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COHOMOLOGY OF LINE BUNDLES ON G/B

BY HENNING HAAHR ANDERSEN

Let G be a connected algebraic group over a field of characteristic $p>0$. Denote by B a Borel subgroup of G. Let χ be a character of B and consider the induced line bundle $L(\chi)$ on G/B. This paper deals with the questions:

(*a*) when is $H^i(G/B, L(\chi)) \neq 0$?

(*b*) what is the structure (dimension, trace, G-composition factors) of $H^i(G/B, L(\chi))$?

An important partial answer to (*a*) is contained in the following Theorem due to G. Kempf [9]:

If $H^0(G/B, L(\chi)) \neq 0$ (i. e. if χ is dominant) then $H^i(G/B, L(\chi))=0$ for $i>0$. (For another proof *see* [2].)

For non-dominant weights, however, very little is known about the vanishing behaviour of the cohomology of $L(\chi)$. In fact the only group for which there has been given a complete answer to (*a*) is SL(3) (W. L. Griffith [5]) but even in this case (*b*) is wide open.

Our main results in this paper are (*see* Section 1 for notation):

(2.3) and (2.9). — If $0 \leqq \langle \alpha^{\vee}, \chi \rangle \leqq p-1$ or $\langle \alpha^{\vee}, \chi \rangle = ap^n - 1$ with $a<p$ and $n \in \mathbf{N}$ then

$$H^i(G/B, L(\chi)) \simeq H^{i+1}(G/B, L(s_\alpha(\chi+\rho)-\rho)).$$

(3.1). — If there exist positive integers a, n with $a<p$ and two distinct simple roots α, β such that $\langle \alpha^{\vee}, \chi \rangle + ap^n + 1 \leqq 0$ and $\langle \beta^{\vee}, \chi \rangle + ap^n \langle \beta^{\vee}, \alpha \rangle + 1 \leqq 0$ then $H^1(G/B, L(\chi))=0$.

(4.5). — For non-dominant characters the condition in 3.1 is also necessary for vanishing of $H^1(G/B, L(\chi))$ when G has semi-simple rank 2.

The proofs of (2.3) and (2.9) are based on M. Demazure's simple proof of Bott's Theorem [4]. As a Corollary we in fact get that Bott's Theorem holds in characteristic p for line bundles induced by "small" characters (2.4) as well as the Steinberg characters (2.10) but we show also that there always exist line bundles having at least 2 non-vanishing cohomology groups (2.7). Using (2.9) together with a couple of Lemmas about the restriction of line bundles to codimension 1 Schubert varieties in G/B we are able to obtain the sufficient condition for vanishing of $H^1(G/B, L(\chi))$ in (3.1). We point out in (3.2) exactly what we need to show in order to get that this condition is also necessary. *Via* a Theorem of C. S. Seshadri [10] we then use this in Section 4 to handle the semi-simple rank 2 case. This

Section contains also some results about the G-module structure of some of the $H^i(G/B, L(\chi))$'s, which though very far from a complete answer to question (*b*) above, may indicate a little about what kind of results to expect. As C. S. Seshadri's work [10] is not generally available we have included an Appendix containing a brief outline of his proof of the above mentioned Theorem.

I would like to thank J. E. Humphreys for some very helpful and stimulating discussions on the problems treated in this paper.

1. Preliminaries

NOTATION. — G will denote a connected reductive algebraic group over a field k. We will assume k is algebraically closed and of positive characteristic p. We fix a maximal torus $T \subset G$ and a Borel subgroup B containing T. R will denote the set of roots of G with respect to T, R_- the set of roots of B and $R_+ = -R_-$. $S \subset R_+$ will be the set of simple roots and W the Weyl group.

When $\alpha \in R$ we let U_α denote the corresponding unipotent subgroup of G and we fix an isomorphism $\theta_\alpha : G_a \to U_\alpha$ satisfying $t\,\theta_\alpha(z)\,t^{-1} = \theta_\alpha(\alpha(t)z)$, $t \in T$, $z \in k$.

The Schubert variety in G/B associated to an element $w \in W$ is defined to be the closure of the cell BwB/B. We denote it X_w.

The character group of T will be denoted X(T). If $\eta \in X(T)$ or more generally if η is any linear representation of B on a vector space E we let $L(\eta)$ [or sometimes $L(E)$] denote the induced locally free sheaf on G/B, i. e. the sheaf whose Sections over an open subset $U \subset G/B$ are the regular functions $\varphi : \pi^{-1}(U) \to E$ satisfying the relation $\varphi(xb) = \eta(b)^{-1}\varphi(x)$, $x \in \pi^{-1}(U)$, $b \in B$. Here π is the canonical morphism $G \to G/B$.

From [4] we recall the following crucial but simple Lemma valid in all characteristics.

LEMMA 1.1. — *Let $\alpha \in S$ and let P_α denote the minimal parabolic subgroup of G having α as only positive root. Let $\eta : B \to GL(E)$ be a linear representation of B and $\mu \in X(T)$. If η extends to P_α and $\langle \alpha^\vee, \mu \rangle = -1$ then*

$$H^i(G/B, L(\eta) \otimes L(\mu)) = 0 \quad \textit{for all } i.$$

In order to apply this Lemma we will need a detailed knowledge of some representations of P_α:

LEMMA 1.2. — *Let $\chi \in X(T)$, $\alpha \in S$ and suppose $r = \langle \alpha^\vee, \chi \rangle \geqq 0$. Then $V_\chi^\alpha = H^0(P_\alpha/B, L(\chi))$ has a basis $v_0, v_1, \ldots, v_r$ with the following properties:*

(*a*) *v_i is a T-semi-invariant of weight $s_\alpha(\chi) + i\alpha$, $i = 0, 1, \ldots, r$;*

(*b*) $\theta_{-\alpha}(z)\,v_j = \sum_{i=0}^{j} \binom{j}{i} z^{j-i} v_i$, $z \in k$, $j = 0, 1, \ldots, r$.

Proof. — Elementary exercise (*see* [1], example 3.6).

Finally we shall need some criteria for vanishing of $H^0(X_w, L(\chi)|_{X_w})$:

LEMMA 1.3. — *Let* f_α: $G/B \to G/P_\alpha$ *denote the canonical morphism. Then* $f_\alpha|_{X_w}$ *is a* P^1*-fibration if and only if* $l(ws_\alpha)=l(w)-1$. [Here $l(w)$ denotes the length of w.]

Proof. — The statement follows from the Bruhat decomposition, *see* [9], Lemma 2.1.

LEMMA 1.4. — If $H^0(X_w, L(\chi)|_{X_w})\neq 0$ *then* $H^0(X_w, L(\chi)|_{X_w})$ *contains a unique* B*-stable line. The weight in question is* $w(\chi)$.

Proof. — The Lemma is well known for $X_w=G/B$ (i. e. for $w=w_0$, the element in W with maximal length), *see* [8], Theorem 8.3. The same proof applies.

COROLLARY 1.5. — *Let* $\alpha\in S$. *Then* $H^0(X_{w_0 s_\alpha}, L(\chi)|_{X_{w_0 s_\alpha}})=0$ *if either:*
(*a*) $\exists\beta\in S-\{\alpha\}: \langle \beta^\vee, \chi\rangle <0$ *or*
(*b*) $\exists\beta\in S-\{\alpha\}: \langle \beta^\vee, s_\alpha(\chi)\rangle <0$.

Proof. — Note that if $\alpha, \beta\in S$, $\alpha\neq\beta$ then

$$l(w_0 s_\alpha)=l(w_0)-1 \qquad \text{and} \qquad l(w_0 s_\alpha s_\beta)=l(w_0)-2.$$

Hence by Lemma 1.3 $f_\beta|_{X_{w_0 s_\alpha}}$ is a $\mathbf{P}^1$-fibration. The fibers are isomorphic to P_β/B and if $\langle \beta^\vee, \chi\rangle <0$ we have $H^0(P_\beta/B, L(\chi))=0$. Hence $f_{\beta *}L(\chi)=0$ and the Leray spectral sequence

$$H^p(f_\beta(X_{w_0 s_\alpha}), R^q f_{\beta *}L(\chi)) \Rightarrow H^{p+q}(X_{w_0 s_\alpha}, L(\chi)),$$

shows then that $H^0(X_{w_0 s_\alpha}, L(\chi))=0$. We are done if χ satisfies condition (*a*). Suppose now that χ satisfies condition (*b*) and assume $H^0(X_{w_0 s_\alpha}, L(\chi))\neq 0$. Set $\gamma=-w_0(\beta)$. Then $l(s_\gamma w_0 s_\alpha)=l(w_0 s_\beta s_\alpha)=l(w_0)-2$. Hence the closure of $B_{w_0 s_\alpha}B$ is P_γ-stable (under multiplication on the left) [3], Proposition 1.4. Hence P_γ acts on $H^0(X_{w_0 s_\alpha}, L(\chi))$. As $w_0 s_\alpha(\chi)$ is a T-weight here so is therefore $s_\gamma w_0 s_\alpha(\chi)$. But

$$s_\gamma w_0 s_\alpha(\chi)=w_0 s_\alpha(\chi)-\langle \gamma^\vee, w_0 s_\alpha(\chi)\rangle\gamma=w_0 s_\alpha(\chi)+\langle \beta^\vee, s_\alpha(\chi)\rangle\gamma.$$

Now in general if $B\to GL(E)$ is any linear representation of B and $e\in E$ is a T-semi-invariant whose corresponding weight is maximal among the T-weights of E [with respect to the order of X(T) induced by B] then the line generated by e is B-stable. But by Lemma 1.4 $H^0(X_{w_0 s_\alpha}, L(\chi))$ has only one B-stable line and the weight in question is $w_0 s_\alpha(\chi)$. Therefore $w_0 s_\alpha(\chi)+\langle \beta^\vee, s_\alpha(\chi)\rangle\gamma$ cannot be bigger than $w_0 s_\alpha(\chi)$. This contradicts condition (*b*).

2. Line bundles induced by small characters

Let $\alpha\in S$, $\chi\in X(T)$ and suppose $r=\langle\alpha^\vee, \chi\rangle\geqq 0$. From Lemma 1.2 we see that we have the following sequences of B-modules

$$(2.1) \qquad \begin{cases} 0\to K_\chi^\alpha\to V_\chi^\alpha\to k_\chi\to 0, \\ 0\to k_{s_\alpha(\chi)}\to K_\chi^\alpha\to \overline{V}_\chi^\alpha\to 0, \end{cases}$$

where k_χ denotes the 1-dimensional B-representation $\chi: B \to GL(k)$ and where $\overline{\overline{V}}{}^\alpha_\chi$ has a basis $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{r-1}\}$ with the properties:

(i) $\bar{v}_i$ is a T-semi-invariant of weight $s_\alpha(\chi)+i\alpha$;

(ii) $\theta_{-\alpha}(z)\bar{v}_j = \sum_{i=1}^{j} \binom{j}{i} z^{j-i}\bar{v}_i,\ z\in k, j=1, 2, \ldots, r-1.$

Let $H^\alpha_\chi: \overline{\overline{V}}{}^\alpha_\chi \to V^\alpha_{\chi-\alpha}$ denote the map that takes $\bar{v}_i$ into iv_{i-1}. It is easy to see that this is a B-equivariant map. The kernel $\overline{\overline{K}}{}^\alpha_\chi$ of H^α_χ has a basis $\{e_1, e_2, \ldots, e_a\}$,

$$a=\max\{n \mid np < \{\alpha^\vee, \chi\rangle\}$$

with the properties:

(i) e_i is a T-semi-invariant of weight $s_\alpha(\chi)+pi\alpha$;

(ii) $\theta_{-\alpha}(z)e_j = \sum_{i=1}^{j} \binom{j}{i} z^{p(j-i)} e_i,\ z\in k.$

Similarly the cokernel Q^α_χ of H^α_χ has a basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_a\}$ with the properties:

(i) $\bar{e}_i$ is a T-semi-invariant of weight $s_\alpha(\chi)+pi\alpha$.

(ii) $\theta_{-\alpha}(z)\bar{e}_j = \sum_{i=1}^{j} \binom{pj-1}{pi-1} z^{p(j-i)}\bar{e}_i,\ z\in k.$

We derive the following properties of these B-representations when $\langle \alpha^\vee, \chi)$ is "small":

If $0 < \langle \alpha^\vee, \chi\rangle \leqq p$ then $\overline{\overline{V}}{}^\alpha_\chi \simeq V_{\chi-\alpha}$ (*via* H_χ).

If $ap < \langle \alpha^\vee, \chi\rangle \leqq (a+1)p$ for some $a<p$ then

$$\overline{\overline{K}}{}^\alpha_\chi \simeq Q^\alpha_\chi \simeq (V^\alpha_{(a-1)\omega_\alpha})^{(p)} \otimes k_{s_\alpha(\chi-p(\alpha+(a-1)\omega_\alpha))}, \tag{2.2}$$

where ω_α is the fundamental weight corresponding to α.

[When E is a B-representation we denote by $E^{(p)}$ the same representation raised to the p'th power.]

Let now ρ = half the sum of the positive roots ($=\sum_{\alpha\in S}\omega_\alpha$). Combining (2.1), (2.2) and Lemma 1.1 we get.

THEOREM 2.3. – *Let* $\alpha\in S$:

(i) *If* $\langle \alpha^\vee, \chi+\rho\rangle \geqq 0$ *then*

$$H^i(G/B, L(\chi)) \simeq H^{i+1}(G/B, L(K^\alpha_{\chi+\rho}) \otimes L(-\rho));$$

(ii) *if* $0 \leqq \langle \alpha^\vee, \chi+\rho\rangle \leqq p$ *then*

$$H^i(G/B, L(\chi)) \simeq H^{i+1}(G/B, L(s_\alpha(\chi+\rho)-\rho));$$

(iii) *if* $ap < \langle \alpha^\vee, \chi+\rho\rangle \leqq (a+1)p$ *for some* $1 \leqq a < p$ *then we have two long exact sequences:*

$$(a)\quad \ldots \to H^{i+1}(G/B, L(s_\alpha(\chi+\rho)-\rho)) \to H^i(G/B, L(\chi)) \to H^{i+1}(G/B, L(\overline{\overline{V}}{}^\alpha_{\chi+\rho}) \otimes L(-\rho)) \to \ldots$$

(b) $\ldots \to H^{i+1}(G/B, L((V^{\alpha}_{(a-1)\omega_\alpha})^{(p)})$

$$\otimes L(s_\alpha(\chi+\rho-p(\alpha+(a-1)\omega_\alpha))-\rho)) \to H^{i+1}(G/B, L(\overline{V}^{\alpha}_{\chi+\rho})$$

$$\otimes L(-\rho)) \to H^i(G/B, L((V^{\alpha}_{(a-1)\omega_\alpha})^{(p)}) \otimes L(s_\alpha(\chi+\rho)-p(\alpha+(a-1)\omega_\alpha)-\rho)) \to \ldots$$

COROLLARY 2.4. — (i) *if* $\langle \alpha^\vee, \chi+\rho \rangle \geqq 0$ *and* $H^{i+1}(G/B, L(s_\alpha(\chi+\rho)-\rho+n\alpha))=0$ *for* $n=0, 1, \ldots, \langle \alpha^\vee, \chi \rangle$ *then* $H^i(G/B, L(\chi))=0$.

(ii) *if* $w \in W$ *and* $0 \leqq \langle \alpha^\vee, \chi+\rho \rangle \leqq p$ *for all* $\alpha \in R_+ \cap w^{-1} R_-$ *then*

$$H^i(G/B, L(\chi)) \simeq H^{i+l(w)}(G/B, L(w(\chi+\rho)-\rho));$$

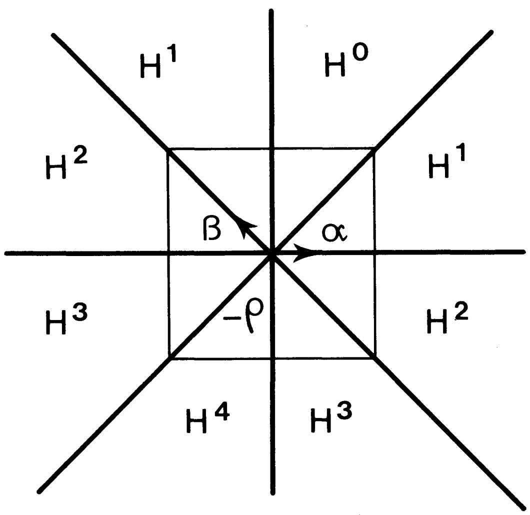
(iii) *let* A_0 *denote the interior of the bottom alcove in the dominant chamber. If* $w \in W$ *and* $\chi+\rho \in w^{-1}(A_0)$ *then*

$$H^i(G/B, L(\chi)) = \begin{cases} 0 & \textit{for} \quad i \neq l(w), \\ H^0(G/B, L(w(\chi+\rho)-\rho)) & \textit{for} \quad i=l(w). \end{cases}$$

Proof. — (i) follows from Theorem 2.3 (i) by considering the exact sequences arising when one takes a B-filtration of $K^{\alpha}_{\chi+\rho}$ with 1-dimensional factors. (ii) and (iii) follow by repeated use of Theorem 2.3 (ii).

Let us illustrate the results in Corollary 2.4 by an.

Example 2.5. — Let G be of type B_2. Then X(T) has rank 2. In the Figure below we have named the Weyl chambers in X(T) H^0, H^1, H^2, H^3 and H^4 in such a way that if the characteristic is zero then $\chi \in H^i$ if and only if $H^j(G/B, L(\chi))=0$ for $j \neq i$, $i=0, 1, \ldots, 4$.



By 2.4 (i) it is easy to see that if $\chi \in H^1$ then $H^3(G/B, L(\chi))=0$ (compare observation below) and by Serre duality we therefore get that $H^1(G/B, L(\chi))=0$ for $\chi \in H^3$. On the

Figure we have also indicated the bottom alcoves. By 2.4 (iii) the vanishing behaviour for line bundles induced by characters from these alcoves is as in characteristic 0.

Set $N=\dim G/B$. It is easy to see that the canonical sheaf on G/B is $L(-2\rho)$ and hence $H^N(G/B, L(\chi))\simeq H^0(G/B, L(-\chi-2\rho))$. As $H^0(G/B, L(\chi))\neq 0$ if and only if χ is dominant (i.e. $\langle \alpha^\vee, \chi\rangle \geqq 0$ for all $\alpha\in S$) we find $H^N(G/B, L(\chi))\neq 0$ if and only if $\langle \alpha^\vee, \chi\rangle \leqq -2$ for all $\alpha\in S$.

Suppose now $\chi\in X(T)$ satisfies $p\leqq\langle \alpha^\vee, \chi\rangle\leqq 2p-2$ for some $\alpha\in S$ and suppose there exists $\beta\in S-\{\alpha\}$ such that $\langle \beta^\vee, s_\alpha(\chi+\rho)-\rho\rangle > -2$. By the above observation $H^N(G/B, L(s_\alpha(\chi+\rho)-\rho))=0$ and from Theorem 2.3 (iii) we get

$$H^{N-1}(G/B, L(\chi))\simeq H^N(G/B, L(\overline{V}^\alpha_{\chi+\rho})\otimes L(-\rho))$$

and

$$\ldots \to H^N(G/B, L(s_\alpha(\chi+\rho)-\rho+p\alpha)) \to H^N(G/B, L(\overline{V}^\alpha_{\chi+\rho}) \otimes L(-\rho)) \to H^{N-1}(G/B, L(s_\alpha(\chi+\rho)-\rho+p\alpha)) \to 0.$$

Now $\langle \alpha^\vee, s_\alpha(\chi+\rho)+p\alpha\rangle = -\langle \alpha^\vee, \chi\rangle -1+2p$ which by assumption lies between 0 and p. Hence by Theorem 2.3 (ii) we have

$$H^i(G/B, L(s_\alpha(\chi+\rho)-\rho+p\alpha))\simeq H^{i+1}(G/B, L(\chi-p\alpha))$$

and we find by inserting this above

$$(2.6) \qquad H^{N-1}(G/B, L(\chi)\simeq H^N(G/B, L(\chi-p\alpha)).$$

If therefore $-\chi+p\alpha-2\rho$ is dominant we can conclude that $H^{N-1}(G/B, L(\chi))\neq 0$. But we assumed above that $s_\alpha(\chi+\rho)-\rho$ does not belong to the Weyl chamber which in characteristic zero contains the characters whose line bundles have $H^{N-1}\neq 0$. If χ therefore is non-singular and satisfies the above conditions then $L(\chi)$ will have at least two non-vanishing cohomology groups. To be precise what we have proved is the following.

COROLLARY 2.7. — *If χ satisfies the conditions:*

(a) there exist two distinct simple roots α, β such that $p\leqq\langle \alpha^\vee, \chi\rangle\leqq 2p-2$ and $\langle \beta^\vee, s_\alpha(\chi+\rho)-\rho\rangle \geqq -1$;

(b) $\langle \gamma^\vee, \chi-p\alpha\rangle\leqq -2$ for all $\gamma\in S$;

then $H^{N-2}(G/B, L(\chi))$ and $H^{N-1}(G/B, L(\chi))$ are both $\neq 0$.

Remark 2.8. — From the Dynkin diagrams of the various root systems one can easily see that the conditions in Corollary 2.7 will always be satisfied for some $\chi\in X(T)$ unless G is of type A_1. In all other cases there will therefore exist line bundles on G/B with at least 2 (in case G_2 at least 3) non-vanishing cohomology groups.

Let now S^α_χ denote the unique simple submodule of V^α_χ. With notation as in Lemma 1.2 we have $S^\alpha_\chi=\operatorname{span}\left\{ v_i \,\middle|\, \binom{r}{i}\neq 0\right\}$. Suppose now that $\langle \alpha^\vee, \chi\rangle = ap^n-1$ for some $a<p$,

$n \in \mathbf{N}$. Then $S^{\alpha}_{\chi+\rho} = \operatorname{span}\{v_0, v_{p^n}, \ldots, v_{ap^n}\}$ and it is easy to see that we have exact sequences

$$0 \to c^{\alpha}_{\chi+\rho} \to S^{\alpha}_{\chi+\rho} \to k_{\chi+\rho} \to 0,$$
$$0 \to k_{s_\alpha(\chi+\rho)} \to c^{\alpha}_{\chi+\rho} \to S^{\alpha}_{\chi+\rho-p^n\alpha} \to 0.$$

From the corresponding exact sequences of locally free sheaves we get by tensoring with $L(-\rho)$ and using Lemma 1.1:

THEOREM 2.9. — *If* $\langle \alpha^{\vee}, \chi \rangle = ap^n - 1$ *with* $a < p$ *then*

$$H^i(G/B, L(\chi)) \simeq H^{i+1}(G/B, L(s_\alpha(\chi+\rho)-\rho)).$$

COROLLARY 2.10. — *Let* $q = p^n$ *and set* $\chi_q = (q-1)\rho$. *If* $p > \langle \alpha^{\vee}, \rho \rangle$ *for all* $\alpha \in R_+$ *then*

$$H^i(G/B, L(w(\chi_q+\rho)-\rho)) = \begin{cases} 0 & \text{for } i \neq l(w), \\ H^0(G/B, L(\chi_q)) & \text{for } i = l(w) \end{cases}$$

for all $w \in W$.

Proof. — Apply Theorem 2.9.

Remark. — The χ_q's are known as the Steinberg weights and $H^0(G/B, L(\chi_q))$ as the Steinberg modules. It is a fact that the Steinberg modules are irreducible [6] and so are therefore $H^{l(w)}(G/B, L(w(\chi_q+\rho)-\rho))$ under the assumption of Corollary 2.10.

3. Vanishing of H^1

THEOREM 3.1. — *Let* $\chi \in X(T)$ *and suppose there exist* $\alpha, \beta \in S$, $\alpha \neq \beta$, $a, n \in \mathbf{N}$ *with* $a < p$ *such that*

$$\langle \alpha^{\vee}, \chi \rangle + ap^n + 1 \leqq 0 \quad \text{and} \quad \langle \beta^{\vee}, \chi \rangle + a\langle \beta^{\vee}, \alpha \rangle p^n + 1 \leqq 0.$$

Then $H^1(G/B, L(\chi)) = 0$.

Proof. — We first note that if α is any simple root then the divisor $X_{w_0 s_\alpha}$ in G/B is defined by the invertible sheaf $L(\omega_\alpha)$, i.e. we have the exact sequence

$$0 \to L(-\omega_\alpha) \to 0_{G/B} \to 0_{X_{w_0 s_\alpha}} \to 0.$$

Tensoring this sequence with $L(\lambda)$ and looking at the associated cohomology sequence we see that the induced map $H^1(G/B, L(\lambda-\omega_\alpha)) \to H^1(G/B, L(\lambda)$ is injective if $H^0(X_{w_0 s_\alpha}, L(\lambda)) = 0$. To prove the Theorem it will therefore be enough to find two non-negative integers r and s with the properties:

(i) $$H^1(G/B, L(\chi + r\omega_\alpha + s\omega_\beta)) = 0$$

and

(ii) $$H^0(X_{w_0 s_\alpha}, L(\chi + n\omega_\alpha + m\omega_\beta)) = H^0(X_{w_0 s_\beta}, L(\chi + n\omega_\alpha + m\omega_\beta)) = 0$$

for all (n, m) with $0 \leqq n \leqq r$ and $0 \leqq m \leqq s$.

We claim that

$$(r, s) = (-ap^n - \langle \alpha^\vee, \chi \rangle - 1, \ -a \langle \beta^\vee, \alpha \rangle p^n - \langle \beta^\vee, \chi \rangle - 1)$$

will do the job: Choose $\chi_1 \in X(T)$ such that

$$s_\alpha(\chi_1 + \rho) - \rho = \chi + r\omega_\alpha + s\omega_\beta.$$

Easy computations show that

$$\langle \alpha^\vee, \chi_1 \rangle = ap^n - 1 \qquad \text{and} \qquad \langle \beta^\vee, \chi_1 \rangle = -1.$$

By Theorem 2.9 we have $H^1(G/B, L(\chi + r\omega_\alpha + s\omega_\beta)) \simeq H^0(G/B, L(\chi_1))$ and the latter is zero by Lemma 1.1. Hence (r, s) satisfies (i). To see that (ii) is also satisfied note first that when $n \leqq r$ we have

$$\langle \alpha^\vee, \chi + n\omega_\alpha + m\omega_\beta \rangle = \langle \alpha^\vee, \chi \rangle + n \leqq \langle \alpha^\vee, \chi \rangle + r = -ap^n - 1 < 0.$$

According to Corollary 1.5 $H^0(X_{w_0 s_\beta}, L(\chi + n\omega_\alpha + m\omega_\beta))$ therefore vanishes. The same Corollary shows that we are done if we also show that

$$\langle \beta^\vee, s_\alpha(\chi + n\omega_\alpha + m\omega_\beta) \rangle < 0 \qquad \text{for} \quad 0 \leqq n \leqq r, \quad 0 \leqq m \leqq s.$$

But

$$s_\alpha(\chi + n\omega_\alpha + m\omega_\beta) = \chi + n\omega_\alpha + m\omega_\beta - (\langle \alpha^\vee, \chi \rangle + n)\alpha$$

and so

$$\begin{aligned} \langle \beta^\vee, s_\alpha(\chi + n\omega_\alpha + m\omega_\beta) \rangle &= \langle \beta^\vee, \chi \rangle + m - (\langle \alpha^\vee, \chi \rangle + n) \langle \beta^\vee, \alpha \rangle \\ &\leqq \langle \beta^\vee, \chi \rangle + s - (\langle \alpha^\vee, \chi \rangle + r) \langle \beta^\vee, \alpha \rangle \\ &= -a \langle \beta^\vee, \alpha \rangle p^n - 1 + (ap^n + 1) \langle \beta^\vee, \alpha \rangle = -1 + \langle \beta^\vee, \alpha \rangle < 0. \end{aligned}$$

Remark 3.2. — If χ is dominant then all the higher cohomology groups of $L(\chi)$ are zero ([9], [2]). It seems very likely that for non-dominant weights the condition in Theorem 3.1 is also necessary for vanishing of H^1. In fact this will be true if we have non-vanishing at the edge points $-(a+1)p^n\omega_\alpha + \sum_{\beta \neq \alpha} a \langle -\beta^\vee, \alpha \rangle p^n \omega_\beta$: Let namely χ be a non-dominant weight and choose $\alpha \in S$ such that $\langle \alpha^\vee, \chi \rangle < 0$. Let $a < p$, $n \in \mathbf{N}$ be determined by $-(a+1)p^n \leqq \langle \alpha^\vee, \chi \rangle \leqq -ap^n - 1$. If we assume that the condition in Theorem 3.1 is not satisfied then $\langle \beta^\vee, \chi \rangle \geqq -a \langle \beta^\vee, \alpha \rangle p^n$ for all $\beta \in S - \{\alpha\}$. We want to show $H^1(G/B, L(\chi)) \neq 0$. This follows by semi-continuity if χ belongs to one of the chambers where $H^1 \neq 0$ in characteristic zero, i. e. if there exists $\gamma \in S$: $s_\gamma(\chi + \rho) - \rho$ is dominant. So assume that χ does not belong to any of these chambers. We claim that then $H^0(X_{w_0 s_\gamma}, L(\sum_\beta n_\beta \omega_\beta)) = 0$ for all (n_β) with $-(a+1)p^n \leqq n_\alpha \leqq \langle \alpha^\vee, \chi \rangle$ and $-a \langle \beta^\vee, \alpha \rangle p^n \leqq n_\beta \leqq \langle \beta^\vee, \chi \rangle$: $\beta \neq \alpha$. For $\gamma \neq \alpha$ this follows *via* Corollary 1.5 from the fact $\langle \alpha^\vee, \sum n_\beta \omega_\beta \rangle < 0$, and for $\gamma = \alpha$ it follows from the assumption that $s_\beta(\chi + \rho) - \rho$ is not dominant. This assumption implies namely that $\langle \alpha^\vee, s_\beta(\chi + \rho) - \rho \rangle < 0$ and hence

$$\begin{aligned} \langle \alpha^\vee, s_\beta(\sum_\gamma n_\gamma \omega_\gamma) \rangle &= \langle \alpha^\vee, s_\beta(\chi + \rho + (\sum_\gamma n_\gamma \omega_\gamma - \chi - \rho) \rangle \\ &= \langle \alpha^\vee, s_\beta(\chi + \rho) - \rho \rangle + 1 + \langle \alpha^\vee, \sum n_\gamma \omega_\gamma - \chi - \rho - (n_\beta - \langle \alpha^\vee, \chi \rangle - 1)\beta \rangle \\ &< 1 + n_\alpha - \langle \alpha^\vee, \chi \rangle - 1 - (n_\beta - \langle \alpha^\vee, \chi \rangle - 1) \langle \alpha^\vee, \beta \rangle \leqq 0. \end{aligned}$$

As in the proof of Theorem 3.1 we conclude that

$$H^1(G/B, L(\sum_\beta n_\beta \omega_\beta - \omega_\gamma)) \to H^1(G/B, L(\sum_\beta n_\beta \omega_\beta))$$

is injective for all $\gamma \in S$. Repeated use of this shows that

$$H^1(G/B, L(-(a+1)p^n \omega_\alpha - \sum a \langle \beta^\check{}, \alpha \rangle p^n \omega_\beta))$$

is injected into $H^1(G/B, L(\chi))$.

4. Semi-simple rank 2

We first recall a Theorem due to Seshadri (valid without any assumption on the semi-simple rank).

THEOREM 4.1 (C. S. Seshadri [10]). — *There exists a locally free sheaf* M *on* G/B *with the properties:*

(*a*) *for any vector bundle* V *on* G/B *there is a long exact sequence*

$$\ldots \to H^i(G/B, V) \overset{F}{\to} H^i(G/B, V^{(p)}) \to H^i(G/B, V \otimes M) \to \ldots$$

(*b*) $M^{(p)} \simeq L(Q)$ *where* Q *is a* B-*representation whose set of* T-*weights is*

$$\{-\sum_{\alpha \in R_+} n_\alpha \alpha \,|\, 0 \leqq n_\alpha \leqq p-1\} - \{0\}.$$

(*For construction and further details about* M see *the Appendix.*)

We shall need the following easy consequence of Seshadri's Theorem.

COROLLARY 4.2. — *Let* $\chi \in X(T)$:
(*a*) *the Frobenius* $H^1(G/B, L(\chi)) \to H^1(G/B, L(p\chi))$ *is injective if*

$$H^0(G/B, L(Q) \otimes L(p\chi)) = 0;$$

(*b*) $H^0(G/B, L(Q) \otimes L(p\chi)) = 0$ *if the weights* $p\chi - \sum_{\alpha \in R_+} n_\alpha \alpha$, $0 \leqq n_\alpha \leqq p-1$ *all are non-dominant.*

Proof. — By Theorem 4.1 (*a*) $H^1(G/B, L(\chi)) \overset{F}{\to} H^1(G/B, L(p\chi))$ is injective if $H^0(G/B, M \otimes L(\chi)) = 0$. Applying Theorem 4.1 (*a*) with $V = M \otimes L(\chi)$ we get (*a*) while (*b*) follows by taking a full filtration of Q.

Remark 4.3. — With notation as in Remark 3.2 observe that if $a \in \mathbf{N}$ then

$$s_\alpha(-(a+1)\omega_\alpha - \sum_{\beta \neq \alpha} a \langle \beta^\vee, \alpha \rangle \omega_\beta + \rho) - \rho$$

is dominant, i.e. $-(a+1)\omega_\alpha - \sum_{\beta \neq \alpha} a \langle \beta^\vee, \alpha \rangle \omega_\beta$ belongs to the H^1-chamber. By semi-continuity $H^1(G/B, L(-(a+1)\omega_\alpha - \sum_{\beta\neq\alpha} a\langle\beta^\vee, \alpha\rangle\omega_\beta))\neq 0$. One way of showing non-vanishing at the edge points mentioned in Remark 3.2 would therefore be to show that the Frobenius

$$H^1(G/B, L(-(a+1)\omega_\alpha - \sum_{\beta\neq\alpha} a\langle\beta^\vee, \alpha\rangle\omega_\beta)) \xrightarrow{F^n} H^1(G/B, L(-(a+1)p^n\omega_\alpha - \sum_{\beta\neq\alpha} a\langle\beta^\vee, \alpha\rangle p^n\omega_\beta))$$

is injective. By Corollary 4.2 a sufficient condition for this is:

$$(4.4)\quad \left\{\begin{array}{l} \sum_{\alpha\in R_+} n_\alpha\alpha-(a+1)p^n\omega_\alpha - \sum_{\beta\neq\alpha} a\langle\beta^\vee, \alpha\rangle p^n\omega_\beta \\ \text{is non-dominant for all } n_\alpha,\ a,\ n \text{ with} \\ 0\leqq n_\alpha\leqq p-1, \quad 0\leqq a<p \quad \text{and} \quad n\geqq 0. \end{array}\right.$$

Unfortunately (4.4) does not hold for all groups and so this method is not fine enough for proving that the condition stated in Theorem 3.1 is both necessary and sufficient for vanishing of H^1. However we shall now see that it does work for groups of semi-simple rank 2:

THEOREM 4.5. — (i) *Let* G *be of type* A_2 *and denote by* α *and* β *the two simple roots. If* $\chi\in X(T)$ *is non-dominant and does not belong to any of the* H^1*-chambers then* $H^1(G/B, L(\chi))\neq 0$ *if and only if there exist* $a, n\in\mathbb{N}$ *with* $a<p$ *such that either* $\langle\alpha^\vee, \chi\rangle\geqq -(a+1)p^n$ *and* $\langle\beta^\vee, \chi\rangle\geqq ap^n$ *or* $\langle\beta^\vee, \chi\rangle\geqq -(a+1)p^n$ *and* $\langle\alpha^\vee, \chi\rangle\geqq ap^n$;

(ii) *Let* G *be of type* B_2 *and denote by* α *and* β *the two simple roots with* $\langle\alpha^\vee, \beta$ *the two simple roots with* $\langle\alpha^\vee, \beta\rangle=-2$. *If* $\chi\in X(T)$ *is non-dominant and does not belong to any of the* H^1*-chambers then* $H^1(G/B, L(\chi))\neq 0$ *if and only if there exist* $a, n\in\mathbb{N}$ *with* $a<p$ *such that*

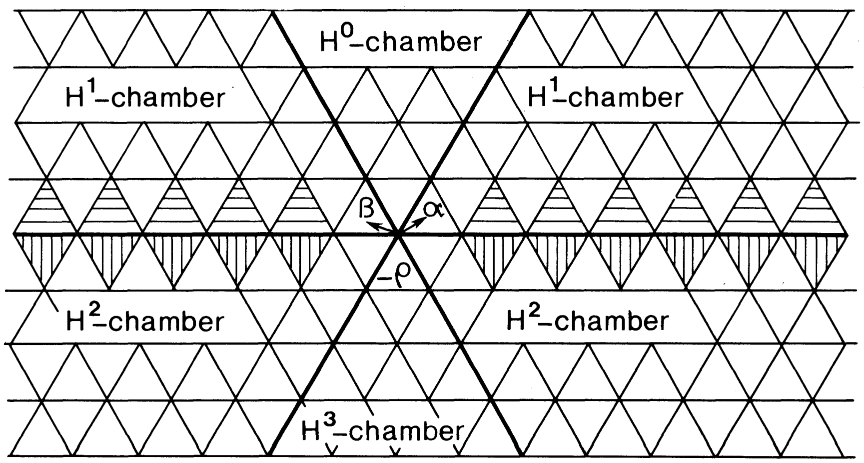


Fig. 1. — Type A_2.

either $\langle \alpha^\vee, \chi \rangle \geqq -(a+1)p^n$ *and* $\langle \beta^\vee, \chi \rangle \geqq ap^n$ *or* $\langle \beta^\vee, \chi \rangle \geqq -(a+1)p^n$ *and* $\langle \alpha^\vee, \chi \rangle \geqq 2ap^n$;

(iii) *Let* G *be of type* G_2 *and denote by* α *and* β *the* 2 *simple roots with* $\langle \alpha^\vee, \beta \rangle = -3$. *If* $\chi \in X(T)$ *is non-dominant and does not belong to any of the* H^1*-chambers then* $H^1(G/B, L(\chi)) \neq 0$ *if and only if there exist* $a, n \in \mathbb{N}$ *with* $a < p$ *such that either* $\langle \alpha^\vee, \chi \rangle \geqq -(a+1)p^n$ *and* $\langle \beta^\vee, \chi \rangle \geqq ap^n$ *or* $\langle \beta^\vee, \chi \rangle \geqq -(a+1)p^n$ *and* $\langle \alpha^\vee, \chi \rangle \geqq 3ap^n$.

Remark 4.6. — *Via* Serre-duality we obtain corresponding necessary and sufficient conditions for vanishing of $H^{N-1}(G/B, L(\chi))$, $N = \dim G/B$. The exact formulation of these conditions is left to the reader. On Figure 1-3 below we have illustrated the results of 4.5 by shading the alcoves with the property that if χ belongs to the interior of such an alcove then $H^1(G/B, L(\chi)) \neq 0$ [resp. $H^{N-1}(G/B, L(\chi)) \neq 0$] in positive characteristic but $=0$ in characteristic zero.

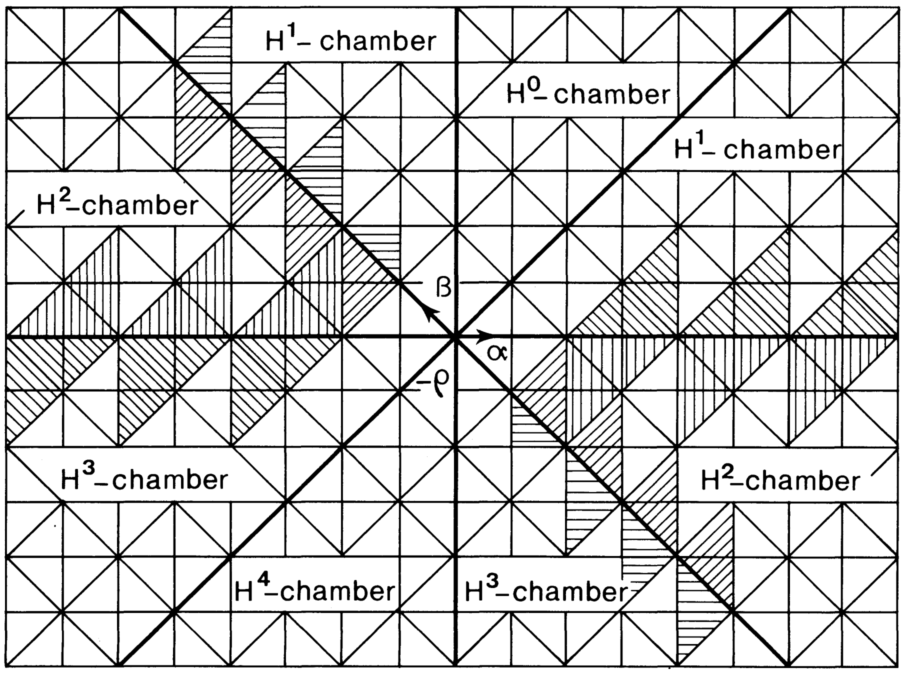


Fig. 2. — Type B_2.
[On this figure we have also indicated the expected vanishing behaviour of H^2.]

Proof of Theorem 4.5. — Theorem 3.1 shows that the conditions are necessary and by Remark 4.3 we are done if we show that (4.4) holds in each of the 3 cases. For type A_2 this was already done by Seshadri in [10]. The method is the same in the other cases. Let us treat the first of the two type G_2-cases and leave the others to the reader: For type G_2 we have $R_+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$ and some easy computations show that

$$n_1\alpha + n_2\beta + n_3(\alpha+\beta) + n_4(2\alpha+\beta) + n_5(3\alpha+\beta) + n_6(3\alpha+2\beta)$$
$$= (2n_1 - 3n_2 - n_3 + n_4 + 3n_5)\,\omega_\alpha + (-n_1 + 2n_2 + n_3 - n_5 + n_6)\,\omega_\beta.$$

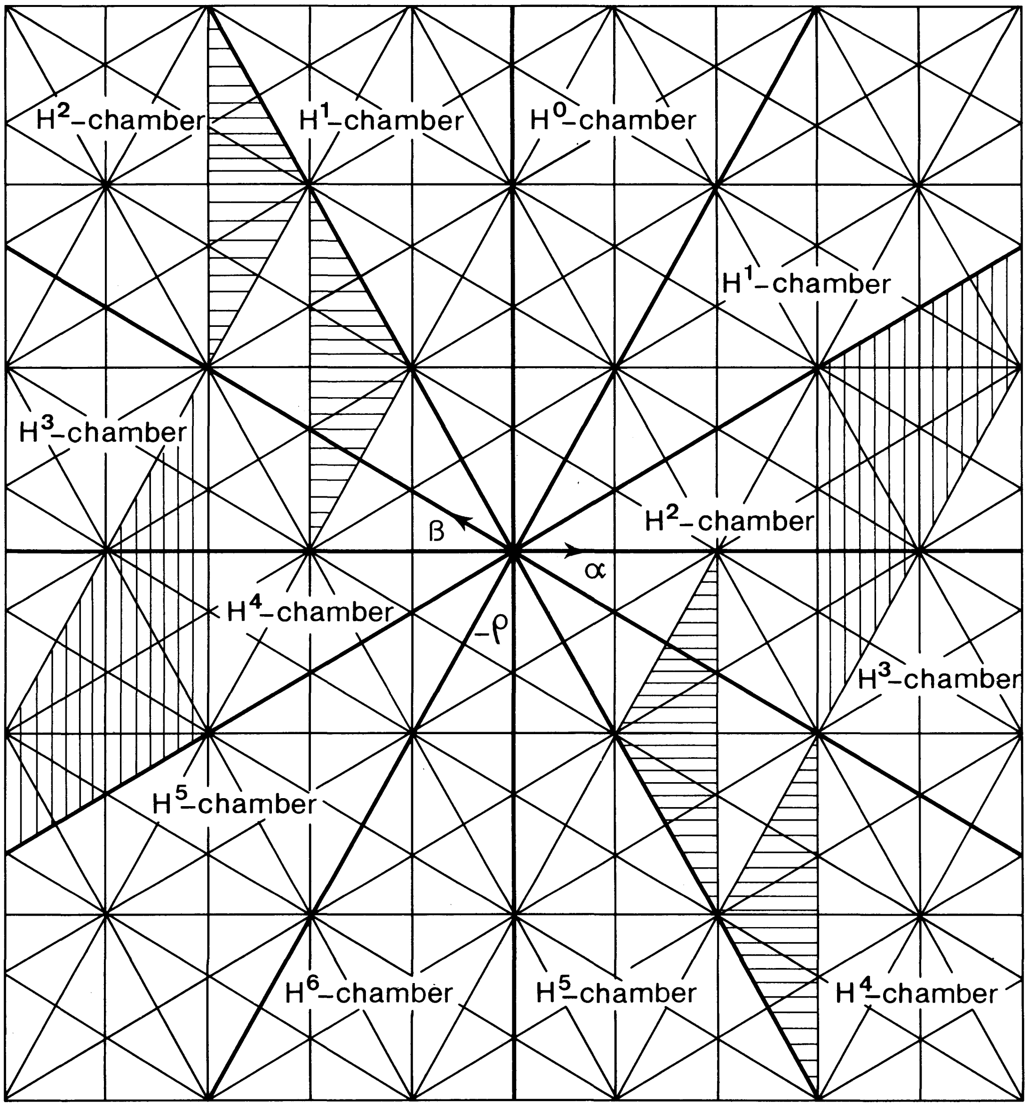


Fig. 3. — Type G_2.

What we have to check is therefore that the inequalities

$$-(2n_1-3n_2-n_3+n_4+3n_5)-(a+1)pn \geqq 0,$$
$$-(-n_1+2n_2+n_3-n_5+n_6)+ap^n \geqq 0,$$

cannot both be satisfied if $0 \leqq n_i \leqq p-1$, $i=1, 2, \ldots, 6$. Assume that they are satisfied. From the second we get $2n_2+n_3 \leqq n_1+n_5-n_6+ap^n$ and by inserting in the first we conclude that the left hand side of this is $\leqq -n_1+n_2-n_4-2n_5-n_6-p^n \leqq n_2-p^n<0$. We have reached a contradiction.

Remark. — The interested reader may check that the above method can be used to prove that condition (4.4) is satisfied also for all groups of semi-simple rank 3 but *not* e. g. for groups of type A_n if $n \geqq 6$!

We will now take a closer look at some of the $H^1(G/B, L(\chi))$'s which "are zero in characteristic 0 but non-zero in characteristic p".

Let G be of type A_2. Suppose $\chi \in X(T)$ is such that $ap \leqq \langle \alpha^\vee, \chi \rangle \leqq (a+1)p-2$ and $-(a+1)p \leqq \langle \beta^\vee, \chi \rangle \leqq -\langle \alpha^\vee, \chi \rangle - 2$ for some $a<p$ [i. e. χ belongs to one of the bottom p^2-alcoves in H^2 where $H^1(G/B, L(\chi)) \neq 0$, Theorem 4.5 (i)]. Then

PROPOSITION 4.7. — *When* χ *is as above* $H^1(G/B, L(\chi))$ *is irreducible with highest weight* $\chi + ap\beta$. (*Here highest weight is with respect to* $\overline{B}$ = *the opposite Borel-subgroup of* B = *the Borel subgroup corresponding to the set of positive roots.*)

Proof. — For $a=1$ this follows *via* Serre-duality from (2.6) and the fact that if λ belongs to the bottom alcove in the dominant chamber then $H^0(G/B, L(\lambda))$ is irreducible ([6], 4.1). We now use induction on a: Set $\lambda = -\chi - 2\rho$. Note first that $s_\beta(\lambda+\rho)-\rho$ does not belong to the H^3-chamber, and that $\langle \beta^\vee, s_\beta(\lambda+\rho-p(\beta+(a-1)\omega_\beta))-\rho \rangle \geqq 0$. Hence

$$H^3(G/B, L(s_\beta(\lambda+\rho)-\rho))=0$$

and

$$H^1(P_\beta/B, L(V^\beta_{(a-1)\omega_\beta})^{(p)} \otimes L(s_\beta(\lambda+\rho-p(\beta+(a-1)\omega_\beta))-\rho))=0.$$

[$L(V^\beta_{(a-1)\omega_\beta})$ is constant on P_β/B.] From the Leray spectral sequence relative to the canonical morphism $G/B \to G/P_\beta$ we conclude that also

$$H^3(G/B, L(V^\beta_{(a-1)\omega_\beta})^{(p)} \otimes L(s_\beta(\lambda+\rho-p(\beta+(a-1)\omega_\beta))-\rho))=0$$

and Theorem 2.3 (iii) gives therefore

$$\begin{aligned} H^2(G/B, L(\lambda)) &\simeq H^3(G/B, L(\overline{V}^\beta_{\lambda+\rho}) \otimes L(-\rho)) \\ &\simeq H^2(G/B, L(V^\beta_{(a-1)\omega_\beta})^{(p)} \otimes L(s_\beta(\lambda+\rho-p(\beta+(a-1)\omega_\beta))-\rho)). \end{aligned}$$

Let now V_n denote the B-submodule of $(V^\beta_{(a-1)\omega_\beta})^{(p)}$ spanned by the first $(n+1)$-basis vectors (*see* Lemma 1.2). From the exact sequences of locally free sheaves associated to the sequences of B-modules

$$(4.8) \qquad 0 \to V_{n-1} \to V_n \to k_{ps_\beta((a-1)\omega_\beta)+np\beta} \to 0,$$

we get for $n=a-1$ using the above observations

$$\ldots \to H^2(G/B, \mathscr{V}_{a-2}) \to H^2(G/B, L(\lambda)) \to H^2(G/B, L(s_\beta(\lambda+\rho)+ap\beta-\rho)) \to H^3(G/B, \mathscr{V}_{a-2}) \to 0,$$

where we for convenience have set

$$\mathscr{V}_{a-2} = L(V_{a-2}) \otimes L(s_\beta(\lambda+\rho-p(\beta+(a-1)\omega_\beta))-\rho).$$

For any $\mu \in X(T)$ the alternating sum $\sum_{i=0}^{N} (-1)^i \operatorname{Tr} H^i (G/B, L(\mu))$ is independent of the characteristic. Using this fact and observing that the induction hypothesis tells us that the highest weight of $H^1 (G/B, L(s_\beta(\lambda+\rho)+ap\beta-\rho))$ is $s_\beta(\lambda+\rho)+ap\beta-\rho+(a-1)p\alpha$ we find that the highest weight of $H^2 (G/B, L(s_\beta(\lambda+\rho)+ap\beta-\rho))$ is the same as it would be in characteristic zero, namely equal to $s_\beta s_\alpha (s_\beta(\lambda+\rho)+ap\beta)-\rho=w_0(\lambda-ap\beta)-2\rho=-w_0(\chi+ap\beta)$. We claim now:

(i) $-w_0(\chi+ap\beta)$ is not a weight of $H^3(G/B, (\mathscr{V}_{a-2}))$;

(ii) $\dim H^2(G/B, (\mathscr{V}_{a-2}))-\dim H^3(G/B, \mathscr{V}_{a-2})$

$$+\dim H^2(G/B, L(s_\beta(\lambda+\rho)+ap\beta-\rho))=\dim S(-w_0(\chi+ap\beta)).$$

[Here $S(-w_0(\chi+ap\beta))$ denotes the irreducible G-module with highest weight $-w_0(\chi+ap\beta)$.]

Let us first see that this claim implies the proposition: The long exact sequence above together with (i) shows that $-w_0(\chi+ap\beta)$ is a weight of $H^2(G/B, L(\lambda))$ and together with (ii) that $\dim H^2(G/B, L(\lambda))\leqq \dim S(-w_0(\chi+ap\beta))$. Hence

$$H^2 (G/B, L(\lambda))\simeq S(-w_0(\chi+ap\beta)).$$

By Serre-duality $H^2(G/B, L(\lambda))$ is isomorphic to the dual of $H^1(L(\chi))$ and hence $H^1(L(\chi))\simeq S(-w_0(\chi+ap\beta))^*\simeq S(\chi+ap\beta)$.

To prove the claim we will use the filtration of V_{a-2} given by (4.8): (i) follows easily just by looking at the weights that occur. To prove (ii) we first note that the weights $s_\beta(\lambda+\rho)-\rho+kp\beta$, $k=1, 2, \ldots, a-1$ all lie outside the region where H^1 (and H^0) is non-zero. Hence the filtration of $\mathscr{V}_{a-2}$ given by (4.8) shows that $H^1(G/B, (\mathscr{V}_{a-2}))=0$, and we conclude that $\dim H^2(G/B, \mathscr{V}_{a-2})-\dim H^3(G/B, \mathscr{V}_{a-2})=$ the Euler characteristic of $\mathscr{V}_{a-2}=$ the sum of the Euler characteristics of $L(s_\beta(\lambda+\rho)-\rho+kp\beta)$, $k=1, 2, \ldots, a-1$. These we compute by Weyl's character formula: The Euler characteristic of $L(\mu)$ is

$$\prod_{\alpha\in R_+} \langle \alpha^\vee, \mu+\rho\rangle/\langle \alpha^\vee, \rho\rangle.$$

By induction hypothesis $\dim H^2(G/B, L(s_\beta(\lambda+\rho)+ap\beta-\rho))=$ Euler characteristic of

$$L(s_\beta(\lambda+\rho)+ap\beta-\rho)-\dim S(s_\beta(\lambda+\rho)+ap\beta-\rho+(a-1)p\alpha).$$

The dimension of $S(\mu)$ can be computed by Steinberg's twisted tensor product Theorem ([6], 2.1). Explicity we get (setting $r=\langle \alpha^\vee, \lambda\rangle$ and $s=\langle \beta^\vee, \lambda\rangle$) that the Euler characteristic of $L(s_\beta(\lambda+\rho)-\rho+kp\beta)$:

$$\text{equals}\begin{cases}\frac{1}{2}(r+s+2-kp)(-s-1+2kp)(r+1+kp) & \text{for } k=1, 2, \ldots, a-1,\\ \frac{1}{2}(r+s+2-ap)(-s-1+2ap)(r+1+ap) & \\ \quad +\frac{1}{4}a(a-1)(r+s+2)(-s-1+(a+1)p)(r+1+(a+1)p) & \text{for } k=a.\end{cases}$$

The equality we have to check then reads

$$\frac{1}{2}\sum_{k=1}^{a}(r+s+2-kp)(-s-1+2kp)(r+1+kp)$$
$$+\frac{1}{4}a(a-1)(r+s+2)(-s-1+(a+1)p)(r+1+(a+1)p)$$
$$=\frac{1}{4}a(a+1)(-r-1-ap)(-s-1+(a+1)p)(-r-s-2+p).$$

This equality can be checked e. g. by comparing the coefficients to r^2, r^1 and r^0 on the two sides. We leave the details to the reader.

Remark 4.9. — In [7] J. E. Humphreys points out that there seems to be a correlation between the non-vanishing of the cohomology groups of certain $L(\chi)$'s and the composition behaviour of Weyl modules. Theorem 4.5 and Proposition 4.7 support the existence of such a correlation.

APPENDIX

In this Appendix we briefly outline C. S. Seshadri's proof of Theorem 4.1.

Let X be a smooth variety over k with functionfield $k(X)$. By $k(X)^{1/p}$ we denote the field extension of $k(X)$ obtained by adjoining all p'th roots and we let $f:\overline{X}\to X$ be the normalisation of X in this extension. The sheaf M is then defined as the quotient of O_X in $f_* O_{\overline{X}}$, i. e.:

$$0\to O_X\to f_* O_{\overline{X}}\to M\to 0.$$

It is easy to see that M is a locally free sheaf of rank $p^{\dim X}-1$. One then checks that if V is a vectorbundle on X then the Frobenius homomorphism $H^i(X, V)\to H^i(X, V^{(p)})$ can be identified with the map occurring in the long exact cohomology sequence associated with the above short exact sequence tensored by V [note that since f is affine we have $H^i(X, V^{(p)})\simeq H^i(\overline{X}, f^* V)\simeq H^i(X, f_* f^* V)$]. In the case where $X=G/B$ the sheaf $f_* O_X^{(p)}$ is homogeneous. The B-module by which it is induced is $O_{G/B,e}/O^p_{G/B,e}$. A good description of the actual B-module structure of this representation seems to be hard to obtain (although such a description certainly would be very useful). However, the action of T is easy to find: As the big cell $w_0 B w_0 B$ is an open T-stable neighbourhood of the identity in G we see that the set of weights in $O_{G/B,e}/O^p_{G/B,e}$ consists of all weights of the form $-\sum n_\alpha \alpha$, where the sum is over all positive roots and $0<n_\alpha<p$. The statements in Theorem 4.1 follow.

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