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STEPHEN GELBART
HERVÉ JACQUET

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A RELATION BETWEEN AUTOMORPHIC REPRESENTATIONS OF GL (2) AND GL (3)*

BY STEPHEN GELBART** AND HERVÉ JACQUET

Some notations

We will denote by G_r the group GL (r), by B_r the standard Borel subgroup of GL (r) (upper triangular matrices), by N_r the unipotent radical of B_r, and by Z_r the center of G_r. On the other hand, we will denote by G the group SL (2), by B its standard Borel subgroup, by $N = N_2$ the unipotent radical group of B, and by A the subgroup of diagonal matrices in G. Finally P will be the standard parabolic subgroup of type (2, 1) in G_3 and U its unipotent radical.

If F is a local non archimedean field, we denote by $\mathfrak{R}_F$ or $\mathfrak{R}$ the ring of integers in F, by $\mathfrak{P}_F$ or $\mathfrak{P}$ its maximal ideal, and by $\tilde{\omega}$ a generator of $\mathfrak{P}$. Finally we set $K_r = \text{GL}(r, \mathfrak{R}_F)$, $K = \text{SL}(2, \mathfrak{R}_F)$. If F is archimedean the same notations will be used for the standard maximal compact subgroups. When F is local, we will often write $G_r, B_r, \ldots$ for $G_r(F)$, $B_r(F) \ldots$

Introduction

The purpose of this paper is to establish a relation between automorphic forms on GL (2) and GL (3). To formulate our main result more precisely we need first to recall some basic facts.

Suppose F is an A-field and π is an irreducible unitary representation of GL $(r, \mathbf{A})$. Then π can be written as an infinite tensor product $\pi = \bigotimes_v \pi_v$ where π_v is an irreducible representation of $G_v = \text{GL}(r, F_v)$ for each place v. Moreover, for almost all finite v the representation π_v is unramified, i. e., contains the trivial representation of a maximal compact subgroup of G_v. But such representations are well known to be parameterized by semi-simple conjugacy classes in GL $(r, \mathbf{C})$. Thus one can associate to π a family of semi-simple conjugacy classes (a_v) in GL $(r, \mathbf{C})$, a_v being defined for almost all finite v.

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Now suppose σ is an automorphic representation of GL (2, A), i. e., σ occurs in the space of automorphic forms on GL (2). If (b_v) is the corresponding family of conjugacy classes in GL (2, C), define a family of conjugacy classes in GL (3, C) by

$$a_v = \begin{pmatrix} mn^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m^{-1}n \end{pmatrix} \quad \text{if} \quad b_v = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}.$$

Our main result then asserts that there is an essentially unique automorphic representation of GL (3, A) – the "lift of σ – whose corresponding family is (a_v).

To see how this "lifting" appears as a special case of the more general conjectures of Langlands the reader can consult [Bo]. Roughly speaking, if W'_v denotes the Weil-Deligne group of F_v, and $\sigma = \otimes \sigma_v$, then each σ_v corresponds (sometimes only conjecturally) to a two-dimensional representation φ_v of W'_v. So if p denotes the map

$$\begin{array}{ccc} GL(2, \mathbf{C}) & \xrightarrow{\;p\;} & GL(3, \mathbf{C}) \\ & \searrow \quad \nearrow & \\ & PGL(2, \mathbf{C}) & \end{array}$$

determined by the adjoint action of PGL (2, C), then (according to Langland's general philosophy) the resulting three dimensional representation

$$p \circ \varphi_v : \quad W'_v \to GL(3, \mathbf{C})$$

should correspond to an irreducible admissible representation $\pi_v = p^* (\sigma_v)$ of GL (3, F_v). This is the local "lifting" of σ_v to GL (3). Globally, the product $\pi = \underset{v}{\otimes} \pi_v$ should (and does!) define an automorphic representation of GL (3, A).

The idea of our proof can be explained as follows. Following [GoJa] one can attach to each irreducible representation π of GL (r, A) an infinite Euler product

$$L(s, \pi) = \prod_v L(s, \pi_v),$$

with

$$L(s, \pi_v) = \det(1 - q_v^{-s} a_v)^{-1},$$

for almost all v. This product converges in some right half-plane. Moreover, if π is automorphic cuspidal, i. e., π occurs in the space of cusp forms, then L (s, π) continues to a holomorphic function of s and satisfies a functional equation. For $r = 2$ or 3 there is also a converse; (*cf.* [JPSS]). For each character χ of the idèle class group consider the representation $g \mapsto (g) \chi (\det (g))$ (also denoted $\pi \otimes \chi$). Then if L (s, $\pi \otimes \chi$) satisfies the above conditions for each χ, π *must* be automorphic cuspidal.

Now recall that to each *pair* of irreducible representations (π_1, π_2) of GL (2, A) one can attach an infinite Euler product

$$L(s, \pi_1 \times \pi_2) = \prod_v L(s, \pi_{1,v} \times \pi_{2,v}),$$

with

$$L(s, \pi_{1,v} \times \pi_{2,v}) = \det(1 - q_v^{-s} a_{1,v} \otimes a_{2,v})^{-1},$$

for almost all v; (*cf.* [Ja]). If π_1 and π_2 are automorphic then this infinite product continues to a meromorphic function of s and satisfies a functional equation In particular, if $\pi_1 = \sigma \otimes \chi$ and $\pi_2 = \tilde{\sigma}$ (the contragredient), then:

$$L(s, (\sigma \otimes \chi) \times \tilde{\sigma}) = L_2(s, \sigma, \chi) L(s, \chi),$$

where $L(s, \chi)$ is the usual Hecke L-function attached to χ and $L_2(s, \sigma, \chi)$ is a new Euler product of degree 3. If π is automorphic, then $L_2(s, \sigma, \chi)$ is also meromorphic and satisfies a functional equation. These facts are discussed in Sections 1 and 2.

In Section 3 we introduce a representation π of GL (3, A)—the "lift" of σ—by requiring that

$$L(s, \pi \otimes \chi) = L_2(s, \sigma, \chi).$$

At almost all places, σ_v is unramified, and the relation between π_v and σ_v is the one already explained. The situation at the remaining places is discussed in Section 3.

The fact that $L_2(s, \sigma, \chi)$ has no poles at all is proved using the method of [Sh]. Our main theorem then results from the converse theorem for GL (3) already alluded to.

To establish the holomorphy of $L_2(s, \sigma, \chi)$ we generalize [Sh] by considering integrals on the metaplectic group Mp (A). This group is an extension of SL (2, A) by the torus **T** which splits over SL (2, F). Thus any form on SL (2, A) may be regarded as a form on Mp (A). The "main integral" we consider is an integral over **T** SL (2, F)$\backslash$(Mp (A)) of the product a form on GL (2, A) belonging to π, an Eisenstein series on Mp (A), and a form on the metaplectic group given by a theta series. In Sections 5 and 6 we show that this integral decomposes as a product of integrals over local metaplectic groups and equals $L_2(s, \sigma, \chi)$ for a suitable choice of functions involved. Thus we need only analytically continue the Eisenstein series. This is carried out in Section 8 after analyzing the "constant term" in Section 7.

Actually, our paper deviates a bit from the simple scheme just outlined—for technical reasons discussed within the text. We also have to complete the results of [JPSS] in Section 4.

Many of the results of this paper were announced earlier by us in [GeJ 2]. We are grateful to Arletta Havlik, Joanne Martin, Margeret Reif and Diana Shye for their expeditious typing of the manuscript. The second-named author would also like to thank R. P. Langlands for pointing out the results of Corollary 1.7, and W. Casselman for bringing to his attention the results of Theorem 2.2.

1. GL (2) × GL (2): local theory

In this Section (and the next) we complete the theory of automorphic forms on GL (2) × GL (2) developed in [Ja]. In particular, we describe the poles of $L(s, \pi_1 \times \pi_2)$ both locally and globally. Although some of our results are not used later in this paper they all seem to be of independent interest.

(1.1) Let F denote a local non-archimedean field, q the cardinality of the residue class field, v_F or v the normalized valuation on F, and $| \ |_F$ or $| \ |$ the corresponding absolue value. Then $|x|_F = q^{-v(x)}$. When convenient, we also let let α_F or α denote the absolute value. By ψ_F or ψ we denote a fixed non-trivial additive character of F.

If (π_1, π_2) is a pair of irreducible admissible infinite-dimensional representations of GL (2, F) then $L(s, \pi_1 \times \pi_2)$ will denote the Euler-factor defined in [Ja].

In this section we shall recall the definition of $L(s, \pi_1 \times \pi_2)$ and complete the results of [Ja] by computing this factor in all cases.

Let $\mathscr{W}(\pi_i; \psi_F)$ denote the *Whittaker model* of π_i (with respect to ψ_F). Recall that this is a space of functions W on GL (2, F) satisfying the following properties:

(1.1.1) $\mathscr{W}(\pi_i; \psi_F)$ is invariant for the right action of $G_2(F)$ and the resulting representation of $G_2(F)$ on $\mathscr{W}(\pi_i; \psi_F)$ is equivalent to π_i;

(1.1.2) for each W in $\mathscr{W}(\pi_i; \psi_F)$:

$$W\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \psi_F(x) W[g] \quad \text{for each } x \in F \quad \text{and} \quad g \in G_2(F).$$

The existence and uniqueness of such a space is proved in Chapter I of [JL].

For $W_i \in \mathscr{W}(\pi_i; \psi)$, Φ a Schwartz-Bruhat function on F^2, and $s \in \mathbf{C}$, set

$$(1.1.3) \qquad \Psi(s, W_1, W_2, \Phi) = \int_{N_2(F)\backslash G_2(F)} W_1(g) W_2(\eta g) \Phi[(0, 1) g] |\det g|^s dg,$$

where

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the factor $L(s, \pi_1 \times \pi_2)$ arises as the "g. c. d." of these integrals. More precisely, these integrals converge for Re (s) sufficiently large and define rational functions of q^{-s}; since the subvector space of $\mathbf{C}(q^{-s})$ spanned by them is a fractional ideal of $\mathbf{C}[q^{-s}, q^s]$ which contains 1 it has a unique generator of the form $P^{-1}(q^{-s})$ with $P \in \mathbf{C}[q^{-s}]$ and $P(0) = 1$: this generator is—by definition—$L(s, \pi_1 \times \pi_2)$.

Recall also that if W is in $\mathscr{W}(\pi_1; \psi_F)$ then the function W defined by

$$(1.1.4) \qquad \tilde{W}(g) = W[w^t g^{-1}], \qquad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

belongs to the Whittaker model $\mathscr{W}(\tilde{\pi}_i; \psi_F)$ of the representation $\tilde{\pi}_i$ contragredient to π_i.

Finally the factor $\varepsilon(s, \pi_1 \times \pi_2; \psi_F)$ — which is a monomial in q^{-s} — is defined by the functional equation:

$$\text{(1.1.5)} \qquad \begin{aligned} &\Psi(1-s, \tilde{W}_1, \tilde{W}_2, \hat{\Phi})/L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2) \\ &= \varepsilon(s, \pi_1 \times \pi_2; \psi_F)\Psi(s, W_1, W_2, \Phi)/L(s, \pi_1 \times \pi_2). \end{aligned}$$

Here

$$\text{(1.1.6)} \qquad \Phi^{\wedge}(x, y) = \iint \Phi(u, v)\psi_F(ux+vy)\,du\,dv$$

is the Fourier transform of Φ (with respect to the self-dual Haar measure). Denote by ω_i the central quasi-character of π_i.

PROPOSITION (1.2). — *Suppose π_2 is supercuspidal. Then the poles of* $L(s, \pi_1 \times \pi_2)$ *are simple. If $\alpha^s \otimes \pi_1$ denotes the representation $g \mapsto |\det g|^s \pi_1(g)$ then s_0 is a pole of* $L(s, \pi_1 \times \pi_2)$ *if and only if $\alpha^{s_0} \otimes \pi_1$ is equivalent to $\tilde{\pi}_2$.*

Proof. — It is clear that s_0 is a pole of $L(s, \pi_1 \times \pi_2)$ if and only if it is a pole $\Psi(s, W_1, W_2, \Phi)$ for at least one choice of W_1, W_2, and Φ. Moreover, the multiplicity of s_0 as a pole of $L(s, \pi_1 \times \pi_2)$ is the maximum multiplicity of s_0 as a pole of each of these integrals.

By Iwasawa's decomposition,

$$\Psi(s, W_1, W_2, \Phi) = \int_K \int_{F^\times} W_1\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right] W_2\left[\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} k\right] f(k)\,|a|^{s-1}\,d^\times a\,dk,$$

with

$$K = GL(2, R_F) \qquad \text{and} \qquad f(g) = |\det g|^s \int \Phi[(0, t)g]\,|t|^{2s}\omega_1\omega_2(t)\,d^\times t.$$

But π_2 is supercuspidal. Therefore ([JL], Prop. (2.16)) the function

$$a \mapsto W_2\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right]$$

has compact support in $F^\times$. Thus every pole of Ψ must come from the integral defining f. In particular, if s_0 is a pole, then s_0 is a simple pole, and $\alpha^{2s_0} = (\omega_1\omega_2)^{-1}$.

So suppose now that s_0 satisfies $\alpha^{2s_0} = (\omega_1\omega_2)^{-1}$. Then

$$\lim_{s \to s_0} (s-s_0) \int \Phi[(0, t)]\,|t|^{2s}(\omega_1\omega_2)^{-1}(t)\,d^\times t = c\,\Phi(0, 0),$$

with c a non-zero constant independent of Φ. It follows that

$$\lim_{s \to s_0} (s-s_0)\Psi(s, W_1, W_2, \Phi) = c\,\Phi(0, 0)\,b(W_1, W_2),$$

with

$$b(W_1, W_2) = \int_{Z_2(F) N_2(F) \backslash G_2(F)} W_1(g) W_2(\eta g) \, |\det g|^{s_0} \, dg.$$

Thus s_0 is a pole of $L(s, \pi_1 \times \pi_2)$ if and only if the bilinear form b is not identically zero. Note that integrand defining b is indeed invariant on the left by $N_2(F) Z_2(F)$ and compactly supported mod $N_2(F) Z_2(F)$. But since π_i acts by right translations in $\mathcal{W}(\pi_i; \psi)$,

$$b(\pi_1(g) W_1, \pi_2(g) W_2) = |\det g|^{-s_0} b(W_1, W_2).$$

Thus if s_0 is a pole of $L(s, \sigma_1 \times \sigma_2)$, b is non-zero and $\alpha^{s_0} \otimes \pi_1$ is contragredient to π_2.

Conversely, suppose $\alpha^{s_0} \otimes \pi_1 \simeq \tilde{\pi}_2$. Since we may replace the pair (π_1, π_2) by $(\pi_1 \otimes \alpha^t, \pi_2 \otimes \alpha^{-t})$, we may assume without loss of generality that ω_2 is unitary. Then π_2 is preunitary, and $\alpha^{s_0} \otimes \pi_1$ is imaginary conjugate to π_2. In particular, if W_1 is in $\mathcal{W}(\pi_1; \psi_F)$, then the function $g \mapsto |\det g|^{\bar{s}_0} \overline{W}_1(g)$ belongs to $\mathcal{W}(\pi_2; \overline{\psi}_F)$. But the function $g \mapsto W_2[\eta g]$ also belongs to $\mathcal{W}(\pi_2; \overline{\psi}_F)$. Therefore b is not identically zero, and s_0 is a pole.

Q.E.D.

COROLLARY (1.3). — *Suppose π is a supercuspidal representation and η is the unramified quadratic character of* $F^\times$. *If π is not equivalent to $\pi \otimes \eta$, then:*

$$L(s, \pi \times \tilde{\pi}) = (1 - q^{-s})^{-1};$$

otherwise

$$L(s, \pi \times \tilde{\pi}) = (1 - q^{-2s})^{-1}.$$

Proof. — The number s_0 is a pole of $L(s, \pi \times \tilde{\pi})$ if and only if $\pi \otimes \alpha^{s_0} \approx \pi$. But this relation implies $\alpha^{2s_0} = 1$, that is $\alpha^{s_0} = \eta$ or 1. Since the poles of L are simple, the Corollary follows.

For completeness, and future reference, we now give the values of L and ε when π_2 is *not* supercuspidal.

PROPOSITION (1.4). — *Suppose $\pi_2 = \pi(\mu_2, \nu_2)$ (using the notation of* [JL], § 3). Then:

$$(1.4.1) \qquad \begin{cases} L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \otimes \mu_2) L(s, \pi_1 \otimes \nu_2), \\ L(s, \tilde{\pi}_1 \times \tilde{\pi}_2) = L(s, \tilde{\pi}_1 \otimes \mu_2^{-1}) L(s, \tilde{\pi}_1 \otimes \nu_2^{-1}) \end{cases}$$

and

$$\varepsilon(s, \pi_1 \times \pi_2; \psi) = \varepsilon(s, \pi_1 \otimes \mu_2; \psi_F) \varepsilon(s, \pi_1 \otimes \nu_2; \psi_F).$$

Suppose π_2 is the special representation $\sigma(\mu_2, \nu_2)$ with $\mu_2 \nu_2^{-1} = \alpha$ but π_1 is not special (loc. cit.). *Then:*

$$(1.4.2) \qquad L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \otimes \mu_2), \; L(s, \tilde{\pi}_1 \otimes \tilde{\pi}_2) = L(s, \tilde{\pi}_1 \otimes \nu_2^{-1})$$

and

$$\varepsilon(s, \pi_1 \times \pi_2; \psi_F) = \varepsilon(s, \pi_1 \otimes \mu_2; \psi_F)\varepsilon(s, \pi_1 \otimes \nu_2; \psi_F)\frac{L(1-s, \tilde{\pi}_1 \otimes \mu_2^{-1})}{L(s, \pi_1 \otimes \nu_2)}.$$

Finally suppose $\pi_i = \sigma(\mu_i, \nu_i)$, $\mu_i . \nu_i^{-1} = \alpha_F$, $i = 1,2$. *Then:*

(1.4.3)
$$\begin{cases} L(s, \pi_1 \times \pi_2) = L(s, \mu_1 \mu_2) L(s, \nu_1 \mu_2), \\ L(s, \tilde{\pi}_1 \times \tilde{\pi}_2) = L(s, \nu_1^{-1} \nu_2^{-1}) L(s, \mu_1^{-1} \nu_2^{-1}), \end{cases}$$

and

$$\varepsilon(s, \pi_1 \times \pi_2; \psi_F) = \varepsilon(s, \pi_1 \otimes \mu_2; \psi_F)\varepsilon(s, \pi_1 \otimes \nu_2; \psi_F).$$

Here the L *and* ε *factors on the right are those of* [JL].

Proof. — For the first two assertions *see* Theorem 15.1 in [Ja]. We prove the third one since—contrary to what is implied in [Ja]— Theorem 15.1 does not apply to this case.

Recall that $\mathscr{W}(\pi_1; \psi_F)$ is a subspace of codimension one of the space denoted $\mathscr{W}(\mu_1, \nu_1; \psi_F)$ in [JL] (§ 3, p. 94). Although the representation of $G_2(F)$ in this larger space is reducible the assertions recalled in (1.1) still apply to this space. Similarly (1.4.1) and (1.4.2) apply with the obvious modifications. In particular the "g. c. d." of the integrals $\Psi(s, W_1, W_2, \Phi)$ with $W_1 \in \mathscr{W}(\mu_1, \nu_1; \psi_F)$ and $W_2 \in \mathscr{W}(\pi_2; \psi_F)$ is

$$L(s, \pi_2 \otimes \mu_1) L(s, \pi_2 \otimes \nu_1) = L(s, \mu_1 \mu_2) L(s, \nu_1 \mu_2).$$

Similarly the "g. c. d." of the integrals $\Psi(s, \tilde{W}_1, \tilde{W}_2, \Phi)$ with W_1 and W_2 in the same spaces is

$$L(s, \tilde{\pi}_2 \otimes \mu_1^{-1}) L(s, \tilde{\pi}_2 \otimes \nu_1^{-1}) = L(s, \mu_1^{-1} \nu_2^{-1}) L(s, \nu_1^{-1} \nu_2^{-1})$$

and there is a similar assertion for the ε-factor.

We conclude that the quotients

$$L(s, \pi_1 \times \pi_2)/L(s, \mu_1 \mu_2) L(s, \nu_1 \mu_2)$$

and

$$L(s, \tilde{\pi}_1 \times \tilde{\pi}_2)/L(s, \nu_1^{-1} \nu_2^{-1}) L(s, \mu_1^{-1} \nu_2^{-1})$$

are polynomials. To see that they are identically one—as we must— we observe that appropriate functional equations imply that the quotients $L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2)/L(s, \pi_1 \times \pi_2)$ and

$$L(1-s, \nu_1^{-1} \nu_2^{-1}) L(1-s, \mu_1^{-1} \nu_2^{-1})/L(s, \mu_1 \mu_2) L(s, \nu_1 \mu_2)$$

can differ only by a constant times a power of q^{-s}. But the second quotient has coprime numerator and denominator. Thus the required identities follow.

Remark (1.4.4). — If π is the one-dimensional representation $g \mapsto \mu(\det g)$ then it can be denoted $\pi(\mu\alpha^{1/2}, \mu\alpha^{-1/2})$ as in [JL] (p. 104). Thus one can use (1.4) to define the factors L and ε for *all* pairs of irreducible admissible representations of $G_2(F)$.

(1.5) Recall that (conjecturally) there is a bijection between the set of classes of 2 dimensional representations of the Weil-Deligne group of F and the set of classes of admissible irreducible representations of $G_2(F)$. This bijection, denoted

$$\sigma \mapsto \pi(\sigma),$$

should satisfy the following properties:

$$(1.5.1) \qquad \pi(\sigma)^{\sim} = \pi(\tilde{\sigma}), \qquad \pi(\sigma \otimes \chi) = \pi(\sigma) \otimes \chi,$$

$$(1.5.2) \qquad \det(\sigma) = \text{central quasi-character of } \pi(\sigma),$$

$$(1.5.3) \qquad L(s, \pi(\sigma) \otimes \chi) = L(s, \sigma \otimes \chi),$$

$$(1.5.4) \qquad L(s, \tilde{\pi}(\sigma) \otimes \chi) = L(s, \tilde{\sigma} \otimes \chi),$$

and

$$(1.5.5) \qquad \varepsilon(s, \pi(\sigma) \otimes \chi; \psi) = \varepsilon(s, \sigma \otimes \chi; \psi);$$

(*cf.* [De]); the factors on the left are those of [JL] and the ones on the right are those of [De], [La 2].

According to the conjectures of Langlands one should also have

$$(1.5.6) \qquad L(s, \pi(\sigma) \times \pi(\tau)) = L(s, \sigma \otimes \tau),$$

and

$$(1.5.7) \qquad \varepsilon(s, \pi(\sigma) \times \pi(\tau); \psi_F) = \varepsilon(s, \sigma \otimes \tau; \psi_F).$$

The factors on the left are now the ones described in (1.1) and the ones on the right are again those of [De], [La 2].

We note that (1.5.6) and (1.5.7) follow from (1.4) except when *both* σ and τ are irreducible representations of the Weil-group. In that case the existence of $\pi(\sigma)$ and $\pi(\tau)$ is perhaps not yet known. Nevertheless we have the following proposition (which completes Corollary 19.16 of [Ja]):

PROPOSITION (1.6). — *Regard* F *as the completion at some place* v *of some global field* K. *Let* σ *and* τ *be two-dimensional irreducible representations of the Weil-group* W_F *satisfying the following properties:*

(i) *there exist two dimensional representations* Σ *and* T *of the Weil-group* W_K *whose compositions with the morphism* $W_F \to W_K$ *are* σ *and* τ;

(ii) *for each place* w *of* K *let* Σ_w *and* T_w *denote the compositions of* Σ *and* T *with the morphism* $W_w \to W_K$*; then* $\pi(\Sigma_w)$ *and* $\pi(T_w)$ *exist;*

(iii) *the representations* $\otimes\, \pi(\Sigma_w)$ *and* $\otimes\, \pi(T_w)$ *are automorphic cuspidal.*

Then (1.5.6) *and* (1.5.7) *hold.*

Proof. — By Corollary 19.16 of [Ja],

$$L(1-s, \tilde{\pi}(\sigma)\times\tilde{\pi}(\tau))\,\varepsilon(s, \pi(\sigma)\times\pi(\tau), \psi_F)/L(s, \pi(\sigma)\times\pi(\tau))$$
$$= L(1-s, \tilde{\sigma}\otimes\tilde{\tau})\,\varepsilon(s, \sigma\otimes\tau; \psi_F)/L(s, \sigma\otimes\tau).$$

Thus it suffices to prove (1.5.6). Since there is no harm in assuming σ and τ unitary and irreducible, $\pi(\sigma)$ and $\pi(\tau)$ can also be taken to be preunitary and super cuspidal.

If s_0 is a pole of $L(s, \sigma\otimes\tau)$ then the representation $\alpha^{s_0}\otimes\sigma$ is equivalent to the representation $\tilde{\tau}$ ([JL], Lemma 12.4); thus s_0 is purely imaginary, and the rational fraction

$$L(s, \sigma\otimes\tau)/L(1-s, \tilde{\sigma}\otimes\tilde{\tau})$$

is irreducible in the ring $\mathbf{C}[q^{-s}, q^{s}]$. But by Proposition (1.2) the same is true of

$$L(s, \pi(\sigma)\times\pi(\tau))/L(1-s, \tilde{\pi}(\sigma)\times\tilde{\pi}(\tau)),$$

and since these fraction differ only by a unit $\mathbf{C}[q^{-s}, q^{s}]$, they have in fact the same numerator.

Q.E.D.

COROLLARY (1.7). — *Suppose σ and τ are as in* (1.6). *If $\pi(\sigma)\simeq\pi(\sigma)$, then $\sigma\simeq\tau$.*

Proof. — Indeed 0 is a pole of $L(s, \pi(\sigma)\times\tilde{\pi}(\tilde{\tau}))$. Since $\tilde{\pi}(\tau) = \pi(\tilde{\tau})$ and the pair $(\sigma, \tilde{\tau})$ satisfies the assumptions of (1.6) we find that 0 is also a pole of $L(s, \sigma\otimes\tilde{\tau})$. Thus our conclusion follows again from [JL] (Lemma 12.4).

(1.8) Suppose σ and τ are "dihedral representations" of the Weil group W_F. This means that there are separable quadratic extensions K and H of F, quasi-characters χ and θ of $K^{\times}$ and $H^{\times}$ so that

(1.8.1) $$\sigma = \mathrm{Ind}(W_F, W_K, \chi) \quad \text{and} \quad \tau = \mathrm{Ind}(W_F, W_H, \theta).$$

Then $\pi(\sigma)$ and $\pi(\tau)$ exist and (1.5.6) and (1.5.7) are satisfied. This is clear by (1.4.1) if either σ on τ is reducible. Otherwise it follows from (1.6).

In particular, if the residual characteristic of F is not 2, the bijection $\sigma\mapsto\pi(\sigma)$ exists and the relations [(1.5.6), (1.5.7)] apply to all pairs of representations.

(1.9) Now we complete the results of [Ja] (§ 20) (although this will not be needed later in the paper). Let K be a separable quadratic extension of F and set $\psi_K = \psi_F\circ\mathrm{Tr}_{K/F}$. Then for any quasi-character θ of $F^{\times}$,

(1.9.1) $$L(s, \theta\circ N_{K/F}) = L(s, \theta)\,L(s, \theta\zeta),$$

and

1.9.2) $$\lambda(K/F, \psi_F)\,\varepsilon(s, \theta\circ N_{K/F}; \psi_K) = \varepsilon(s, \theta; \psi_F)\,\varepsilon(s, \theta\zeta; \psi_F).$$

Here ζ is the quadratic character of $F^{\times}$ attached to K and $\lambda(K/F, \psi_F)$ is a contant whichs depends on the extension and the choice of ψ_F (*cf.* [JL], p. 6).

If χ is any character of K^x let σ_χ be the representation defined by (1.8.1). If σ is any admissible irreducible reprensentation of $G_2(F)$ and π is an admissible irreducible representation of $G_2(K)$ we way that *π is a base change lifting of* σ (a strict lifting in the terminology of [Ja]) if the following conditions are satisfied:

(1.9.3) the central quasi-characters ω and ω' of σ and π satisfy the relation

$$\omega' = \omega \circ N_{K/F};$$

(1.9.4) for any quasi-character χ of K^x,

$$L(s, \sigma \times \pi(\sigma_\chi)) = L(s, \pi \otimes \chi)$$
$$L(s, \tilde{\sigma} \times \tilde{\pi}(\sigma_\chi)) = L(s, \tilde{\pi} \otimes \chi)$$

and

$$\varepsilon(s, \sigma \times \pi(\sigma_\chi); \psi_F) = \lambda(K/F, \psi_F)^2 \varepsilon(s, \pi \otimes \chi; \psi_K).$$

We note that if σ exists at all it is unique. Moreover, suppose $\sigma = \pi(\tau)$ where τ is a 2-dimensional representation of the Weil-Deligne group of F. Suppose also that (1.5.6) and (1.5.7) are true for each pair $(\pi(\sigma_\chi), \pi(\tau))$. Let τ' be the restriction of τ to the Weil-Deligne group of K. If $\pi(\tau')$ exists it is clearly a lifting of $\pi(\tau)$. Thus the existence of a lifting is established in all cases except the following: the residual characteristic of F is 2 and π is a supercuspidal representation not of the form $\pi(\sigma_\chi)$, χ a quasi-character of K^x. Nevertheless the following Lemma holds:

LEMMA (1.9.5). — *Suppose σ is a supercuspidal representation of* $G_2(F)$ *not of the form* $\pi(\sigma_\chi)$. *Suppose π is an irreducible admissible representation of* $G_2(K)$ *satisfying* (1.9.3) *and the following condition:*

$$\varepsilon(s, \sigma \times \pi(\sigma_\chi); \psi_F) L(1-s, \tilde{\sigma} \times \tilde{\pi}(\sigma_\chi))/L(s, \sigma \times \pi(\sigma_\chi)),$$
$$= \varepsilon(s, \pi \otimes \chi; \psi_K) L(1-s, \tilde{\pi} \otimes \chi^{-1})/L(s, \pi \otimes \chi).$$

Then π is a supercuspidal representation of $G_2(F)$ *and a lifting of* σ.

Proof. — By Proposition (1.2):

$$L(s, \sigma \times \pi(\sigma_\chi)) = L(s, \tilde{\sigma} \times \tilde{\sigma}(\sigma_\chi)) = 1.$$

Thus

$$L(1-s, \tilde{\pi} \otimes \chi^{-1})/L(s, \pi \otimes \chi)$$

is monomial for all χ. Since this can happen only if π is supercuspidal ([JPSS], § 7),

$$L(s, \pi \otimes \chi) = L(s, \tilde{\pi} \otimes \chi^{-1}) = 1,$$

and (1.9.4) is satisfied.

Examples (1.9.6). — In the Table below, the representation on the right is a base change lifting of the representation on the left.

$$\sigma = \pi(\mu_1, \mu_2), \qquad \pi = \pi(\mu_1 \circ N_{K/F}, \mu_2 \circ N_{K/F}),$$

$$\sigma = \sigma(\mu_1, \mu_2), \qquad \mu_1 . \mu_2^{-1} = \alpha_F, \qquad \pi = \sigma(\mu_1 \circ N_{K/F}, \mu_2 \circ N_{K/F}),$$

$\sigma = \pi(\sigma_\chi)$, χ quasi-character of K^x, $\pi = \pi(\chi, \chi')$ where χ' is the conjugate of χ by the non-trivial element of Gal (K/F).

(1.10) The discussion above applies to the case of an archimedean field too. In this case the bijection $\sigma \mapsto \pi(\sigma)$ is known and the factors $L(s, \pi_1 \times \pi_2)$—defined by (1.5.6) and (1.5.7)—are still related to integrals like (1.1.3) (*cf.* [Ja], § 17, § 18). The notion of a base-change lifting is also defined and the first and third examples of (1.9.6) now cover all cases.

Remark (1.10.1). — For all fields F the representation $\sigma(\mu_1, \mu_2)$ where $\mu_1 . \mu_2^{-1} = \alpha_F$ can be defined as the infinite dimensional component of the induced representation:

$$\mathrm{Ind}(G_2, B_2, \mu_1, \mu_2).$$

Of course if F = **R** or **C** then $\sigma(\mu_1, \mu_2) = \pi(\lambda)$ for a suitable λ.

2. GL (2) × GL (2): global theory

(2.1) In this section F will denote a global field and $\psi = \prod_v \psi_v$ a non-trivial additive character of A/F. By π_i, $i = 1,2$, we denote an irreducible admissible representation of (the Hecke algebra of) $G_2(\mathbf{A})$, with central quasi-character ω_i. We assume ω_i is trivial on F^x. Then $\pi_i = \otimes \pi_{i,v}$ and we set:

(2.1.1) $$L(s, \pi_1 \times \pi_2) = \prod_v L(s, \pi_{1,v} \times \pi_{2,v}),$$

(2.1.2) $$L(s, \tilde{\pi}_1 \times \tilde{\pi}_2) = \prod_v L(s, \tilde{\pi}_{1,v} \times \tilde{\pi}_{2,v}),$$

(2.1.3) $$\varepsilon(s, \pi_1 \times \pi_2) = \prod_v \varepsilon(s, \pi_{1,v} \times \pi_{2,v}; \psi_v).$$

For almost all v the representation π_i, v is unramified and has the form

(2.1.4) $$\pi_{i,v} = \pi(\mu_{i,v}, \nu_{i,v}), \ \mu_{i,v}(x) = |x|_v^{s_{i,v}}, \ \nu_{i,v}(x) = |x|^{t_{i,v}}.$$

We assume also that there is a constant c such that

(2.1.5) $$-c \leqq \mathrm{Re}(s_{i,v}) \leqq c, \ -c \leqq \mathrm{Re}(t_{i,v}) \leqq c$$

for almost all v. Note that this condition is always satisfied (with $c = 1/2$) if π_1 and π_2 are preunitary. In any case (2.1.5) implies (2.1.1) and (2.1.2) converge absolutely in some half-plane $\mathrm{Re}(s) > s_0$. As for (2.1.3), it is actually independent of ψ and almost all its factors are equal to 1; in particular, as a function of s, it is just a constant times an exponential function of s.

THEOREM (2.2). — (1) *Suppose* π_1 *and* π_2 *are automorphic. Then the functions* $L(s, \pi_1 \times \pi_2)$ *and* $L(s, \tilde{\pi}_1 \times \tilde{\pi}_2)$ — *originally defined only in some half-plane — continue as meromorphic functions to* $\mathbf{C}$.

(2) *If* F *is a number field these functions have only finitely many poles and are bounded at infinity in vertical strips. If* F *is a function field whose field of constants has* Q *elements then they are rational functions of* Q^{-s}.

(3) *These functions satisfy the functional equation*

$$L(s, \pi_1 \times \pi_2) = \varepsilon(s, \pi_1 \times \pi_2) L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2).$$

(4) *If* π_1 *is cuspidal but* π_2 *is not then* $L(s, \pi_1 \times \pi_2)$ *is entire. If* π_1 *and* π_2 *are cuspidal then the poles of* $L(s, \pi_1 \times \pi_2)$ *are simple. Moreover* s_0 *is a pole if and only if*

$$\alpha^{s_0} \otimes \pi_1 \simeq \tilde{\pi}_2 \qquad \text{or} \qquad \alpha^{1-s_0} \otimes \tilde{\pi}_1 = \pi_2.$$

Proof. — Suppose π_2 is not cuspidal. Then by Theorem (10.10) of [JL] there are two quasi-characters μ, ν of $F_A^\times/F^\times$ such that for all v, $\pi_{2,v}$ is a component of the representation $\rho(\mu_v, \nu_v)$ of $G_{2,v}$ induced by the quasi-character of $B_{2,v}$ given by the pair (μ_v, ν_v) (*loc. cit.*). Actually, if $\rho(\mu_v, \nu_v)$ is irreducible then $\pi(\mu_v, \nu_v) = \rho(\mu_v, \nu_v)$; if not, then $\rho(\mu_v, \nu_v)$ has two irreducible components $\pi(\mu_v, \nu_v)$ and $\sigma(\mu_v, \nu_v)$. If μ_v, ν_v are unramified then $\pi(\mu_v, \nu_v)$ is unramified. In any case, *for almost all* v,

$$(2.2.5) \qquad \left\{\begin{array}{l} \pi_{2,v} = \pi(\mu_v, \nu_v), \\ L(s, \pi_{1,v} \times \pi_{2,v}) = L(s, \pi_{1,v} \otimes \mu_v) L(s, \pi_{1,v} \otimes \nu_v), \\ L(s, \tilde{\pi}_{1,v} \times \tilde{\pi}_{2,v}) = L(s, \tilde{\pi}_{1,v} \otimes \mu_v^{-1}) L(s, \tilde{\pi}_{1,v} \otimes \nu_v^{-1}), \end{array}\right.$$

and

$$\varepsilon(s, \pi_{1,v} \times \pi_{2,v}; \psi_v) = \varepsilon(s, \pi_{1,v} \otimes \mu_v; \psi_v) \varepsilon(s, \pi_{1,v} \otimes \nu_v; \psi_v).$$

On the other hand, for all v, the ratios

$$L(s, \pi_{1,v} \times \pi_{2,v})/L(s, \pi_{1,v} \otimes \mu_v) L(s, \pi_{1,v} \otimes \nu_v),$$
$$L(s, \tilde{\pi}_{1,v} \times \tilde{\pi}_{2,v})/L(s, \tilde{\pi}_{1,v} \otimes \mu_v^{-1}) L(s, \tilde{\pi}_{1,v} \otimes \nu_v^{-1})$$

are entire and

$$\begin{aligned} &\varepsilon(s, \pi_{1,v} \times \pi_{2,v}; \psi_v) L(1-s, \pi_{1,v} \times \tilde{\pi}_{2,v})/L(s, \pi_{1,v} \times \pi_{2,v}) \\ &\quad = \varepsilon(s, \pi_{1,v} \otimes \mu_v; \psi_v) L(1-s, \tilde{\pi}_{1,v} \otimes \mu_v^{-1})/L(s, \pi_{1,v} \otimes \mu_v) \\ &\qquad \times \varepsilon(s, \pi_{1,v} \otimes \nu_v; \psi_v) L(1-s, \tilde{\pi}_{1,v} \otimes \nu_v^{-1})/L(s, \pi_{1,v} \otimes \nu_v), \end{aligned}$$

as can be checked directly. Thus our assertion follows from the known analytic properties of $L(s, \pi_1 \otimes \mu)$ and $L(s, \pi_1 \otimes \nu)$. (Corollary 11.2 *in* [JL].)

Suppose now both π_1 and π_2 are cuspidal. By Theorem 19.14 of [Ja] it suffices to prove (4). For $W_i \in \mathscr{W}(\pi_i; \psi)$ (notations of [JL], § 10 and § 11) and Φ a Schwartz-Bruhat function on A^2, define an integral $\Psi(s, W_1, W_2, \Phi)$ by (1.1.3) the integral now being on $N_2(A)\backslash G_2(A)$. These integrals converge in some right half-space and $L(s, \pi_1 \times \pi_2)$ is a linear combinaison of integrals Ψ. Conversely, at least for Φ in a certain dense subset, each integral Ψ is equal to the L-factor times an entire function of s.

Now set

$$\varphi_i(g) = \sum_{\alpha \in F^\times} W_i\left[\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right].$$

Then φ_i is a cusp form which belongs to a space realizing the representation π_i. Consider also for each quasi-character ω of $F_A^\times/F^\times$ the Eisenstein series $E(g, \Phi, \omega, s)$ defined by

$$E(g, \Phi, \omega, s) = \sum f(\gamma g), \qquad \gamma \in B_2(F)\backslash G_2(F),$$

$$f(g) = \int_{F_A^\times} \Phi[(0, t) g] \, |t|^{2s} \omega(t) \, d^\times t.$$

Then, for Res large enough,

$$\Psi(s, W_1, W_2, \Phi) = \int \varphi_1(g) \varphi_2(g) E(g, \Phi, \omega_1 \omega_2, s) \, dg,$$

$$g \in Z_2(A) G_2(F)\backslash G_2(A).$$

(*cf.* [Ja], § 19.) This shows the left hand side is meromorphic. Moreover, if s_0 is a pole of the left hand-side, it is a pole of the Eisenstein-series. Thus s_0 is a simple pole and $\alpha^{2-2s_0} = \omega_1 \omega_2$ or $\alpha^{2s_0} = \omega_1^{-1} \omega_2^{-1}$. In the first case

$$\lim_{s \to s_0} (s - s_0) \Psi(s, W_1, W_2, \Phi)$$
$$= c_1 \Phi(0, 0) \int |\det g|^{s_0 - 1} \varphi_1(g) \varphi_2(g) \, dg, \quad g \in G_2(F) Z_2(A)\backslash G_2(A).$$

In the second case

$$\lim_{s \to s_0} (s - s_0) \Psi(s, W_1, W_2, \Phi)$$
$$= c_2 \Phi(0, 0) \int |\det g|^{s_0} \varphi_1(g) \varphi_2(g) \, dg, \quad g \in G_2(F) Z_2(A)\backslash G_2(A).$$

Here c_1, c_2 are non zero constants.

Arguing as in the local case we find first that any pole s_0 of $L(s, \pi_1 \times \pi_2)$ is simple and such that $\alpha^{2s_0} = (\omega_1 \omega_2)^{-1}$ or $\alpha^{2-2s_0} = (\omega_1 \omega_2)^{-1}$. If s_0 is any number such that $\alpha^{2s_0} = (\omega_1 \omega_2)^{-1}$ it will be a pole of L if and only if the bilinear form

$$(\varphi_1, \varphi_2) \mapsto \int |\det g|^{s_0} \varphi_1(g) \varphi_2(g) \, dg$$

is not identically zero, that is, $\alpha^{s_0} \otimes \pi_1$ is equivalent to $\tilde{\pi}_2$. Conversely if s_0 is any number such that $\alpha^{s_0} \otimes \pi_1 \simeq \tilde{\pi}_2$ then $\alpha^{s_0} = (\omega_1 \omega_2)^{-1}$ and the above argument applies. The other case is handled similarly.

(2.3) We can now complete the global results of [Ja] (§ 20). Suppose π is an admissible irreducible representation of the (Hecke algebra) of $G_2(F_A)$ and K is a separable quadratic extension of F. Then an admissible irreducible representation Π of $G_2(K_A)$ is said to be a *base-change lifting* of π if the following conditions are satisfied:

(2.3.1) Suppose v is a place of F which splits in K, with w_1, w_2 the corresponding places of K, so that $F_v \simeq K_{w_1} \simeq K_{w_2}$ and $G_{2,v} \simeq G_{2,w_1} \simeq G_{2,w_2}$; then $\pi_v \simeq \Pi_{w_1} \simeq \Pi_{w_2}$.

(2.3.2) Suppose v is a place of F which does not split in K, and w is the place of K above v, so that K_w is a quadratic extension of F_v; then Π_w is a base change lifting of π_v in the sense of (1.9) above.

Suppose χ is a character of $K_A^\times/K^\times$. If v is a place of F which does not split let $\pi_v = \pi(\sigma_{\chi_w})$ [using the notation of (2.3.2)]; otherwise let $\pi_v = \pi(\chi_{v_1}, \chi_{v_2})$ [notation of (2.3.1)]. Then set $\pi(\sigma_\chi) = \otimes \pi_v$. It is clear that $\pi(\sigma_\chi)$ admits a lifting Π; indeed $\Pi_w = \pi(\chi_w, \chi'_w)$ where χ' is the conjugate of χ under the action of the Galois group. One can show that both $\pi(\sigma_\chi)$ and Π are automorphic; ([La 5]). If π is automorphic cuspidal but not of the form $\pi(\sigma_\chi)$ then the following is true:

PROPOSITION (2.3.3). — *Let π be an automorphic cuspidal representation of $G_2(F_A)$ not of the form $\pi(\sigma_\chi)$. Then π admits a base change lifting Π to $G_2(K_A)$ and Π is automorphic cuspidal.*

Proof. — Let S be the set (finite and possibly empty) of places v of F of residual characteristic 2 where π_v is supercuspidal but not of the form $\pi(\sigma_\chi)$ [here χ is a quasi-character of $K_w^\times$ and we use the notations of (2.3.2)]. If v does not split and is not in S then π_v admits a lifting Π_w. If v is in S (and does not split) then by Theorem (20.6) of [Ja] we at least know there exists a representation Π_w satisfying the conditions of Lemma (1.9.5). So by this same Lemma, Π_w is actually a lifting of π_w and we conclude π has a lifting Π. Since we may assume π to be preunitary it is easily checked that Π is preunitary and the Π_w are infinite-dimensional. Moreover (*cf.* (20.6) of [Ja]),

$$L(s, \Pi \otimes \chi) = L(s, \pi \times \pi(\sigma_\chi)),$$
$$L(s, \tilde{\Pi} \otimes \chi^{-1}) = L(s, \tilde{\pi} \times \tilde{\pi}(\sigma_\chi)),$$

and

$$\varepsilon(s, \Pi \otimes \chi) = \varepsilon(s, \pi \times \pi(\sigma_\chi)).$$

It follows now from (2.2) that L $(s, \Pi \otimes \sigma_\chi)$ is entire, bounded in vertical strips if F is a number field, and satisfies

$$L(s, \Pi \otimes \chi) = \varepsilon(s, \Pi \otimes \chi) L(1-s, \tilde{\Pi} \otimes \chi^{-1}).$$

Thus our conclusion follows from [JL] (Th. 11.3).

Remark (2.3.4). — From the "strong multiplicity" one theorem for G_2 it follows that Π is indeed the base change lifting of π as defined in [La 4].

3. The notion of lifting

Henceforth, σ will be used to denote a typical representation of GL (2), and π a representation of GL (3).

In this Section we shall introduce the notion of lifting a representation σ of GL (2) to GL (3). We do this in terms of the L-functions

$$L_2(s, \sigma, \chi) = \frac{L(s, (\sigma \otimes \chi) \times \tilde{\sigma})}{L(s, \chi)}.$$

(3.1) Let F denote a local field and $\psi_F = \psi$ a non-trivial additive character of F. For each irreducible admissible representation σ of G_2 (F), set

(3.1.1) $$L_2(s, \sigma, \chi) = L(s, (\sigma \otimes \chi) \times \tilde{\sigma})/L(s, \chi)$$

and

(3.1.2) $$\varepsilon_2(s, \sigma, \chi; \psi) = \varepsilon(s, (\sigma \otimes \chi) \times \tilde{\sigma}; \psi)/\varepsilon(s, \chi; \psi).$$

We note that the factors on the right side have been defined for all σ and all F [(1.4.4) and (1.10)].

Definition 3.1.3. — *Let* π *be an admissible irreducible representation of*

$$G_3(F) = GL(3, F)).$$

We shall say that π *is a lift of* σ *if the following conditions are satisfied:*

(i) *the central quasi-character of* π *is trivial;*

(ii) $\pi \simeq \tilde{\pi}$;

(iii) *for any quasi-character* χ *of* $F^{\times}$*;*

$$L(s, \pi \otimes \chi) = L_2(s, \sigma, \chi), \qquad \varepsilon(s, \pi \otimes \chi; \psi) = \varepsilon_2(s, \sigma, \chi; \psi).$$

We note that if the last condition is satisfied for one choice of ψ, then by (i) it is satisfied for all choices of ψ. Moreover if π is a lift of σ, it is also a lift of $\sigma \otimes \chi$ for any χ and, in particular, of $\tilde{\sigma} \simeq \sigma \otimes \omega^{-1}$, ω denoting the central quasi-character of σ.

PROPOSITION 3.2. — *Suppose* F *is archimedean. Then any* σ *admits a lift* π, *unique up to equivalence.*

Proof. — Let us denote by $\tau \mapsto \pi(\tau)$ the "natural" bijection between the semi-simple representations of degree n of the Weil-group W_F and the irreducible admissible representations of $G_n(F)$. Write accordingly

$$\sigma = \pi(\tau), \qquad \deg(\tau) = 2. \tag{3.2.1}$$

Then:

$$\tau \otimes \tilde{\tau} = \lambda \oplus 1, \quad \text{where} \quad \deg(\lambda) = 3. \tag{3.2.2}$$

Set

$$\pi = \pi(\lambda). \tag{3.2.3}$$

We contend that π is a lift of σ. Indeed

$$(\tau \otimes \tilde{\tau})^{\sim} = \tilde{\tau} \otimes \tau = \tau \otimes \tilde{\tau}. \tag{3.2.4}$$

Moreover $\det(\tau \otimes \tilde{\tau}) = 1$. Therefore

$$\det(\lambda) = 1, \tag{3.2.5}$$

and—since the central quasi-character of $\pi(\lambda)$ is $\det(\lambda)$—we see that condition (i) is satisfied. From (3.2.4) one concludes that

$$\lambda^{\sim} = \lambda. \tag{3.2.6}$$

So since $\pi(\lambda^{\sim}) = \pi(\lambda)^{\sim}$, we also see that condition (ii) is satisfied. Now note

$$\begin{aligned} L(s, (\pi \otimes \chi) \times \tilde{\pi}) &= L(s, (\tau \otimes \chi) \otimes \tilde{\tau}), \\ \varepsilon(s, (\pi \otimes \chi) \times \tilde{\pi}; \psi) &= \varepsilon(s, (\tau \otimes \chi) \otimes \tilde{\tau}; \psi). \end{aligned} \tag{3.2.6}$$

But

$$(\tau \otimes \chi) \otimes \tilde{\tau} = (\tau \otimes \tilde{\tau}) \otimes \chi = (\lambda \otimes \chi) \oplus \chi. \tag{3.2.7}$$

Thus:

$$\begin{aligned} L(s, (\tau \otimes \chi) \otimes \tilde{\tau}) &= L(s, \lambda \otimes \chi) L(s, \chi), \\ \varepsilon(s, (\tau \otimes \chi) \otimes \tilde{\tau}; \psi) &= \varepsilon(s, \lambda \otimes \chi; \psi) \varepsilon(s, \chi; \psi). \end{aligned} \tag{3.2.8}$$

So comparing (3.2.6), (3.2.8), (3.1.1), and (3.1.2) we see that condition (iii) is also satisfied.

The uniqueness follows from the following Lemma, whose tedious but straightforward proof is left to the reader.

LEMMA (3.2.9). — *Suppose that* λ *and* λ' *are semi-simple representations of degree* 3 *of* W_F. *Suppose moreover that* $\det(\lambda) = \det(\lambda')$,

$$L(s, \lambda \otimes \chi) = L(s, \lambda' \otimes \chi), \qquad L(s, \tilde{\lambda} \otimes \chi) = L(s, \tilde{\lambda}' \otimes \chi),$$

and

$$\varepsilon(s, \lambda \otimes \chi; \psi) = \varepsilon(s, \lambda' \otimes \chi; \psi) \quad \textit{for all } \chi.$$

Then $\lambda' \simeq \lambda$.

In the non-archimedean case we will eventually manage to prove a result analogous to Proposition 3.2. For the time being, however, we content ourselves with the following weaker result.

PROPOSITION 3.3. — *Suppose* F *is non-archimedean.*

(1) *If a lift of* σ *exists, it is unique up to equivalence.*

(2) *If* σ *is not an "extraordinary representation", then* σ *admits a lift.*

(3) *If* σ *is extraordinary and admits a lift* π, *then* π *is cuspidal.*

Proof. — The first assertion follows from Definition (3.1.3), and Lemma (7.5.3) of [JPSS].

Recall that an extraordinary representation is a cuspidal representation which is not of the form $\pi(\tau)$ where τ is a two-dimensional induced representation of W_F. If σ is not extraordinary, then $\sigma = \pi(\tau)$ where τ is a two-dimensional representation of the Weil-Deligne group. Then (3.2.4) to (3.2.5), as well as (3.2.7) and (3.2.8) still hold. In principle, the lift π of σ is given by (3.2.3). But since $\pi(\lambda)$ has not yet been defined, we give now an explicit description of the lift of σ.

Suppose first that τ is a representation of the Weil group. Suppose moreover that

(3.3.4) $$\tau = \mu_1 \oplus \mu_2, \quad \mu_i = \chi_i \alpha^{t_i}, \quad \text{with} \quad t_i \in \mathbf{R}, \quad \chi_i \bar{\chi}_i = 1.$$

In other words, suppose $\sigma = \pi(\mu_1, \mu_2)$. Then:

(3.3.5) $$\mu_1 . \mu_2^{-1} = \chi_1 \chi_2^{-1} \alpha^{t_1 - t_2}, \qquad \mu_2 . \mu_1^{-1} = \chi_2 . \chi_1^{-1} \alpha^{t_2 - t_1}.$$

Let us form the induced representation

(3.3.6) $$\xi = \operatorname{Ind}(G_3, B_3; \mu_1 . \mu_2^{-1}, 1, \mu_2 . \mu_1^{-1}).$$

If $t_1 - t_2 = 0$, ξ is a unitary irreducible (principal series) representation. We contend then that $\pi = \xi$ is a lift of σ. Indeed the central quasi-character of ξ is $\mu_1 . \mu_2^{-1} . \mu_2 \mu_2^{-2} = 1$. Now

(3.3.7) $$\tilde{\xi} = \operatorname{Ind}(G_3, B_3; \mu_2 . \mu_1^{-1}, 1, \mu_1 . \mu_2^{-1}),$$

and this representation has the same character as ξ. Thus $\xi = \tilde{\xi}$ or $\tilde{\pi} = \pi$. Finally we have to check that, with the notations of (3.2.2),

(3.3.8) $$L(s, \pi \otimes \chi) = L(s, \lambda \otimes \chi) \quad \text{and} \quad \varepsilon(s, \pi \otimes \chi; \psi) = \varepsilon(s, \lambda \otimes \chi; \psi).$$

Since $\lambda = \mu_2 . \mu_1^{-1} \oplus 1 \oplus \mu_1 . \mu_2^{-1}$ this follows from [Go Ja] [Theorem (3.4)].

Now suppose $t_1 - t_2 \neq 0$. We may assume $t_1 > t_2$ and then form the representation (3.3.6). It may fail to be irreducible, but it always admits a maximal subrepresentation ξ'.

The quotient representation $\pi = \xi/\xi'$ is then irreducible (*see* [Si] for instance) and again we contend that π is a lift of σ. As before, (i) is satisfied. Now $\tilde{\pi}$ can also be obtained as the unique irreducible quotient of

$$\rho = \operatorname{Ind}(G_3, {}^t B_3; \mu_2 . \mu_1^{-1}, 1, \mu_1 . \mu_2^{-1}). \tag{3.3.9}$$

(*Cf.* [J 1] for instance). If

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

then $g \mapsto w g w^{-1}$ takes ${}^t B$ to B and ρ to ξ. We conclude that $\pi \simeq \tilde{\pi}$. Finally, since (3.3.8) has also been established in [J 1] [Prop. (3.4)], (iii) is satisfied.

Now suppose that

$$\tau = \operatorname{Ind}(W_F, W_K, \chi), \tag{3.3.10}$$

where K is a separable quadratic extension of F and χ is a quasi-character of $K^{\times}$, i. e., $\sigma = \pi(\sigma_\chi)$. Let also χ' be the quasi-character of χ conjugate to χ by the action of the nontrivial element of the Galois group. Then:

$$\tau = \operatorname{Ind}(W_F, W_K, \chi'), \qquad \tilde{\tau} = \operatorname{Ind}(W_F, W_K, \chi^{-1}), \qquad \text{and} \qquad \tilde{\tau} \mid W_K = \chi^{-1} \oplus \chi'^{-1}.$$

Thus:

$$\begin{aligned} \tau \otimes \tilde{\tau} &\simeq \operatorname{Ind}(W_F, W_K, \chi \otimes (\tilde{\tau} \mid W_K)) = \operatorname{Ind}(W_F, W_K, 1 \oplus \chi . \chi'^{-1}) \\ &= \operatorname{Ind}(W_F, W_K, 1) \oplus \operatorname{Ind}(W_F, W_K, \chi . \chi'^{-1}) \\ &= 1 \oplus \eta \oplus \operatorname{Ind}(W_F, W_K, \chi . \chi'^{-1}), \end{aligned} \tag{3.3.11}$$

where η is the quadratic character attached to K, i. e.,

$$\tau \otimes \tilde{\tau} = \lambda \oplus 1, \quad \text{where} \quad \lambda = \mu \oplus \eta \quad \text{and} \quad \mu = \operatorname{Ind}(W_F, W_K, \chi . \chi'^{-1}). \tag{3.3.12}$$

But $\chi . \chi'^{-1}$ is a character. Thus μ is unitary. Moreover $(\chi . \chi'^{-1})' = (\chi . \chi'^{-1})^{-1}$, and

$$\tilde{\mu} = \operatorname{Ind}(W_F, W_K, (\chi . \chi'^{-1})^{-1}) = \operatorname{Ind}(W_F, W_K, (\chi . \chi'^{-1})') = \mu. \tag{3.3.13}$$

The representation $\pi(\mu)$ is therefore unitary and isomorphic to its contragredient. Its central character is η and it is either in the principal series or cuspidal. It follows that the induced representation

$$\pi = \operatorname{Ind}(G_3, P; \pi(\mu), \eta), \tag{3.3.14}$$

where P is the parabolic subgroup of type (2, 1), is unitary irreducible. (*Cf.* for instance [JPSS], § 6.) We contend it is a lift of σ. Indeed its central quasi-character is $\eta^2 = 1$, and since $\tilde{\eta} = \eta$, and $\tilde{\pi}(\mu) = \pi(\mu)$, we get $\pi = \tilde{\pi}$. Finally (*cf.* [J 1]):

$$L(s, \pi \otimes \chi) = L(s, \pi(\mu) \otimes \chi) L(s, \eta\chi) = L(s, \mu \otimes \chi) L(s, \eta\chi) = L(s, \lambda \otimes \chi).$$

Since a similar relation is true for ε, (3.3.8) [and hence condition (iii)] is satisfied.

Of course all our discussion so far applies to the archimedean case as well. Suppose now τ is a 2-dimensional representation of the Weil-Deligne group which is not a representation of the Weil group. Then σ is special, that is of the form

$$\sigma(\chi\alpha^{1/2}, \chi\alpha^{-1/2}) = \chi \otimes \sigma(\alpha^{1/2}, \alpha^{-1/2}).$$

Let π be the Steinberg representation of $G_3(F)$, that is, the square integrable component of

$$\mathrm{Ind}(G_3, B_3; \alpha, 1, \alpha^{-1}).$$

That π satisfies (i) and (ii) is well known. As for condition (iii), it follows at once from Theorem 7.11 of [Go J] and (1.4.3) of this paper.

Next suppose σ is an extraordinary representation. We want to show first that

$$L_2(s, \sigma, \chi) = 1.$$

For this we appeal to the fact that $\sigma \otimes \chi \not\simeq \sigma$ if $\chi \neq 1$. (*Cf.* [La 4], Lemma 5.16.) Thus s_0 is a pole of $L(s, (\sigma \otimes \chi) \times \tilde{\sigma})$ if and only if $\alpha^{s_0}\chi = 1$, that is, s_0 is a pole of $L(s, \chi)$. Our conclusion then follows. This being so, if π is a lift of σ, we get

$$L(s, \pi \otimes \chi) = L(s, \tilde{\pi} \otimes \chi) = 1,$$

and it is easily checked that this can happen only if π is cuspidal. (*Cf.* [JPSS], § 7).

(3.4) Suppose σ is unitary, but not one dimensional, the field F being archimedean or not. For our purposes, it will be important to determine when π is unitary "generic"; this means (*cf.* [JPSS], § 6 and § 10) that the restriction of π to the subgroup

(3.4.1) $$P^1 = \left\{ \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

is equivalent to the irreducible representation of P^1 induced by the character θ of N_3 defined by

(3.4.2) $$\theta \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \psi(x+y).$$

If F is non-archimedean, σ is extraordinary, and σ admits a lift π, then π is cuspidal and has a trivial central quasi-character; thus π is unitary generic (*cf.* [JPSS]). Similarly, if F is non-archimedean and σ is special, its lift is special and again unitary generic (*loc. cit.*).

The remaining discussion will apply to the archimedean and non-archimedean case as well. We write $\sigma = \pi(\tau)$ where τ is a two-dimensional representation of W_F. If τ has the form (3.3.10), then π is given by (3.3.14) which is irreducible generic (*loc. cit.*). If τ has the form (3.3.4) and $t_1 = t_2$ then π is given by (3.3.6) which is irreducible generic. So it remains only to examine the case where π is in the complementary series, that is τ has the form (3.3.4) where χ is a character and $t_1 = -t_2 = t$, $0 < t < 1/2$. Then (3.3.6) takes the form

$$\xi = \mathrm{Ind}(G_3, B_3; \alpha^{2t}, 1, \alpha^{-2t}).$$

Now the maximal subrepresentation ξ' of ξ is nothing but the kernel of the intertwining operator

$$f \mapsto \int f(wng)\,dn, \qquad n \in N_3.$$

It is then easy to determine ξ' and π. One finds that ξ is irreducible unless $t = 1/4$ in which case

$$\pi = \mathrm{Ind}(G_3, P; 1_2, 1)$$

where 1_2 is the trivial representation of G_2 and 1 the trivial representation of $F^\times$. The representation π is then unitary but not generic.

If $0 < t < 1/4$, then the representation $\pi(\alpha^{2t}, \alpha^{-2t})$ is unitary in the complementary series. Since ξ is irreducible,

$$\pi = \xi = \mathrm{Ind}(G_3, B_3; \alpha^{2t}, \alpha^{-2t}, 1) = \mathrm{Ind}(G_3, P, \pi(\alpha^{2t}, \alpha^{-2t}), 1)$$

and this shows that π unitary generic (*loc. cit.*) If $1/4 < t < 1/2$ then $\pi = \xi$ is not unitary.

(3.5) Suppose F is non-archimedean and σ "quasi-unramified". This means that $\sigma = \sigma_0 \otimes \chi$ where σ_0 is unramified, that is contains the trivial representation of $K_2 = GL(2, R_F)$. Then $\sigma = \pi(\alpha^{s_1}\chi, \alpha^{s_2}\chi) = \pi(\tau)$, where $\tau = \alpha^{s_1}\chi \oplus \alpha^{s_2}\chi$. As we have seen, σ admits a lift $\pi = \pi(\lambda)$ where $\lambda = \alpha^{s_1-s_2} \oplus 1 \oplus \alpha^{s_2-s_1}$. More precisely, π is the only irreducible quotient of

$$\xi = \mathrm{Ind}(G_3, B_3; \alpha^{s_1-s_2}, 1, \alpha^{s_2-s_1}),$$

and π contains the trivial representation of $K_3 = GL(3, R_F)$, that is, π is unramified. Note that if χ is also unramified then $\chi = \alpha^t$ and

(3.5.1) $$L_2(s, \sigma, \chi) = L(s, \pi \otimes \chi) = [(1-q^{-s-t+s_1-s_2})(1-q^{-s-t})(1-q^{-s-t+s_2-s_1})]^{-1}$$

and if ψ has exponent zero,

(3.5.2) $$\varepsilon(s, \sigma, \chi; \psi) = \varepsilon(s, \pi \otimes \chi; \psi) = 1.$$

(3.6) Let now F be an A-field and σ an admissible irreducible representation of $G_2(\mathbf{A})$ (or rather of its Hecke algebra). Then we say that an irreducible admissible representation π of $G_3(\mathbf{A})$ is a *lift* of σ if, for all places v, π_v is a lift of σ_v. One of the purposes of this paper is to show that if σ is automorphic cuspidal then σ admits a lift π and π is automorphic (although perhaps not cuspidal).

(3.7) There is already one case we can dispose of immediately. Indeed suppose σ is automorphic cuspidal and there is a character χ of $F_{\mathbf{A}}^{\times}/F^{\times}$ so that $\chi \neq 1$ and $\sigma \otimes \chi \simeq \sigma$. Then $\chi^2 = 1$, and χ determines a quadratic extension H of F. According to the main result of [LL] there is a quasi-character Ω of $H_{\mathbf{A}}^{\times}/H^{\times}$ so that σ is the automorphic representation attached to Ω. More precisely, if $\chi_v \neq 1$ then v does not split in H and $\sigma_v = \pi(\tau_v)$ with $\tau_v = \operatorname{Ind}(W_{F_v}, W_{H_w}, \Omega_w)$ and w the unique place of H above v. Then the lift π_v of σ_v is given by

$$\pi_v = \operatorname{Ind}(G_{3v}, P_v; \pi'_v, \chi_v),$$

where

$$\pi'_v = \pi(\tau'_v),\ \tau'_v = \operatorname{Ind}(W_{F_v}, W_{H_w}, \Omega_w.\Omega'^{-1}_w),$$

and Ω'_w is the quasi-character conjugate to Ω_w. On the other hand, if $\chi_v = 1$, then v splits and, if w_1, w_2 are the two places of H above v, we have

$$\sigma_v = \pi(\tau_v) \qquad \text{with} \quad \tau_v = \Omega_{w_1} \oplus \Omega_{w_2}.$$

The lift π_v of σ_v is then given by

$$\pi_v = \operatorname{Ind}(G_{3,v}, P_v; \pi'_v, \chi_v),$$

where

$$\pi'_v = \pi(\tau'_v),\ \tau'_v = \Omega_{w_1}.\Omega_{w_2}^{-1} \oplus \Omega_{w_2}.\Omega_{w_1}^{-1}.$$

Thus we see that $\pi' = \otimes\, \pi'_v$ is the representation attached to the character $\Omega.\Omega'^{-1}$ of $H_{\mathbf{A}}^{\times}/H^{\times}$, where Ω' is the conjugate of Ω. If $\Omega.\Omega'^{-1} = \eta \circ N_{K/F}$ where η is some character of $F_{\mathbf{A}}^{\times}/F^{\times}$, then

$$\pi'_v = \pi(\eta_v, \eta_v \chi_v),$$

$$\pi_v = \operatorname{Ind}(G_{3,v}, B_{3,v}; \eta_v, \eta_v \chi_v, \chi_v),$$

and $\pi = \otimes\, \pi_v$ is automorphic by [La 5]. Otherwise $\pi' = \otimes\, \pi'_v$ is automorphic cuspidal and again $\pi = \otimes\, \pi_v$ is automorphic (*loc. cit.*).

(3.8) Suppose σ is arbitrary automorphic cuspidal. Set

(3.8.1) $$L_2(s, \sigma, \chi) = \prod_v L_2(s, \sigma_v, \chi_v)$$

and

(3.8.2) $$\varepsilon_2(s, \sigma_\chi) = \prod_v \varepsilon_2(s, \sigma_v, \chi_v, \psi_v).$$

It follows from (3.5.1) that (3.8.1) converges in some right half-plane. On the other hand almost all factors in (3.8.2) are one, and their product is an exponential function of s which does not depend on ψ. This being so, the analytic properties of $L(s, (\sigma \otimes \chi) \times \sigma)$ and $L(s, \chi)$ imply that $L_2(s, \sigma, \chi)$ continues to a meromorphic function of s and satisfies the functional equation

$$L_2(s, \sigma, \chi) = \varepsilon_2(s, \sigma, \chi) L_2(1-s, \tilde{\sigma}, \chi^{-1}).$$

On the other hand, if π is a lift of σ, then

$$L(s, \pi \otimes \chi) = L_2(s, \sigma, \chi) \quad \text{and} \quad \varepsilon(s, \pi \otimes \chi) = \varepsilon_2(s, \sigma, \chi).$$

4. Complements for GL (3, F) (F archimedean)

This Section collects some technical results about Whittaker models for representations of GL (3, F) when F is archimedean. Thus $F = \mathbf{R}$ or $\mathbf{C}$ in this Section.

(4.1) In [JPSS] there is attached to every unitary *generic* representation π of $G_3(F)$ a space $\mathscr{W}(\pi; \psi_F)$. However, in lifting unitary representations from $G_2(F)$ to $G_3(F)$ irreducible representations arise which are not unitary generic. As we have seen, these are precisely the representations of the form

(4.1.1) $$\pi = \pi(\lambda) \quad \text{where} \quad \lambda = \alpha^t \oplus 1 \oplus \alpha^{-t} \quad \text{and} \quad \frac{1}{2} \leqq t < 1.$$

More explicitly, such a representation is the unique irreducible quotient of the induced representation

(4.1.2) $$\xi = \operatorname{Ind}(G_3, B_3; \alpha^t, 1, \alpha^{-t}).$$

What we are going to do now is define a space $\mathscr{W}(\pi; \psi_F)$ for representations of the form (4.1.1).

Let $\mathscr{V}$ be the space of C^∞ functions f on $G_3(F)$ such that

(4.1.3) $$f\left[\begin{pmatrix} a_1 & x & z \\ 0 & a_2 & y \\ 0 & 0 & a_3 \end{pmatrix} g\right] = |a_1|_F^{t+1} |a_3|^{-t-1} f(g).$$

For f in V set

(4.1.4) $$W_f(g) = \int_{N_3(F)} f(wng)\bar{\theta}(n)\, dn,$$

where θ is defined by (3.4.2) and

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

this integral converges absolutely and uniformly for g in a compact set. We let $\mathscr{W}(\pi; \psi)$ be the space spanned by the functions (4.1.4) for $f \in V$.

LEMMA (4.1.5). — *The map $f \mapsto W_f$ is a bijection.*

Proof. — Suppose $W_f = 0$. Then from (4.1.4) we get

$$\int \psi(-\alpha x)\,dx \int_{U(F)} f\left[w\begin{pmatrix}1 & x & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}ug\right]\theta(u)\,du = 0$$

for $\alpha = 1$. Changing g to diag $(\alpha, 1, 1)\,g$ we get the same identity for $\alpha \neq 0$. By continuity it is therefore true for $\alpha = 0$. Thus we get from the uniqueness of the Fourier transform,

$$\int_{U(F)} f[wug]\,\bar{\theta}(u)\,du = 0.$$

Changing g into

$$\begin{pmatrix}1 & 0 & 0\\ \alpha & \beta & 0\\ 0 & 0 & 1\end{pmatrix}^{-1} g$$

we find

$$\iint f\left[w\begin{pmatrix}1 & 0 & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}g\right]\psi(\alpha x+\beta y)\,dz\,dy = 0$$

for all $\alpha \in F$, $\beta \in F^{\times}$. As before, we find $f = 0$, which concludes the proof of (4.1.5).

Note that $\pi \simeq \tilde{\pi}$. Moreover, as in [JPSS], if W is in $\mathscr{W}(\pi; \psi)$, then the function $\tilde{W}$ defined by

$$\tilde{W}(g) = W[w^t g^{-1}] \tag{4.1.6}$$

is still in $\mathscr{W}(\pi; \psi) = \mathscr{W}(\tilde{\pi}; \psi)$; this follows from thef act that the automorphism $g \mapsto w^t g^{-1} w^{-1}$ leaves invariant N_3, B_3, θ, and ξ.

We also denote by $\mathscr{W}_0(\pi; \psi)$ the space of K_3-finite elements in $\mathscr{W}(\pi; \psi)$. It is clear that this space is invariant under right convolution by the elements of the Hecke-algebra and the representation of the Hecke algebra on $\mathscr{W}_0(\pi; \psi)$ is equivalent to ξ; moreover, since π is the unique irreducible quotient of ξ, the representation ξ is semi-simple if and only if it is irreducible. Then $\xi = \pi$ and $t \neq 1/2$. If in addition ξ is unitary generic then $0 < t < (1/2)$. As in [JPSS] we set

$$\begin{aligned} \|g\| &= (\Sigma g_{ij}^2)^{1/2}\,|\det g|^{-3/2} && \text{if } F = \mathbf{R},\\ &= (\Sigma g_{ij}.\bar{g}_{ij})(\det g\bar{g})^{-3/2} && \text{if } F = \mathbf{C}. \end{aligned} \tag{4.1.7}$$

LEMMA (4.1.8). — *There is a number $r > 0$ such that any element W of $\mathscr{W}_0(\pi, \psi)$ is dominated by $g \mapsto \|g\|^r$. Moreover any element of $\mathscr{W}_0(\pi; \psi)$ is dominated by a gauge.*

Proof. — The notion of gauge was introduced in [JPSS] (§ 8) and the second assertion follows from the first (*loc. cit.*). To prove the first we observe that, given $f \in \mathcal{V}$ and a compact set Ω of $G_3(F)$, there is a $c > 0$ such that

$$|f(gh)| \leqq c f_0(g) \qquad \text{for} \quad h \in \Omega;$$

here f_0 is the element of $\mathcal{V}$ such that $f_0(k) = 1$ for $k \in K_3$. Note that $f_0 > 0$. Thus

$$|W_f(g)| \leqq f_1(g), \qquad f_1(g) = \int f_0(wng)\, dn.$$

Now f_1 is given by

$$f_1\left[\begin{pmatrix} a_1 & x & z \\ 0 & a_2 & y \\ 0 & 0 & a_3 \end{pmatrix} k\right] = |a_1|^{-t+1} |a_3|^{t+1} f_1(e).$$

So $|f_1(g)| \leqq f_1(e) \| g \|^r$, if r is large enough, and our assertion follows.

We want also to replace π by a representation $\pi' = \pi \otimes \chi$ where χ is a character. Then $\tilde{\pi}' = \pi \otimes \chi^{-1}$. We define $\mathcal{W}(\pi'; \psi)$ to be the space spanned by the functions

$$W \otimes \chi : \ g \mapsto W(g)\chi(\det g), \ W \in \mathcal{W}(\pi; \psi).$$

PROPOSITION (4.1.9). — *Suppose π' is any representation of the form $\pi \otimes \chi$ where π is as in* (4.1.1) *and χ is a character. Then the assertions of theorems* (9.2) *and* (11.2) *of* [JPSS] *apply to the space $\mathcal{W}(\pi'; \psi)$.*

Proof. — Although ξ may fail to be irreducible, the assertions of Theorem (8.7) of [Go Ja] apply to ξ with

$$L(s, \xi \otimes \chi) = L(s, \pi \otimes \chi) \qquad \text{and} \qquad \varepsilon(s, \xi \otimes \chi; \psi) = \varepsilon(s, \pi \otimes \chi; \psi).$$

We note that if W is in $\mathcal{W}_0(\pi; \psi)$ and η is a K_3-finite function on K_3 then the function φ defined by

$$\varphi(g) = \int_{K_3} \eta(k) W(kg)\, dk$$

is a (bi-K_3-finite) matrix coefficient of ξ. Indeed for f in $\mathcal{V}$, set

$$\lambda(f) = \int_{K_3} \eta(k)\, dk \int_{N_3} f(wnk)\bar{\theta}(n)\, dn.$$

It will suffice to show that this is a continuous linear form on $\mathcal{V}$. [Note $\varphi(g) = \lambda(\xi(g)f)$ if $W = W_f$.] Of course $\|f\| = \operatorname{Sup}_{K_3} |f|$ defines the topology of $\mathcal{V}$. So the function f_0 being as in the proof of (4.1.8), we have

$$|f| \leqq f_0 \|f\|$$

and

$$|\lambda(f)| \leqq \|f\| \iint |\eta(k)| f_0(wn)\, dn\, dk = \|f\| \|\eta\|_1 f_1(e).$$

Thus λ is continuous.

Combining these remarks with lemmas (4.1.5) and (4.1.8) it becomes clear that the proof of (9.2) and (11.2) in [JPSS] applies to the case at hand.

5. Product decomposition of the main integral

Our purpose in this section is to decompose into local factors an integral on the metaplectic group involving an Eisenstein series, a theta-function and a cusp form on G_2. Accordingly F will denote an A-field of characteristic not equal to 2, and G will denote the group SL (2) regarded as an algebraic group over F.

(5.1) By Mp (A) we denote the (global) metaplectic group introduced in [We]. This is a group of unitary operators in $L^2(\mathbf{A})$ which fits into the sequence

(5.1.1) $$1 \to \mathbf{T} \to \mathrm{Mp}(\mathbf{A}) \overset{pr}{\to} \mathrm{G}(\mathbf{A}) \to 1.$$

Here $\mathbf{T} = \{ z \in \mathbf{C} \mid z\bar{z} = 1 \}$ is regarded as the group of unitary scalar operators $\Phi \mapsto \lambda\,\Phi$ in $L^2(\mathbf{A})$; it is central in Mp (A) and G (A) is (topologically) isomorphic to Mp (A)/T. In other words, the sequence is exact.

If Φ is $L^2(\mathbf{A})$ and g is in Mp (A) we write $g.\Phi$ for the image of Φ by the unitary operator g. According to paragraph 41 of [We] the sequence (5.1.1) splits over the subgroup G (F), the splitting homomorphism

(5.1.2) $$r_{\mathrm{F}}: \quad \mathrm{G}(\mathrm{F}) \to \mathrm{Mp}(\mathbf{A})$$

being determined by the condition

(5.1.3) $$\sum_{\xi \in \mathrm{F}} (r_{\mathrm{F}}(\gamma)\Phi)(\xi) = \sum_{\xi \in \mathrm{F}} \Phi(\xi) \qquad \text{for} \quad \Phi \in \mathscr{S}(\mathbf{A}), \quad \gamma \in \mathrm{G}(\mathrm{F}).$$

The sequence also splits over $\mathscr{B}$ (A) or, what amounts to the same thing, over $\mathscr{N}$ (A) and $\mathscr{A}$ (A). The corresponding splitting homomorphisms

(5.1.4) $$t: \ \mathscr{N}(\mathbf{A}) \to \mathrm{Mp}(\mathbf{A}) \qquad \text{and} \qquad d: \ \mathscr{A}(\mathbf{A}) \to \mathrm{Mp}(\mathbf{A})$$

are given by

(5.1.5) $$t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.\Phi(y) = \psi\left(\frac{1}{2}xy^2\right)\Phi(y),$$

and

$$d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\Phi(y) = |a|^{1/2}\,\Phi(ay).$$

Note that t agrees with r_{F} on N (F) and d agrees with r_{F} on A (F). Finally, if $w_0.\Phi(\mathrm{X}) = \hat{\Phi}(-x)$ where $\hat{\Phi}$ is the Fourier transform of Φ with respect to ψ, then $\mathrm{pr}(w_0) = w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $r_{\mathrm{F}}(w) = w_0$.

As in [Ge] we say that a function φ on Mp (**A**) is *genuine* if $\varphi(\lambda g) = \lambda\varphi(g)$ for all $g \in$ Mp (**A**) and $\lambda \in$ **T**. Thus, if φ_1 and φ_2 are genuine functions on Mp (**A**), the product $\varphi_1 . \bar{\varphi}_2$ is invariant under **T**, and there is a functions f on G (**A**) such that $\varphi_1 . \bar{\varphi}_2(g) = f(\mathrm{pr}(g))$. Similarly $|\varphi_1|$ may be regarded as a function on G (**A**).

To prove ultimately that cuspidal representations of GL (2) lift to GL (3) we first need to prove the following:

THEOREM 8.1. — *Suppose* σ *is cuspidal representation of* GL (2, **A**) *and* χ *is a "highly ramified" character of* $F^{x}\backslash F^{x}_{\mathbf{A}}$ (*see* (5.3) below). *Then the Euler product of degree* 3 *defined by*

$$L_2(\sigma, \chi, s) = \frac{L(s, (\sigma \otimes \chi) \times \tilde{\sigma})}{L(s, \chi)}$$

is entire and bounded in vertical strips of finite width.

The proof of this Theorem will be lengthy. Thus it might help to explain it classical significance as well as the ideas behind its proof. Suppose

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is a holomorphic cusp form of weight k and character ω. Suppose also that ψ is a primitive Dirichlet character of **Z**,

$$\Sigma a_n n^{-s} = \Pi[(1-\alpha_p p^{-s})(1-\beta_p p^{-s})]^{-1},$$

and $\psi\omega(-1) = 1$. Then in [Sh] Shimura proves that the Euler product

$$\begin{aligned} L_2(s, f, \psi) = \pi^{-3s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-k+2}{2}\right) \\ \times\prod_p[(1-\psi(p)\alpha_p^2 p^{-s})(1-\psi(p)\alpha_p\beta_p p^{-s})(1-\psi(p)\beta_p^2 p^{-s})]^{-1} \end{aligned}$$

is holomorphic everywhere except possibly at $s = k$ or $k-1$.

Now suppose that F = **Q**, σ is the cuspidal representation of GL (2, **A**) generated by f, and χ is the character $\omega\psi$. Then:

$$L_2(s+k-1, f, \psi) = L_2(s, \sigma, \chi) = \frac{L(s, (\sigma \otimes \chi) \times \tilde{\sigma})}{L(s, \chi)}.$$

But in Section 9 we characterize those σ whose corresponding L-functions have no poles. Thus our results refine as well as generalize Shimura's.

The idea behind the proof of Theorem 8.1 is to generalize the Rankin-Selberg method used in [Sh]. More precisely, Shimura exploits the fact that $L_2(s, f, \psi)$ has an integral representation of the form

$$(\star) \qquad L_2(s, f, \psi) = \iint_{\Gamma_0(N)\backslash H} f(z)\bar{\theta}(z)E(z, s)\frac{dx\,dy}{y^2}.$$

Here N is an integer depending on the level of f and ψ,

$$\theta(z)=\sum_{n=-\infty}^{\infty}\psi(n)\,e^{2\pi i n^2 z},$$

and E (z, s) is a certain real analytic Eisenstein series of "half-integral weight". Since both sides of ($\star$) define holomorphic functions of s for Re (s) sufficiently large, the problem of analytically continuing $L_2(s, f, \psi)$ is therefore reduced to the problem of analytically continuing E (z, s).

To generalize Shimura's method we introduce functions on the metaplectic group Mp (**A**) which generalize the functions $f(z)$, $\theta(z)$, and E (z, s) in (*). We denote these functions by $\varphi(g)$, $\theta_\Psi(g)$, and E (g, s) respectively. The functions φ, θ_Ψ and E are invariant by $r_F(G(F))$ and our generalization of the integral in ($\star$) is

$$I(s, \chi, F, \Psi, \varphi)=\int_{r_F(G(F))\backslash Mp(\mathbf{A})}\varphi(g)\,\bar{\theta}_\Psi(g)\,E(g, s)\,dg.$$

As already indicated, our purpose in this Section is to prove that this "main integral" can be expressed as the product of certain local integrals $I_v(s, F_v, \Psi_v, W_v)$.

(5.2) Let MK denote the subgroup of Mp (**A**) which projects to the standard maximal compact subgroup K of G (**A**). Let χ be a character of $F_{\mathbf{A}}^{x}/F^{x}$ and F a continuous function on $\mathbf{C}\times Mp(\mathbf{A})$. We will often write $F_s(g)$ in place of F (s, g). We assume that for each $g \in Mp(\mathbf{A})$, $s \mapsto F_s(g)$ is holomorphic and F_s is MK-finite on the right, uniformly with respect to s; in other words, there is a MK-finite function ξ on MK such that

$$\int F_s(gk)\,\xi(k)\,dk=F_s(g),$$

for all s and all g. Finally we assume that

(5.2.1) $$F_s\left[\lambda\, d\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} t\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} g\right]=\lambda\chi(a)\,|a|^{s+1/2}\,F_s(g),$$

for all $g \in Mp(\mathbf{A})$, $\lambda \in \mathbf{T}$.

The *Eisenstein series* corresponding to F is

(5.2.2) $$E(g, s)=\sum_{B(F)\backslash G(F)} F_s[r_F(\gamma)\,g].$$

As explained above, the function $|F_s|$ may be regarded as a function on G (**A**). As such, it is majorized by a function h on G (**A**) which is K-invariant and such that

$$h\left[\begin{pmatrix} a & x\\ 0 & a^{-1}\end{pmatrix} g\right]=|a|^{Re(s)+1/2}\,h(g).$$

Thus (5.2.2) is dominated by the ordinary Eisenstein series

(5.2.3) $$\sum_{B(F)\backslash G(F)} h(\gamma g),$$

and hence (5.2.2) converges absolutely for $\operatorname{Re} s > (3/2)$, uniformly for $(\operatorname{Re}(s), g)$ in a compact set. Consequently this series defines a continuous function on the set of pairs (s, g) with $\operatorname{Re} s > (3/2)$; the resulting function $s \mapsto E(g, s)$ is holomorphic, and

$$E(r_F(\gamma) g, s) = E(g, s) \quad \text{for} \quad \gamma \in G(F).$$

As on p. 329 of [JL] we may introduce the notion of a slowly increasing (resp. rapidly decreasing) function on $G(F)\backslash G(\mathbf{A})$. If f is a genuine function on $\mathrm{Mp}(\mathbf{A})$ which is invariant under $r_F(G(F))$ on the left, we will say that f is slowly increasing (resp. rapidly decreasing) if $|f|$—regarded as a function on $G(\mathbf{A})$—is slowly increasing (resp. rapidly decreasing). For $\operatorname{Re} s > (3/2)$, it is well known that the series (5.2.3) defines a slowly increasing function on $G(F)\backslash G(\mathbf{A})$. Thus (5.2.2) is, for $\operatorname{Re} s > (3/2)$, a slowly increasing function of g on $r_F(G(F))\backslash \mathrm{Mp}(\mathbf{A})$. In fact this condition is satisfied uniformly for $\operatorname{Re}(s)$ in a compact set.

Now suppose Ψ belongs to the Schwartz-Bruhat space $\mathscr{S}(\mathbf{A})$ and set

$$\theta_\Psi(g) = \sum_{\xi \in F} (g.\Psi)(\xi). \tag{5.2.4}$$

Since $(\lambda g).\Phi = \lambda(g.\Phi)$ for $\lambda \in \mathbf{T}$, this "theta-series" is a genuine function on $\mathrm{Mp}(\mathbf{A})$. By the very definition of r_F (5.1.3), it is also invariant on the left by $r_F(G(F))$. Finally it is slowly increasing (*cf.* [We]). Thus for $\operatorname{Re} s > (3/2)$,

$$g \mapsto \bar{\theta}_\Psi(g) E(g, s)$$

may be regarded as a slowly increasing function on $G(F)\backslash G(\mathbf{A})$.

(5.3) Let σ be an automorphic cuspidal representation of $G_2(\mathbf{A})$ whose central character ω has module one. In particular, σ is unitary. If φ is a (K_2-finite) form belonging to the corresponding space of cusp-forms, then φ is left $G_2(F)$-invariant, rapidly decreasing, and

$$\varphi\left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g\right] = \omega(z)\varphi(g) \quad \text{for} \quad z \in F_{\mathbf{A}}^{\times}, \quad g \in G_2(\mathbf{A}). \tag{5.3.1}$$

The main integral we wish to consider is

$$I(s, \chi, F, \Psi, \varphi) = \int_{G(F)\backslash G(\mathbf{A})} \varphi(h)\bar{\theta}_\Psi(g) E(g, s)\, dg, \quad pr(g) = h. \tag{5.3.2}$$

If $\operatorname{Re} s > (3/2)$, the series (5.2.2) converges absolutely, and the integral (5.3.2) converges absolutely precisely because φ is rapidly decreasing and $\theta_\Psi E$ is slowly increasing. The convergence is even uniform for $\operatorname{Re}(s)$ on a compact set; thus the integral is a holomorphic function of s in the half-plane $\operatorname{Re}(s) > (3/2)$.

Now let S be the finite set of finite places v of F where σ_v is not quasi-unramified in the sense of paragraph (3.5). Let χ be a character of $F_{\mathbf{A}}^{\times}/F^{\times}$ whose ramification at each place v in S is so high that

$$L(s, \chi_v) = 1 \quad \text{and} \quad L(s, (\sigma_v \otimes \chi_v) \times \tilde{\sigma}_v) = L(s, (\tilde{\sigma}_v \otimes \chi_v^{-1}) \times \sigma_v) = 1. \tag{5.3.3}$$

Given σ and χ satisfying the above conditions we shall show in paragraph 6 that

$$\text{(5.3.4)} \qquad I(s, \chi, F, \Psi, \varphi) = L_2(s, \sigma, \chi),$$

for F, Ψ, and φ appropriately chosen, and Re (s) sufficiently large. We shall also show in paragraphs 7 and 8 that E (g, s) extends to an entire function of s if—in addition—χ^2 is ramified at a least one place. Thus it will easily follow that $L_2(s, \sigma, \chi)$ itself is entire when χ satisfies the above conditions.

In order to establish (5.3.4) we shall need to write the main integral as a product of local ones. Although these integrals are taken over the local groups $G_v = SL(2, F_v)$, we still need to introduce the local metaplectic groups before defining them.

(5.4) Let F now be a *local* field. The (local) *metaplectic group* Mp (F_v) is a group of unitary operators on $L^2(F_v)$ which fits into the exact sequence

$$\text{(5.4.1)} \qquad 1 \to \mathbf{T} \to \mathrm{Mp}(F) \xrightarrow{pr_F} G(F) \to 1.$$

Here $\mathbf{T}$ is again the ordinary torus regarded as a group of operators in $L^2(F)$ and the sequence (5.4.1) again splits over B (F), N (F), and A (F). If $g.\Phi$ again denotes the image of Φ in $L^2(F)$ under g in Mp (F), the splitting homomorphisms $t_F : N(F) \to \mathrm{Mp}(F)$ and $d_F : A(F) \to \mathrm{Mp}(F)$, noted also t and d, are defined by

$$\text{(5.4.2)} \qquad \left[t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right].\Phi(y) = \psi_F\left[\frac{1}{2}xy^2\right]\Phi(y)$$

and

$$\text{(5.4.3)} \qquad d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.\Phi(y) = |a|^{1/2}\Phi(ay).$$

Also we define w_0 in Mp (F) and w in G_2 (F) by

$$w_0.\Phi(x) = \hat{\Phi}(-x) = \int \Phi(y)\psi_F(-yx)\,dy, \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that pr $(w_0) = w$.

If F does not have residual characteristic 2, and if the conductor of ψ_F is $\mathfrak{R}_F$, then the sequence (5.4.1) splits over $K = SL(2, R_F)$. The corresponding homomorphism $r_{\mathfrak{R}_F} : K \to \mathrm{Mp}(F)$ is determined by the condition

$$\int_{\mathfrak{R}_F} r_{\mathfrak{R}_F}(g).\Phi(x)\,dx = \int_{\mathfrak{R}_F} \Phi(x)\,dx \qquad \text{for} \quad \Phi \in \mathscr{S}(F);$$

(*cf.* [We], § 19). In fact if Φ^0 denotes the characteristic function of $\mathfrak{R}_F$, then

$$r_{\mathfrak{R}_F}(g)\Phi^0 = \Phi^0 \quad \text{for all } g \in K,$$

and it is easy to check that

$$r_{\mathfrak{R}_F}(w) = w_0.$$

It is also clear that $r_{\mathfrak{R}}$ coincide with t and d on N (F) $\cap$ K and A (F) $\cap$ K respectively.

(5.5) Now let F be an A-field. For each place v of F we write t_v, d_v, pr_v for t_{F_v}, d_{F_v}, pr_{F_v}; similarly if v is finite, does not have characteristic 2, and ψ_v has exponent zero, we write r_v for $r_{\mathfrak{R}_v}$. Then there is a natural injection

(5.5.1) $$i_v:\quad G_v \to G(\mathbf{A})$$

of G_v as a pseudo-factor. Similarly there is a homomorphism

(5.5.2) $$\xi_v:\quad \mathrm{Mp}(F_v) \to \mathrm{Mp}(\mathbf{A})$$

determined by the condition

(5.5.3) $$\xi_v(g_v).\Phi(x) = g_v.\Phi_v(x_v).\prod_{w \neq v} \Phi_w(x_w)$$

for each $g_v \in \mathrm{Mp}(F_v)$ and $\Phi = \prod_w \Phi_w$ in $\mathscr{S}(\mathbf{A})$. The diagram

(5.5.4) $$\begin{array}{ccc} \mathrm{Mp}(\mathbf{A}) & \xrightarrow{pr} & G(\mathbf{A}) \\ \uparrow{\scriptstyle \xi_v} & & \uparrow{\scriptstyle i_v} \\ \mathrm{Mp}(F_v) & \xrightarrow{pr_v} & G_v \end{array}$$

is commutative.

Let $h = (h_v)$ in $G(\mathbf{A})$. For each v choose g_v in $\mathrm{Mp}(F_v)$ so that $\mathrm{pr}_v(g_v) = h_v$. If h_v is in K_v, and r_v is defined, we may take $g_v = r_v(h_v)$; let us agree to take this this choice for almost all v. Then there is now exactly one element g of $\mathrm{Mp}(\mathbf{A})$ such that

(5.5.5) $$g.\Phi(x) = \prod_v g_v.\Phi_v(x_v), \qquad \Phi = \prod_v \Phi_v.$$

Moreover $\mathrm{pr}(g) = h$. We write (g_v) for g [even though $\mathrm{Mp}(\mathbf{A})$ is not a restricted product of the groups $\mathrm{Mp}(F_v)$].

(5.6) We are now ready to return to the main integral (5.3.2). Let us take $\mathrm{Re}(s)$ sufficiently large and compute formally. We can write the integral (5.3.2) as an integral over $\mathbf{T}\, r_F(G(F))\backslash \mathrm{Mp}(\mathbf{A})$. Then we can replace E by its expression (5.2.2) to get

$$\begin{aligned} I(s, \chi, F, \Psi, \varphi) &= \int_{\mathbf{T} r_F(G(F))\backslash Mp(\mathbf{A})} \varphi(\mathrm{pr}(g))\, \Sigma\, F_s(r(\gamma) g)\, \bar{\theta}_\Psi(g)\, dg \\ &= \int_{\mathbf{T} r_F(B(F))\backslash Mp(\mathbf{A})} \varphi(\mathrm{pr}(g))\, F_s(g)\, \bar{\theta}_\Psi(g)\, dg \\ &= \int_{B(F)\backslash G(\mathbf{A})} \varphi(h)\, F_s(g)\, \bar{\theta}_\Psi(g)\, dh, \end{aligned}$$

where g is any element which projects to h. But by Iwasawa's decomposition,

(5.6.1) $$\begin{aligned} I(s, \chi, F, \Psi, \varphi) = \int \varphi &\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k\right] \\ &\times F_s\left[t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} m\right] \\ &\times \bar{\theta}_\Psi\left[t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} m\right] dx\, \frac{d^\times a}{|a|^2}\, dk, \end{aligned}$$

where $m \in \mathrm{Mp}(\mathbf{A})$ projects to k, k is integrated over K, x over $F_{\mathbf{A}}$ and a over $F_{\mathbf{A}}^{\times}$. In view of (5.2.1), this reads:

(5.6.2)
$$\int F_s[m]\chi(a)|a|^{s-3/2}\,d^{\times}a\,dk$$
$$\times\int\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}k\right]\bar{\theta}_{\Psi}\left[t\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}d\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}m\right]dx.$$

Now we recall that the function

$$x\mapsto\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right]$$

has no constant Fourier coefficient, i. e.,

(5.6.3)
$$\int_{\mathbf{A}/\mathbf{F}}\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h\right]dx=0.$$

Moreover, if

(5.6.4)
$$\mathrm{W}(h)=\int\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h\right]\psi(-x)\,dx,$$

then the ξ-th Fourier coefficient of this function is

(5.6.5)
$$\mathrm{W}\left[\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}h\right]=\int\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h\right]\psi(-x\xi)\,dx.$$

Taking (5.1.5) and (5.2.4) into account, we get that—for $h=\mathrm{pr}(g)$—

(5.6.6)
$$\int\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h\right]\bar{\theta}_{\Psi}\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right]dx$$
$$=\int\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h\right]\sum_{\xi\in\mathrm{F}}\bar{\psi}\left(\frac{1}{2}\xi^2x\right)\overline{g.\Psi(\xi)}\,dx$$
$$=\sum_{\xi\in\mathrm{F}}g.\overline{\Psi}(\xi)\int\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}h\right]\bar{\psi}\left(\frac{1}{2}\xi^2x\right)dx$$
$$=\sum_{\xi\in\mathrm{F}^{\times}}g.\overline{\Psi}(\xi)\,\mathrm{W}\left[\begin{pmatrix}\frac{1}{2}\xi^2 & 0\\ 0 & 1\end{pmatrix}h\right]$$
$$=\sum_{\xi\in\mathrm{F}^{\times}}g.\overline{\Psi}(\xi)\,\mathrm{W}\left[\begin{pmatrix}\frac{1}{2}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}h\right].$$

Thus (5.6.2) gives

$$(5.6.7)\qquad I(s,\chi,F,\Psi,\varphi)=\int_K F_s(m)\,dk\int_{F_{\mathbf{A}}^\times/F^\times}\chi(a)\,|a|^{s-3/2}\,d^\times a$$

$$\times\sum_{\xi\in F^\times} d\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} m.\overline{\Psi}(\xi)\,W\left[\begin{pmatrix}\frac{1}{2}\xi a & 0\\ 0 & \xi^{-1}a^{-1}\end{pmatrix}k\right]$$

$$=\int_K F_s(m)\,dk\int_{F_{\mathbf{A}}^\times/F^\times}\chi(a)\,|a|^{s-1}\,d^\times a\sum_{\xi\in F^\times}\overline{m.\Psi(a\xi)}\,W\left[\begin{pmatrix}\frac{1}{2}\xi a & 0\\ 0 & \xi^{-1}a^{-1}\end{pmatrix}k\right]$$

$$(5.6.8)\qquad =\int_{K\times F_{\mathbf{A}}^\times} F_s(m)\,\overline{m.\Psi(a)}\,W\left[\begin{pmatrix}\frac{1}{2}a & 0\\ 0 & a^{-1}\end{pmatrix}k\right]\chi(a)\,|a|^{s-1}\,dk\,d^\times a.$$

Finally, let ω be the central character of σ so that W transforms according to ω. Then:

$$(5.6.9)\quad I=\int_{K\times F_{\mathbf{A}}^\times} F_s(m)\,\overline{m.\Psi(a)}\,W\left[\begin{pmatrix}\frac{1}{2}a^2 & 0\\ 0 & 1\end{pmatrix}k\right]\chi\omega^{-1}(a)\,|a|^{s-1}\,d^\times a\,dk,\ \operatorname{pr}(m)=k.$$

Before justifying our steps, let us specify F_s. For each place v we select a function $F_{v,s}(g_v)$ on $Mp(F_v)\times\mathbf{C}$ such that

$$(5.6.10)\qquad F_{v,s}\left[t_v\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}d_v\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}g_v\right]=\chi_v(a)\,|a|_v^{s+1/2}\,F_{v,s}(g_v).$$

For each g_v in $Mp(F_v)$ the ratio

$$F_{v,s}(g_v)/L(2s,\chi^2)$$

will be an entire function of s, in fact a polynomial in q_v^{-s}, q_v^{s} if v is finite (with module q_v) and a sum of functions $as^i e^{bs}$ if v is infinite. Moreover, let K_v be the standard maximal compact subgroup of G_v and MK_v its inverse image in $M_p(F_v)$. Then for each s,

$$k_v\mapsto F_{v,s}(k_v)/L(2s,\chi^2)$$

belongs to a fixed finite dimensional right-invariant subspace of continuous functions on MK_v.

Finally, for almost all v, we select

$$(5.6.11)\qquad F_{v,s}(r_v(k_v))=L(2s,\chi^2)\qquad\text{if}\quad k_v\in K_v.$$

Then if $g=(g_v)$ as in (5.5.5), we set

$$(5.6.12)\qquad F_s(g)=\prod_v F_s(g_v).$$

Since almost all these factors are equal to $L(2s, \chi_v^2)$, the product in (5.6.12) converges absolutely for Res large enough. In fact it continues to an holomorphic function of s since χ^2 (by assumption) is not principal (it is ramified at some place v). We observe also that

$$F_s(k)/L(2s, \chi^2)$$

is for any $k \in MK$ a sum of functions of the form $as^i e^{bs}$ and for a fixed s belongs to a finite dimensional space of right-invariant functions on MK, independent of s.

Let us recall also that W belongs to the space $\mathscr{W}(\sigma; \psi)$ spanned by functions of the form

$$g \mapsto \prod_v W_v(g_v)$$

where W_v is in $\mathscr{W}(\sigma_v; \psi_v)$ for all v, and for almost all v is the unique element of $\mathscr{W}(\sigma_v; \psi_v)$ invariant under (and equal to one on) $K_{2,v} = GL(2, \mathscr{R}_v)$.

This being so, for $F_{v,s}$ as above, Ψ_v in $\mathscr{S}(F_v)$, and W_v in $\mathscr{W}(\sigma_v; \psi_v)$, we introduce the local integral

(5.6.13)
$$\begin{aligned} I_v &= I(s, F_v, \Psi_v, W_v) \\ &= \int_{K_v \times F_v^\times} |a_v|^{s-1} \chi_v \omega_v^{-1}(a_v) F_{v,s}(m_v) \\ &\quad \times \overline{m_v . \Psi_v(a_v)}\, W_v\left[\begin{pmatrix} \frac{1}{2}a_v^2 & 0 \\ 0 & 1 \end{pmatrix} k_v\right] dk_v\, d^\times a_v, \\ &\qquad \mathrm{pr}_v(m_v) = k_v. \end{aligned}$$

Then we have:

PROPOSITION (5.6.14). — *Suppose* $\Psi = \Pi \Psi_v$, $W = \Pi W_v$, *and* $F_s = \Pi F_{v,s}$ *as above. Then for* Re (s) *large enough, each one of the integrals* (5.6.13) *converges absolutely, their infinite product converges absolutely, and*

$$I(s, F, \Psi, \varphi) = \prod_v I(s, F_v, \Psi_v, W_v).$$

Proof. — We can choose the element m in Mp (A) which projects to $k = (k_v)$ as in (5.5). Then $m = (m_v)$ and $\mathrm{pr}_v(m_v) = k_v$. It is then formally clear that (5.6.9) is equal to the product of the integrals (5.6.13). We need however to justify all our steps.

Fix s in a compact set with Re s large. For each place v there is a number c_v and a function $\Psi_v^1 > 0$ in $\mathscr{S}(F_v)$ so that

$$|F_{v,s}(m)(m . \overline{\Psi^1})(a)| \leqq c_v |L(2s, \chi_v^2)| \Psi_v^1(a).$$

There is also a function $\Psi_v^2 > 0$ and $t > 0$ independent of v such that

$$\left| W_v\left[\begin{pmatrix} a_v b_v & 0 \\ 0 & b_v \end{pmatrix} k_v\right]\right| \leqq |a_v|^{-t} \Psi_v^2(a_v).$$

(*Cf.* [JPSS], § (2.3) and (8.3.3).) Furthermore, we may assume $c_v = 1$ s real, $\chi_v^2 = 1$, and $\Psi_2^1 = \Psi_v^2 =$ characteristic function of $\mathfrak{R}_v$ for almost all v. Then the integral (5.6.13) is dominated by

$$c_v \, L(2s, \chi_v^2) \int_{K_v \times F_v^\times} |a_v|^{s-1-t} \Psi_v^1(a) \Psi_v^2\left(\frac{1}{2} a_v^2\right) d^\times a_v \, dk_v.$$

This shows that when Re s is large enough, the integrals (5.6.13) converge absolutely. Similarly the integral on the right hand side of (5.6.9) is dominated by

$$\Pi c_v L(2s, \chi^2) \int_{K \times F_{\mathbb{A}}^\times} |a_v|^{s-1-t} \Psi_v^1(a) \Psi_v^2\left(\frac{1}{2} a^2\right) d^\times a \, dk$$

with $\Psi^1 = \Pi \Psi_v^1$, $\Psi^2 = \Psi_v^2$. Thus it converges absolutely and is equal to the product of the local integrals for Re s large enough.

Now we have to prove identity (5.6.9), for Re s large enough. Certainly we may assume

$$|m.\Psi|(x) \leqq \Psi^1(x)$$

for all $m \in$ MK. Then:

$$\left|\theta_\Psi\left[t\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} d\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} m\right]\right| \leqq |a|^{1/2} \sum_{\xi \in F} \Psi^1(a\xi).$$

Going back over our computations we see that all our steps will be justified once we show that

$$\int_{A/F \times F_{\mathbb{A}}^\times \times K} |\varphi| \left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} k\right] \sum_{\xi \in F} \Psi^1(a\xi) |a|^{s-1/2} \, dx \, d^\times a \, dk$$

is finite for s real and large enough. But there is a $c > 0$ such that

$$\sum_{\xi \in F} \Psi^1(a\xi) \leqq c |a|^{-1}.$$

Thus it suffices to show that

$$\int |\varphi| \left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} k\right] |a|^{s-3/2} \, dx \, d^\times a < +\infty$$

for s real and large. Since φ is bounded and for any r

$$\varphi\left[\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} k\right] = 0(|a|^{-r})$$

for $|a|$ large, this is clear.

Q.E.D.

6. Local computations for the main integral: identification with $L_2(s, \sigma, \chi)$

In this Section we prove that

$$(\star)\qquad I(s, \chi, F, \Psi, \varphi) = \frac{L(s, (\sigma \otimes \chi) \times \tilde{\sigma})}{L(s, \chi)} (= L_2(s, \sigma, \chi))$$

for Re s sufficiently large, χ "highly ramified", and F_s, Ψ and φ appropriately chosen. But both sides of $(\star)$ decompose as products over the places of F (the left side by Propositions (5.6.14), the right side by definition). Therefore $(\star)$ is actually a local assertion.

(6.1) Throughout this section, F will denote a local field of characteristic not two and ψ_F or ψ will denote a fixed non-trivial character of F. If $F = \mathbf{R}$ we take $\psi(x) = \exp(2i\pi x)$ and if $F = \mathbf{C}$ we take $\psi(z) = \exp[2i\pi(z+\bar{z})]$. By σ we denote a fixed irreducible unitary generic representation of $G_2(F)$, ω its central character, and χ a character of $F^\times$.

If F is non-archimedean and σ is not quasi-unramified we assume that the ramification of χ is so high that

$$(6.1.1)\qquad L(s, \chi) = L(s, \chi^{-1}) = 1 \quad \text{and} \quad L_2(s, \sigma, \chi) = L_2(s, \sigma, \chi^{-1}) = 1.$$

We will consider functions $F(s, g) = F_s(g)$ of $s \in \mathbf{C}$ and $g \in \mathrm{Mp}(F)$ satisfying the following conditions:

$$(6.1.2)\qquad F_s\left[d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \chi(a)|a|^{s+1/2} F_s[g]$$

$$\text{for all } a \in F^\times, \quad x \in F, \quad g \in \mathrm{Mp}(F);$$

(6.1.3) there is an entire function H(s) with values in a fixed, finite dimensional right invariant space of continuous functions on MK such that

$$F_s(k) = L(2s, \chi^2) H(s)(k) \quad \text{for all } k \in \mathrm{MK}.$$

If $F = \mathbf{R}$ or $\mathbf{C}$, H(s) is a polynomial in s times an exponential factor. If F is non-archimedean H(s) is a polynomial in q^{-s}, q^s. In fact in all cases but the complex case F_s will have the form

$$(6.1.4)\qquad F_s(g) = f_s(\mathrm{pr}(g))(g.\Phi)(0),$$

where Φ is in $\mathscr{S}$(F) (and MK finite) and f_s is a function of $s \in \mathbf{C}$ and $g \in G(F)$ such that

$$(6.1.5)\qquad f_s\left[\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} g\right] = |a|^s \chi(a) f_s(g)$$

and a condition analogous to (6.1.3) holds with MK replaced by K.

For F_s as above, W in $\mathscr{W}(\sigma; \psi)$, and Ψ in $\mathscr{S}$(F), we set

$$(6.1.6)\qquad I = I(s, F, \Psi, W) = \int_{N(F)\backslash G(F)} F_s(h)\overline{(h.\Psi)}(1) W\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} g\right] dg,$$

where pr $(h) = g$.

When F has the form (6.1.4) we also write

$$I = I(s, f, \Phi, \Psi, W).$$

By Iwasawa's decomposition, we have then:

(6.1.7)
$$I = \int_{F^\times \times K} |a|^{s-1} \chi\omega^{-1}(a) f_s(k)(m.\Phi)(0) \times m.\overline{\Psi}(a) W\left[\begin{pmatrix} \frac{1}{2}a^2 & 0 \\ 0 & 1 \end{pmatrix} k\right] dk\, d^\times a, \quad \mathrm{pr}(m) = k.$$

Now we have already observed in (5.6) that such an integral converges for Re (s) sufficiently large. Our purpose in this section is to prove:

(6.2) PROPOSITION. — *For* Re (s) *sufficiently large, and appropriately chosen* F_s *(or* f_s, Φ *if* $F \neq \mathbf{C}$), Ψ, *and* W, *the integral* (6.1.6) *equals* $L_2(s, \sigma, \chi)$ *times an exponential factor.*

The exact form of F_s (or f_s and Φ) will be needed for later reference. Moreover, a more precise result will be needed in the "unramified situation" (see below). In the meantime we make two useful remarks.

(6.2.1) *Remark.* — Replace σ by $\sigma \otimes \eta$. Then the L_2-factor does not change. The space $\mathscr{W}(\sigma; \psi)$, however, does change in general; it is replaced by the space spanned by the functions

$$W \otimes \eta : \quad g \mapsto W(g)\eta(\det g), \qquad W \in \mathscr{W}(\sigma; \psi).$$

However, the functions W enters the integral only through their restrictions to the set

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} G(F).$$

It follows that in proving (6.2) we may, without loss of generality, replace σ by $\sigma \otimes \eta$.

(6.2.2) *Remark.* — Replace ψ by ψ' where $\psi'(x) = \psi(ax)$ and $a \neq 0$. Then the group of unitary operators Mp (F) does not change. The projection pr, however, is replaced by another map pr′, and there is a commutative diagram

(6.2.3)
$$\begin{array}{ccc} \mathrm{Mp}(F) & \xrightarrow{pr'} & G(F) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow \\ \mathrm{Mp}(F) & \xrightarrow{pr} & G(F) \end{array}$$

where the second vertical arrow is the restriction to G (F) of the inner automorphism

(6.2.4)
$$g \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

of G_2 (F). Similarly there is a commutative diagram

(6.2.5)
$$\begin{array}{ccc} N(F) & \xrightarrow{t} & Mp(F) \\ \downarrow & & \downarrow id \\ N(F) & \xrightarrow{t'} & Mp(F) \end{array}$$

where t (resp. t') is the splitting morphism defined by (5.4.3) and $\psi_F = \psi$ (resp. $\psi_F = \psi'$), and the first vertical arrow is induced by (6.2.4). Note that (6.2.4) induces the identity on A (F) and d does not depend on ψ. Moreover the inner automorphism (6.2.4) leaves σ invariant and takes the function W in $\mathscr{W}(\pi; \psi)$ to the function

$$g \mapsto W\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right]$$

which is in $\mathscr{W}(\pi; \psi')$. Thus it follows that in proving (6.2) we may replace ψ by ψ'.

Our proof will now proceed case by case.

(6.3) F *is non-archimedean,* σ *is quasi-unramified,* χ *is unramified.*

By remark (6.2.1) we may assume that σ is actually unramified. Suppose for the time being the residual characteristic of F is not two. By remark (6.2.2) we may also assume that the conductor of ψ is $\mathfrak{R}_F$; the situation at hand is then the "unramified one". Let W_0 be the element of $\mathscr{W}(\sigma, \psi)$ which is one on K_2 and invariant under K_2. Take $\Phi = \Psi$ to be the characteristic function Φ_0 of $\mathfrak{R}_F$. Finally define f by

(6.3.1)
$$f_s\left[\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} k\right] = |a|^s \chi(a) L(2s, \chi^2)$$

LEMMA (6.3.2) ("unramified situation"). — *Assume the Haar measures of* $F^\times$ *and* K *are so chosen that* Vol $(\mathfrak{R}^\times) = 1$, Vol (K) $= 1$. *Then* f_s *being given by* (6.3.1):

$$I(s, f, \Phi_0, \Phi_0, W_0) = L_2(s, \sigma, \chi).$$

Proof. — The extension splits over K and Φ_0 is r_F (K)-invariant. On the other hand f_s and W_0 are K-invariant. So from (6.1.7) we get

$$I = L(2s, \chi^2) \int |a|^{s-1} \chi\omega^{-1}(a) W_0 \begin{pmatrix} \frac{1}{2} a^2 & 0 \\ 0 & 1 \end{pmatrix} d^\times a.$$

Now

$$W\begin{pmatrix} ab & 0 \\ 0 & 1 \end{pmatrix} = W\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad a \in \mathfrak{R}^\times.$$

So I is the product of L $(2s, \chi^2)$ and the series

(6.3.3)
$$\sum_{n \geq -\infty} X^n q^n \chi\omega^{-1}(\tilde{\omega}^n) W_0 \begin{pmatrix} \tilde{\omega}^{2n} & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = q^{-s},$$

and what remains to be shown is that the sum of this series is $L_2(s, \sigma, \chi) . L(2s, \chi^2)^{-1}$.

Since σ is unramified, we have $\sigma = \pi(\mu, \nu)$ for some choice of quasi-characters μ, ν of $F^\times$. Then, by Proposition (1.4),

$$L_2(s, \sigma, \chi) = (1-\mu\nu^{-1}\chi(\tilde{\omega})q^{-s})^{-1}(1-\chi(\tilde{\omega})q^{-s})^{-1}(1-\mu^{-1}\nu\chi(\tilde{\omega})q^{-s})^{-1}.$$

On the other hand ([JL]), Prop. (9.5)):

$$\int_{F^\times} W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a = \Sigma X^n q^{n/2} W\begin{pmatrix} \tilde{\omega}^n & 0 \\ 0 & 1 \end{pmatrix} = L(s, \sigma) = L(s, \mu) L(s, \nu),$$

while

$$L(2s, \chi^2) = (1+\chi(\tilde{\omega})q^{-s})^{-1}(1-\chi(\tilde{\omega})q^{-s})^{-1}.$$

Thus our assertion follows from the following Lemma, whose proof is left to the reader:

LEMMA (6.3.4). — *Let* X *denote an indeterminate. If*

$$\sum_{n \geq 0} c_n X^n = (1-\alpha X)^{-1}(1-\beta X)^{-1}$$

then:

$$\sum_{n \geq 0} c_{2n} X^n = (1+\alpha\beta X)(1-\alpha^2 X)^{-1}(1-\beta^2 X)^{-1}.$$

Suppose now F has residual characteristic 2. Let ψ_0 be a character whose conductor is $\mathfrak{R}$. We may assume $\psi(x) = \psi_0(2x)$ [Remark (6.2.2)]. Let f_s, Φ_0, W_0 be as before.

LEMMA (6.3.5). — *There is* $c > 0$ *so that:*

$$\int_K (m.\Phi_0)(0)\overline{m.\Phi_0(a)}\,dk = c\,\Phi_0(a), \qquad (\mathrm{pr}(m) = k).$$

Proof. — In the integral above let us change m into

$$t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.m$$

with $x \in \mathfrak{R}$ and let us integrate over $\mathfrak{R}$. We find that the integral gets multiplied by

$$\int_{\mathfrak{R}} \psi_0(xa^2)\,dx.$$

Thus the integral vanishes unless a is in $\mathfrak{R}$. For $a = 0$, it has some positive value c. So it will suffice to show that the integral does not depend on a, provided a is in $\mathfrak{R}$.

So assume a is in $\mathfrak{R}$. Then the integrand is invariant under $t(N \cap K)$ on the left. On the other hand, Φ_0 is invariant under $t(N \cap K)$ and $d(A \cap K)$, so the integrand too is invariant on the right for these groups. Thus there are two positive constants c_1, c_2 so that the integral is:

$$(6.3.6) \qquad c_1(w_0.\Phi_0)(0)\overline{w_0.\Phi_0(a)} + c_2\int_{\mathfrak{P}}\left(w_0 t\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} w_0^{-1}\right).\Phi(0)$$
$$\times \overline{\left(w_0 t\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} w_0^{-1}\right).\Phi(a)}\,dz \qquad \text{where} \quad \mathrm{pr}(w_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We also have

$$w_0 . \Phi(x) = \hat{\Phi}(-x).$$

Thus:

$$w_0 . \Phi_0(x) = \Phi_0(2x)|2|^{1/2}$$

and the first term in (6.3.6) is indeed independent of a. We also have

$$\left(t\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}w_0^{-1}\right).\Phi_0(x) = \Phi_0(2x)\psi_0(x^2 z)|2|^{1/2}.$$

So the second term can be written as

$$c_2|4|\int_{\mathfrak{P}} dz \int \Phi_0(2x_1)\psi_0(x_1^2 z)\,dx_1 \int \Phi_0(2x_2)\psi_0(x_2^2 z - 2x_2 a)\,dx_2.$$

In this integral $x_2 \in 1/2\,\mathfrak{R}$. So $2x_2 a$ is in $\mathfrak{R}$ and the integral does not depend on a.

The Lemma being proved, set

$$W(g) = W_0\left[\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}g\right].$$

Then W is in $\mathscr{W}(\sigma;\psi)$ and invariant under K. From (6.1.7) and (6.3.5) we seet hat the integral $I(s, f, \Phi_0, \Phi_0, W)$ is thus equal to (6.3.3) and our assertion is proved in the case at hand just as in the previous case.

(6.4) F *is nonarchimedean,* σ *is not quasi-unramified, or* χ *is ramified.*

If σ is quasi-unramified and χ ramified then

$$L(s, (\sigma\otimes\chi)\times\tilde{\sigma}) = L_2(s, \sigma, \chi) = L(s, \chi) = 1. \tag{6.4.1}$$

On the other hand recall that if σ is not quasi-unramified then we assume that χ is so highly ramified that (6.4.1) holds. Thus Proposition (6.2) amounts to the assertion that

$$I(s, f, \Phi, \Psi, W) = 1 \tag{6.4.2}$$

for appropriately chosen f, Φ, Ψ, and W.

To prove this we introduce the function

$$H(k, s) = (m.\Phi)(0)\int_{F^\times} |a|^{s-1}\chi\omega^{-1}(a)\overline{(m.\Psi)}(a) \times W\left[\begin{pmatrix}\frac{1}{2}a^2 & 0\\ 0 & 1\end{pmatrix}k\right]d^\times a \quad \text{where} \quad \mathrm{pr}(m) = k. \tag{6.4.3}$$

Then:

$$H\left[\begin{pmatrix}\varepsilon & \eta\\ 0 & \varepsilon^{-1}\end{pmatrix}k, s\right] = \chi^{-1}(\varepsilon)H(k, s), \quad \text{for } \varepsilon\in\mathfrak{R}_F^\times \text{ and } \eta\in\mathfrak{R}_F \tag{6.4.4}$$

and

$$(6.4.5)\qquad I(s, f, \Phi, \Psi, W) = \int_{B\cap K\backslash K} H(k, s) f_s(k)\, dk.$$

Choose n so large that χ identically 1 on $1+\mathfrak{P}_F^n$ and $H(k, s)$ is right invariant by the congruence subgroup

$$(6.4.6)\qquad K' = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid a \equiv d \equiv 1,\ b \equiv c \equiv 0 \bmod \mathfrak{P}^n\right\}.$$

Let also

$$K'' = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid c \equiv 0 \bmod \mathfrak{P}^n\right\}.$$

Then $K'' = (B \cap K).K'$. Define

$$f_s(k) = \begin{cases} \chi(a) & \text{if } k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K'', \\ 0 & \text{otherwise.} \end{cases}$$

For this choice of f_s, it follows from (6.4.4) that

$$I(s, f, \Phi, \Psi, W) = \int_{B\cap K\backslash K''} H(k, s) f_s(k)\, dk = c\, H(e, s), \qquad c > 0.$$

Thus it remains to prove we can choose Φ, Ψ, and W so that $H(e, s)$ is a non-zero constant, So choose $\Psi(a)$ equal to $\chi\omega^{-1}(a)$ if $a \in \mathfrak{R}^\times$ and zero otherwise. We assume $\Phi(0) \neq 0$ and we choose W in such a way that

$$a \mapsto W\begin{pmatrix} \frac{1}{2}a & 0 \\ 0 & 1 \end{pmatrix}$$

is the characteristic function of $\mathfrak{R}^\times$ in $F^\times$ (*cf.* [JL], Lemma (2.16.1)). Then:

$$H(e, s) = \Phi(0) \int_{F^\times} |a|^{s-1} \chi\omega^{-1}(a) \overline{\Psi}(a)\, W\begin{pmatrix} \frac{1}{2}a^2 & 0 \\ 0 & 1 \end{pmatrix} d^\times a = \mathrm{Vol}(\mathfrak{R}^\times)\, \Phi(0),$$

and we are done.

(6.5) *Real case.*

If $F \neq \mathbf{C}$, Mp (F) reduces to a non-trivial extension of $G(F) = SL(2, F)$ by $\{1, -1\}$. We denote the corresponding two sheeted covering of G (F) by $Sp_2(F)$. When $F = \mathbf{R}$, $Sp_2(\mathbf{R})$ is therefore the unique two-sheeted covering of $G(\mathbf{R})$ given by the theory of Lie-groups. Its Lie-algebra is the same as the Lie-algebra of $G(\mathbf{R})$: it consists of all real two by two matrices of trace zero. Recall $\psi(x) = \exp(2 i \pi x)$.

For our purposes it is convenient to single out the one parameter subgroups of $Sp_2(\mathbf{R})$ corresponding to the Lie algebra elements

(6.5.1) $$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are the subgroups

(6.5.2) $$N_* = \left\{ t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbf{R}, \right\} \quad A_* = \left\{ d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a > 0 \right\}.$$

(*Cf.* (5.4.2) and (5.4.3).) Recall also that $K = SO(2, \mathbf{R})$ [resp. $MK \cap Sp_2(\mathbf{R})$] is the subgroup of $G(\mathbf{R})$ [resp. $Sp_2(\mathbf{R})$] corresponding to the Lie algebra element

(6.5.3) $$U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

As defined, the group $Sp_2(\mathbf{R})$ is provided with a unitary representation on $L^2(\mathbf{R})$; the elements of $\mathscr{S}(\mathbf{R})$ are C^∞-vectors and the corresponding action of the Lie-algebra on them is given by

(6.5.4) $$X_+ . \Phi(y) = i\pi y^2 \Phi(y),$$

(6.5.5) $$H . \Phi(y) = y\Phi'(y) + \frac{1}{2}\Phi(y),$$

and

(6.5.6) $$U . \Phi(y) = i\pi y^2 \Phi(y) - i\frac{1}{4\pi}\frac{d^2}{dy^2}\Phi(y).$$

Suppose now Φ is an eigenvector for U, that is,

(6.5.7) $$U . \Phi = in . \Phi$$

or, as we shall say, Φ is of weight n. Then Φ satisfies the second order differential equation

$$\frac{d^2\Phi}{dy^2}(y) = (4\pi^2 y^2 - 4n\pi)\Phi(y).$$

It follows that an orthogonal basis of eigenvectors for U is provided by the classical Hermite functions

(6.5.8) $$\Phi_m(x) = e^{-\pi x^2} H_m(x\sqrt{2\pi}),$$

where H_m denotes the classical Hermite polynomial of degree m. The corresponding weight is

(6.5.9) $$\frac{n}{2} = m + 1/2.$$

Thus if $\Phi = \Phi_m$ we find that

$$\exp(t\,U).\Phi = \exp\left(i\frac{n}{2}t\right)\Phi.$$

Similarly if φ is a C^∞-function on $G(\mathbf{R})$ or $G_2(\mathbf{R})$ let us say it is of weight n if

$$\frac{d}{dt}\varphi[g\exp(t\,U)]\Big|_{t=0} = in\,\varphi(g). \tag{6.5.10}$$

In proving (6.2) we may—without loss of generality—assume $\chi = 1$ or $\chi = \text{sgn}$. Indeed χ can be replaced by $\chi\alpha^t$ by changing s to $s-t$. We may also replace σ by $\sigma \otimes \mu$ and assume (whenever convenient) that the central character ω of σ is either 1 or sgn. Thus the cases we have to examine are the following:

TABLE (6.5.11)

Case	σ	χ	$L_2(s, \sigma, \chi)$
a	$\pi(\mu_1, \mu_2)$ $\mu_2 = \alpha^{\sigma_1}$, $\mu_2 = \alpha^{\sigma_2}$	1	$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+\sigma_1-\sigma_2}{2}\right)\Gamma\left(\frac{s+\sigma_2-\sigma_1}{2}\right)$
b	$\pi(\mu_1, \mu_2)$ $\mu_1 = \alpha^{\sigma_1}$, $\mu_2 = \alpha^{\sigma_2}$	sgn	$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+1+\sigma_1-\sigma_2}{2}\right)\Gamma\left(\frac{s+1+\sigma_2-\sigma_1}{2}\right)$
c	$\pi(\mu_1, \mu_2)$ $\mu_1 = \alpha^{\sigma_1}\,\text{sgn}$ $\mu_2 = \alpha^{\sigma_2}$	1	$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1+\sigma_1-\sigma_2}{2}\right)\Gamma\left(\frac{s+1+\sigma_2-\sigma_1}{2}\right)$
d	$\pi(\mu_1, \mu_2)$ $\mu_1 = \alpha^{\sigma_1}\,\text{sgn}$ $\mu_2 = \alpha^{\sigma_2}$	sgn	$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+\sigma_1-\sigma_2}{2}\right)\Gamma\left(\frac{s+\sigma_2-\sigma_1}{2}\right)$
e	σ_Ω $\Omega(z) = \dfrac{z^m}{(z\bar{z})^{m/2}}$	1	$\Gamma\left(\frac{s+1}{2}\right)\Gamma(s+m)$
f	σ_Ω $\Omega(z) = \dfrac{z^m}{(z\bar{z}^{\,m/2})}$	sgn	$\Gamma\left(\frac{s}{2}\right)\Gamma(s+m)$

In this Table, the appropriate values of $L(s, \sigma, \chi)$ are given within an exponential factor (*cf.* § 3 and [Ja], § 17). The function Φ (resp. Ψ) will be one of the functions (6.5.8) and so will be determined by its weight. Similarly f_s will be given by

$$f_s\left[\begin{pmatrix} a & x \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\right] = \chi(a)\,|\,a\,|^s e^{in\theta}\,\mathrm{L}(2s,\chi^2)\,\mathrm{P}(s), \tag{6.5.12}$$

where P is a polynomial; so f_s will be determined by its weight and the polynomial P. Finally, since the restriction of σ to $\mathrm{K} = \mathrm{SO}(2, \mathbf{R})$ decomposes with multiplicity one, W will also—up to a scalar factor—be determined by its weight. Thus the appropriate choice of f, Φ, Ψ, W, and P is given by the following Table:

TABLE (6.5.13)

Case	wgt (f_s)	P	wgt (Φ)	wgr (Ψ)	wgt (W)
a.	0	1	1/2	1/2	0
b.	-1	s	5/2	3/2	0
c.	0	1	1/2	3/2	1
d.	-1	1	1/2	1/2	1
e.	0	1	1/2	$m+3/2$	$m+1$
f.	1	1	1/2	$m+1/2$	$m+1$

In each case we have

$$\mathrm{wgt}(f_s)+\mathrm{wgt}(\Phi)-\mathrm{wgt}(\Psi)+\mathrm{wgt}(\mathrm{W}) = 0.$$

Thus formula (6.1.7) reduces to

$$\mathrm{I}(s, f, \Phi, \Psi, \mathrm{W}) = \mathrm{I} = \Phi(0)\,\mathrm{L}(2s,\chi^2)\,\mathrm{P}(s) \tag{6.5.14}$$

$$\times\int_0^\infty |\,a\,|^{s-1}\chi\omega^{-1}(a)\,\overline{\Psi}(a)\,\mathrm{W}\begin{pmatrix} \frac{1}{2}a^2 & 0 \\ 0 & 1\end{pmatrix} d^\times a.$$

Since $\Phi(0) \neq 0$, up to an exponential factor this is

$$\mathrm{L}(2s,\chi^2)\,\mathrm{P}(s)\int_0^\infty |\,a\,|^{(1/2)(s-1)}\chi\omega^{-1}(\sqrt{a})\,\overline{\Psi}(\sqrt{2a})\,\mathrm{W}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} d^\times a. \tag{6.5.15}$$

We treat this integral case by case in (6.6) and (6.7).

(6.6) *Cases a to d.*

We need to compute the Mellin transform of the functions involved in (6.5.15). For $a \mapsto \overline{\Psi}(\sqrt{2a})$ this is easy enough. Indeed, up to scalar factors,

$$\overline{\Psi}(\sqrt{2x}) = \begin{cases} \exp(-2\pi x) & \text{cases } a \text{ and } d, \\ x\exp(-2\pi x) & \text{cases } b \text{ and } c. \end{cases} \tag{6.6.1}$$

Thus:

$$(6.6.2)\qquad \int_0^{+\infty} \overline{\Psi}(\sqrt{2a})\,|a|^{s-1/2}\,d^\times a \begin{cases} \pi^{-s}\Gamma\left(\dfrac{s+(1/2)}{2}\right)\Gamma\left(\dfrac{s-(1/2)}{2}\right) & \text{cases } a,\ d, \\ \pi^{-s-1}\Gamma\left(\dfrac{s+(3/2)}{2}\right)\Gamma\left(\dfrac{s+(1/2)}{2}\right) & \text{cases } b \text{ and } c. \end{cases}$$

To compute the other Mellin transform, recall that for any infinite dimensional representation σ of the form $\pi(\mu_1, \mu_2)$, the space $\mathscr{W}(\sigma;\psi)$ can be described in the following way: for each Φ in $\mathscr{S}(F^2)$ of the form $\exp(-\pi(x^2+y^2))\,P(x, y)$ — where P is a polynomial — set

$$(6.6.3)\qquad f(g) = f_\Phi(g) = \int \Phi[(0, t)g]\,\mu_1\mu_2^{-1}(t)\,|t|\,d^\times t\,\mu_1(\det g)\,|\det g|^{1/2}.$$

Then the elements of $\mathscr{W}(\sigma;\psi)$ are exactly the functions of the form

$$(6.6.4)\qquad W(g) = \int f\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right]\psi(-x)\,dx.$$

The integral may be divergent but may be given a meaning as in [JL] (§ 5). In particular

$$(6.6.5)\qquad W\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} = \int \Phi'[tx, x^{-1}]\,\mu_1.\mu_2^{-1}(x)\,\mu_1(t)\,|t|^{1/2}\,d^\times x,$$

where

$$(6.6.6)\qquad \Phi'(x, y) = \int \Phi(x, v)\,\psi[-yv]\,dv$$

(in the notation of [JL], $W = W_{\Phi'}$).

Thus:

$$(6.6.7)\qquad \int_0^{+\infty} W\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}|t|^{s-1/2}\,d^\times t$$

$$= \frac{1}{2}\left[\int_{-\infty}^{+\infty} W\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}|t|^{s-1/2}\,d^\times t + \int_{-\infty}^{+\infty} W\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}|t|^{s-1/2}\frac{t}{|t|}\,d^\times t\right]$$

$$= \frac{1}{2}\iint \Phi'(x, y)\,\mu_1(x)\,|x|^s\,\mu_2(y)\,|y|^s\,s^\times x\,d^\times y$$

$$+ \frac{1}{2}\iint \Phi'(x, y)\,\mu_1(x)\,|x|^s\frac{x}{|x|}\times\mu_2(y)\,|y|^s\frac{y}{|y|}\,d^\times x\,d^\times y.$$

To get the right weight we take

$$\Phi(x, y) = \begin{cases} \exp(-\pi(x^2+y^2)) & \text{in cases } a,\ b, \\ \exp(-\pi(x^2+y^2))(x+iy) & \text{in cases } c,\ d. \end{cases}$$

Then:

$$\Phi'(x, y) = \begin{cases} \exp(-\pi(x^2+y^2)) & \text{in cases } a, b, \\ \exp(-\pi(x^2+y^2))(x+y) & \text{in cases } c, d \end{cases}$$

and, up to scalar factors

$$\int_0^{+\infty} W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a = \pi^{-(s+\sigma_1)/2}\Gamma\left(\frac{s+\sigma_1}{2}\right)\pi^{-(s+\sigma_2)/2}\Gamma\left(\frac{s+\sigma_2}{2}\right)$$

in cases a, b

$$= \pi^{-(s+\sigma_1+1)/2}\Gamma\left(\frac{s+\sigma_1+1}{2}\right)\Gamma\left(\frac{s+\sigma_2}{2}\right)$$

$$+\pi^{-(s+\sigma_2+1)/2}\Gamma\left(\frac{s+\sigma_1}{2}\right)\Gamma\left(\frac{s+\sigma_2+1}{\sigma}\right)$$

in cases c, d.

To continue we use a Barnes Mellin Lemma as formulated on page 77 of [Ja]. The result then follows immediately.

(6.7) *Cases e and f.*

This is a new situation, since σ belongs to the "discrete series". More precisely, it is the representation in the discrete series with central character $(\text{sgn})^{m+1}$ and lowest positive weight $m+1$. In particular $\mathscr{W}(\sigma; \psi)$ contains—up to a scalar factor—exactly one element W of weight $m+1$. It is given by

$$W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} e^{-2\pi a} a^{(m+1)/2} & \text{if } a > 0, \\ 0 & \text{if } a < 0. \end{cases}$$

Then:

$$\int_0^{+\infty} W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a = (2\pi)^{-s}\Gamma\left(s+\frac{m}{2}\right);$$

(*cf.* Section 5 of [JL]).

Now consider case (e) where $\chi = 1$. Then $\omega = (\text{sgn})^{m+1}$,

$$\Phi(x) = e^{-\pi x^2} \quad \text{and} \quad \Psi(x) = e^{-\pi x^2} H_{m+1}(x\sqrt{2\pi}).$$

Our main integral reduces to

$$I = L(2s, 1)\int_{-\infty}^{+\infty} \left(\frac{a}{|a|}\right)^{m+1} |a|^{s-1} W\begin{pmatrix} \frac{1}{2}a^2 & 0 \\ 0 & 1 \end{pmatrix} e^{-\pi a^2} H_{m+1}(a\sqrt{2\pi})\, d^\times a.$$

But

$$H_{m+1}(-a) = (-1)^{m+1} H_{m+1}(a).$$

Therefore:

$$I = 2L(2s, 1)\int_0^{+\infty} a^{s-1} W\left[\begin{pmatrix} \frac{1}{2}a^2 & 0 \\ 0 & 1 \end{pmatrix}\right] \times e^{-\pi a^2} H_{m+1}(a\sqrt{2\pi})\, d^\times a.$$

So replacing W by its expression gives

$$I = L(2s, 1)\mathscr{I}_{m+1}(s),$$

with

$$\mathscr{I}_k(s) = \int_0^{\infty} a^{s+k-1} e^{-a^2} H_k(a)\, d^\times a.$$

(Recall that we are ignoring exponential factors).

To continue, we need to compute $\mathscr{I}_k(s)$. This is done by integrating by parts using the Rodriguez formula

$$\frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n e^{-x^2} H_n(x).$$

The end result is that

$$\mathscr{I}_k(s) = (s+k-2)(s+k-3)\ldots s\,\Gamma\left(\frac{s+1}{2}\right).$$

Therefore:

$$I = \Gamma(s)(s+m-1)(s+m-2)\ldots s\,\Gamma\left(\frac{s+1}{2}\right)$$

$$= \Gamma(s+m)\Gamma\left(\frac{s+1}{2}\right) = L_2(s, \sigma, \chi).$$

Case (f) is entirely similar and we leave it to the reader.

(6.8) *Complex case.*

In this case Mp (C) is a trivial extension of SL (2, C) by **T**. In other words, there is a splitting homomorphism $r: G(\mathbf{C}) \to \mathrm{Mp}(\mathbf{C})$ which provides an (ordinary) representation (also denoted r) of G (C) on L^2 (C). In fact if we set

$$L_+ = \{f \in L^2(\mathbf{C}) \mid f(x) = f(-x)\},$$
$$L_- = \{f \in L^2(\mathbf{C}) \mid f(x) = -f(-x)\},$$

then:

$$L^2(\mathbf{C}) = L_+ \oplus L_-$$

and the spaces L_+, L_- are invariant and irreducible under G (C). For our purposes it is convenient to extend the resulting representations r_+ and r_- to G_2 by taking them to be trivial on

$$\left\{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \middle| \; t > 0\right\}.$$

Then:

$$r_+ = \pi(\alpha_{\mathbf{C}}^{1/4}, \alpha_{\mathbf{C}}^{-1/4}), \qquad r_- = \pi(\xi, 1), \qquad \text{where} \quad \xi(z) = z(z\bar{z})^{-1/2};$$

(*cf.* [Ge], p. 95).

Suppose Φ is in L_+ (resp. L_-) and K-finite. Then Φ is actually in $\mathscr{S}(\mathbf{C})$. Set

$$W^*(g) = \left(r\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} g\right].\Phi\right)(1).$$

Then:

$$W^*\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right] = \psi(x) W^*[g].$$

Moreover

$$W^*\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} = 0(1) \qquad \text{for} \quad |z| \to +\infty.$$

Finally, the space spanned by the $W^{*\prime}s$ realizes the representation r_+ (resp. r_-) of the Hecke algebra. It follows that W^* is arbitrary in $\mathscr{W}(r_+; \psi_{\mathbf{C}})$, [resp. $\mathscr{W}(r_-; \psi_{\mathbf{C}})$].

Since the extension splits, we may regard F_s as a function on G (**C**) such that

$$F_s\left[\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} g\right] = \chi(a) |a|_{\mathbf{C}}^{s+1/2} F_s(g).$$

Our local integral may be written then as an integral over $N(\mathbf{C})\backslash G(\mathbf{C})$:

$$\begin{aligned} I &= I(s, F, \Psi, W) \\ &= \int_{N(F)\backslash G(F)} F_s(g) \overline{r(g).\Psi(1)}\, W\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} g\right] d^\times g \\ &= \int_{N(F)\backslash G(F)} F_s(g) \overline{W^*}\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} g\right] W\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} g\right] d^\times g. \end{aligned}$$

It remains to select F_s. We take

$$F_s(g) = L(s, \chi) \int_{\mathbf{C}^\times} \Phi[(0, t) g] |t|^{s+1/2} \chi(t) d^\times t,$$

where:

$$\Phi(z_1, z_2) = \exp(-2\pi(z_1 \bar{z}_1 + z_2 \bar{z}_2)) P(z_1, \bar{z}_1, z_2, \bar{z}_2),$$

and P is a polynomial.

Then, for $k \in K = SU(2)$,

$$F_s(k) = L(s, \chi) L\left(s + \frac{1}{2}, \chi\right) Q(s)(k)$$

where Q is a polynomial with values in the space of K-finite functions on K. Since

$$L(2s, \chi^2) = L(s, \chi) L\left(s + \frac{1}{2}, \chi\right)$$

up to exponential factors, F_s has the required analytic properties (6.1.3).

Now we want to write I as an integral over $N(\mathbf{C}) \backslash GL(2, \mathbf{C})$. To that end, recall that without loss of generality we may replace σ by $\sigma \otimes \mu$. But this replaces the central character ω of σ by $\omega\mu^2$. Therefore we may assume that we are in one of the following two cases:

Case	$(\chi\omega^{-1})$	χ	Ψ is in
g	1	$\chi = \omega$	L^2_+
h	-1	$\chi = \omega\bar{\xi}$, $\xi(z) = z(z\bar{z})^{-1/2}$	L^2_-

Then in case (g) [resp. (h)] W^* is in $\mathscr{W}(r_+; \psi)$ [resp. $\mathscr{W}(r_-; \psi)$] and transforms under the central character $\omega_+ = 1$ of r_+ (resp. $\omega_- = \xi$ of r_-). Since $\omega\overline{\omega_+} = \chi$ (resp. $\omega\overline{\omega_-} = \chi$) and $G_2(\mathbf{C}) = \mathbf{C}^\times G(\mathbf{C})$ we can write

$$I = L(s, \chi) \cdot \int_{N(\mathbf{C}) \backslash GL(2, \mathbf{C})} \overline{W^*}\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} g\right] W\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} g\right] |\det g|^{(s/2)+(1/4)} \Phi[(0, 1) g] \, d^\times g.$$

Finally $\overline{W^*}$ has the form

$$\overline{W^*}(g) = W_2[\eta g], \qquad \eta = \operatorname{diag}(-1, 1),$$

with W_2 in $\mathscr{W}(\bar{r}_+; \psi)$ [resp. $\mathscr{W}(\bar{r}_-; \psi)$]. In other words

$$I = L(s, \chi) \Psi\left(\frac{s}{2} + \frac{1}{4}, W_1, W_2, \Phi\right).$$

(Notations of § 1.) By [Ja], Proposition 17.4 (*see* also § 1), for appropriate W_1, W_2 and Φ:

$$\Psi(s, W_1, W_2, \Phi) = L(s, \sigma \times \bar{r}_+) \qquad [\text{resp. } L(s, \sigma \times \bar{r}_+)].$$

Taking into account the explicit expansion for all the L-factors concerned, and the duplication formula, we obtain (6.2). This completes the proof of Proposition (6.2) for $= \mathbf{C}$ (and for that matter for arbitrary F).

7. Local computation of the constant term of the Eisenstein series

Having proved that (for appropriate s, F_s, Ψ and φ):

$$L_2(s, \sigma, \chi) = \int_{G(F)\backslash G(\mathbf{A})} \varphi(g)\bar{\theta}_\Psi(g) E(g, s)\, ds$$

it remains to analytically continue $E(g, s)$. This we do by first analytically continuing the constant term of $E(g, s)$ (when F_s is appropriately chosen).

Conveniently, the "intertwining integral" describing the constant term of $E(g, s)$ is a product of local integrals. Since we are therefore dealing with a local problem, we can (and do) keep the notation of Section 6.

(7.1) If F_s is the function on Mp (F) defined in paragraph 6, we want to consider the "intertwining integral"

$$\text{(7.1.1)}\qquad F_s^*(g) = \int F_s\left[w_0 t\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] dx,\ \mathrm{pr}(w_0) = w.$$

Replacing F_s by the function $|F_s|$ [which is a function on G (F)] and applying standard results for G (F) we find that this integral converges absolutely for $\operatorname{Re} s > 1/2$. Also a formal computation shows that

$$\text{(7.1.2)}\qquad F_s^*\left[\lambda d\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} t\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] = \lambda\chi^{-1}(a)|a|^{1/2-s} F_s^*[g], \qquad \lambda\in\mathbf{T},$$

whenever F_s^* is defined.

What we want to prove is that F_s^* satisfies a condition analogous to (6.1.3).

(7.2) PROPOSITION. — *Notations being as above, there is an entire function* H^*, *with values in a fixed finite dimensional right-invariant space of continuous functions on* MK, *such that*

$$F_s^*(k) = L(2s-1, \chi^2) H^*(s)(k), \qquad k\in MK.$$

Moreover, if $F = \mathbf{R}$ *or* $\mathbf{C}$, *then* H^* *is a polynomial in s times an exponential factor; if* F *is non-archimedean of module q, then* H^* *is a polynomial in* q^{-s}, q^s.

The proof will proceed case by case. In the unramified case, additional information will be obtained. Recall also that except in the complex case, F_s is given by (6.1.4).

(7.2) F *is non-archimedean,* χ *is unramified.*

By Remark (6.2.2) we may assume $\mathfrak{R}_F$ is the conductor of ψ_F. Assume also—for the time being—that the residual characteristic of F is not 2. Then we are in the "unramified situation".

LEMME (7.2.1) ("unramified situation"). — *With the notation as above, suppose the Haar measure of* F *is chosen so that* $\mathrm{vol}(\mathfrak{R}_F) = 1$. *Then:*

$$F_s^*(k) = L(2s-1, \chi^2) \quad \text{for} \quad k\in r_{\mathfrak{R}}(K).$$

Since F_s is $r_{\mathfrak{R}}(K)$-invariant, F_s^* has the same property. Thus it suffices to establish this formula for $k = e$. Recall that

$$(7.2.2) \qquad F_s(g) = f_s(\mathrm{pr}\, g)(g.\Phi_0)(0),$$

where Φ_0 is the characteristic function of $\mathfrak{R}_F$. As for the function f_s, it will be convenient to extend it to a function (still denoted by f_s) on $G_2(F)$ defined by

$$(7.2.3) \qquad f_s\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right] = \left|\frac{a}{b}\right|^{1/2} |a|^{s-1} \chi(a) L(2s, \chi^2).$$

Thus:

$$\begin{aligned} F_s^*(e) &= \int f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right]\left(w_0\, t\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).\Phi_0(1)\, dx \\ &= \int f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] dx \int \Phi_0(y)\, \psi\left[\frac{1}{2} y^2 x\right] dy, \end{aligned}$$

since for Φ in $\mathscr{S}(F)$,

$$w_0.\Phi(1) = \int \Phi(y)\, dy.$$

Now

$$\int \left| f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right]\right| dx < +\infty$$

for $\mathrm{Re}\, s > 1$, and since ψ has module one, we see that for $\mathrm{Re}\, s > 1$ (which is sufficient for our purposes), $F_s^*(e)$ is given by the absolutely convergent double integral

$$(7.2.4) \qquad F_s^*(e) = \int_F \int_F f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] \Phi_0(y)\, \psi\left[\frac{1}{2} y^2 x\right] dx\, dy.$$

By a simple change of variables this equals

$$\int_F \int_F f_s\left[w\begin{pmatrix} 1 & y^{-2} x \\ 0 & 1 \end{pmatrix}\right] \Phi_0(y)\, \psi\left[\frac{1}{2} x\right] dx\, |y|^{-2}\, dy$$

or, taking into account the relation

$$w\begin{pmatrix} 1 & y^{-2} x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix},$$

$$\int_F \chi(y)\, |y|^s \Phi_0(y)\, |y|^{-2}\, dy \int_F f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\right] \psi\left[\frac{1}{2} x\right] dx.$$

Now set $\psi'(x) = \psi(-(1/2)\, x)$. Then the conductor of ψ' is still $\mathfrak{R}_F$, and for each s,

$$(7.2.5) \qquad W_s(h) = \int_F f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h\right] \psi\left(\frac{1}{2} x\right) dx$$

is in the space denoted $\mathscr{W}(\chi\alpha^{s-1}, 1; \psi')$ in paragraph 3 of [JL] (if $\chi\alpha^{s-1} \neq \alpha$, this space is also the space $\mathscr{W}(\pi(\chi\alpha^{s-1}, 1); \psi'))$.

A description of that space (valid for all local fields) has been recalled in (6.6) (*cf.* [JL], Prop. 3.2).

Let Φ be the characteristic function of $\mathfrak{R}_F^2$ in F^2. Then let

$$f_{\Phi, s}(g) = \int \Phi[(0, t) g] \chi\alpha^s(t) d^\times t . \chi\alpha^{s-1/2}(\det g) \tag{7.2.6}$$

and

$$W_s^*(g) = \int f_{\Phi, s}\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right]\psi'(-x)\, dx. \tag{7.2.7}$$

This is the unique element of $\mathscr{W}(\chi\alpha^{s-1}, 1; \psi')$ invariant under (and equal to one on) K_2. Note that $f_{\Phi,s}$ is proportional to f_s and

$$f_{\Phi, s}(e) = L(s, \chi); \tag{7.2.8}$$

it follows that

$$\int f_s\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right]\psi\left[\frac{1}{2}x\right] dx = \frac{L(2s, \chi^2)}{L(s, \chi)} W_s^*(g). \tag{7.2.9}$$

Thus:

$$F_s^*(e) = \frac{L(2s, \chi^2)}{L(s, \chi)} \int_{F^\times} W_s^*\begin{pmatrix} y & 0\\ 0 & y^{-1}\end{pmatrix} \Phi_0(y) \chi(y) |y|^{s-2} dy$$

or, since W_s^* transforms under the quasi-character $\omega_s = \chi\alpha^{s-1}$ of the center,

$$\begin{aligned} F_s^*(e) &= \frac{L(2s, \chi^2)}{L(s, \chi)} \int_F W_s^*\left[\begin{pmatrix} y^2 & 0\\ 0 & 1\end{pmatrix}\right] \Phi_0(y) \frac{dy}{|y|} \\ &= \left(\frac{L(2s, \chi^2)}{L(s, \chi)}(1-q^{-1})\right) \int_{F^\times} W_s^*\begin{pmatrix} y^2 & 0\\ 0 & 1\end{pmatrix} d^\times y. \end{aligned} \tag{7.2.9}$$

Here $d^\times y$ is so chosen that vol $(\mathfrak{R}^\times) = 1$ or, what amounts to the same

$$\frac{dy}{|y|} = (1-q^{-1}) d^\times y.$$

Also we have used the fact that

$$W_s^*\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix} = 0 \quad \text{if } |y| > 1.$$

But the convergent integral in (7.2.9) is the value at $t = s$ of the integral

$$\int |y|^{t-1} \chi\omega_s^{-1}(y) W_s^*\begin{pmatrix} y^2 & 0\\ 0 & 1\end{pmatrix} d^\times y.$$

Since the integrand here vanishes for $|y| > 1$, this integral converges for $\operatorname{Re}(t) > s$.

Moreover, for Re (t) sufficiently large, it is given by (6.3.3) with s replaced by t, ω by ω_s, and σ by $\pi(\chi\alpha^{s-1}, 1)$, i. e., its value is

$$L(t, \chi^2\alpha^{s-1})L(t, \chi)L(t, \alpha^{1-s})L(2t, \chi^2)^{-1}.$$

Setting $t = s$ we get

$$\int W_s^*\begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix} d^\times y = \frac{L(2s-1, \chi^2)L(s, \chi)L(1, 1)}{L(2s, \chi^2)} = \frac{1+\chi(\tilde{\omega})q^{-s}}{(1-\chi^2(\tilde{\omega})q^{-2s+1})(1-q^{-1})}.$$

Therefore

(7.2.10) $$F_s^*(e) = L(2s-1, \chi^2)$$

as required. This concludes the proof of (7.2.1).

Suppose now the residual characteristic of F is 2. Then we choose ψ as in paragraph 6: $\psi(x) = \psi_0(2x)$, and $\psi'(x) = \psi_0(-x)$, where ψ_0 has conductor $\mathfrak{R}_F$. Again F_s is defined by (7.2.2) and we extend f_s to $G_2(F)$ by (7.2.3). We check that for all k in MK:

(7.2.11) $$F_s^*(k)/L(2s, \chi^2)$$

is a polynomial in q^{-s}, q^s. As before

$$F_s^*(k) = (1+\chi(\tilde{\omega})q^{-s})(1-q^{-1}).\int W_s^*\begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix} k.\Phi_0(y)\,d^\times y.$$

For $k = e$, this is (7.2.10). Thus it will suffice to show that if

$$U_s(k) = \int_{|y|=1} W_s^*\begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix} k.\Phi_0(y)\,d^\times y$$

then:

$$U_s(k) = U_s(e)\lambda(k),$$

where λ is a character of MK.

Note that any element of MK projects to an element of K of the form

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad x_i \in \mathfrak{R}, \quad a \in \mathfrak{R}^\times$$

or

$$w\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} w^{-1} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad x_1 \in \mathfrak{P}, \quad x_2 \in \mathfrak{R}, \quad a \in \mathfrak{R}^\times.$$

In the first case we may assume

$$k = t\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} w_0\, t\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Then, for $y \in \mathfrak{R}$,

$$k.\Phi_0(y) = |2|^{1/2}\psi_0[x_1^2 y]\Phi_0(2y) = |2|^{1/2}.$$

In the second case we may assume

$$k = w_0\, {}^t\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} w_0^{-1}\, {}^t\begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} d\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

Then for $y \in \mathfrak{R}$,

$$k.\Phi_0(y) = |2|^{1/2}\int \Phi_0(2z)\psi_0[x_1^2 z]\psi_0[-2zy]\,dz.$$

However $\Psi_0(-2zy) = 1$ if $\Phi_0(2z) \neq 0$. Thus $k.\Phi_0(y)$ does not depend in $y \in \mathfrak{R}$, and our conclusion follows.

(7.3) F *is non archimedean*, χ *is ramified.* In this case recall that F_s has the form

(7.3.1) $$F_s(g) = f_s(\mathrm{pr}\, g)(g.\Phi)(0),$$

where f_s satisfies (6.1.5) and $f_s \mid K$ is actually independent of s. Thus it will suffice to prove the following Lemma:

LEMMA (7.3.2). — *Suppose* F_s *is a function of the form* (7.3.1) *where* Φ *is in* $\mathcal{S}$ (F) *and* f_s *is a function on* G_2 (F) *satisfying*

$$f_s\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right] = \left|\frac{a}{b}\right|^{1/2} \chi(a)|a|^{s-1}h(k),$$

where h is a fixed finite function on GL (2, $\mathfrak{R}$). *Then if* F_s^* *is as in* (7.1.1), *for any g*

$$F_s^*(g)/L(2s-1, \chi^2)$$

is a polynomial in q^{-s}.

Proof. — We may assume pr (g) is in K. Replacing F_s by a translate we may even assume $g = e$. As in (7.2) we find that for Re s large enough,

$$F_s^*(e) = \int W_s\left[\begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix}\right]\Phi(y)\,d^\times y,$$

where

$$W_s(g) = \int_F f_s\left[w\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} g\right]\psi\left[\frac{1}{2}x\right]dx.$$

In particular (*cf.* Prop. 3.2 of [JL])

$$W_s\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = |y|^{1/2}\int f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right]\psi\left[\frac{1}{2}xy\right]dx.$$

Since f_s is K-finite, we also see that

$$f_s\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]=|x|^{-s}\chi^{-1}(x)f_s\begin{pmatrix}1 & 0\\ -x^{-1} & 1\end{pmatrix}=|x|^{-s}\chi^{-1}(x)f_s(e)$$

for $|x|$ large enough. Thus

$$f_s\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right]=\sum_{i=1}^{N}\theta_i(x)q^{-is}+\theta_0(x, s),$$

where the θ_i are in $\mathscr{S}$ (F) and $\theta_0(x, s)$ is defined by

$$\theta_0(x, s)=\begin{cases}|x|^{-s}\chi^{-1}(x) & \text{for } |x|\geqq 1\\ 0 & \text{for } |x|<1.\end{cases}$$

But the Fourier transform of $\theta_0(x, s)$ for Re (s) large enough is given by

$$\int\theta_0(x, s)\psi\left[\frac{1}{2}xy\right]dx=\int_{|x|\geqq 1}|x|^{-s}\chi^{-1}(x)\psi\left[\frac{1}{2}xy\right]dx,$$

and since χ is ramified it is easily found that this integral has the form $q^{-ms}\varphi(y)\chi(y)|y|^{s-1}$ with φ in $\mathscr{S}$ (F). Thus:

$$W_s\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}=\sum\hat{\theta}_i(y)|y|^{1/2}q^{-is}+q^{-ms}\chi(y)|y|^{s-1/2}\hat{\varphi}(y),$$

and

$$F_s^*(e)=\sum_i q^{-is}\int\hat{\theta}_i(y)\,dy+q^{-ms}\int\hat{\varphi}(y)\chi^2(y)|y|^{2s-1}\frac{dy}{|y|}.$$

The sum on i is a polynomial in q^{-s}, q^s; the second term is the product of a polynomial in q^{-s}, q^s by L $(2s-1, \chi^2)$ *cf.* ([Ta], [Go Ja]). This concludes the proof of Lemma (7.3.2).

(7.4) *Real case*. In this case:

(7.4.1) $$F_s(g)=f_s(\mathrm{pr}\, g)(g.\Phi)(0),$$

where Φ is an element of $\mathscr{S}$ (F) of a certain weight and f_s is a function on G_2 (F) such that

(7.4.2) $$f_s\left[\begin{pmatrix}a & x\\ 0 & b\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right]=\chi(a)|a|^{s-1}\left|\frac{a}{b}\right|^{1/2}e^{in\theta}L(2s, \chi^2)P(s).$$

The integer n is the weight of f_s and P is a polynomial. The possibilities for χ, P, f_s, and Φ are as follows (*cf.* Tables (6.5.11), (6.5.13) and the remarks made in establishing those tables):

TABLE (7.4.3)

Case	χ	wgt (f_s)	wgt (Φ)	wgt (F_s)	P (s)
(i).......	1	0	1/2	1/2	1
(ii).......	sgn	-1	1/2	$-1/2$	1
(iii)......	sgn	-1	5/2	3/2	s

Since F_s transforms according to a character of MK, to prove Proposition (7.2) for $F = \mathbf{R}$, it will suffices to show there is a polynomial P^* such that

(7.4.4) $$F_s^*(e) = L(2s-1, \chi^2)P^*(s),$$

up to an exponential factor.

Now, as in (7.2),

(7.4.5) $$F_s^*(e) = \int W_s\left[\begin{pmatrix} -\frac{1}{2}y^2 & 0 \\ 0 & 1 \end{pmatrix}\right]\Phi(y)\,d^\times y,$$

where

(7.4.6) $$W_s(g) = \int_{-\infty}^{+\infty} f_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right]\psi(-x)\,dx.$$

Again, for each s (with $\operatorname{Re} s > 1$) W_s is in the space denoted $\mathcal{W}(\chi\alpha^{s-1}, 1; \psi_F)$ in [JL]. Therefore we proceed as in (7.2). We let Φ_1 be an element of $\mathcal{S}(F^2)$ such that the function

(7.4.5) $$f_{\Phi_1}(g) = \int_{\mathbf{R}^\times} \Phi_1[(0, t)g]\chi(t)|t|^s d^\times t\,\chi(\det g)|\det g|^{s-1/2}$$

has the same weight as f_s. Then the function W_s^* defined by

(7.4.6) $$W_s^*(g) = \int f_{\Phi_1}\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right]\psi(-x)\,dx$$

is in $\mathcal{W}(\chi\alpha^{s-1}, 1; \psi_F)$ and proportional to W_s.

More precisely, we take

(7.4.7) $$\Phi_1(x, y) = \begin{cases} \exp(-\pi(x^2+y^2)) & \text{case (i)}, \\ \exp(-\pi(x^2+y^2))(x-iy) & \text{cases (i) and (ii)}. \end{cases}$$

Then Φ_1' [*cf.* (6.6.6)] is given by

(7.4.8) $$\Phi_1'(x, y) = \begin{cases} \exp(-\pi(x^2+y^2)) & \text{case (i)}, \\ \exp(-\pi(x^2+y^2))(x+y) & \text{cases (i) and (ii)}. \end{cases}$$

In all cases

$$f_{\Phi_1}(e) = L(s, \chi)$$

so that

(7.4.9) $$F_s^*(e) = \frac{L(2s, \chi^2)}{L(s, \chi)}P(s)\int_{-\infty}^{+\infty} W_s^*\left[\begin{pmatrix} -\frac{1}{2}y^2 & 0 \\ 0 & 1 \end{pmatrix}\right]\Phi(y)\,d^\times y.$$

Since (in all cases) $\Phi(-y) = \Phi(y)$, up to exponential factors we also have

(7.4.10) $$F_s^*(e) = \frac{L(2s, \chi^2)}{L(s, \chi)}P(s)\int_0^{+\infty} W_s^*\left[\begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix}\right]\Phi(\sqrt{2y})\,d^\times y,$$

(7.4.11) $$W_s^*\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \int \Phi_1'(yt, t^{-1})\chi(t)|t|^{s-1} d^\times t \chi(y)|y|^{s-1/2}.$$

This time

(7.4.12) $$\int_0^{+\infty} W_s^*\left[\begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix}\right]|y|^{t-1/2} d^\times t$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} W_s^*\left[\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right]|y|^{t-1/2} d^\times t - \frac{1}{2}\int_{-\infty}^{+\infty} W_s^*\left[\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right]|y|^{t-1/2}\frac{y}{|y|} d^\times y$$

$$= \frac{1}{2}\iint \Phi_1'(x, y)\chi(x)|x|^{s+t-1}|y|^t d^\times x\, d^\times y$$

$$- \frac{1}{2}\iint \Phi_1'(x, y)\chi(x)\frac{x}{|x|}|x|^{s+t-1}\frac{y}{|y|}|y|^t d^\times x\, d^\times y.$$

Thus this integral can be easily computed.

Similarly it is easy to compute

$$\int_0^{+\infty} \Phi(\sqrt{2y})|y|^{t-1/2} d^\times t$$

since:

$$\Phi(x) = \begin{cases} \exp(-\pi x^2) & \text{cases (i) and (ii),} \\ \exp(-\pi x^2)(8\pi x^2 - 2) & \text{case (ii).} \end{cases}$$

Using a Barnes-Mellin lemma as before we can (for Ret large enough) compute the integral

(7.4.13) $$\int_0^{+\infty} W_s^*\begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix}\Phi(\sqrt{2y})|y|^{t-s} d^\times y.$$

[From (7.4.11) we see this integral converges for Re $(t-s) \geqq 0$.] Thus we may evaluate (7.4.10) by setting $s = t$ in the expression for (7.4.13). This will give the required result [with $P^* \not\equiv 1$ only in case (iii)].

(7.5) *Complex case.* — If $F = \mathbf{C}$ recall that

$$F_s(g) = L(s, \chi)\int \Phi[(0, t)g]|t|^{s+1/2}\chi(t) d^\times t,$$

where Φ is a suitably chosen function of the form

$$\Phi(x, y) = \exp[-2\pi(x\bar{x} + y\bar{y})]P(x, \bar{x}, y, \bar{y}).$$

Now:

$$F_s^*(g) = \int F_s\left[w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right]dx = L(s, \chi)\iint \Phi[(t, tx)g]|t|^{s+1/2}\chi(t) d^\times t\, dx$$

$$= L(s, \chi)\iint \Phi[(t, x)g]|t|^{s-1/2}\chi(t) d^\times t\, dx.$$

As in Section 6, if g is in SU (2),

$$F_s^*(g) = L(s, \chi) L\left(s-\frac{1}{2}, \chi\right) Q(s)(g)$$

where Q is a polynomial. So since

$$L(s, \chi) L\left(s-\frac{1}{2}, \chi\right) = L(2s-1, \chi^2)$$

up to exponentional factors, F_s^* has the required analytic properties, and the proof of (7.2) is complete.

8. Analytic continuation of the main integral

In this Section F is a *number field*. As before we let $\sigma = \otimes_v \sigma_v$ be an automorphic cuspidal representation of $G_2(\mathbf{A})$ and S the (finite) set of finite places v where σ_v is not quasi-unramified. We let χ be a character of $F_{\mathbf{A}}^\times / F_x$ and we assume χ_v is highly ramified for $v \in S$ [*cf.* (5.3.3)]. We assume in addition that there is at least one finite place w such that χ_w^2 is ramified. Since this is so if χ_w is of high enough ramification, we may take w to be in S if $S \neq \emptyset$.

Our purpose in this section is to prove the following:

THEOREM (8.1). — *Suppose* σ *and* χ *satisfy the above conditions. Then* $L_2(s, \sigma, \chi)$ *is entire and bounded in vertical strips of finite width.*

To prove this theorem we need only consider the "main integral" introduced in Section 5. Indeed for each v let $F_{v,s}$ be the function introduced in paragraph 6. (Recall that the choice of $F_{v,s}$ depends on σ_v and χ_v). Define $F_s(g)$ by

$$F_s(g) = \prod_v F_{v,s}(g_v)$$

if $g = (g_v)$ [*cf.* (5.5.5)]. Then:

$$F_s\left[d\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} t\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\right] = \chi(a)\,|a|^{s+1/2} F(g).$$

If we set

$$E(g, s) = \sum_{\gamma \in B(F)\backslash G(F)} F_s(r_F(\gamma) g)$$

then this converges for Re s large enough. Moreover, there is a Ψ in $\mathscr{S}(\mathbf{A})$ and a cusp form φ on $G_2(\mathbf{A})$ belonging to σ such that

$$L_2(s, \sigma, \chi) = \int_{G(F)\backslash G(\mathbf{A})} E(g, s)\overline{\theta_\Psi(h)}\,\varphi(g)\,dg, \qquad (\mathrm{pr}(h) = g)$$

for Re s large enough [*cf.* Prop. (6.2) and (5.6.14)]. Thus Theorem 8.1 reduces to:

PROPOSITION 8.2. — *Suppose* F_s *and* χ *are as above, and* φ *is a rapidly decreasing genuine function on* $\mathrm{Mp}(F)\backslash\mathrm{Mp}(A)$. *Then the function*

$$s \mapsto \int E(h, s)\bar{\varphi}(h)\,dh,$$

$$h \in r_F(G(F))T\backslash \mathrm{Mp}(A),$$

initially defined for Re (s) *sufficiently large, extends to an entire function of* s *which is bounded in vertical strips of finite width.*

This Proposition is tantamount to the analytic continuation of the Eisenstein series $E(g, s)$. Thus we need to recall a crucial fact from the theory of Eisenstein series.

LEMMA (8.2.1). — *Let* $\tilde{F}_s$ *be a function on* $\mathrm{Mp}(A)$ *such that*

$$\tilde{F}_s\left[\lambda\, d\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} t\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} k\right] = \lambda H(k)\,|a|^{s+1/2}\chi(a),$$

where H *is a fixed finite function on* MK. *Then the function* $\tilde{F}_s^*$, *defined for* Re s *large enough by*

$$\tilde{F}_s^*(g) = \int_A \tilde{F}_s\left[r_F(w)\, t\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\right] dx,$$

continues to a meromorphic function of s *without poles on the line* Re $(s) = 1/2$. *Moreover, if* Re $s = 1/2$, *then:*

$$\int_{MK} \tilde{F}_s(k)\bar{\tilde{F}}_s(k)\,dk = \int_{MK} \tilde{F}_s^*(k)\bar{\tilde{F}}_s^*(k)\,dk.$$

We take this Lemma for granted; it follows from the more general results of [La 3]. (Recall F is a number field!)

Now our function F_s is clearly of the form

(8.2.2) $$F_s(g) = L(2s, \chi^2)\sum h_i(s)\tilde{F}_s^i(g),$$

where, for each i, $\tilde{F}_s^i$ satisfies the conditions of the Lemma and $h_i(s)$ is an elementary function of s, i. e., a sum of terms like $as^i e^{bs}$. If we set

(8.2.3) $$F_s^*(g) = \int F_s\left[r_F(w)\, t\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\right] dx,$$

then we have also

(8.2.4) $$F_s^*(g) = L(2s, \chi^2)\sum h_i(s)\tilde{F}_s^{i*}(g).$$

Thus it follows from Lemma (8.2.2) that

(8.2.5) $$\int F_s(k)\overline{F_s}(k)\,dk = \int F_s^*(k)\overline{F_s^*}(k)\,dk \qquad \text{if} \quad \mathrm{Re}\, s = \frac{1}{2}.$$

Actually we have more information about the analytic behavior of F_s^*. Namely (by Proposition 7.2):

$$F_s^*(k) = L(2s-1, \chi^2)\sum_j H_j'(k)\, h_j'(s) \tag{8.2.6}$$

where H_j is some finite function on MK and h_j' is elementary.

To continue, we need to use a familiar "truncation process". (*see* [DL] or [Ge J1] for more details.) For $c > 0$ we let χ_c be the characteristic function of the "Siegel set":

$$\left\{g \mid g = d\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} t \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} k,\ |a| \geqq c, k \in MK\right\}.$$

We set

$$F_s'(g) = E^0(g, s)\chi_c(g), \tag{8.2.7}$$

where E^0 is the constant term of $E(g, s)$:

$$E^0(g, s) = \int_{F\backslash A} E\left[t\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g, s\right] dx. \tag{8.2.8}$$

Then by the Bruhat decomposition,

$$E^0(g, s) = F_s(g) + F_s^*(g). \tag{8.2.9}$$

Thus E^0 has a known analytic behavior. But from reduction theory we hnow that if c is large enough then the series

$$E'(g, s) = \sum_{B(F)\backslash G(F)} F_s'(r_F(\gamma) g) \tag{8.2.10}$$

has only finitely non-zero many terms when g is in a siegel set. Let us study then the scalar product

$$\int E'(h, s)\bar{\varphi}(h)\, dh. \tag{8.2.11}$$

To evaluate this integral consider what happens when h is in a Siegel set $\mathfrak{S}$. There is a compact subset Ω of $\mathfrak{S}$ such that for $h \in \mathfrak{S} - \Omega$,

$$E'(h, s) = E^0(h, s);$$

cf. [Ge J1]. On the other hand, for h in Ω, the series defining E' has only finitely many non zero-terms, so since φ is rapidly decreasing at infinity in $\mathfrak{S}$, it is clear that (8.2.11) converges for all s and defines an entire function of s bounded at infinity in vertical strips.

Now set

$$E''(h, s) = E(h, s) - E'(h, s). \tag{8.2.12}$$

Let also L^2 denote the space of genuine functions on Mp (A) which are $r_F(G(F))$ invariant and such that

$$\int f(h)\bar{f}(h)\,dh < +\infty,\ h \in r_F(G(F))T\backslash Mp(A).$$

Clearly φ is in L^2. Then Proposition (8.2) will follow from:

PROPOSITION 8.3. — *If* Re s *is large enough*, E''_s *is in* L^2, *and the map* $s \mapsto E''_s$ *continues to an entire function of s with values in* L^2. *Moreover, the map* $s \mapsto \| E''_s \|$ *is bounded in vertical strips of finite width.*

Proof. — Again, from the known theory of Eisenstein series, one knows that E''_s is square integrable and $s \mapsto E''_s$ holomorphic, for Re s large enough. Moreover for Re s_1 > Re $s_2 \gg 0$,

$$(8.3.1)\qquad (E''_{s_1}, E''_{\bar{s}_2}) = \int_{|a|<c} |a|^{s_1+s_2-1}\,d^\times a\left(\int_{MK} F_{s_1}(k)\overline{F_{\bar{s}_2}}(k)\,dk\right)$$
$$-\int_{|a|>c} \chi^2(a)|a|^{s_2-s_1}\,d^\times a\int_{MK} F_{s_1}(k)\overline{F^*_{\bar{s}_2}}(k)\,dk$$
$$-\int_{|a|>c} |a|^{1-s_1-s_2}\,d^\times a\int_{MK} F^*_{s_1}(k)\overline{F^*_{\bar{s}_2}}(k)\,dk$$
$$+\int_{|a|<c} \bar{\chi}^2(a)|a|^{s_1-s_2}\,d^\times a\int_{MK} F^*_{s_1}(k)\overline{F_{\bar{s}_2}}(k)\,dk.$$

In each integral above the integration over $F^\times_A$ is taken modulo $F^\times$. But χ^2 is not a principal character of $F^\times\backslash F^\times_A$. Therefore the second and fourth terms drop out, leaving

$$(8.3.2)\qquad (E''_{s_1}, E''_{s_2}) = A(s_1, s_2)(s_1+s_2-1)^{-1},$$

with

$$(8.3.3)\qquad A(s_1, s_2) = c^{s_1+s_2-1}\int_{MK} F_{s_1}(k)\overline{F_{\bar{s}_2}}(k)\,dk - c^{1-s_1-s_2}\int_{MK} F^*_{s_1}(k)\overline{F^*_{\bar{s}_2}}(k)\,dk.$$

Now $A(s_1, s_2)$ is holomorphic in $\mathbf{C}^2$. Moreover, if $s_1 = (1/2)+iy$ and $s_2 = (1/2)-iy$ with $y \in \mathbf{R}$, we find that A vanishes [*cf.* (8.2.5)]. Thus $A(s_1, s_2) = 0$ if $s_1+s_2-1 = 0$, and it follows that

$$(8.3.4)\qquad (E''_{s_1}, E''_{\bar{s}_2})$$

extends to a holomorphic function of (s_1, s_2) in $\mathbf{C}^2$. A standard argument (*cf.* [La 3] or [Ge J 2]) then shows that $s \mapsto E''_s$ continues to an entire function (with values in L^2), and by analytic continuation, formula (8.3.2) is still valid when $s_1+s_2-1 \neq 0$.

To complete the proof of Proposition 8.3 (and hence Theorem 8.1) we have to show that (E''_s, E''_s) is bounded in vertical strips. For this we write $s = (1/2)+x+iy$ with

$-a \leqq x \leqq a$ and $y \in \mathbf{R}$. Then:

$$(8.3.5) \qquad (E_s'', E_s'') = A\left(\frac{1}{2}+x+iy, \frac{1}{2}+x-iy\right)(2x)^{-1} = B(x, y)(x)^{-1},$$

where

$$B(x, y) = \frac{1}{2} A\left(\frac{1}{2}+x+iy, \frac{1}{2}+x-iy\right).$$

Now $B(0, y) = 0$. Thus:

$$B(x, y) = x \int_0^1 \frac{\partial B}{\partial x}(tx, y)\, dt$$

and

$$(E_s'', E_s'') = \int_0^1 \frac{\partial B}{\partial x}(tx, y)\, dt.$$

But

$$\frac{\partial B}{\partial x}(x, y) = \frac{1}{2}\, \frac{\partial A}{\partial s_1}\left(\frac{1}{2}+x+iy, \frac{1}{2}+x-iy\right) + \frac{1}{2}\, \frac{\partial A}{\partial s_2}\left(\frac{1}{2}+x+iy, \frac{1}{2}+x-iy\right).$$

So since the partial derivatives of A can be expressed in terms of elementary functions and L-functions and their derivatives, it easily follows that $\partial B/\partial x$ is bounded for $-a \leqq x \leqq a$, and the same is true of (E_s'', E_s'').

Q.E.D.

9. The Main Theorem

Again F is a number field.

(9.1) We first review and complete the results of [JPSS]. We let T be a finite set of places (possibly empty). For each $v \notin T$ we let π_v be an admissible irreducible representation of $G_{3, v}$ (or its Hecke algebra). In addition we make the following assumptions (and fix notation as follows):

(9.1.1) For each v, the representation π_v has trivial central quasi-character;

(9.1.2) Suppose v is finite; if π_v is generic, i. e. admits a "Whittaker model" in the sense of [JPSS] (§ 2), then we set $\xi_v = \pi_v$; if π_v is not generic, then it is a quotient of a certain induced representation ξ_v which is said to be attached to π_v (*loc. cit.*) for instance, we may have

$$(9.1.2.1) \qquad \pi_v = \operatorname{Ind}(G_{3, v}, P_v; 1_2, 1),$$

where P_v is the parabolic subgroup of type (2.1), $1_2 = \pi(\alpha_v^{1/2}, \alpha_v^{-1/2})$ the trivial representation of $G_{2, v}$, and

$$(9.1.2.2) \qquad \xi_v = \operatorname{Ind}(G_{3, v}, B_{3, v}; \alpha_v^{1/2}, 1, \alpha_v^{-1/2}).$$

(9.1.3) Suppose v is archimedean; then either π_v is unitary generic, in which case we set $\xi_v = \pi_v$, or π_v is one of the representations considered in paragraph 4, namely:

$$\pi_v = \mathrm{Ind}(G_{3,v}, B_{3,v}; \alpha_v^t, 1, \alpha_v^{-t}), \frac{1}{2} < t < 1, \tag{9.1.3.1}$$

in which case $\pi_v = \xi_v$, or π_v is (9.1.2.1), in which case ξ_v is (9.1.2.2).

In all cases, π_v is a quotient of ξ_v, and the representation ξ_v is semi-simple only if $\xi_v = \pi_v$.

(9.1.4) For almost all finite v, π_v is unramified, thus of the form

$$\pi_v = \pi(\mu_{1,v}, \mu_{2,v}, \mu_{3,v}),$$

where $\mu_{i,v}(x) = |x|^{t_{i,v}}$: this means that π_v is the only unramified component of the induced representation

$$\mathrm{Ind}(G_{3,v}, B_{3,v}; \mu_{i,v}, \mu_{2,v}, \mu_{3,v});$$

we also assume there is a $t > 0$ (independent of v) such that

$$-t \leqq t_{i,v} \leqq t \qquad (i = 1, 2, 3, \text{ almost all } v).$$

Given the notation and conditions above, we can now form the representations

$$\pi^T = \bigotimes_{v \notin T} \pi_v, \qquad \xi^T = \bigotimes_{v \notin T} \xi_v$$

of the Hecke-algebra $\mathscr{H}^T$ of the restricted product group

$$G^T = \prod G_v \qquad (v \notin T)$$

and form for each character χ of $F_A^\times/F^\times$ the infinite products

$$L(s, \pi^T, \chi) = \prod_{v \notin T} L(s, \pi_v, \chi_v),$$

and

$$L(s, \tilde{\pi}^T, \chi) = \prod_{v \notin T} L(s, \tilde{\pi}_v, \chi_v^{-1}).$$

These products converge absolutely for Re (s) sufficiently large.

(9.2) Suppose that the above functions are entire, bounded in vertical strips, and satisfy

$$L(s, \pi^T, \chi) = \prod_{v \notin T} \varepsilon(s, \pi_v \otimes \chi_v, \psi_v) \prod_{v \in T} \varepsilon(s, \chi_v, \psi_v)^3 L(1-s, \tilde{\pi}^T, \chi^{-1}) \tag{9.2.1}$$

provided the ramification of χ is sufficiently high at each place v in T. Then we conclude as in [JPSS] (§ 13) that there is a space $\mathscr{V}$ of C^∞ functions on $G_3(F) Z_3(A) \backslash G_3(A)$ which is invariant on the right under convolution by $\mathscr{H}^T$ and realizes the representation ξ^T. Moreover, the elements of $\mathscr{V}$ are "slowly increasing". The various possibilities for T and $\mathscr{V}$ are now discussed case by case.

(i) Suppose first that $T = \emptyset$. Then we set

$$\pi = \pi^T = \otimes \pi_v, \qquad \xi = \xi^T = \otimes \xi_v,$$

so that

$$L(s, \pi^T, \chi) = L(s, \pi \otimes \chi), \qquad L(s, \tilde{\pi}^T, \chi) = L(s, \tilde{\pi} \otimes \chi),$$

and the functional equation reads

$$L(s, \pi \otimes \chi) = \varepsilon(s, \pi \otimes \chi) L(1-s, \tilde{\pi} \otimes \chi^{-1}). \tag{9.2.2}$$

In this case the elements of $\mathcal{V}$ are cuspidal, thus cusp-forms.

Thus each ξ_v is semi-simple and unitary generic; in particular $\pi_v = \xi_v$ for all v, π_v is unitary generic, and π_v cannot be a representation of the form (9.1.3.1) or (9.1.2.1).

(ii) Now suppose $T \neq \emptyset$ but the elements of $\mathcal{V}$ are cuspidal. Then again they are cusp forms. For $v \notin T$, we again find $\pi_v = \xi_v$ is unitary generic. On the other hand, for $v \in T$, there is a unitary generic representation π_v with trivial central character such that $\pi = \otimes \pi_v$ (all v) is automorphic cuspidal. Moreover (9.2.2) is satisfied.

(iii) Next suppose the elements of $\mathcal{V}$ have zero constant terms along

$$N_3 = \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

but not along

$$U = \left\{ \begin{pmatrix} 1 & 0 & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \right\};$$

(*cf.* [JPSS], § 13).

Then (*cf.* the proof of [13.8] in [JPSS]), there is a quasi-character μ of $F_A^\times/F^\times$ and an automorphic cuspidal representation τ of $G_2(A)$ with central quasi-character μ^{-1} satisfying the following conditions: for any v set

$$\eta_v = \mathrm{Ind}(G_{3,v}, P_v; \tau_v, \mu_v);$$

then for all $v \notin T$ the representations ξ_v and η_v have a common irreducible component; moreover, for almost all finite v both ξ_v and η_v contain a (unique) component which is unramified and that component is the same for both representations (and equal to π_v). Note that μ need not be a character and therefore τ need not be unitary. (The precise definition of the induced representation η_v—for v infinite—is irrelevant since allt hat matters to us is the infinitesimal class of η_v and its components.) If v is finite then τ_v is generic and η_v has exactly one generic component. Therefore, for $v \in T$, let π'_v be that component and form the representation

$$\pi' = (\bigotimes_{v \in T} \pi'_v) \otimes \pi^T.$$

We will write also $\pi' = \otimes \pi'_v$ so that $\pi'_v \simeq \pi_v$ for $v \notin T$. Then (again *see* the proof of (13.8) in [JPSS]) $L(s, \pi' \otimes \chi)$ is meromorphic and satisfies (9.2.2) for all χ.

(iv) Suppose now that the elements of $\mathscr{V}$ have zero constant terms along N but not along

$$V = \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Then a similar conclusion can be reached.

(v) Finally suppose that the elements of $\mathscr{V}$ have non zero-constant term along N. Then there are 3 quasi-characters ν_i of $F_A^\times/F^\times$ such that $\nu_1 \nu_2 \nu_3 = 1$ and, if we set,

$$\eta_v = \operatorname{Ind}(G_{3,v}, B_{3,v}; \nu_{1v}, \nu_{2v}, \nu_{3v}),$$

then for all $v \notin T$ the representations ξ_v and η_v have a common component. For almost all v they contain also the same unramified component which is then π_v. For $v \in T$ let π'_v be the unique generic component of η_v and set

$$\pi' = \bigotimes_{v \in T} \pi'_v \otimes \pi^T.$$

Then $L(s, \pi' \otimes \chi)$ satisfies (9.2.2) for all χ (*loc. cit.*).

We can now formulate the main theorem of this paper.

(9.3) THEOREM. — *Let σ be a unitary irreducible representation of* $G_2(A)$ *which is automorphic cuspidal. Assume that for any character χ of* $F_A^\times/F^\times$, $\chi \neq 1$, *the representations σ and $\sigma \otimes \chi$ are inequivalent. Then:*

(1) *for any* χ, $L_2(s, \sigma, \chi)$ *is entire;*

(2) *for any place v the representation σ_v admits a lift π_v to* $G_{3,v}$*;*

(3) *set* $\pi = \bigotimes_v \pi_v$ *(all v). Then π is automorphic cuspidal;*

(4) *no representation of the form* $\pi(\lambda\alpha_v^t, \lambda\alpha_v^{-t})$ *with* $(1/4) \leqq t$ *occurs as a component of* σ.

The proof will occupy the rest of paragraph 9. (The case when $\sigma \otimes \chi \approx \sigma$ for some χ has already been discussed in Section 3.7; *see* also Remark 9.9).

(9.4) We prove first (9.3.2). Let S_0 be the set of places where σ_v is extraordinary. There is nothing to prove if $S_0 = \emptyset$. So assume $S_0 \neq \emptyset$, and let $S \supset S_0$ be the set of places v where σ_v is not quasi-unramified. For $v \notin S$, let π_v be the lift of σ_v. By (8.1) the assumptions of (9.2) are satisfied with $T = S$. From (9.2) we conclude that there is, for each $v \in S$, a generic representation π'_v, trivial on the center, such that, if

$$\pi' = \bigotimes_{v \in s} \pi'_v \otimes \bigotimes_{v \notin s} \pi_v,$$

hen $L(s, \pi' \otimes \chi)$ is meromorphic and satisfies the functional equation

$$L(s, \pi' \otimes \chi) = \varepsilon(s, \pi' \otimes \chi) L(1-s, \pi' \otimes \chi^{-1})$$

for all χ. On the other hand, $L_2(s, \sigma, \chi)$ is meromorphic and

$$L_2(s, \sigma, \chi) = \varepsilon_2(s, \sigma, \chi) L_2(1-s, \tilde{\sigma}, \chi^{-1})$$

for all χ. Since the local L and ε factors in both equations are in fact the same for all $v \notin S$ we conclude that

$$(9.4.1) \qquad \prod_{v \in S} L(1-s, \tilde{\pi}'_v \otimes \chi_v^{-1}) \varepsilon(s, \pi'_v \otimes \chi_v; \psi_v)/L(s, \pi'_v \otimes \chi_v)$$
$$= \prod_{v \in S} L_2(1-s, \tilde{\sigma}_v, \chi_v^{-1}) \varepsilon_2(s, \sigma_v, \chi_v; \psi_v)/L_2(s, \sigma_v, \chi_v).$$

Now fix $w \in S_0$ and let η be a character of $F_w^\times$. Choose χ in such a way that $\chi_w = \eta$ and χ_v is highly ramified for $v \in S$, $v \neq w$. Then for $v \in S$, $v \neq w$,

$$(9.4.2) \qquad \begin{cases} L(s, \pi_v \otimes \chi_v) = L(s, \tilde{\pi}_v \otimes \chi_v^{-1}) = 1, \\ \varepsilon(s, \pi'_v \otimes \chi_v; \psi_v) = \varepsilon(s, \chi_v; \psi_v)^3, \end{cases}$$

and

$$(9.4.3) \qquad \begin{cases} L_2(s, \sigma_v, \chi_v) = L_2(s, \tilde{\sigma}_v, \chi_v^{-1}) = 1, \\ \varepsilon_2(s, \sigma_v, \chi_v; \psi_v) = \varepsilon(s, \chi_v; \psi_v)^3; \end{cases}$$

[*cf.* [JPSS] Theorem (5.6) and [Ja] Theorem (16.1)]. We conclude that

$$(9.4.4) \qquad \varepsilon(s, \pi'_w \otimes \eta; \psi_w) L(1-s, \tilde{\pi}'_w \otimes \eta^{-1})/L(s, \pi'_w \otimes \eta)$$
$$= \varepsilon_2(s, \sigma_w, \eta; \psi_w) L_2(1-s, \tilde{\sigma}_w, \eta^{-1})/L_2(s, \sigma_w, \eta).$$

But now σ_w is extraordinary. Therefore

$$L_2(s, \sigma_w, \eta) = L_2(s, \tilde{\sigma}_w, \eta^{-1}) = 1$$

and the left hand side of (9.4.4) is a nomomial in q_w^{-s}. This, however, can happen only if π'_w is super cuspidal ([JPSS], § 7). Therefore

$$L(s, \pi'_w \otimes \eta) = L_2(s, \sigma_w, \eta) \quad (=1),$$

and

$$\varepsilon(s, \pi'_w \otimes \eta; \psi_w) = \varepsilon_2(s, \sigma_w, \eta; \psi_w).$$

Note that the functional equation for π' also reads

$$L(s, \tilde{\pi}' \otimes \chi) = \varepsilon(s, \tilde{\pi}' \otimes \chi^{-1}) L(1-s, \pi' \otimes \chi^{-1}),$$

so since $\pi'_v \simeq \pi_v \simeq \tilde{\pi}_v \simeq \tilde{\pi}'_v$ for $v \notin S$, we find

$$\prod_{v \in S} L(1-s, \tilde{\pi}'_v \otimes \chi_v^{-1}) \varepsilon(s, \tilde{\pi}'_v \otimes \chi_v; \psi_v)/L(s, \pi'_v \otimes \chi_v)$$
$$= \prod_{v \in S} L(1-s, \pi'_v \otimes \chi_v^{-1}) \varepsilon(s, \pi'_v \otimes \chi_v; \psi_v)/L(s, \tilde{\pi}'_v \otimes \chi_v).$$

Threfore we conclude as before that

$$L(1-s, \tilde{\pi}'_w \otimes \eta^{-1})\,\varepsilon(s, \tilde{\pi}'_w \otimes \eta; \psi_w)/L(s, \pi'_w \otimes \eta)$$

is equal to the same expression for $\tilde{\pi}'_w$. But as π'_w is trivial on the center, this implies $\pi'_w \simeq \tilde{\pi}'_w$ (*cf.* (7.5.3) of [JPSS]). Thus π'_w is indeed the lift of σ_w.

(9.5) Now that (9.3.2) is proved, for each v we let π_v denote the lift of σ_v. For $v \in S$ [S being as in (9.4)], π_v is unitary generic and we set $\xi_v = \pi_v$. For $v \notin S$ we define ξ_v as in (9.2). Note that the only places where $\xi_v \not\simeq \pi_v$ are those for which

(9.5.1) $$\sigma_v = \pi(\lambda_v \alpha^{1/4}, \lambda_v \alpha^{-1/4})$$

in which case π_v is given by (9.1.2.1) and ξ_v by (9.1.2.2). Then ξ_v has another component ξ'_v which is unitary generic, namely

(9.5.2) $$\xi'_v = \operatorname{Ind}(G_{3,v}, P_v; \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}), 1_v),$$

where $\sigma(\mu_1, \mu_2)$ is defined as in (1.10.1).

Now set $\pi = \otimes\, \pi_v$ (all v). Then:

(9.5.3) $$\begin{cases} L(s, \pi \otimes \chi) = L_2(s, \sigma, \chi), \\ \varepsilon(s, \pi \otimes \chi) = \varepsilon_2(s, \sigma, \chi), \end{cases}$$

and $L(s, \pi \otimes \chi)$ is meromorphic and satisfies the functional equation

(9.5.4) $$L(s, \pi \otimes \chi) = \varepsilon(s, \tilde{\pi} \otimes \chi)\, L(1-s, \tilde{\pi} \otimes \chi^{-1}).$$

Finally, set $T = S$ if $S \neq \emptyset$, and $T = \{ w \}$ (where w is any finite place) if $S = \emptyset$.

If χ is *sufficiently ramified at each place of* T, then we know [*cf.* (8.1)] that $L(s, \pi \otimes \chi)$ is entire and bounded in vertical strips; moreover (*cf.* (5.1) of [JPSS]),

(9.5.5) $$\varepsilon(s, \pi_v \otimes \chi_v, \psi_v) = [\varepsilon(s, \chi_v, \psi_v)]^3, \quad \text{and} \quad L(s, \pi_v \otimes \chi_v) = 1 \quad \text{for} \quad v \in T.$$

Therefore the discussion in (9.2) applies. In particular, we are in one of the cases 9.2 (ii) to 9.2 (v). What we are going to show is that 9.2 (ii) is the only possibility and this will imply the theorem.

(9.6) Suppose we are in case (iii). With the notations of (iii) we know therefore that the representations ξ_v and η_v have a common irreducible component for all $v \notin T$. Although the representations η_v, ξ_v, $\eta = \otimes\, \eta_v$, $\xi = \otimes\, \xi_v$ may fail to be irreducible, the factors $L(s, \xi_v)$, $L(s, \eta_v)$, $L(s, \xi)$, $L(s, \eta)$, $\varepsilon(s, \xi_v; \psi_v)$, $\varepsilon(s, \eta_v; \psi_v)$, $\varepsilon(s, \xi)$, and $\varepsilon(s, \eta)$ are defined. Moreover, for all character χ of $F_A^\times/F^\times$,

$$L(s, \xi \otimes \chi) = \varepsilon(s, \xi \otimes \chi)\, L(1-s, \tilde{\xi} \otimes \chi^{-1}),$$

with a similar statement for $L(s, \eta \otimes \chi)$.

Arguing as we did in (9.4) we conclude that if v is any place in T and λ is a character of $F_v^\times$, then:

$$\varepsilon(s, \xi_v \otimes \lambda; \psi_v) L(1-s, \tilde{\xi}_v \otimes \lambda^{-1})/L(s, \xi_v \otimes \lambda)$$
$$= \varepsilon(s, \eta_v \otimes \lambda; \psi_v) L(1-s, \tilde{\eta}_v \otimes \lambda^{-1})/L(s, \eta_v \otimes \lambda).$$

But ξ_v and η_v have unique irreducible generic components ξ'_v and η'_v and the previous identity is true for these pairs too. So once more we conclude that $\xi'_v \simeq \eta'_v$. In particular, *for all* v, ξ_v and η_v have a common irreducible component. Recall also that for almost all finite v, π_v is unramified and equal to the common unramified component of ξ_v and η_v. To prove that (iii) is impossible we first suppose that μ is a character.

(9.6.1) If μ is a character then τ is unitary, each representation τ_v unitary generic, and it follows that η_v is irreducible (in fact unitary generic, *cf.* [JPSS], § 6). Thus η_v is an irreducible component of ξ_v for all v (at the infinite places in the infinitesimal sense only). If, in addition, ξ_v is irreducible then $\eta_v = \xi_v = \pi_v$ and

(9.6.1.1) $$L(s, \pi_v \otimes \mu_v^{-1}) = L(s, \eta_v \otimes \mu_v^{-1}) = L(s, \tau_v \otimes \mu_v^{-1}) L(s, 1_v).$$

On the other hand, if ξ_v is not irreducible, then π_v is given by (9.1.2.1), ξ_v by (9.1.2.2) (v being finite or not), and ξ_v admits another irreducible component which is unitary generic, namely ξ'_v (9.5.2). Since η_v is unitary generic, we find

$$\eta_v = \xi'_v;$$

in particular, η_v is ramified if v is finite. Thus the set E of places where ξ_v is reducible (and $\eta_v = \xi'_v$) is *finite*, and for $v \in E$ we find

(9.6.1.2) $$L(s, \pi_v \otimes \mu_v^{-1}) = L(s, \tau_v \otimes \mu_v^{-1}) L(s, 1_v) . \frac{L(s, \pi(\alpha_v^{1/2}, \alpha_v^{-1/2}))}{L(s, \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}))}.$$

The factor on the extreme right is $(1-q_v^{1/2-s})^{-1}$ if v is finite, and $ca^s(s-(1/2))^{-1}$ if v is infinite.

In either case, it *does not vanish* for $s = 1$. Thus we find from (9.6.1.1) and (9.6.1.2) that

$$L(s, \pi \otimes \mu^{-1}) = L(s, \tau \otimes \mu^{-1}) L(s, 1) \times \prod_{v \in E} \frac{L(s, \pi(\alpha_v^{1/2}, \alpha_v^{-1/2}))}{L(s, \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}))}$$

and

$$L(s, (\sigma \otimes \mu^{-1}) \times \tilde{\sigma}) = L(s, \tau \otimes \mu^{-1}) L(s, 1) L(s, \mu^{-1}) \times \prod_{v \in E} \frac{L(s, \pi(\alpha_v^{1/2}, \alpha_v^{-1/2}))}{L(s, \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}))}.$$

But if μ is note a principal character then $L(1, \mu^{-1}) \neq 0$. Also $L(1, \tau \otimes \mu^{-1}) \neq 0$ by the main result of [J Sh]. So since the finite product above does not vanish at 1, we find that $L(s, (\sigma \otimes \mu^{-1}) \times \tilde{\sigma})$ has a pole at $s = 1$. This means $\sigma \otimes \mu^{-1} \simeq \sigma$, which is a contradiction. Thus μ cannot be a character.

(9.6.2) Now suppose $\mu = \alpha^{2t}\mu_0$ where μ_0 is a character and t is real and $\neq 0$. For almost all finite v, μ_0 is unramified and the representation τ_v is of the form $\tau_v = \pi(\alpha_v^{a_v}, \alpha_v^{b_v})$ The unramified component of η_v is therefore the representation $\pi(\alpha_v^{a_v}, \alpha_v^{b_v}, \alpha^{2t}\mu_{0,v})$ which must (for almost all v) coincide with the unramified component of ξ_v which is π_v. Now π_v has the form $\pi(\alpha_v^{p_v}, 1, \alpha_v^{-p_v},)$ and so the triples $(\alpha_v^{a_v}, \alpha_v^{b_v}, \alpha_v^{c_v})$ and $(\alpha_v^{p_v}, 1, \alpha_v^{-p_v})$ must be the same (up to order).

From the discussion of paragraph 3.3 and 3.4 it follows that σ_v must be in the complementary series for almost all finite v and, after a change of notations,

$$a_v = -2t, \qquad b_v = 0, \qquad \mu_{0,v} = 1, \qquad 0 < |t| < \frac{1}{2}.$$

Thus $\tau_v = \pi(\alpha_v^{-2t}, 1)$ for almost all v. This situation, however, is impossible. Indeed it implies the existence of a unitary cuspidal automorphic representation ρ of $G_2(A)$ with $\rho_v = \pi(\alpha_v^t, \alpha_v^{-t})$, $0 < t < (1/2)$, for all v not in some finite set of places E. This in turn implies [by (9.7)] that

$$L(s, \rho\times\tilde{\rho}) = L(s+t, 1)L(s-t, 1)L(s, 1)^2 \times \prod_{v\in E'} \frac{L(s, \rho_v\times\tilde{\rho}_v)}{L(s+t, 1_v)L(s-t, 1_v)L(s, 1_v)^2}.$$

But at $s = t$, $L(s+t, 1)$ and $L(s, 1)$ do not vanish. Neither does the finite product, as is easily checked. Thus $L(s, \rho\times\tilde{\rho})$ has a pole at $s = t \neq 1$, an obvious contradiction.

In summary, possibility (iii) cannot occur; similar arguments show that (iv) too is impossible.

(9.7) Suppose finally we are on case (v). Using the notations of (v) we see (just as before) that η_v and ξ_v have a common irreducible component for all v.

(9.7.1) Suppose first that the ν_i are characters. Then for each v the representation η_v is irreducible unitary generic ([JPSS], § 6 and 10). Thus it is a component of ξ_v. If ξ_v is irreducible we find $\xi_v = \pi_v = \eta_v$ and

(9.7.1.1) $$L(s, \pi_v\otimes\chi_v) = L(s, \eta_v\otimes\chi_v) = \prod_i L(s, \nu_{i,v}\chi_v).$$

If ξ_v is not irreducible then, since π_v is the lift of σ_v, π_v is given by (9.1.2.1) and ξ_v by (9.1.2.2). Since π_v is not unitary generic, η_v must be the other component of ξ_v [*cf.* (9.5.2)] i. e.:

$$\eta_v = \mathrm{Ind}(G_{3,v}, P_v; \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}), 1_v).$$

On the other hand,

$$\eta_v = \mathrm{Ind}(G_{3,v}, B_{3,v}; \nu_{1,v}, \nu_{2,v}, \nu_{3,v})$$

and—except in the complex case—these two equalities are incompatible.

Thus we let E denote the set of (complex) places where this happens. For $v \in E$,

$$(9.7.1.2)\quad L(s, \pi_v \otimes \chi_v) = L(s, \xi_v \otimes \chi_v) = \prod_{i=1}^{3} L(s, \nu_{i,v}\chi_v) \cdot \frac{L(s+1/2, \chi_v) L(s, \chi_v) L(s-1/2, \chi_v)}{\prod L(s, \nu_{i,v}\chi_v)}.$$

Moreover we then have (up to order):

$$\nu_{1,v}(z) = z(z\bar{z})^{-1/2}, \qquad \nu_{2,v} = 1, \qquad \nu_{3,v} = \bar{z}(z\bar{z})^{-1/2}$$

and

$$\sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}) = \pi(\nu_{1,v}, \nu_{3,v}).$$

From (9.7.1.1) and (9.7.1.2) we find that

$$L(s, (\sigma \otimes \chi) \times \tilde{\sigma}) = \prod_{1=1}^{3} L(s, \nu_i \chi) L(s, \chi) \times \prod_{v \in E'} \frac{L(s+1/2, \chi_v) L(s, \chi_v) L(s-1/2, \chi_v)}{\prod L(s, \nu_{i,v}\chi_v)}.$$

Now set $\chi = \nu_1^{-1}$. Then $L(s, \nu_1 \chi)$ has a pole at $s = 1$ which cannot be compensated for by a zero of the other L-factors (nor by a zero of the finite product, as is easily checked). Thus we get a pole for $L(s, (\sigma \otimes \nu_1^{-1}) \times \tilde{\sigma})$ at $s = 1$. But if $\nu_1 \neq 1$ this contradicts our hypothesis (since it implies $\sigma \simeq \sigma \otimes \nu_1^{-1}$). On the other hand, if $\nu_1 = 1$, we get a pole of order at least two, and this is also a contradiction.

(9.7.2) Now suppose the ν_i are not all characters. Using again the fact that ξ_v and η_v have (for almost all v) the same unramified component (namely π_v) we conclude as before that σ_v is in the complementary series for almost all v and that (changing notation if necessary):

$$\nu_1 = \alpha^{2t} \nu'_1, \qquad \nu_3 = \alpha^{-2t} \nu'_3, \qquad \nu_2 = \nu'_2,$$

where ν'_i is a character and $t \neq 0$ is real. In fact we even find $\nu'_{i,v} = 1$ for almost all v so that $\nu_1 = \alpha^{2t}$, $\nu_3 = \alpha^{-2t}$, $\nu_2 = 1$. Moreover $0 < t < (1/2)$.

Suppose first that $t \neq (1/4)$. Then:

$$\eta_v = \mathrm{Ind}(G_{3,v}, B_{3,v}, \alpha_v^{2t}, 1, \alpha_v^{-2t})$$

is irreducible for all v and is a component of ξ_v, for all v. If ξ_v is also irreducible then $\xi_v = \pi_v = \eta_v$. On the other hand, if ξ_v is not irreducible then ξ_v has (as seen before) 2 components; one is π_v, the other

$$\xi'_v = \mathrm{Ind}(G_{3,v}, P_v; \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}), t_v).$$

But ξ'_v does not contain the trivial representation of $K_{3,v}$ and η_v does. Thus $\pi_v = \eta_v$ again.

Summing up, if $t \neq 1/4$, then for all v,

$$L(s, \pi_v) = L(s, \eta_v) = L(s+2t, 1_v)L(s, 1_v)L(s-2t, 1_v)$$

and

$$L(s, \sigma\times\tilde{\sigma}) = L(s, \pi)L(s, 1) = L(s+2t, 1)L(s-2t, 1)L(s, 1)^2.$$

Again the pole of $L(s-2t, 1)$ at $2t$ is not cancelled by a zero and we conclude $L(s, \sigma\times\tilde{\sigma})$ has a pole at $s = 2t \neq 1$, a contradiction.

Now suppose $t = 1/4$. Then η_v has always two irreducible components:

$$\eta'_v = \operatorname{Ind}(G_{3,v}, P_v; 1_{2,v}, 1_v),$$
$$\eta''_v = \operatorname{Ind}(G_{3,v}, P_v; \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}), 1_v).$$

One of these is always a component of ξ_v. Since η'_v is unramified if v is finite, we find $\eta'_v = \pi_v$ for almost all v. If ξ_v is irreducible then either η'_v or η''_v is a component of ξ_v. If ξ_v is reducible then $\xi_v = \eta_v$ and $\pi_v = \eta'_v$. In conclusion, for almost all v, $\eta'_v = \pi_v$ and

$$L(s, \pi_v) = L(s, \eta'_v) = L\left(s+\frac{1}{2}, 1_v\right)L(s, 1_v)L\left(s-\frac{1}{2}, 1_v\right).$$

However, at a finite set of places (say E) we may have $\eta''_v = \pi_v$ and

$$L(s, \pi_v) = L\left(s+\frac{1}{2}, 1_v\right)L(s, 1_v)L\left(s-\frac{1}{2}, 1_v\right)\frac{L(s, \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}))}{L(s, 1_{2,v})}.$$

So

$$L(s, \sigma\times\tilde{\sigma}) = L\left(s+\frac{1}{2}, 1\right)L\left(s-\frac{1}{2}, 1\right)L(s, 1)^2\prod_{v\in E}\frac{L(s, \sigma(\alpha_v^{1/2}, \alpha_v^{-1/2}))}{L(s, 1_{2,v})}.$$

Again the pole of $L(s-(1/2), 1)$ at $s = 3/2$ cannot be cancelled by a zero of the other L-factors or the finite product and we again get a contradiction.

(9.8) We have now shown that the only possibility is (ii). Thus $\pi_v = \xi_v$ is unitary generic for any $v \notin T$. If $S \neq \emptyset$, $S = T$ and the same conclusion is true for $v \in S$ [*cf.* (3.41]). On the other hand, if $s = \emptyset$, then $T = \{w\}$. Since w is really arbitrary the same conclusion is true for all w. Thus $\pi_v = \xi_v$ is unitary generic *for all v*.

We have seen in [9.2 (ii)] that there is a cuspidal automorphic representation π' with trivial central character and $\pi'_v \simeq \pi_v$ for $v \notin S$. Now for any χ:

(9.8.1) $$L(s, \pi'\otimes\chi) = \varepsilon(s, \pi'\otimes\chi)L(1-s, \tilde{\pi}'\otimes\chi^{-1}).$$

Similarly $L(s, \pi\otimes\chi)$ satisfies (9.5.4). So using an argument used before, these two equations imply that for any $v \in T$ and any character λ of $F_v^\times$,

$$L(1-s, \tilde{\pi}_v\otimes\lambda^{-1})\varepsilon(s, \pi_v\otimes\chi; \psi_v)/L(s, \pi_v\otimes\lambda)$$
$$= L(1-s, \tilde{\pi}'_v\otimes\lambda^{-1})\varepsilon(s, \pi'_v\otimes\lambda; \psi_v)/L(s, \pi'_v\otimes\lambda).$$

However, since both π_v and π'_v are generic, this implies $\pi_v \simeq \pi'_v$. Thus $\pi = \otimes \pi_v$ is automorphic cupsidal and we have proved part (3) of Theorem (9.3). But part 1 follows from the identity $L_2(s, \sigma, \chi) = L_2(s, \pi \otimes \chi)$, and since the lift π_v of σ_v is always unitary generic, the last assertion of the Theorem also follows.

Remark (9.9). — If $\sigma \simeq \sigma \otimes \eta$, with $\eta \neq 1$, then $\eta^2 = 1$. Let H be the corresponding quadratic extension of F. Recall that there is a character Ω of $H_A^\times/H^\times$ such that σ is the automorphic representation $\pi(\sigma_\Omega)$ of $G_2(A)$ attached to Ω. Let Ω' be the conjugate of Ω. Then the lift π of σ is

$$\pi = \mathrm{Ind}(G_3(A), P(A); \pi(\sigma_{\Omega\Omega'^{-1}}), \eta)$$

and

$$L(s, \pi \otimes \chi) = L_2(s, \sigma, \chi) = L(s, \Omega\Omega'^{-1}. \chi \circ N_{H/F}) L(s, \eta\chi).$$

Recall that π is automorphic. On the other hand, $L(s, \pi \otimes \chi)$ has some pole (for $\chi = \eta$ for instance), so π is *not* cuspidal.

Concluding Remark. — We wish to comment finally on the classical significance of part (4) of Theorem 9.3. Suppose

$$f(z) = \prod_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is a cusp form of weight k on $\Gamma_0(N)$ (which is not a theta-series with grossencharakter à la Hecke-Maass). By the Ramanujan-Petersson conjecture recently established by Deligne we know that

$$(\star) \qquad a_n = 0(n^{k/2-1/2+\varepsilon}).$$

On the other hand, if $F = \mathbf{Q}$ and π corresponds to the form f above, then part (4) of Theorem (9.3) implies that

$$(\star\star) \qquad a_n = O(n^{k/2-1/4+\varepsilon}).$$

This estimate is stronger than the one made by Rankin in his well-known work [Ra] but obviously weaker than ($\star$). According to [Se] ($\star\star$) also results from Weil's work on exponential sums (*cf.* [W 2]). For number fields, however, our result seems to be new. Indeed the generalized Ramanujan-Petersson conjecture—though established for function fields now by Drinfeld—has not yet been proven in general.

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S. GELBART,
Cornell University,
Ithaca, New York 14853,
U.S.A.