tau_j} \theta(s)\,dx(s)=X(\tau_j) \qquad \text{and} \qquad E\left[\int_0^{\tau_j} \theta(s)^2\,ds\right]<\infty.$$

In particular,

$$E^{\mathscr{W}}\left[\int_0^\infty |\theta(s)|^2\,ds\right]=\lim_{j\uparrow\infty} E^{\mathscr{W}}\left[\int_0^{\tau_j} |\theta(s)|^2\,ds\right]=\lim_{j\uparrow\infty} E[(X(\tau_j))^2]=E^{\mathscr{W}}[X^2]$$

and obviously

$$X=\int_0^\infty \theta(s)\,dx(s),$$

which is exactly what Itô's theorem asserts.

For future reference, we will want a generalization of the line of reasoning just carried out to complete Remark (1.2).

THEOREM (1.5). — *Let* $(\Omega, \mathscr{M}, P)$ *be a probability space and let* $\{\mathscr{M}_t : t \geqq 0\}$ *be a non-decreasing family of sub* σ*-algebras of* $\mathscr{M}$ *such that* $\mathscr{M}=\bigvee_{t\geqq 0}\mathscr{M}_t$.

Let $X=(X_1, \ldots, X_d) : [0, \infty)\times\Omega \to R^d$ *be a continuous* $(\mathscr{M}_t)$*-adapted function such that* $E^P[|X(t)|^2]<\infty$ *for all* $t \geqq 0$ *and* $(X_i(t), \mathscr{M}_t, P)$ *is a martingale for* $1 \leqq i \leqq d$. *If* $\mathscr{S}(X)$ *denotes the set of all functions*

$$Y=\int_0^\infty \langle \theta(s), dX(s)\rangle$$

where $\theta : [0, \infty)\times\Omega \to R^d$ *is previsible and*

$$E^P\left[\sum_{i,j=1}^d \int_0^\infty (\theta_i\theta_j)(s)\,d\langle X_i, X_j\rangle_s\right]<\infty \quad (^1)$$

(1) It is well known that the measure $\sum_{i,j} \theta_i\theta_j(s)\,d\langle X_i, X_j\rangle_s$ is positive.

then

$$\overline{\mathcal{S}(X)}^{L^1(P)} \cap L^2(P) = \mathcal{S}(X).$$

Proof. – The argument is essentially the same as the one given in the completion of Remark (1.2).

Section (2)

In this section we apply the ideas of section (1) to the study of "martingale problems". To begin with, we will work with a general formulation of a martingale problem, later we will specialize.

Let E be a Polish space (i. e. a complete separable metric space) and let $\Omega = D([0, \infty), E)$ be the Skorohod space of right-continuous functions $\omega : [0, \omega) \to E$ having left-limits. For $t \geqq 0$, $x(t, \omega)$ will denote the position of ω at time t. $\mathcal{M}$ is the Borel field over Ω and $\mathcal{M}_t = \sigma(x(s) : 0 \leqq s \leqq t)$, $t \geqq 0$. Let $\underline{\Psi}$ be a countable collection of right-continuous $\mathcal{M}_t$-progressively measurable functions $X : [0, \infty) \times \Omega \to R^1$ such that $\sup_{\substack{0 \leqq t \leqq T \\ \omega \in \Omega}} |X(t, \omega)| < \infty$, $T > 0$. Given $s \geqq 0$, we define $\mathcal{P}_s(\underline{\Psi})$ to be the set of all probability measures P on $(\Omega, \mathcal{M})$ such that $(X(t \vee s), \mathcal{M}_{t \vee s}, P)$ is a martingale. For $(s, x) \in [0, \infty) \times E$, let $\mathcal{P}_{s, x}(\underline{\Psi}) = \{ P \in \mathcal{P}_s(\underline{\Psi}) : P(x(s) = x) = 1 \}$. The following theorem can be easily deduced from Theorem (1.2.10) in [*S. and V.*].

THEOREM (2.1). – *Let $\tau : \Omega \to [s, \infty)$ be an $\mathcal{M}_t$-stopping time (i.e. $\{\tau \leqq t\} \in \mathcal{M}_t$ for $t \geqq s$). If $P \in \mathcal{P}_s(\underline{\Psi})$ and $\{P_\omega\}$ is an r. c. p. d. of $P | \mathcal{M}_\tau$* (cf. p. 34 of [*S. and V.*]), *then there is a P-null set $N \in \mathcal{M}_\tau$ such that $P_\omega \in \mathcal{P}_{\tau(\omega), x(\tau(\omega), \omega)}(\underline{\Psi})$ for all $\omega \notin N$. Conversely, if $\omega \to Q_\omega \in \mathcal{P}_{\tau(\omega), x(\tau(\omega), \omega)}$ is an $\mathcal{M}_\tau$-measurable map, then $P \otimes_{\tau(.)} Q_. \in \mathcal{P}_s(\underline{\Psi})$* (cf. Thm. (6.1.2) *of* [*S. and V.*] *for the definition of* $P \otimes_{\tau(.)} Q_.$).

Now define

$$(2.2) \qquad \mathcal{F}_s(\underline{\Psi}) = \{ (X(t_2) - X(t_1)) \chi_A : s \leqq t_1 < t_2,\ X \in \underline{\Psi}, \text{ and } A \in \mathcal{M}_{t_1} \}.$$

Then it is clear that $\mathcal{P}_s(\underline{\Psi}) = \mathcal{N}(\mathcal{F}_s(\underline{\Psi}))$. We can use this fact to study $\operatorname{ext}(\mathcal{P}_s(\underline{\Psi}))$ [the set of extreme elements of $\mathcal{P}_s(\underline{\Psi})$].

THEOREM (2.3). – *$P \in \operatorname{ext}(\mathcal{P}_s(\underline{\Psi}))$ if and only if $1 \oplus \operatorname{span}(\mathcal{F}_s(\underline{\Psi}))$ is dense in $L^1(P)$. Moreover, if $P \in \operatorname{ext}(\mathcal{P}_s(\underline{\Psi}))$, then for every $Y \in L^1(P)$, $E^P[Y | \mathcal{M}_{. \vee s}]$ admits a right-continuous adapted version; and if, in addition, $X(. \vee s)$ is P-a. s. continuous for all $X \in \underline{\Psi}$, then this version of $E^P[Y | \mathcal{M}_{. \vee s}]$ will also be P-a. s. continuous. In particular, if $\tau : \Omega \to [s, \infty)$ is an extended stopping time (i.e. $\{\tau < t\} \in \mathcal{M}_t$ for $t \geqq 0$), then $\mathcal{M}_\tau^{(-)} = \mathcal{M}_{\tau+0}$ (a.s., P), where $\mathcal{M}_\tau^{(-)} \equiv \sigma[\tau$ and $x(t \wedge \tau)$, $t \geqq 0]$ and $\mathcal{M}_{\tau+0} \equiv \bigcap_{\varepsilon > 0} \mathcal{M}_{\tau+\varepsilon}$.*

Proof. — Since $\mathscr{P}_s(\overline{\Psi})=\mathscr{N}(\mathscr{F}_s(\overline{\Psi}))$, the characterization of $\text{ext}(\mathscr{P}_s(\overline{\Psi}))$ is just the content of Douglas's theorem. In the proof of the rest of this theorem, we may and will assume that $s=0$.

To prove that $E^P[Y|\mathscr{M}_.]$ admits a right-continuous progressively measurable version for all $Y\in L^1(P)$, first note that this is obvious when $Y\in 1\oplus\text{span}(\mathscr{F}_0(\overline{\Psi}))$. Moreover, if $Y_n:[0,\infty)\times\Omega\to R^1$, $n\geqq 1$, is a right-continuous progressively measurable function and if $\lim_{m\to\infty}\sup_{n\geqq m}P(\sup_{t\geqq 0}|Y_n(t)-Y_m(t)|\geqq\varepsilon)=0$ for all $\varepsilon>0$, then, by Lemma 4.3.3 in [*S. and V.*], there exists a right-continuous progressively measurable $Y:[0,\infty)\times\Omega\to R^1$ such that $P(\sup_{t\geqq 0}|Y_n(t)-Y(t)|\geqq\varepsilon)\to 0$ as $n\to\infty$ for each $\varepsilon>0$. Thus, by Doob's inequality plus the density of $1\oplus\text{span}(\mathscr{F}_0(\overline{\Psi}))$ in $L^1(P)$, $E^P[Y|\mathscr{M}_.]$ admits a right-continuous progressively measurable version for all $Y\in L^1(P)$. Furthermore, it is clear that if $X(.)$ is P-a.s. continuous for each $X\in\overline{\Psi}$, then the same argument shows that our version of $E^P[Y|\mathscr{M}_.]$ is also P-a.s. continuous.

Finally, let $\tau:\Omega\to[0,\infty)$ be an extended stopping time. Given $A\in\mathscr{M}_{\tau+0}$, let $Y(.)$ be a right-continuous progressively measurable version of $P(A|\mathscr{M}_.)$. Then there is a $\{t_n\}_1^\infty\subseteq[0,\infty)$ and a measurable $F:[0,\infty)\times R^{Z^+}\to[0,\infty)$ such that

$$Y(t,\omega)=F(t,x(t_1\wedge t,\omega),\ldots,x(t_n\wedge t,\omega),\ldots),\qquad (t,\omega)\in[0,\infty)\times\Omega.$$

Thus:

$$\chi_A=\lim_{n\to\infty}Y(\tau+1/n)=Y(\tau)=F(\tau,x(t_1\wedge\tau),\ldots,x(t_n\wedge\tau),\ldots)\quad(\text{a.s., P})$$

and so $A\in\mathscr{M}_\tau^{(-)}$. That is, $\mathscr{M}_{\tau+0}\subseteq\mathscr{M}_\tau^{(-)}$ (a.s., P). Since the opposite inclusion is obvious, the proof is complete.

Q.E.D.

We next want to show that $\text{ext}(\mathscr{P}_s(\overline{\Psi}))$ is closed under the same operations of conditioning and splicing as is $\mathscr{P}_s(\overline{\Psi})$.

THEOREM (2.4). — *Let* $\tau:\Omega\to[s,\infty)$ *be a stopping time. If* $P\in\text{ext}(\mathscr{P}_s(\overline{\Psi}))$ *and* $\{P_\omega\}$ *is a r.c.p.d. of* $P|\mathscr{M}_\tau$, *then there is a* P-*null set* $N\in\mathscr{M}_\tau$ *such that* $P_\omega\in\text{ext}(\mathscr{P}_{\tau(\omega),x(\tau(\omega),\omega)}(\overline{\Psi}))$ *for each* $\omega\notin N$. *Conversely, if* $P\in\text{ext}(\mathscr{P}_s(\overline{\Psi}))$ *and* $\omega\to Q_\omega\in\text{ext}(\mathscr{P}_{\tau(\omega),x(\tau(\omega),\omega)}(\overline{\Psi}))$ *is* $\mathscr{M}_\tau$-*measurable, then* $P\otimes_{\tau(.)}Q_.\in\text{ext}(\mathscr{P}_s(\overline{\Psi}))$.

Proof. — Since it is clear that

$$\text{ext}(\mathscr{P}_{t,y}(\overline{\Psi}))=(\text{ext}(\mathscr{P}_t(\overline{\Psi})))\cap\mathscr{P}_{t,y}(\overline{\Psi}),\qquad (t,y)\in[0,\infty)\times E,\tag{2.5}$$

the first assertion will follow if we can show that there is a P-null set $N\in\mathscr{M}_\tau$ such that $1\oplus\text{span}(\mathscr{F}_{\tau(\omega)}(\overline{\Psi}))$ is dense in $L^1(P_\omega)$ for $\omega\notin N$. Furthermore, since $\mathscr{M}$ is countably generated, this will be accomplished if we show that for each bounded $\mathscr{M}$-measurable Y with

$E[Y]=0$ there is a $\{Y_n\}_1^\infty \subseteq \operatorname{span}(\mathscr{F}_s(\overline{\Psi}))$ such that

$$\lim_{n\to\infty} E^{P_\omega}[|(Y_n - E^{P_\omega}[Y_n]) - (Y - E^{P_\omega}[Y])|]=0$$

for P-almost all ω. To this end, choose $\{Y_n\}_1^\infty \subseteq \operatorname{span}(\mathscr{F}_s(\overline{\Psi}))$ so that $\sum_1^\infty E^P[|Y_n - Y|]<\infty$. Then, since

$$P(\{\omega : E^{P_\omega}[|(Y_n - E^{P_\omega}[Y_n]) - (Y - E^{P_\omega}[Y])|]\geqq \varepsilon\})$$
$$\leqq P(\{\omega : E^{P_\omega}[|Y_n - Y|]\geqq \varepsilon/2\}) + P(|E^P[Y_n|\mathscr{M}_\tau] - E^P[Y|\mathscr{M}_\tau]|\geqq \varepsilon/2)\leqq \frac{4}{\varepsilon}E^P[|Y_n - Y|],$$

it follows that $E^{P.}[|(Y_n - E^{P.}[Y_n]) - (Y - E^{P.}[Y])|]\to 0$ (a.s., P).

To prove the second assertion, we begin by showing that if $\mathscr{P}_s(\overline{\Psi}^\tau)$ is the set of probability measures Q on $(\Omega, \mathscr{M}_\tau)$ such that $(X((t\wedge\tau)\vee s), \mathscr{M}_{(t\wedge\tau)\vee s}, Q)$ is a martingale for all $X\in\overline{\Psi}$, then $P\in \operatorname{ext}(\mathscr{P}_s(\overline{\Psi}))$ implies $P|_{\mathscr{M}_\tau}\in \operatorname{ext}(\mathscr{P}_s(\overline{\Psi}^\tau))$. To this end, first note that, by Doob's stopping time theorem, $P|_{\mathscr{M}_\tau}\in\mathscr{P}_s(\overline{\Psi}^\tau)$. Second, observe that since $1\oplus\operatorname{span}(\mathscr{F}_s(\overline{\Psi}))$ is dense in $L^1(P)$ and $E^P[Y|\mathscr{M}_.]$ admits a right-continuous progressively measurable version for each $Y\in L^1(P)$, $\{E[Y|\mathscr{M}_\tau] : Y\in 1\oplus\operatorname{span}(\mathscr{F}_s(\overline{\Psi}))\}$ is dense in $\{Y\in L^1(P) : Y$ is $\mathscr{M}_\tau$-measurable$\}$. Hence, by Douglas's theorem, $P|_{\mathscr{M}_\tau}\in\operatorname{ext}(\mathscr{P}_s(\overline{\Psi}^\tau))$.

Now set $R = P\otimes_{\tau(.)}Q_.$ and suppose that $R=\theta R_1+(1-\theta)R_2$ for some $0<\theta<1$ and $R_1, R_2\in\mathscr{P}_s(\overline{\Psi})$. Then, since

$$R|_{\mathscr{M}_\tau}=P|_{\mathscr{M}_\tau}\in\operatorname{ext}(\mathscr{P}_s(\overline{\Psi}^\tau)) \quad\text{and}\quad R_1|_{\mathscr{M}_\tau}, R_2|_{\mathscr{M}_\tau}\in\mathscr{P}_s(\overline{\Psi}_\tau),$$

$$R_1|_{\mathscr{M}_\tau}=R_2|_{\mathscr{M}_\tau}=P|_{\mathscr{M}_\tau}.$$

But this means that $Q_.=\theta(R_1)_.+(1-\theta)(R_2)_.$ (a.s., P), where $\{(R_i)_\omega\}$ is a r.c.p.d. of $R_i|_{\mathscr{M}_\tau}$ $(i=1, 2)$. Since $Q_\omega\in\operatorname{ext}(\mathscr{P}_{\tau(\omega)}(\overline{\Psi}))$ for P-almost all ω and $(R_i)_\omega\in\mathscr{P}_{\tau(\omega)}(\overline{\Psi})$ for P-almost all ω, we conclude that $(R_1)_.=(R_2)_.$ (a.s., P) and therefore $R_1=R_2$.

Q.E.D.

The rest of this section is devoted to the following special case of the preceding set-up. In the first place we will take $\Omega = C([0, \infty), R^d)$. More important we will assume that elements of $\overline{\Psi}$ are of the form

$$X_\varphi(t)=\varphi(x(t))-\int_0^t L\varphi(x(s))\,ds$$

where $\varphi \in C_0^\infty(R^d)$ and

$$L = 1/2 \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x_i}.$$

is a second order (degenerate) elliptic operator with bounded measurable coefficients.

(In order to insure that $\underline{\Psi}$ is countable, we can restrict φ to a countable dense subset of $C_0^\infty(R^d)$, although it is clear that $\mathscr{P}_s(\underline{\Psi})$ will be the same whether we allow φ to range over all of $C_0^\infty(R^d)$ or restrict φ to a dense subset.) For $(s, x) \in [0, \infty) \times R^d$, define $\mathscr{C}_L(s, x)$ to be the set of $P \in \mathscr{P}_s(\underline{\Psi})$ such that $P(x(t) = x, 0 \leqq t \leqq s) = 1$. Then it is clear that $\mathscr{C}_L(s, x)$ is convex and that

$$\text{ext}(\mathscr{C}_L(s, x)) = (\text{ext}(\mathscr{P}_s(\underline{\Psi}))) \cap \mathscr{C}_L(s, x). \tag{2.6}$$

Furthermore, by exactly the same reasoning as we used in Remark (1.2), one can prove the following special case of a theorem due to Jacod [13] (*cf.* also Jacod and Yor [14]).

THEOREM (2.7). — *$P \in \text{ext}(\mathscr{C}_L(s, x))$ if and only if $P \in \mathscr{C}(s, x)$ and for every $\Phi \in L^2(P)$ there is a previsible $\theta : [0, \infty) \times \Omega \to R^d$ such that*

$$E^P \left[\int_s^\infty \langle \theta(u), a(x(u)) \theta(u) \rangle du \right] < \infty \quad \textit{and} \quad \Phi = E[\Phi] + \int_s^\infty \langle \theta(u), d\bar{x}(u) \rangle,$$

where

$$\bar{x}(.) = x(.) - \int_s^{.} b(x(u)) du.$$

If for each (s, x), $\mathscr{C}_L(s, x)$ contains exactly one element $P_{s,x}$, then one can show that $\{P_{s,x} : (s, x) \in [0, \infty) \times R^d\}$ is a measurable, strong Markov (with respect to non-extended stopping times), time-homogeneous family such that

$$E^{P_{s,x}}[\varphi(x(t))] - \varphi(x) = E^{P_{s,x}}\left[\int_s^t L\varphi(x(u)) du \right], \qquad \varphi \in C_b^2(R^d).$$

For details *see* Theorem (6.2.2) and the discussion following that theorem in [*S. and V.*]. However, there are many reasonable choices of L for which $\mathscr{C}_L(s, x)$ will not consist of exactly one element. There are even examples for which $\mathscr{C}(s, x)$ will be empty: *see* exercise 6.7.6 in [*S. and V.*]. We will devote the rest of this section to a discussion of the case in which $\mathscr{C}(s, x)$ contains many elements. The starting point of our study lies in some ideas of N. Krylov [2] (*cf.* also Chapt. XII of [*S. and V.*]).

Let $M(\Omega)$ denote the set of all probability measures on $(\Omega, \mathscr{M})$ endowed with the Lévy metric for weak convergence. By comp$(M(\Omega))$ we mean the space of non-empty compact subsets of $M(\Omega)$ and we think of comp$(M(\Omega))$ as a metric space with the Hausdorff metric. Given a collection $\mathscr{K} = \{\mathscr{C}(s, x) : (s, x) \in [0, \infty) \times R^d\}$ of subsets of $M(\Omega)$, we will say that $\mathscr{K}$ is a *Krylov system* if:

(*a*) $(s, x) \to \mathscr{C}(s, x)$ is a measurable map into comp$(\mathrm{M}(\Omega))$;

(*b*) $\mathrm{P} \in \mathscr{C}(0, x)$ if and only if $\mathrm{P} \circ \mathrm{T}_s^{-1} \in \mathscr{C}(s, x)$ where $\mathrm{T}_s : \Omega \to \Omega$ is given by $x(t, \mathrm{T}_s \omega) = x((t-s) \vee 0, \omega)$, $t \geqq 0$;

(*c*) if $\mathrm{P} \in \mathscr{C}(0, x)$, $\tau : \Omega \to [0, \infty)$ is a stopping time, and $\{\mathrm{P}_\omega\}$ is a r. c. p. d. of $\mathrm{P} \,|\, \mathscr{M}_\tau$, then there is a P-null set $\mathrm{N} \in \mathscr{M}_\tau$ such that $\delta_{x(\tau(\omega), \omega)} \otimes_{\tau(\omega)} \mathrm{P}_\omega \in \mathscr{C}(\tau(\omega), x(\tau(\omega), \omega))$ for each $\omega \notin \mathrm{N}$;

(*d*) if $\mathrm{P} \in \mathscr{C}(0, x)$, $\tau : \Omega \to [0, \infty)$ is a stopping time, and $\omega \to \mathrm{Q}_\omega \in \mathscr{C}(\tau(\omega), x(\tau(\omega), \omega))$ is an $\mathscr{M}_\tau$-measurable map, then $\mathrm{P} \otimes_{\tau(.)} \mathrm{Q}_. \in \mathscr{C}(0, x)$.

By Theorem (2.1) and equation (2.6), it is easy to check that $\mathscr{K}_\mathrm{L} = \{\mathscr{C}_\mathrm{L}(s, x) : (s, x) \in [0, \infty) \times \mathrm{R}^d\}$ is a Krylov system if and only if $\mathscr{K}_\mathrm{L}$ satisfies (*a*). Furthermore, if the coefficients of L are continuous, it is easy to see that $\mathscr{K}_\mathrm{L}$ satisfies (*a*).

LEMMA (2.8). — *Let $\mathscr{K} = \{\mathscr{C}(s, x) : (s, x) \in [0, \infty) \times \mathrm{R}^d\}$ be a Krylov system and let $\lambda > 0$ and $f : \mathrm{R}^d \to \mathrm{R}^1$, a bounded upper semi-continuous function, be given. Define*

$$u(s, x) = \sup_{\mathrm{P} \in \mathscr{C}(s, x)} \mathrm{E}^\mathrm{P}\left[\int_0^\infty e^{-\lambda t} f(x(t+s))\, dt\right].$$

Then

$$u(s, x) = u(0, x), \qquad s \geqq 0;$$

and if

$$\mathrm{C}'(s, x) = \left\{\mathrm{P} \in \mathscr{C}(s, x) : \mathrm{E}^\mathrm{P}\left[\int_0^\infty e^{-\lambda t} f(x(t+s))\, dt\right] = u(s, x)\right\}$$

then

$$\mathscr{K}' = \{\mathscr{C}'(s, x) : (s, x) \in [0, \infty) \times \mathrm{R}^d\}$$

is a Krylov system.

Proof. — *See* Lemma 12.2.2 in [*S. and V.*].

LEMMA (2.9). — *Let everything be the same as in Lemma* (2.8), *only this time assume that $f \in \mathrm{C}_b(\mathrm{R}^d)$ and that $\mathscr{C}(s, x) \subseteqq \mathscr{C}_\mathrm{L}(s, x)$, $(s, x) \in [0, \infty) \times \mathrm{R}^d$, where L has continuous coefficients. Then, $u(s, x)$ is upper semi-continuous. Moreover, if $\tau : \Omega \to [0, \infty)$ is an extended stopping time, $\mathrm{P} \in \mathscr{C}'(0, x)$, and $(t, y) \to \mathrm{P}_{t, y} \in \mathscr{C}'(t, y)$ is measurable, then*

$$\mathrm{Q} \equiv \mathrm{P} \otimes_{\tau(.)} \mathrm{P}_{\tau(.), x(\tau(.), .)} \in \mathscr{C}_\mathrm{L}(0, x)$$

and

$$u(0, x) \leqq \mathrm{E}^\mathrm{Q}\left[\int_0^\infty e^{-\lambda t} f(x(t))\, dt\right].$$

Proof. — The upper semi-continuity of $u(s, x)$ is obvious. To prove the second assertion, define $\tau_n = \tau + 1/n$, $n \geqq 1$. Then τ_n is a stopping time. Next, set

$$a_n(t) = \begin{cases} a(x(t)) & \text{if } t < \tau_n, \\ a(x(t) + \Delta_n) & \text{if } t \geqq \tau_n \end{cases}$$

and

$$b_n(t)=\begin{cases} b(x(t)) & \text{if} \quad t<\tau_n, \\ b(x(t)+\Delta_n) & \text{if} \quad t\geqq\tau_n \end{cases}$$

where

$$\Delta_n = x(\tau_n)-x(\tau).$$

If $Q^n_\omega=\delta_\omega\otimes_{\tau_n(\omega)}(P_{\tau_n(\omega),\, x(\tau(\omega),\,\omega)}\circ S^{-1}_{\Delta_n(\omega)})$, where $S_y:\Omega\to\Omega$ is given by $x(t, S_y\omega)=y+x(t,\omega)$, then it is not hard to check that $Q^n\equiv P\otimes_{\tau_n(.)}Q^n_.\in\mathscr{P}_{0,x}(\underline{\Psi}_n)$, where $\underline{\Psi}_n$ consists of functions of the form

$$\varphi(x(t))-\int_0^t\left[\sum_1^d a_n^{ij}(s)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x(s))+\sum_1^d b_n^i(s)\frac{\partial\varphi}{\partial x_i}(x(s))\right]ds$$

(*cf.* Thm. 6.1.2 in [*S. and V.*] with $\varphi\in C_0^\infty(R^d)$.. In particular, since $Q^n_\omega\to\delta_\omega\otimes_{\tau(\omega)}P_{\tau(\omega),\,x(\tau(\omega),\,\omega)}$ as $n\to\infty$ for each ω and therefore $Q^n\to Q$, we see that $Q\in\mathscr{C}_L(0, x)$.

We must still show that

$$u(0, x)\leqq E^Q\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right].$$

But

$$E^Q\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right]=\lim_{n\to\infty}E^{Q^n}\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right]$$
$$=E^P\left[\int_0^\tau e^{-\lambda t}f(x(t))\,dt\right]+\lim_{n\to\infty}E^{Q^n}\left[\int_{\tau_n}^\infty e^{-\lambda t}f(x(t))\,dt\right]$$

and

$$E^{Q^n}\left[\int_{\tau_n}^\infty e^{-\lambda t}f(x(t))\,dt\right]=E^{Q^n}\left[\int_{\tau_n}^\infty e^{-\lambda t}f(x(t)-\Delta_n)\,dt\right]$$
$$+E^{Q^n}\left[\int_{\tau_n}^\infty e^{-\lambda t}(f(x(t))-f(x(t)-\Delta_n))\,dt\right]$$
$$=E^P[e^{-\lambda\tau_n}u(\tau_n, x(\tau))]+E^{Q^n}\left[\int_{\tau_n}^\infty e^{-\lambda t}(f(x(t))-f(x(t)-\Delta_n)\,dt\right]$$
$$\to E^P[e^{-\lambda\tau}u(\tau, x(\tau))]$$

since $u(\tau_n, .)=u(\tau, .)$ and $f\in C_b(R^d)$. Hence

$$E^Q\left[\int_0^\infty e^{-\lambda t}f(x(t))\,dt\right]=E^P\left[\int_0^\tau e^{-\lambda t}f(x(t))\,dt\right]+E^P[e^{-\lambda\tau}u(\tau, x(\tau))].$$

Finally:

$$e^{-\lambda\tau} u(\tau, x(\tau)) \geqq \overline{\lim_{n\to\infty}}\, e^{-\lambda\tau_n} u(\tau_n, x(\tau_n))$$

$$= \overline{\lim_{n\to\infty}}\, E^{P^n_\cdot}\left[\int_{\tau_n(\cdot)}^{\infty} e^{-\lambda t} f(x(t))\,dt\right] = E^P\left[\int_{\tau}^{\infty} e^{-\lambda t} f(x(t))\,dt \,\middle|\, \mathscr{M}_\tau\right] \quad \text{(a.s., P)}$$

where $\{P^n_\omega\}$ is a r.c.p.d. of $P|\mathscr{M}_{\tau_n}$ and we have used the fact that

$$\delta_{x(\tau_n(\omega),\,\omega)} \otimes_{\tau_n(\omega)} P^n_\omega \in \mathscr{C}'(\tau_n(\omega),\, x(\tau_n(\omega),\, \omega)) \quad \text{(a.s., P)}.$$

Combining this with the preceding, we conclude that

$$E^Q\left[\int_0^{\infty} e^{-\lambda t} f(x(t))\,dt\right] \geqq E^P\left[\int_0^{\infty} e^{-\lambda t} f(x(t))\,dt\right] = u(0, x).$$

Q.E.D.

THEOREM (2.10). — *Suppose that $\mathscr{K}_L$ is a Krylov system. Then for each $\lambda > 0$ and each bounded, upper semi-continuous function $f: R^d \to R^1$, there is a measurable map $(s, x) \to P_{s,x} \in \text{ext}(\mathscr{C}_L(s, x))$ such that $\{P_{s,x} : (s, x) \in [0, \infty) \times R^d\}$ forms a time-homogeneous strong Markov family (w.r.t. non-extended stopping times) and*

$$E^{P_{s,x}}\left[\int_0^{\infty} e^{-\lambda t} f(x(t+s))\,dt\right] = \sup_{P \in \mathscr{C}(s,x)} E^P\left[\int_0^{\infty} e^{-\lambda t} f(x(t+s))\,dt\right], \qquad (s, x) \in [0, \infty) \times R^d.$$

Moreover, if the coefficients of L are continuous and $f \in C_b(R^d)$ then for each extended stopping time $\tau: \Omega \to [0, \infty)$, $\{\delta_\omega \otimes_{\tau(\omega)} P_{\tau(\omega),\, x(\tau(\omega),\, \omega)}\}$ is a conditional probability distribution of $P_{0,x}$ given $\mathscr{M}_{\tau+0}$.

Proof. — The first part of this theorem is due to Krylov [2] (*cf.* Thm. (12.2.3) in [*S. and V.*]); we will say how the proof runs because we use it in the proof of the second part. Given λ and f, define $(\lambda_0, \varphi_0) = (\lambda, f)$. Next choose $\{(\lambda_n, \varphi_n)\}_1^\infty \subseteq (0, \infty) \times C_0(R^d)$ to be a dense set. Let

$$u_0(s, x) = \sup_{P \in \mathscr{C}_L(s,x)} E^P\left[\int_0^{\infty} e^{-\lambda_0 t} \varphi_0(x(t+s))\,dt\right]$$

and

$$\mathscr{C}_0(s, x) = \left\{P \in \mathscr{C}_L(s, x) : E^P\left[\int_0^{\infty} e^{-\lambda_0 t} \varphi_0(x(t+s))\,dt\right] = u_0(s, x)\right\}.$$

By induction, define u_n and $\mathscr{C}_n$ for $n \geqq 1$ by:

$$u_n(s, x) = \sup_{P \in \mathscr{C}_{n-1}(s,x)} E^P\left[\int_0^{\infty} e^{-\lambda_n t} \varphi_n(x(t+s))\,dt\right]$$

and

$$\mathscr{C}_n(s, x)=\left\{P\in\mathscr{C}_{n-1}(s, x): E^P\left[\int_0^\infty e^{-\lambda_n t}\varphi_n(x(t+s))\,dt\right]=u_n(s, x)\right\}.$$

By repeated use of Lemma (2.8), one sees that $\mathscr{K}_n=\{\mathscr{C}_n(s, x):(s, x)\in[0, \infty)\times R^d\}$ is a Krylov system for each $n\geqq 0$.

Hence if $\mathscr{C}_\infty(s, x)=\bigcap_0^\infty \mathscr{C}_n(s, x)$, then $\mathscr{K}_\infty=\{\mathscr{C}_\infty(s, x):(s, x)\in[0, \infty)\times R^d\}$ is a Krylov system. Moreover, there is exactly one element $P_{s, x}$ in each $\mathscr{C}_\infty(s, x)$, and from this it is obvious that $\{P_{s, x}:(s, x)\in[0, \infty)\times R^d\}$ is a time-homogeneous strong Markov family. Finally, to see that $P_{s, x}\in \text{ext}(\mathscr{C}_L(s, x))$, suppose that $P_{s, x}=\theta Q_1+(1-\theta)Q_2$ where $0<\theta<1$ and $Q_1, Q_2\in\mathscr{C}_L(s, x)$. Then, by induction, one sees that $Q_1, Q_2\in\mathscr{C}_n(s, x)$, for all $n\geqq 0$. Hence, $Q_1, Q_2\in\mathscr{C}_\infty(s, x)$, and so $Q_1=P=Q_2$.

Now suppose that $f\in C_b(R^d)$ and that the coefficients of L are continuous. Set $Q_\omega=\delta_\omega\otimes_{\tau(\omega)}P_{\tau(\omega), x(\tau(\omega), \omega)}$ and $Q=\int Q_\omega P_{0, x}(d\omega)$. Using induction plus Lemma (2.9), check that $Q\in\mathscr{C}_n(0, x)$ for all $n\geqq 1$. Hence, $Q=P_{0, x}$. Thus, since $\mathscr{M}_{\tau+0}=\mathscr{M}_\tau^{(-)}$ (a.s., $P_{0, x}$), we will know that $\{Q_\omega\}$ is a conditional probability distribution of P given $\mathscr{M}_{\tau+0}$ once we show that $P(A\cap B)=E^P[Q_{.}(B), A]$ for all $A\in\mathscr{M}_\tau^{(-)}$ and $B\in\mathscr{M}$. To this end, we must show that $Q_{.}(A)=\chi_A$ (a.s., P) if $A\in\mathscr{M}_\tau^{(-)}$; and this will be proved if we show that $Q_\omega(\tau=\tau(\omega))=1$ for $P_{0, x}$-almost all ω. But because $\{\tau<t\}\in\mathscr{M}_t$, $t\geqq 0$:

$$\chi_{[0, t)}(\tau(\omega'))=F(t, x(t_1\wedge t, \omega'), \ldots, x(t_n\wedge t, \omega'), \ldots), \qquad \omega'\in\Omega,$$

where $F:[0, \infty)\times R^{Z^+}\to[0, 1]$ is measurable and $\{t_n\}_1^\infty\subseteq[0, \infty)$.

Hence $\tau(\omega')\geqq\tau(\omega)$ if $x(t\wedge\tau(\omega), \omega')=x(t\wedge\tau(\omega), \omega)$ for all $t\geqq 0$. Thus $Q_\omega(\tau<\tau(\omega))=0$. On the other hand

$$E^{P_{0, x}}[e^{-\tau}]=E^{P_{0, x}}[E^{Q_{.}}[e^{-\tau}]],$$

and so we conclude that $Q_\omega(\tau=\tau(\omega))=1$ (a.s., $P_{0, x}$).

Q.E.D.

Theorem (2.10) shows that, under mild assumptions on L, the time-homogeneous strong Markov selections $(s, x)\to P_{s, x}\in\text{ext}(\mathscr{C}_L(s, x))$ are important. Indeed, Theorem (2.10) tells us that such selections exist under very general conditions. Furthermore, it is easy to show from Theorem (2.10) that if $\mathscr{K}_L$ is a Krylov system, then card $(\mathscr{C}_L(s, x))=1$ for all (s, x) if and only if there is precisely one such selection (*cf.* Thm. (12.2.4) in [*S. and V.*]). (If the coefficients of L are continuous, then one can use Theorem (2.10) to show that card $(\mathscr{C}_L(s, x))=1$ for all (s, x) if and only if there is precisely one time-homogeneous selection $(s, x)\to P_{s, x}\in\text{ext}(\mathscr{C}_L(s, x))$ such that $\{P_{s, x}:(s, x)\in[0, \infty)\times R^d\}$ is strong Markov with respect to extended stopping times). Thus, it is possible to tell something about the structure of $\mathscr{K}_L$ from a knowledge of such selections. Of course, by Choquet's theorem, ext $(\mathscr{C}_L(s, x))$ completely determines $\mathscr{C}_L(s, x)$ if $\mathscr{K}_L$ is a Krylov system.

However, in general not every element of $\text{ext}(\mathscr{C}_L(s, x))$ will be a member of a time-homogeneous strong Markov selection. To see this, simply splice two such selections together at a stopping time (for instance, the second hitting time of some closed set) and apply Theorem (2.4) to conclude that the resulting measure is again extreme, but clearly it is not part of a time-homogeneous strong Markov selection. Nonetheless, we can recapture all of $\mathscr{C}_L(s, x)$ from time-homogeneous strong Markov selections in the following sense.

THEOREM (2.11). — *Assume that* $\mathscr{K}_L$ *is a Krylov system. Let* $\mathscr{A}$ *index the set of all time-homogeneous, strong Markov selections* $(s, x) \to \text{ext}(\mathscr{C}_L(s, x))$ *in the sense that for each* $\alpha \in \mathscr{A}$ *there is exactly one such selection* $\{ P^{(\alpha)}_{s,x} : (s, x) \in [0, \infty) \times R^d \}$. *Denote by* $\mathscr{D}_L(x)$ *the smallest set of* P'*s such that:*

(i) $\{ P^{(\alpha)}_{0,x} : \alpha \in \mathscr{A} \} \subseteqq \mathscr{D}_L(x)$;

and

(ii) *if* $P \in \mathscr{D}_L(x)$, $t > 0$, *and* $\omega \to P^{(\alpha(\omega))}_{t, x(t, \omega)}$ *is an* $\mathscr{M}_t$*-measurable map into* $M(\Omega)$, *then* $P \otimes_t P^{(\alpha(.))}_{t, x(t, .)} \in \mathscr{D}_L(x)$. *Then* $\mathscr{D}_L(x) \subseteqq \text{ext}(\mathscr{C}_L(0, x))$ *and* $\mathscr{C}_L(0, x)$ *coïncides with the closed convex hull of* $\mathscr{D}_L(x)$.

Proof. — Everything but the inclusion $\mathscr{D}_L(x) \subseteqq \text{ext}(\mathscr{C}_L(0, x))$ is proved in Theorem (12.3.1) of [*S. and V.*]. However, this inclusion is an immediate consequence of Theorem (2.4).

Remark (2.12). — Theorem (2.11) tells us that we will have a reasonable good grasp of $\mathscr{K}_L$ once we classify all time-homogeneous, strong Markov selections $(s, x) \to P_{s,x} \in \text{ext}(\mathscr{C}_L(s, x))$. As we will see in section (4) below, such a classification is often possible to carry out, although a general classification procedure is still waiting to be found. One of the most serious difficulties that we have encountered is that we have not yet discovered a really good practical criterion for determining when the members of a time-homogeneous selection are extreme. Some progress in this direction is made in section (3) [*cf.* Thms. (3.17) and (3.19) and Cor. (3.20)]; but the natural conjecture that every such selection consists of extreme elements is false, as the following example demonstrates.

Example (2.13). — Let $d = 1$ and $L = b(x)\partial/\partial x$, where $b(x) = |x|^{1/2} \wedge 1$. We are going to construct a time-homogeneous, strong Markov selection $(s, x) \to P_{s,x} \in \mathscr{C}_L(s, x)$ such that $P_{0,0} \notin \text{ext}(\mathscr{C}_L(0, x))$.

Define $u : [0, \infty) \times R^1 \to R^1$ so that

$$u(t, x) = \left\{ \begin{array}{ll} x + t, & 0 \leqq t \leqq -x+1 \\ -\left(1 - \dfrac{t+x+1}{2}\right)^2, & -x-1 \leqq t \leqq -x+1 \\ (t+x-1)^2/4, & -x+1 \leqq t \leqq -x+3 \\ t+x-2, & t \geqq -x+3 \end{array} \right\} \quad \text{and} \quad x < -1$$

$$= \left\{ \begin{array}{ll} -((-x)^{1/2} - t/2)^2, & 0 \leqq t \leqq 2(-x)^{1/2} \\ (t - 2(-x)^{1/2})/4, & 2(-x)^{1/2} \leqq t \leqq 2(-x)^{1/2} + 2 \\ t - 2(-x)^{1/2} - 1, & t \geqq 2(-x)^{1/2} + 2 \end{array} \right\} \quad \text{and} \quad -1 \leqq x \leqq 0$$

$$= \left\{ \begin{array}{ll} (x^{1/2}+t/2)^2, & 0 \leqq t \leqq 2(1-x^{1/2}) \\ t+2x^{1/2}-1, & t \geqq 2(1-x^{1/2}) \end{array} \right\} \quad \text{and} \quad 0<x<1$$

$$= t+x, \quad t \geqq 0 \quad \text{and} \quad x \geqq 1.$$

Next define $\tau_0 : (-\infty, 0] \to [0, \infty)$ by

$$\tau_0(x) = \left\{ \begin{array}{ll} -x+1 & \text{if} \quad x<-1, \\ 2(-x)^{1/2} & \text{if} \quad -1 \leqq x \leqq 0. \end{array} \right.$$

Then it is easy to check the following facts:

(i) $u \in C^{1,0}([0, \infty) \times R^1)$ and $u(t, y) \geqq u(s, x)$ if $t \geqq s$ and $y \geqq x$;

(ii) $\partial u/\partial t(t, x) = b(u(t, x))$, $t \geqq 0$ and $x \in R^1$, and $u(0, x) = x$;

(iii) $u(t+s, x) = u(t, u(s, x))$, $s, t \geqq 0$ and $x \in R^1$;

(iv) $\tau_0(x) = \min\{t \geqq 0 : u(t, x) = 0\}$ if $x \leqq 0$.

On $[0, \infty) \times R^1 \times \mathscr{B}_{R^1}$, we now define

$$P(s, x, \Gamma) = \left\{ \begin{array}{l} \chi_\Gamma(u(s, x)) \quad \text{if} \quad x \leqq 0 \quad \text{and} \quad 0 \leqq s < \tau_0(x), \\ e^{-(s-\tau_0(x))} \chi_\Gamma(0) + \displaystyle\int_{\tau_0(x)}^{s} e^{-(\sigma-\tau_0(x))} \chi_\Gamma(u(s-\sigma, 0)) \, d\sigma \\ \qquad \text{if} \quad x \leqq 0 \quad \text{and} \quad s \geqq \tau_0(x), \\ \chi_\Gamma(u(s, x)) \quad \text{if} \quad x>0 \quad \text{and} \quad s \geqq 0. \end{array} \right.$$

Using the preceding facts, one can easily check that $P(s, x, \Gamma)$ is a transition probability function and that

$$(2.14) \qquad \int \varphi(y) P(s, x, dy) - \varphi(x) = \int_0^s dt \int L\varphi(y) P(t, x, dy), \qquad \varphi \in C_0(R^d).$$

Also, one can easily construct a time-homogeneous Markov family $\{P_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ on $(\Omega, \mathscr{M})$ having $P(s, x, \Gamma)$ as its transition probability function; and because of (2.14), it is clear that $P_{s,x} \in \mathscr{C}_L(s, x)$ for all (s, x).

Finally, if P^t, $t \geqq 0$, is defined by

$$P^t = \delta_{u((.-t) \vee 0, 0)},$$

then $P^t \in \mathscr{C}_L(0, 0)$ for each $t \geqq 0$ and

$$P_{0,0} = \int_0^\infty e^{-t} P^t \, dt.$$

Thus we will have our example if we can show that $\{P_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ is strong Markov.

To prove the strong Markov property for $\{P_{s,x}:(s,x)\in[0,\infty)\times R^1\}$, we consider the following alternate construction of the $P_{s,x}$'s. Let $E=(-\infty,0]\cup(1,\infty)$ and on $[0,\infty)\times\overline{E}\times\mathscr{B}_{\overline{E}}$, define

$$\hat{P}(s,x,\Gamma)=\begin{cases}\chi_\Gamma(u(s,x)) & \text{if } x\leqq 0 \text{ and } 0\leqq s<\tau_0(x),\\ e^{-(s-\tau_0(x))}\chi_\Gamma(0)+\displaystyle\int_{\tau_0(x)}^{s} e^{-(\sigma-\tau_0(x))}\chi_\Gamma(u(s-\sigma,0)+1)\,d\sigma & \\ \qquad\text{if } x\leqq 0 \text{ and } s\geqq\tau_0(x), & \\ \chi_\Gamma(u(s,x-1)+1) & \text{if } x\geqq 1 \text{ and } s\geqq 0.\end{cases}$$

Then it is easy to check that $\hat{P}(s,x,\Gamma)$ is a Feller continuous transition function on $\overline{E}$ and that there is a Feller continuous time-homogeneous Markov family $\{\hat{P}_{s,x}:(s,x)\in[0,\infty)\times\overline{E}\}$ on $D([0,\infty),\overline{E})$ having transition probability function $\hat{P}(s,x,\Gamma)$. Furthermore, $\hat{P}_{0,x}$ is concentrated on $C([0,\infty),[1,\infty))$ if $x\geqq 1$; and if $x\leqq 0$, then $\hat{P}_{0,x}$-almost surely $x(.)$ has precisely one jump and the left and right limits at that jump time are 0 and 1, respectively. Finally, because of the Feller continuity, $\{\hat{P}_{s,x}:(s,x)\in[0,\infty)\times\overline{E}\}$ is strong Markov with respect to extended stopping times. Next set $y(t)=x(t-0)$. Observe that

$$\hat{P}_{0,x}(y(t)\in E,\ t\geqq 0)=1,\qquad x\in E,$$

$$\hat{P}_{0,x}((\exists t\geqq 0)\,y(t)\neq x(t)\text{ and }(x(t)\neq 1\text{ or }y(t)\neq 0))=0,\qquad x\in E,$$

$$\hat{P}_{0,x}(y(t)\neq x(t))=0,\qquad t\geqq 0\ \text{ and }\ x\in E.$$

We are now going to show that if $\tau:\Omega\to[0,\infty)$ is a stopping time for $y(.)$, then $\{\delta_\omega\otimes_{\tau(\omega)}\hat{P}_{\tau(\omega),\,y(\tau(\omega),\,\omega)}\}$ is a r.c.p.d. of $\hat{P}_{0,x}|\sigma(y(t\wedge\tau):t\geqq 0)$ for all $x\in E$. To do this, it is enough for us to show that $\hat{P}_{0,x}(x(\tau)=y(\tau))=1$; since, by the strong Markov property, $\{\delta_\omega\otimes_{\tau(\omega)}\hat{P}_{\tau,\,x(\tau(\omega),\,\omega)}\}$ is a r.c.p.d. of $\hat{P}_{0,x}|\mathscr{M}_\tau$. But

$$\tau(\omega)=F(y(t_1\wedge\tau(\omega),\omega),\ \ldots,\ y(t_n\wedge\tau(\omega),\omega),\ \ldots)$$

for some measurable $F:(R^1)^{Z^+}\to[0,\infty)$ and $\{t_n\}_1^\infty\subseteq[0,\infty)$.

Thus if $\zeta(\omega)=\inf\{t\geqq 0:x(t)-y(t)\geqq 1\}$, then for $x\leqq 0$:

$$\begin{aligned}\hat{P}_{0,x}(\tau=\zeta)&\leqq\hat{P}_{0,x}(\zeta=F(y(t_1\wedge\zeta),\ \ldots,\ y(t_n\wedge\zeta),\ \ldots))\\ &=\hat{P}_{0,x}(\zeta=F(u(t_1\wedge\tau_0(x),x),\ \ldots,\ u(t_n\wedge\tau_0(x),x),\ \ldots))=0\end{aligned}$$

since

$$\hat{P}_{0,x}(\zeta>t)=\begin{cases}1 & \text{if } 0\leqq t\leqq\tau_0(x),\\ e^{-(t-\tau_0(x))} & \text{if } t>\tau_0(x)).\end{cases}$$

Also, $\hat{P}_{s,x}(\zeta<\infty)=0$ if $x>1$.

The final step is to define $f:E\to R^1$ by

$$f(x)=\begin{cases}x & \text{if } x\leqq 0,\\ x-1 & \text{if } x>1,\end{cases}$$

and then set $z(.)=f(y(.))$. Then one can easily check that $P_{0,x}$ is the distribution of $z(.)$ under $\hat{P}_{0,f^{-1}(x)}$ for all $x\in R^1$, and now the strong Markov property for $\{P_{s,x}:(s,x)\in[0,\infty)\times R^1\}$ follows immediately from the considerations in the preceding paragraph.

Remark (2.15). — Example (2.13) has a number of interesting properties. In the first place, for all $t\geqq 0$ and $x\in R^1$, $\mathcal{M}_t=\mathcal{M}_{t+0}$ (a.s., $P_{0,x}$). This can be seen from the corresponding fact about $\{\hat{P}_{s,x}:(s,x)\in[0,\infty)\times\overline{E}\}$ plus the equality $\hat{P}_{0,x}(x(t)=y(t))=1$, $t\geqq 0$ and $x\in E$. Secondly, $\{P_{s,x}:(s,x)\in[0,\infty)\times R^1\}$ is an example of a time-homogeneous strong Markov process which is *not* strong Markov with respect to extended stopping times. Indeed, let $\tau'=\inf\{t\geqq 0:x(t)>0\}$. Then $P_{0,x}(x(1+\tau')=u(1,0))=1$. On the other hand

$$E^{P_{0,x}}[P_{\tau'(.),x(\tau'(.))}(x(1+\tau'(.))=u(1,0))]=P_{0,x}(x(1)=u(1,0))=P(1,0,\{u(1,0)\})=0.$$

Thus, the strong Markov property fails for τ'. In particular, this example does not rule out the possibility that any time-homogeneous selection which is strong Markov with respect to extended stopping times must be made up of extreme elements.

Remark (2.16). — Very little change would have been required to prove Theorems (2.10) and (2.11) had we taken $\Omega=D([0,\infty),R^d)$ and L to be a Lévy generator (*cf.* [4]). We have restricted our attention to the diffusion case only because the technical details are fewer and the problems involved are already apparent.

Section (3)

We begin this section with an application of Douglas's theorem to the study of quite general Markov processes. Later we will return to the study of diffusions.

Let E, Ω, $x(t,\omega)$, $\mathcal{M}$, and $\mathcal{M}_t$ be defined as they were at the beginning of section (2) and let $\{P_{s,x}:(s,x)\in[0,\infty)\times E\}$ be a time-homogeneous Markov family of probability measures on $(\Omega,\mathcal{M})$ having transition probability function $P(t,x,\Gamma)$ [i.e. $P_{s,x}(x(t_2)\in\Gamma\,|\,\mathcal{M}_{t_1})=P(t_2-t_1,x(t_1),\Gamma)$ for $0\leqq s\leqq t_1<t_2$]. Denote by D_A and A respectively the domain of the weak generator and the weak generator itself of the semi-group determined by $P(t,x,\Gamma)$; and observe that, because $C_b(E)$ is in the weak center C_A of this semi-group, D_A is weakly dense in $B(E)$ (the space of bounded $\mathcal{B}_E$-measurable $f:E\to R^1$). Next define R_λ, $\lambda>0$, to be the resolvent operator associated with $P(t,x,\Gamma)$ $\left[\text{i.e. } R_\lambda f(x)=\int_0^\infty e^{-\lambda t}dt\int f(y)P(t,x,dy),\ f\in B(E)\right]$. Then $D_A=R_\lambda C_A$ for each $\lambda>0$. In particular, if $\mathcal{D}\subseteq C_b(E)$ is weakly dense, then $\{(R_1 f, AR_1 f):f\in\mathcal{D}\}$ is weakly dense in graph $(A)\equiv\{(f,Af):f\in D_A\}$.

LEMMA (3.1). — *The measure $P_{s,x}$ is uniquely characterized on $(\Omega,\mathcal{M})$ by the fact that $P_{s,x}(x(t)=x,\ 0\leqq t\leqq s)=1$ and that either one of the following holds:*

(*a*) $\left(f(x(t\vee s))-\int_s^{t\vee s}Af(x(u))\,du,\ \mathcal{M}_{t\vee s},\ P_{s,x}\right)$ *is a martingale for all* $f\in D_A$, *or*

(*b*) $\left(e^{-\lambda t \vee s} R_\lambda f(x(t \vee s)) + \int_s^{t \vee s} e^{-\lambda u} f(x(u))\, du,\ \mathscr{M}_{t \vee s},\ P_{s,x}\right)$ *is a martingale for all* $f \in B(E)$.

In fact, in (*a*) D_A *can be replaced by any subset such that* $\{(f, Af) : f \in \mathscr{D}\}$ *is weakly dense in graph* (A)*; and in* (*b*) B(E) *can be replaced by any weakly dense subset of itself.*

Proof. — The proof is easily obtained from the characterization of $P_{s,x}$ as the only P on $(\Omega, \mathscr{M})$ such that $P(x(t)=x, 0 \leqq t \leqq s)=1$ and

$$E^P[f(x(t_2)) \mid \mathscr{M}_{t_1}] = \int f(y) P(t_2 - t_1, x(t_1), dy) \quad (\text{a.s., P})$$

for all $s \leqq t_1 < t_2$ and f in a weakly dense subset of B(E).

Q.E.D.

THEOREM (3.2). — *Let* $\mathscr{D} \subseteqq D_A$ *be a set such that* $\{(f, Af) : f \in \mathscr{D}\}$ *is weakly dense in graph* (A), *and define*

$$\mathscr{F}_s = \left\{ \left(f(x(t_2)) - f(x(t_1)) - \int_{t_1}^{t_2} Af(x(u))\, du \right) \chi_A : f \in \mathscr{D},\ s \leqq t_1 < t_2,\ \textit{and}\ A \in \mathscr{M}_{t_1} \right\}.$$

Then $1 \oplus \operatorname{span}(\mathscr{F}_s)$ *is dense in* $L^1(P_{s,x})$. *Alternatively, let* $\mathscr{D} \subseteqq C_b(E)$ *be weakly dense and let* $\Lambda \subseteqq (0, \infty)$ *be dense. Set*

$$\mathscr{F}_s = \left\{ \left(e^{-\lambda t_2} R_\lambda f(x(t_2)) - e^{-\lambda t_1} R_\lambda f(x(t_1)) + \int_{t_1}^{t_2} e^{-\lambda u} f(x(u))\, du \right) \chi_A : f \in \mathscr{D},\ s \leqq t_1 < t_2\ \textit{and}\ A \in \mathscr{M}_{t_1} \right\}.$$

Then again $1 \oplus \operatorname{span}(\mathscr{F}_s)$ *is dense in* $L^1(P_{s,x})$.

Proof. — Both these facts are easy consequences of Douglas's theorem. Indeed, in either case, we know from Lemma (3.1) that there is exactly one $P \in \mathscr{N}(\mathscr{F}_s)$ such that $P(x(t)=x, 0 \leqq t \leqq s)=1$, namely $P_{s,x}$. Hence $P_{s,x} \in \operatorname{ext}(\mathscr{N}(\mathscr{F}_s))$.

Q.E.D.

Remark (3.3). — The second part of Theorem (3.2) is very close to a result due to Kunita and Watanabe [5]. Indeed, their theorem says that $\mathscr{F}_s$ is dense in $L^2(P_{s,x})$. When $\Omega = C([0, \infty), E)$ and $P(t, x, \Gamma)$ is Feller continuous, one can obtain the K.-W. theorem directly from ours by using Theorem (1.4). Indeed, one then has that $\mathscr{F}_s$ is dense in $L^p(P_{s,x})$ for $1 \leqq p < \infty$.

In general, one can prove the K.-W. theorem using Capon's variant of Douglas's theorem (*cf.* [6]): If $1 < p < \infty$ then $1 \oplus \operatorname{span}(\mathscr{F})$ in dense in $L^p(\mu)$ if and only if for all non-negative $g \in L^{p'}(\mu)((1/p)+(1/p')=1)$ with $E^\mu[g]=1$ the measure μ^g given by $d\mu^g = g\, d\mu$ is extremal in

$\mathcal{N}\left(\left\{f-\int f\,d\mu^g : f\in\mathcal{F}\right\}\right)$ [here the notation is the same as in the first part of section (1)]. Thus we will know that $1\oplus\operatorname{span}(\mathcal{F}_0)$ is dense in $L^p(P_{0,x})$, where $\mathcal{F}_0$ is defined as in the second part of Theorem (3.2), once we show that for every non-negative $g\in L^{p'}(P_{0,x})$, with $E^{P_{0,x}}[g]=1$, $P^g_{0,x}$ is the only $P\in\mathcal{N}(\{f-E^{P^g_{0,x}}[f]:f\in\mathcal{F}_0\})$ such that $P(x(0)=x)=1$. To this end, note that if $P\in\mathcal{N}(\{f-E^{P^g_{0,x}}[f]:f\in\mathcal{F}_0\})$, then for all $h\in C_b(E)$, $0\leqq t_1$, $A\in\mathcal{M}_{t_1}$, and $\lambda>0$:

$$E^P\left[R_\lambda h(x(t_1))-\int_0^\infty e^{-\lambda u}h(x(u+t_1))\,du,\ A\right]$$
$$=E^{P^g_{0,x}}\left[R_\lambda h(x(t_1))-\int_0^\infty e^{-\lambda u}h(x(u+t_1))\,du,\ A\right].$$

Hence, because the Laplace transform is injective

$$E^P\left[\int h(y)P(t,x(s),dy)-h(x(t+s)),\ A\right]=E^{P^g_{0,x}}\left[\int h(y)P(t,x(s),dy)-h(x(t+s)),\ A\right]$$

for all $s,\ t\geqq 0$, $A\in\mathcal{M}_s$. From here it is an easy matter to check that if in addition $P(x(0)=x)=1$, then $P=P^g_{0,x}$.

(The procedure is very much like the proof that a Markov process is determined by its transition probability function.)

THEOREM (3.4). — *Assume that* $P(t,x,\Gamma)$ *is Feller continuous. Then for every* $(s,x)\in[0,\infty)\times E$ *and every* $Y\in L^1(P)$, $E^{P_{s,x}}[Y\,|\,\mathcal{M}_{\cdot\vee s}]$ *admits a right-continuous progressively measurable version. Moreover, if in addition* $\Omega=C([0,\infty),E)$, *then this version will be* $P_{s,x}$-a. s. *continuous.*

Proof. — Let $\underline{\Psi}$ be the set of functions $X(.)$ of the form

$$X(t)=e^{-\lambda t}R_\lambda f(x(t))+\int_0^t e^{-\lambda u}f(x(u))\,du$$

as λ runs over a countable dense set in $(0,\infty)$ and f runs over a countable weakly dense set in $C_b(E)$. By Lemma (3.1), $P_{s,x}\in\operatorname{ext}\mathcal{P}_{s,x}(\underline{\Psi})$. Thus we can apply Theorem (2.3) to finish the proof.

Q.E.D.

Throughout the remainder of this section we will specialize to the case when $E=R^d$ and

$$(3.5)\qquad \int\varphi(y)P(t,x,dy)-\varphi(x)=\int_0^t ds\int L\varphi(y)P(s,x,dy),\qquad \varphi\in C_0^\infty(R^d)$$

where L is a second order (degenerate) elliptic operator of the sort introduced in section (2). From (3.5), it is easy to check the following:

(*a*) $P_{s,x} \in \mathscr{C}_L(s, x)$ for all $(s, x) \in [0, \infty) \times R^d$. In particular we can take $\Omega = C([0, \infty), R^d)$;

(*b*) $C_b^2(R^d) \subseteq D_A$ and $Af = Lf$ for $f \in C_b^2(R^d)$;

(*c*) Let $\hat{B}(R^d) = \{ f \in B(R^d) : \lim_{R \uparrow \infty} \sup_{|x| \geqq R} |f(x)| = 0 \}$ and set $\hat{C}(R^d) = C(R^d) \cap \hat{B}(R^d)$.

Then the semi-group determined by $P(t, x, \Gamma)$ maps $\hat{B}(R^d)$ into itself. In particular, if $P(t, x, \Gamma)$ is Feller continuous, then this semi-group maps $\hat{C}(R^d)$ into itself.

Observe that from (*b*) we know that $C_b^2(R^d)$ is contained in the strong center of the semi-group determined by $P(t, x, \Gamma)$ and therefore $\{ R_1 \varphi : \varphi \in C_0^\infty(R^d) \} \subseteq D_{A_s}$, where D_{A_s} is the domain of the strong generator A_s determined by $P(t, x, \Gamma)$. In particular, we have

$$(3.6) \qquad \overline{\{(f, A_s f) : f \in D_{A_s}\}} \supseteq \operatorname{graph}(A).$$

Combining the preceding, Theorem (2.7) and Lemma (3.1), and the reasoning used in Remark (1.2), we arrive at the next result.

THEOREM (3.7). — *Let* $\bar{x}(t) = x(t) - \int_0^t b(x(u))\,du$ *[$b(.)$ is the vector of first order coefficients of* L].

Then $P_{s,x} \in \operatorname{ext} \mathscr{C}_L(s, x)$ *if and only if for each* $f \in D_A$ *there is a previsible* $\theta : [0, \infty) \times \Omega \to R^d$ *such that*

$$E^{P_{s,x}} \left[\int_s^t \langle \theta(u), a(x(u)) \theta(u) \rangle \, du \right] < \infty, \qquad t \geqq s,$$

and

$$(3.8) \qquad f(x(t)) - f(x(s)) - \int_s^t Af(x(u))\,du = \int_s^t \langle \theta(u), d\bar{x}(u) \rangle, \qquad t \geqq s.$$

In fact, in order that $P_{s,x} \in \operatorname{ext}(\mathscr{C}_L(s, x))$, *it is sufficient that* (3.8) *holds for all* $f \in D_{A_s}$.

For the remainder of this section we will study the problem of determining when (3.8) holds. The first step in our program is to introduce Meyer's notion of the extended generator [3] (*see* also Kunita [35]). To be precise, let $\mathscr{E}$ be the set of pairs (f, g) where $f \in B(R^d)$ and $g : R^d \to R^1$ is a measurable function $R_\lambda(|g|) \in B(R^d)$ for all $\lambda > 0$ and $f = R_\lambda(\lambda f - g)$, $\lambda > 0$. Clearly, $\operatorname{graph}(A) \subseteq \mathscr{E}$ and $\mathscr{E}$ is linear. Denote by D_{A_e} the projection of $\mathscr{E}$ on its first coordinate.

LEMMA (3.9). — *If* $(f, g) \in \mathscr{E}$, *then for all* $T > 0$:

$$\sup_{x \in R^d} E^{P_{0,x}} \left[\left(\int_0^T |g(x(u))|\,du \right)^2 \right] \leqq 2 e^{2T} \| R_1 |g| \|^2 < \infty,$$

and $\left(f(x(t)) - \int_0^t du\, g(x(u)), \mathscr{M}_t, P_{0,x} \right)$ *is a martingale for all* $x \in R^d$. *In particular,* *if* $f \in D_{A_e}$ *and* (f, g_1), $(f, g_2) \in \mathscr{E}$, *then* $\Gamma = \{x : g_1(x) \neq g_2(x)\}$ *has null potential* $\left[\text{i.e. } P_{0,x}\left(\int_0^\infty \chi_\Gamma(x(u))\,du = 0\right) = 1 \text{ for all } x \in R^d\right]$.

Proof. — First note that

$$E^{P_{0,x}}\left[\left(\int_0^\infty e^{-t}|g(x(t))|\,dt\right)^2\right]$$

$$= 2\,E^{P_{0,x}}\left[\int_0^\infty e^{-s}|g(x(s))|\,ds \int_s^\infty e^{-t}|g(x(t))|\,dt\right]$$

$$= 2\,E^{P_{0,x}}\left[\int_0^\infty e^{-2s}|g(x(s))|\,R_1(|g|)(x(s))\,ds\right] \leqq 2\,\|R_1(|g|)\|^2.$$

Hence the first assertion is proved. To prove the second statement, observe that because $f = R_\lambda(\lambda f - g)$:

$$\left(e^{-\lambda t} f(x(t)) + \int_0^t e^{-\lambda u}(\lambda f - g)(x(u))\,du,\ \mathscr{M}_t,\ P_{0,x}\right)$$

is a martingale. Using the estimate just obtained, it is now easy to see that after letting $\lambda \downarrow 0$, we still have a martingale. Thus the second assertion. Finally, if $(f, g_i) \in \mathscr{E}$, $i = 1, 2$, then, by the preceding $\left(\int_0^t (g_1(x(u)) - g_2(x(u)))\,du,\ \mathscr{M}_t,\ P_{0,x}\right)$ is a martingale.

Since the only continuous martingales of bounded variation are constant almost surely, we have now proved that $g_1 = g_2$ except on a set of null potential.

Q.E.D.

The final part of Lemma (3.9) allows us to make the following definition. Given $f \in D_{A_e}$, define $A_e f$ to be set of g such that $(f, g) \in \mathscr{E}$. Since any two elements of $A_e f$ differ on at most a set of null potential, we are justified in identifying $A_e f$ with any element $g \in A_e f$ so long as we only use $A_e f$ in integrals of $A_e f(x(t, \omega))$ with respect to $dt \times dP_{0,x}$. Of course, if $f \in D_A$, we will take $A_e f = A f$.

We will need one small refinement of the last part of Lemma (3.9).

LEMMA (3.10). — *Suppose that* $f \in D_{A_e} \cap C_b(R^d)$ *and that* $f = 0$ *on the open set* $\mathscr{G}$. *Then we can take* $A_e f = 0$ *on* $\mathscr{G}$.

Proof. — Let $x_0 \in \mathscr{G}$ and choose $0 < R_1 < R_2$ so that $\overline{B(x_0, R_2)} \subseteqq \mathscr{G}$. Define $\sigma_0 = \inf\{t \geqq 0 : |x(t) - x_0| \leqq R_1\}$ and

$$\tau_n = \inf\{t \geqq \sigma_{n-1} : |x(t) - x_0| \geqq R_2\},$$
$$\sigma_n = \inf\{t \geqq \tau_n : |x(t) - x_0| \leqq R_1\}.$$

Then $\left(f(x(t\wedge\tau_n))-f(x(t\wedge\sigma_{n-1}))-\int_{\sigma_{n-1}\wedge t}^{\tau_n\wedge t} A_e f(x(u))\,du,\ \mathscr{M}_t,\ P_{0,x}\right)$ is a martingale for all $n\geqq 1$ and $x\in R^d$. Hence

$$\left(\int_0^t \chi_{[\sigma_{n-1},\,\tau_n]}(u)\,A_e f(x(u))\,du,\ \mathscr{M}_t,\ P_{0,x}\right)$$

is a martingale, so

$$\int_0^{\cdot} \chi_{[\sigma_{n-1},\,\tau_n]}(u)\,A_e f(x(u))\,du\equiv 0\quad (\text{a.s., } P_{0,x})\quad \text{for all } n\geqq 0.$$

It follows immediately that

$$\int_0^{\cdot} \chi_{B(x_0,\,R_1)}(x(u))\,A_e f(x(u))\,du\equiv 0\quad (\text{a.s., } P_{0,x}).$$

Q.E.D.

Lemma (3.10) enables us to make the following definition. Let $D_{\mathscr{A}}$ be the set of $f\in C(R^d)$ such that for each $R>0$ there is an $\eta_R\in C_0^\infty(R^d)$ with the properties that $\eta_R\equiv 1$ on $B(0, R)$ and $\eta_R . f\in D_{A_e}$. By Lemma (3.10), $A_e(\eta_R . f)$ on $B(0, R)$ is independent of the choice of η_R up to a set of null potential. Hence we can define $\mathscr{A} f=A_e(\eta_R . f)$ on $B(0, R)$ for all $R>0$.

LEMMA (3.11). — *Suppose that D_{A_e} is an algebra (i.e. $f, g\in D_{A_e}$ implies $f.g\in D_{A_e}$). Then $D_{\mathscr{A}}$ is an algebra. Given $f, g\in D_{A_e}$ $(\in D_{\mathscr{A}})$, define*

$$Q(f, g)=A_e(f.g)-f.A_e g-g A_e f\ (=\mathscr{A}(f.g)-f\mathscr{A} g-g\mathscr{A} f).$$

If $f, g\in D_{A_e}$, then

$$\left(X_f(t)X_g(t)-\int_0^t Q(f, g)(x(u))\,du,\ \mathscr{M}_t,\ P_{0,x}\right)$$

is a martingale for all $x\in R^d$, where

$$X_h(t)\equiv h(x(t))-\int_0^t A_e h(x(u))\,du,\qquad h\in D_{A_e}. \tag{3.12}$$

In fact, this uniquely determines $Q(f, g)$ for $f, g\in D_{A_e}$ up to a set of null potential. In particular, for $f\in D_{A_e}\cup D_{\mathscr{A}}$, $Q(f, f)\geqq 0$ except possibly on a set of null potential.

Proof. — The first assertion is trivial. All of the other assertions follow easily from Roth's article [7] (*cf.* also Kunita's paper [35]).

Q.E.D.

Note that if $f, g\in C^2(R^d)$, then $Q(f, g)=(\nabla f, a\nabla g)$. In particular, if $\chi_i(x)=x_i\,(x\in R^d)$, $Q(\chi_i, \chi_j)=a^{ij}$. Also, from the non-negative definiteness assertion about the quadratic form Q, one can easily deduce

$$Q(f, g)^2 \leqq Q(f, f)\, Q(g, g)$$

up to a set of null potential.

LEMMA (3.13). — *Assume that* D_{A_e} *is an algebra. Given* $f \in D_{A_e} \cap C_b(R^d)$, *define*

$$(3.14) \qquad C_f(x) = \begin{pmatrix} a(x) & Q(\chi, f)(x) \\ Q(\chi, f)^T(x) & Q(f, f)(x) \end{pmatrix} \in R^{d+1} \otimes R^{d+1}, \qquad x \in R^d,$$

where $Q(\chi, f)$ *denotes the vector* $\begin{pmatrix} Q(\chi_1, f) \\ \vdots \\ Q(\chi_d, f) \end{pmatrix}$. *Then for any* $\varphi \in C_0^\infty(R^{d+1})$ *and* $x \in R^d$:

$$(3.15) \quad \varphi(x(t), X_f(t)) - \varphi(x, f(x))$$
$$= \sum_{j=1}^{d} \int_0^t \frac{\partial \varphi}{\partial x_j}(x(u), X_f(u))\, d\bar{x}_j(u) + \int_0^t \frac{\partial \varphi}{\partial x_{d+1}}(x(u), X_f(u))\, dX_f(u)$$
$$+ \int_0^t \mathscr{L}_f \varphi(x(u), X_f(u))\, du \quad (\text{a.s., } P_{0,x})$$

where

$$(3.16) \qquad \mathscr{L}_f = 1/2 \sum_{i,j=1}^{d+1} C_f(x)^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b^i(x) \frac{\partial}{\partial x_i}.$$

In particular, if $\theta : [0, \infty) \times \Omega \to R^d$ *is a progressively measurable function such that* $E^{P_{0,x}}\left[\int_0^t \langle \theta(u), a(x(u))\theta(u) \rangle du\right] < \infty$, *then*

$$(3.17) \qquad E^{P_{0,x}}\left[\left(X_f(t) - f(x) - \int_0^t \langle \theta(u), d\bar{x}(u) \rangle\right)^2\right]$$
$$= E^{P_{0,x}}\left[\int_0^t \begin{pmatrix} \theta(u) \\ -1 \end{pmatrix}^T C_f(x(u)) \begin{pmatrix} \theta(u) \\ -1 \end{pmatrix} du\right].$$

Proof. — The first assertion will be proved once we have shown that

$$\left(\bar{x}_j(t) X_f(t) - \int_0^t Q(\chi_j, f)(x(u))\, du, \mathscr{M}_t, P_{0,x}\right)$$

is a martingale for all $f \in D_{A_e} \cap C_b(R^d)$, $1 \leqq j \leqq d$, and $x \in R^d$.

To this end, choose a sequence $\{\rho_n\}_1^\infty \subseteq C_0^\infty(R^1)$ such that

$$\rho_n(0) = 0, \qquad \rho_n'(x) = 1 \qquad \text{for } |x| \leqq n,$$

and

$$\sup_{n \geqq 1} \sup_{x \in R^1} |\rho_n'(x)| \vee |\rho_n''(x)| < \infty.$$

Let $g_n(x)=\rho_n(x_j)$. Then

$$\left(X_{g_n}(t)X_f(t)-\int_0^t Q(g_n, f)(x(u))\,du,\ \mathcal{M}_t,\ P_{0,x}\right)$$

is a martingale for all $n \geqq 1$. Furthermore, using the estimates $|g_n(x)| \leqq C|x_j|$ and

$$Q(g_n, f)^2 \leqq Q(g_n, g_n)Q(f, f)=(\nabla g_n, a\nabla g_n)Q(f, f) \leqq C\,Q(f, f),$$

one easily sees that the desired result follows upon letting $n \to \infty$. Given the first part of the Lemma, the second part is an immediate consequence of Itô's calculus for stochastic integrals.

Q.E.D.

THEOREM (3.18). — *Assume that* $P(t, x, \Gamma)$ *is Feller continuous. Then* $P_{s,x} \in \text{ext}(\mathscr{C}_L(s, x))$ *for all* $(s, x) \in [0, \infty) \times R^d$ *if and only if each of the following two conditions holds:*

(*a*) D_{A_e} *is an algebra;*

(*b*) *for each* $f \in D_{A_s} \cap C_b(R^d)$, *the set* Γ_f *of* $x \in R^d$ *such that* $\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \text{Range}(C_f(x))$ *has null potential.*

Proof. — The proof of necessity relies heavily on Meyer's theorem [3] which says that D_{A_e} is an algebra if and only if for every continuous square integrable martingale $(M(t), \mathcal{M}_t, P_{0,x})$ there is a progressively measurable $m:[0, \infty) \times \Omega \to [0, \infty)$ such that $\left(M^2(t)-\int_0^t m(u)\,du,\ \mathcal{M}_t,\ P_{0,x}\right)$ is a martingale. With Meyer's theorem, it is clear that the necessity of (*a*) is an immediate consequence of Theorem (2.7). To show the necessity of (*b*), note that if $P_{0,x} \in \text{ext}(\mathscr{C}_L(0, x))$, then for $f \in D_A \cap C_b(R^d)$:

$$f(x(t))-f(x(0))-\int_0^t A_s f(x(u))\,du=\int_0^t \langle \theta(u), d\bar{x}(u) \rangle \quad (\text{a.s., } P_{0,x})$$

with $\theta : [0, \infty) \times \Omega \to R^d$ previsible and satisfying

$$E^{P_{0,x}}\left[\int_0^t \langle \theta(u), a(x(u))\theta(u) \rangle\,du\right] < \infty, \qquad t>0.$$

Thus, by (3.17):

$$E^{P_{0,x}}\left[\int_0^\infty \begin{pmatrix} \theta(u) \\ -1 \end{pmatrix}^T C_f(x(u)) \begin{pmatrix} \theta(u) \\ -1 \end{pmatrix} du\right]=0.$$

But $\complement \Gamma_f=\left\{x:(\exists \theta \in R^d)\begin{pmatrix} \theta \\ -1 \end{pmatrix}^T C_f(x)\begin{pmatrix} \theta \\ -1 \end{pmatrix}=0\right\}$, and so Γ_f has null potential.

Conversely, assume (*a*) and (*b*). Given $f \in D_{A_s} \cap C_b(R^d)$, we can use a standard selection principle to find a measurable $\theta : R^d \to R^d$ such that $\begin{pmatrix} \theta(x) \\ -1 \end{pmatrix}^T C_f(x)\begin{pmatrix} \theta(x) \\ -1 \end{pmatrix}=0$ for

$x \notin \Gamma_f$. Furthermore, by (3.17):

$$E^{P_{0,x}}\left[\left(X_f(t)-f(x)-\int_0^t \langle \theta(x(u)), d\bar{x}(u)\rangle\right)^2\right] \equiv 0.$$

In other words, (3.8) holds for all $f \in D_{A_s} \cap C_b(R^d)$. But because $P(t, x, \Gamma)$ is Feller continuous, $\{(f, A_s f) : f \in D_{A_s} \cap C_b(R^d)\}$ is weakly dense in graph (A). Thus (3.8) holds for all $f \in D_A$, and so $P_{0,x} \in \text{ext}(\mathscr{C}_L(0, x))$.

Q.E.D.

Unfortunately, Theorem (3.18) does not provide a very practical criterion for extremality. Nonetheless, with the help of the next lemma, we can use Theorem (3.17) to arrive at a more workable sufficient condition.

LEMMA (3.19). — *Assume that the second order coefficients $a(x)$ are continuous at x^0. Also, assume that $P(t, x, \Gamma)$ is Feller continuous and that $D_{A_s} \cap C_b(R^d)$ is an algebra. Then for each $f \in D_{A_s} \cap C_b(R^d)$ there is a version of $C_f(.)$ which is bounded everywhere and continuous at x^0. Furthermore, if $f \in D_{A_s} \cap C_b(R^d)$ is Lipschitz continuous at x^0, then $\begin{pmatrix} 0 \\ -1 \end{pmatrix} \notin \text{Range}\, C_f(x^0)$.*

Proof. — The first part follows easily from the fact that

$$A_s : \quad D_{A_s} \cap C_b(R^d) \to C_b(R^d).$$

To prove the second part, define

$$x_\varepsilon(t) = (1/\varepsilon)(x(\varepsilon^2 t) - x^0),$$
$$y_\varepsilon(t) = (1/\varepsilon)(X_f(\varepsilon^2 t) - f(x^0)),$$

and let P^ε on $C([0, \infty), R^{d+1})$ denote the distribution under P_{0,x^0} of $\begin{pmatrix} x_\varepsilon(.) \\ y_\varepsilon(.) \end{pmatrix}$. Then $P^\varepsilon \in \mathscr{C}_{L_\varepsilon}(0, 0)$, where

$$L_\varepsilon = 1/2 \sum_{i,j=1}^{d+1} C_f(x^0+\varepsilon x)^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \varepsilon b^i(x^0+\varepsilon x) \frac{\partial}{\partial x_i}.$$

Hence, it follows that $P^\varepsilon \to P$ as $\varepsilon \downarrow 0$, where P is the distribution under $(d+1)$-dimensional Wiener measure $\mathscr{W}$ of $C_f(x^0)\, x(.)$. In particular, if $\begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \text{Range}\, C_f(x^0)$, then it is easy to check that

$$P(|\hat{x}(1)| < 1 \ \text{ and } \ |x_{d+1}(1)| > M) > 0$$

for all $M > 0$, where $\hat{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$. On the other hand,

$P_\varepsilon(|\hat{x}(1)|<1 \text{ and } |x_{d+1}(1)|>M)$

$$=P_{0,x^0}\left(|x_\varepsilon(1)|<1 \text{ and } \left|\frac{f(\varepsilon x_\varepsilon(1)+x^0)-f(x^0)}{\varepsilon}-\varepsilon\int_0^1 A_s f(x(\varepsilon^2 v)dv\right|>M\right)\to 0$$

as $\varepsilon\downarrow 0$, so long as

$$M>M_0\equiv\lim_{\delta\downarrow 0}\sup_{|y-x^0|<\delta}\frac{|f(y)-f(x^0)|}{|y-x^0|}.$$

Hence

$$P(|\hat{x}(1)|<1 \text{ and } |x_{d+1}(1)|>M)$$

$$\leqq\varliminf_{\varepsilon\downarrow 0}P(|\hat{x}(1)|<1 \text{ and } |x_{d+1}(1)|>M)=0 \qquad \text{for} \quad M>M_0.$$

In particular, $\begin{pmatrix}0\\-1\end{pmatrix}\notin \text{Range}(C_f(x^0))$.

Q.E.D.

THEOREM (3.20). — *Assume that* $P(t, x, \Gamma)$ *is Feller continuous and that* $D_{A_s}\cap\hat{C}(R^d)$ *is an algebra. Also assume that there is a set* Γ_0 *of null potential such that the second order coefficients* $a(.)$ *of* L *are continuous at each* $x\notin\Gamma_0$ *and that every* $f\in D_{A_s}\cap\hat{C}(R^d)$ *is Lipschitz continuous at all points outside a set* Γ_f *of null potential. Then* $P_{s,x}\in\text{ext}(\mathscr{C}_L(s, x))$ *for all* $(s, x)\in[0, \infty)\times R^d$.

Proof. — By Lemma (3.19), the hypotheses guarantee that for each $f\in D_{A_s}\cap\hat{C}(R^d)$, $\begin{pmatrix}0\\-1\end{pmatrix}\notin\text{Range}(C_f(x))$ for x outside a set of null potential.

Once one has this, the rest of the proof is word for word the same as the proof of sufficiency in Theorem (3.18).

Q.E.D.

COROLLARY (3.21). — *Assume that* $a(.)$ *is continuous and that* $P(t, x, \Gamma)$ *is Feller continuous. If*

$$D_{A_s}\cap\hat{C}(R^d)\subseteq\{f\in\hat{C}(R^d)\cap C^1(R^d):(\nabla f, a\nabla f)\in\hat{C}(R^d)\},$$

then $P_{s,x}\in\text{ext}(\mathscr{C}_L(s, x))$ *for each* $(s, x)\in[0, \infty)\times R^d$.

Proof. — In view of Theorem (3.20), we need only check that $D_{A_s}\cap\hat{C}(R^d)$ is an algebra. To this end, we use the results of Dynkin [8] which show that $f\in\hat{C}(R^d)$ is in D_{A_s} if and only if $\lim_{\varepsilon\downarrow 0}(E^{P_{0,x}}[f(x(\tau_\varepsilon))]-f(x))/E^{P_{0,x}}[\tau_\varepsilon]$ exists for all $x\in R^d$ and defines an element of $\hat{C}(R^d)$ (which is, indeed, $A_s f$), where

$$\tau_\varepsilon=\inf\{t\geqq 0:|x(t)-x(0)|\geqq\varepsilon\}.$$

Now suppose that $f \in D_{A_s} \cap \hat{C}(R^d)$. If $E^{P_{0,x}}[\tau_\varepsilon] < \infty$ for $0 < \varepsilon \leqq \varepsilon_0$, then

$$\frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))] - f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]} = \frac{E^{P_{0,x}}[(f(x(\tau_\varepsilon)) - f(x))^2]}{E^{P_{0,x}}[\tau_\varepsilon]} + 2f(x)\frac{E^{P_{0,x}}[f(x(\tau_\varepsilon))] - f(x)}{E^{P_{0,x}}[\tau_\varepsilon]}$$

for $0 < \varepsilon \leqq \varepsilon_0$. The second term on the right tends to $2f(x)(A_s f)(x)$ as $\varepsilon \downarrow 0$. To handle the first term, note that

$$E^{P_{0,x}}[(f(x(\tau_\varepsilon)) - f(x))^2] = \sum_{i,j=1}^{d} \frac{\partial f}{\partial x_i}(x)\frac{\partial f}{\partial x_j}(x) E^{P_{0,x}}[(x_i(\tau_\varepsilon) - x_i)(x_j(\tau_\varepsilon) - x_j)] + o(\varepsilon^2).$$

But

$$\left(|x(t) - x|^2 - \int_0^t \operatorname{Trace} a(x(u))\,du - 2\int_0^t (x(u) - x, b(x(u)))\,du, \mathscr{M}_t, P_{0,x}\right)$$

is a martingale. Thus if $E^{P_{0,x}}[\tau_\varepsilon] < \infty$, then

$$E^{P_{0,x}}\left[\int_0^{\tau_\varepsilon} \operatorname{Trace} a(x(u))\,du\right] = \varepsilon^2 - 2E^{P_{0,x}}\left[\int_0^{\tau_\varepsilon} (x(u) - x, b(x(u)))\,du\right]$$

and therefore $\varliminf_{\varepsilon \downarrow 0} (1/\varepsilon^2) E^{P_{0,x}}[\tau_\varepsilon] > 0$. At the same time

$$\frac{E^{P_{0,x}}[x_i(\tau_\varepsilon) - x_i)(x_j(\tau_\varepsilon) - x_j)]}{E^{P_{0,x}}[\tau_\varepsilon]} \to a^{ij}(x) \quad \text{as} \quad \varepsilon \downarrow 0.$$

Hence

$$\frac{E^{P_{0,x}}[(f(x(\tau_\varepsilon)) - f(x))^2]}{E^{P_{0,x}}[\tau_\varepsilon]} \xrightarrow{\varepsilon \downarrow 0} (\nabla f, a \nabla f)(x);$$

and so

$$\lim_{\varepsilon \downarrow 0} \frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon)) - f^2(x)]}{E^{P_{0,x}}[\tau_\varepsilon]} = (\nabla f, a \nabla f)(x) + 2f(x) A_s f(x)$$

at x such that $E^{P_{0,x}}[\tau_\varepsilon] < \infty$ for small enough ε. Next suppose that $E^{P_{0,x}}[\tau_\varepsilon] = \infty$ for all $\varepsilon > 0$. Then, according to Dynkin's theory, $P_{0,x}(x(t) = x, t \geqq 0) = 1$. Hence in this case $a(x) = 0$ and $A_s f(x) = 0$. Thus once again

$$\lim_{\varepsilon \downarrow 0} \frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))] - f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]} = (\nabla f, a \nabla f)(x) + 2f(x) A_s f(x).$$

Since $(\nabla f, a \nabla f) + 2f A_s f \in \hat{C}(R^d)$, this completes the proof.

Q.E.D.

Section (4)

This section contains several examples of L's for which it is possible to classify all the extreme, strong Markov, time-homogeneous selections from $\mathscr{K}_L$. Unfortunately, no general schema has grown out of these examples. In fact, we find these examples to be convincing evidence that a general procedure is going to be hard to come by.

Example (4.1). — Let $d=1$ and let $a:\mathrm{R}^1 \to [0, \infty)$ be a bounded continuous function whose zeroes are isolated. Assume further that

$$\sup_{|x| \leq \mathrm{R}} \int_0^{\mathrm{T}} dt \int_{\mathrm{R}^1} g(t, y-x) \frac{1}{a(y)} dy < \infty, \qquad \mathrm{T}>0 \text{ and } \mathrm{R}>0, \tag{4.2}$$

where

$$g(t, x) = \frac{1}{(2\pi t)^{1/2}} e^{-x^2/2t}.$$

We are going to study $\mathscr{H}_{\mathrm{L}}$ with $\mathrm{L} = 1/2\, a(x)\, \partial^2/\partial x^2$. In fact, what we are going to do is show that every time-homogeneous strong Markov selection from $\mathscr{H}_{\mathrm{L}}$ is Feller continuous and that every Feller continuous time-homogeneous strong Markov selection consists of extreme elements. Combining these facts with Theorem (2.11), we will have thereby ended up with a reasonably satisfactory description of $\mathscr{H}_{\mathrm{L}}$.

To prove that every time-homogeneous strong Markov selection $\{\mathrm{P}_{s, x} : (s, x) \in [0, \infty) \times \mathrm{R}^1\}$ is Feller continuous, we proceed as follows.Let $\beta(.)$ be a 1-dimensional Brownian motion. Using (4.2), one can easily check that if $\tau_x(.)$ is defined by

$$\int_0^{\tau_x(t)} \frac{1}{a(x+\beta(s))} ds = t, \qquad t \geqq 0,$$

then, almost surely, τ_x is a continuous increasing function such that $\tau_x(0)=0$ and $\tau_x(t) \uparrow \infty$ as $t \uparrow \infty$. Moreover one can see that if $\mathrm{P}^0_{s, x}$ on $(\Omega, \mathscr{M})$ is the distribution of $x+\beta(\tau_x((.-s) \vee 0))$, then $\mathrm{P}^0_{s, x} \in \mathscr{C}_{\mathrm{L}}(s, x)$ (*cf.* Thm. 6.5.2 in [*S. and V.*]). In fact, if $\xi<\eta$ and $a(.)>0$ on (ξ, η), then for any $\mathrm{P} \in \mathscr{C}_{\mathrm{L}}(0, x)$, P equals $\mathrm{P}^0_{0, x}$ on $\mathscr{M}_{\tau_{(\xi, \eta)}}$ where $\tau_{(\xi, \eta)} = \inf\{t \geqq 0 : x(t) \notin (\xi, \eta)\}$ (*cf.* section (6.6) of [*S. and V.*]). In particular, we have that if $f \in \mathrm{C}_b(\mathrm{R}^1)$ and $u(t, x) = \mathrm{E}^{\mathrm{P}_{0,x}}[f(x(t))]$, then

$$u(t, x) = \mathrm{E}^{\mathrm{P}^0_{0,x}}[f(x(t)), \tau_{(\xi, \eta)} > t] + \mathrm{E}^{\mathrm{P}^0_{0,x}}[u(t-\tau_{(\xi, \eta)}, x(\tau_{(\xi, \eta)})), \tau_{(\xi, \eta)} \leqq t]$$

so long as $a(.)>0$ on (ξ, η). But it is a simple matter to check that for each $\varepsilon>0$:

$$\lim_{x \downarrow \xi} \mathrm{P}^0_{0, x}(\tau_{(\xi, \eta)} > \varepsilon \text{ or } x(\tau_{(\xi, \eta)}) \neq \xi) = \lim_{x \uparrow \eta} \mathrm{P}^0_{0, x}(\tau_{(\xi, \eta)} > \varepsilon \text{ or } x(\tau_{(\xi, \eta)}) \neq \eta) = 0.$$

Hence, for any $\xi<\eta$ such that $a(.)>0$ on (ξ, η),

$$u(t, x) \to \begin{cases} u(t, \xi) & \text{as } x \downarrow \xi, \\ u(t, \eta) & \text{as } x \uparrow \eta. \end{cases}$$

Because the zeroes of $a(.)$ are isolated, we have now proved that $\{\mathrm{P}_{s, x} : (s, x) \in [0, \infty) \times \mathrm{R}^1\}$ is Feller continuous.

Our next step is to prove that if A_s is the strong generator of $\{P_{s,x}:(s, x)\in[0, \infty)\times R^1\}$, then every $f\in D_{A_s}\cap\hat{C}(R^1)$ is locally Lipschitz continuous, $f\in C^2(\{x:a(x)>0\})$, and $A_s f=Lf$ on $\{x:a(x)>0\}$. To this end, it is enough to prove the second and third properties of f, since the local Lipschitz continuity will then be obvious from (4.2). But $(x(t), \mathcal{M}_t, P_{0,x})$ and $\left((x(t)-x)^2-\int_0^t a(x(u))\,du, \mathcal{M}_t, P_{0,x}\right)$ are martingales.

Thus, if $\tau_\varepsilon=\inf\{t\geqq 0:|x(t)-x(0)|\geqq\varepsilon\}$ and $P_{0,x}(\tau_\varepsilon<\infty)=1$, then

$$(4.3)\qquad P_{0,x}(x(\tau_\varepsilon)=x+\varepsilon)=1/2 \quad\text{and}\quad E^{P_{0,x}}\left[\int_0^{\tau_\varepsilon} a(x(u))\,du\right]=\varepsilon^2.$$

In particular, if $a(x)>0$, then we have:

$$P_{0,x}(x(\tau_\varepsilon)=x+\varepsilon)=1/2 \quad\text{and}\quad E^{P_{0,x}}[\tau_\varepsilon]=\varepsilon^2/a(x)+o(\varepsilon^2)$$

for small $\varepsilon>0$. Hence, by Dynkin's formula

$$(4.4)\qquad A_s f(x)=\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[f(x(\tau_\varepsilon))-f(x)]}{E^{P_{0,x}}[\tau_\varepsilon]}=\frac{1}{2}a(x)\lim_{\varepsilon\downarrow 0}\frac{f(x+\varepsilon)+f(x-\varepsilon)-2f(x)}{\varepsilon^2},$$

when $a(x)>0$.

This proves both that $f\in C^2(\{x:a(x)>0\})$ and that $A_s f=Lf$ on $\{x:a(x)>0\}$.

We now want to show that $P_{s,x}\in\text{ext}(\mathcal{C}_L(s, x))$. Suppose that we have done this under the additional assumption that $a(x)\geqq\varepsilon$, $|x|\geqq R$, for some $\varepsilon>0$ and $R>0$. Then, using Theorem (2.4) together with an easy localization argument, we will have proved the general case. Thus we will assume that $a(x)\geqq\varepsilon>0$ for $|x|\geqq R$. By the preceding paragraph, if $f\in D_{A_s}\cap\hat{C}(R^1)$, then $f\in C^2(\{x:a(x)>0\})$ and

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))]-f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]}\\
&\qquad=\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[(f(x(\tau_\varepsilon))-f(x))^2]}{E^{P_{0,x}}[\tau_\varepsilon]}+2f(x)\frac{E^{P_{0,x}}[f(x(\tau_\varepsilon))]-f(x)}{E^{P_{0,x}}[\tau_\varepsilon]}\\
&\qquad=a(x)(f'(x))^2+2f(x)Lf(x)=a(x)(f'(x))^2+2f(x)A_s f(x)
\end{aligned}$$

when $a(x)>0$. Next suppose that $a(x)=0$. We distinguish two cases: x is absorbing and x is not absorbing. In the absorbing case

$$\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))]-f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]}=0=2f(x)A_s f(x).$$

In the non-absorbing case, there is an $\varepsilon_0>0$ such that $E^{P_{0,x}}[\tau_\varepsilon]<\infty$ for all $0<\varepsilon\leqq\varepsilon_0$. Thus, by (4.3),

$$\frac{1}{\varepsilon^2}E^{P_{0,x}}\left[\int_0^{\tau_\varepsilon} a(x(u))\,du\right]=1 \qquad \text{as} \quad \varepsilon\leqq\varepsilon_0.$$

Since $a(x)=0$ it follows that

$$\frac{1}{\varepsilon^2}E^{P_{0,x}}[\tau_\varepsilon]\to\infty \qquad \text{as} \quad \varepsilon\downarrow 0.$$

Hence, since f is locally Lipschitz continuous

$$\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))]-f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]}=\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[(f(x(\tau_\varepsilon))-f(x))^2]}{E^{P_{0,x}}[\tau_\varepsilon]}+2f(x)A_s f(x)=2f(x)A_s f(x),$$

We have therefore shown that for $f\in D_{A_s}\cap\hat{C}(R^1)$:

$$\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))]-f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]}=\begin{cases}2f(x)A_s f(x)+a(x)(f'(x))^2 & \text{if} \quad a(x)>0,\\ 2f(x)A_s f(x) & \text{if} \quad a(x)=0.\end{cases}$$

Since f is locally Lipschitz continuous, we conclude that

$$F(x)\equiv\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))]-f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]}\in C(R^1)$$

when $f\in D_{A_s}\cap\hat{C}(R^1)$. Finally since $A_s f\to 0$ at infinity and $a(x)\geqq\varepsilon>0$ for $|x|\geqq R$, we see that $f''(x)\to 0$ as $|x|\to\infty$ and that $F(x)-a(x)(f'(x))^2\to 0$ as $|x|\to\infty$. But $f\to 0$ and $f''\to 0$ at ∞ imply $f'\to 0$ at ∞, and so $F\to 0$ at ∞. In other words

$$\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,x}}[f^2(x(\tau_\varepsilon))]-f^2(x)}{E^{P_{0,x}}[\tau_\varepsilon]}\in\hat{C}(R^1).$$

By Dynkin's theory, it follows that $f^2\in D_{A_s}\cap\hat{C}(R^1)$.

To summarize, we have now shown that every time-homogeneous strong Markov selection $\{P_{s,x}:(s,x)\in[0,\infty)\times R^1\}$ of $\mathscr{K}_L$ is Feller continuous and consists of extreme elements. One can go further. Namely, it is possible to describe the generators of all these Feller selections. In a particular case, this was done by Girsanov [9]; and the general case can be deduced from the particular one by localization.

Example (4.5). — An interesting feature of the preceding example is that all time-homogeneous strong Markov selections turn out to be Feller continuous. We now give an example in which things are even better: namely, there exist precisely two time-homogeneous strong Markov selections both of which are strongly Feller continuous.

Let $d=1$, $G=R^1\setminus\{0\}$ and define

$$L=1/2\,\chi_G(x)\frac{\partial^2}{\partial x^2}+\chi_{\{0\}}(x)\frac{\partial}{\partial x}.$$

Since the coefficients of this L are discontinuous, we must first check that $\mathscr{K}_L$ is a Krylov system [i.e. condition (*a*) preceding Lemma (2.8) is fulfilled].

LEMMA (4.6). — *For each $x\in R^1$, $\mathscr{W}_{0,x}\in\mathscr{C}_L(0, x)$, where $\mathscr{W}_{0,x}$ denotes 1-dimensional Wiener measure starting from x at time 0. Furthermore, if $x_n\to x$ and $P_n\in\mathscr{C}_L(0, x_n)$, then $\{P_n\}_1^\infty$ is relatively compact and every limit is an element of $\mathscr{C}_L(0, x)$.*

Proof. — Since $\mathscr{W}_{0,x}\left(\int_0^\infty \chi_{\{0\}}(x(t))\,dt=0\right)=1$ for all $x\in R^1$, it is clear that $\mathscr{W}_{0,x}\in\mathscr{C}_L(0, x)$.

To prove the second assertion, first note that $\{P_n\}_1^\infty$ is relatively compact because the coefficients of L are bounded. Next note that $P\in\mathscr{C}_L(0, x)$ if and only if $P(x(0)=x)=1$ and for all $f\in C_0^{1,2}([0, \infty)\times R^1)$ satisfying $((\partial f/\partial t)+(\partial f/\partial x))(t, 0)\geqq 0$:

$$\left(f(t, x(t))-\int_0^t \chi_G(x(u))\left(\frac{\partial f}{\partial t}+1/2\frac{\partial^2 f}{\partial x^2}\right)(u, x(u))\,du, \mathscr{M}_t, P\right)$$

is a submartingale. The "only if" statement is easy.

To see the "if" direction, suppose that P satisfies the submartingale condition and that $P(x(0)=x)=1$. It is then easy to check that for every $\lambda>0$:

$$(e\lambda x(t)-(\lambda^2/2)t+e-\lambda x(t)-(\lambda^2/2)t, \mathscr{M}_t, P)$$

is a supermartingale. In particular, $E^P[\sup_{0\leqq t\leqq T} e\lambda|x(t)|]<\infty$ for all $\lambda>0$ and $T>0$. Using this estimate, it is now a simple matter to check that

$$(x^2(t)+x(t)-t, \mathscr{M}_t, P) \quad \text{and} \quad \left(x^2(t)-\int_0^t \chi_G(x(u))\,du, \mathscr{M}_t, P\right)$$

are martingales. Thus $\left(x(t)-\int_0^t \chi_{\{0\}}(x(u))\,du, \mathscr{M}_t, P\right)$ is also a martingale. Now, let $f\in C_0^\infty(R^1)$ and define $\bar f(x)=f(x)-f'(0)x$. Then

$$\left(\bar f(x(t))-\int_0^t(\chi_G\,1/2 f'')(x(u))\,du, \mathscr{M}_t, P\right)$$

is a martingale; and so

$$f(x(t))-\int_0^t L f(x(u))\,du=\bar f(x(t))-\int_0^t(\chi_G\,1/2 f'')(x(u))\,du +f'(0)\left(x(t)-\int_0^t \chi_{\{0\}}(x(u))\,du\right)$$

is a P-martingale. In other words, $P\in\mathscr{C}_L(0, x)$.

In order to complete the proof, suppose that $P_n \to P$. Clearly $P(x(0)=x)=1$, and so we need only check that P satisfies the submartingale condition. This in turn will be done if we show that for $f \in C_b(R^1)$, $0 \leqq t_1 < t_2$ and bounded continuous $\mathscr{M}_{t_1}$-measurable $F:\Omega \to R^1$:

$$E^P\left[F\int_{t_1}^{t_2}(\chi_G f)(x(u))\,du\right] = \lim_{n\to\infty} E^{P_n}\left[F\int_{t_1}^{t_2}(\chi_G f)(x(u))\,du\right].$$

In other words, we need only check that

$$\sup_n E^{P_n}\left[\int_0^t \chi_{(0,\varepsilon)}(|x(u)|)\,du\right] \to 0$$

as $\varepsilon \downarrow 0$ for each $t \geqq 0$. For this purpose, choose $\eta \in C_0^\infty(R^1)$ so that $0 \leqq \eta \leqq 1$, $\eta \equiv 1$ on $[-1, 1]$, and $\eta \equiv 0$ off of $[-2, 2]$.

Set $\psi(x) = \int_0^x dy \int_0^y \eta(t)\,dt$ and $\psi_\varepsilon(x) = \psi(x/\varepsilon)$. Then $|\psi_\varepsilon(x)| \leqq (c/\varepsilon)|x|$ and therefore

$$E^{P_n}\left[\int_0^t \chi_{(0,\varepsilon)}(|x(u)|)\,du\right] \leqq E^{P_n}\left[\int_0^t (\chi_G\,\eta)(x(u)/\varepsilon)\,du\right]$$

$$= 2\varepsilon^2 E^{P_n}\left[\int_0^t (\chi_G\, 1/2\,\psi_\varepsilon'')(x(u))\,du\right]$$

$$= 2\varepsilon^2 (E^{P_n}[\psi_\varepsilon(x(t)] - \psi_\varepsilon(x_n)) \leqq C(t)\,\varepsilon.$$

Q.E.D.

We now know that $\mathscr{K}_L$ is a Krylov system and therefore that the results of section (2) apply. Furthermore, $\{\mathscr{W}_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ is one time-homogeneous strong Markov selection from $\mathscr{K}_L$ consisting of extreme elements. We will now construct a second one. Namely, from results in [10], there is for each $(s, x) \in [0, \infty) \times [0, \infty)$ precisely one $Q_{s,x} \in \mathscr{C}_L(s, x)$ such that $Q_{s,x}(x(t) \geqq 0$ for $t \geqq s) = 1$. We now define $Q_{s,x}$ for $(s, x) \in [0, \infty) \times (-\infty, 0)$ by

$$Q_{s,x} = \mathscr{W}_{s,x} \otimes_{\tau_0(.)} Q_{\tau_0(.),0}, \qquad \text{where} \quad \tau_0(.) = \inf\{t \geqq 0 : x(t) = 0\}.$$

Then, since for any $x \in R^1$ and $P \in \mathscr{C}_L(0, x)$, P equals $\mathscr{W}_{0,x}$ on $\mathscr{M}_{\tau_0}$, it is clear that for each $(s, x) \in [0, \infty) \times R^1$, $Q_{s,x}$ is the only $P \in \mathscr{C}_L(s, x)$ such that $P(x(t) \geqq 0, t \geqq \tau_0) = 1$. From this uniqueness property, it is easy to see that $\{Q_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ is a second time-homogeneous strong Markov selection from $\mathscr{K}_L$. In particular,

$$E^{Q_{0,x}}[f(x(t))] = E^{\mathscr{W}_{0,x}}[f(x(t)), \tau_0 > t] + E^{\mathscr{W}_{0,x}}[E^{Q_{\tau_0(.),0}}[f(x(t - \tau_0(.)))], \tau_0(.) \leqq t].$$

Hence, $\{Q_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ is not only Feller continuous, it is strongly Feller continuous. Finally, $Q_{s,x} \in \text{ext}(\mathscr{C}_L(s, x))$ for all (s, x) since if $Q_{s,x} = \theta P_1 + (1-\theta) P_2$ with $0 < \theta < 1$ and $P_1, P_2 \in \mathscr{C}_L(s, x)$, then $P_1(x(t) \geqq 0, t \geqq \tau_0) = P_2(x(t) \geqq 0, t \geqq \tau_0) = 1$ and so $P_1 = P_2 = Q_{s,x}$.

LEMMA (4.7). — *Let* $\{P_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ *be a time-homogeneous strong Markov selection from* $\mathcal{K}_L$. *If* $P_{0,0}\left(\int_0^\infty \chi_{\{0\}}(x(u))\,du > 0\right) > 0$, *then* $P_{s,x} = Q_{s,x}$ *for all* $(s, x) \in [0, \infty) \times R^1$. *If* $P_{0,0}\left(\int_0^\infty \chi_{\{0\}}(x(u))\,du > 0\right) = 0$, *then* $P_{s,x} = \mathcal{W}_{s,x}$ *for all* (s, x).

Proof. — We prove the last part first. Indeed, by time-homogeneity and the strong Markov property, $P_{0,0}\left(\int_0^\infty \chi_{\{0\}}(x(u))\,du > 0\right) = 0$ implies $P_{s,x}\left(\int_0^\infty \chi_{\{0\}}(x(u))\,du > 0\right) = 0$ for all (s, x). But this means that $\left(f(x(t \vee s)) - \int_s^{t \vee s} du\, 1/2 f''(x(u)), \mathcal{M}_t, P_{s,x}\right)$ is a martingale for all $f \in C_0^\infty(R^1)$, and so $P_{s,x} = \mathcal{W}_{s,x}$.

We next observe that $\{P_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ is Feller continuous. [The proof is exactly the same as the one that we just gave for $\{Q_{s,x} : (s, x) \in [0, \infty) \times R^1\}$]. Thus if A_s and A_s^0 denote the strong generators of $\{P_{s,x} : (s, x) \in [0, \infty) \times R^1\}$ and $\{Q_{s,x} : (s, x) \in [0, \infty) \times R^1\}$, respectively, then we will be done once we show that

$$P_{0,0}\left(\int_0^\infty \chi_{\{0\}}(x(u))\,du > 0\right) > 0$$

implies that $D_{A_s} \cap \hat{C}(R^1) \subseteq D_{A_s^0} \cap \hat{C}(R^1)$ and $A_s f = A_s^0 f$ for $f \in D_{A_s} \cap \hat{C}(R^1)$.

Assume that $P_{0,0}\left(\int_0^\infty \chi_{\{0\}}(x(u))\,du > 0\right) > 0$ and set

$$\zeta = \inf\left\{ t \geqq 0 : \int_0^t \chi_{\{0\}}(x(u))\,du > 0 \right\}.$$

Then, $P_{0,0}(\zeta < \infty) > 0$. Moreover, by the Blumenthal 0-1 law, $P_{0,0}(\zeta > 0) \in \{0, 1\}$. If $P_{0,0}(\zeta > 0) = 1$,

$$\begin{aligned} P_{0,0}(\zeta < \infty) &= P_{0,0}\left(\int_0^{\zeta+\varepsilon} \chi_{\{0\}}(x(u))\,du > 0, \zeta < \infty\right) \\ &= P_{0,0}(\zeta < \infty)\, P_{0,0}\left(\int_0^\varepsilon \chi_{\{0\}}(x(u))\,du > 0\right) \\ &\xrightarrow{\varepsilon \downarrow 0} P_{0,0}(\zeta < \infty)\, P_{0,0}(\zeta = 0) = 0. \end{aligned}$$

Thus, $P_{0,0}(\zeta > 0) = 0$. Next set $\varphi(x) = x^2 + x$. Then $L\varphi \equiv 1$ and so for all $R > 0$:

$$E^{P_{0,0}}[\tau_R \wedge t] = E^{P_{0,0}}[\varphi(x(\tau_R \wedge t))] \leqq R^2 + R, \qquad t \geqq 0,$$

where $\tau_R = \inf\{ t \geqq 0 : |x(t)| \geqq R \}$. From this it follows that

$$E^{P_{0,0}}[\tau_R] = R^2 + E^{P_0}[x(\tau_R)]. \tag{4.8}$$

Since $\left(x(t)-\int_0^t \chi_{\{0\}}(x(u))\,du,\ \mathcal{M}_t,\ P_{0,0}\right)$ is a martingale, we have:

$$\text{(4.9)}\qquad E^{P_{0,0}}[x(\tau_R)]=E^{P_{0,0}}\left[\int_0^{\tau_R}\chi_{\{0\}}(x(u))\,du\right]\equiv\alpha_R>0$$

because $P_{0,0}(\zeta=0)=1$. Now define $p_R=P_{0,0}(x(\tau_R)=R)$. Then for $0<R_1<R_2$:

$$\begin{aligned}p_{R_2}&=P_{0,0}(x(\tau_{R_2})=R_2)\\&=p_{R_1}P_{0,R_1}(x(\tau_{R_2})=R_2)+(1-p_{R_1})P_{0,-R_1}(x(\tau_{R_2})=R_2)\\&=p_{R_1}\left(\frac{R_2-R_1}{R_2}p_{R_2}+\frac{R_1}{R_2}\right)+(1-p_{R_1})\frac{R_2-R_1}{R_2}p_{R_2}=\frac{R_1}{R_2}p_{R_1}+\left(1-\frac{R_1}{R_2}\right)p_{R_2}\end{aligned}$$

and so $p_{R_1}=p_{R_2}$. (We have used here the fact that $P_{0,x}$ equals $\mathcal{W}_{0,x}$ on $\mathcal{M}_{\tau_0}$). Thus, from (4.9),

$$\text{(4.10)}\qquad \alpha_R=\alpha_1 R,\qquad R>0$$

where

$$\text{(4.11)}\qquad 0<\alpha_1=2p_1-1.$$

We can now compute $A_s f$ for $f\in D_{A_s}\cap\hat{C}(R^1)$. For $x\neq 0$, it is clear from Dynkin's formula that $A_s f(x)=1/2 f''(x)$. Thus $f\in C_b^2(R^1\setminus\{0\})$, $\lim_{x\to 0} f''(x)$ exists and $f''\to 0$ at ∞. In particular, $f\in C_b^1(R^1)$. Moreover,

$$\begin{aligned}A_s f(0)&=\lim_{\varepsilon\downarrow 0}\frac{E^{P_{0,0}}[f(x(\tau_\varepsilon))]-f(0)}{E^{P_{0,0}}[\tau_\varepsilon]}=\lim_{\varepsilon\downarrow 0}\frac{p_1 f(\varepsilon)+(1-p_1)f(-\varepsilon)-f(0)}{\varepsilon^2+\alpha_1\varepsilon}\\&=\lim_{\varepsilon\downarrow 0}\frac{1/2f(\varepsilon)+1/2f(-\varepsilon)-f(0)}{\varepsilon^2+\alpha_1\varepsilon}+1/2\,\frac{\alpha_1 f(\varepsilon)-\alpha_1 f(-\varepsilon)}{\varepsilon^2+\alpha_1\varepsilon}=f'(0).\end{aligned}$$

Thus $D_{A_s}\cap\hat{C}(R^1)\subseteq\mathcal{F}\equiv\{f\in\hat{C}(R^1):\ f\in C_b^2(R^1\setminus\{0\}),\ \lim_{|x|\to\infty}f''(x)=0,$ and $f'(0)=\lim_{\substack{x\to 0\\ x\neq 0}}1/2f''(x)\}$ and for $f\in D_{A_s}\cap\hat{C}(R^1)$:

$$A_s f(x)=\begin{cases}1/2f''(x) & \text{if } x\neq 0,\\ f'(0) & \text{if } x=0.\end{cases}$$

Since it is simple to show that $D_{A_s^0}\cap\hat{C}(R^1)=\mathcal{F}$, the proof is now complete.

Q.E.D.

We have now shown that $\{\mathcal{W}_{s,x}: (s,x)\in[0,\infty)\times R^1\}$ and $\{Q_{s,x}: (s,x)\in[0,\infty)\times R^1\}$ are the only time-homogeneous strong Markov selections from $\mathcal{K}_L$. Note that both these are strongly Feller continuous. Finally, all elements of $\mathscr{C}_L(s, x)$ can be obtained from these two selections by the procedure described in Theorem (2.11).

Example (4.12). — In the preceding examples, the time-homogeneous strong Markov selections of extreme elements all turn out to be Feller continuous or better. We now give an example in which this is not the case.

Let $d \geqq 2$ and let $a: \mathbb{R}^d \to [0, \infty)$ be a bounded continuous function such that $a(0)=0$, $a(.)$ is uniformly positive on $\mathbb{R}^d \setminus B(0, \varepsilon)$ for each $\varepsilon > 0$, and $\int_{B(0,1)} (1/a(x))\, dx < \infty$. Set $L = 1/2\, a(x)\, \Delta$. We will show that there are exactly two time-homogeneous strong Markov selections of extreme elements from $\mathscr{H}_L$, one of which is Feller continuous and the other one is not.

Note that for all $x \in \mathbb{R}^d$,

$$\sup_x E^{\mathscr{W}_{0,x}} \left[\int_0^t \frac{1}{a(x(u))}\, du \right] < \infty \quad \text{for all } t > 0$$

and

$$\int_0^t \frac{1}{a(x(u))}\, du \uparrow \infty \qquad \text{as} \quad t \uparrow \infty \quad (\text{a.s.}, \mathscr{W}_{0,x}),$$

where $\mathscr{W}_{0,x}$ is d-dimensional Wiener measure starting from x at time 0.

Thus if $\tau(.)$ is defined by

$$\int_0^{\tau(t)} \frac{1}{a(x(u))}\, du = t, \qquad t \geqq 0,$$

then $\mathscr{W}_{0,x}$-almost surely: $\tau(0)=0$, $\tau(.)$ is continuous, and $\tau(t) \uparrow \infty$ as $t \uparrow \infty$. Finally, if $Q_{s,x}$ on $(\Omega, \mathscr{M})$ is the distribution of $x(\tau(.-s) \vee 0))$ under $\mathscr{W}_{0,x}$, then $\{Q_{s,x} : (s, x) \in [0, \infty) \times \mathbb{R}^d\}$ is a Feller continuous, time-homogeneous, strong Markov selection from $\mathscr{H}_L$.

We next show that if $x \neq 0$, then $\mathscr{C}_L(s, x) = \{Q_{s,x}\}$. Indeed, by an easy random time change argument, one sees that any $P \in \mathscr{C}_L(s, x)$ equals $Q_{s,x}$ on $\mathscr{M}_{\tau_0}$, where $\tau_0 = \inf\{t \geqq 0 : x(t) = 0\}$. But $Q_{s,x}(\tau_0 < \infty) = \mathscr{W}_{s,x}(\tau_0 < \infty) = 0$, and so we conclude that $P = Q_{s,x}$.

LEMMA (4.13). — *Let* $\sigma_0^s = \inf\{t \geqq s : x(t) \neq 0\}$ *and let* $P^0 \in M(\Omega)$ *be the measure such that* $P^0(x(t)=0,\ t \geqq 0)=1$. *Then for any* $P \in \mathscr{C}_L(s, 0)$:

$$P = P(\sigma_0^s = \infty) P^0 + \int_s^\infty P^0 \otimes_t Q_{t,0}\, P(\sigma_0^s \in dt). \tag{4.14}$$

In particular, $P \in \mathscr{C}_L(s, 0)$ *equals* $Q_{s,0}$ *if and only if* $P(\sigma_0^s = s) = 1$; *and therefore* $Q_{s,0}$ *is extreme. Also,* $P \in \mathscr{C}_L(s, 0)$ *is extreme if and only if there is a* $t \in [s, \infty]$ *such that* $P(\sigma_0^s = t) = 1$.

Proof. — We may and will assume that $s=0$, and we will drop the superscript on σ_0^s.

To prove (4.14), define $\sigma_\varepsilon=\inf\{t\geqq 0: |x(t)|\geqq\varepsilon\}$. Then $\sigma_\varepsilon\downarrow\sigma_0$. Given a bounded continuous $\Phi: \Omega\to \mathrm{R}^1$:

$$\mathrm{E}^{\mathrm{P}}[\Phi\,|\,\mathscr{M}_{\sigma_\varepsilon\wedge t}]=\mathrm{E}^{\mathrm{P}^0\otimes_{\sigma_\varepsilon(.)\wedge t}\mathrm{Q}_{\sigma_\varepsilon(.),\,x(\sigma_\varepsilon(.),\,.)}}[\Phi]\quad(\text{a.s., P})$$

for all $\varepsilon>0$ and $t\geqq 0$. Letting $\varepsilon\downarrow 0$ and then $t\uparrow\infty$, we obtain:

$$\mathrm{E}^{\mathrm{P}}[\Phi\,|\,\mathscr{M}_{\sigma_0}]=\chi_{\{\infty\}}(\sigma_0(.))\,\mathrm{E}^{\mathrm{P}^0}[\Phi]+\chi_{(0,\,\infty)}(\sigma_0(.))\,\mathrm{E}^{\mathrm{P}^0\otimes_{\sigma_0(.)}\mathrm{Q}_{\sigma_0(.),\,0}}[\Phi];$$

and clearly (4.14) follows from this.

Next, it is obvious from (4.14) that $\mathrm{P}=\mathrm{Q}_{0,\,0}$ if and only if $\mathrm{P}(\sigma_0=0)=1$; and the extremality of $\mathrm{Q}_{0,\,0}$ is immediate from this.

Hence $\mathrm{Q}_{t,\,0}\in\operatorname{ext}(\mathscr{C}_{\mathrm{L}}(t,\,0))$ for all $t\geqq 0$; and so, by Theorem (2.4), $\mathrm{P}^0\otimes_t\mathrm{Q}_{t,\,0}\in\operatorname{ext}(\mathscr{C}_{\mathrm{L}}(0,\,0))$ for all $t\geqq 0$ [since P^0 is obviously in $\operatorname{ext}(\mathscr{C}_{\mathrm{L}}(0,\,0))$]. From here plus (4.14), it is clear that $\mathrm{P}\in\mathscr{C}_{\mathrm{L}}(0,\,0)$ is extreme if and only if $\mathrm{P}(\sigma_0=t)=1$ for some $t\in[0,\,\infty]$.

We now have the following facts:

(i) $\{\mathrm{Q}_{s,\,x}:(s,x)\in[0,\,\infty)\times\mathrm{R}^d\}$ is one time-homogeneous strong Markov, Feller continuous selection of extreme elements from $\mathscr{K}_{\mathrm{L}}$;

(ii) if $\mathrm{P}\in\operatorname{ext}(\mathscr{C}_{\mathrm{L}}(s,\,0))$, then either $\mathrm{P}=\mathrm{P}^0$ or $\mathrm{P}=\mathrm{P}^0\otimes_t\mathrm{Q}_{t,\,0}$ for some $t\in[s,\,\infty)$;

(iii) if $x\neq 0$, then $\mathscr{C}_{\mathrm{L}}(s,\,x)=\{\mathrm{Q}_{s,\,x}\}$. Thus if $\{\mathrm{P}_{s,\,x}:(s,\,x)\in[0,\,\infty)\times\mathrm{R}^d\}$ is a second time-homogeneous strong Markov selection of extreme elements from $\mathscr{K}_{\mathrm{L}}$, then $\mathrm{P}_{s,\,x}=\mathrm{Q}_{s,\,x}$ for $x\neq 0$ and either $\mathrm{P}_{s,\,0}=\mathrm{P}^0$ or $\mathrm{P}_{s,\,0}=\mathrm{P}^0\otimes_{t_0+s}\mathrm{Q}_{t_0+s,\,0}$ for some $t_0\in(0,\,\infty)$. But if $\mathrm{P}_{s,\,0}=\mathrm{P}^0\otimes_{t_0+s}\mathrm{Q}_{t_0+s,\,0}$ for some $t_0\in(0,\,\infty)$, then by the Markov property

$$\begin{aligned}\mathrm{P}_{0,\,0}(\sigma_0=t_0)&=\mathrm{P}_{0,\,0}(\sigma_0=t_0,\,x(t_0)=0)=\mathrm{E}^{\mathrm{P}_{0,\,0}}[\mathrm{P}_{t_0,\,x(t_0)}(\sigma_0=t_0),\,x(t_0)=0]\\&=\mathrm{P}_{t_0,\,0}(\sigma_0=t_0)\ \mathrm{P}_{0,\,0}(x(t_0)=0)=0\end{aligned}$$

since $\mathrm{P}_{t_0,0}(\sigma_0=t_0)=P^0\otimes_{2t_0}\mathrm{Q}_{2t_0,0}(\sigma_0=t_0)=0$. On the other hand,

$$\mathrm{P}_{0,\,0}(\sigma_0=t_0)=\mathrm{P}^0\otimes_{t_0}\mathrm{Q}_{t_0,0}(\sigma_0=t_0)=1.$$

Thus it must be that $\mathrm{P}_{s,\,0}=\mathrm{P}^0$.

We have therefore shown that there exist exactly two time-homogeneous strong Markov selections $\{\mathrm{P}_{s,\,x}:(s,\,x)\in[0,\,\infty)\times\mathrm{R}^d\}$ of extreme elements from $\mathscr{K}_{\mathrm{L}}$. In both selections, $\mathrm{P}_{s,\,x}=\mathrm{Q}_{s,\,x}$ for $x\neq 0$; and in one of the selections $\mathrm{P}_{s,\,0}=\mathrm{Q}_{s,\,0}$, while in the second one $\mathrm{P}_{s,\,0}=\mathrm{P}^0$. In particular, since $\{\mathrm{Q}_{s,\,x}:(s,\,x)\in[0,\,\infty)\times\mathrm{R}^d\}$ is Feller continuous, the second selection cannot be. Perhaps it is worth noting at this point that exercise 12.4.2 in [*S. and V.*] provides an example of a continuous coefficient L for which there are *no* Feller continuous selections from $\mathscr{K}_{\mathrm{L}}$.

Example (4.15). — We present one last example. This example displays no new phenomena but it does show just how complicated the structure of $\mathscr{K}_{\mathrm{L}}$ can be even for rather simple L'*s*.

Let $d=2$ and define

$$a(z)=\begin{pmatrix} 1 & \dfrac{2xy}{x^2+y^2} \\ \dfrac{2xy}{x^2+y^2} & 1 \end{pmatrix}$$

for $z=\begin{pmatrix} x \\ y \end{pmatrix}\in\mathrm{R}^2$. We are going to study $\mathscr{K}_{\mathrm{L}}$ when

$$\mathrm{L}=1/2\left(a^{11}(z)\frac{\partial^2}{\partial x^2}+2a^{12}(z)\frac{\partial^2}{\partial x\,\partial y}+a^{22}(z)\frac{\partial^2}{\partial y^2}\right).$$

Note that since $a(.)$ is discontinuous at 0, it is not altogether obvious that $\mathscr{K}_{\mathrm{L}}$ is a Krylov system. On the other hand, for any $z\in\mathrm{R}^2$, $x(.)$ under $\mathrm{P}(\in\mathscr{C}_{\mathrm{L}}(0, z))$ is a 1-dimensional Brownian motion starting from x. Hence, $\lim_{\varepsilon\downarrow 0}\sup_{z\in\mathrm{R}^2}\sup_{\mathrm{P}\in\mathscr{C}_{\mathrm{L}}(0,z)}\mathrm{E}^{\mathrm{P}}\left[\int_0^t \chi_{(0,\varepsilon)}(|z(u)|)\,du\right]=0$ for all $t>0$ [which-in particular-allows us to have neglected defining $a(0)$!], and from this fact it is easy to see that $\mathscr{K}_{\mathrm{L}}$ is a Krylov system (*cf.* the proof of Lemma 4.6).

In fact, if $\mathrm{P}_n\in\mathscr{C}_{\mathrm{L}}(s_n, z_n)$ and $(s_n, z_n)\to(s, z)$, then every limit of $\{\mathrm{P}_n\}$ is in $\mathscr{C}_{\mathrm{L}}(s, z)$. Set $\mathrm{E}=\{z\in\mathrm{R}^2\colon x=y \text{ or } x=-y\}$ and define

$$\tau_{\mathrm{E}}=\inf\{t\geqq 0\colon z(t)\in\mathrm{E}\}.$$

Since $a(.)$ is continuous and non-degenerate on $\mathrm{R}^2\setminus\mathrm{E}$, it is a simple matter to show that for each $z\in\mathrm{R}^2\setminus\mathrm{E}$, all $\mathrm{P}\in\mathscr{C}_{\mathrm{L}}(0, z)$ agree on $\mathscr{M}_{\tau_{\mathrm{E}}}$.

LEMMA (4.16). — *For all* $z^0\in\mathrm{R}^2\setminus\mathrm{E}$ *and all* $\mathrm{P}\in\mathscr{C}_{\mathrm{L}}(0, z^0)$:

$$\mathrm{P}(\tau_{\mathrm{E}}\leqq t)=\left(\frac{2}{\pi}\right)^{1/2}\int_{|x^0|\vee|y^0|/t^{1/2}}^{\infty} e^{-u^2/2}\,du \qquad \textit{and} \qquad \mathrm{P}(z(\tau_{\mathrm{E}})=0)=1.$$

For all $z^0\in\mathrm{E}$, $\mathrm{P}(z(t)\in\mathrm{E},\ t\geqq 0)=1$ *if* $\mathrm{P}\in\mathscr{C}_{\mathrm{L}}(0, z^0)$.

Proof. — By symmetry, it suffices for us to look at $z^0\in\mathrm{R}^2\setminus\mathrm{E}$ such that $0<|y^0|<x^0$. For $\mathrm{R}>x^0$, define $\mathscr{G}_{\mathrm{R}}=\{z\colon 0<|y|<x<\mathrm{R}\}$ and let ζ_{R} and σ_{R} be the first exit times from $\mathscr{G}_{\mathrm{R}}$ and $\{z\colon 0<x<\mathrm{R}\}$, respectively. Then $\zeta_{\mathrm{R}}\leqq\sigma_{\mathrm{R}}$ for all $\mathrm{R}>0$, $\zeta_{\mathrm{R}}\uparrow\tau_{\mathrm{E}}$ as $\mathrm{R}\uparrow\infty$, and because $x(.)$ is a 1-dimensional Brownian motion under P:

$$\lim_{\mathrm{R}\uparrow\infty}\mathrm{P}(\sigma_{\mathrm{R}}\leqq t)=\left(\frac{2}{\pi}\right)^{1/2}\int_{x^0/t^{1/2}}^{\infty} e^{-u^2/2}\,du, \qquad t>0.$$

We now want to show that $\mathrm{P}(\zeta_{\mathrm{R}}=\sigma_{\mathrm{R}})=1$ for all $\mathrm{R}>x^0$. To this end, let $\varphi(z)=x^2-y^2$ and note that $\mathrm{E}=\{z\colon\varphi(z)=0\}$, $\nabla\varphi\neq 0$ on $\mathrm{E}\setminus\{0\}$, and $\mathrm{L}\varphi=\langle\nabla\varphi, a\nabla\varphi\rangle=0$ on E. Thus, by Lemma (7.2) in [11], $\mathrm{P}(z(\tau_{\mathrm{R}})\in\mathrm{E}\setminus\{0\})=0$, and clearly the equality $\mathrm{P}(\zeta_{\mathrm{R}}=\sigma_{\mathrm{R}})=1$ follows from this. Finally,

$$\mathrm{P}(z(\zeta_{\mathrm{R}})=0)=\mathrm{P}(x(\sigma_{\mathrm{R}})=0)=\frac{\mathrm{R}-x_0}{\mathrm{R}}\to 1 \qquad \text{as} \quad \mathrm{R}\uparrow\infty.$$

Thus the first part of the lemma has been proved.

To prove the second part, let $z^0 \in E$ be given and define φ as above and $\eta(t)=\varphi(z(t))$. Then, for any $P \in \mathscr{C}_L(0, z^0)$, $(\eta(t), \mathscr{M}_t, P)$ is a martingale. Furthermore

$$\left(\eta^2(t)-4\int_0^t (\eta^2(u)/|z(u)|^2)\,du,\ \mathscr{M}_t,\ P\right)$$

is also a martingale. Hence there is a 1-dimensional Brownian motion $\beta(.)$ and a progressively measurable function $\sigma(.)$ such that $\eta(.)$ under P has the same distribution as $\xi(.)=\int_0^{.} \sigma(u)\,d\beta(u)$ and $\sigma^2(.)\leqq 4|\xi(.)|$. But, by the result of Watanabe and Yamada [12], this means that $\xi(.)\equiv 0$ a.s.; and so $P(\eta(t)=0,\ t\geqq 0)=1$.

Q.E.D.

LEMMA (4.17). — *Let* $\tau_0=\inf\{t\geqq 0: z(t)=0\}$. *Then for every* $z\in R^2$ *and* $P\in\mathscr{C}_L(0, z)$, P *is uniquely determined on* $\mathscr{M}_{\tau_0}$. *Furthermore, for all* $z\in R^2$ *and* $P\in\mathscr{C}_L(0, z)$, $\tau_0(.)$ *is continuous at* P-*almost all* ω *and*

$$P(\tau_0\leqq t)=\left(\frac{2}{\pi}\right)^{1/2}\int_{|x|\vee|y|/t^{1/2}}^{\infty} e^{-u^2/2}\,du.$$

Proof. — The uniqueness statement when $z^0\notin E$ is obvious from Lemma (4.16) plus the uniqueness of $P\in\mathscr{C}_L(0, z^0)$ on $\mathscr{M}_{\tau_E}$. When $z^0\in E$, then the uniqueness statement is a consequence of the fact that $P(x(t)=\operatorname{sgn}(x^0y^0)\,y(t),\ 0\leqq t<\tau_0)=1$ plus the fact that $x(.)$ is a 1-dimensional Brownian motion starting at x^0 under all $P\in\mathscr{C}_L(0, z^0)$. Furthermore the distribution of τ_0 under $P\in\mathscr{C}_L(0, z^0)$ is an easy consequence of Lemma (4.16).

To prove the almost sure continuity of $\tau_0(.)$ under any P from $\mathscr{K}_L$, assume that $P\in\mathscr{C}_L(0, z^0)$ where $0\leqq|y^0|\leqq x^0$. Then, by Lemma (4.16), $\tau_0=\sigma_0$ (a.s., P) where $\sigma_0=\inf\{t\geqq 0: x(t)\leqq 0\}$.

But $\sigma_0(.)$ is P-almost surely continuous because $x(.)$ under P is a 1-dimensional Brownian motion and therefore $\sigma_0=\inf\{t\geqq 0: x(t)<0\}$ (a.s., P).

LEMMA (4.18). — *Let* $\{P_{s,z}: (s, z)\in[0, \infty)\times R^2\}$ *be a time-homogeneous strong Markov selection from* $\mathscr{K}_L$. *Then* $\{P_{s,z}: (s, z)\in[0, \infty)\times R^2\}$ *is Feller continuous. Furthermore, two such selections* $\{P_{s,z}: (s, z)\in[0, \infty)\times R^2\}$ *and* $\{Q_{s,z}: (s, z)\in[0, \infty)\times R^2\}$ *are equal if and only if* $P_{0,0}=Q_{0,0}$.

Proof. — Clearly for any $f\in C_b(R^d)$ and $t>0$:

$$E^{P_{0,z}}[f(z(t))]=E^{P_{0,z}}[f(z(t)),\ \tau_0>t]+E^{P_{0,z}}[E^{P_{\tau_0(.),0}}[f(z(t-\tau_0(.))],\ \tau_0(.)\leqq t].$$

Since $P_{0,z}$ is uniquely determined on $\mathscr{M}_{\tau_0}$ and $P_{s,0}$, $s>0$, is uniquely determined by $P_{0,0}$, the last assertion is obvious. Furthermore, from the expression for the distribution of $\tau_0(.)$ given in Lemma (4.17), it is clear that the second term on the right is continuous as a function of z. To prove that the first term on the right is continuous with respect to z, note that:

$$E^{P_{0,z}}[f(z(t)),\ \tau_0>t]=E^{P_{0,z}}[\bar{f}(z(t\wedge\tau_0))]+f(0)\,P_{0,z}(\tau_0>t)$$

where $\bar{f}(.)=f(.)-f(0)$. Now suppose that $z_n \to z$. Then by the discussion preceding Lemma (4.16), every limit of $\{P_{0,z_n}\}_{n=1}^{\infty}$ is an element of $\mathscr{C}_L(0, z)$. Hence, because all elements of $\mathscr{C}_L(0, z)$ coïncide on $\mathscr{M}_{\tau_0}$, $E^{P_{0,z_n}}[\Phi] \to E^{P_{0,z}}[\Phi]$ for all bounded $\mathscr{M}_{\tau_0}$-measurable $\Phi: \Omega \to R^1$ which are $P_{0,z}$-almost surely continuous. But by Lemma (4.17), $\bar{f}(z(t \wedge \tau_0))$ is such a Φ, and so the proof is complete, as $P_{0,.}(\tau_0 > t)$ is continuous [again by Lemma (4.17)].

Q.E.D.

LEMMA (4.19). — *Let $\gamma_\varepsilon = \inf\{t \geq 0: |x(t)| \geq \varepsilon\}$. Then for all $P \in \mathscr{C}_L(0, 0)$ and $\varepsilon > 0$, $E^P[\gamma_\varepsilon] = \varepsilon^2$. Furthermore, for each $0 \leq \alpha \leq 1$, there is a unique time-homogeneous strong Markov selection $\{P^\alpha_{s,z}: (s, z) \in [0, \infty) \times R^2\}$ such that $P^\alpha_{0,0}(x(\gamma_1) = y(\gamma_1)) = \alpha$.*

Proof. — The first assertion is trivial, since $x(.)$ is a 1-dimensional Brownian motion starting at 0 under any $P \in \mathscr{C}_L(0, 0)$.

To prove the existence of $\{P^\alpha_{s,z} : (s, z) \in [0, \infty) \times R^2\}$, it suffices to construct $\{P^\alpha_{s,z} : (s, z) \in [0, \infty) \times E\}$, since we can then define $P^\alpha_{s,z}$ for $z \notin E$ by: $P^\alpha_{s,z} = P \otimes_{\tau_0(.)} P^\alpha_{\tau_0(.),0}$, where P is any element of $\mathscr{C}_L(s, z)$.

To construct $\{P^\alpha_{s,z}: (s, z) \in [0, \infty) \times E\}$, let $0 \leq \alpha \leq 1$ be given and set $\bar{\alpha} = 1 - \alpha$. For $x, \xi \geq 0$, define

$$
\begin{aligned}
p_\alpha(t, (\pm x, \pm x), (\pm\xi, \pm\xi)) &= g(t, x-\xi) - \bar{\alpha} g(t, x+\xi),\\
p_\alpha(t, (\pm x, \pm x), (\mp\xi, \mp\xi)) &= \alpha g(t, x+\xi),\\
p_\alpha(t, (\pm x, \pm x), (\pm\xi, \mp\xi)) &= \bar{\alpha} g(t, x+\xi),\\
p_\alpha(t, (\mp x, \pm x), (\mp\xi, \pm\xi)) &= g(t, x-\xi) - \alpha g(t, x+\xi),\\
p_\alpha(t, (\mp x, \pm x), (\pm\xi, \mp\xi)) &= \bar{\alpha} g(t, x+\xi),\\
p_\alpha(t, (\mp x, \pm x), (\pm\xi, \pm\xi)) &= \alpha g(t, x+\xi);
\end{aligned}
$$

where $g(t, \eta) = (1/(2\pi t)^{1/2})\, e^{-\eta^2/2t}$. Next, for $z \in E$ and $\Gamma \in \mathscr{B}_E$, define

$$
\begin{aligned}
P_\alpha(t, z, \Gamma) = &\int_{\{\xi \geq 0:\, (\xi,\xi) \in \Gamma\}} p_\alpha(t, z, (\xi, \xi))\, d\xi + \int_{\{\xi \geq 0:\, (-\xi,\xi) \in \Gamma\}} p_\alpha(t, z, (-\xi, \xi))\, d\xi\\
&+ \int_{\{\xi \geq 0:\, (\xi,-\xi) \in \Gamma\}} p_\alpha(t, z, (\xi, -\xi))\, d\xi + \int_{\{\xi \geq 0:\, (-\xi,-\xi) \in \Gamma\}} p_\alpha(t, z, (-\xi, -\xi))\, d\xi.
\end{aligned}
$$

It is then easy to check that there is a Feller continuous time-homogeneous Markov family $\{P^\alpha_{s,z}: (s, z) \in [0, \infty) \times E\}$ of probability measures on $C([0, \infty), E)$ having $P_\alpha(t, z, \Gamma)$ as its transition probability function. Furthermore $P^\alpha_{s,z} \in \mathscr{C}_L(s, z)$ for each $(s, z) \in [0, \infty) \times E$. Finally, if $\alpha \in \{0, 1\}$, then it is clear that $P^\alpha_{0,0}(x(\gamma_\varepsilon) = y(\gamma_\varepsilon)) = \alpha$, $\varepsilon > 0$. If $\alpha \in (0, 1)$, set $f(\xi, \xi) = \bar{\alpha}|\xi|$ and $f(\xi, -\xi) = -\alpha|\xi|$ for $\xi \in R^1$. Then $\int f(\zeta)\, P_\alpha(t, z, d\zeta) = f(z)$ for all $(t, z) \in [0, \infty) \times E$, and so $(f(z(t)), \mathscr{M}_t, P^\alpha_{0,0})$ is a martingale. In particular

$$\bar{\alpha}\varepsilon\, P^\alpha_{0,0}(x(\gamma_\varepsilon) = y(\gamma_\varepsilon)) - \alpha\varepsilon\, P^\alpha_{0,0}(x(\gamma_\varepsilon)) = -y(\gamma_\varepsilon)) = 0$$

and so $P^\alpha_{0,0}(x(\gamma_\varepsilon) = y(\gamma_\varepsilon)) = \alpha$.

To prove uniqueness, it is enough for us to show that if $\{P_{s,z}: (s, z)\in[0, \infty)\times E\}$ is any Feller continuous, time-homogeneous strong Markov family such that $P_{s,z}\in\mathscr{C}_L(s, z)$ for all $(s, z)\in[0, \infty)\times E$ and $P_{0,0}(x(\gamma_1)=y(\gamma_1))=\alpha$, then $A_s=A_s^\alpha$ on $D_{A_s^\alpha}\cap\hat{C}(E)$, where A_s and A_s^α are, respectively, the strong generators of $\{P_{s,z}: (s, z)\in[0, \infty)\times E\}$ and $\{P_{s,z}^\alpha: (s, z)\in[0, \infty)\times E\}$. The first step is to show that if $\alpha_\varepsilon=P_{0,0}(x(\gamma_\varepsilon)=y(\gamma_\varepsilon))$, then $\alpha_\varepsilon=\alpha$ for all $\varepsilon>0$. But a simple argument shows that:

$$\alpha_{\varepsilon_1+\varepsilon_2}=\alpha_{\varepsilon_1}\left(\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}+\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}\alpha_{\varepsilon_1}\right)+(1-\alpha_{\varepsilon_1})\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}\alpha_{\varepsilon_1}=\alpha_{\varepsilon_1}.$$

We next show that

$$P_{0,0}(x(\gamma_\varepsilon)=y(\gamma_\varepsilon)=\varepsilon)=P_{0,0}(x(\gamma_\varepsilon)=y(\gamma_\varepsilon)=-\varepsilon)=\alpha/2$$

while

$$P_{0,0}(x(\gamma_\varepsilon)=-y(\gamma_\varepsilon)=\varepsilon)=P_{0,0}(x(\gamma_\varepsilon)=-y(\gamma_\varepsilon)=-\varepsilon)=(1-\alpha)/2.$$

To this end, note that $(x(t)+y(t), \mathscr{M}_t, P_{0,0})$ is a martingale and so

$$0=E^{P_{0,0}}[x(\gamma_\varepsilon)+y(\gamma_\varepsilon)]=2\varepsilon P_{0,0}(x(\gamma_\varepsilon)=y(\gamma_\varepsilon)=\varepsilon)-2\varepsilon P_{0,0}(x(\gamma_\varepsilon)=y(\gamma_\varepsilon)=-\varepsilon)$$

and clearly this shows that

$$P_{0,0}(x(\gamma_\varepsilon)=y(\gamma_\varepsilon)=\varepsilon)=P_{0,0}(x(\gamma_\varepsilon)=y(\gamma_\varepsilon)=-\varepsilon)=\alpha/2.$$

The proof of the other equality is similar. Using Dynkin's theory, one can now easily show that $D_{A_s}\cap\hat{C}(E)$ consists of $f\in\hat{C}(E)$ such that the functions $f_\pm(x)=f(x, \pm x)$ are in $C_b^2(R^1)$, $\lim_{|x|\to\infty} f_\pm''(x)=0$, and

$$\lim_{\substack{x\to 0\\ x\neq 0}}\frac{1}{2}f_\pm''(x)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^2}\left(\alpha\frac{f_+(\varepsilon)+f_+(-\varepsilon)}{2}+(1-\alpha)\frac{f_-(\varepsilon)+f_-(-\varepsilon)}{2}-f(0)\right);$$

and that for $f\in D_{A_s}\cap\hat{C}(E)$:

$$A_s f(z)=\begin{cases}1/2 f_\pm''(x) & \text{if } z=(x, \pm x),\\ \lim_{\varepsilon\downarrow 0}\dfrac{1}{\varepsilon^2}\left(\alpha\dfrac{f_+(\varepsilon)+f_+(-\varepsilon)}{2}+(1-\alpha)\dfrac{f_-(\varepsilon)+f_-(-\varepsilon)}{2}-f(0)\right) & \text{if } z=0\end{cases}$$

Q.E.D.

LEMMA (4.20). — *For each $0\leqq\alpha\leqq 1$ the family $\{P_{s,z}^\alpha: (s, z)\in[0, \infty)\times R^2\}$ consists of extreme elements of $\mathscr{K}_L$.*

Proof. — Because $P^{\alpha}_{s,z}=P^{\alpha}_{s,z}\otimes_{\tau_0(.)}P^{\alpha}_{\tau_0(.),0}$ and all elements of $\mathscr{C}_L(s,z)$ coïncide on $\mathscr{M}_{\tau_0}$, it suffices for us to show that $P^{\alpha}_{0,0}\in \text{ext}(\mathscr{C}_L(0,0))$. Furthermore, since $P^{\alpha}_{0,z}(z(t)\in E,\ t\geqq 0)=1,\ z\in E$, we will know that $P^{\alpha}_{0,0}\in\text{ext}(\mathscr{C}_L(0,0))$ once we have shown that for all $f\in D_{A^{\alpha}_s}\cap\hat{C}(E)$ and $t\geqq 0$:

$$f(z(t))-f(0)-\int_0^t A^{\alpha}_s f(z(u))\,du$$

is in $L^1(P^{\alpha}_{0,0})$ closure of

$$\left\{\chi_A(\varphi(z(t_2))-\varphi(z(t_1))-\int_{t_1}^{t_2}L\varphi(z(u))\,du\colon 0\leqq t_1<t_2\leqq t,\ \varphi\in C_0^{\infty}(R^2)\text{ and }A\in\mathscr{M}_{t_1}\right\}.$$

We will do this by proving that

$$f(z(t))=f(0)+\int_0^t\theta_f(u)\,dx(u)+\int_0^t A_s f(z(u))\,du\quad(\text{a.s., }P^{\alpha}_{0,0})\tag{4.21}$$

where

$$\theta_f(t)=\begin{cases}f'_+(z(t)) & \text{if } x(t)=y(t),\\ f'_-(z(t)) & \text{if } x(t)=-y(t)\end{cases}$$

and $f_{\pm}$ are as in the proof of Lemma (4.19). [Recall that $x(.)$ under $P^{\alpha}_{0,0}$ is a Brownian motion with respect to $\{\mathscr{M}_t\colon t\geqq 0\}$ and so the stochastic integral in (4.21) is well-defined].

The proof of (4.21) is an easy extension of Itô's formula. In fact, since $f_{\pm}\in C_b^2(R^1\setminus\{0\})$, it is clear that for all $n\geqq 1$ and $\varepsilon>0$:

$$\begin{aligned}f(z(\tau^{\varepsilon}_n\wedge t))&-f(z(\sigma^{\varepsilon}_n\wedge t))\\&=\int_{\sigma^{\varepsilon}_n\wedge t}^{\tau^{\varepsilon}_n\wedge t}f'_{\text{sgn}(x(\sigma^{\varepsilon}_n).y(\sigma^{\varepsilon}_n))}(x(u))\,dx(u)+\int_{\sigma^{\varepsilon}_n\wedge t}^{\tau^{\varepsilon}_n\wedge t}1/2 f''_{\text{sgn}(x(\sigma^{\varepsilon}_n).y(\sigma^{\varepsilon}_n))}(x(u))\,du\end{aligned}$$

where $\tau^{\varepsilon}_0=0$ and

$$\begin{aligned}\sigma^{\varepsilon}_n&=\inf\{t\geqq\tau^{\varepsilon}_{n-1}\colon |x(t)|=\varepsilon\},\quad n\geqq 1,\\ \tau^{\varepsilon}_n&=\inf\{t\geqq\sigma^{\varepsilon}_n\colon x(t)=0\},\quad n\geqq 1.\end{aligned}$$

Let $N_{\varepsilon}(t)=\max\{n\geqq 1\colon \sigma^{\varepsilon}_n\leqq t\}$. Then since $\bigcup_1^{N_{\varepsilon}(t)}[\tau^{\varepsilon}_{n-1},\sigma^{\varepsilon}_n]\subseteqq\{u\leqq t\colon |x(u))|\leqq\varepsilon\}$ and $E^{P^{\alpha}_{0,0}}\left[\int_0^t\chi_{[0,\varepsilon]}(|x(u)|)\,du\right]\to 0$ as $\varepsilon\downarrow 0$, we have

$$\sum_1^{N_{\varepsilon}(t)}\int_{\sigma^{\varepsilon}_n}^{\tau^{\varepsilon}_n\wedge t}f'_{\text{sgn}(x(\sigma^{\varepsilon}_n).y(\sigma^{\varepsilon}_n))}(x(u))\,dx(u)\to\int_0^t\theta_f(u)\,dx(u)$$

and

$$\sum_{1}^{N_\varepsilon(t)} \int_{\sigma_n^\varepsilon}^{\tau_n^\varepsilon \wedge t} 1/2\, f''_{\operatorname{sgn}(x(\sigma_n^\varepsilon)).\, y(\tau_n^\varepsilon))}(x(u))\,du \to \int_0^t A_s f(x(u))\,du$$

in $L^2(P^\alpha_{0,0})$. At the same time

$$f(z(t))-f(0)=\sum_{1}^{N_\varepsilon(t)}(f(z(\tau_n^\varepsilon\wedge t))-f(z(\sigma_n^\varepsilon)))+\sum_{1}^{N_\varepsilon(t)+1}(f(z(\sigma_n^\varepsilon\wedge t))-f(z(\tau_{n-1}^\varepsilon\wedge t)).$$

Thus, we only have to show that the second sum on the right tends to zero in $L^2(P^\alpha_{0,0})$ as $\varepsilon\downarrow 0$. Since each summand in this sum is dominated in absolute value by ε, we are left with proving that

$$E^{P^\alpha_{0,0}}\left[\left(\sum_{1}^{N_\varepsilon(t)}(f(z(\sigma_n^\varepsilon))-f(z(\tau_{n-1}^\varepsilon)))\right)^2\right]\to 0.$$

But

$$E^{P^\alpha_{0,0}}\left[\left(\sum_{1}^{N_\varepsilon(t)}(f(z(\sigma_n^\varepsilon))-f(z(\tau_{n-1}^\varepsilon)))\right)^2\right]=\varepsilon^2 E^{P^\alpha_{0,0}}[N_\varepsilon(t)]$$

$$+2\sum_{1\leqq m<n} E^{P^\alpha_{0,0}}[(f(z(\sigma_n^\varepsilon))-f(z(\tau_{n-1}^\varepsilon)))(f(z(\sigma_m^\varepsilon))-f(z(\tau_{m-1}^\varepsilon))),\ \sigma_n^\varepsilon\leqq t]$$

and the second sum on the right can be written

$$\sum_{1\leqq m<n} E^{P^\alpha_{0,0}}\left[\int_{\tau_{n-1}^\varepsilon}^{\sigma_n^\varepsilon} A_s^\alpha f(z(u))\,du\,(f(z(\sigma_m^\varepsilon))-f(z(\tau_{m-1}^\varepsilon))),\ \tau_{n-1}^\varepsilon\leqq t\right]$$

$$-\sum_{1\leqq m<n} E^{P^\alpha_{0,0}}[(f(z(\sigma_n^\varepsilon))-f(z(\tau_{n-1}^\varepsilon)))(f(z(\sigma_m^\varepsilon))-f(z(\tau_{m-1}^\varepsilon))),\ \tau_{n-1}^\varepsilon\leqq t<\sigma_n^\varepsilon]$$

$$\leqq C(\varepsilon^3 E^{P^\alpha_{0,0}}[(N_\varepsilon(t))^2]+\varepsilon^2 E^{P^\alpha_{0,0}}[N_\varepsilon(t)])$$

since $E^{P^\alpha_{0,0}}[\sigma_n^\varepsilon-\tau_{n-1}^\varepsilon \mid \mathscr{M}_{\tau_{n-1}^\varepsilon}]=\varepsilon^2$.

Thus all that we need to know is that $\overline{\lim}_{\varepsilon\downarrow 0} E^{P^\alpha_{0,0}}[(\varepsilon N_\varepsilon(t))^2]<\infty$. But $\varepsilon N_\varepsilon(t)\xrightarrow{L^2(P^\alpha_{0,0})} l_0(t)$, the local time of $x(.)$ at 0 up to time t; and so we are done.

Q.E.D.

To summarize, we have now shown that every time-homogeneous strong Markov selection from $\mathscr{K}_L$ is one of the processes $\{P^\alpha_{s,z}: (s, z)\in[0, \infty)\times R^2\}$ and that for each $0\leqq\alpha\leqq 1$ the process $\{P^\alpha_{s,z}: (s, z)\in[0, \infty)\times R^2\}$ is Feller continuous and consists of extreme elements from $\mathscr{K}_L$.

Section (5)

In this section, which begins the second part of our paper, we first study the transformation of continuous martingales under changes of time, and derive from there the main properties of a particular class of continuous martingales, called pure martingales.

$(\Omega, \mathcal{M}, P)$ is the basic probability space, endowed with a right-continuous, $(\mathcal{M}, P)$ complete, filtration $(\mathcal{M}_t)_{t \geqq 0}$ [which may be changed, in the sequel, into $(\mathcal{F}_t)$, or $(\mathcal{G}_t) \ldots$]. $\mathcal{O}(\mathcal{M}.)$, resp.: $\mathcal{P}(\mathcal{M}.)$ denotes the optional, resp.: predictable σ-field on $\Omega \times \mathbb{R}_+$, associated with $(\mathcal{M}_t)$.

In agreement with N. Kazamaki [25], we define a $(\mathcal{M}_t)$-change of time as a family $T = (\tau_t)_{t \geqq 0}$ of *finite valued*, $(\mathcal{M}_t)$ stopping times such that, for almost all ω, the trajectory $\tau_.(\omega)$ is increasing and right-continuous.

Given a $(\mathcal{M}_t)$ change of time $T = (\tau_t)$, and a $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable process $(X_t)_{t \geqq 0}$, we denote by $T(\mathcal{M}.)$, resp.: $T(X)$, the filtration $(\mathcal{M}_{\tau_t})$, resp.: the process (X_{τ_t}).

If $X = (X_t)_{t \geqq 0}$ is a (real-valued) process defined on $(\Omega, \mathcal{M}, P)$, $\mathcal{M}(X) = (\mathcal{M}(X)_t, t \geqq 0)$ is the smallest right-continuous filtration, composed of $(\mathcal{M}, P)$ complete σ-fields, with respect to (in short: w.r.t.) which the process X is adapted. A $\mathcal{M}(X)$-change of time is also called a X-adapted change of time. A $(\mathcal{M}_t)$ change of time $T = (\tau_t)$ is said to be X-continuous if, outside an evanescent set, X is constant on each interval $[\tau_{t-}, \tau_t]$, and on $[0, \tau_0]$.

In the following, X is always a continuous $(\mathcal{M}_t)$ local martingale with $X_0 = 0$. The interest of X-continuous changes of time appears in the next.

PROPOSITION (5.1). — *Let* $T = (\tau_t)$ *be a* $(\mathcal{M}_t)$ *change of time. The following assertions are equivalent:*

(i) T *is* X-*continuous;*

(ii) $T(X)$ *is a* $T(\mathcal{M}_t)$ *local continuous martingale, with increasing process* $T(\langle X \rangle)$.

Moreover, if Y *is another continuous* $(\mathcal{M}_t)$ *local martingale, such that* T *is* X- *and* Y-*continuous, the only* $T(\mathcal{M}_.)$-*adapted, continuous process with bounded variation, associated to the product* $T(X)T(Y)$ *is given by*

$$\langle T(X), T(Y) \rangle = T(\langle X, Y \rangle). \tag{5.2}$$

The main ingredient in the proof of Proposition (5.1) is the following

LEMMA (5.3) (Getoor-Sharpe [22]). — X *and* $\langle X \rangle$ *have the same intervals of constancy, almost surely.*

Proof of proposition (5.1). — (i) $\Rightarrow$ (ii). — Kazamaki proved in Proposition 1 of [25] that if T is X-continuous, then $T(X)$ is a local martingale w.r.t. $T(\mathcal{M}_.)$.

From lemma (5.3), T is also $\langle X \rangle$-continuous; therefore, T is Y-continuous, where $Y = X^2 - \langle X \rangle$, so that, using again Kazamaki's result, $T(Y) = (T(X))^2 - T(\langle X \rangle)$ is a $T(\mathcal{M}_.)$ local martingale. As $T(\langle X \rangle)$ is a continuous, $T(\mathcal{M}_.)$ adapted, increasing process, this proves that

$$\langle T(X) \rangle = T(\langle X \rangle).$$

(ii) $\Rightarrow$ (i). — In particular, $T(\langle X \rangle)$ is continuous, i.e.: T is $\langle X \rangle$-continuous; therefore, from lemma (5.3), T is X-continuous.

To show (5.2), we need only remark that if T is X- and Y-continuous, it is $(X+\lambda Y)$-continuous for any $\lambda \in \mathbb{R}$. Thus, from our previous results, we get: $\langle T(X+\lambda Y) \rangle = T(\langle X+\lambda Y \rangle)$. Developing w.r.t. λ, we obtain (5.2).

We investigate now the effects of X-continuous changes of time on stochastic integrals w.r.t. X.

PROPOSITION (5.4). — *Let X be a continuous $(\mathscr{M}_t)$ local martingale, with $X_0=0$, and $T=(\tau_t)$ a X-continuous change of time.*

If C is a $(\mathscr{M}_t)$ optimal process such that: $\forall\, t, \int_0^t C_s^2\, d\langle X \rangle_s < \infty$ a.e., we denote by C.X the stochastic integral $\int_0^{\cdot} C_s\, dX_s$.

Then, the process T(C) is $T(\mathscr{M}_.)$ optional, (TC).(TX) *is well defined, and*

$$T(C.X) = (TC).(TX). \tag{5.5}$$

Proof. — 1) If C is right-continuous, and $(\mathscr{M}_t)$ adapted, T(C) is right-continuous, $T(\mathscr{M}_.)$ adapted, and consequently $T(\mathscr{M}_.)$-optional. Thus, by the monotone class theorem, if C is $(\mathscr{M}_t)$ optional, T(C) is $T(\mathscr{M}_.)$ optional;

2) Suppose $(\gamma_t)_{t \geqq 0}$ is a continuous increasing process, not necessarily $(\mathscr{M}_t)$ adapted, but such that T is γ-continuous. Then, for any positive Borel function u: $[0, \infty[\, \to \mathbb{R}_+$, one has

$$\forall\, t, \quad \int_0^{\tau_t} u_s\, d\gamma_s = \int_0^t u_{\tau_s}\, d(\gamma_{\tau_s}),$$

which, in short, may be written as

$$T(u.\gamma) = (Tu).(T\gamma); \tag{5.5'}$$

3) As a consequence of 2), and of Proposition (1.1) (ii), the finiteness of $\int_0^t C_s^2\, d\langle X \rangle_s$, for every t, implies that of $\int_0^t (TC)_s^2\, d\langle TX \rangle_s$, for every t, and so (TC).(TX) is well defined;

4) To prove (5.5), we only have to show that:

$$I \overset{\text{def}}{=} \langle T(C.X) - (TC).(TX) \rangle = 0$$

[remark that, from lemma (5.3), T is $\langle C.X \rangle$–, and thus (C.X)-continuous; consequently, Proposition (5.1) may be applied to T(C.X)]. Developing I, one gets

$$I = \langle T(C.X) \rangle - 2(TC).\langle TX, T(C.X) \rangle + (TC)^2.(T(\langle X \rangle)).$$

From formula (5.2), we deduce

$$I=T(\langle C.X\rangle)-2(TC).T(\langle X, C.X\rangle)+(TC)^2.T(\langle X\rangle)$$
$$=2(TC)^2.T(\langle X\rangle)-2(TC)^2.T(\langle X\rangle)=0. \quad \square$$

The above preliminaries on changes of time will play an important part in the sequel. But, even now, they are helpful to sketch a proof of the Dambis-Dubins-Schwarz (D.D.S., from now on) result already alluded to in the introduction and to draw several conclusions from it.

So, let X be a $(\mathcal{M}_t)$ continuous local martingale such that $X_0=0$, and $\langle X\rangle_\infty=\infty$ a.e. Let $\tau_t=\inf\{s/\langle X\rangle_s>t\}$. $T=(\tau_t)$ is a $(\mathcal{M}_t)$ change of time, which is X-adapted [as the process $\langle X\rangle$ is adapted to $\mathcal{M}$ (X)], and even X-continuous, as $T(\langle X\rangle)_t=t$ [then, use lemma (5.3)]. As a consequence of Proposition (5.1), T(X) is a $T(\mathcal{M}_.)$ local continuous martingale with increasing process t, i.e.: a $T(\mathcal{M}_.)$-Brownian motion, from Paul Lévy's theorem. We shall call $\beta \overset{\text{def}}{=} T(X)$ the D.D.S Brownian motion attached to X, and we note $\beta=\beta(X)$. Moreover, as T is X-continuous, we also have

$$(5.6) \qquad X_t=\beta_{\langle X\rangle_t}.$$

The effects of changes of time on the D.D.S Brownian motion are studied in the next

LEMMA (5.7). — *Let* X *be a continuous* $(\mathcal{M}_t)$ *local martingale, with* $X_0=0$, *and* $\langle X\rangle_\infty=\infty$ *a.e. Let* $R=(\rho_t)_{t\geq 0}$ *be a* $(\mathcal{M}_t)$ *change of time, which is* X-*continuous, and such that* $\rho_\infty=\infty$ *a.e.*

Then, one has

$$(5.8) \qquad \beta(R(X))=\beta(X).$$

Proof. — Note that, from Proposition (5.1), $Y=R(X)$ is a $R(\mathcal{M}_.)$ continuous local martingale, with $\langle Y\rangle=R(\langle X\rangle)$, and so, from formula (5.6):

$$Y_t=\beta_{R(\langle X\rangle)_t}=\beta_{\langle Y\rangle_t},$$

where $\beta=\beta(X)$. From this equality, we finally deduce: $\beta(Y)=\beta$. $\square$

In the D.D.S result, we may regard $(\langle X\rangle_t)$ as a $(\mathcal{M}_{\tau_t})$ change of time, since: $\forall t: \langle X\rangle_t=\inf\{s/\tau_s>t\}$. Thus, in lemma (1.4), we have composed changes of time. We shall need the following general result concerning this situation.

LEMMA (5.9). — *Let* $T=(\tau_t)$ *be a* $(\mathcal{M}_t)$ *change of time, and* $S=(\sigma_t)$ *be a* $(\mathcal{M}_{\tau_t})$ *change of time.*

Then, $ST \overset{\text{def}}{=} (\tau_{\sigma_t})$ *is a* $(\mathcal{M}_t)$ *change of time, and*

$$(5.10) \qquad \forall t, \quad (\mathcal{M}_{\tau_.})_{\sigma_t}=\mathcal{M}_{\tau_{\sigma_t}}.$$

The first part of the lemma has been proved by Kazamaki ([25], lemma 2) and here is a sketch of the proof of (5.10): we recall that if $(\mathcal{G}_t)_{t\geq 0}$ is a "usual" filtration, and u a finite $(\mathcal{G}_t)$

stopping time, any $(\mathscr{G}_u)$ measurable random variable may be expressed as: Z_u, where Z is a $(\mathscr{G}_t)$ optional process. Suppose now that Z is a $(\mathscr{M}_t)$ optional process. Then, $Z_{\tau_{\sigma_t}}=(Z_{\tau.})_{\sigma_t}$. Moreover, $(Z_{\tau_t})_{t\geqq 0}$ is a $(\mathscr{M}_{\tau_t})$ optional process, and this proves:

$$\mathscr{M}_{\tau_{\sigma_t}}\subseteqq(\mathscr{M}_{\tau.})_{\sigma_t}. \tag{5.10'}$$

Conversely, a monotone class argument shows that any $(\mathscr{M}_{\tau_t})$ optional process may be written as: $f(t;\tau_t,\omega)$, where f is a $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{O}(\mathscr{M}.)$ measurable function. This implies the equality

$$(\mathscr{M}_{\tau.})_{\sigma_t}=\mathscr{M}_{\tau_{\sigma_t}}\vee\sigma\{\sigma_t\}. \tag{5.10''}$$

Thus, to prove (5.10), one needs only show that σ_t is $\mathscr{M}_{\tau_{\sigma_t}}$-measurable.

This last result is easily obtained, using the dyadic approximations from above of σ_t, and the right-continuity of (τ_t). □

We are now ready to recall the definition of a pure continuous martingale which is—again—due to Dubins and Schwarz [20].

Let (X_t) be a continuous $(\mathscr{M}_t)$ local martingale, with $X_0=0$, and $A=\langle X\rangle$ its increasing process, such that $A_\infty=\infty$ a.e. Note β the D.D.S Brownian motion attached to X, and $\tau_t=\inf\{s/A_s>t\}$ $(t\geqq 0)$.

Remark (5.11). — It is worth pointing out here that, with the previous notations, the following equality always obtains

$$\forall t,\quad \mathscr{M}(X)_{\tau_t}=\mathscr{M}(\beta;\tau)_t. \tag{5.12}$$

The inclusion $\mathscr{M}(\beta;\tau)_t\subseteqq\mathscr{M}(X)_{\tau_t}$ is obvious; conversely, as $\mathscr{M}(X)_{\tau_t}=\lim_{\varepsilon\downarrow 0}\downarrow\{\mathscr{M}(X)_{(\tau_{t+\varepsilon})-}\}$, one needs only show: $\mathscr{M}(X)_{(\tau_t)-}\subseteqq\mathscr{M}(\beta;\tau)_t$, for every t. The σ-field $\mathscr{M}(X)_{(\tau_t)-}$ is equal to $\mathscr{M}(X)_0\vee\sigma\{X_{s\wedge\tau_t},\ s\geqq 0;\ \tau_t\}$ up to P negligible sets. Moreover, one has

$$X_{(s\wedge\tau_t)}=\beta_t 1_{(\tau_t<s)}+X_s 1_{(s\leqq\tau_t)}.$$

Thus, all reduces to showing that for $s>0$, $X_s 1_{(s\leqq\tau_t)}$ is $\mathscr{M}(\beta;\tau)_t$-measurable, which follows from the equality

$$X_s 1_{(s\leqq\tau_t)}=\lim_{(n\to\infty)}\beta_{(A_{s-(1/n)})}1_{(A_{s-(1/n)}\leqq t)}$$

and the fact that (A_t) is a $\mathscr{M}(\beta;\tau)$-change of time. □

Now, by definition, X is pure iff it satisfies

$$\mathscr{M}(X)_\infty=\mathscr{M}(\beta)_\infty, \tag{5.13}$$

a condition which is easily shown (*see* [32] for example) to be equivalent to

$$\forall t,\quad \mathscr{M}(X)_{\tau_t}=\mathscr{M}(\beta)_t \tag{5.13'}$$

or to

(5.13″) [compare with (5.12)] $A=(A_t)$ is a $\mathcal{M}(\beta)$-change of time.

Also remark that, if $\mathscr{B}\overset{\text{def}}{=}\mathcal{M}(\beta)$, and X is pure, then

(5.14) $$\forall t, \quad \mathcal{M}(X)_t=\mathscr{B}_{A_t}.$$

The importance of pure martingales originates from the following remark, made initially by Dubins and Schwarz [20]: if $\mathscr{C}_{\underline{X}}^{\text{loc}}$ is the (convex) set of probability measures on $(\Omega, \mathcal{M}(X)_\infty)$ under which X is a continuous local martingale (w. r. t. $\{\mathcal{M}(X)_t\}$), and P is an element of $\mathscr{C}_{\underline{X}}^{\text{loc}}$ such that X is pure (under P), then P is extremal in $\mathscr{C}_{\underline{X}}^{\text{loc}}$.

[*Nota bene.* — In the following, we shall rephrase the property "P is extremal in $\mathscr{C}_{\underline{X}}^{\text{loc}}$" by the slightly less correct expression: "X is extremal", without mentioning the ever present probability P...]

We now sketch the proof of the Dubins-Schwarz remark: as $\mathscr{C}_{\underline{X}}^{\text{loc}}$ is convex, it is obviously sufficient to show that if Q belongs to $\mathscr{C}_{\underline{X}}^{\text{loc}}$, and $Q\sim P$, then: $Q=P$. As Q is equivalent to P, the increasing processes of X under Q and P are indistinguishable as are, therefore, the D.D.S Brownian motions attached to X under P and Q. Thus, under Q, $\mathcal{M}(X)_\infty=\mathcal{M}(\beta)_\infty$ obtains. Finally, the Brownian distribution being unique, one has: $P=Q$.

Conversely, as already indicated in the Introduction, a continuous local martingale X, with $X_0=0$, and $\langle X\rangle_\infty=\infty$ a. e. need not be pure to be extremal (*see* Dubins-Schwarz [20] and Yor [32] for counter-examples).

We now discuss some examples of pure martingales:

— it has already been remarked in [32] (p. 193) that if (Z_t) is a continuous local martingale, with $Z_0=0$, $\langle Z\rangle_\infty=\infty$ a. e, and $\langle Z\rangle_t=\int_0^t a(Z_s)\,ds$, with $a:\mathbb{R}\to\mathbb{R}_+\setminus\{0\}$, Borel, locally bounded, then Z is pure. This remark is, in fact, at the heart of the powerful method of time-substitution, used — for instance — in Examples (4.1) and (4.12) of this paper;

— we have also tried to study the purity of stochastic integrals

$$X_t^h=\int_0^t h(s,\omega)\,dB_s,$$

with (B_t) a real-valued Brownian motion, and $h\in\mathscr{P}(\mathcal{M}(B))$ satisfying $\int_0^t h^2(s)\,ds<\infty$ for finite t's, and $\int_0^\infty h^2(s)\,ds=\infty$ a. e. It is already very difficult to decide, in general, whether X^h is extremal or not; nonetheless, the following partial result is easily obtained:

(5.15) if h is strictly positive, except possibly on a Lebesgue negligible set (which may depend on ω), then X^h is extremal.

Indeed, as

$$h=\sqrt{\frac{d\langle X^h\rangle_t}{dt}} \quad \text{and} \quad B_t=\int_0^t 1/h_s\,d(X_s^h),$$

one sees that: $\mathcal{M}(B)=\mathcal{M}(X^h)$. From Ito's theorem, every $\mathcal{M}(X^h)$ martingale may be written as

$$c+\int_0^t u_s\,dB_s=c+\int_0^t \left\{\frac{u_s}{h_s}\right\} d(X_s^h),$$

with $u\in\mathcal{P}(\mathcal{M}(X^h))$ and $c\in\mathbb{R}$.

So, X^h possesses the "predictable representation property", and is, therefore, from sections (1) and (2) of this paper, extremal.

Remark. — If in (5.15), one replaces the strict positivity of h by: $h\geqq 0$, then X^h may not be extremal. For instance, if T is a *non-constant, integrable*, Brownian stopping time, and $h_s^{(T)}\overset{\text{def}}{=}1_{[T,\infty[}(s)$, then

$$Y_t=X_t^{h(T)}=(B_t-B_T)\,1_{(T\leqq t)}$$

is not extremal [indeed, $T=\inf\{t/Y_t\neq 0\}$ is a $\mathcal{M}(Y)$ stopping time, but T cannot be expressed as a constant plus a stochastic integral w.r.t.Y].

Another — certainly more interesting — example is that of $M\overset{\text{def}}{=}\int_0^{\cdot} 1_{(0,\infty)}(B_s)\,dB_s$, which the authors have shown to be non-extremal ([33]). In fact, there are even purely discontinuous $\mathcal{M}(M)$-martingales. □

Once (5.15′) has been obtained, it is natural to study, for $h>0$ (except possibly on a Lebesgue negligible set), the purity of X^h. Apart from the obvious case where h is deterministic (then, X^h is pure, no matter h is positive or not!), we have only been able to settle the case, which we partially discuss now, of one other type of martingale, namely: $\int_0^t B_s^n\,dB_s$ (n odd) is pure (the proof of this will be published in [33]). Remark that this martingale may be written as $\int_0^t |B_s|^n\{\operatorname{sgn}(B_s)\,dB_s\}$, and since $\mathcal{M}(|B|)=\mathcal{M}\left(\left(\int_0^{\cdot}\operatorname{sgn}(B_u)\,dB_u\right)\right.$ (*see*, for example, [32]), $\int_0^t B_s^n\,dB_s$ is of the form X^h, with $h(s,\omega)=|B_s(\omega)|^n$, the "basic" Brownian motion being here $\left(\int_0^t \operatorname{sgn}(B_s)\,dB_s\right)$.

An open question. — Does there exist a $\mathcal{M}(B)$ predictable process $h>0$ (except possibly on a Lebesgue negligible set), with $\int_0^t h^2(s)\,ds$ infinite iff $t=\infty$, such that X^h is not pure ? □

We now come back to the general study of pure martingales.

In the following proposition, we show that pure martingales are left stable under "nice" changes of time.

Proposition (5.16). — *Let* $M=(M_t)_{t\geq 0}$ *be a pure continuous* $(\mathcal{M}_t)$ *local martingale, with* $M_0=0$, *and* $\langle M\rangle_\infty=\infty$ a.e.

Then, if $T=(\tau_t)_{t\geq 0}$ *is a* M*-adapted, and* M*-continuous, change of time, increasing to* $+\infty$ *as* $t\uparrow+\infty$, $N=T(M)$ *is a pure local martingale.*

Proof. – From proposition (5.1), N is a $T(\mathscr{M})$ local continuous martingale, with increasing process $T(\langle M \rangle)$.

From lemma (5.7), the D.D.S Brownian motions attached to M and N are equal. Denote this (common) process by β, and $\mathscr{B} = \mathscr{M}(\beta)$.

So, to prove that N is pure, we need only show, from (5.13″), that $T(\langle M \rangle)$ is a $\mathscr{B}$-change of time. But, as M is pure, $\langle M \rangle$ is a $\mathscr{B}$-change of time; from (5.14), T is a $(\mathscr{B}_{\langle M \rangle})$ change of time, and, from lemma (5.9), $T(\langle M \rangle)$ is a $\mathscr{B}$-change of time.

Remark. – For a converse to Proposition (5.16), the reader is invited to look ahead to theorem (7.3), which gives a new characterization of pure martingales.

The following statement, suggested to us by L. Dubins [21], is also a characterization of pure martingales; it can be considered as an extension of the D.D.S theorem, when one replaces the Wiener measure (i.e.: the Brownian distribution) by pure distributions (i.e.: distributions of pure martingales).

THEOREM (5.17). – *Let* P *be the distribution of a continuous local martingale* X, *such that* $X_0 = 0$, *and* $\langle X \rangle_\infty = \infty$ a.e.

Then, P *is pure iff, for any local continuous martingale* M, *defined on some filtered probability space* $(\Omega', \mathscr{M}', (\mathscr{M}'_t), P')$, *with* $M_0 = 0$, *and* $\langle M \rangle_\infty = \infty$ a.e., *there exists a* M-*adapted, and* M-*continuous change of time* $L = (\lambda_t)_{t \geqq 0}$ *such that the distribution of* L(M) *is* P. [We note $P = \mathscr{L}((M_{\lambda_t})_{t \geqq 0})$].

Proof. – 1) Suppose the condition holds. It holds in particular when M is the real-valued Brownian motion $(B_t)_{t \geqq 0}$, with $B_0 = 0$.

Let $R = (\rho_t)_{t \geqq 0}$ be a B-adapted, and B-continuous (this amounts here to be continuous, from lemma (5.3), and the fact that $\langle B \rangle_t = t$) change of time such that $P = \mathscr{L}((B_{\rho_t})_{t \geqq 0})$. Then, from Proposition (5.1), $Y = B_{\rho_.}$ is a continuous local martingale w.r.t. $\mathscr{M}(Y)$, and $\langle Y \rangle = \rho$. The D.D.S Brownian motion attached to Y is obviously B, and from (5.13″), Y is pure;

2) Conversely, suppose P is a pure distribution. By definition, $P = \mathscr{L}(Y)$, where Y is a continuous local martingale, which may be written as $Y = B_{\rho_.}$, with B a Brownian motion, and $R = (\rho_t)_{t \geq 0}$ a B-adapted, and continuous, change of time.

Now, let M be a continuous local martingale, defined on a filtered probability space $(\Omega', \mathscr{M}', (\mathscr{M}'_t), P')$, with $M_0 = 0$, and $\langle M \rangle_\infty = \infty$, P′ a.s.

Note $\Lambda_t = \inf \{ s / \langle M \rangle_s > t \}$ $(t \geqq 0)$. Then, from the D.D.S theorem, $(\beta_t = M_{\Lambda_t},\ t \geqq 0)$ is a real valued Brownian motion.

By "transport" of the B-adapted, and continuous, change of time $R = (\rho_t)_{t \geqq 0}$ on the probability space where M is defined, there exists a β-adapted change of time $R' = (\rho'_t)_{t \geqq 0}$ such that $P = \mathscr{L}((\beta_{\rho'_t})_{t \geqq 0})$.

Now, we may write: $\forall\, t$, $M_{\Lambda_{\rho'_t}} = \beta_{\rho'_t}$, and finally, we only need to show that $(\Lambda_{\rho'_t})_{t \geqq 0}$ is a M-adapted, and M-continuous change of time: it is M-adapted, as a consequence of lemma (5.9), because (Λ_t) is a $\mathscr{M}(M)$-change of time, and $R' = (\rho'_t)$ is a $\mathscr{M}(\beta)$-, and therefore a $\{ \mathscr{M}(M)_{\Lambda_t} \}$-change of time; it is M-continuous, as (Λ_t) is $\langle M \rangle$-, and thus M-continuous, and ρ' is continuous.

Section (6)

The aim of this section is to obtain a general method for the construction of extremal, but not pure continuous martingales. Some examples are then developed, making use of certain stochastic differential equations having only weak solutions. Our main tool will be the characterization of extremal martingales given in the next theorem; there, X is a continuous local martingale, with $X_0=0$, $\langle X\rangle_\infty=\infty$ a.e., β is the D.D.S Brownian motion attached to X, and $\tau_t=\inf\{s/\langle X\rangle_s>t\}$ $(t\geqq 0)$.

THEOREM (6.1) ([32], th. 2). — *X is extremal iff* $\beta=X_{\tau_.}$ *has the (predictable) representation property w.r.t. the filtration* $\{\mathcal{M}(X)_{\tau_t}\}$, i.e.: every $\{\mathcal{M}(X)_{\tau_t}\}$ *(local) martingale* M *may be written as*

$$M_t=c+\int_0^t \Phi(s)\,d\beta_s,$$

where $c\in\mathbb{R}$, *and* Φ *is a* $\{\mathcal{M}(X)_{\tau_t}\}$ *predictable process such that:*

$$\forall t,\quad \int_0^t \Phi^2(s)\,ds<\infty \quad a.e.$$

For completedness, we sketch the proof of this result: it has been shown in the first part of the paper [*see* sections (1) and (2)] that X is extremal iff it has the predictable representation property w.r.t. $\{\mathcal{M}(X)_t\}$. This is equivalent to the following: for every $Y\in L^2(\mathcal{M}(X)_\infty)$, there exist $c\in\mathbb{R}$, and Φ a predictable process w.r.t. $\mathcal{M}(X)$, such that

$$E\left(\int_0^\infty \Phi^2(s)\,d\langle X\rangle_s\right)<\infty \qquad\text{and}\qquad Y=c+\int_0^\infty \Phi(s)\,dX_s.$$

But, if this property is true, we have, from Proposition (5.4), as $\tau_\infty=\infty$,

$$\int_0^\infty \Phi(s,\omega)\,dX_s=\int_0^\infty \Phi(\tau_s\omega,\omega)\,d\beta_s=\int_0^\infty \psi(s,\omega)\,d\beta_s,$$

where ψ is the $L^2(\pi, ds, dP)$-projection of $\Phi(\tau_.,.)$, and $\pi=\mathcal{P}(\mathcal{M}(X)_{\tau_.})$.

The converse is also easily obtained from Proposition (5.4), once the following remarks are made:

(i) $(\langle X\rangle_t)$ is a continuous $\{\mathcal{M}(X)_{\tau_t}\}$ change of time

and

(ii) if ψ is a π-measurable process, then: $(t,\omega)\to\psi(\langle X\rangle_t(\omega),\omega)$ is predictable w.r.t. $\mathcal{M}(X)$.

Remark. — Here is a different proof of theorem (6.1). — "X is extremal" means, by definition, that P is an extremal point of the convex set of all probabilities Q on $(\Omega,\mathcal{M}(X)_\infty)$ for which X is a local martingale. Then, with the help of Proposition (5.1), this is equivalent to the extremality of P among all probabilities $Q(\ll P$ on $\mathcal{M}(X)_\infty)$ such that β is a $(\mathcal{M}(X)_{\tau_t})$ martingale.

Finally, from part I of this paper, this is again equivalent to the predictable representation property of β w.r.t. $(\mathcal{M}(X)_{\tau_t})$ under P. □

We now develop the promised "constructive" method. Let $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ be a given "usual" filtered probability space such that:

(A1) $L^1(\Omega, \mathcal{G}_\infty, P)$ *is separable;*

(A2) *there exists a* $(\mathcal{G}_t)$ *Brownian motion* $(\beta_t)_{t \geqq 0}$ (with $\beta_0 = 0$).

Dellacherie and Stricker ([36]) have remarked that the assumption (A1) is equivalent to the existence of an increasing process (which is far from unique!) $(\tau_t)_{t \geqq 0}$ such that $\mathcal{G}_t = \mathcal{M}(\tau_.)_t$, for every t. Of course, (τ_t) can be chosen to be strictly increasing, continuous, and $\tau_t \geqq t$.

We will always assume that these properties are satisfied.

Now, define $A_t = \inf\{s/\tau_s > t\}$ $(t \geqq 0)$. From the inequality: $A_t \leqq t$, and Doob's optional sampling theorem, we deduce that $X_t = \beta_{A_t}$ is a $(\mathcal{G}_{A_t})$ continuous martingale, with increasing process $\langle X \rangle = A$, and associated Brownian motion β.

The following theorem is devoted to the discussion of the extremality and/or the purity of this martingale X in terms of β. Although quite easy to obtain, these results will be very helpful in the sequel of the paper.

THEOREM (6.2). — X, *defined through the previous construction, satisfies one, and only one, of the following properties:*

(i) X *is pure iff:* $\mathcal{M}(\beta)_\infty = \mathcal{G}_\infty$ (1);

(ii) X *is extremal, but not pure iff:* β *has the predictable representation property w.r.t.* $(\mathcal{G}_t)$ *but* $\mathcal{M}(\beta)_\infty \subsetneqq \mathcal{G}_\infty$;

(iii) X *is not extremal iff:* β *does not have the predictable representation property w.r.t.* $(\mathcal{G}_t)$.

Theorem (6.2) is a simple consequence of theorem (6.1), once one remarks that

$$\forall t, \quad \mathcal{G}_t = \mathcal{M}(X)_{\tau_t}. \tag{6.3}$$

Indeed, the obvious inclusion $\mathcal{M}(X)_{\tau_t} \supseteq \mathcal{M}(\tau_.)_t = \mathcal{G}_t$ is verified for all t's.

Conversely, from (5.12), one has: $\forall t$, $\mathcal{M}(X)_{\tau_t} = \mathcal{M}(\beta; \tau)_t \subset \mathcal{M}(\tau_.)_t = \mathcal{G}_t$, as β is adapted to $(\mathcal{G}_t)$. (6.3) follows. □

The assertion (ii) of theorem (6.2) provides an easy tool to construct extremal but not pure, continuous martingales. We examine now how it may be applied in the setting of one-dimensional stochastic differential equations: let Ω_c be the set of all continuous maps: $w: \mathbb{R}_+ \to \mathbb{R}$, and define $\mathcal{C}_t = \sigma\{w \to w(s);\ s \leqq t\}$; suppose $b: (s, w) \to b(s, w)$, and $\sigma: (s, w) \to \sigma(s, w)$ are two $(\mathcal{C}_t)$ predictable, real-valued, uniformly bounded applications, and moreover, that σ never vanishes. On the other hand, suppose that, on a given filtered probability space $(\Omega, \mathcal{M}, (\mathcal{M}_t), P)$, there exist a real-valued $(\mathcal{M}_t)$ Brownian motion $(\beta_t)_{t \geqq 0}$,

(1) It may be worth recalling here that all considered σ-fields are supposed to be $(\mathcal{G}_\infty, P)$ complete, and so, no more completion operations are needed.

with $\beta_0=0$, as well as a $(\mathcal{M}_t)$ adapted solution $(Y_t)_{t\geqq 0}$ to the stochastic differential equation

$$\text{(E)}\qquad \begin{cases} dY_s=b(s,\, Y_.(\omega))\,ds+\sigma(s,\, Y_.(\omega))\,d\beta_s(\omega), \\ Y_0=y\,(\in \mathbb{R}). \end{cases}$$

The hypothesis: $\sigma(s, w)\neq 0$ for all $(s, w)'s$ implies that (β_t) is $\{\mathcal{M}(Y)_t\}$ adapted, as:

$$\beta_t=\int_0^t (1/\sigma(s,\, Y_.(\omega)))\{dY_s-b(s,\, Y_.(\omega))\,ds\}$$

[consequently, (β_t) is a $\{\mathcal{M}(Y)_t\}$ Brownian motion].

Conversely, if Y is adapted (resp.: is *not* adapted) to $\{\mathcal{M}(\beta)_t\}$, (Y_t) is said to be a *strong* (resp.: a *weak*) solution of (E).

In accordance with the construction preceding Theorem (6.2), we define (for simplicity) the process

$$\tau_t=\int_0^t \left\{2+\frac{Y_s}{1+|Y_s|}\right\} ds,$$

which is strictly increasing, continuous; it satisfies: $\tau_t\geqq t$, as well as $\mathcal{M}(Y)_t=\mathcal{M}(\tau_.)_t$, for every t. Let $A_t=\inf\{s/\tau_s>t\}$ $(t\geqq 0)$, and finally note: $X_t=\beta_{A_t}$, which is, once again, a $\{\mathcal{M}(Y)_{A_t}\}$ continuous martingale.

Before discussing the purity and/or extremality of X, we still need to introduce the set π_E of the distributions [on $(\Omega_c, \mathcal{C}_\infty)$] of *all* continuous processes $(Y_t)_{t\geqq 0}$, defined on any probability space, and such that: $Y_0=y$ and $Y_t-\int_0^t b(s,\, Y_.(\omega))\,ds$ is a $\{\mathcal{M}(Y)_t\}$ martingale with increasing process $\int_0^t \sigma^2(s,\, Y_.(\omega))\,ds$.

Now, we have the following

THEOREM (6.4) (We use the previously defined notations). — X *satisfies one, and only one, of the following properties:*

(i) X *is pure iff:* (Y_t) *is a strong solution of* (E);

(ii) X *is extremal, but not pure iff:* (Y_t) *is a weak solution of* (E), *but its distribution is extremal in* π_E;

(iii) X *is not extremal iff the distribution of* (Y_t) *is not extremal in* π_E.

COROLLARY (6.5). — *Suppose* π_E *consists of only one probability. Then,* X *is extremal, and it is pure iff* (Y_t) *is a strong solution of* (E).

The corollary follows immediately from the theorem, so that we need only give a

Proof of theorem (6.4). — (i) is obvious; to show (ii), and (iii), we need only prove that X is extremal iff the distribution of (Y_t) is extremal in π_E. But, from part I, the distribution of (Y_t) is extremal in π_E iff the martingale

$$M_t=Y_t-\int_0^t b(s;\, Y_.(\omega))\,ds$$

has the representation property w.r.t. $\{\mathcal{M}(Y)_t\}$. As

$$M_t = \int_0^t \sigma(s; Y_.(\omega))\, d\beta_s,$$

and $\sigma(s, w) \neq 0$, for all (s, w)'s, (M_t) has the representation property w.r.t. $\{\mathcal{M}(Y)_t\}$ iff β has it.

The proof is now completed by an application of theorem (6.2). □

Remark (6.6). — Another proof of theorem (6.4) can be given, which avoids the use of the representation property, as one may also see directly that X is extremal iff the distribution of (Y_t) is extremal in π_E. □

Now, Theorem (6.4) tells us that to exhibit some extremal, but non-pure continuous martingales, we only need to give some examples of equations (E) which do not possess strong solutions, and then pick an extremal point in π_E.

The first (and easier) example of this is:

$$(E_1) \qquad \begin{cases} dY_s = \operatorname{sgn}(Y_s)\, d\beta_s, \\ Y_0 = 0. \end{cases}$$

Indeed, if on the given probability space, there exists a solution to (E_1), then (Y_t) is a real-valued Brownian motion, as $\langle Y, Y \rangle_t = t$; therefore, π_{E_1} consists only in the Wiener measure.

On the other hand, for every t, one has: $\mathcal{M}(\beta)_t = \mathcal{M}(|Y|)_t$, so that (Y_t) is always a weak solution.

Finally, for our purpose, we need to construct a solution to (E_1), and (luckily!) β does not have to be given *a priori*. So, simply take a real-valued Brownian motion (B_t), with $B_0 = 0$, and $\beta_t \overset{\text{def}}{=} \int_0^t \operatorname{sgn}(B_s)\, dB_s$. Then, obviously, (B_t) is a solution of (E_1).

It had already been remarked in [32] that the martingale $X_t = \beta_{A_t}$, where (A_t) is the inverse process of

$$\tau_t = \int_0^t \left(2 + \frac{B_s}{1 + |B_s|}\right) ds,$$

is extremal but not pure. □

Remarks (6.7). — (*a*) Leaving aside the theory of stochastic differential equations, and looking again at theorem (6.2), we see that, more generally than in the previous example, any stochastic integral $B^z = \int_0^{\cdot} z_s\, dB_s$, with z a $\mathcal{M}(B)$-predictable process, valued in $\{\pm 1\}$, such that $\mathcal{M}(B^z)_\infty \subsetneq \mathcal{M}(B)_\infty$ will provide us with a real-valued Brownian motion which has the representation property w.r.t. $\{\mathcal{M}(B)_t\}$ although verifying the previous strict inclusion;

(*b*) Note that (E_1) is closely related to the last example in section (4). □

The "sign" example may be considerably generalized: let u and v be two positive real numbers such that: $0 < u < v < \infty$, and $\varphi(s, \omega)$ a $\mathcal{M}(B)$-predictable process, taking only the values ± 1, and such that

$$\varphi(s, \omega)\, 1_{[u, v]}(s) = \operatorname{sgn}(B_s)\, 1_{[u, v]}(s).$$

We claim that $\beta_t^\varphi = \int_0^t \varphi(s, \omega)\, dB_s$ satisfies the strict inclusion

$$\mathcal{M}(\beta^\varphi)_\infty \subsetneq \mathcal{M}(B)_\infty. \tag{6.8}$$

Indeed, suppose that the equality $\mathcal{M}(\beta^\varphi)_\infty = \mathcal{M}(B)_\infty$ holds.

As, for every t, $(\beta_{t+h}^\varphi - \beta_t^\varphi,\ h \geqq 0)$ is independant of $\mathcal{M}(B)_t$, the σ-fields $\mathcal{M}(\beta^\varphi)_\infty$ and $\mathcal{M}(B)_t$ are conditionally independant, given $\mathcal{M}(\beta^\varphi)_t$.

Thus, for every t, $\mathcal{M}(\beta^\varphi)_t = \mathcal{M}(B)_t \cap \mathcal{M}(\beta^\varphi)_\infty$. Now, our supposition implies

$$\forall t, \quad \mathcal{M}(\beta^\varphi)_t = \mathcal{M}(B)_t. \tag{6.9}$$

On the other hand, for any t such that: $0 < u < t < v$, one has

$$\beta_t^\varphi - \beta_u^\varphi = \int_u^t \operatorname{sgn}(B_s)\, dB_s.$$

Thus, for such t's, one gets

$$\mathcal{M}(\beta^\varphi)_t \subseteq \mathcal{M}(B)_u \vee \sigma\left\{ \int_u^h \operatorname{sgn}(B_s)\, dB_s;\ u \leqq h \leqq t \right\}.$$

First, Tanaka's formula, and then, Trotter's theorem tell us that

$$|B_h| = |B_u| + \int_u^h \operatorname{sgn}(B_s)\, dB_s + \{ L_h^0 - L_u^0 \},$$

and

$$L_h^0 - L_u^0 = \lim_{(\varepsilon \to 0)} \frac{1}{2\varepsilon} \int_u^h ds\, 1(|B_s| \leqq \varepsilon).$$

Finally, $\mathcal{M}(\beta^\varphi)_t \subseteq \mathcal{M}(B)_u \vee \sigma\{ |B_h|;\ u \leqq h \leqq t \}$, but this contradicts (6.9) since the random variable: $\operatorname{sgn}(B_t)$ is certainly *not* measurable w.r.t.

$$\mathcal{M}(B)_u \vee \sigma\{ |B_h|;\ u \leqq h \leqq t \}.$$

Therefore, our initial supposition was wrong, and (6.8) is true. □

A second (and certainly deeper) example of a stochastic differential equation not possessing a strong solution is due to Tsirel'son [29].

Tsirel'son showed that if one takes $\sigma \equiv 1$, and

$$b(s, w) = \sum_{k \in (-\mathbb{N})} \left[\frac{w(t_k) - w(t_{k-1})}{t_k - t_{k-1}} \right] 1_{]t_k, t_{k+1}]}(s), \tag{6.10}$$

where $\{t_k\}_{k\in(-\mathbb{N})}$ is a sequence of real numbers strictly decreasing from 1 to 0, as $k\downarrow -\infty$, and $[x]$ is the fractional part of $x\in\mathbb{R}$, then the equation (E_2) associated with this particular pair (σ, b) has no strong solution; the existence of a weak solution follows from Girsanov's theorem, which also shows that π_{E_2} consists of precisely one probability.

Remarks (6.11). — (*a*) We recall here that if, instead of Tsirel'son's drift, one takes $b(t, w)=b(t, w(t))$, with $b(t, x)$ a bounded, Borel function defined on $\mathbb{R}_+\times\mathbb{R}$, Zvonkin [34] proved that any solution of (E) (with $\sigma\equiv 1$) is strong.

(*b*) Apart from Tsirel'son's original proof showing that (E_2) has no strong solution, V. Benes [17] has given a measure-theoretical one, as well as several extensions of Tsirel'son's example. A proof due to N. Krylov appears in Lipcer and Shyriaev's book [28]. Although very much inspired by Krylov's, the proof we give below does not proceed "par l'absurde", and provides some additional information. □

Thus, suppose that (β_t) is a $(\mathscr{M}_t)$ real-valued Brownian motion, and (Y_t) is a $(\mathscr{M}_t)$ adapted solution of

$$(E_2)\qquad \begin{cases} dY_s=b(s;\, Y_.(\omega))\,ds+d\beta_s, \\ Y_0=0, \end{cases}$$

where b is Tsirel'son's drift [which is given by (6.10)]. We now introduce the following notations

$$\eta_t=\frac{Y_t-Y_{t_{k-1}}}{t-t_{k-1}};\qquad \varepsilon_t=\frac{\beta_t-\beta_{t_{k-1}}}{t-t_{k-1}}\qquad \text{for}\quad t\in]t_{k-1}, t_k]$$

and, to simplify again: $\eta_k=\eta_{t_k}$; $\varepsilon_k=\varepsilon_{t_k}$, for $k\in(-\mathbb{N})$.

Remark that, by hypothesis, one has, for any $t\in]t_{k-1}, t_k]$:

$$Y_t-Y_{t_{k-1}}=\left[\frac{Y_{t_{k-1}}-Y_{t_{k-2}}}{t_{k-1}-t_{k-2}}\right](t-t_{k-1})+(\beta_t-\beta_{t_{k-1}}),$$

so that

$$(6.12)\qquad \eta_t=[\eta_{k-1}]+\varepsilon_t\qquad (t\in]t_{k-1}, t_k]),$$

and, in particular

$$(6.12')\qquad \eta_k=[\eta_{k-1}]+\varepsilon_k\qquad (k\in(-\mathbb{N})).$$

The weakness of (Y_t), as a solution of (E_2), will be *a fortiori* obtained, once the following is established.

PROPOSITION (6.13). — *For any* $t\in]0, 1]$, $[\eta_t]$ *is independant of* $\mathscr{M}(\beta)_\infty$, *and is uniformly distributed on* $[0, 1]$.

Proof. — (*a*) For any $k\in(-\mathbb{N})$, set $d_k=E(e^{2i\pi\eta_k})$. From (6.12'), one has: $d_k=E(e^{2i\pi\eta_{k-1}}\,e^{2i\pi\varepsilon_k})$. As (β_t) is a $\{\mathscr{M}(Y)_t\}$ Brownian motion, the variables η_{k-1} and ε_k are independant.

Therefore

$$d_k = d_{k-1}\,\mathrm{E}(e^{2i\pi\varepsilon_k}) = d_{k-1}\,e^{-2\pi^2/(t_k - t_{k-1})}.$$

This implies

$$|d_k| \leqq |d_{k-1}|\,e^{-2\pi^2},$$

and by iteration,

$$|d_k| \leqq |d_{k-n}|\,e^{-2\pi^2 n} \quad \text{for any } n \in \mathbb{N}, \qquad \leqq e^{-2\pi^2 n} \xrightarrow[(n\to\infty)]{} 0.$$

Thus, for any $k \in (-\mathbb{N})$, $d_k = 0$.

(*b*) For any couple (t, t') of real numbers such that: $0 \leqq t \leqq t'$, let

$$\mathscr{B}_{(t,\,t')} \overset{\text{def}}{=} \sigma\{\beta_u - \beta_v;\ t \leqq u \leqq v \leqq t'\}.$$

For any $k \in (-\mathbb{N})$, $n \in \mathbb{N}$, one has

$$\mathrm{E}(e^{2i\pi\eta_k} \,|\, \mathscr{B}_{(t_{k-n},\,t_k)}) = e^{2i\pi\varepsilon_k}\,\mathrm{E}(e^{2i\pi\eta_{k-1}} \,|\, \mathscr{B}_{(t_{k-n},\,t_k)}) = e^{2i\pi(\varepsilon_{k+1-n} + \ldots + \varepsilon_k)}\,\mathrm{E}(e^{2i\pi\eta_{k-n}} \,|\, \mathscr{B}_{(t_{k-n},\,t_k)}).$$

Again, η_{k-n} is independant from the process $(\beta_{t_{k-n}+u} - \beta_{t_{k-n}},\ u \geqq 0)$, and therefore from $\mathscr{B}_{(t_{k-n},\,t_k)}$. This implies that

$$\mathrm{E}\,(e^{2i\pi\eta_k} / \mathscr{B}_{(t_{k-1},\,t_k)}) = e^{2i\pi(\varepsilon_{k+1-n} + \ldots + \varepsilon_k)}\,d_{k-n} = 0.$$

Now, letting n increase to $+\infty$, one gets

$$\mathrm{E}(e^{2i\pi\eta_k} / \mathscr{M}(\beta)_{t_k}) = 0.$$

Moreover, as $\mathscr{B}_{(t_k,\,\infty)}$ and $\mathscr{M}(\mathrm{Y})_{t_k}$ are independant, one finally gets

$$\mathrm{E}(e^{2i\pi\eta_k} / \mathscr{M}(\beta)_\infty) = 0.$$

(*c*) The arguments we have just used for the random variables $(2\pi\eta_k)_{k\in(-\mathbb{N})}$ are also valid for $(2\pi p\,\eta_k)_{k\in(-\mathbb{N})}$, for any $p \in \mathbb{Z}\setminus\{0\}$.

Therefore, for any $k \in (-\mathbb{N})$, and any $p \in \mathbb{Z}\setminus\{0\}$, one has:

$$\mathrm{E}(e^{2ip\pi[\eta_k]} / \mathscr{M}(\beta)_\infty) = 0.$$

This exactly proves, from Stone-Weierstrass theorem, the wanted result for each of the variables $([\eta_k])_{k\in(-\mathbb{N})}$.

(*d*) With the help of (6.12), one has, for any $t \in]t_{k-1}, t_k]$, and any $p \in \mathbb{Z}\setminus\{0\}$:

$$\mathrm{E}(e^{2i\pi p[\eta_t]} / \mathscr{M}(\beta)_\infty) = e^{2i\pi p\varepsilon_t}\,\mathrm{E}(e^{2i\pi p[\eta_{k-1}]} / \mathscr{M}(\beta)_\infty) = 0,$$

which finishes the proof. □

The following corollary provides us with a lot of (complicated!) stochastic differential equations which have no strong solution.

COROLLARY (6.14). — *Let $c:(s, w) \to c(s, w)$ be a $(\mathscr{C}_t)$ predictable real-valued, uniformly bounded application, such that $c(s, w) \equiv 0$, for $s \geqq 1$. Define $d:(s, w) \to d(s, w)$ by the formula*

$$d(s, w) = b(s, w) + c\left(s;\ w - \int_0^{\cdot} b(u, w)\, du\right),$$

where b is Tsirel'son's drift, which is given by (6.10).

Suppose that $(\gamma_t)_{t \geqq 0}$ is a $(\mathscr{M}_t, P)$ real-valued Brownian motion, and $(Y_t)_{t \geqq 0}$ is a $(\mathscr{M}_t)$ adapted solution of

$$(E_3) \qquad \begin{cases} dY_s = d(s;\ Y_{\cdot}(\omega))\, ds + d\gamma_s, \\ Y_0 = 0. \end{cases}$$

Then, Y *is a weak solution of* (E_3).

Proof. — Let $\beta_t = Y_t - \int_0^t b(s;\ Y_{\cdot}(\omega))\, ds$.

[Beware: (β_t) is *not* a Brownian motion w. r. t. the underlying probability P!]. As (Y_t) is a solution of (E_3), we have

$$\beta_t = \int_0^t c(s;\ \beta_{\cdot}(\omega))\, ds + \gamma_t(\omega).$$

Consequently, for every t, one has: $\mathscr{M}(\gamma)_t \subseteq \mathscr{M}(\beta)_t$.

Define now a new probability Q by

$$Q = \exp\left\{ - \int_0^1 c(s;\ \beta_{\cdot}(\omega))\, d\gamma_s - \frac{1}{2}\int_0^1 c^2(s;\ \beta_{\cdot}(\omega))\, ds \right\} . P.$$

Girsanov's theorem (if necessary, look at the very beginning of section (8), or at Wong and Van Schuppen [30], or at Girsanov [23]) tells us that $\beta_t = \gamma_t + \int_0^t c(s;\ \beta_{\cdot}(\omega))\, ds$ is a $(\mathscr{M}_t, Q)$ Brownian motion.

Thus, as from the definition of (β_t), (Y_t) is a solution of equation (E_2) associated with (β_t), i. e.:

$$Y_t = \int_0^t b(s;\ Y_{\cdot}(\omega))\, ds + \beta_t,$$

we have, from (6.13): $\forall\, t > 0$, $\mathscr{M}(\beta)_t \subsetneqq \mathscr{M}(Y)_t$, where the two filtrations in question are taken to be complete w. r. t. $(\mathscr{M}_\infty, Q)$, which amounts to being complete w. r. t. $(\mathscr{M}_\infty, P)$, as P and Q are equivalent.

So, finally, one has: $\forall\, t > 0$, $\mathscr{M}(\gamma)_t \subsetneqq \mathscr{M}(Y)_t$. □

Tsirel'son's construction originated as a (negative) answer to the well-known Innovation Problem in filtering theory. This problem is worth describing here, as it has some close connections with our study of pure continuous martingales.

So, suppose that, on a given filtered probability space $(\Omega, \mathscr{M}, (\mathscr{M}_t), P)$, there exists a real-valued $(\mathscr{M}_t)$ Brownian motion $(B_t)_{t\geqq 0}$, with $B_0=0$.

Let $(h_t)_{t\geqq 0}$ be a uniformly bounded $(\mathscr{M}_t)$ predictable process. Now define

$$Y_t=\int_0^t h_s\,ds+B_t.$$

In general, h is not adapted to $\{\mathscr{M}(Y)_t\}$. So, one considers $(\hat{h}_t)$, which is the $\{\mathscr{M}(Y)_t\}$ predictable projection of (h_t). Then, it is not difficult to see that the process (β_t) defined by

$$Y_t=\int_0^t \hat{h}_s\,ds+\beta_t$$

is a $\{\mathscr{M}(Y)_t\}$ Brownian motion.

Filtering theory (more exactly: Girsanov's theorem again; *see* lemma (8.1) for instance) tells us that (β_t) has the predictable representation property w.r.t. $\{\mathscr{M}(Y)_t\}$.

The Innovation Problem consists in deciding whether: $\forall\, t, \mathscr{M}(Y)_t=\mathscr{M}(\beta)_t$ or not (as seen previously, Tsirel'son's example proves that the answer to the Innovation Problem is not always positive).

We conclude this paragraph by noting that, in this general filtering setting, the martingale $X_t=\beta_{A_t}$, defined above theorem (6.4) is, from the previous remarks, extremal, and it is pure iff the answer to the Innovation Problem is positive.

Section (7)

In this section, we study the representation property with respect to different filtrations, and obtain from there another characterization of pure continuous martingales, thus completing section (5).

On a given probability space $(\Omega, \mathscr{M}, P)$, let $(\mathscr{G}_t)$ and $(\mathscr{F}_t)$ be two (different!) usual filtrations such that: $\forall\, t, \mathscr{G}_t\subseteq\mathscr{F}_t\subseteq\mathscr{M}$, and M a $(\mathscr{F}_t)$ continuous local martingale, which is adapted to $(\mathscr{G}_t)$.

There are plenty of examples where M has the representation property w.r.t. $(\mathscr{G}_t)$, but not w.r.t. $(\mathscr{F}_t)$: for instance, using the notations of theorem (6.1), if β is the D. D. S. Brownian motion attached to X, a non-extremal continuous local martingale, then β has the representation property w.r.t. $(\mathscr{M}(\beta)_t)$ (from Ito's theorem), but not w.r.t. $(\mathscr{M}(X)_{\tau_t})$ [from theorem (6.1)].

It is also natural to ask about the converse statement: if M has the representation property w.r.t. $(\mathscr{F}_t)$, is it also true w.r.t. $(\mathscr{G}_t)$?

As we shall see further on, the general answer to this question is negative. Nonetheless, the following equivalences may be noteworthy.

PROPOSITION (7.1) (We use the previous notations). — *Suppose that* M *has the representation property w.r.t.* $(\mathcal{F}_t)$. *The following assertions are equivalent:*

(i) M *has the representation property w.r.t.* $(\mathcal{G}_t)$;

(ii) *every* $(\mathcal{G}_t)$ *martingale is a* $(\mathcal{F}_t)$ *martingale;*

(iii) *every* $(\mathcal{G}_t)$ *martingale is a continuous* $(\mathcal{F}_t)$ *semi-martingale.*

Proof. — (i) ⇒ (ii). Every $(\mathcal{G}_t)$ martingale (N_t) may be written as $N_t = c + \int_0^t \varphi_s \, dM_s$, where $c \in \mathbb{R}$, and φ is predictable w.r.t. $(\mathcal{G}_t)$, and therefore w.r.t. $(\mathcal{F}_t)$. So, $\left(\int_0^t \varphi_s \, dM_s, \ t \geqq 0\right)$ is a $(\mathcal{F}_t)$ martingale, and so is N.

(ii) ⇒ (iii). — As M has the representation property w.r.t. $(\mathcal{F}_t)$, every $(\mathcal{G}_t)$ martingale, which is — by hypothesis — a $(\mathcal{F}_t)$ martingale, may be written as a stochastic integral w.r.t. M, and so, is a continuous $(\mathcal{F}_t)$ martingale.

(iii) ⇒ (i). — Let N be a $(\mathcal{G}_t)$ martingale. By hypothesis, N may be written as $N_t = c + \int_0^t \varphi_s \, dM_s + A_t$, with $c \in \mathbb{R}$, φ a $(\mathcal{F}_t)$ predictable process, and A a $(\mathcal{F}_t)$ adapted, continuous process, with finite variation.

Then, as M is continuous, the process $[N, M]_t = \int_0^t \varphi_s \, d\langle M, M\rangle_s$ is continuous, and $(\mathcal{G}_t)$ adapted. Thus, $\varphi = d[N, M]/d\langle M, M\rangle$ may be chosen to be $(\mathcal{G}_t)$ predictable. So, $N - \left(c + \int_0^{\cdot} \varphi_s \, dM_s\right)$ is a $(\mathcal{G}_t)$ continuous martingale, equal to A, a process with finite variation. This implies: $A \equiv 0$.

Q.E.D.

Remark. — In fact, the above proof shows the slightly more general equivalence: (i′) ⇔ (iii′), with:

(i′) every continuous $(\mathcal{G}_t)$ martingale is a stochastic integral w.r.t. M (with a $(\mathcal{G}_t)$ predictable integrand);

(iii′) every $(\mathcal{G}_t)$ continuous martingale is a $(\mathcal{F}_t)$ semi-martingale. □

We still work in the setting described at the very beginning of this chapter. M may or may not have the representation property w.r.t. $(\mathcal{F}_t)$. However, the following lemma shows *a fortiori* that there exists a third filtration $(\mathcal{H}_t)$ such that: $\forall t, \mathcal{G}_t \subseteqq \mathcal{H}_t \subseteqq \mathcal{F}_t$, but w.r.t. which M does not have the representation property.

LEMMA (7.2). — *If* $(\mathcal{G}_t)$ *and* $(\mathcal{F}_t)$ *are two different filtrations such that, for every* $t, \mathcal{G}_t \subseteqq \mathcal{F}_t$, *there exists a third filtration* $(\mathcal{H}_t)$ *such that:*

(i) $\forall t, \mathcal{G}_t \subseteqq \mathcal{H}_t \subseteqq \mathcal{F}_t$;

(ii) *there exists at least a purely discontinuous* $(\mathcal{H}_t)$ *martingale, whose only jump occurs at a deterministic time;*

(iii) *if* $L^1(\mathcal{G}_\infty, P)$ *is separable,* $L^1(\mathcal{H}_\infty, P)$ *is also separable.*

Proof. — As $(\mathcal{G}_t)$ and $(\mathcal{F}_t)$ are different filtrations, there exists $t_0>0$ such that $\mathcal{G}_{t_0} \subsetneq \mathcal{F}_{t_0}$. Consequently, there is a r. v. h which belongs to $L^2(\mathcal{F}_{t_0})$, and is orthogonal to $L^2(\mathcal{G}_{t_0})$.

Define the process $H_t = h\, 1_{(t_0 \leqq t)}$, and the filtration $(\mathcal{H}_t)$ generated by $(\mathcal{G}_t)$ and the process H. It is clear that $\mathcal{G}_{t_0^-} = \mathcal{H}_{t_0^-}$.

Moreover, H is a $(\mathcal{H}_t)$ martingale, which is obviously purely discontinuous. Indeed, all we have to show is that for any bounded $\mathcal{H}$-predictable process $(Z_t)_{t \geqq 0}$, one has: $E\left[\int_0^\infty Z_s\, dH_s\right] = 0$.

But, $E\left[\int_0^\infty Z_s\, dH_s\right] = E[Z_{t_0} h] = 0$, as $\sigma(Z_{t_0}) \subseteqq \mathcal{G}_{t_0^-} \subseteqq \mathcal{G}_{t_0}$.

The properties (i) and (iii) are obviously satisfied. □

An open question. — In lemma (7.2), we exhibited a *discontinuous* $(\mathcal{H}_t)$ martingale. So, in relation with, and in the setting of, proposition (7.1), it is natural to ask whether there exists an example of a continuous martingale M which has the representation property w. r. t. $(\mathcal{F}_t)$, but not w. r. t. $(\mathcal{G}_t)$, and at the same time, all $(\mathcal{G}_t)$ martingales are continuous ?

Here is an example which illustrates lemma (7.2): take for $(\mathcal{F}_t)$ the natural filtration of a real-valued Brownian motion $(B_t)_{t \geqq 0}$, with $B_0 = 0$, and for $(\mathcal{G}_t)$ the natural filtration of $(|B_t|,\ t \geqq 0)$. Let $t_0 > 0$. Then, $h = \mathrm{sgn}(B_{t_0})$ is independant of $(|B_t|,\ t \geqq 0)$, and in consequence, orthogonal to $L^2(\mathcal{G}_{t_0})$. As in lemma (7.2), $(\mathcal{H}_t)$ denotes the filtration generated by $\mathcal{G}_t$ and $H_t = \mathrm{sgn}(B_{t_0})\, 1_{(t_0 \leqq t)}$. Finally, take $M_t = \int_0^t \mathrm{sgn}(B_s)\, dB_s$. This process is a $(\mathcal{F}_t)$ Brownian motion, which has the representation property w. r. t. $(\mathcal{F}_t)$; moreover, as M and $|B|$ have the same natural filtration, M is adapted to $(\mathcal{G}_t)$, and even has the representation property w. r. t. $(\mathcal{G}_t)$. Nonetheless, as a consequence of the lemma, M does not have the representation property w. r. t. $(\mathcal{H}_t)$.

A slight change in the presentation of this example provides us with a continuous $(\mathcal{F}_t)$ martingale N, which has the representation property w. r. t. $(\mathcal{F}_t)$, but not w. r. t. $(\mathcal{M}(N)_t)$: this is the case for the $(\mathcal{F}_t)$ martingale

$$N_t = \int_0^t [1_{(s<t_0)} + (2 + \mathrm{sgn}(B_{t_0}))\, 1_{(t_0 \leqq s)}]\, \mathrm{sgn}(B_s)\, dB_s$$

as $\mathcal{M}(N)_t = \mathcal{H}_t$, with the notations of the previous example.

D. Lane [21] has also given some examples of martingales $M_t^f = \int_0^t f(B_s)\, dB_s$ with f continuous, such that M^f has the representation property w. r. t. $(\mathcal{F}_t)$ [a necessary and sufficient condition for this to happen is that the Lebesgue measure of $\{x / f(x) = 0\}$ is 0], but the natural filtration of M^f supports discontinuous martingales [this is the case, for instance, if $f(x) = x \wedge a,\ a > 0$].

We now give another characterization of pure continuous martingales, the proof of which relies mainly on lemma (7.2).

THEOREM (7.3). — *Let* (X_t) *be a continuous local martingale such that* $X_0=0$, *and* $\langle X, X\rangle_\infty=\infty$ *a.s. Then,* X *is pure iff: for any* X*-continuous, and* X*-adapted change of time* $T=(\tau_t)_{t\geq 0}$, *such that* $\tau_t\uparrow\infty$, *as* $t\uparrow\infty$, *the local martingale* T(X) *is extremal.*

Proof. — (i) The condition is obviously necessary from Proposition (5.16), and the fact that a pure martingale is extremal;

(ii) Conversely, suppose that the condition holds, but that X is not pure. Note $\tau_t=\inf\{s/\langle X\rangle_s>t\}$ $(t\geqq 0)$, and $\beta_t=X_{\tau_t}$ the D. D. S. Brownian motion attached to X.

Then, as X is supposed to be non-pure, we have the strict inclusion

$$\mathcal{M}(\beta)_\infty \subsetneqq \mathcal{M}(X)_{\tau_\infty} \ (=\mathcal{M}(X)_\infty).$$

From lemma (7.2) above, there exists a filtration $(\mathcal{H}_t)_{t\geqq 0}$ such that

(*a*) $\forall t$, $\mathcal{M}(\beta)_t \subseteqq \mathcal{H}_t \subseteqq \mathcal{M}(X)_{\tau_t}$;

(*b*) $L^1(\mathcal{H}_\infty, P)$ is separable;

(*c*) β *does not* have the representation property w.r.t. $(\mathcal{H}_t)$.

From Dellacherie and Stricker [36], there exists a continuous, strictly increasing process $(\sigma_t)_{t\geqq 0}$ such that: $\sigma_0=0$, $\sigma_t\geqq t$, and, for every $t:\mathcal{H}_t=\mathcal{M}(\sigma_.)_t$. Then, the part (iii) of Theorem (6.2) tells us that if $\Lambda_t=\inf\{s/\sigma_s>t\}$ $(t\geqq 0)$, the martingale $\beta_{\Lambda_t}=X_{(\tau_{\Lambda_t})}$ is not extremal.

Moreover, from lemma (5.9), the process $(\tau_{\Lambda_t})_{t\geqq 0}$ is a X-adapted change of time; it is also X-continuous, as (Λ_t) is continuous, with $\Lambda_0=0$, and $\Lambda_\infty=\infty$ P a.e., and (τ_t) is X-continuous.

Finally, we have obtained a contradiction with our hypothesis. Thus, X is pure.

Remark. — Theorem (7.3) opens a possibility of defining the notion of purity for right-continuous local martingales. Indeed, by analogy with the continuous case, we may define a pure right-continuous local martingale (X_t), with $X_0=0$, as a martingale which satisfies the condition stated in Theorem (7.3).

This general definition of purity has been partially investigated in [37], and, at least in some cases, the situation appears to be completely different from that of continuous martingales: indeed, it is shown in [37] that if (M_t) is a purely discontinuous $((\mathcal{F}_t), P)$ local martingale such that:

(*a*) $M_0=0$; $\langle M, M\rangle_\infty=\infty$;

(*b*) its jumps are identically equal to 1;

(*c*) its jump times are totally unpredictable,

then M is pure iff it is extremal.

Section (8)

We recall here some — now classical — results on (Girsanov's) transformations of local martingales under changes of probability, and show that the set of pure, resp.: extremal martingale distributions is not left invariant under these transformations.

Let $(\Omega, \mathcal{M}, (\mathcal{F}_t), P)$ be a usual filtered probability space. Consider Q, a second probability on $(\Omega, \mathcal{M})$, supposed to be equivalent to P on $\mathcal{M}$; note $L = dQ/dP$, and $(L_t)_{t \geqq 0}$ a right continuous version of $(E_P(L/\mathcal{F}_t);\ t \geqq 0)$.

The following result, due to Wong and van Schuppen [30], gives the canonical decomposition of continuous P-martingales as Q-semi-martingales; it is an extension of an older theorem of Girsanov [23], concerned only with Brownian motion: *if* X *is a* P *local continuous martingale, then* $\tilde{X} = X - \int_0^{\cdot} (1/L_s)\, d\langle X, L\rangle_s$ *is a* Q *local continuous martingale.*

[Girsanov's theorem also states that if X is a $(P, \mathcal{F}_t)$ Brownian motion, $\tilde{X}$ is a $(Q, \mathcal{F}_t)$ one; but, this is also immediate from Paul Levy's characterization of the Brownian motion, as then: $\langle \tilde{X}, \tilde{X}\rangle_t = \langle X, X\rangle_t = t$].

We call $\tilde{X}$ the Girsanov transform of X (given the ordered pair (P, Q) of equivalent probabilities). It may be worth to emphasize here that the Girsanov transform of $\tilde{X}$ [given the ordered pair (Q, P)] is X.

The following result has been obtained by many authors (for instance, Jacod-Memin [24], Kunita [26], Yoeurp-Yor [31]), using different methods, and will be very useful in the sequel.

LEMMA (8.1) (We use the previous notations). — X *has the representation property w.r.t.* $((\mathcal{F}_t), P)$ *iff* $\tilde{X}$ *has the representation property w.r.t.* $((\mathcal{F}_t), Q)$.

We specialize now to the case where $(\mathcal{F}_t)$ is the natural filtration of a real-valued Brownian motion (B_t) (under the probability P). As a consequence of lemma (8.1), $\tilde{B}_t = B_t - \int_0^t (1/L_s)\, d\langle B, L\rangle_s$ is a $((\mathcal{F}_t), Q)$ Brownian motion, which has the representation property w.r.t. $((\mathcal{F}_t), Q)$.

But, Tsirel'son's example [*cf.* section (6)] — which is now looked at from the point of view of Girsanov's transform — shows that there exist some densities L such that

$$\mathcal{M}(\tilde{B})_\infty \subsetneq \mathcal{M}(B)_\infty. \tag{8.2}$$

Indeed, take

$$L(\omega) = \exp\left\{ \int_0^1 b(s;\ B_.(\omega))\, dB_s(\omega) - \frac{1}{2}\int_0^1 b^2(s;\ B_.(\omega))\, ds \right\} \tag{8.3}$$

where b is Tsirel'son's drift, which is given by (6.10).

Then, one has

$$\tilde{B}_t = B_t - \int_0^{t \wedge 1} b(s;\ B_.(\omega))\, ds, \tag{8.4}$$

so that (B_t) appears to be a solution of (E_2), for the given Brownian motion $(\tilde{B}_t)$. (8.2) follows then from proposition (6.13).

Take again for $(\tau_t)_{t \geqq 0}$ the process

$$\tau_t = \int_0^t \left\{ 2 + \frac{B_s}{1 + |B_s|} \right\} ds$$

and

$$A_t = \inf\{s/\tau_s > t\} \qquad (t \geqq 0).$$

Our previous arguments, and theorem (6.2), or theorem (6.4), show that the $(\mathscr{F}_{A_t}, Q)$ martingale $\tilde{X}_t = \tilde{B}_{A_t}$ is *extremal, but not pure*.

On the other hand, $X_t = B_{A_t}$ is a $(\mathscr{F}_{A_t}, P)$ martingale, which is pure, by construction, and, from (5.12), one has

$$\forall\, t, \quad \mathscr{M}(X)_t = \mathscr{F}_{A_t} = \mathscr{M}(\tilde{X})_t.$$

Moreover, $\tilde{X}$ is the Girsanov transform of X, with respect to the ordered pair (P, Q) and the filtration $(\mathscr{M}(X)_t = \mathscr{F}_{A_t})$; this fact may be easier to visualize when observing the following commutative diagram [the commutativity property is easily obtained from Proposition (5.1)].

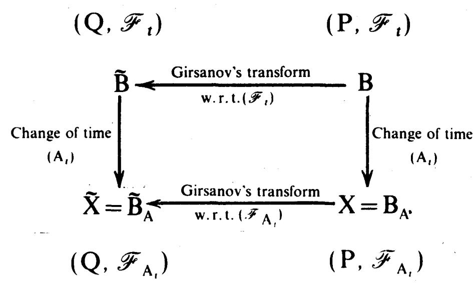

Fig. 1

Therefore, not only does Tsirel'son's construction provide us with an example of extremal, but not pure continuous martingale, but it (with the help of Girsanov's theorem) also shows that:

(8.5) *the set of pure martingales* (or distributions) *is not invariant under Girsanov's transforms*.

Nonetheless, in the example we are after giving, $\tilde{X}$ is extremal; worse things may happen, as the following proposition shows.

PROPOSITION (8.6). — *There exist a pure martingale* X, *and a Girsanov transform of* X, *which is not even extremal.*

Nota bene. — In (8.5) and Proposition (8.6), the Girsanov's transforms are *of course* relative to the natural filtration of the original pure martingale.

Proof of the proposition. — Let $(\mathscr{F}_t)$ be the natural filtration of a real-valued Brownian motion $(B_t)_{t \geqq 0}$, and $(\tilde{B}_t)_{t \geqq 0}$ the Q-Brownian motion [formula (8.4)] obtained *via* the Girsanov's transform associated with the density L given by (8.3), and (6.10).

As, for every t, $\mathscr{M}(\tilde{B})_t \subseteq \mathscr{F}_t (= \mathscr{M}(B)_t)$, but $\mathscr{M}(\tilde{B})_\infty \subsetneqq \mathscr{M}(B)_\infty$, we can exhibit, from lemma (7.2), a filtration $(\mathscr{H}_t)$ such that:

(*a*) for every t, $\mathscr{M}(\tilde{B})_t \subseteq \mathscr{H}_t \subseteq \mathscr{M}(B)_t$;

(*b*) $\tilde{B}$ does not have the representation property w.r.t. $((\mathscr{H}_t), Q)$;

(*c*) $L^1(\Omega, \mathscr{H}_\infty, Q)$ is separable.

Using (*c*), we can now apply the construction done before Theorem (6.2), i.e.: there exists a continuous, strictly increasing process (τ_t), with $\tau_0=0$, $\tau_t \geqq t$, and $\mathscr{H}_t=\mathscr{F}(\tau_.)_t$, for every t. Note $A_t=\inf\{s/\tau_s>t\}$.

Then, from Theorem (6.2) (iii), $\tilde{X}=\tilde{B}_A$ *is not extremal*.

But, as $\mathscr{H}_t \subseteqq \mathscr{F}_t$, for every t, the $(\mathscr{F}_{A_t}, P)$ martingale $X=B_A$ *is pure* [and: $\mathscr{M}(X)_t=\mathscr{F}_{A_t}$, from (5.14)].

Finally, $\tilde{X}$ appears as a Girsanov's transform of X, as the diagram in (*Fig.* 1) (which needs no change, for we have kept the same notations) is still commutative. □

Remark (8.7). — Note that in the example right above (8.5), one has:

$$\forall t, \quad \mathscr{M}(X)_t=\mathscr{M}(\tilde{X})_t.$$

As a consequence of lemma (8.1), this can no longer be true in the example developed in the proof of Proposition (8.6). Indeed, we have, in this case

$$\forall t, \quad \mathscr{M}(X)_t=\mathscr{F}_{A_t},$$

and so

$$\mathscr{M}(X)_{\tau_t}=\mathscr{F}_t;$$

on the other hand, we have shown in (6.3), that: $\mathscr{M}(\tilde{X})_{\tau_t}=\mathscr{H}_t$.

Thus, the filtrations $\{\mathscr{M}(X)_t\}$ and $\{\mathscr{M}(\tilde{X})_t\}$ differ. □

Conversely, let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a usual filtered probability space, $Q \simeq P$ on $\mathscr{F}$, and X a $(\mathscr{F}_t, P)$ continuous martingale, with $X_0=0$, and $\langle X, X\rangle_\infty=\infty$ a.e., which is pure, but such that there exists a Girsanov transform of X [w.r.t. the ordered pair (P, Q), and the filtration $\mathscr{M}(X)_t$], say $\tilde{X}$, which is not pure. As $\langle X, X\rangle = \langle \tilde{X}, \tilde{X}\rangle$, it is easy to see that the D.D.S. Brownian motion attached to $\tilde{X}$ (w.r.t. Q) is the Girsanov transform of the D.D.S. Brownian motion attached to X (w.r.t. P), the Girsanov transform being relative to (P, Q), and the filtration $\mathscr{M}(X)_{\tau_t}=\mathscr{M}(\beta)_t$, where $\tau_t=\inf\{s/\langle X\rangle_s>t\}$. As $\tilde{X}$ is not pure, we have

$$\mathscr{M}(\tilde{\beta})_\infty \subsetneqq \mathscr{M}(\tilde{X})_\infty \subseteqq \mathscr{M}(X)_\infty=\mathscr{M}(\beta)_\infty.$$

Thus, the filtration of $\tilde{\beta}$ is strictly smaller than that of β, e.g.: we have a Tsirel'son-type example.

We end this section by drawing a table in which we summarize some of the main results obtained up to now: this table indicates whether or not certain martingale properties are left stable under either Girsanov's transforms or changes of time. Our presentation is probably over concise, but we hope this will make no confusion, as the exact references in the text are given.

Nonetheless, we precise that the Girsanov transforms (resp.: changes of time) of a martingale X, considered here, are supposed to be relative to $(\mathscr{M}(X)_t)$ (resp.: to be X-continuous, and X-adapted).

Transformations	Martingale properties: Representation property w.r.t. $(\mathscr{F}_t)$	Extremality	Purity
Girsanov's transform.	*Stable* [Lemma (8.1)]	*Unstable* [Proposition (8.6)]	*Unstable* [example (8.5)]
Change of time..	*Stable* [same proof as for Theorem (6.1)]	*Unstable* [Theorem (7.3)]	*Stable* [Proposition (5.16)]

Fig. 2

Section (9)

This final section consists of several questions, which seem (to us!) to be the main open problems concerning extremal continuous martingales.

Although some of the theorems in the second part of this paper appear to give – at least in our opinion – a more precise view of the differences which exist between extremal, and pure continuous martingales, the main question asked in [32] has not yet been answered. Let us recall it here:

(B) *If* $(X_t)_{t \geqq 0}$ *is a continuous local martingale, with* $X_0 = 0$, *and* $\langle X, X \rangle_\infty = \infty$ *a.e., whose law is extremal among the laws of continuous local martingales, is* $\{ \mathscr{M}(X)_{\tau_t} \}$ (2) *the natural filtration of a real-valued Brownian motion* $(B_t)_{t \geqq 0}$, *with* $B_0 = 0$?

It is possible to ask a (seemingly) more general question, which involves only a real-valued Brownian motion:

(S) *if* $(\mathscr{G}_t)$ *is a usual filtration such that:*

(*a*) $L^1(\Omega, \mathscr{G}_\infty, P)$ *is separable;*

and

(*b*) *there exists a* $(\mathscr{G}_t)$ *real-valued Brownian motion* $(\beta_t)_{t \geqq 0}$, *with* $\beta_0 = 0$, *such that* β *has the representation property w.r.t.* $(\mathscr{G}_t)$, *is* $(\mathscr{G}_t)$.

Is $(\mathscr{G}_t)$ *the natural filtration of a real-valued Brownian motion?*

Finally, we recall another question raised in [32]:

(B′) *If* (Z_t) *is a local continuous martingale, such that* $d_s \langle Z, Z \rangle_s(\omega)$ *is equivalent to* (ds), *and Z is extremal, then, is* $\{ \mathscr{M}(Z)_t \}$ *the natural filtration of a real-valued Brownian motion?*

Remarks (9.1). – 1) The letters B and S are here respectively for "Brownian" and "separable".

2) It is easy to see that, conversely, if the property concluding any of the questions (B), (B′) or (S) is satisfied, then the hypothesis made in the corresponding question is verified. □

(2) We denote, as usual, $\tau_t = \inf \{ s / \langle X \rangle_s > t \}$ $(t \geqq 0)$.

We note $(Q)_+$ if the answer to question (Q) is always yes. We now prove

$$(B)_+ \Leftrightarrow (S)_+ \Leftrightarrow (B')_+ \tag{9.2}$$

e. g.: *the* 3 *questions are "equivalent"*.

$(B)_+ \Rightarrow (S)_+$. — From the construction preceding theorem (6.2), hypothesis (S) implies the existence of a continuous, extremal, martingale X, with $X_0=0$, $\langle X, X \rangle_\infty = \infty$ a.e., such that: $\mathcal{G}_t = \mathcal{M}(X)_{\tau_t}$ (where $\tau_t = \inf\{s / \langle X \rangle_s > t\}$).

$(S)_+ \Rightarrow (B)_+$. — We know that, if X is extremal, $\beta_t = X_{\tau_t}$ has the representation property w.r.t. $\{\mathcal{M}(X)_{\tau_t}\}$.

$(S)_+ \Rightarrow (B')_+$. — As $d_s \langle Z, Z \rangle_s(\omega)$ is equivalent to (ds), we can write

$$Z_t = \int_0^t z_s \, d\beta_s,$$

with z a $(\mathcal{M}(Z)_t)$ predictable process, and β a $(\mathcal{M}(Z)_t)$ Brownian motion, which has the representation property w.r.t. $(\mathcal{M}(Z)_t)$, as Z has it.

$(B')_+ \Rightarrow (S)_+$. — As $(\mathcal{G}_t)$ satisfies (a), there exists a strictly increasing, continuous $(\mathcal{G}_t)$ adapted process $(A_t)_{t \geq 0}$ such that $\mathcal{G}_t = \mathcal{M}(A)_t$, for every t (Dellacherie and Stricker [36]). It is then easy to show that, for every $t \geqq 0$, $\mathcal{G}_t = \mathcal{M}(Z)_t$, where $Z_t \overset{\text{def}}{=} \int_0^t A_s \, d\beta_s$ (*see* [38]).

Moreover, $d_s \langle Z, Z \rangle_s(\omega)$ is equivalent to (ds), and Z is extremal, as β has the representation property w.r.t. $\mathcal{M}(Z)(=\mathcal{G})$. □

Finally, we add two more questions to our list; it will turn out that $(S)_+$ implies that the answers to these last questions are positive, but we do not know anything about the converse.

The first question seems very tentative:

(G) *If* $(B_t)_{t \geqq 0}$ *is a real-valued Brownian motion, with* $B_0 = 0$, *defined on* $(\Omega, \mathcal{F}, P)$, *and* $Q \sim P$ *on* $\mathcal{M}(B)_\infty$, *does there exist a real-valued process* $(C_t)_{t \geqq 0}$, *such that:*

(a) $\forall t$, $\mathcal{M}(B)_t = \mathcal{M}(C)_t$;

(b) C *is a Brownian motion, with* $C_0 = 0$, *under* Q?

Recall that the density dQ/dP, on $\mathcal{M}(B)_\infty$, may be written as

$$\exp\left\{\int_0^\infty \varphi(s) \, dB_s - \frac{1}{2}\int_0^\infty \varphi_s^2 \, ds\right\}, \tag{9.3}$$

with φ a $(\mathcal{M}(B)_t)$ predictable process.

For bounded φ's, whose support is contained in $[0, T]$, for some $T > 0$, we write (G_b) instead of (G).

The second question originates from filtering theory [look at the end of section (6), to see how this question arises].

(I') *If, on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, $(B_t)_{t \geqq 0}$ *is a real-valued* $(\mathcal{F}_t)$ *Brownian motion, with* $B_0 = 0$, *and* (h_t) *is a uniformly bounded* $(\mathcal{F}_t)$ *predictable process, with support* *contained in* $[0, T]$, *for some* $T>0$, *is the natural filtration of*

$$Y_t \overset{\text{def}}{=} B_t + \int_0^t h_s\, ds \tag{9.4}$$

that of a real-valued Brownian motion $(\beta_t)_{t\geqq 0}$, *with* $\beta_0=0$?

Notation. — The letters G and I are here respectively for "Girsanov" and "Innovation".

Here is the list of implications we have obtained between these questions and (S) for instance

$$\begin{array}{ccc} (S)_+ & \Longrightarrow & (G)_+ \\ \Downarrow & & \Downarrow \\ (I')_+ & \Longleftrightarrow & (G_b)_+ \end{array} \tag{9.5}$$

$(S)_+ \Rightarrow (G)_+$. — From lemma (8.1), the Q-Brownian motion $\left(\tilde{B}_t = B_t - \int_0^t \varphi_s\, ds,\ t\geqq 0\right)$ has the representation property w.r.t. $(\mathcal{F}(B)_t)$.

$(G)_+ \Rightarrow (G_b)_+$, obviously.

We make some remarks on (I'): Let h be the $(\mathcal{M}(Y)_t)$ predictable projection of h. Then, we may write (9.4) as

$$Y_t = \beta_t + \int_0^t \hat{h}_s\, ds, \tag{9.4'}$$

where (β_t) is a $\{\mathcal{M}(Y)_t\}$ Brownian motion.

Under $Q \overset{\text{def}}{=} \exp\left[-\int_0^\infty \hat{h}_s\, d\beta_s - \frac{1}{2}\int_0^\infty (\hat{h}_s)^2\, ds\right].P$, (Y_t) is a $\{\mathcal{M}(Y)_t\}$ Brownian motion, with $Y_0=0$, and so, has the representation property w.r.t. $(\mathcal{M}(Y)_t, Q)$. (β_t) is the Girsanov transform of (Y_t), for the ordered pair (Q, P), and so, from lemma (8.1): $(S)_+ \Rightarrow (I')_+$. From the definition of (G_b), we also have $(G_b)_+ \Rightarrow (I')_+$.

$(I')_+ \Rightarrow (G_b)_+$. — If dQ/dP is given by (9.3), with φ uniformly bounded, and with compact support, we may write, under Q:

$$B_t = \tilde{B}_t + \int_0^t \varphi_s\, ds,$$

which is the equality (9.4), where Y has been changed in B, B in $\tilde{B}$, and $h=\hat{h}$ in φ.

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