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C^{∞} APPROXIMATIONS OF CONVEX, SUBHARMONIC, AND PLURISUBHARMONIC FUNCTIONS (1)

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Introduction

Most methods for the study of the behavior of functions on Riemannian manifolds apply directly only to functions which have some degree of differentiability. On the other hand, many functions which arise naturally from the geometry of the manifolds are in general at best continuous. Thus it is important to have in hand mechanisms of constructing smooth approximations of continuous functions. The stardard mechanism, the use of partitions of unity combined with smoothing by convolution in local coordinate systems, tends to obliterate geometrically meaningful properties and is thus unsatisfactory for many geometric problems. The purpose of the present paper is to present a mechanism of smooth approximation which tends to preserve geometric properties and is thus broadly applicable to geometric questions. Many of the results of this paper were announced by the authors in [5] (*c*), and some specific applications of the general methods here presented were discussed in [5] (*b*) and [5] (*e*).

The paper is organized as follows: paragraph 1 contains a discussion of the smoothing method in terms of smooth approximations of sections of subsheaves of the sheaf of germs of continuous functions on a Riemannian manifold; being able to carry out approximation of continuous sections by C^{∞} sections of the same subsheaf corresponds to being able to preserve the geometric structure in the approximation procedure. Paragraph 2 discusses the specific cases of Lipschitz continuous and convex functions, and paragraph 3 that of subharmonic functions. Paragraph 4 discusses a method of establishing the hypotheses of the theorems of paragraph 1 for certain specific subsheaves, in particular, the sheaf of germs of strictly plurisubharmonic functions on a complex manifold and certain other related sheaves. A synopsis of the results of this paper is given in a Table at the end of the paper.

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equations; and to Y.-T. Siu for pointing out the relevance of Richberg's work [11]. We acknowledge the assistance of all of these with thanks. We also thank S.-T. Yau, who, while making use of our announced results [5] (*c*) in his article [12], encouraged us to make available in published form the proofs presented in this paper. Finally, we thank the referee of this paper for several very helpful suggestions, which led to a considerable clarification of the first part of paragraph 3 (on the various concepts of subharmonicity), among other improvements.

1. Global approximation theorems

It is well known that a continuous real-valued function defined on a C^∞ Riemannian manifold can be approximated by C^∞ functions in the strongest reasonable sense of the word "approximation": namely, given such a continuous function f and an everywhere positive continuous function g, there exists a C^∞ function F such that $|f-F|<g$ everywhere. The purpose of the present section of this paper is to state and prove similar approximation results for certain subsets of the set of all continuous functions. These results will be applied later to the specific subsets: the set of convex functions, the set of subharmonic functions, and the set of plurisubharmonic functions (in case the manifold is a complex manifold). However, it is convenient to unify the exposition of these approximation results by proving the theorems in a general setting which incorporates the features common to all these specific cases.

Let $\mathscr{S}$ be a subsheaf of the sheaf $\mathscr{C}$ of germs of continuous real-valued functions on a (fixed) C^∞ Riemannian manifold M, i. e. $\mathscr{S}$ is to be an open subset of $\mathscr{C}$ such that $\pi | \mathscr{S} : \mathscr{S} \to M$ is surjective, where $\pi : \mathscr{C} \to M$ is the standard projection. Note that $\mathscr{S}$ is not required to be closed under the algebraic operations of $\mathscr{C}$. The elements of $\mathscr{C}$ (or of $\mathscr{S}$) will be denoted by $[f]_p$ where $p \in M$ and f is a continuous function defined in a neighborhood of p. The set $(\pi | \mathscr{S})^{-1}(p)$ will be denoted by $\mathscr{S}_p$ and the set of continuous functions $f: U \to R$ defined on an open subset U of M with the property that $[f]_p \in \mathscr{S}_p$ for every $p \in U$ will be denoted by $\Gamma(\mathscr{S}, U)$. This notation is consistent with the usual notation for sections of sheaves since $\Gamma(\mathscr{S}, U)$ as just defined is naturally identified with the set of all sections of $\mathscr{S}$ over U.

DEFINITION 1.1. — $\mathscr{S}$ has the $\vee$*-closure property* (read maximum-closure property) if for any two germs $[f]_p, [g]_p \in \mathscr{S}_p$, $p \in M$, the germ $[f \vee g]_p$ is in $\mathscr{S}_p$, where $f \vee g$ at each point is the maximum of the values of f and of g at that point.

DEFINITION 1.2. — $\mathscr{S}$ has the *convex composition property* if for any $[f]_p \in \mathscr{S}_p$, $p \in M$, and for any function $\chi : R \to R$ which is convex and (strictly) increasing in a neighborhood of $f(p)$, the germ $[\chi \circ f]_p$ is in $\mathscr{S}_p$.

The next definitions and the proofs of the Theorems of this section will depend on some standard function space topology concepts, which will now be summarized. For further details, one can consult [10] or [8] for instance.

Let K be a compact subset of M; let $C^\infty(K)$ denote the set of function $F : U \to \mathbf{R}$ where U is an open subset of M containing K and f is a C^∞ function on U. Choose a fixed covering of K by a finite number of (open) coordinate systems, say $x^{(\lambda)} : V_\lambda \to R^n$, $n = \dim M$, $\lambda \in \Lambda$, where Λ is a finite set. Choose then for each $\lambda \in \Lambda$ an open set V'_λ having compact closure contained in V_λ in such a way that $K \subset \bigcup_{\lambda \in \Lambda} V'_\lambda$. These choices are possible by the compactness of K. Then for each positive integer i and each $f \in C^\infty(K)$ the supremum

$$\sup_{\lambda \in \Lambda}\left[\sup_{p \in V'_\lambda \cap K}\begin{pmatrix}\text{maximum of the } x^{(\lambda)}\text{-coordinate system partial} \\ \text{derivatives of order } i \text{ at } p\end{pmatrix}\right]$$

is finite. This supremum will be denoted by $\| f |_{K, i}$. Define $\| f \|_{K, 0}$ for $f \in C^\infty(K)$ to be $\sup_{p \in K} | f(p) |$. The function $d_K : C^\infty(K) \times C^\infty(K) \to R$ defined by

$$d_K(f, g) = \| f - g \|_{K,0} + \sum_{i=1}^{+\infty} \frac{1}{2^i} \min(1, \| f - g \|_{K,i})$$

is a (finite-valued) pseudometric on $C^\infty(K)$. The topology on $C^\infty(K)$ that it determines is independent of the choices made in defining the pseudometric d_K even though d_K itself is not independent of these choices. In the following discussions, the notation d_K will be used without explicitly noting the assumption that appropriate choices of $\{ \Lambda, V_\lambda, V'_\lambda \}$ have to be made. In all cases, these choices may be made arbitrarily except for the conditions already given.

DEFINITION 1.3. — $\mathscr{S}$ has the C^∞ *stability property* if: when U is an open subset of M, K is a compact subset of U, and $f : U \to R$ is a function such that $[f]_p \in \mathscr{S}_p$ for every $p \in U$, then there exists an $\varepsilon > 0$ such that every function $g \in C^\infty(K)$ with $d_K(0, g) < \varepsilon$ has the property that $[f + g]_p \in \mathscr{S}_p$ for all $p \in K$.

DEFINITION 1.4. — $\mathscr{S}$ has the *semilocal* C^∞ *approximation property* if the following condition holds: Let U be an open subset of M and K be a compact subset of U and f be a function in $\Gamma(\mathscr{S}, U)$ such that f is C^∞ in a neighborhood of a (possibly empty) compact subset K_1 of K; then there exists an open subset V of U with $K \subset V$ such that for every $\varepsilon > 0$ there exists a C^∞ function $F \in \Gamma(\mathscr{S}, V)$ such that:

(*a*) $\sup_{p \in K} | f(p) - F(p) | < \varepsilon$;

(*b*) $d_{K_1}(f, F) < \varepsilon$.

The following Theorems give circumstances under which the approximations in neighborhoods of compact sets given by the semilocal approximation property can be extended to all of M. For the statement of these Theorems recall that the C^0 *fine topology* on the set $\Gamma(\mathscr{C}, M)$ of continuous functions on M is by definition the topology generated by the sets

$$\{ F \in \Gamma(\mathscr{C}, M) \mid | f(p) - F(p) | < g(p), p \in M \},$$

where $f \in \Gamma(\mathscr{C}, M)$, $g \in \Gamma(\mathscr{C}, M)$, and g is positive everywhere on M. The C^0 *coarse topology* (or *compact-open topology*) on $\Gamma(\mathscr{C}, M)$ is the topology generated by the sets

$$\{ F \in \Gamma(\mathscr{C}, M) \mid | f(p) - F(p) | < \varepsilon, p \in K \}$$

where $f \in \Gamma(\mathscr{C}, M)$, ε is a positive real number and K is a compact subset of M. These topologies on $\Gamma(\mathscr{C}, M)$ induce topologies on $\Gamma(\mathscr{S}, M)$, which will again be called the C^0 fine and C^0 coarse (or compact-open) topologies respectively.

The fine C^0 and coarse C^0 topologies are special cases of the fine C^r and coarse C^r ($0 \leqq r \leqq \infty$) topologies in the terminology of differential topology (*cf.* [10]). The terminology strong C^r (instead of fine C^r) and weak C^r (instead of coarse C^r) is also used (*cf.* [8]). It would thus seem logical to replace entirely the phrase "compact-open topology" and the often used equivalent phrase "compact convergence" by one of the phrases "coarse C^0 topology" or "weak C^0 topology". However, out of deference to strong tradition, the coarse C^∞ topology is hereafter in this paper usually called the compact-open topology. Though consequently unreinforced by suggestive terminology, the contrast between the fineness of the fine C^0 topology and the coarseness of the compact-open topology is nonetheless of great importance throughout and should be carefully noted.

THEOREM 1.1. — *If $\mathscr{S}$ has the C^∞ stability property, the $\vee$-closure property and the semilocal approximation property, then the set of C^∞ functions in $\Gamma(\mathscr{S}, M)$ is dense in $\Gamma(\mathscr{S}, M)$ in the C^0 fine topology.*

A function $f \in \Gamma(\mathscr{C}, M)$ is an *exhaustion function* if for every real number c the set $\{p \in M \mid f(p) \leqq c\}$ is a compact subset of M. The set of exhaustion functions in $\Gamma(\mathscr{C}, M)$ will be denoted by $\Gamma_E(\mathscr{C}, M)$ and the set $\Gamma_E(\mathscr{C}, M) \cap \Gamma(\mathscr{S}, M)$ by $\Gamma_E(\mathscr{S}, M)$.

THEOREM 1.2. — *If $\mathscr{S}$ has the $\vee$-closure property, the convex composition property, and the semilocal approximation property, and if $\Gamma(\mathscr{S}, M)$ is closed in $\Gamma(\mathscr{C}, M)$ in the compact-open topology, then the set of C^∞ functions in $\Gamma_E(\mathscr{S}, M)$ is dense in $\Gamma_E(\mathscr{S}, M)$ in the compact-open topology.*

Before giving the proofs of Theorems 1.1 and 1.2, we now present three examples with $M = R$, which, though perhaps too simple to be intrinsically interesting, nonetheless suffice to illustrate the crucial aspects of the definitions and the proofs of the theorems which will be given later:

Let $\mathscr{S}_1$ = the sheaf over R which is defined by setting $(\mathscr{S}_1)_p = \{[f]_p \mid f$ is convex in a neighborhood of $p\}$. Here a function's being convex on an open subset U of R has the usual meaning that for each x_1, x_2 in U such that (x_1, x_2) is in U:

$$f(\lambda x_1 + (1-\lambda) x_2) \leqq \lambda f(x_1) + (1-\lambda) f(x_2),$$

for all λ satisfying $0 \leqq \lambda \leqq 1$. A function f defined on an open subset U of R is convex on U if and only if it is convex in a neighborhood of each point of U, and a convex function on U is necessarily continuous. Thus $\mathscr{S}_1$ is a subsheaf of $\mathscr{C}$, and $\Gamma(\mathscr{S}_1, U)$ = the set of convex functions on U for any open subset U of R.

Let $\mathscr{S}_2$ be defined by $(\mathscr{S}_2)_p = \{[f]_p \mid \exists\, \varepsilon > 0 \ni f - \varepsilon e^x$ is convex in a neighborhood of $p\}$. $\mathscr{S}_2$ is a subsheaf of $\mathscr{S}_1$ and thus of $\mathscr{C}$. $\mathscr{S}_2$ may be considered to be the sheaf of germs of functions which are strictly convex in the sense that they are locally "more convex" than εe^x for sufficiently small $\varepsilon > 0$. If f is C^∞ in a neighborhood of $p \in R$ then $[f]_p \in \mathscr{S}_2$ if and only if $f''(p) > 0$.

The following properties of $\mathscr{S}_1$ and $\mathscr{S}_2$ are easily verified: Both $\mathscr{S}_1$ and $\mathscr{S}_2$ have the $\vee$-closure property, the convex composition property, and the semilocal approximation property (That $\mathscr{S}_1$ and $\mathscr{S}_2$ have the semilocal approximation property follows easily from the application of the standard convolution smoothing process). However, $\mathscr{S}_1$ fails to have the C^∞ stability property as consideration of the germs of the identity function shows. $\mathscr{S}_2$ on the other hand does have the C^∞ stability property, as is easily verified. $\Gamma(\mathscr{S}_1, R)$ is closed in $\Gamma(\mathscr{C}, R)$ in the compact-open topology but $\Gamma(\mathscr{S}_2, R)$ is not. Thus, $\mathscr{S}_1$ satisfies the hypotheses of Theorem 1.2 but not of Theorem 1.1 while $\mathscr{S}_2$ satisfies the hypotheses of Theorem 1.1 but not of Theorem 1.2.

As a final example, let $\mathscr{S}_3$ be the sheaf determined by $(\mathscr{S}_3)_p = \{ [f]_p \in (\mathscr{S}_2)_p \mid f$ is Lipschitz continuous with Lipschitz constant < 2 in a neighborhood of $p \}$. Then $\mathscr{S}_3$ has the $\vee$-closure property, the C^∞ stability property, and the semilocal approximation property (again by convolution smoothing); thus $\mathscr{S}_3$ satisfies the hypotheses of Theorem 1.1. $\mathscr{S}_3$ fails to have the convex composition property; and $\Gamma(\mathscr{S}_3, R)$ is not closed in $\Gamma(\mathscr{C}, R)$ with the compact-open topology.

We return now to the situation where M is an arbitrary C^∞ Riemannian manifold. In the following lemma and throughout the remainder of this section (§ 1), let $\{ K_i \mid i \in \mathbf{Z}^+ \}$ be a sequence of compact subsets of M such that $K_i \subset \mathring{K}_{i+1}$ for all $i \in \mathbf{Z}^+$ and $\bigcup_{i \in \mathbf{Z}^+} K_i = M$. The lemma follows from the Weierstrass theorem that convergence uniformly on each compact set of a sequence of C^1 functions and of their first derivatives implies that the limit function is C^1 and that the first derivatives of the sequence converge to the corresponding first derivatives of the limit.

LEMMA 1.1. — *If* $\{ f_i \mid i \in \mathbf{Z}^+ \}$ *is a sequence of functions on* M *such that* $f_i \in C^\infty(K_i)$ *for each* $i \in \mathbf{Z}^+$ *and if for each* $i \in \mathbf{Z}^+$ *the sequence* $\{ f_j \mid j \geqq 1 \}$ *is a Cauchy sequence in the* d_{K_i}-*pseudometric, then the sequence* f_i *converges on* M *to a function* $f : M \to R$ *and* f *is a* C^∞ *function. Moreover, for each i, the sequence* $\{ f_j \mid j \geqq i \}$ *converges in the* d_{K_i}-*pseudometric to* f.

The next lemma gives an approximation construction, iteration of which will yield the proof of Theorem 1.1:

LEMMA 1.2. — *Suppose that the sheaf* $\mathscr{S}$ *satisfies the hypotheses of Theorem* 1.1 *and that* $\varphi \in \Gamma(\mathscr{S}, M)$. *If* A_1, A_2, *and* A_3 *are compact subsets of* M *with* $A_1 \subset A_2 \subset \mathring{A}_3$, *if* $\varphi \in C^\infty(A_1)$, *and if* ε *is any positive real number, then there exists a function* $\psi \in \Gamma(\mathscr{S}, M)$ *such that:*

(*a*) $\psi \in C^\infty(A_2)$;

(*b*) $\psi \mid (M - A_3) = \varphi \mid (M - A_3)$;

(*c*) $d_{A_1}(\varphi - \psi) < \varepsilon$;

(*d*) $\sup_{p \in A_3} | \varphi(p) - \psi(p) | < \varepsilon$.

Proof of Lemma 1.2. — Let A_4 be a compact set such that $A_2 \subset \mathring{A}_4 \subset A_4 \subset \mathring{A}_3 \subset A_3$. By a standard construction, one obtains a C^∞ function $\rho : M \to R$ such that $\rho \equiv +1$ in a neighborhood of A_2 and $\rho \equiv -1$ on $M - A_4$. By virtue of the C^∞ stability property of $\mathscr{S}$, there exists a positive constant η_1 such that, for all $\eta \in [0, \eta_1)$, $[\varphi + \eta\rho]_p \in \mathscr{S}_p$ for all $p \in A_3$. With this η_1 fixed, there exists a function $\tau \in C^\infty(A_3)$ such that:

(1) τ is greater than φ in a neighborhood of A_2 but less than φ in a neighborhood of the boundary ∂A_3 of A_3

and

(2) $[\tau]_p \in \mathcal{S}_p$ for all $p \in A_3$;

namely, one need only choose τ to be a sufficiently good approximation (in the sense of uniform approximation of functional values) of $\varphi + \eta\rho$ on A_3 which also satisfies (2). This choice of τ is possible because $\mathcal{S}$ has the semilocal approximation property. Then the function ψ defined by

$$\psi = \max(\tau, \varphi) \quad \text{on } A_3,$$
$$\psi = \varphi \quad \text{on } M - A_3$$

is in $\Gamma(\mathcal{S}, M)$: That $[\psi]_p \in \mathcal{S}_p$ for $p \in \mathring{A}_3$ follows from the hypothesis that $\mathcal{S}$ has the $\vee$-closure property. That $[\psi]_p \in \mathcal{S}_p$ for $p \in \partial A_3$ follows from the fact that, in a neighborhood of ∂A_3, $\tau < \varphi$ so that $\psi = \varphi$ in that neighborhood and hence $[\psi]_p = [\varphi]_p \in \mathcal{S}_p$ for $p \in \partial A_3$. That $[\psi]_p \in \mathcal{S}_p$ for $p \in M - A_3$ is clear from the openness of $M - A_3$. The function ψ is in $C^\infty(A_2)$ since $\tau > \varphi$ in a neighborhood of A_2. If $\eta \in (0, \eta_1)$ is chosen sufficiently small and if τ is a sufficiently good approximation of φ in the senses of $\sup_{A_2} |\varphi - \tau|$ and $d_{A_1}(\varphi, \tau)$ being sufficiently small (which is possible in both senses by virtue of the semilocal approximation property of $\mathcal{S}$), then the corresponding ψ will satisfy the approximation requirement of the lemma.

Proof of Theorem 1.1. — Let f be a function in $\Gamma(\mathcal{S}, M)$ and g be a continuous function on M which is everywere positive. Then to prove the theorem one needs to show that there is a C^∞ function $F \in \Gamma(\mathcal{S}, M)$ such that $|f(p) - F(p)| < g(p)$ for all $p \in M$.

Define $\varepsilon_i = \inf_{K_i} g$, $i \in \mathbf{Z}^+$. Then $\varepsilon_i > 0$ for all $i \in \mathbf{Z}^+$ and $\varepsilon_1 \geqq \varepsilon_2 \geqq \varepsilon_3 \ldots$ Moreover, if $\sup_{K_1} |f - F| < \varepsilon_1$ and $\sup_{K_{i+1} - K_i} |f - F| \leqq \varepsilon_{i+1}$ for all $i \in \mathbf{Z}^+$ then $|f(p) - F(p)| < g(p)$ for all $p \in M$. Now define inductively sequences $\{F_i : M \to R \mid i \in \mathbf{Z}^+\}$ of functions and $\{\eta_i \mid i \in \mathbf{Z}^+\}$ of real numbers as follows: let F_1 be any function in $C^\infty(K_1) \cap \Gamma(\mathcal{S}, M)$ with $\sup_{K_2} |f - F_1| < \varepsilon_2/2$ and $F_1 = f$ on $M - K_2$, and let η_1 be any positive number with $\eta_1 \leqq \varepsilon_1$ and with the property that if $G \in C^\infty(K_1)$ and $d_{K_1}(F_1, G) < 2\eta_1$ then $[G]_p \in \mathcal{S}_p$ for all $p \in K_1$. The existence of such a function F_1 follows from Lemma 1.2; the existence of such an η_1 follows from the C^∞ stability property of $\mathcal{S}$. Now, to complete the inductive definitions, suppose that $F_1, \ldots, F_i$ and $\eta_1, \ldots, \eta_i$ are determined. Then let F_{i+1} be any element of $C^\infty(K_{i+1}) \cap \Gamma(\mathcal{S}, M)$ with

$$\sup_{K_{i+1}} |F_{i+1} - F_i| < \frac{\varepsilon_{i+2}}{2^{i+1}},$$

$$d_{K_j}(F_{i+1}, F_i) < \frac{\eta_j}{2^{i+1}} \quad \text{for all } j \leqq i,$$

$$F_{i+1} = f \quad \text{on } M - K_{i+2}.$$

And let η_{i+1} be a positive number which is less than min $(\varepsilon_{i+2}, \eta_i)$ and which has the property that if $G \in C^\infty(K_{i+1})$ and $d_{K_{i+1}}(F_{i+1}, G) \leqq \eta_{i+1}$ then $[G]_p \in \mathscr{S}_p$ for all $p \in K_{i+1}$. Again the possibility of so choosing F_{i+1} follows from Lemma 1.2, together with the fact that if $\xi > 0$ is sufficiently small then $d_{K_i}(F_{i+1}, F_i) < \xi$ implies that $d_{K_j}(F_{i+1}, F_i) < \eta_j/2^{i+1}$ for all $j \leqq i$ since $K_j \subset \mathring{K}_i$ for $j < i$. The possibility of making the required choice of η_{i+1} follows from the C^∞ stability property of $\mathscr{S}$.

Now Lemma 1.1 implies that the sequence $\{ F_i \mid i \in \mathbf{Z}^+ \}$ converges to a C^∞ function F. Clearly $\sup_{K_1} | F - f | < \varepsilon_1$ since

$$\sup_{K_1} | F - f | \leqq \sup_{K_1} | F_1 - f | + \sum_{j=1}^{+\infty} \sup_{K_1} | F_{j+1} - F_j | < \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2^2} + \ldots = \varepsilon_1.$$

Similarly, for $i \in \mathbf{Z}^+$:

$$\sup_{K_{i+1}-K_i} | F - f | \leqq \sup_{K_{i+1}-K_i} | F_1 - f | + \sum_{j=1}^{+\infty} \sup_{K_{i+1}-K_i} | F_{j+1} - F_j | < \frac{\varepsilon_{i+1}}{2^i} + \frac{\varepsilon_{i+1}}{2^{i+1}} \ldots \leqq \varepsilon_{i+1}$$

because: (*a*) $F_j = f$ on $K_{i+1} - K_i$ for $j \leqq i-1$ so $F_{j+1} - F_j \equiv 0$ and (*b*) for $j \geqq i-1$,

$$\sup_{K_{i+1}-K_i} | F_{j+1} - F_j | \leqq \sup_{K_{j+2}} | F_{j+1} - F_j | < \frac{\varepsilon_{j+2}}{2^{j+1}} \leqq \frac{\varepsilon_{i+2}}{2^{j+1}}.$$

Finally, $[F]_p \in \mathscr{S}_p$ for all $p \in M$. To derive this conclusion, note that $p \in K_i$ for some i and

$$d_{K_i}(F, F_i) \leqq \sum_{j=i}^{+\infty} d_{K_i}(F_{j+1}, F_j) < \frac{\eta_i}{2^{i+1}} + \frac{\eta_i}{2^{i+2}} + \ldots \leqq \eta_i.$$

Then by the choice of η_i, $[F]_p \in \mathscr{S}_p$. Thus F is the C^∞ function in $\Gamma(\mathscr{S}, M)$ and approximating f which was required.

The next lemma will play a role in the proof of Theorem 1.2 similar to that played by Lemma 1.2 in the proof of Theorem 1.1 [a construction related to that of the following lemma but used for a somewhat different purpose is given in [5] (*e*)].

LEMMA 1.3. — *Suppose that $\mathscr{S}$ satisfies the hypotheses of Theorem 1.2 and that $\varphi \in \Gamma_E(\mathscr{S}, M) \cap C^\infty(A_1)$, where A_1 is a compact subset of M. Suppose also that c is a real number with the property that the set $A_2 = \{ p \in M \mid \varphi(p) \leqq c \}$ contains A_1. Then, if ε and λ are positive numbers ε, there exists a function $\psi \in \Gamma_E(\mathscr{S}, M)$ such that*

(*a*) $\psi \in C^\infty(A_2)$;

(*b*) $| \psi - \varphi | < \varepsilon$ *on* $\{ p \in M \mid \varphi(p) \leqq c + \lambda \}$;

(*c*) $d_{A_1}(\psi, \varphi) < \varepsilon$;

(*d*) *if $p \in M$ has the property that $\varphi(p) \geqq c + \lambda$ then $\psi(p) \geqq \varphi(p)$; also if $p, q \in M$ have the property that $\varphi(p) = \varphi(q) \geqq c + \lambda$, then $\psi(p) = \psi(q)$.*

Proof of Lemma 1.3. — For notational convenience, let $A_3 = \{ p \in M \mid \varphi(p) \leqq c+\lambda \}$. Then A_1, A_2, and A_3 are compact, and $A_1 \subset A_2 \subset \mathring{A}_3$. For any $\eta > 0$ (and $\varepsilon > 0$), there is a function ψ, in $C^\infty(A_3)$ such that:

$$[\psi_1]_p \in \mathscr{S}_p \quad \text{for all } p \in A_3,$$

$$\sup_{A_3} |\varphi - \psi_1| < \eta,$$

$$d_{A_1}(\psi_1, \varphi) < \varepsilon.$$

The existence of such a function ψ_1 is a consequence of the semilocal approximation property of $\mathscr{S}$.

For any $\eta > 0$, there exists a convex (strictly) increasing function $\chi : R \to R$ such that:

$$\chi(t) = \lambda - 2\eta \quad \text{if } t \leqq c,$$

$$|\chi(t) - \lambda| < 2\eta \quad \text{if } c < \lambda < c+\lambda,$$

$$\chi(t) \geqq \lambda + 2\eta \quad \text{if } t \geqq c+\lambda.$$

Then $\chi \circ \varphi \in \Gamma(\mathscr{S}, M)$ because $\mathscr{S}$ has the convex composition property. Also, $\chi \circ \varphi > \varphi + \eta$ near the boundary of A_3 since φ is near $c+\lambda$ near the boundary of A_3; but $\chi \circ \varphi < \varphi - \eta$ near A_2 since $\varphi \leqq c$ on A_2. Thus the function defined by

$$\psi(p) = \max\{(\chi \circ \varphi)(p), \psi_1(p)\}, \quad p \in A_3,$$

$$\psi(p) = (\chi \circ \varphi)(p), \quad p \in M - A_3$$

is in $\Gamma(\mathscr{S}, M)$: for $p \in M - A_3$, $[\psi]_p = [\chi \circ \varphi]_p \in \mathscr{S}_p$; for $p \in \mathring{A}_3$, $[\psi]_p \in \mathscr{S}_p$ by the $\vee$-closure property of $\mathscr{S}$; for $p \in \partial A_3$, $\chi \circ \varphi > \psi_1$ near p so $[\psi]_p = [\psi \circ \varphi]_p \in \mathscr{S}_p$. Moreover, $\psi = \psi_1$ in a neighborhood of A_2 since $\chi \circ \varphi < \varphi - \eta < \psi_1$ on (and hence in a neighborhood of) A_2. Finally, if $p \in A_3$, then $\psi_1(p) \leqq \psi(p) \leqq \varphi(p) + 2\eta$. Hence if $0 < \eta < \varepsilon/2$, the function ψ has the properties required by the Lemma. □

If $\gamma > 0$, then the function $\chi : R \to R$ in the proof of Lemma 1.3 can be chosen to the Lipschitz continuous with Lipschitz constant $\leqq 1+\gamma$, provided that $\eta > 0$ is chosen sufficiently small. This fact will be used later in deriving a variant (Theorem 1.2′) of Theorem 1.2.

Proof of Theorem 1.2. — To establish the theorem, one needs to show that, for each compact set $K \subset M$ and each real number $\varepsilon > 0$, there exists a C^∞ function F in $\Gamma_E(\mathscr{S}, M)$ such that $\sup |F - f| < \varepsilon$. Let $c = \sup f$. Then $K \subset \{ p \in M \mid f(p) \leqq c \}$. Define, for each $i \in \mathbf{Z}^+$, $K_i = \{ p \in M \mid f(p) \leqq c+i-1 \}$. Because f is an exhaustion function, each K_i is compact. Also, for each $i \in \mathbf{Z}^+$, $K_i \subset \mathring{K}_{i+1}$; and $\bigcup_{i \in \mathbf{Z}^+} K_i = M$. Thus these specific K_i's satisfy the general requirements previously imposed. Now choose successively functions F_i, $i \in \mathbf{Z}^+$, as follows: Let F_1 be a function in $C^\infty(K_1) \cap \Gamma(\mathscr{S}, M)$ as provided by Lemma 1.3 such that $\sup_{K_1} |F_1 - f| < \varepsilon/2^2$. Then, to complete the inductive definition, the $F_1, \ldots, F_i$ being chosen, let F_{i+1} be a function in $C^\infty(K_{i+1}) \cap \Gamma(\mathscr{S}, M)$ as provided by Lemma 1.3 with

$$d_{K_j}(F_{i+1}, F_i) < \varepsilon/2^{i+2},$$

for all $j \leqq i$. The possibility of so choosing the F_i's follows from Lemma 1.3: Lemma 1.3 applies immediately to choosing F_1; and, for each $i \in \mathbf{Z}^+$, there exist real numbers λ_1, λ_2 with $\lambda_1 < \lambda_2$ such that $K_{i+1} = \{ p \in M \mid F_i(p) \leqq \lambda_1 \}$ and $K_{i+2} = \{ p \in M \mid F_i(p) \leqq \lambda_2 \}$. Hence Lemma 1.3 can be applied with $\varphi = F_i$, $c = \lambda_1$ and $\lambda = \lambda_2 - \lambda_1$ to produce F_{i+1}.

The sequence $\{ F_i \mid i \in \mathbf{Z}^+ \}$ converges to a C^∞ function F by Lemma 1.1, since for each $i \in \mathbf{Z}^+$ $\{ F_j \mid j \geqq i \}$ is clearly a Cauchy sequence in the d_{K_i} metric. Moreover,

$$\sup_K |F-f| \leqq \sup_{K_1} |F-f| \leqq \sup_{K_1} |F_1 - f| + \sum_{j=1} \sup_{K_1} |F_{j+1} - F_j|$$

$$\leqq \sup_{K_1} |F_1 - f| + \sum_{j=1}^{+\infty} \sup_{K_j} |F_{j+1} - F_j|$$

$$\leqq \sup_{K_1} |F_1 - f| + \sum_{j=1}^{+\infty} d_{K_j}(F_{j+1}, F_j) \leqq \frac{\varepsilon}{2^2} + \sum_{j=1}^{+\infty} \frac{\varepsilon}{2^{j+2}} < \varepsilon.$$

That $F \in \Gamma(\mathscr{S}, M)$ follows from the hypothesis that $\Gamma(\mathscr{S}, M)$ is closed in $\Gamma(\mathscr{C}, M)$ in the compact-open topology together with the facts that the sequence $\{ F_i \mid i \in \mathbf{Z}^+ \}$ converges to $\Gamma(\mathscr{S}, M)$ in that topology and $F_i \in \Gamma(\mathscr{S}, M)$ for each $i \in \mathbf{Z}^+$. The fact that $F \geqq f - \varepsilon$ outside K_1 implies that F is an exhaustion function. Thus $F \in \Gamma_E(\mathscr{S}, M)$. $\square$

The methods used to establish Theorems 1.1 and 1.2 can be used to prove other similar theorems. For instance, the full force of the hypothesis that $\Gamma(\mathscr{S}, M)$ be closed in $\Gamma(\mathscr{C}, M)$ was not needed. And a slightly stronger theorem can be obtained (by essentially the same proof) in which this hypothesis is replaced by the hypothesis that, for any sequence

$$\{ F_i \in \Gamma(\mathscr{S}, M) \cap C^\infty(K_i) \mid i \in \mathbf{Z}^+ \}$$

with the property that for each $i \in \mathbf{Z}^+$ $\{ F_j \mid j \geqq i \}$ is a Cauchy sequence in the d_{K_i} metric, the limit F of $\{ F_i \mid i \in \mathbf{Z}^+ \}$ is in $\Gamma(\mathscr{S}, M)$. Even more generally, these convergence hypotheses may be expressed in a localized version (i.e., in terms of properties of the individual stalks $\mathscr{S}_p$). We shall not undertake to list exhaustively all the possible variations on these constructions. However, one variant of Theorem 1.2 plays a role in the applications that we shall discuss later. To state this variant, an additional definition is needed.

DEFINITION. – $\mathscr{S}$ has the *Lipschitz semilocal approximation property* if it has the semilocal approximation and when f is Lipschitz continuous on U (in Definition 1.4) with Lipschitz constant $\leqq \lambda$ then the set V can be so chosen that the approximating functions F can be taken to Lipschitz continuous on V with Lipschitz constant $\leqq \lambda + \varepsilon$.

THEOREM 1.2'. – *Suppose that $\mathscr{S}$ has the $\vee$-closure property, the convex composition property, and the Lipschitz semilocal approximation property. Then for any $\varepsilon > 0$, the closure in $\Gamma_E(\mathscr{S}, M)$ with the compact open topology of the set of C^∞ functions in $\Gamma_E(\mathscr{S}, M)$ which are Lipschitz continuous with Lipschitz constant $\leqq \Lambda + \varepsilon$ contains the set of functions in $\Gamma_E(\mathscr{S}, M)$ which are Lipschitz continuous with Lipschitz constant Λ.*

The modifications of the proof of Theorem 1.2 needed to prove Theorem 1.2' are as follows: One needs to show that given a compact set K and a positive number ε, that there is a C^∞ function $F \in \Gamma_E(\mathscr{S}, M)$ which is Lipschitz continuous with constant $\leqq \Lambda + \varepsilon$ and which satisfies $\sup_K |F-f| < \varepsilon$. Such an F is obtained by a construction almost identical to that used to produce the corresponding function F in the proof of Theorem 1.2. It suffices to impose the additional condition that each F_i, $i \in \mathbf{Z}^+$, be Lipschitz continuous with Lipschitz constant $\leqq \Lambda + \varepsilon(1-1/2i)$. If the F_i be so chosen, then the limit F of the F_i is Lipschitz continuous with Lipschitz constant $\leqq \Lambda + \varepsilon$. That the F_i can be so chosen is a consequence of the following variant of Lemma 1.3:

LEMMA 1.3′. — *Suppose that* $\mathscr{S}$ *satisfies the hypotheses of Theorem* 1.2 *and that* $\varphi \in \Gamma_E(\mathscr{S}, M) \cap C^\infty(A_1)$, *where* A_1 *is a compact subset of* M. *Suppose also that* φ *is Lipschitz continuous with Lipschitz constant* $\leqq \Sigma$; *and that c is a real number with the property that the set* $A_2 = \{ p \in M \mid \varphi(p) \leqq c \}$ *contains* A_1. *Then, if* ε *and* λ *are positive real numbers, there exists a function* $\psi \in \Gamma(\mathscr{S}, M)$ *such that:*

(*a*) $\psi \in C^\infty(A_2)$;

(*b*) $|\psi - \varphi| < \varepsilon$ *on* $\{ p \in M \mid \varphi(p) \leqq c + \lambda \}$;

(*c*) $d_{A_1}(\psi, \varphi) < \varepsilon$;

(*d*) *if* $p \in M$ *has the property that* $\varphi(p) \geqq c + \lambda$, *then* $\psi(p) \geqq \varphi(p)$; *also if p and* $q \in M$ *have the property that* $\varphi(p) = \varphi(q) \geqq c + \lambda$, *then* $\psi(p) = \psi(q)$;

(*e*) ψ *is Lipschitz continuous with Lipschitz constant* $\leqq \Sigma + \varepsilon$.

The proof of Lemma 1.3′ follows the pattern of the proof of Lemma 1.3 exactly except that one makes use of the remark immediately following the proof of Lemma 1.3 (that for $\gamma > 0$, χ' can be made $\leqq 1 + \gamma$) together with the following observations: (*a*) If ξ is a Lipschitz continuous function (on any metric space) with Lipschitz constant $\leqq \Sigma$ and $\chi : R \to R$ is Lipschitz continuous with Lipschitz constant $\leqq 1 + \gamma$ then $\chi \circ \xi$ has Lipschitz constant $\leqq (1+\gamma)\Sigma$; (*b*) If ξ_1 and ξ_2 are Lipschitz continuous functions (on any metric space) with Lipschitz constants $\leqq \Sigma$, then $\max(\xi_1, \xi_2)$ is Lipschitz continuous with Lipschitz constant $\leqq \Sigma$. □

2. Smooth approximation of Lipschitz continuous functions and convex functions on Riemannian manifolds

The purpose of this section is to discuss a method of approximation of continuous functions by C^∞ ones on Riemannian manifolds which can be used to establish the semilocal approximation property (defined in paragraph 1) for certain geometrically significant sheaves. The usual method for constructing such approximations is as follows (*see* for instance [10] for a more detailed discussion): If $f : M \to R$ is a continuous function, then by a partition-of-unity procedure one may express f as a locally finite sum $\sum_{\lambda \in \Lambda} f_\lambda$, where each f_λ has compact support inside some coordinate open set: specifically, for this procedure, one chooses a locally finite cover of M by coordinate open sets and takes a partition of unity subordinate to this cover. Then each f_λ can be considered to be a function with compact support on a euclidean space, and thus each f_λ can be approximated by a family $(f_\lambda)_\varepsilon$, $\varepsilon \to 0^+$, of C^∞ functions by the convolution smoothing process. Moreover, the approximating functions $(f_\lambda)_\varepsilon$ can be chosen to have support in the image of the λth coordinate open set so that these functions can be considered (by extension by 0) to be defined (and C^∞) on all of M. The sum $\sum_{\lambda \in \Lambda} (f_\lambda)_{\varepsilon_\lambda}$ will again be a locally finite sum and thus this sum defines a C^∞ function on M. If the convolution smoothing parameters $\varepsilon_\lambda > 0$ are correctly chosen, then the sum $\sum_{\lambda \in \Lambda} (f_\lambda)_{\varepsilon_\lambda}$ will be in any preassigned C^0 fine neighborhood of f. This method of showing the density of the C^∞ functions in the continuous functions on M in the C^0 fine topology suffers from the disavantage that it ignores the Riemannian metric structure of the manifold M, so that the geometric behavior of the approximating functions is not closely related to the geometric behavior of the function being approximated. The approximation process to be discussed in this section, called Riemannian convolution smoothing, behaves better in terms of preserving geometric properties than does the coordinate system smoothing process just outlined. A detailed discussion of the Riemannian convolution smoothing process was given by the authors in [5] (*a*, *b* and *e*); only the definition of the process and a summary of its properties which are relevant to the application of paragraph 1 will be given here.

To define Riemannian convolution smoothing on a Riemannian manifold M of dimension n, let $\varkappa : \mathbf{R} \to \mathbf{R}$ be a nonnegative C^∞ function that has its support contained in $[-1, 1]$, is constant in a neighborhood of 0, and has the property that $\int_{v \in \mathbf{R}^n} \varkappa(\|v\|) = 1$. If K is a compact subset of M then there is a positive number ε_K such that, for all p in K and all $v \in TM_p$ (= the tangent space of M at p) with $\|v\| < \varepsilon_K$, $\exp_p v$ is defined. Now given a continuous function $\tau : M \to \mathbf{R}$ define for each positive ε less than $\varepsilon_K/3$ the function τ_ε by

$$\tau_\varepsilon(q) = \frac{1}{\varepsilon^n} \int_{v \in TM_q} \varkappa\left(\frac{\|v\|}{\varepsilon}\right) \tau(\exp_q v),$$

where the integral is taken relative to the Lebesgue measure on TM_q determined by the Riemannian metric at q. The notation τ_ε will be used in this sense throughout the remainder of this section. Then there is a neighborhood U of K on which the functions τ_ε are all defined; if U is chosen, as it may always be, to have compact closure in M, then for all sufficiently small positive ε, the functions τ_ε will be C^∞ on U. Also $\tau_\varepsilon \to \tau$ uniformly on U as $\varepsilon \to 0^+$, and if τ is C^∞ in a neighborhood of a subset K_1 of K then $d_{K_1}(\tau, \tau_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (the notation d_{K_1} was introduced in paragraph 1). These two properties of the Riemannian convolution process are essentially standard facts about convolution smoothing with a kernel [*see* [5] (*a*)]. It follows that the approximation estimates (*a*) and (*b*) in the definition 1.4 in paragraph 1 of the semilocal approximation property hold if the τ_ε are used as approximations. Whether or not τ_ε will be, for all sufficiently small ε, a section of a subsheaf $\mathscr{S}$ of the sheaf of germs of continuous functions when τ is a section of $\mathscr{S}$ depends of course on which subsheaf $\mathscr{S}$ is. This property does hold for some geometrically interesting subsheaves $\mathscr{S}$; some of these will be discussed in the following paragraphs.

The first of these subsheaves to be considered is the sheaf of germs of functions which are locally Lipschitz continuous with Lipschitz constant less than B. To make a precise definition, let f be a function defined in a neighborhood of p; f is by definition *Lipschitz continuous at p with Lipschitz constant less than* B if there are positive numbers r and B_r,

$B_r < B$, such that f is defined on the open ball about p of radius r and for every q_1, q_2 in that open ball $|f(q_1)-f(q_2)| \leqq B_r \operatorname{dis}_M(q_1, q_2)$. Here dis_M is the Riemannian distance function. Then the subsheaf $\mathscr{S}_{LcB}$ to be considered is the sheaf of germs $\{[f]_p : p \in M$ and f is Lipschitz continuous at p with Lipschitz constant less than $B\}$. The relevance of this subsheaf to standard ideas of Lipschitz continuity is explained in the following lemma [*cf.* [5] (*e*)].

LEMMA 2.1. — *A continuous function* $f: M \to R$ *is a section of* $\mathscr{L}_{LcB}$ *if and only if for each compact set* K *in* M *there is a number* $B_K < B$ *such that, for all* $p, q \in K$, $|f(p)-f(q)| < B_K \operatorname{dis}_M(p, q)$.

Proof. — Suppose f satisfies the latter condition. For each $p \in M$ there is an open ball around p with compact closure K. Then for any q_1, q_2 in this open ball $|f(q_1)-f(q_2)| \leqq B_K \operatorname{dis}_M(q_1, q_2)$. Hence $[f]_p \in \mathscr{S}_{LcB}$ for each $p \in M$ and f is a section of $\mathscr{S}_{LcB}$. Conversely, suppose f is a section of $\mathscr{S}_{LcB}$ and suppose that K is a compact subset of M for which no constant B_K having the property required exists. Then there are sequences $\{p_i\}$ and $\{q_i\}$ of points in K and $\{B_i\}$ of real numbers such that $\lim\inf B_i \geqq B$ and $p_i \neq q_i$ for any i and $|f(p_i)-f(q_i)| \geqq B_i \operatorname{dis}_M(p_i, q_i)$. By passing to a subsequence if necessary assume $p_i \to p$ and $q_i \to q$, $p, q \in K$. Two cases might arise: $p=q$ or $p \neq q$. If $p=q$, then choose $r > 0$ and $B_r > 0$ as in the definition of Lipschitz continuity with constant less than B at p. If i is then so large that

$$\operatorname{dis}_M(p, q_i) < r \quad \text{and} \quad \operatorname{dis}(p, p_i) < r, \qquad |f(p_i)-f(q_i)| \leqq B_r \operatorname{dis}_M(p_i, q_i)$$

contradicting the facts that $\lim\inf B_i \geqq B$ and $|f(p_i)-f(q_i)| \geqq B_i \operatorname{dis}(p_i, q_i)$. So it must be that $p \neq q$. By continuity considerations $|f(p)-f(q)| \geqq B \operatorname{dis}_M(p, q)$. Let $C : [0, 1] \to M$ be a rectifiable curve of length $l(C)$ less than $\varepsilon + \operatorname{dis}_M(p, q)$ with $C(0)=p$ and $C(1)=q$: such a curve exists for any positive number ε. Choose r and $B_r(< B)$ as in the definition of Lipschitz continuity at p with constant less than B. Assume as is always possible that $r <$ the length of C. Now choose $\delta \in [0, 1]$ such that $C([0, \delta])$ is contained in the open ball of radius r around p and $\operatorname{dis}_M(p, C(\delta))=r/2$. For instance

$$\delta=\inf\{t \in [0, 1] : \operatorname{dis}_M(p, C(t)) \geqq r/2\}$$

would do. Let $C_1 : [\delta, 1] \to M$ be $C|[\delta, 1]$. Note that a standard Lebesgue number argument shows that there is a partition $\delta=t_1 < \ldots < t_k=1$ of $[\delta, 1]$ such that the C_1-image $C_1([t_i, t_{i+1}])$ is contained in an open ball on which f is Lipschitz continuous with Lipschitz constant less than B so that

$$|f(C(t_i)) - f(C(t_{i+1}))| \leqq B \operatorname{dis}_M(C(t_i), C(t_{i+1})) \leqq B\, l(C_1 | [t_i, t_{i+1}]).$$

Adding these inequalities yields

$$|f(C(\delta)) - f(C(1))| \leqq B\, l(C_1).$$

Thus

$$|f(p)-f(q)| \leqq |f(p)-f(C(\delta))| + |f(C(\delta))-f(q)| \leqq \frac{r}{2} B_r + B\, l(C_1).$$

Clearly $l(C) \geqq (r/2) + l(C_1)$ so

$$\varepsilon + \text{dis}_M(p, q) \geqq \frac{r}{2} + l(C_1)$$

and

$$\frac{r}{2} B_r + B\, l(C_1) \leqq \frac{r}{2} B_r + B\left(\varepsilon + \text{dis}_M(p, q) - \frac{r}{2}\right).$$

But

$$\lim_{\varepsilon \to 0^+} \left[\frac{r}{2} B_r + B\left(\varepsilon + \text{dis}_M(p, q) - \frac{r}{2}\right)\right] = B \text{ dis } (p, q) - \frac{r}{2}(B - B_r) < B \text{ dis}_M(p, q)$$

so $|f(p) - f(q)| < B \text{ dis}_M(p, q)$. This contradiction of the previously obtained estimate $|f(p) - f(q)| \geqq B \text{ dis}_M(p, q)$ completes the proof.

The sheaf $\mathscr{S}_{LcB}$ has the semilocal approximation property: it was shown in [5] (*b*) that if τ is a section of $\mathscr{S}_{LcB}$ in a neighborhood of a compact set K then the τ_ε obtained from the Riemannian convolution smoothing process are, for all sufficiently small (positive) ε, also sections of $\mathscr{S}_{LcB}$ in a neighborhood of K. That $\mathscr{S}_{LcB}$ has the maximum closure and C^∞ stability properties is clear from the definition. Thus one obtains from Theorem 1.1 the following proposition directly by noting that the C^∞ sections of $\mathscr{S}_{LcB}$ are precisely those C^∞ functions f such that $\|\text{grad } f\| < B$ everywhere.

PROPOSITION 2.1. — *The C^∞ function $f : M \to \mathbf{R}$ such that $\|\text{grad } f\| < B$ everywhere on* M *are dense in the C^0 fine topology in the set of all sections of $\mathscr{S}_{LcB}$, i. e. the set of all continuous functions on* M *which are locally Lipschitz continuous on* M *with local Lipschitz constants less than* B.

COROLLARY [4]. — *There is a C^∞ function f on a Riemannian manifold* M *with $\|\text{grad } f\| < 1$ everywhere on* M *and with $f^{-1}((-\infty, \alpha])$ compact for all $\alpha \in \mathbf{R}$ if and only if* M *is complete.*

Proof of the Corollary. — If such a function $f : M \to \mathbf{R}$ exists, then, for any $p \in M$ and any $r \geqq 0$, the set $\{q \in M \mid \text{dis}(p, q) \leqq r\}$ is a closed subset of the set $\{q \in M \mid |f(p) - f(q)| \leqq r\}$ and hence of the compact set $f^{-1}((-\infty, |f(p)| + r))$. Thus $\{q \in M \mid \text{dis}(p, q) \leqq r\}$ is compact so M is complete by the Hopf-Rinow Theorem. Conversely, if M is any Riemannian manifold and p is a point of M then the function $q \to (1/2) \text{ dis}(p, q)$ is a section of $\mathscr{S}_{Lc1}$. There is a C^∞ function f such that $\|\text{grad } f\| < 1$ and $|f(q) - (1/2) \text{ dis}(p, q)| < 1$ for all $q \in M$, according to the Proposition. Then $f^{-1}((-\infty, \alpha])$ is a closed subset of $\{q \in M \mid (1/2) \text{ dis } (p, q) \leqq 1 + \alpha\}$. If M is complete, the latter set is compact and hence $f^{-1}((-\infty, \alpha])$ is also compact. □

The second class of geometrically significant subsheaves $\mathscr{S}$ for which the semilocal approximation property can be established using the Riemannian convolution smoothing process is the class of sheaves of germs of functions satisfying a particular local lower bound on their convexity. To describe these precisely, the following definitions are useful.

DEFINITION. — Let $f : M \to \mathbf{R}$ be a continuous function on a Riemannian manifold M; f is *convex* if, for any geodesic $C : [-\lambda, \lambda] \to M$, $2f(C(0)) \leqq f(C(-\lambda)) + f(C(\lambda))$.

Convexity is a local property: a function $f : M \to \mathbf{R}$ is convex if and only if it is convex in a neighborhood of each point of M. This fact follows immediately from the elementary observation that a function on an interval in $\mathbf{R}$ is convex if and only if it is locally convex. Thus if $\mathscr{S}_c$ = the sheaf on M of germs of locally convex functions, then the sections of $\mathscr{S}_c$ are exactly the convex functions on M.

DEFINITION. — Let $f : M \to \mathbf{R}$ be a continuous function on a Riemannian manifold M and ξ be a real number. The function f is ξ-*convex* at a point $p \in M$ if there is a positive constant δ such that the function $q \to f(q) - (1/2)(\xi + \delta)\, \mathrm{dis}^2(p, q)$ is convex in a neighborhood of p. (Here dis = Riemannian distance.) If $\eta : M \to \mathbf{R}$ is a continuous function, then f is η-*convex* on M if, for each $p \in M$, f is $\eta(p)$ convex at p.

The property of being η-convex is a local property from its very definition. Thus if $\mathscr{S}_{\eta c}$ is the sheaf of germs of locally η-convex functions, then the sections of $\mathscr{S}_{\eta c}$ on M are precisely the η-convex functions on M.

The ξ-convexity of a C^2 function f at a point p means exactly that the second derivative of f at p along every are length parameterized geodesic issuing from p is greater than ξ at p. Similarly, η-convexity means that those second derivatives at p are greater than $\eta(p)$ for each p in M. A function is *strictly convex* [*cf.* [5] (*e*)] if it is locally the sum of a convex function and a C^∞ function which has positive second derivatives along geodesics. It is easy to see that a (continuous) function f on M is strictly convex if and only if it is 0-convex on M in the sense of the previous definition. The notation $\mathscr{S}_{sc}$ will be used for the sheaf of germs of strictly convex functions; as noted, $\mathscr{S}_{sc} = \mathscr{S}_{0c}$, where $0 : M \to \mathbf{R}$ is the zero function.

It was shown in [5] (*a* and *e*) that if τ is a strictly convex function on M and K is a compact subset of M then there is a neighborhood of K and a positive number ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$, τ_ε is (C^∞ and) strictly convex on a fixed neighborhood of K where τ_ε is the function obtained from τ by Riemannian convolution, as defined at the beginning of this section. The same proof applies to show that if η is a continuous function on M, if τ is an η-convex function on M, and if K is again a compact subset of M, then there is a neighborhood of K and an ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, τ_ε is η-convex on the ceighborhood. As an alternative to rephrasing the entire proof, one could deduce the η-convexity statement from the strict convexity one as follows: An η-convex function is, for each $p \in M$, the sum in a neighborhood of p of a strictly convex function τ_1 and a C^∞ function τ_2 namely

$$\tau(q) = \left(\tau(q) - \frac{1}{2}\left(\eta(p) + \frac{1}{2}\delta\right)\mathrm{dis}^2(p, q)\right) + \frac{1}{2}\left(\eta(p) + \frac{1}{2}\delta\right)\mathrm{dis}^2(p, q).$$

Here δ is as in the definition of $\eta(p)$-convexity at p, and the function

$$q \to \tau(q) - \frac{1}{2}(\eta(p) + \frac{1}{2}\delta)\ \mathrm{dis}^2(p, q)$$

is strictly convex near p because it is the sum of the convex function

$$q \to \tau(q) - \frac{1}{2}(\eta(p)+\delta)\ \mathrm{dis}^2(p, q)$$

and the C^∞ function

$$q \to \left(\frac{1}{2}\delta\right)\frac{1}{2}\ dis^2\ (p, q),$$

which has positive second derivatives along geodesics near p. For sufficiently small ε, $(\tau_1)_\varepsilon$ is strictly convex in a fixed neighborhood of p. Since τ_2 is C^∞ near p, the second derivatives along geodesics of $(\tau_2)_\varepsilon$ converge uniformly in a fixed neighborhood of p to those of τ_2 as $\varepsilon \to 0^+$. The second derivatives of τ_2 along (arc-length parameterized) geodesics are $\eta(p)+(1/2)\delta$ at p and so in some neighborhood of p they are greater than $\eta(p)+(1/4)\delta$. Thus in some (slightly smaller) neighborhood of p, the second derivatives of $(\tau_2)_\varepsilon$ are greater than $\eta(p)+(1/4)\delta$ for all sufficiently small ε. Since $\tau_\varepsilon=(\tau_1)_\varepsilon+(\tau_2)_\varepsilon$, one concludes that there is a neighborhood of p such that for all sufficiently small ε, τ_ε is $[(1/4)\delta+\eta(p)]$-convex at every point of the neighborhood. But $(1/4)\delta+\eta(p) > \eta(q)$ for all q sufficiently near p. Hence, on some neighborhood of p, τ_ε is η-convex for all sufficiently small ε. The required conclusion about τ_ε in a neighborhood of K now follows by covering K with finitely many such neighborhoods of points $p \varepsilon K$.

Since the approximation properties of the τ_ε relative to τ are, as observed earlier in this section, automatic, it follows that each $\mathscr{S}_{\eta c}$ and in particular $\mathscr{S}_{sc}$ has the semilocal approximation property. It is easy to check that $\mathscr{S}_{\eta c}$ has the C^∞ stability and maximum closure properties so the hypotheses of Theorem 1.1 are satisfied and the following Proposition follows. The second statement was established in [5] (*e*), where numerous geometric applications are also given.

PROPOSITION 2.2. — *For any continuous function* $\eta : M \to \mathbf{R}$, *the* C^∞ *sections of* $\mathscr{S}_{\eta c}$ *are dense in the* C^0 *fine topology in the set of all sections. In particular, the* C^∞ *strictly convex functions on any Riemannian manifold* M *are dense in the* C^0 *fine topology in the set of continuous strictly convex functions.*

If $\mathscr{S}_1$ and $\mathscr{S}_2$ are two subsheaves of the sheaf of germs of continuous functions on a manifold M each of which has the maximum closure property then $\mathscr{S}_1 \cap \mathscr{S}_2$ also has the maximum closure property: this fact is an immediate consequence of the definition of the property. Also, if $\mathscr{S}_1$ and $\mathscr{S}_2$ have the C^∞ stability property, then $\mathscr{S}_1 \cap \mathscr{S}_2$ does; this is also obtainable immediately from the definition. But if $\mathscr{S}_1$ and $\mathscr{S}_2$ has the semilocal approximation property then *a priori* the C^∞ approximations for $\mathscr{S}_1$ sections might be obtained by an entirely different process from that used for the $\mathscr{S}_2$ sections so that no conclusion could be drawn about whether or not $\mathscr{S}_1 \cap \mathscr{S}_2$ would also have the semilocal approximation property. If, however, the semilocal approximations for both $\mathscr{S}_1$ and $\mathscr{S}_2$ sections are obtainable by the Riemannian convolution smoothing process, then it is again immediate that $\mathscr{S}_1 \cap \mathscr{S}_2$ has the semilocal approximation property. In particular, if $\mathscr{S}_{\eta c B}$ is defined to be $\mathscr{S}_{LcB} \cap \mathscr{S}_{\eta c}$ ($\eta : M \to \mathbf{R}$ a continuous function, B a positive number) then $\mathscr{S}_{\eta c B}$ has the semilocal approximation property, as well as the maximum closure and C^∞ stability properties. From Theorem 1.1, one then obtains a proposition of a by-now familiar form. (In [5] (*b*), a special case of this proposition is established, and some differential geometric applications are discussed.)

PROPOSITION 2.3. — *The* C^∞ *sections of* $\mathscr{S}_{LcB} \cap \mathscr{S}_{\eta c}$ *are dense in the* C^0 *fine topology in the set of all sections of* $\mathscr{S}_{LcB} \cap \mathscr{S}_{\eta c}$.

3. Subharmonic Functions

A C^∞ function f on a Riemannian manifold M is *subharmonic* by definition if Δf is nonnegative everywhere on M. If $V_1, \ldots, V_n$ is an orthonormal frame in the tangent space TM_p of M at a point p, then $\Delta f|_p = \sum_{i=1}^{n} D_f^2(V_i, V_i)$ where $D_f^2(V_i, V_i)$ = the second derivative of f at p along the geodesic through p having tangent V_i. Thus a C^∞ convex function is necessarily subharmonic. But of course the class of subharmonic functions is in general much larger than that of convex functions.

It is natural to try to extend the definition of subharmonicity to continuous, not necessarily C^∞ functions. Unfortunately, to do so in terms of the behavior of the function along geodesics is difficult. What is needed is a characterizing property of C^∞ subharmonic functions that is easily extended to the case of continuous functions. One such property is that of being a subsolution of Dirichlet problems in the sense of the following definition:

DEFINITION. — A continuous function $f: M \to \mathbf{R}$ on a Riemannian manifold M is *a subsolution of the Dirichlet problem determined by its boundary values* for an open set U in M with compact closure, if f has the following property: if h is any function harmonic on U and continuous on $\overline{U}$ with $h(q) \geqq f(q)$ for all $q \in \overline{U} - U$ then $h(q) \geqq f(q)$ for all $q \in U$.

A C^∞ function is subharmonic if and only if it is a subsolution of the Dirichlet problems determined by its boundary values for all compact closure open sets U in M: this fact is well known and follows in any case from results later in this section.

A second characterizing property of subharmonicity of C^∞ functions, which also extends immediately to continuous functions, is the satisfying of suitable local maximum principle.

DEFINITION. — A continuous function $f: M \to \mathbf{R}$ *satisfies the local harmonic maximum principle* if for any point $p \in M$ and any (C^∞) function h defined and harmonic in a neighborhood of p the function $f - h$ has a local maximum at p only if $f - h$ is constant in a neighborhood of p.

A C^∞ function is subharmonic if and only if it satisfies the local harmonic maximum principle. This fact is again well known and also is a special case of results proved later.

That a continuous function $f: M \to \mathbf{R}$ which satisfies the local harmonic maximum principle is a subsolution of the Dirichlet problems determined by its boundary values for all compact closure open sets U is well known and easily established by elementary considerations (*see* e.g., [1], pp. 135-137) for a proof in case $M = \mathbf{R}^2$, which proof is easily extended to the general case). The converse holds, but a stronger result than the direct converse also holds: If a continuous function $f: M \to \mathbf{R}$ is a subsolution of the Dirichlet problems for all a collection of compact closure open sets U which together form a basis for the topology of M then f satisfies the local harmonic maximum principle. (The proof of this essentially standard fact is again a straightforward extension of the proof for $M=\mathbf{R}^2$ in [1].)

Harmonic functions on a Riemannian manifold can be treated from the viewpoint of the axiomatic potential theory of Brelot (*see* [3]; also [6]): the "harmonic functions" of the theory are just to be the harmonic functions in the usual sense of solutions of $\Delta=0$, and the "regular open sets" of the theory to be (sufficiently small) open balls $B(x;\varepsilon)$, $0<\varepsilon<\varepsilon(x)$, $x\in M$, where $\varepsilon: M\to\mathbf{R}$ is an arbitrary positive function. It is immediate that the axioms are satisfied. The theory then provides a definition of subharmonicity of a continuous function: $f: M\to\mathbf{R}$ is subharmonic if and only if, for all regular open sets U and points $x\in U$, $f(x)\leqq\int f d\rho_x^U$, where ρ_x^U is the harmonic measure determined by U and x. $\Big($More generally, a not necessarily continuous function $f: M\to\mathbf{R}\cup\{-\infty\}$ is defined to be subharmonic if it is upper semicontinuous, not $\equiv -\infty$ on any component of M, and satisfies $f(x)\leqq\int f d\rho_x^U$ for all regular open sets U and $x\in U$, as before.$\Big)$ Since the function $x\to\int f d\rho_x^U$ is the solution of the Dirichlet problem determined by the boundary values of (the continuous function) f, it is clear that a continuous function f is subharmonic in the sense of this definition if and only if f is a subsolution of the Dirichlet problems determined by its boundary values for all regular open sets forms U. Since the set of regular open sets forms a basis, it follows that all three concepts of subharmonicity—the subsolution property, the local harmonic maximum principle property, and the axiomatic theory definition—are equivalent (for continuous functions).

The following definition/lemma summarizes these considerations:

Definition. — A continuous function $f: M\to\mathbf{R}$ on a Riemannian manifold M is *subharmonic* if it has any (and hence all) of the following three properties:

(i) It is a subsolution of the Dirichlet problems determined by its boundary values for all compact closure open sets $U\subset M$.

(ii) It satisfies the local harmonic maximum principle.

(iii) It is subharmonic in the sense of axiomatic potential theory with "harmonic functions" being solutions of $\Delta=0$ and "regular open sets" being sufficiently small balls around each point.

Clearly subharmonicity in the sense of this definition is a local property: (ii), for instance, is obviously a local condition, and (iii) is shown to be local in the development in [3]. Set $\mathscr{S}_{sh}$ = the sheaf of germs of continuous, locally subharmonic functions. Then a continuous function $f: M\to\mathbf{R}$ is subharmonic if and only if it is a section of $\mathscr{S}_{sh}$.

If f is a C^∞ function on M and g is a C^∞ function of compact support on M, then $\int_M(\Delta f)g=\int f(\Delta g)$. Thus one is led to define the Laplacian of a continuous function f on M as a distribution by

$$(\Delta f)g=\int f\,\Delta g,$$

for g, C^∞ compact support. One says that Δf is nonnegative if $(\Delta f)g$ is nonnegative for all nonnegative C^∞ functions g of compact support on M; if f is a C^∞ subharmonic function on M then $(\Delta f)g=\int_M g(\Delta f)$ so Δf is nonnegative in the distribution sense. Moreover, it follows from the definition that if f is any function (not necessarily C^∞) which is a limit uniformly on compact sets of functions with nonnegative distribution Laplacian then f has a nonnegative distribution Laplacian. Thus in investigating the approximation properties of $\mathscr{S}_{sh}$ it is reasonable to consider first the distribution Laplacian of the section of $\mathscr{S}_{sh}$.

LEMMA 3.1. — *A continuous function f: $M \to R$ is subharmonic if and only if the distribution Laplacian Δf is nonnegative.*

It is possible to prove this result by direct argument: the fact that nonnegativity of the distribution Laplacian implies subharmonicity is proved directly in [9] and a direct proof of the converse can also be given. However, if one takes for granted the machinery of axiomatic potential theory, in particular the results of [7], then a very short proof of the lemma becomes possible: For a discussion of this and other, related results (in the more general context of not necessarily continuous functions) *see*, for instance, ([6] pp, 1-13); the results there are for M = an open subset of $\mathbf{R}^n$, but the proofs are essentially in the context of the axiomatic theory so that they apply in the present situation. For the convenience of the reader, an outline of the proof is given here:

Outline of Proof of Lemma 3.1. — Since both properties — subharmonicity and nonnegativity of the distribution Laplacian — are local, it is sufficient to consider the question of their equivalence locally. Let p be any point in M and B be an open ball about p of radius so small that $\overline{B}$ is compact and that $(B, \overline{B})$ is diffeomorphic to (open unit ball, closed unit ball) in $\mathbf{R}^n$. There exists a Green's function $g : B \times B \to (0, +\infty]$, with the property that $g(x, y)$ is for each fixed $y \in B$ asymptotically equal to $(1/\alpha_n)(\operatorname{dis}(x, y))^{2-n}$ as $x \to y$ where α_n is Poisson's constant (for $n \neq 2$; for $n=2$, the asymptotic behavior is $(2\pi)^{-1} \log \operatorname{dis}(x, y)$). Then g is a fundamental solution of Laplace's equation, i. e. :

$$\int g(x, y)\,\Delta\varphi\, dx = -\varphi(y)$$

for any C^2 function φ with compact support in B, where $\int dx$ is integration relative to the Riemannian volume measure for the point x. If μ is a nonnegative Radon measure of compact support on B with the property that $G\mu \overset{\text{def}}{\equiv} \int g(.,y)\,d\mu(y)$ is not $\equiv +\infty$, then by Fubini's theorem

$$\int G\mu\,\Delta\varphi\, dx = -\int \varphi\, d\mu,$$

for all C^2 functions of compact support in B. With these notations in mind, suppose that f: $M \to \mathbf{R}$ is a continuous subharmonic function. Then the results of [7] applied to the "harmonic space" in the Brelot sense determined by taking as "harmonic functions" the solutions of $\Delta=0$ and as "regular open sets" sufficiently small balls, as before, yield that $f \mid B$ may be written as $-p+h$, where is harmonic and $p=G\mu$, for some nonnegative μ (this is the generalized Riesz decomposition theorem). Then $\Delta f=\Delta h-\Delta p=-\Delta p$ and

$$(\Delta f)\varphi=(-\Delta p)\varphi=\int \varphi\, d\mu \geqq 0,$$

for any C^2 compact-support-in-B function φ so that Δf is a nonnegative distribution (on B). Conversely, suppose that f is a continuous function with nonnegative distribution Laplacian $u=\Delta f$. Let D be any open set with compact closure in B and set $u_D=u\times$ characteristic function of D. Then $\Delta(G u_D)$, in the distribution sense, is equal to $-u_D$ since as noted earlier (with φ as then):

$$\int (G u_D)(\Delta\varphi)=-\int \varphi\, u_D.$$

Thus $\Delta(f+G u_D)$ is zero on D as a distribution. By the ellipticity of $\Delta, f+G u_D$ is equal almost everywhere to a (C^∞) harmonic function on D. Since $-G u_D$ is subharmonic and since the sum of a subharmonic and a harmonic function is by trivial considerations subharmonic, it follows that f is equal a. e. to a subharmonic functions. But a continuous function which is equal almost everywhere to a subharmonic function is itself subharmonic. □

The sheaf $\mathscr{S}_{sh}$ has the maximum closure property: this follows easily from the (subsolution-property) definition of subharmonicity. It will be shown shortly that $\mathscr{S}_{sh}$ also has the semilocal and convex composition properties. But $\mathscr{S}_{sh}$ does not have the C^∞ stability property, e. g., 0 is a section of $\mathscr{S}_{sh}$ but obviously there are arbitrarily small (in the C^∞ sense) perturbations of 0 that are not subharmonic. Definitions of strict subharmonicity, and, more generally, η-subharmonicity, analogous to the definitions of strict convexity and η-convexity, lead to sheaves having the C^∞ stability property in addition to the maximum closure and semilocal approximation properties.

DEFINITION. — If $\eta : M\to \mathbf{R}$ and $f : M\to\mathbf{R}$ are continuous functions, then f is η-*subharmonic* if for each point p in M there is a positive number δ such that the function

$$q\to f(q)-\left(\delta+\frac{1}{2n}\eta(p)\right)\operatorname{dis}^2(p, q)$$

is subharmonic in a neighborhood of p. A continuous function $f : M\to\mathbf{R}$ is *strictly subharmonic* if it is 0-subharmonic.

Strict subharmonicity and more generally η-subharmonicity are obviously local properties. Set $\mathscr{S}_{\eta sh}$ = the sheaf of germs of (continuous) η-subharmonic functions and $\mathscr{S}_{ssh}=\mathscr{S}_{0sh}$ = the sheaf of germs of (continuous) strictly subharmonic functions. Then of course a continuous function $f : M\to\mathbf{R}$ is a section of $\mathscr{S}_{\eta sh}(\mathscr{S}_{ssh})$ if and only if f is η-subharmonic (resp., strictly suharmonic).

A C^∞ function $f : M \to \mathbf{R}$ is η-subharmonic if and only if $(\Delta f)(p) > \eta(p)$ for all $p \in M$. This fact is an immediate consequence of the elementary formula $\Delta \operatorname{dis}^2(.,p)|_p = 2n$. Lemma 3.1 implies immediately that a continuous function $f : M \to \mathbf{R}$ is η-subharmonic if and only if (in the notation of the definition of η-subharmonicity) $\Delta f - (\delta + \eta(p))$ is a nonnegative distribution in a neighborhood of p. From this one concludes by a standard argument the following Lemma:

LEMMA 3.2. — *A continuous function $f : M \to \mathbf{R}$ is η-subharmonic on M if and only if there is a C^∞ function $\eta_1 : M \to \mathbf{R}^+$ such that everywhere $\Delta f \geqq \eta_1$ (in the sense that $\Delta f - (\eta_1)$ is a nonnegative distribution on M) and such that $\eta_1 > \eta$ on M.*

It is evident from Lemma 3.2 (or in fact directly from the definition) that $\mathscr{S}_{\eta sh}$ and in particular $\mathscr{S}_{ssh}$ have the C^∞ stability property. It is also easy to see that each $\mathscr{S}_{\eta sh}$ has the maximum closure property: if

$$f_1 - \left(\delta_1 + \frac{1}{2n}\eta(p)\right) \operatorname{dis}^2(\ ,p) \qquad \text{and} \qquad f_2 - \left(\delta_2 + \frac{1}{2n}\eta(p)\right) \operatorname{dis}^2(\ ,p)$$

are subharmonic in a neighborhood of p, then one checks easily using the maximum closure property for subharmonic functions that

$$\max(f_1, f_2) - \left(\min(\delta_1, \delta_2) + \frac{1}{2n}\eta(p)\right) \operatorname{dis}^2(\ ,p)$$

is subharmonic in a neighborhood of p. It remains to investigate the semilocal approximation property for $\mathscr{S}_{\eta sh}$ and $\mathscr{S}_{sh}$.

THEOREM 3.1. — (i) *For any continuous $\eta : M \to \mathbf{R}$, $\mathscr{S}_{\eta sh}$ has the semilocal approximation property.* (ii) $\mathscr{S}_{sh}$ has the semilocal approximation property.

COROLLARY 1. — *The C^∞ sections of $\mathscr{S}_{\eta sh}$ are dense in the sections of $\mathscr{S}_{\eta sh}$ in the C^0 fine topology. The C^∞ sections of $\mathscr{S}_{sh}$ are dense in the section of $\mathscr{S}_{sh}$ in the compact-open topology.*

The first statement follows from the present theorem combined with Theorem 1.1 and the observations of the paragraph preceding the present theorem. The second statement is trivial if M is compact for then any section of $\mathscr{S}_{sh}$ is constant by the maximum principle. If M is noncompact, there exists a C^∞ function $g : M \to \mathbf{R}$ such that $\Delta g = 1$ on M. The existence of such a function g is a consequence of the local solvability of elliptic equations together with the Lax-Malgrange theorem on approximation of local solutions by global ones in the case of second order elliptic equations (or more generally arbitrary order with adjoint having the unique continuation property: *see* [5] (*d*) for details and the original references). If f is a section of $\mathscr{S}_{sh}$ and K a compact subset of M, then $f + \lambda g$ with λ a sufficiently small positive number approximates f in the C^0 sense near K. Now $f + \lambda g$ is a section of $\mathscr{S}_{ssh}$ so $f + \lambda g$ can be approximated in the C^0 sense near K (in fact, globally) by a C^∞ strictly subharmonic function, which is necessarily a section of $\mathscr{S}_{sh}$.

COROLLARY 2. — *$\mathscr{S}_{sh}$ has the convex composition property, and the C^∞ subharmonic exhaustion functions are dense in the compact-open topology in the continuous subharmonic exhaustion functions.*

The second part follows from Theorem 1.2, the first part, and the maximum closure property of $\mathscr{S}_{sh}$. For the first part, note that if f is a section of $\mathscr{S}_{sh}$ in a neighborhood of $p \in M$, then by the theorem there is a sequence $\{ f_i \}$ of C^∞ subharmonic functions defined in a neighborhood of p with $f_i \to f$ uniformly. If $\chi : \mathbf{R} \to \mathbf{R}$ is an increasing convex function, then there exists a sequence of C^∞ functions $\{ \chi_i \}$ defined and convex in a neighborhood of $f(p)$ and converging uniformly on a neighborhood of $f(p)$ to χ. Direct computation shows that $\Delta(\chi_i \circ f_i)$ is nonnegative in a neighborhood of p. Since $\chi_i \circ f_i \to \chi \circ f$ uniformly on a neighborhood of p and since the uniform limit of subharmonic functions is subharmonic, $\chi \circ f$ is subharmonic in a neighborhood of p.

The corollary just established shows that $\mathscr{S}_{sh}$ satisfies the hypotheses of Theorem 1.2. The corollary holds also for $\mathscr{S}_{ssh}$ in place of $\mathscr{S}_{sh}$. But, in terms of construction of approximations, this fact is not needed since $\mathscr{S}_{ssh}$ has the C^∞ stability property and so Theorem 1.1 applies, whereas for $\mathscr{S}_{sh}$ one has only the weaker approximation conclusion of Theorem 1.2.

Proof of Theorem 3.1. — Part (ii) follows from part (i) by an argument similar to the deduction of the second statement of Corollary 1; namely, suppose that f is a continuous subharmonic function defined on an open neighborhood U of a compact set K in M. If U is compact (i. e. if M is compact and U = M), then f is constant by the maximum principle and there is nothing to prove. If U is noncompact, there exists a C^∞ function $g : U \to \mathbf{R}$ such that $\Delta g = 1$ on U (the existence of g is obtained by applying the argument already given to the noncompact manifold U). The function $f + \lambda g$, with a sufficiently small positive number λ, approximates f in the C^0 sense near K and in the C^∞ sense on that part of K on which f is C^∞, and $f + \lambda g$ is strictly subharmonic. Assuming part (i) for the moment, one concludes that $f + \lambda g$ may be approximated by a C^∞ strictly subharmonic function near K in the required C^0 and C^∞ senses and thus that there is a C^∞ (strictly) subharmonic approximation of f of the required sort.

Before undertaking the proof of part (i) of Theorem 3.1, note that the deduction of part (ii) from part (i) depends in an essential way upon the existence of a strictly subharmonic function defined in a whole neighborhood of K, not just in neighborhoods of different points of K. As noted, such strictly subharmonic functions always exist (except in the trivial case of U compact). But given a compact set K it may very well be the case that no strictly *convex* function exists on any neighborhood of K. This is the case if, for instance, K contains a closed geodesic. It is for this reason that the sheaf of germs of convex functions cannot be shown by the pressent methods to have the semilocal approximation property even though it was shown in paragraph 2 that the sheaf of germs of strictly convex functions does. It is at present unknown to the authors whether or not the sheaf of germs of convex functions has in general the semilocal approximation property.

Returning now to the proof of part (i) of Theorem 3.1, suppose that K is a compact subset, that U is a neighborhood of K, and that f is a continuous η-subharmonic function on U. The case K = U = M, which can occur if M is compact and η is negative somewhere on M, requires special consideration and will be dealt with later. Suppose now that $K \neq M$ so that, by choosing a smaller U if necessary, U may be taken to be noncompact. Choose an open set V with C^∞ boundary and with $K \subset V \subset \overline{V} \subset U$. Set N be the double of V; $N = \overline{V} \cup \overline{V}_1$ where $\overline{V} \cap \overline{V}_1 =$ the boundary of V. By standard extension theorems, there exists a Riemannian metric on N with the property that in a neighborhood W of K, with $\overline{W} \subset V$, this metric equals the Riemannian metric of M restricted to W. Also there exists a continuous function $\tilde{f} : N \to \mathbf{R}$ such that $\tilde{f} \mid W = f$. Choose any such $\tilde{f}$. Then $\tilde{f}$ is η-subharmonic on W, but not of course on all of N in general (indeed, η-subharmonicity is not at the moment and need not ever be defined on N as a whole).

Now consider the function $F(x, t) : N \times \{ t \in \mathbf{R} \mid t \geqq 0 \} \to \mathbf{R}$ obtained by solving the heat equation

$$\frac{\partial F}{\partial t} = \Delta F$$

on N with the initial condition $F(x, 0) = \tilde{f}(x)$, $x \in N$. By standard theorems there is a function F which is such a solution in the sense that: (1) F is continuous on $N \times \{ t \in \mathbf{R} \mid t \geqq 0 \}$ (2) F is C^∞ on $N \times \{ t \in \mathbf{R} \mid t > 0 \}$ and $\partial F / \partial t = \Delta F$ on that set and (3) $F(x, 0) = \tilde{f}(x)$. Moreover, F has the property that (4) if $\tilde{f}$ is C^∞ in a neighborhood of a compact set $K_1 \subset N$ then the derivatives of all orders of $F(\ , t) : N \to \mathbf{R}$ converge uniformly on K_1 as $t \to 0^+$ to the corresponding derivatives of $\tilde{f}$. Thus, the functions $F(\ , t) : N \to \mathbf{R}$, $t \to 0^+$, form a family that when restricted to a neighborhood of K approximate f in the sense of the definition of the semilocal approximation property. To check that $F(\ , t)$ is η-subharmonic in a neighborhood of K, some further properties of the heat equation need to be used.

For notational convenience, write $H_t \tilde{f} : N \to \mathbf{R}$ for $F(\ , t) : N \to R$. It is again a standard result that even if g is only a distribution, rather than a continuous function, $H_t g$ is still defined in the sense that there is a unique C^∞ function $G : N \times \{ t \in \mathbf{R} \mid t > 0 \} \to \mathbf{R}$ satisfying $\partial G / \partial t = \Delta G$ and having the property that $G(\ , t) \to g$ as $t \to 0^+$, where convergence of $G(\ , t)$ is in the sense of distributions. In fact, it is known [2] that the heat operator is given by convolution with a kernel, in the sense that there is a C^∞ function $k : N \times N \times \{ t \in \mathbf{R} \mid t > 0 \} \to \mathbf{R}$ such that

$$(H_t h)(x) = \int_{y \in N} h(y) k(x, y, t) dy, \qquad t > 0,$$

for any continuous function $h : N \to \mathbf{R}$; more generally, if h is a distribution on N then $(H_t h)(x)$ is h operating on $k(x, y, t)$ with respect to the y variable. Also, k has the properties that, as $t \to 0^+$, $k(x, y, t)$ converges to 0 uniformly in the C^∞ topology on any compact subset of $N \times N - \{ (x, x) \mid x \in N \}$ and that there exists a neighborhood of $\{ (x, x) \mid x \in N \}$ in $N \times N$ such that, for all sufficiently small positive t, $k(x, y, t)$ is positive for all (x, y) in the neighborhood. Finally, it is known that H_t commutes with Δ_N, i. e. for any distribution h:

$$\Delta (H_t h) = H_t (\Delta h) \quad \text{for all } t > 0.$$

From this fact, one can deduce that $H_t \tilde{f}$ is η-subharmonic on some neighborhood of K for all sufficiently small t, in the following way.

First, choose $\delta > 0$ so small that $\tilde{f}$ is η-subharmonic on the set $\{ x \in N \mid \text{dis}(x, K) < 4\delta \}$ and that for all $x, y \in N$ with dis $(x, y) < 2\delta$, $k(x, y, t)$ is positive for all sufficiently small t. By a partition of unity on $N \times N$, k may be expressed as $k_1 + k_2$ where k_1 and k_2 are C^∞ functions on $N \times N \times \{ t \in \mathbf{R} \mid t > 0 \}$ with the properties that

$$\text{support of } k_1 \subset \{ (x, y) \in N \times N \mid \text{dis}(x, y) < 2\delta \} \times \{ t \in \mathbf{R} \mid t > 0 \}$$

and

$$\text{support of } k_2 \subset \{ (x, y) \in N \times N \mid \text{dis}(x, y) > \delta \} \times \{ t \in \mathbf{R} \mid t > 0 \}.$$

Then $k_2(\; , \; , t)$ converges to 0 in the C^∞ topology on $N \times N$ as $t \to 0^+$, *and* $k_1 \geqq 0$ for small t.

Now let $g = \Delta f$ (the distribution Laplacian) and let η_1 be a C^∞ function on the neighborhood W of K such that $g > \eta_1$ on W and $\eta_1 > \eta$ on W; the existence of η_1 is guaranteed by Lemma 3.2. As would in fact be the case for any distribution, $g(y) k_2(x, y, t) \to 0$ as $t \to 0^+$ uniformly on N, where $g(y) k_2(x, y, t)$ is the function of x resulting from applying the distribution g to the y variable of $k_2(x, y, t)$ for each fixed t. On the other hand, if η_2 is a C^∞ function on N agreeing with η_1 on $\{ x \in N \mid \text{dis}(x, K) < 3\delta \}$, then for all $x \in N$ such that dis $(x, K) \leqq \delta$:

$$g(y) k_1(x, y, t) \geqq \int \eta_2(y) k_1(x, y, t)\, dy,$$

since $k_1(x, y, t) = 0$ if $\text{dis}(x, y) \geqq 2\delta$ and $g(y) > \eta_1(y)$ for all y with $\text{dis}(y, K) < 3\delta$. Now $\int \eta_2(y) k_1(x, y, t)\, dy \to \eta_2(x)$ uniformly on any compact subset of N as $t \to 0^+$ since $\int \eta_2(y) k(x, y, t)\, dy = H_t \eta_2 \to \eta_2$ uniformly and $\int \eta_2(y) k_2(x, y, t) \to 0$ uniformly as $t \to 0^+$. In particular, for all sufficiently small t :

$$\int \eta_2(y) k_1(x, y, t)\, dy > \frac{1}{2} \eta(x) + \frac{1}{2} \eta_1(x),$$

for all x such that $\text{dis}(x, K) \leqq \delta$. So

$$g(y) k_1(x, y, t) > \frac{1}{2} \eta(x) + \frac{1}{2} \eta_1(x),$$

for those t and x. Since

$$g(y) k(x, y, t) = g(y) k_1(x, y, t) + g(y) k_2(x, y, t),$$

and since $g(y) k_2(x, y, t) \to 0$ uniformly on N as $t \to 0^+$, there is a positive number t_0 such that for all $t \in (0, t_0)$ and all x such that $\text{dis}(x, K) < \delta$:

$$g(y) k(x, y, t) > \frac{3}{4} \eta(x) + \frac{1}{4} \eta_1(x).$$

Now the facts that $g(y)k(x, y, t) = H_t(\Delta \tilde{f})(x) = \Delta(H_t \tilde{f})(x)$ and that $\eta_1(x) > \eta(x)$ for the points x in question imply that, for $t \in (0, t_0)$, $H_t f$ is η-subharmonic on $\{x \in N \mid \text{dis}(x, K) < \delta\}$.

The special case $K = U = M$, consideration of which was postponed at the beginning of the proof, can also be treated by the heat equation method just discussed. One omits the passage to the double N of $\overline{V}$ and uses instead the family $H_t f$ obtained by operating on f by the heat operator H_t of the compact manifold M itself. The remainder of the proof is very similar to that just given and is omitted for brevity.

In geometric applications, Lipschitz continuous subharmonic functions play a special role [*see* [5] (*b*, *e*)]. So it is natural to inquire whether the semilocal approximation property holds for $\mathscr{S}_{sh} \cap \mathscr{S}_{LcB}$. In fact, it does, and this fact can be established by the same constructions used to establish the semilocal approximation for $\mathscr{S}_{sh}$ itself. A similar situation occurs in the cases of $\mathscr{S}_{ssh} \cap \mathscr{S}_{LcB}$ and $\mathscr{S}_{\eta sh} \cap \mathscr{S}_{LcB}$. In certain special applications (*see* [12]) it is useful to have a more refined version of these facts which allows for variation of the local Lipschitz constant. To formulate this refinement, an additional definition is useful :

DEFINITION. — If $\beta : M \to \mathbf{R}$ is a positive continuous function, then a function $f : M \to \mathbf{R}$ is *locally Lipschitz continuous with variable local Lipschitz constant* β if for each $x \in M$ f is a section of $\mathscr{S}_{Lc\beta(x)}$ in a neighborhood of x.

The sheaf of germs of such functions will be denoted by $\mathscr{S}_{Lc\beta}$.

THEOREM 3.2. — *If* $\beta : M \to \mathbf{R}$ *is a positive continuous function and* $\eta : M \to \mathbf{R}$ *is a continuous function on a Riemannian manifold* M, *then* $\mathscr{S}_{sh} \cap \mathscr{S}_{Lc\beta}$ *and* $\mathscr{S}_{\eta sh} \cap \mathscr{S}_{Lc\beta}$ *have the semilocal approximation property. In particular,* $\mathscr{S}_{ssh} \cap \mathscr{S}_{Lc\beta}$ *has the semilocal approximation property.*

COROLLARY 1. — *The* C^∞ *sections of* $\mathscr{S}_{\eta sh} \cap \mathscr{S}_{Lc\beta}$ *(in particular, of* $\mathscr{S}_{ssh} \cap \mathscr{S}_{Lc\beta}$*) are dense in the* C^0 *fine topology in the sections of* $\mathscr{S}_{\eta sh} \cap \mathscr{S}_{Lc\beta}$ *(respectively of* $\mathscr{S}_{ssh} \cap \mathscr{S}_{Lc\beta}$*).*

COROLLARY 2. — *For any positive constants* B *and* ε, *the* C^∞ *subharmonic exhaustion functions with gradient everywhere of length less than* $B + \varepsilon$ *are dense in the compact-open topology in the subharmonic exhaustion functions with local Lipschitz constant less than* B.

As in previous cases, Corollary 1 follows by combining the present theorem with Theorem 1.1. Corollary 2 follows from the present theorem combined with Theorem 1.2′ and the convex combination property of $\mathscr{S}_{sh}$ (Corollary 2 of Theorem 3.1).

Proof of Theorem 3.2. — Postpone consideration of the case $K = U = M$. In case $K \neq M$, perform the construction used in the proof of Theorem 3.1 of the double N of the closure of the neighborhood V of K. Choose the extensions $\tilde{f}$ and $\tilde{G}$ to N of $f \mid W$ and the Riemannian metric G of W so that $\tilde{f}$ is a section of $\mathscr{S}_{Lc\tilde{\beta}}$ for some positive continuous function $\tilde{\beta} : N \to \mathbf{R}$ such that $\tilde{\beta} \mid W = \beta$, $\overline{W} \subset V$. These choices are possible for elementary reasons. In either of the cases, $K = U = M$ or $K \neq M$, it becomes sufficient to prove that if $\beta : N \to \mathbf{R}$ is a positive continuous function on a compact Riemannian manifold and $f : N \to \mathbf{R}$ is a section of $\mathscr{S}_{Lc\beta}$ then $H_t f$ is a section of $\mathscr{S}_{Lc\beta}$ for all sufficiently small t. (That $H_t f$ will be a section of $\mathscr{S}_{\eta sh}$ near K was established in the proof of Theorem 3.1.)

Now let p be a point of N; there exists a finite set $X_1, \ldots, X_n$ of vector fields each with length $\leqq 1$ everywhere on N and length $=1$ in a neighborhood of p such that, on some fixed neighborhood of p, any C^∞ function h with $|X_i h| < \beta(p)$, $i=1, \ldots, n$, is a section of $\mathscr{S}_{Lc\beta}$ on that neighborhood. Specifically, choose $X_1, \ldots, X_n$ to be an orthonormal basis at p and in a neighborhood of p and (for later purposes) to have $DX_i = 0$ at p for all i. Such a choice is possible by standard constructions. For instance, the X_i near p can be determined by Gram-Schmidt orthonormalization of the coordinate vector fields of a Riemannian normal coordinate system centered at p. The conclusion that h is a section of $\mathscr{S}_{Lc\beta}$ near p under the circumstances indicated follows from estimating the derivative of h along any smooth arc near p. A standard compactness argument now shows that to complete the proof it is enough to establish the following: Suppose that X is a vector field of length $\leqq 1$ on N satisfying $DX(p)=0$ for some $p \in N$ and that f is a section on N of $\mathscr{S}_{Lc\beta}$. Then there is a positive number t_0 and a neighborhood U of p such that if $t \in (0, t_0)$ then $|X(H_t f)(x)| < \beta(p)$ for all $x \in U$.

The estimate to be established is of course equivalent to

$$\frac{d}{ds}(H_t f)(\varphi_s(x))\big|_{s=0} < \beta(p),$$

where $\varphi_s : N \times \mathbf{R} \to N$, $s \in \mathbf{R}$, is the one parameter group of diffeomorphism of N determined by the vector field X. In these terms, it is enough to show that there is a constant β_1, less than $\beta(p)$, such that, for all $t \in (0, t_0)$ and $x \in U$, $|(H_t f)(\varphi_s(x)) - (H_t f)(x)| < s\beta_1$ for all sufficiently small s. For notational simplicity, write ${}_sH_t =$ the heat operator at "time" t for the metric $\varphi_s^* \tilde{G}$ on N, i. e. the metric induced on N from $\tilde{G}$ by the map $\varphi_s : N \to N$. Then $(H_t f)(x) = ({}_sH_t(\varphi_s^* f))(x)$ since the heat equation is invariant under isometry of N. Hence

$$(\bigstar) \qquad |(H_t f)(\varphi_s(x)) - (H_t f)(x)| \leqq |{}_sH_t(\varphi_s^* f)(x) - H_t(\varphi_s^* f)(x)| + |H_t(\varphi_s^* f)(x) - H_t f(x)|.$$

Now there is a constant $\beta_2 < \beta(p)$ such that f is a section of $\mathscr{S}_{Lc\beta_2}$ in a neighborhood of p, in particular on $\{x \in N \mid \mathrm{dis}(x, p) < \delta\}$ for some positive number δ. Then, for x in the set $\{x \in N \mid \mathrm{dis}(x, p) < \delta/2\}$ and for any (positive) $s < \delta/2$, $|(\varphi_s^* f)(x) - f(x)| < \beta_2 s$. Thus $(1/s)|(\varphi_s^* f)(x) - f(x)| < \beta_2$. Now $(1/s)|(\varphi_s^* f)(x) - f(x)|$ is for all s bounded on N by $\sup_N \beta$; and, as observed, near p it is, for all sufficiently small s, bounded by β_2. A straighthorward application of the argument used previously in which the heat kernel k is written as $k_1 + k_2$ shows that for $s < \delta/2$ and for all t sufficiently small

$$\left| H_t\left(\frac{1}{s}[(\varphi_s^* f)(x) - f(x)]\right)\right| < \beta_2 \quad \text{for all } x \text{ with } \mathrm{dis}(x, p) \leqq \delta/4,$$

the smallness of t required being uniform in s as $s \to 0^+$. To estimate

$$\frac{1}{s}|(H_t f)(\varphi_s(x)) - (H_t f)(x)|,$$

it remains to estimate the first term $|{}_sH_t(\varphi_s^* f)(x) - H_t(\varphi_s^* f)(x)|$ of the right hand side of the inequality ($\bigstar$).

To obtain the required estimate it is necessary to use more detailed information about the heat kernel k than has been used up to now. Specifically, the existence of an asymptotic expansion for $k(x, y, t)$ in the following form is needed [2] :

$$k(x, y, t)=(4\pi t)^{-n/2}[\exp(-r^2(x, y)/4t)](u_0(x, y)+tu_1(x, y, t)).$$

Here $n=$ dimension N, $r=$ the Riemannian distance function on $N\times N$ (previously denoted by dis), u_0 is a C^∞ function on $N\times N$, and u_1 is a continuous function on $N\times N\times\{t\in\mathbf{R}\,|\,t>0\}$ which is bounded on $N\times N$ uniformly in t as $t\to 0^+$. For notational convenience, write k_s, r_s, ${}_su_0$, and ${}_su_1$ for the functions corresponding to k, r, u_0, and u_1 when the Riemannian metric G of N is replaced by $\varphi_s^* G$. Also, let $v_s : N\to\mathbf{R}$ be the positive C^∞ function which is the ratio of the n-dimensional measure determined by $\varphi_s^* G$ on N to that determined by G; then for any function (or distribution) $h : N\to\mathbf{R}$:

$$({}_sH_t h)(x)=\int_N k_s(x, y, t)h(y)v_s(y)\,dy,$$

where the integral is taken relative to the measure determined by G.

To estimate $|{}_sH_t(\varphi_s^* f)(x)-H_t(\varphi_s^* f)(x)|$ is to estimate

$$\left|\int_N k_s(x, y, t)(\varphi_s^* f)(y)v_s(y)\,dy-\int_N k(x, y, t)(\varphi_s f)(y)\,dy\right|.$$

Now

$$\begin{aligned}
&\left|\int_N k_s(x, y, t)(\varphi_s^* f)(y)v_s(y)\,dy-\int_N k(x, y, t)(\varphi_s^* f)(y)\,dy\right|\\
&\quad=(4\pi t)^{-n/2}\left|\int_N [\exp(-r_s^2(x, y)/4t)]({}_su_0(x, y)+t\,{}_su_1(x, y, t))(\varphi_s^* f)(y)v_s(y)\,dy\right.\\
&\qquad\left.-\int_N [\exp(-r^2(x, y)/4t)](u_0(x, y)+tu_1(x, y, t))(\varphi_s^* f)(y)\,dy\right|\\
&\quad\leqq(4\pi t)^{-n/2}\left\{\left|\int_N [\exp(-r_s^2(x, y)/4t)](\varphi_s^* f)(y)({}_su_0(x, y)+t\,{}_su_1(x, y, t))(v_s(y)-1)\,dy\right|\right.\\
&\quad+\left|\int_N [\exp(-r_s^2(x, y)/4t)-\exp(-r^2(x, y)/4t)](\varphi_s^* f)(y)({}_su_0(x, y)+t\,{}_su_1(x, y, t))\,dy\right|\\
&\qquad+\left|\int_N [\exp(-r^2(x, y)/4t](\varphi_s^* f)(y)({}_su_0(x, y)-u_0(x, y))\,dy\right|\\
&\qquad\left.+\left|\int_N [\exp(-r^2(x, y)/4t)](\varphi_s^* f)(y)\,t\,({}_su_1(x,y, t)-u_1(x, y, t))\,dy\right|\right\}.
\end{aligned}$$

This last (four term) sum will be referred to as (†). It will now be shown that for suitable choice of t_0 and neighborhood of p each of the terms in the four term sum (†) is $<(\varepsilon/4)s$ if

$t \in (0, t_0)$, if x and y are in the neighborhood, and if s is sufficiently small. To do this for the first term, let $(x_1, \ldots, x_n)$ be a local coordinate system around $p \in N$. Then

$$\left|\left(\frac{d}{ds} v_s\right)(p)\right|_{s=0} = \| L_X \omega \| = \| d(i_X \omega) \|$$

$$= \left\| \sum_{i=1}^{n} D_{\partial/\partial x_i}(i_X \omega) \wedge dx_i \right\| = \left\| \sum_{i=1}^{n} \left(i_{D_{\partial/\partial x_i} X} \omega \right) \wedge dx_i \right\| = 0,$$

where $\omega =$ the volume form of N determined by G and $D_{\partial/\partial x_i} X = 0$ at p because $DX = 0$ at p by choice of X. (Since the calculation is local, it is valid even if N is nonorientable so that ω is not globally definable.) Hence there is a neighborhood, say $\{ x \in N \mid r(x, p) < \delta \}$, of p in which $|(d/ds) v_s|_{s=0}| < \xi$, where ξ is a (small) positive constant to be specified later. Let $C = 2 \sup_N |(d/ds) v_s|_{s=0}|$. Then the first term of the sum (†) is, for small enough s and for x with $r(x, p) < \delta/2$, less than or equal to

$$s \eta \int_{y \ni r(x, y) < \delta} |(4 \pi t)^{-n/2} \exp(-r_s^2(x, y)/4t) (\varphi_s^* f)(y)$$
$$\times ({}_s u_0(x, y) + t\, {}_s u_1(x, y, t))| \, dy$$
$$+ C s \int_{y \ni r(x, y) \geqq \delta} |(4 \pi t)^{-n/2} (\exp(-r_s^2(x, y)/4t) (\varphi_s^* f)(y)$$
$$\times ({}_s u_0(x, y) + t\, {}_s u_1(x, y, t))| \, dy.$$

Now $|\varphi_s^* f|$ is bounded uniformly on N independently of s. Thus the second of the two integrals here goes to 0 uniformly in s and x at $t \to 0^+$ in view of the properties of $k(x, y, t)$, as discussed earlier. Also, when δ is chosen sufficiently small, the first integral is bounded uniformly for all x and sufficiently small s as $t \to 0^+$. That is because, if δ is so small that $k_s(x, y, t) \geqq 0$ for all x, y with $r(x, y) < \delta$ for all t sufficiently small, then

$$\int_{y \ni r(x, y) < \delta} |k_s(x, y, t)| \, dy = \int_{y \ni r(x, y) < \delta} k_s(x, y, t) \, dy \quad = [{}_s H_t(1)](x) \to (1)(x) = 1 \quad \text{as } t \to 0^+,$$

so that

$$\int |(4 \pi t)^{-n/2} \exp(-r_s^2(x, y)/t\, ({}_s u_0(x, y) + t\, {}_s u_1(x, y, t))| \, v_s(y) \, dy \to 1$$

as $t \to 0^+$; but $v_s \leqq 2$ everywhere on N if s is small enough so

$$\overline{\lim_{t \to 0^+}} \int |(4 \pi t)^{-n/2} \exp(-r_s^2(x, y)/t) (\varphi_s^* f)(y)$$
$$\times ({}_s u_0(x, y) + t\, {}_s u_1(x, y, t))| \, v_s(y) \, dy$$
$$\leqq 2 \sup_N |(\varphi_s^* f)| \leqq 2 \sup_N |f|.$$

It now follows by choosing η sufficiently small that for t_0 sufficiently small the first term of (†) is $\leqq (\varepsilon/4) s$ for all $t \in (0, t_0)$ and all x sufficiently close to p.

The argument establishing the required estimate for the third and fourth terms of (†) is quite similar to that just given and will be only briefly indicated. The estimation of the second term requires a different procedure; consideration of that term is being deferred for the moment. For the third term estimate, note [2] that ${}_s u_0 (x, x) = 1$ for all s and all $x \in \mathrm{N}$ so that $(d/ds)\, {}_s u_0 (x, x) = 0$. The function ${}_s u_0 (x, y)$ being C^∞ in all three of variables, it follows that, given a positive number ξ (to be specified later), there is a $\delta > 0$ such that for x and y with $r(x, y) < \delta$ and for sufficiently small s, $|(d/ds)\, {}_s u_0 (x, y)| < \xi$. In particular, for x and y near p and s small

$$|{}_s u_0 (x, y) - u_0 (x, y)| < \xi s.$$

Also, $|(d/ds)\, {}_s u_0 (x, y)|$ is bounded uniformly for s small over $\mathrm{N} \times \mathrm{N}$ so that for some C_1:

$$|{}_s u_0 (x, y) - u_0 (x, y)| \leqq C_1 s$$

for all $x, y \in \mathrm{N}$ and s sufficiently small. The required estimate on the third term of (†) is now obtainable by the same procedure as before (splitting the integral into two parts, one over $y \ni r(x, y) < \delta$, one over $y \ni r(x, y) \geqq \delta$). For the estimation of the fourth term of (†), note that there is a constant C_2 such that

$$|{}_s u_1 (x, y, t) - u_1 (x, y, t)| \leqq C_2 s,$$

for all $x, y \in \mathrm{N}$ and s sufficiently small; this fact follows from the determination of u_1 in [2]. Then the fourth term is less than or equal to

$$C_1 st \int |(4 \pi t)^{-n/2} \exp(-r^2 (x, y)/t)(\varphi_s^* f)(y)| \, dy.$$

The integral is uniformly bounded as $t \to 0^+$ so that for t and s sufficiently small the fourth term is less than or equal to $(C_2 t) s$, where C_2 is independent of s, t, and x. Since, for small t, $C_2 t < \varepsilon/4$, the required estimate follows.

It remains to estimate the second term. For this purpose, a preliminary estimate is needed on the behavior of $r_s (x, y)$ for x and y near p. Namely, given a positive number ξ (to be specified later), there is a neighborhood of p such that for all sufficiently small s:

$$|r_s (x, y) - r(x, y)| < \xi s r(x, y),$$

for all x and y in the neighborhood. To verify that this is so, let x and y be points near p and let $c : [0, r(x, y)] \to \mathrm{N}$ be an arc length parameter minimizing geodesic from x to y. Every point of c if near p is x and y are sufficiently near p. Now let c_s be the variation of c defined by $c_s (u) = \varphi_s (c(u))$, $u \in [0, r(x, y)]$. The first-variation-of-arc-length formula gives (letting $l(c_s) =$ length of $c(s)$):

$$\left| \frac{d}{ds} l(c_s) \right|_{s=0} = |G(X(y), \dot{c}(r(x, y))) - G(X(x), \dot{c}(0))|.$$

Now

$$\frac{d}{dt}G(X(c(t)), \dot{c}(t)) = G(D_{\dot{c}(t)}X, \dot{c}(t)),$$

so the right hand side of the first variation formula is

$$\leqq r(x, y) \sup_t G(D_{\dot{c}(t)}X, \dot{c}(t)) \leqq r(x, y) \sup_t \| D_{\dot{c}(t)}X \|.$$

Since $DX=0$ at p, for x and y and hence c near enough to p, $\sup_t \| D_{\dot{c}(t)}X \|$ can be made arbitrarily small, in particular $< \xi/2$. Then $|(d/ds)\, l(c_s)| < (\xi/2)\, r(x, y)$. It follows that, for small s, $|l(c_s) - l(c)| < (\xi/2)\, sr(x, y)$. But $l(c_s) \geqq r_s(x, y)$ and $l(c) = r(x, y)$ so that

$$r_s(x, y) \leqq r(x, y) + \frac{\xi}{2} sr(x, y).$$

A symmetric argument using the flow of $-X$ shows that

$$r(x, y) \leqq r_s(x, y) + \frac{\xi}{2} sr_s(x, y)$$

and hence

$$r(x, y) \leqq r_s(x, y) + \frac{\xi}{2} sr(x, y) + \frac{\xi^2}{2} sr^2(x, y).$$

If ξ is chosen small enough that $\xi r(x, y) < 1$ for all x, y then

$$r(x, y) \leqq r_s(x, y) + \xi\, sr(x, y) \qquad \text{and} \qquad |r_s(x, y) - r(x, y)| < \xi\, sr(x, y).$$

It is easy to check by similar reasoning that there exists a constant C_3 such that for all $x, y \in N$:

$$|r_s(x, y) - r(x, y)| < C_3\, s,$$

for all s sufficiently small. Now, since $|e^z - 1| \leqq 2|z|$ for real numbers z near 0, it follows that, for s small

$$|e^{-r_s^2(x, y)/t} - e^{-r^2(x, y)/t}| \leqq e^{-r^2(x, y)/t} \left| \frac{2}{t}(r_s^2(x, y) - r^2(x, y)) \right|.$$

Since

$$|r_s^2(x, y) - r^2(x, y)| = |r_s(x, y) + r(x, y)| . |r_s(x, y) - r(x, y)|,$$

for all x, y:

$$e^{-r^2(x, t)/t} \left| \frac{2}{t}(r_s^2(x, y) - r^2(x, y)) \right| \leqq e^{-r^2(x, y)/t} . \frac{2}{t} C_3\, sr(x, y) . 3\, r(x, y);$$

here the estimate $|r_s(x, y)-r(x, y)| < C_3 s$ has been combined with its consequence that, for sufficiently small s, $r_s(x, y) < 2r(x, y)$ so that

$$r_s(x, y)+r(x, y) < 3r(x, y).$$

Moreover for x and y sufficiently near p:

$$e^{-r^2(x, y)/t}\left|\frac{2}{t}(r_s^2(x, y)-r^2(x, y))\right| \leqq e^{-r^2(x, y)/t}\, 2\xi sr(x, y) \,.\, 3r(x, y).$$

The required estimate of the second term can now be established: Using the fact that $\varphi_*^s f$ is uniformly bounded on N and that ${}_su_0(x, y)+t\,{}_su_1(x, y, t)$ is uniformly bounded on $N \times N$ and s near 0 as $t \to 0^+$, it now suffices to show that for any positive constant ζ:

$$\int |(4\pi t)^{-n/2}\{\exp(-r_s^2(x, y)/t)-\exp(-r^2(x, y)/t)\}|\, dy \leqq \zeta s,$$

for all t and s sufficiently small. To estimate this last integral, it is convenient to express it in terms of integration in geodesic coordinates as follows: Consider the exponential map $\mathrm{Exp}_x : TN_x \to N$. Exp_x maps a starshaped (relative to 0) bounded region U of TN_x diffeomorphically onto a region of N the complement of which has measure zero. Thus one can express integration over N by integration over the bounded region U of TN_x. Specifically

$$\int_N h(y)\,dy = \int_{v \in U} h(\mathrm{Exp}_x v)\,\mu(v)\,dv,$$

where $\mu(v)$ is a nonnegative function on U and dv denotes the measure induced on TN_x by the inner product on TN_x determined by the Riemannian metric of N. Using polar coordinates on TN_x, r = distance from 0, $\theta \in S^{n-1} \subset TN_x$, gives

$$\int_N h(y)\,dy = \int_{(r, \theta) \in U} h(\exp_x(r, \theta))\,\nu(r, \theta)\,r^{n-1}\,dr\,d\theta,$$

where $d\theta$ = the standard measure on S^{n-1} and $\nu(r, \theta)$ is again a nonnegative function. Then $\nu(r, \theta)$ is a bounded function on U. To see this, note that $\nu(r, \theta)$ is the $(n-1)$-dimensional volume multiplication factor of $(\mathrm{Exp}_x)_*$ at $(r, \theta) \in TN_x$ on the orthogonal complement of the radial direction. Thus $\nu(r, \theta)$ is bounded by $1/r^{n-1} \times$ the $(n-1)$st power of the supremum at distance r of the length of Jacobi fields J along geodesics emanating from x and having $J(x)=0$, $\|\dot{J}(x)\|=1$, and $\dot{J}(x)$ perpendicular to the geodesic. The quantity $1/r^{n-1} \times$ this supremum is uniformly bounded for $r \in [0, r_0]$ for any r_0, and since N has bounded diameter, the boundedness of $\nu(r, \theta)$ on U follows. Explicitly, a bound on $\nu(r, \theta)$ can be obtained by comparison of N with a constant negative curvature manifold of curvature $-\max$ (| sectional curvature of N |). This comparison shows that $\nu(r, \theta)$ is in fact bounded on U by a bound which may be chosen to be independent of the choice of x. Now let s be sufficiently small. It follows from all these observations that there is a constant C_4 independent of x such that

$$\int \left| (4\pi t)^{-n/2} \{ \exp(-r_s^2(x, y)/t - \exp(-r^2(x, y)/t) \} \right| dy$$
$$\leqq C_4 \int (4\pi t)^{-n/2} e^{-r^2/t} \sup_{r(x,y)=r} \left| \frac{2}{t} \{ r_s^2(x, y) - r^2(x, y) \} \right| r^{n-1} dr$$
$$\leqq 6\xi C_4 s \int_0^\delta (4\pi t)^{-n/2} e^{-r^2/t} \frac{r^2}{t} r^{n-1} dr$$
$$+ 6 s C_3 C_4 \int_\delta^{+\infty} (4\pi t)^{-n/2} e^{-r^2/t} \frac{r^2}{t} r^{n-1} dr, \quad (\dagger\dagger)$$

where δ is chosen such that $r(x, y) < \delta$ implies $|r_s(x, y) - r(x, y)| < s\xi r(x, y)$. Now it can be shown directly that

$$\int_0^\delta (4\pi t)^{-n/2} e^{-r^2/t} \frac{r^2}{t} r^{n-1} dt$$

is bounded uniformly as $t \to 0^+$: namely, make the change of variable $\alpha = r/\sqrt{t}$. Then

$$\int_0^\delta (4\pi t)^{-n/2} e^{-r^2/t} \frac{r^2}{t} r^{n-1} dr = \int_0^{\delta/\sqrt{t}} e^{-\alpha^2} \alpha^2 d\alpha < \int_0^{+\infty} e^{-\alpha^2} \alpha^2 < +\infty.$$

Thus, if ξ is chosen $\leqq \left[6 C_4 \int_0^\infty e^{-\alpha^2} \alpha^2 d\alpha \right]^{-1} \varepsilon/8$, the first term of $(\dagger\dagger)$ is less than $(\varepsilon/8) s$.

The same change of variable shows that

$$\int_\delta^{+\infty} (4\pi t)^{-n/2} e^{-r^2/t} \frac{r^2}{t} r^{n-1} dr = \int_{\delta/\sqrt{t}}^{+\infty} e^{-\alpha^2} \alpha^2 d\alpha.$$

Now $\int_{\delta/\sqrt{t}}^{+\infty} e^{-\alpha^2} \alpha^2 d\alpha \to 0$ faster than any power of t as $t \to 0^+$. Thus the second term of $(\dagger\dagger)$ is less than $(\varepsilon/8) s$ for all t sufficiently small. The required estimate on the fourth term of $(\dagger)$ is thus established, and the proof of Theorem 3.2 is complete. □

For certain purposes (*cf.* [12]), it is useful to be able to carry out approximation of subharmonic functions which are Lipschitz continuous in a neighborhood of a closed set—but not, perhaps, elsewhere—by C^∞ such functions. To formulate a refinement of Theorem 3.2 appropriate for this purpose, define, for any closed set L in M and any subsheaf $\mathcal{S}$ of germs of continuous functions on a neighborhood of L, the sheaf $\mathcal{S} | L$ to be the sheaf on M the stalk of which is, for each $p \in L$, the corresponding stalk of $\mathcal{S}$ and is, for each $p \in M - L$, the set of all germs of continuous functions at p. The desired refinement of Theorem 3.2 is the following:

THEOREM 3.2′. — *Let* L *be a closed subset of* M *and* β *a positive continuous function on a neighborhood of* L. *Then:*

(*a*) $\mathscr{S}_{sh} \cap (\mathscr{S}_{Lc\beta} | L)$ *has the semilocal approximation property;*

(*b*) $\mathscr{S}_{\eta\, sh} \cap (\mathscr{S}_{Lc\beta} | L)$ *has, for any continuous function* $\eta : M \to \mathbf{R}$, *the semilocal approximation property.*

$\mathscr{S}_{\eta sh} \cap (\mathscr{S}_{Lc\beta} | L)$ clearly has the maximum closure and C^∞ stability properties. Thus from the present theorem combined with Theorem 1.1 follows the usual corollary that the C^∞ sections of $\mathscr{S}_{\eta\, sh} \cap (\mathscr{S}_{Lc\beta} | L)$ are dense in the C^0 fine topology in the continuous sections. Also the same reasoning that led to Corollary 1 of Theorem 3.1 from that theorem yields in the present case that the C^∞ sections of $\mathscr{S}_{sh} \cap (\mathscr{S}_{Lc\beta} | L)$ are dense in the compact-open topology in the continuous sections.

Proof of Theorem 3.2′. — As in Theorems 3.1 and 3.2, it suffices to establish the semilocal approximation property for $\mathscr{S}_{\eta\, sh} \cap (\mathscr{S}_{Lc\beta} | L)$. Let f be a section there of in a neighborhood of a compact set K which section is C^∞ in a neighborhood of a (perhaps empty) compact set K_1. Set $K_2 = K_1 \cup (L \cap K)$. Clearly there is a positive continuous function β_2 defined in a neighborhood of K_2 and having the properties that $\beta = \beta_2$ in a neighborhood of $K \cap L$ and that f is a section of $\mathscr{S}_{Lc\beta_2}$ in a neighborhood of K_2. By Theorem 3.2, there is a C^∞ section f_1 *of* $\mathscr{S}_{\eta\, sh} \cap (\mathscr{S}_{Lc\beta_2} | K_2)$ in a neighborhood of K_2 which approximates f in the C^0 sense near K_2 and in the C^∞ sense near K_1. An application of the technique used in the proof of Lemma 1.1 combines f_1 and f to yield a section f_2 *of* $\mathscr{S}_{\eta\, sh}$ on a neighborhood of K such that f_2 is an approximation of f on K, f_2 is C^∞ near K_2, and f_2 is an approximation in the C^∞ sense of f_1 on K_2. The two approximation requirements are consistent because f_1 is a C^0 approximation of f near K_2.

An application of Theorem 3.1 yields a C^∞ section f_3 of $\mathscr{S}_{\eta\, sh}$ on a neighborhood of K such that f_3 is an approximation in the C^0 sense of f_2 in a neighborhood of K and an approximation in the C^∞ sense of f_2 in a neighborhood of K_2. Since f_2 is a C^∞ section of $\mathscr{S}_{\eta\, sh} \cap (\mathscr{S}_{Lc\beta} | L)$ near $K \cap L$, f_3 is by C^∞ stability a section of $\mathscr{S}_{\eta\, sh} \cap (\mathscr{S}_{Lc\beta} | L)$ near $K \cap L$ if the C^∞ approximation of f_2 by f_3 near K_2 is sufficiently good. Thus if all approximations are chosen to be sufficiently good, f_3 will approximate f in the C^0 sense on K and in the C^∞ sense on K_1, and f_3 will be a section of $(\mathscr{S}_{Lc\beta} | L)$ in a neighborhood of $L \cap K$ and a section of K. Hence f_3 will be a section of $\mathscr{S}_{\eta\, sh} \cap (\mathscr{S}_{Lc\beta} | L)$ in a neighborhood of K approximating f in the senses required in the definition of the semilocal approximation property.

4. Obtaining Semi-local Approximations from Local Ones

To apply Theorem 1.1 it is necessary to have a method of constructing smooth approximations of sections of the sheaf $\mathscr{S}$ in neighborhoods of arbitrary compact subsets of the manifold. Since it is not generally possible to engulf an arbitrary compact subset in a single coordinate system, one is thus, at least at first sight, restrained from using smoothing processes which are defined in such a way as to depend upon coordinate choice. In some cases, the only obvious candidate for a smoothing process does depend upon a coordinate system choice; for instance, the approximation of (continuous) strictly plurisubharmonic functions on a complex manifold by smooth strictly plurisubharmonic functions is obtainable easily only within a given complex coordinate system, in which the usual convolution process may be used. Thus it would be advantageous to obtain a version of Theorem 1.1 in which one needed smooth approximation only in neighborhoods of given points, not of given compact sets. Such a result for the specific case of global smooth approximation of strictly plurisubharmonic functions was obtained by Richberg [11]. The purpose of this section is to formulate and prove a theorem closely related to that of [11] but general in character dealing with the passage from local to global approximations. To state the theorem, one needs first to make precise the type of local approximations needed.

DEFINITION. — Let $\mathscr{S}$ be a subsheaf of the sheaf of germs of continuous functions on a manifold M. $\mathscr{S}$ has the *local approximation property* if for each point p of M there is an open neighrohood U_p of p with the property that $\mathscr{S} \mid U_p$ considered as a subsheaf of the sheaf of germs of continuous functions on U_p (U_p being considered as a manifold) has the semilocal approximation property.

Note that the local approximation property just described is obtained when a suitable convolution smoothing process is available in coordinate systems. For instance, the sheaf of germs of continuous plurisubharmonic functions on a complex manifold has the local approximation property: the neighborhood U_p can be taken to be the domain of a complex coordinate system containing p and the required approximations in a neighborhood of a given compact subset of U_p are obtained by convolution smoothing in the coordinate system using kernels of small support as usual.

On a Hermitian manifold M with Hermitian metric given in local coordinate form by $\{ g_{\alpha\beta} \mid \alpha, \beta = 1, \ldots, \dim_{\mathbf{C}} M \}$, one can define a concept of η-plurisubharmonicity analogous to the η-convexity introduced in paragraph 2, η being as before a continuous function on M. Namely, a continuous function φ is said to be η-*plurisubharmonic* if for each p in M there exists a C^∞ function τ defined in a neighborhood of p such that $\varphi - \tau$ is plurisubharmonic in that neighborhood and such that the eigenvalues of the form

$$\sum_{\alpha, \beta = 1}^{n} g^{\alpha\beta} \frac{\partial^2 \tau}{\partial z_\alpha \partial \bar{z}_\beta} \, dz_\alpha \, d\bar{z}_\beta$$

are greater than $\eta(p)$ at p, where $(z_1, \ldots, z_n)$ is a complex coordinate system defined in a neighborhood of p. A continuous function is *strictly plurisubharmonic* if it is 0-plurisubharmonic. It is easy to check that the strict plurisubharmonicity property is actually independent of the choice of Hermitian metric g; in particular, a function is strictly plurisubharmonic if and only if it is strictly plurisubharmonic in each local coordinate system. By contrast, η-plurisubharmonicity, $\eta \not\equiv 0$ depends on g.

The convolution smoothing process already discussed can be used to show that the definitions just given coincide with the obvious definitions in terms of Oka's modulus of plurisubharmonicity, namely, that the modulus be greater than η or greater than some positive continuous function, respectively [*cf.* [5] (*f*)]. More to the present purpose, the application of the convolution smoothing process yields immediately that for each continuous function $\eta : M \to R$ the sheaf of germs of η-plurisubharmonic functions has the local approximation property. In particular, the sheaf of germs of strictly plurisubharmonic functions has the local approximation property. These observations illustrate the relevance of the following theorem to the construction of smooth approximations of plurisubharmonic functions.

THEOREM 4.1. — *If $\mathscr{S}$ is a subsheaf of the sheaf of germs of continuous functions on a manifold* M *and if $\mathscr{S}$ has the local approximation, the* C^∞ *stability, and the maximum closure properties, then $\mathscr{S}$ has the semi-local approximation property also.*

COROLLARY 1. — *If $\mathscr{S}$ has the local approximation,* C^∞ *stability, and maximum closure properties then the global sections of $\mathscr{S}$ which are* C^∞ *are dense in the global sections of $\mathscr{S}$ in the* C^0 *fine topology.*

COROLLARY 2. — *If* M *is a complex manifold then the* C^∞ *strictly plurisubharmonic functions are dense in the continuous strictly plurisubharmonic functions in the* C^0 *fine topology. If* M *is an Hermitian manifold then for any continuous function* $\eta : M \to R$ *the* C^∞ η*-plurisubharmonic functions are dense in the (continuous)* η*-plurisubharmonic functions in the* C^0 *fine topology.*

Corollary 1 follows directly from combining Theorems 1.1 and 4.1. Corollary 2 follows from Corollary 1 and the remarks preceding the statement of Theorem 4.1. The first half of Corollary 2 is given in [11].

The following lemma will be used to establish Theorem 4.1:

LEMMA 4.1. — *Let $\mathscr{S}$ be a subsheaf of the sheaf of germs of continuous functions on a manifold* M, *and suppose that $\mathscr{S}$ has the maximum closure,* C^∞ *stability and local approximation properties. Suppose also that* L_1 *and* L_2 *are compact subsets of* M, *that* f *is a section of $\mathscr{S}$ on some neighborhood* W *of* $L_1 \cup L_2$, *that* f *is* C^∞ *in a neighborhood* W_1 *of* L_1, *and that there is a neighborhood* W_2 *of* L_2 *such that* $\mathscr{S} \mid W_2$ *has the semilocal approximation property. Then there exists a section* f_1 *of* $\mathscr{S} \mid W$ *such that* f *approximates* f_1 *everywhere on* W *in the* C^0*-fine sense, such that* f_1 *approximates* f *in a neighborhood of* L_1 *in the* C^∞ *sense, and such that* f_1 *is* C^∞ *in a neighborhood of* $L_1 \cup L_2$.

Proof. — Choose open sets W_3 and W_4 having compact closures and having $L_1 \subset \overline{W}_3 \subset W_1$, $L_2 \subset \overline{W}_4 \subset W_2$. Let ρ_1 be a C^∞ function which is identically 1 on W_3 and which has its support contained in W_1. Let g be a continuous function on W which: (*a*) is a C^∞ section of $\mathscr{S}$ in a neighborhood of W_4; (*b*) approximates f in the C^0 sense on $\overline{W}_4$; (*c*) approximates f in the C^∞ sense on a neighborhood of the intersection of $\overline{W}_4$ and the support of ρ_1. The existence of g follows from the semilocal approximation property of $\mathscr{S}$ on W_2 and standard extension properties of continuous functions. Set $h=(1-\rho_1)g+\rho_1 f$. This function h is C^∞ in a neighborhood of $\overline{W}_3 \cup \overline{W}_4$ and approximates f there. Also if g was a sufficiently good approximation of f in the C^∞ sense on the neighborhood of the intersection of $\overline{W}_4 \cap$ the support of ρ_1, then by the C^∞ stability property, h will be a section of $\mathscr{S}$ on a neighborhood of $\overline{W}_3 \cup \overline{W}_4$. Finally $h=f$ on W_3.

Now choose a C^∞ function ρ_2 which is identically 0 in a neighborhood of $L_1 \cup L_2$ and which is identically 1 in a neighborhood of $M-(W_3 \cup W_4)$. Let ξ be a (small) positive number, and let f_1 be defined as follows: $f_1 = f$ on $W-(W_3 \cup W_4)$; $f_1 = \max(f, h+\xi-2\xi\rho_2)$ on $(W_3 \cup W_4)-(L_1 \cup L_2)$ and $f_1 = h$ on $L_1 \cup L_2$. By the maximum closure property, f_1 is a (continuous) section of $\mathcal{S}$ on W provided that h is a sufficiently good approximation of f on $\overline{W}_3 \cup \overline{W}_4$. By choosing ξ sufficiently small and h such a sufficiently good approximation, one obtains that f_1 has the properties required.

Proof of Theorem 4.1. — Let K be a compact subset of M and f a section of $\mathcal{S}$ in a neighborhood U of K. To construct a C^∞ section of $\mathcal{S}$ defined in a neighborhood of K and approximating f on K in the senses of the definition of the semilocal approximation property, chose open subsets $U_1, \ldots, U_l$ of U such that $K \subset \bigcup_{j=1}^{l} U_j$ and $\mathcal{S} \mid U_j$ has the semilocal approximation property for each $j = 1, \ldots, l$. This choice is possible since $\mathcal{S}$ has the local approximation property. Let $V_1, \ldots, V_l$ be open subsets of M each with compact closures $\overline{V}_j \subset U_j$ and with $K \subset \bigcup_{j=1}^{l} V_j$. The existence of such V_j is a standard fact. Now repeatedly apply Lemma 4.1: first with $L_1 =$ the compact subset $K_1 \subset K$ in a neighborhood of which f is C^∞ (*see* the definition of the semilocal approximation property in paragraph 1: K_1 may be empty), with $W = U$ and with $L_2 = \overline{V}_1$; then with $L_1 = K_1 \cup \overline{V}_1$ and $L_2 = \overline{V}_2$, and $W = U$ still; then with $L_1 = K_1 \cup \overline{V}_1 \cup \overline{V}_2$ and $L_2 = \overline{V}_3$ and still $W = U$. Continuing in this fashion, one obtains after l steps, a C^∞ section of $\mathcal{S}$ in a neighborhood $\cup \overline{V}_l$ and so in a neighborhood of K, which will approximate f near K in the sense of the semilocal approximation property provided that at each of the l steps the approximation of the immediately previous step was chosen to be sufficiently good. □

Being defined independently of the choice of a metric on the manifold, the coordinate system smoothing process used to construct C^∞ strictly plurisubharmonic local approximations to continuous strictly plurisubharmonic functions (or C^∞ η-plurisubharmonic approximations to continuous η-plurisubharmonic functions) naturally does not bear any relationship to the metric properties of the manifold when a metric is chosen. However, it is sometimes important to know that the approximations so obtained do in fact preserve certain metric-related properties of the functions being approximated. The case of Lipschitz continuity is disposed of by the following lemma:

LEMMA. — *Let $(z_1, \ldots, z_n)$ be a complex coordinate system defined on an open subset* U *of an Hermitian manifold* M, K *be a compact subset of* U *and* $f : U \to \mathbf{R}$ *be a continuous function on* U *which is locally Lipschitz on* U *with local Lipschitz constants less than* B, *i.e. f is a section on* U *of $\mathcal{S}_{LCB} \mid U$ (in the notation of paragraph* 2). *Let f_ε, $\varepsilon \in (0, \varepsilon_0)$, be the family of functions, defined and C^∞ in a fixed neighborhood* V *of* K, *obtained by coordinate convolution smoothing with kernel $\varkappa$ relative to the $(z_1, \ldots, z_n)$ coordinate system. Then, for all sufficiently small ε, $\|(\operatorname{grad} f_\varepsilon)(p)\| < B$ for all $p \in K$.*

Proof. — Choose a neighborhood W of K such that $\overline{W}$ is a compact subset of V. $\overline{W}$ is covered by a finite number of open subsets U_i of U such that there are constants $B_i < B$ with

$|f(q_1)-f(q_2)|<B_i$ dis (q_i, q_2) for all $q_1, q_2 \in B_i$. It follows that there is a number B_W such that $B_W<B$ and $f|W$ is locally Lipschitz continuous with local Lipschitz constant less than B_W, i.e. $f|W$ is a section of $\mathscr{S}_{LcB_W}$ on W. Choose now a neighborhood W_1 of K with $\overline{W}_1 \subset W$. By paragraph 2, with τ there being taken to be $f|W$, there exists a family $\tau_\delta : W_1 \to R$ of C^∞ functions such that τ_δ converges uniformly to f on W_1 as $\delta \to 0^+$ and such that, for all sufficiently small δ, τ_δ is a section of $\mathscr{S}_{LcB_W}$ on W_1. Of course, the τ_δ are necessarily closely related to the f_ε obtained from coordinate convolution smoothing.

Now let p be any point of K and V be a vector in TM_p with $\|V\|=1$. There exists a unique C^∞ vector field $\tilde{V}$ on U such that $\tilde{V}$ has constant coefficients in the real coordinate system associated to the complex coordinate system $(z_1, \ldots, z_n)$ and such that $\tilde{V}(p)=V$. As usual

$$(V f_\varepsilon)(p)=\frac{1}{\varepsilon^n} V \int_{\|v\|\leq\varepsilon} \varkappa\left(\frac{\|v\|}{\varepsilon}\right) f(p+v)$$

where addition of vectors means addition relative to the (real) coordinate structure. Since $\tau_\delta \to f$ uniformly on compact sets as $\delta \to 0^+$:

$$\frac{1}{\varepsilon^n} V \int_{\|v\|\leq\varepsilon}\left[\varkappa\left(\frac{\|v\|}{\varepsilon}\right) f(p+v)\right]=\lim_{\delta\to 0^+}\frac{1}{\varepsilon^n} V \int_{\|v\|\leq\varepsilon} \varkappa\left(\frac{\|v\|}{\varepsilon}\right)\tau_\delta(p+v)$$
$$=\lim_{\delta\to 0^+}\frac{1}{\varepsilon^n}\int_{\|v\|\leq\varepsilon} V\left[\varkappa\left(\frac{\|v\|}{\varepsilon}\right)\tau_\delta(p+v)\right]=\lim_{\delta\to 0^+}\frac{1}{\varepsilon^n}\int_{\|v\|\leq\varepsilon}\varkappa\left(\frac{\|v\|}{\varepsilon}\right)(\tilde{V}\tau_\delta)(p+v),$$

where the V and $\tilde{V}$ differentiate the p variable. Now

$$\frac{1}{\varepsilon^n}\int_{\|v\|\leq\varepsilon}\varkappa\left(\frac{\|v\|}{\varepsilon}\right)(\tilde{V}\tau_\delta)(p+v)\leq \sup_{\|v\|\leq\varepsilon}|(\tilde{V}\tau_\delta)(p+v)|.$$

As $\varepsilon \to 0^+$, $\sup_{\|v\|\leq\varepsilon}\|\tilde{V}(p+v)\| \to 1$. Moreover, this convergence is uniform in variation of the direction (at p) of the unit vector $V\in TM_p$. Since $B_W<B$, it follows that for ε sufficiently small, the smallness required being uniform in p and $V \in TM_p$ with $\|V\|=1$, that

$$|(V f_\varepsilon)(p)|<B.$$

If $(\text{grad } f_\varepsilon)(p)\neq 0$, take $V=(\text{grad } f_\varepsilon)/\|\text{grad } f_\varepsilon\|$ to obtain $\|\text{grad } f_\varepsilon\|<B$. If $\text{grad } f_\varepsilon=0$, then $\|\text{grad } f_\varepsilon\|<B$ trivially, and the proof is complete.

Tabular Summary of Results

NOTATION:

sheaf of germs of:

convex functions. . . . . $\mathscr{S}_c$
strictly convex functions. . . . . $\mathscr{S}_{sc}$
η-convex functions. . . . . $\mathscr{S}_{\eta c}$
subharmonic functions. . . . . $\mathscr{S}_{sh}$

strictly subharmonic functions. . . . . $\mathscr{S}_{ssh}$
plurisubharmonic functions. . . . . $\mathscr{S}_{psh}$
strictly plurisubharmonic functions. . . . . $\mathscr{S}_{spsh}$
η-plurisubharmonic functions.. . . . . $\mathscr{S}_{\eta psh}$
Lipschitz continuous functions with local Lipschitz constant $<$ B (B constant). . $\mathscr{S}_{\mathrm{LcB}}$
Lipschitz continuous functions with local Lipschitz constant $< \beta$ (β possibly variable). . . . . $\mathscr{S}_{\mathrm{Lc}\beta}$

Preliminary remarks on the table:

(1) In the cases where the entries involving $\mathscr{S}_{\mathrm{Lc}\beta}$ would be the same whether β is constant or variable the two cases have not been listed separately. In the one case where they differ ($\mathscr{S}_{sh} \cap \mathscr{S}_{\mathrm{LcB}}$ vs. $\mathscr{S}_{sh} \cap \mathscr{S}_{\mathrm{Lc}\beta}$) they have been given separate listing.

(2) The entries for the sheaves $\mathscr{S}_{\eta c}$, $\mathscr{S}_{\eta sh}$, and $\mathscr{S}_{\eta psh}$ and for their intersections with $\mathscr{S}_{\mathrm{Lc}\beta}$ would be the same as the corresponding entries for, respectively, $\mathscr{S}_{sc}$, $\mathscr{S}_{ssh}$, and $\mathscr{S}_{spsh}$ and their intersections with $\mathscr{S}_{\mathrm{Lc}\beta}$. These duplicated sets of entries are omitted for brevity.

(3) Certain results (e. g. Theorem 3 . 2′ and its corollaries) which did not fit conveniently in the table are omitted.

(4) Blank spaces represent cases which are to the authors' knowledge unsettled.

Sheaf of functions germs	Properties								
	Coarse C^0 approximation	Fine C^0 approximation	Coarse C^0 approximation of exhaustions by C^∞ exhaustions	Maximum closure	Semilocal approximation	Lipschitz semilocal approximation	Convex composition	C^∞ stability	Local approximation property
β-Lipschitz ($\mathscr{S}_{\mathrm{Lc}\beta}$).	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes
Convex ($\mathscr{S}_c$)..				Yes			Yes	No	Yes
Strictly convex ($\mathscr{S}_{sc}$).	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Convex, β-Lipschitz ($\mathscr{S}_c \cap \mathscr{S}_{\mathrm{Lc}\beta}$).				Yes			No	No	Yes
Strictly convex, β-Lipschitz ($\mathscr{S}_{sc\beta} \cap \mathscr{S}_{\mathrm{Lc}\beta}$).	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	Yes
Subharmonic ($\mathscr{S}_{sh}$).	Yes		Yes	Yes	Yes	Yes	Yes	No	Yes
Subharmonic, B-Lipschitz, B constant ($\mathscr{S}_{sh} \cap \mathscr{S}_{\mathrm{LcB}}$).			Yes	Yes	Yes	Yes	No	No	Yes
Subharmonic, β-Lipschitz ($\mathscr{S}_{sh} \cap \mathscr{S}_{\mathrm{Lc}\beta}$).				Yes	Yes	Yes	No	No	Yes
Strictly subharmonic ($\mathscr{S}_{ssh}$).	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Strictly subharmonic, β-Lipschitz ($\mathscr{S}_{ssh} \cap \mathscr{S}_{\mathrm{Lc}\beta}$).	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Plurisubharmonic ($\mathscr{S}_{psh}$).				Yes			Yes	No	Yes
Plurisubharmonic, β-Lipschitz ($\mathscr{S}_{psh} \cap \mathscr{S}_{\mathrm{Lc}\beta}$).				Yes			No	No	Yes
Strictly plurisubharmonic ($\mathscr{S}_{spsh}$).	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Strictly plurisubharmonic, β-Lipschitz ($\mathscr{S}_{spsh} \cap \mathscr{S}_{\mathrm{Lc}\beta}$).	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

Added in proof: R. L. Bishop has informed us in a letter that considerations akin to the convex composition property were taken up in his joint paper with S. ALEXANDER, *Convex-Supporting Domains on Spheres* (*Ill. J. Math.*, 18, 1974, pp. 31-47).

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