ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 11, nº 1 (1978), p. 71-92
<http://www.numdam.org/item?id=ASENS_1978_4_11_1_71_0>

Ann. scient. Éc. Norm. Sup.,
4ᵉ série, t. 11, 1978, p. 71 à 92.

CURVATURE OPERATORS : PINCHING ESTIMATES AND GEOMETRIC EXAMPLES [1]

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1. Summary of results

1.1. The most frequently used and best motivated curvature assumptions in qualitative Riemannian geometry are *bounds on the sectional curvatures* of the Riemann tensor R. Associated with R is the so-called *curvature operator* $\hat{R} : \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$ ($T_p M = V$ henceforth). The symmetries of R (not including the Bianchi identity) imply that $\hat{R}$ is a symmetric operator. Special assumptions on $\hat{R}$ also have geometric consequences, for example:

(i) a compact, connected and oriented Riemannian manifold with positive curvature operator $\hat{R}$ has the rational homology of the sphere (this is a nice theorem of D. Meyer proved in [4]);

(ii) the positivity of $\hat{R}$ implies that the Gauss-Bonnet integrand is positive (*cf.* [12]).

In both cases positive sectional curvature is known not to be sufficient (*cf.* [5] and [10] for (ii)).

1.2. The linear map $\hat{R}$ has more geometric invariants than just its spectrum, since the *geometrically relevant* orthogonal group which acts on all occurring tensor spaces is O (V). For example the Bianchi identity for $\hat{R}$ does not make sense for the action of O $(\Lambda^2 V)$. Further O (V)-invariants are connected with the rank of the eigenvectors. Notice then the following geometric examples:

(i) the eigenvector associated with the largest eigenvalue of $\hat{R}$ for the standard complex projective space is the Kähler form;

[1] This article has been written under the program SFB 40 "Theoretische Mathematik" at the University of Bonn (F.R.G.).

(ii) if one has a basis of eigenvectors which are forms of rank two, then the Pontrjagin forms vanish (for further consequences of this assumption, *see* [13]).

The Bianchi identity is crucial for these further invariants.

1.3. The curvature tensor also acts on *symmetric two-tensors* (this fact was used in [3] for example, but does not seem to have been studied systematically). Again the symmetries of R, not including the Bianchi identity, imply that $\mathring{R}$ is a symmetric map of $S^2 V$ into itself. The new feature is that $S^2 V$ is not irreducible under the action of O (V).

In this paper we are concerned with both $\hat{R}$ and $\mathring{R}$ since, when one relates various second order differential operators, they are the only curvature actions which occur.

We aim at results which from sectional curvature bounds give information about the spectral and other invariants of $\hat{R}$ and $\mathring{R}$. As far as possible we develop algebraic and geometric examples which illustrate the sharpness of our estimates and the use of curvature operators.

1.4. Also, every curvature tensor admits a decomposition into O (V)-irreducible components, namely its *scalar*, *Ricci-traceless* and *Weyl* components. In some geometric applications not all components are relevant, for example:

(i) the Pontrjagin forms depend only on the Weyl component;

(ii) on an Einstein manifold the Ricci-traceless component vanishes and the Weyl component is harmonic as a vector-valued two-form (*see* [2]).

Therefore we extend the estimates to the operators deduced from the various irreducible components of the curvature tensor.

1.5. The principal results are the following (throughout m denotes the dimension of V):

PROPOSITION 3.3. — *Let $\hat{r}_{\min}$ and $\hat{r}_{\max}$ be respectively the smallest and the largest eigenvalues of $\hat{R}$. Then the eigenvalues of $\hat{R}-1/2\,\rho\,\hat{\wedge}\,g$ lie in the interval $[-(m-2)\,\hat{r}_{\max}, -(m-2)\,\hat{r}_{\min}]$ (the bounds are achieved in $\mathbf{C}\,P^n$,* see 5.3). *In particular if $\hat{R}$ is a positive operator, then $\hat{R}-1/2\,\rho\,\hat{\wedge}\,g$ is negative (here ρ is the Ricci curvature and $\wedge$ is defined in 2.2).*

This is precisely what was needed to prove 1.1 (i).

PROPOSITION 3.8. — *If the sectional curvature $\overline{R}$ of R satisfies $\delta \leqq \overline{R} \leqq \Delta$ and if an eigenvector of $\hat{R}$ for the eigenvalue $\hat{r}$ has rank $2k$, then*

$$(\delta+\Delta)-((4k-1)/3)(\Delta-\delta) \leqq \hat{r} \leqq (\delta+\Delta)+((4k-1)/3)(\Delta-\delta).$$

In general the eigenvalues of $\hat{R}$ lie in the interval $[(\delta+\Delta)-((4\,[m/2]-1)/3)\,(\Delta-\delta), (\delta+\Delta)+((4\,[m/2]-1)/3)\,(\Delta-\delta)]$. Moreover the last upper bound is achieved on $\mathbf{C}\,P^n$, see 5.2.

PROPOSITION 4.3. — *If $\delta \leqq \overline{R} \leqq \Delta$, then all eigenvalues of $\mathring{R}$ but one lie in the interval $[-(1/2)\,((\delta+\Delta)+(m-1)\,(\Delta-\delta)), -(1/2)\,((\delta+\Delta)-(m-1)\,(\Delta-\delta))]$, the other lying in the interval $[(m-1)\,\delta, (m-1)\,\Delta]$. The first upper bound is sharp for $\mathbf{C}\,P^2$ and the last lower bound almost sharp for $\mathbf{C}\,P^n$,* see 5.3.

PROPOSITION 7.5. — *The curvature tensor* $R_\alpha = R_{S^6} - \alpha R_{\mathbf{C}P^3} + 3\alpha R_{G^{2,5}}$ *for* $\alpha \geqq 0$ (*where* $G^{2,5}$ *is the Grassmann manifold of two-planes in* $\mathbf{R}^5$) *has the sectional curvature range* $[1-\alpha, 1+2\alpha]$. *The Euler integrand* $E(R_\alpha)$ *is given by* $E(R_\alpha) = 18(5+3\alpha+12\alpha^2-28\alpha^3)$ *and is negative for* $\alpha \geqq \alpha_0 = 0.8227$. *The pinching at* α_0 *is* $1/15$ (*hence new counter-examples to the algebraic Hopf conjecture*).

1.6. The paper splits into two parts. Part One is purely algebraic and is divided into three sections (§ 2 to 4): in section 2 we fix up our notations and conventions and we study how a curvature tensor R acts on tensors. We establish our pinching estimates for $\hat{R}$ in section 3 and the ones for $\mathring{R}$ in section 4.

Parts Two presents geometric situations which are enlightened by the consideration of curvature operators. On the way we find global examples where our pinching estimates are sharp. Part two is divided into three sections (§ 5 to 7): in section 5 we detail the case of the projective spaces and their duals. The distance spheres in these spaces turn out to be easily describable in terms of curvature operators and to provide nice global examples of sharpness of our estimates: they are developed in section 6. We end up in section 7 by studying the Euler integrand in our formalism, which provides insight in Geroch's counterexample to the algebraic Hopf conjecture.

1.7. Some of our conventions are not the standard ones for various tensor operations, but they have been chosen to minimize the occurrence of denominators in the formulas.

PART ONE: PINCHING ESTIMATES

2. Operations of a curvature tensor on tensors

2.1. Let (V, g) be a Euclidean vector space of dimension m. *Via* the metric g we always identify V and V*. We denote the (once) *contraction* by γ, the *composition of linear maps* by $\circ$ and their *restriction* by $\restriction$. The extension of the scalar product to tensor spaces is still *denoted by* g or $(\ , \ )$. Throughout the discussion (e_i) is an orthonormal basis of V.

Our conventions on exterior products and symmetric products are, for a and b in V,

$$a \wedge b = a \otimes b - b \otimes a, \qquad a \bigcirc b = a \otimes b + b \otimes a.$$

The extension to higher order tensors is made in order to preserve associativity.

2.2. We will be interested in the subspace $\mathscr{R}$ of $\otimes^4 V$ consisting of *curvature tensors*, i. e. tensors satisfying the following identities: for a, b, c, d in V

$$R(a, b, c, d) = -R(b, a, c, d) = -R(a, b, d, c) = R(c, d, a, b) \tag{2.3}$$

and also the *Bianchi identity:*

$$R(a, b, c, d) + R(b, c, a. d) + R(c, a, b, d) = 0.$$

Simple examples of curvature tensors which turn out to be important are the following: let s and t be two-tensors, we *define* $s \wedge t$ to be the four-tensor whose value on a, b, c, d is

$$(s \wedge t)(a, b, c, d) = s(a, c)\,t(b, d) + s(b, d)\,t(a, c) - s(a, d)\,t(b, c) - s(b, c)\,t(a, d).$$

If s and t are both symmetric, then $s \wedge t$ is a curvature tensor (*cf.* [12]). In particular $g \wedge g$ is the curvature tensor of a Riemannian manifold (M, g) with constant sectional curvature $+2$. This operation can be used to make $\bigoplus_{k=1}^{m} S^2 \Lambda^k V$ into a commutative algebra (*cf.* [12] again).

For a given curvature tensor R we define symmetric actions on $\Lambda^2 V$ and on $S^2 V$ as follows:

(i) for ω in $\Lambda^2 V$ put

$$\hat{R}(\omega)(a, b) = \sum_{i, j=1}^{m} R(e_i, e_j, a, b)\,\omega(e_i, e_j),$$

so that $(s \mathbin{\hat{\wedge}} t)(\omega) = 2(t \circ \omega \circ s^* + s \circ \omega \circ t^*)$ where $*$ denotes the adjoint of a map; in particular

$$g \mathbin{\hat{\wedge}} g = 4\,\mathrm{Id}_{\Lambda^2 V}.$$

From (2.3) one easily sees that the map $\hat{}$ imbeds $\mathscr{R}$ into $S^2 \Lambda^2 V$;

(ii) for h in $S^2 V$, put

$$\mathring{R}(h)(a, b) = \sum_{i, j=1}^{m} R(e_i, a, e_j, b)\,h(e_i, e_j),$$

so that $(s \mathbin{\mathring{\wedge}} t)(h) = \operatorname{trace}(s \circ h)\,t + \operatorname{trace}(t \circ h)\,s - t \circ h \circ s - s \circ h \circ t$.

We also have $\mathring{R}(g) = \gamma(R) = \rho$, the Ricci tensor.

Again from (2.3) follows that the map $\mathring{}$ sends $\mathscr{R}$ into $S^2 S^2 V$. The Bianchi identity implies that $\mathring{}$ is an embedding.

2.4. We now consider the various ways in which the curvature tensor can operate on tensors. Since we want these actions to be O (V)-invariant, by H. Weyl's theorem (*cf.* [16], p. 64), they must be expressible by means of contractions.

On the scalars the curvature can only operate *via its scalar curvature* $u = \gamma(\rho)$, since there is, up to sign, only one nontrivial way of contracting twice a curvature tensor.

On V [on which O (V) acts irreducibly] two operations appear: one is *via the Ricci curvature* ρ, the other is diagonal *via* $u.g$ (here we again use that, up to sign, there is only one nontrivial way of contracting R).

In $\otimes^2 V$ appears a new phenomenon: the group O (V) does not act irreducibly. The space $\otimes^2 V$ splits into $\mathbf{R}.g \oplus S_0^2 V \oplus \Lambda^2 V$ (where $S_0^2 V$ *denotes* the space of traceless symmetric two-tensors). If we restrict ourselves to contractions between the curvature tensor R and an element t of $\otimes^2 V$, because of the symmetries of R, only two possibilities are left: either the contraction takes place on the first indices of R and t and the second indices of R and t, i. e., for a, b in V, we set

$$R^{1,2}(t)(a, b) = \sum_{i,j=1}^{m} R(e_i, e_j, a, b)\, t(e_i, e_j),$$

or the contraction takes place on the first indices of R and t and on the third index of R and the second of t, i. e., for a, b in V, we have

$$R^{1,3}(t)(a, b) = \sum_{i,j=1}^{m} R(e_i, a, e_j, b)\, t(e_i, e_j).$$

From the skewsymmetry of R in its two first arguments follows that $S^2 V$ lies in the kernel of $R^{1,2}$ and that $R^{1,2} \upharpoonright \Lambda^2 V = \hat{R}$. If we decompose orthogonally $t = \omega + h$ with ω in $\Lambda^2 V$ and h in $S^2 V$, using the Bianchi identity we get

$$\begin{aligned} R^{1,3}(t)(a, b) = -\sum_{i,j=1}^{m} [&(R(a, e_j, e_i, b) + R(e_i, e_j, a, b))\,\omega(e_i, e_j) \\ &- R(e_i, a, e_j, b)\, h(e_i, e_j)], \end{aligned}$$

so that using the symmetries of R and ω we obtain $R^{1,3} \upharpoonright S^2 V = \mathring{R}$ together with $R^{1,3} \upharpoonright \Lambda^2 V = -1/2\, \hat{R}$.

2.5. For the action of $\mathscr{R}$ on higher order tensors we will use the geometry as leading path: we want to apply our estimates mainly to integrands in formulas relating various second order differential operators naturally defined in terms of the Riemannian metric. These operators will be assumed to operate on a space of tensor fields of a fixed type: as examples think of the various Laplace operators that one can define on the space of exterior differential k-forms by considering, for $0 \leqq l \leqq k$, an exterior differential k-form as an exterior differential l-form with values in the bundle of $(k-l)$-forms.

The curvature appears there as the skewsymmetric part of the double covariant derivative: as such, the curvature operates as a derivation and is contracted once with the tensor field under consideration. If we want to get a tensor field of the same type as the given one, one has to contract once more. Then only three possibilities are left: either the second contraction is taken on the tensor field itself (in this case the curvature operates as a derivation on a lower order tensor field), or the contraction is taken on the derivation symbols themselves (in this case the skewsymmetric contribution drops out: the influence of the curvature is then cancelled) or the contraction is taken on one derivation symbol and the tensor field (in this case the curvature acts *via* its actions on two-tensor fields).

To summarize this discussion let us say that in formulas relating second order differential operators acting on a space of tensor fields of a fixed type, the curvature operates as endomorphisms of this space of tensors necessarily *via* $\hat{R}$ or $\mathring{R}$ (for illustrations, *see* [2]).

2.6. If we concentrate on O (V)-invariant actions one has also to decompose the space $\mathscr{R}$ of curvature tensors into O (V)-irreducible subspaces. Using H. Weyl's invariant theory one proves (*cf.* [12] for example) that R splits into $R = U+Z+W$ where, if

$$\rho = \gamma(R) = (u/m)\, g + z,$$

we set

$$U = (u/2\,m(m-1))\, g \wedge g, \qquad Z = 1/(m-2)\ z \wedge g.$$

The various components U, Z and W are successively refered to as the *scalar*, the *Ricci-traceless* and the *Weyl* parts of R.

The orthogonal complement of $\mathscr{R}$ imbedded by $\hat{}$ in $S^2 \Lambda^2 V$ is $\Lambda^4 V$, the Bianchi identity expressing that an element of $\mathscr{R}$ lies in the kernel of the orthogonal projector β of $S^2 \Lambda^2 V$ onto $\Lambda^4 V$, where, for A in $S^2 \Lambda^2 V$ and a, b, c, d in V,

$$3\,\beta(A)(a, b, c, d) = A(a, b, c, d) + A(b, c, a, d) + A(c, a, b, d).$$

Notice that trace $\hat{R}$ = trace $\hat{U} = u$.

2.7. We summarize our notations on curvature tensors: $\hat{R}$ denotes R operating on skew-forms, $\mathring{R}$ on symmetric forms and $\overline{R}$ the sectional curvature viewed as a function on the Grassmann manifold of two-planes.

3. Pinching estimates on $\hat{R}$

3.0. The Bianchi identity will play a decisive but subtle role in this section. We start by clarifying its influence on the respective size of the extreme eigenvalues of $\hat{R}$ (this is just what was needed in [4]), then proceed to estimating components of the tensor R and eigenvalues of $\hat{R}$ in terms of bounds of the sectional curvature. We finally deduce the estimates on the irreducible components of $\hat{R}$.

3.1. The simplest elements in $S^2 \Lambda^2 V$ are projectors on lines. If ω in $\Lambda^2 V$ has length one, then $\omega \otimes \omega$ is the projection onto $\mathbf{R}.\omega$ and since $24\,\beta(\omega \otimes \omega) = \omega \wedge \omega$, we see that $\omega \otimes \omega$ is a curvature tensor (i. e. satisfies the Bianchi identity) if and only if ω is decomposable. Therefore if R is a curvature tensor and $\hat{R} = \sum_\nu r_\nu\, \omega_\nu \otimes \omega_\nu$ is the spectral decomposition of $\hat{R}$, then its eigenvectors ω_ν and eigenvalues r_ν satisfy the relation

$$\sum_\nu \hat{r}_\nu\, \omega_\nu \wedge \omega_\nu = 0. \tag{3.2}$$

To ω in $\Lambda^2 V$ associate the element $\omega \wedge \omega$ of $S^2 \Lambda^2 V$. Since $6\,\beta(\omega \wedge \omega) = -\omega \wedge \omega$,

$$24\,\omega \otimes \omega = 4(4\,\omega \otimes \omega + \omega \wedge \omega) - \omega \wedge \omega,$$

is the decomposition of $24\,\omega \otimes \omega$ into a curvature tensor and a four-form. If ω is the complex structure of a complex vector space, then $4\,\omega \otimes \omega + \omega \wedge \omega$ is the nondiagonal part of the curvature tensor of complex projective space.

The proof of the next Proposition is inspired by some work of H. Maillot (for generalizations to p-forms and geometric applications, *see* [4]):

3.3. PROPOSITION. — *Let $\hat{r}_{\min}$ and $\hat{r}_{\max}$ be respectively the smallest and the largest eigenvalues of $\hat{R}$. Then the eigenvalues of $\hat{R} - 1/2\,\rho \,\hat{\wedge}\, g$ lie in the interval $[-(m-2)\,\hat{r}_{\max}, -(m-2)\,\hat{r}_{\min}]$ (the bounds are achieved on* $\mathbf{C}\,P^n$; see 5.3).

In particular if $\hat{R}$ is a positive operator, then $\hat{R} - 1/2\,\rho \,\hat{\wedge}\, g$ is a negative operator.

Proof. — Notice first that for ω in Λ^2 V and A in $\otimes^2 \Lambda^2$ V, we have

$$\begin{aligned}(\mathrm{A}, g \wedge \omega) &= \sum_{i,j,k,l=1}^{m} \mathrm{A}(e_i, e_j, e_k, e_l) \\ &\quad \times (g(e_i, e_k)\,\omega(e_j, e_l) + g(e_j, e_l)\,\omega(e_i, e_k) - g(e_i, e_l)\,\omega(e_j, e_k) - g(e_j, e_k)\,\omega(e_i, e_l)) \\ &= 4 \sum_{i,j,k,l=1}^{m} \omega(e_i, e_j)\,\mathrm{A}(e_k, e_i, e_k, e_j) \\ &= 4\ (\omega, \gamma^{1,3}(\mathrm{A})).\end{aligned}$$

In $(\omega, \gamma^{1,3}(\mathrm{A}))$ only the skewsymmetric part of $\gamma^{1,3}(\mathrm{A})$ contributes: we *denote it* by $\gamma^{sk}(\mathrm{A})$.

We now consider the sequence of maps

$$\Lambda^2 \mathrm{V} \xrightarrow{g\wedge} \otimes^2 \Lambda^2 \mathrm{V} \xrightarrow{(\hat{\mathrm{R}})\circ} \otimes^2 \Lambda^2 \mathrm{V} \xrightarrow{\gamma^{sk}} \Lambda^2 \mathrm{V}.$$

For any nonzero element ω in Λ^2 V, we have

$$(\gamma^{sk}(\hat{\mathrm{R}} \circ (g \hat{\wedge} \omega)), \omega) = 2(\hat{\mathrm{R}} \circ (g \hat{\wedge} \omega), g \wedge \omega). \tag{3.4}$$

We first consider the case where $\hat{\mathrm{R}}$ is a positive operator (i. e. $\hat{r}_{\min} > 0$). Then the right hand side of (3.4) is positive since it is trace $(\hat{\mathrm{R}} \circ (g \hat{\wedge} \omega)^2)$.

On the other hand a straightforward computation using the Bianchi identity (compare [4], p. 264) gives

$$\gamma^{sk}(\hat{\mathrm{R}} \circ (g \hat{\wedge} \omega)) = (\rho \hat{\wedge} g - 2\hat{\mathrm{R}})(\omega).$$

According to (3.4) by taking the scalar product with ω, we get

$$(\hat{\mathrm{R}} \circ (g \hat{\wedge} \omega), g \wedge \omega) = -((\hat{\mathrm{R}} - 1/2\, \rho \hat{\wedge} g)(\omega), \omega)$$

which proves that $\hat{\mathrm{R}} - 1/2\, \rho \hat{\wedge} g$ is negative when $\hat{r}_{\min} > 0$.

In the general case $\hat{\mathrm{R}} - 1/4\, \hat{r}_{\min}\, g \hat{\wedge} g$ is positive and has $\rho - \hat{r}_{\min}(1/2)(m-1)\, g$ as Ricci curvature. A direct check gives the expected bound. ■

3.5. Corollary. — *Let* W *be a Weyl tensor,* $\hat{w}_{\min}$ *and* $\hat{w}_{\max}$ *be respectively the smalles and largest eigenvalues of* $\hat{\mathrm{W}}$. *Then we have*

$$-(m-2)\,\hat{w}_{\max} \leqq \hat{w}_{\min} \leqq 0 \leqq \hat{w}_{\max} \leqq -(m-2)\,\hat{w}_{\min}.$$

The last estimate is sharp for **C** $\mathbf{P}^n$, see 5.2.

3.6. If R is a curvature tensor its *sectional curvature* function is denoted by $\overline{\mathrm{R}}$. Moreover if $\delta \leqq \overline{\mathrm{R}} \leqq \Delta$, it is very convenient to center R by setting

$$\mathrm{R}_0 = \mathrm{R} - (\Delta + \delta)/4\ g \wedge g.$$

In particular $|\mathrm{R}_0| \leqq (\Delta - \delta)/2$.

In the rest of the paragraph we will suppose that we are in this situation. For all proofs one can assume that Δ and δ are sharp bounds and allow larger bounds only in the final statements.

If all eigenvalues of $\hat{R}$ lie in $[\hat{r}_{\min}, \hat{r}_{\max}]$, then the sectional curvature $\overline{R}$ satisfies $1/2\, \hat{r}_{\min} \leqq \overline{R} \leqq 1/2\, \hat{r}_{\max}$ (the factor 1/2 comes from our conventions). The bounds are achieved if a corresponding eigenvector is a decomposable form, this occurs in particular for hypersurfaces in spaces of constant curvature.

The next Lemma is essential for the proof of Propositions 3.8 and 3.16. Notice that the Bianchi identity will be used only here.

3.7. LEMMA. — *For unit vectors a, b, c, d in* V, *the following estimates hold:*

$$|R_0(a, b, c, d)| \leqq 2(\Delta-\delta)/3, \qquad |R(a, b, c, d)| \leqq \operatorname{Max}(\Delta, -\delta, 2(\Delta-\delta)/3)$$

and the estimates are sharp for $\mathbf{C}\,P^n$, see 5.1.

Proof. — By using the Bianchi identity one proves that, for a, b, c, d in V,

$$\begin{aligned} 6R(a, b, c, d) = {} & R(a, b+c, b+c, d) - R(b, a+c, a+c, d) \\ & - R(a, b-c, b-c, d) + R(b, a-c, a-c, d). \end{aligned}$$

Each term on the right hand side is the difference of two sectional curvatures up to scalar factors like $(b+c, b+c)$.

Applied to R_0 for which we know that $|\overline{R}_0| \leqq (\Delta-\delta)/2$, we get

$$6|R(a, b, c, d)| \leqq ((b+c, b+c)+(a+c, a+c)+(a-c, a-c)+(b-c, b-c))(\Delta-\delta).$$

Hence, if a, b, c and d are unit vectors, we have

$$|R_0(a, b, c, d)| \leqq 2(\Delta-\delta)/3$$

and

$$|R(a, b, c, d)| \leqq 4/3 \max(\Delta, -\delta).$$

To improve the estimate on $|R(a, b, c, d)|$ we first choose unit vectors b′ and c' in the planes $P = \{a, b\}$ and $Q = \{c, d\}$ such that (b', c') is maximal. We then choose unit vectors a' and d' such that $(a', b') = 0$, $(c', d') = 0$ and $(a', d') \geqq 0$. The special choice of b' and c' forces a' to be orthogonal to c'. *Setting* $\cos x = (b', c')$ and $\cos y = (a', d')$ we have

$$R(a, b, c, d) = R(a', b', c', d') = R_0(a', b', c', d') - (\delta+\Delta)/2 \ \cos x \cos y.$$

But for any real numbers λ and μ we have

$$\begin{aligned} 6R_0(a', b', c', d') &= 6R_0(\lambda a', \mu^{-1} b', \mu c', \lambda^{-1} d') \\ &= R_0(a', b'+c', b'+c', d') - R_0(a', b'-c', b'-c', d') \\ &\quad - R_0(\mu^{-1} b', \lambda a'+\mu c', \lambda a'+\mu c', \lambda^{-1} d') \\ &\quad + R_0(\mu^{-1} b', \lambda a'-\mu c', \lambda a'-\mu c', \lambda^{-1} d'). \end{aligned}$$

The first two terms together are again bounded by $2(\Delta-\delta)$.

The arguments in the alternating pairs of the last two terms are not orthogonal, therefore choosing $\lambda^2 = \sin x$ and $\mu^2 = \sin y$ improves their estimate to $(\Delta-\delta)(1+\sin x \sin y)$. ■

3.8. PROPOSITION. — *If $\delta \leqq \overline{R} \leqq \Delta$ and if an eigenvector of $\hat{R}$ for the eigenvalue $\hat{r}$ has rank $2k$, then*

$$(\Delta+\delta)-((4k-1)/3)(\Delta-\delta) \leqq \hat{r} \leqq (\Delta+\delta)+((4k-1)/3)(\Delta-\delta).$$

In general the eigenvalues of $\hat{R}$ lie in the interval $[(\delta+\Delta)-((4[m/2]-1)/3)(\Delta-\delta), (\delta+\Delta)+((4[m/2]-1)/3)(\Delta-\delta)]$. Moreover the upper bound is achieved on $\mathbf{C}\,P^n$, see 5.2.

3.9. COROLLARY. — *The curvature operator $\hat{R}$ is positive as soon as the sectional curvature is α-pinched with $\alpha = 1-(3/(2[m/2]+1))$ (recall that this means that $\alpha \operatorname{Max} \overline{R} \leqq \overline{R} \leqq \operatorname{Max} \overline{R}$). The bound is almost sharp for distance spheres*, see 6.5.

Proof. — We denote by $\hat{r}$ an eigenvalue of $\hat{R}$. If ω is an eigenvector of $\hat{R}$ [i. e. $\hat{R}(\omega) = \hat{r}\,\omega$], then we have

$$\hat{R}_0(\omega) = \hat{r}_0\,\omega, \qquad \text{with} \quad r_0 = \hat{r}-(\Delta+\delta). \tag{3.10}$$

For any skewsymmetric two-form ω of rank $2k$ there exists (*see* [15], p. 24) some orthonormal basis (e_i) of V in which one can write

$$\omega = \sum_{i=1}^{k} \omega_i e_i \wedge e_{i'} \qquad (\text{where } i' = i+k).$$

With respect to this basis, (3.10) reads:

$$2\sum_{i=1}^{k} \omega_i R_0(e_i, e_{i'}, e_j, e_{j'}) = \hat{r}_0\,\omega_j \qquad (j = 1, \ldots, k).$$

We single out the scalar equation for which $|\omega_j|$ is maximal. Using Lemma 3.7 for $i \neq j$ we get

$$|\hat{r}_0| \leqq 2\sum_{i=1}^{k} |\omega_i/\omega_j| . |R_0(e_i, e_{i'}, e_j, e_{j'})| \leqq (\Delta-\delta)((4/3)(k-1)+1)$$

and the proposition follows. ■

3.11. Our method is in spirit an L^∞-method. Therefore we cannot get any information on the distributions of the eigenvalues in the range that we described. The L^2-estimate of [8] (in $m^{3/2}$) tells us that not too many eigenvalues have the size allowed by the L^∞-estimate.

3.12. We now come to estimating the operators associated to the irreducible components U, Z and W of R. For the Ricci curvature $\rho = \gamma(R)$, we get

$$(m-1)\delta \leqq \overline{\rho} \leqq (m-1)\Delta$$

(here $\bar{\rho}$ denotes the function on the unit sphere defined by ρ), so that

(3.13) $$|\bar{\rho}_0| \leqq (m-1)(\Delta-\delta)/2, \qquad |\bar{z}| \leqq (m-1)(\Delta-\delta) \quad (\textit{see } 2.6)$$

and

$$|u_0| \leqq m(m-1)(\Delta-\delta)/2, \qquad |\hat{u}_0| \leqq (\Delta-\delta).$$

3.14. *Remark.* — One can actually improve the estimate on u_0 a bit to

$$|u_0| \leqq (m(m-1)-2)(\Delta-\delta)/2$$

(choose a basis spanning two-planes on which the extremal values of the sectional curvature are achieved).

3.15. If (e_i) is a basis which diagonalizes z (the eigenvalues of z are *denoted by* z_i) one easily sees that $\hat{Z} = 1/(m-2)\ z \wedge g$ operates diagonally in the basis $(e_i \wedge e_j)$ of $\Lambda^2 V$ with eigenvalues $(2/(m-2))(z_i+z_j)$. In particular all eigenvectors of $\hat{Z}$ have rank two. From (3.13) we deduce that

$$|\hat{z}| \leqq 4(m-1)(\Delta-\delta)/(m-2).$$

To estimate the eigenvalues $\hat{w}$ of $\hat{W}$ it is more convenient to use the formula $W = R_0-(U_0+Z)$. Notice that $U_0+Z = 1/(m-2)\ (\rho_0 \wedge g-(u_0/2(m-1))\, g \wedge g)$, so that, using the same basis as above and (3.13), we have the inequality

$$|\hat{u}_0+\hat{z}| \leqq (3m-2)(\Delta-\delta)/(m-2).$$

Therefore we get

$$|\hat{w}| \leqq (4[m/2]/3+8/3+4/(m-2))(\Delta-\delta),$$

giving the following:

3.16. PROPOSITION. — *If the sectional curvature* $\bar{R}$ *satisfies* $\delta \leqq \bar{R} \leqq \Delta$. *then the eigenvalues of the operators* $\hat{U}$, $\hat{Z}$ *and* $\hat{W}$ *associated to the irreducible components of* $R\ (R = U+Z+W)$ *satisfy*

(3.17) $$\begin{cases} 2\delta \leqq u \leqq 2\Delta, \qquad |\hat{z}| \leqq 4(1+1/(m-2))(\Delta-\delta), \\ \qquad |\hat{w}| \leqq (4[m/2]/3+8/3+4/(m-2))(\Delta-\delta). \end{cases}$$

Moreover if the curvature tensor R *is Einstein* (*i. e. if* $Z = 0$), *the estimate* (3.17) *for* $\hat{w}$ *can be sharpened as follows*:

(3.17′) $$|\hat{w}| \leqq (2/3)(2[m/2]+1)(\Delta-\delta) \quad (\textit{for an application}, \text{ see } [2]).$$

This estimate is asymptotically sharp on $\mathbf{C}\,\mathbf{P}^n$.

3.18. *Remark.* — If we know that $\hat{\delta} \leqq \hat{r} \leqq \hat{\Delta}$, then the estimates are unchanged for $\hat{u}$ and $\hat{z}$ (with $\hat{\Delta}/2$ replacing Δ and $\hat{\delta}/2$ replacing δ as mentionned in 3.6) but the estimate for $\hat{w}$ becomes

$$|\hat{w}| \leqq 5(1+1/(m-2))(\hat{\Delta}-\hat{\delta}).$$

4. Pinching estimates on $\mathring{R}$

4.1. Two new features appear in the study of $\mathring{R}$:

(i) the map $R \mapsto \mathring{R}$ from $S^2 \Lambda^2 V$ to $S^2 S^2 V$ is injective only on $\mathscr{R}$ (indeed for a four-form Ω, $\mathring{\Omega} = 0$) and the orthogonal complement to its image is not easy to describe;

(ii) the space $S^2 V$ on which $\mathring{R}$ acts is not irreducible under the action of $O(V)$; for the decomposition of R into irreducible components $R = U+Z+W$, only $\mathring{U}$ and $\mathring{W}$ preserve the irreducible subspaces of $S^2 V$. The constant sectional curvature case $g \mathbin{\mathring{\wedge}} g$ has already two distinct eigenvalues: $2(m-1)$ on the line $\mathbf{R}.g$ and -2 on the other irreducible subspace consisting of the traceless symmetric two-tensors. The description of $\mathring{R}$ in the general case will also be more involved than that of $\hat{R}$.

We now estimate the eigenvalues of $\mathring{R}$ and $\mathring{R}_0$ (compare 3.8) in terms of sectional curvature bounds:

4.2. Lemma. — *Let h be an eigenvector of $\mathring{R}_0$, say $\mathring{R}_0(h) = \mathring{r}_0 h$, which as linear map on V has rank k. Then*

$$|\mathring{r}_0| \leqq (k-1) \max |\overline{R}_0|.$$

Proof. — We choose an orthonormal basis (e_i) in which h is diagonal, i. e. $h = \sum_{i=1}^{k} h_i e_i \otimes e_i$.

From the system of k equations

$$(\mathring{R}_0(h))_j = \sum_{i=1}^{k} h_i R_0(e_i, e_j, e_i, e_j) = \mathring{r}_0 h_j \qquad (j = 1, \ldots, k),$$

we single out the equation in which $|h_j|$ is maximum (hence $\neq 0$) to get the estimate for $\mathring{r}_0$. ■

4.3. Proposition. — *If the sectional curvature of* R *satisfies $\delta \leqq \overline{R} \leqq \Delta$, then all eigenvalues of $\mathring{R}$ but one lie in the interval*

$$[-(1/2)((\Delta+\delta)+(m-1)(\Delta-\delta)),\ -(1/2)((\Delta+\delta)-(m-1)(\Delta-\delta))],$$

the other lying in the interval $[(m-1)\delta, (m-1)\Delta]$.

The first upper bound is sharp for $\mathbf{C}P^2$ *and the last lower bound almost sharp for* $\mathbf{C}P^n$, see 5.3.

Proof. — The result follows from Lemma 4.2, the knowledge of the eigenvalues of $\mathring{R}-\mathring{R}_0$ together with the following:

4.4. Lemma ([9] p. 126). — *Let* A *and* B *be two symmetric linear maps on a Euclidean vector space* F *with eigenvalues respectively* (α_i) *and* (β_i) *naturally ordered. Then if* $C = B-A$ *has eigenvalues* (γ_i) *we have*

$$\sup_i |\beta_i - \alpha_i| \leqq \sup_i |\gamma_i|. \quad ■$$

4.5. We have the following partial result on the rank of an eigenvector h of $\mathring{R}$ for the eigenvalue $\mathring{r}_{\max}$ [with value $2(m-1)$ for $g \wedge g$]: h has maximal rank if the sectional curvature is α-pinched with $\alpha \geqq 1-(2/m)-(3/m^2)$.

This follows by estimating angles between h and g.

4.6. PROPOSITION. — *If the sectional curvature of* R *satisfies* $\delta \leqq \overline{R} \leqq \Delta$, *then the eigenvalues of the irreducible component* $\mathring{Z}$ *of* R *acting on symmetric two-tensors satisfy*

$$|\mathring{z}| \leqq ((m-1)(m+2)/(m-2))(\Delta-\delta)$$

and the eigenvalues of $\mathring{U}$ *and* $\mathring{W}$ *on traceless symmetric two-tensors satisfy*

$$(4.7) \qquad -2\Delta \leqq \mathring{u} \leqq -2\delta, \qquad |\mathring{w}| \leqq 1/2 \ (m+4+(4/(m-2)))(\Delta-\delta).$$

Moreover if the curvature tensor R *is Einstein,*

$$(4.7') \qquad |\mathring{w}| \leqq (m/2)\ (\Delta-\delta).$$

Proof. — We first notice that

$$\mathring{Z}(h) = (1/(m-2))((g, h)z+(z, h)g-z\circ h-h\circ z).$$

If (e_i) is an orthonormal basis of V in which z is diagonal [we set $z(e_i, e_i) = z_i$], then, for $i \neq j$, $e_i \bigcirc e_j$ is an eigenvector of $\mathring{Z}$ for the eigenvalue $-2/(m-2)\ (z_i+z_j)$.

On the m-dimensional space $\bigoplus_{i=1}^{m} \mathbf{R}.(e_i \otimes e_i)$ we consider $(m-2)\mathring{Z}$ as the sum of the rank-two operator $h \mapsto (g, h)z+(z, h)g$ and of the operator $h \to z\circ h+h\circ z$.

Using the estimates (3.13) we get that any eigenvalue $\mathring{z}$ of $\mathring{Z}$ on $\bigoplus_{\substack{i,j=1\\ i\neq j}}^{m} \mathbf{R}.(e_i \bigcirc e_j)$ satisfies

$$|\mathring{z}| \leqq 2(m-1)/(m-2)\ (\Delta-\delta).$$

On the other hand on $\bigoplus_{i=1}^{m} \mathbf{R}.(e_i \otimes e_i)$, we have only

$$|\mathring{z}| \leqq ((m+2)(m-1)/(m-2))\ (\Delta-\delta).$$

Since g lies in the kernel of $\mathring{W}$, we only have to estimate $\mathring{W}$ on traceless symmetric two-tensors. We avoid using the previous estimate by noticing that on traceless symmetric tensors the quadratic form $h \mapsto (m-2)(\mathring{Z}(h), h)$ reduces to $h \mapsto -2(z\circ h, h)$.

If h is a unit eigenvector of $\mathring{W}$ for the eigenvalue $\mathring{w}$, we have

$$\mathring{w} = (\mathring{W}(h), h) = (\mathring{R}_0(h), h)+(2/(m-2))(z\circ h, h)+u_0/m(m-1),$$

so that

$$|\mathring{w}| \leqq 1/2\ (m-1+(4(m-1)/(m-2)))(\Delta-\delta)+u_0/m(m-1).$$

Therefore using (3.13) and Lemma 4.2 we have

$$|\mathring{w}| \leqq 1/2\ (m+4+(4/(m-2)))(\Delta-\delta)$$

and if R is Einstein

$$|\mathring{w}| \leqq (m/2)(\Delta-\delta). \quad \blacksquare$$

4.8. If R is Einstein, we can bring into the estimates the *Einstein constant* k related to the scalar curvature by $m\,k = u$. In this case $\mathring{R}$ preserves the decomposition of $S^2\,V$ into irreducible subspaces. The next proposition was suggested to us by [11].

4.9. PROPOSITION. – *Let* R *be an Einstein curvature tensor with Einstein constant k and with sectional curvature satisfying* $\delta \leqq \overline{R} \leqq \Delta$. *Then the eigenvalues* $\mathring{r}$ *of* $\mathring{R}$ *on traceless symmetric tensors satisfy*

$$k-m\,\Delta \leqq \mathring{r} \leqq k-m\,\delta.$$

Proof. – After introducing $R' = R-\delta/2\; g \wedge g$ so that $0 \leqq \overline{R}'$, the proof goes as in Lemma 4.2, the positivity of $\overline{R}'$ replacing the estimate on $\overline{R}_0$. $\blacksquare$

PART TWO: GEOMETRIC EXAMPLES

5. Projective spaces and their duals

5.0. Surprisingly the sharpness of many of the previous inequalities can already be demonstrated with the curvature tensors of the simplest symmetric spaces, namely the projective spaces, or of simple examples derived from these.

5.1. We consider the *complex projective space* and *its dual.*

In terms of the complex structure J, we have for the curvature tensor $R_{\mathbf{CP}^n}$ and the operators $\hat{R}_{\mathbf{CP}^n}$ and $\mathring{R}_{\mathbf{CP}^n}$ (opposite signs for the dual)

$$2\,R_{\mathbf{CP}^n} = g \wedge g + J \wedge J + 4\,J \otimes J$$

(with this normalization the sectional curvature ranges between 1 and 4), for ω in $\Lambda^2\,V$

$$\hat{R}_{\mathbf{CP}^n}(\omega) = 2(\omega - J \circ \omega \circ J - (J,\,\omega)\,J)$$

and for h in $S^2\,V$,

$$\mathring{R}_{\mathbf{CP}^n}(h) = -h + \operatorname{trace}(h)\,g - 3\,J \circ h \circ J.$$

Of course $R_{\mathbf{CP}^n}$ is Einstein (so that $Z_{\mathbf{CP}^n} = 0$) and its diagonal component is

$$U_{\mathbf{CP}^n} = (n+1)/(2\,n-1) \quad g \wedge g.$$

As four-tensor applied to a, b, c, d in V, we get

$$\begin{aligned} R_{\mathbf{CP}^n}(a,\,b,\,c,\,d) = {} & (a,\,c)(b,\,d)-(a,\,d)(b,\,c) \\ & +(J\,a,\,c)(J\,b,\,d)-(J\,a,\,d)(J\,b,\,c)+2(J\,a,\,b)(J\,c,\,d). \end{aligned}$$

5.2. We give the spectral data for $\hat{R}_{CP^n}$ (notice that, for u and v in V,

$$J \circ (u \wedge v) \circ J = -J u \wedge J v) :$$

Eigenspace	Dimension	Eigenvalue
$R.J$	1	$4n+4 = 2(\dim \mathbf{C}P^n + 2)$ (*cf.* Prop. 3.8)
span $\{ u \wedge v - J u \wedge J v \}$ (u, v **C**-linearly independent)	$n(n-1)$	0
span $\{ u \wedge v + J u \wedge J v \}$ (u, v **C**-linearly independent)	$n(n-1)$	4
span $\{ u \wedge J u - v \wedge J v \}$	$n-1$	4

The curvature operator $\hat{R}_{CP^n}$ has many eigenvectors of rank 4 which makes it very special. Notice also the large multiplicities.

5.3. We give the spectral data of $\mathring{R}_{CP^n}$ (notice that for u and v in V,

$$J \circ (u \bigcirc v) \circ J = -J u \bigcirc J v) :$$

Eigenspace	Dimension	Eigenvalue
$R.g$	1	$2(n+1)$ (*cf.* Prop. 4.3)
span $\{ u \circ v - J u \circ J v \}$	$n(n+1)$	-4
span $\{ u \circ v + J u \circ J v \}$ (v orthogonal to u)	$n(n-1)$	2
span $\{ u \circ u + J u \circ J u - v \circ v - J v \circ J v \}$ [with $(u, u) = (v, v)$]	$n-1$	2

The curvature operator $\mathring{R}_{CP^n}$ has also many eigenvectors of rank 4 and eigenspaces with large multiplicities.

5.4. The curvature tensor R_{HP^n} of the quaternionic projective space is in terms of three orthonormal imaginary quaternions I, J, K given as follows:

$$2 R_{HP^n} = g \wedge g + I \wedge I + 4 I \otimes I + J \wedge J + 4 J \otimes J + K \wedge K + 4 K \otimes K.$$

The main difference between the operators $\hat{R}_{HP^n}$ and $\mathring{R}_{HP^n}$ and the operators $\hat{R}_{CP^n}$ and $\mathring{R}_{CP^n}$ lies in the dimensions of the eigenspaces: for example $\hat{R}_{HP^n}$ has the three-dimensional eigenspace span $\{ I, J, K \}$ with the "large" eigenvalue $8n$ ($= 2 \dim \mathbf{H} P^n$) not quite as large in terms of dimension as in the complex case.

6. Distance sphere in the projectives spaces and their duals

6.0. The distance spheres in the complex projective space have frequently been treated since they furnish the famous counterexamples for lower bounds on the injectivity radius of the exponential map. The distance spheres in the dual give almost sharp *geometric* examples for the inequality of Corollary 3.10.

6.1. We determine the curvature tensor of the distance spheres with the Gauss-equation. The *shape operator* S (or second fundamental tensor) of the distance spheres is obtained as follows: call N the outer unit normal field of the spheres. In the complex case we have a distinguished tangent vector JN and its orthogonal complement, in the quaternionic case we have a distinguished three-dimensional tangent subspace span $\{ \mathrm{IN}, \mathrm{JN}, \mathrm{KN} \}$ and its orthogonal complement. Jacobi fields along the radial geodesics with initial data in the distinguished subspaces behave as in the case of constant curvature 4 (-4 in the dual), since in these symmetric spaces the curvature tensor and hence the distinguished subspaces are parallel; the Jacobi fields orthogonal to these subspaces behave as in the constant curvature 1 (-1 in the dual) case.

6.2. Now let $r \mapsto j(r)$ be a normal Jacobi field with $j(0) = 0$ (i. e. coming from a variation by radial geodesics), then $j(r)$ is a tangent vector to the sphere of radius r. Now the Weingarten formula gives the shape operator of the distance spheres in terms of these Jacobi fields:

$$\mathrm{S}_r(j(r)) = -\mathrm{D}_{j(r)}\mathrm{N}(r)$$

[where $\mathrm{D}_{j(r)}\mathrm{N}(r) = \mathrm{D}/(dr)\, j(r)$] since $\mathrm{N}(r)$ is tangent to the radial geodesics.

We list the spectral data for the shape operator S_r of the distance sphere of radius r:

Eigenspace	Dimension	Eigenvalue	
		Compact case	*Dual case*
Complex case :			
span $\{ \mathrm{JN} \}$	1	$2 \cot 2r$	$2 \coth 2r$
$\{ \mathrm{N}, \mathrm{JN} \}^{\perp}$	$2n-2$	$\cot r$	$\coth r$
Quaternionic case			
span $\{ \mathrm{IN}, \mathrm{JN}, \mathrm{KN} \}$	3	$2 \cot 2r$	$2 \coth 2r$
$\{ \mathrm{N}, \mathrm{IN}, \mathrm{JN}, \mathrm{KN} \}^{\perp}$	$4n-4$	$\cot r$	$\coth r$

6.3. In the complex case, trace $\mathrm{S}_r = (2n-1) \cot r - \tan r$ and in the quaternionic case trace $\mathrm{S}_r = (4n-1) \cot r - 3 \tan r$ so that in these two cases there is one and only one distance sphere which is minimal. Its radius depends on the dimension. No such minimal sphere exists in the dual spaces for obvious reasons.

6.4. The Gauss equation for a hypersurface M with shape operator S in a manifold $\tilde{\mathrm{M}}$ is

$$\mathrm{R} = \tilde{\mathrm{R}}^{\tan} + 1/2 \quad \mathrm{S} \wedge \mathrm{S},$$

where, in terms of the orthogonal projection $\mathrm{P} : \mathrm{T}_p \tilde{\mathrm{M}} \to \mathrm{T}_p \mathrm{M}$

$$(\tilde{\mathrm{R}}^{\tan}(a, b)c, d) = (\tilde{\mathrm{R}}(\mathrm{P}a, \mathrm{P}b)\mathrm{P}c, \mathrm{P}d).$$

This enables us to get very easily our hands on the curvature tensor of the distance spheres. In the following list we give only sectional curvatures of special two-planes, but it is easily seen that they contain the maximum and the minimum of the sectional curvature (we restrict ourselves to the complex case):

Two-plane	Compact case	Dual case	$\lim_{r \to \infty}$
$u \wedge v$ ($u, v \perp$ JN; $u \perp v$, Jv)	$1+\cot^2 r$	$-1+\coth^2 r$	-0
$u \wedge$ Ju ($u \perp$ N, JN)	$4+\cot^2 r$	$-4+\coth^2 r$	-3
$u \wedge$ JN ($u \perp$ N)	$\cot^2 r$	$\coth^2 r$	1
Maximum of the sectional curvature	$4+\cot^2 r$	$\coth^2 r$	
Minimum of the sectional curvature	$\cot^2 r$	$-4+\coth^2 r$	
Length of the shortest closed geodesic	$\pi \sin 2r$		

Note that $\pi \sin 2r < 2\pi (4+\cot^2 r)-1/2$ as soon as $\tan^2 r > 2$, these are the famous "small injectivity radius" examples. Similar examples fail to exist in the quaternionic case, since the sectional curvature for planes in the distinguished 3-dimensional tangent subspace is equal to $4+(2\cot 2r)^2$, i. e. is larger than $4+\cot^2 r$ if $\tan^2 r > 2$. Exactly at the radius at which the examples start to appear in the complex case does, in the quaternionic case, the sectional curvature of distinguished planes become too large.

6.5. Since, in the complex case, $\tilde{R}^{\tan}$ differs only slightly from the curvature tensor of $\mathbf{C}\,P^{n-1}$ (resp. its dual), namely for u orthogonal to N and JN

$$\tilde{R}^{\tan}(u, JN)v = (JN, v)u-(u, v)JN,$$

one can write down the eigenvectors of $\hat{\tilde{R}}^{\tan}$ as in 5.2 and of $1/2\, S \mathbin{\hat{\wedge}} S$ from the spectral data of S quite easily. We list the spectral data of $\hat{R}$ for the distance spheres in the complex projective space:

Eigenspace	Dimension	Eigenvalues: *Compact case*	Eigenvalues: *Dual case*
span $\{ J^{\tan} = P \circ J \circ P \}$	1	$4n+2\cot^2 r$	$-4n+2\coth^2 r$
span $\{ u \wedge v - J u \wedge J v \}$ ($u, v \perp$ N, JN; u, v linearly independent over **C**)	$(n-1)(n-2)$	$2\cot^2 r$	$2\coth^2 r$
span $\{ u \wedge v + J u \wedge J v \}$ ($u, v \perp$ N, JN; u, v linearly independent over **C**)	$(n-1)(n-2)$	$4+2\cot^2 r$	$-4+2\coth^2 r$
span $\{ u \wedge J u \}$ ($u \perp$ N, JN)	$n-2$	$4+2\cot^2 r$	$-4+2\coth^2 r$
span $\{ u \wedge JN \}$ ($u \perp$ N, JN)	$2n-2$	$2\cot^2 r$	$2\coth^2 r$

The lowest eigenvalue is never 0 in $\mathbf{C}\,P^n$, showing a global situation where a one-parameter family of curvature operators is positive as long as the sectional curvature is. The lowest eigenvalue is 0 in the dual situation if $\coth^2 r = 2n$. At this radius the pinching of the sectional curvature is $1-(2/n) = 1-(4/(1+\dim S^{2n-1}))$; in other words we get negative eigenvalues already for rather narrow pinching. This is also almost sharp in view of Corollary 3.9 [the best possible is $1-(3/1+m))$].

6.6. Finally it is also interesting to look at the Ricci curvature of these examples. Again we use the Gauss equation. The contribution to the Ricci tensor from the shape operator is well known:

$$\gamma(\mathrm{S} \wedge \mathrm{S}) = (\operatorname{trace} \mathrm{S})\,\mathrm{S} - \mathrm{S}^2.$$

The contribution from $\tilde{\mathrm{R}}^{\tan}$ (as follows easily from the explicit formula for $\tilde{\mathrm{R}}$) has JN as eigenvector (eigenvalue $2n-2$) and $\{\mathrm{N}, \mathrm{JN}\}^{\perp}$ as eigenspace (eigenvalue $2n+1$) in the complex case (take the opposite in the dual case) and IN, JN, KN as eigenvectors (eigenvalue $8+4n-4$) and eigenspace $\{\mathrm{N}, \mathrm{IN}, \mathrm{JN}, \mathrm{KN}\}^{\perp}$ (eigenvalue $12+4n-5$) in the quaternionic case. This gives the following spectral data for the Ricci tensor of the distance spheres:

Eigenspace	Dimension	Eigenvalues: *Compact case*	Eigenvalues: *Dual case*
Complex case :			
span $\{\mathrm{JN}\}$	1	$(2n-2)\cot^2 r$	$(2n-2)\coth^2 r$
$\{\mathrm{JN}\}^{\perp}$	$2n-2$	$2n+(2n-2)\cot^2 r$	$-2n+(2n-2)\coth^2 r$
Quaternionic case :			
span $\{\mathrm{IN}, \mathrm{JN}, \mathrm{KN}\}$	3	$(4n-2)\cot^2 r + 2\tan^2 r + 4$	$(4n-2)\coth^2 r + 2\tanh^2 r - 4$
$\{\mathrm{IN}, \mathrm{JN}, \mathrm{KN}\}^{\perp}$	$4n-4$	$(4n-2)\cot^2 r + 4n + 4$	$(4n-2)\coth^2 r - 4n - 4$

We have here four one-parameter families of examples in which the Ricci tensor has only two distinct eigenvalues: in three of them the eigenvalues remain far apart, but for the distance spheres in $\mathbf{H}\,\mathrm{P}^n$ with radius determined by $\tan^2 r = 2n$ we recover Jensen's examples (*cf.* [7]) of non-standard Einstein metrics on the spheres S^{4n-1}. The eigenvalues of S are $1/(\sqrt{2n})-\sqrt{2n}$ on span $\{\mathrm{IN}, \mathrm{JN}, \mathrm{KN}\}$ and $1/(\sqrt{2n})$ on $\{\mathrm{IN}, \mathrm{JN}, \mathrm{KN}, \mathrm{N}\}^{\perp}$. The sectional curvatures vary between $1/2n$ and $(1/2n)+4$ so that the pinching goes to 0 when n goes to infinity.

Similarly the 15-dimensional distance sphere in the Cayley projective plane with $\tan^2 r = 8/3$ is also a non-standard Einstein space.

6.7. The spheres in the noncompact case have for large r a very negative Ricci tensor, namely all but one eigenvalue negative (all but three in the quaternionic case) and negative scalar curvature for still larger r. This is as negative as it is possible for a homogeneous space, since negative Ricci curvature on a compact manifold allows only a finite isometry group.

7. The Euler integrand

7.0. The systematic use of the curvature operator enables us to present a simple construction of six-dimensional counterexamples to the conjecture that "positive sectional curvature implies positive Euler integrand" (the first one was given by Geroch in [5], *see* also [10]). For that we study curvature tensors R which are linear combinations of the curvature tensors of the only irreducible 6-dimensional symmetric spaces S^6, $\mathbf{C}\,P^3$ and $G^{2,5}$, the Grassmann manifold of two-planes in $\mathbf{R}^5$. Following R. S. Kulkarni (*cf.* [12]), we compute the Euler integrand of R in a diagonalizing basis for $\hat{R}$. To determine the range of $\overline{R}$, we need detailed information about $G^{2,5}$; this is again obtained by using curvature operator techniques.

7.1. Both $G^{2,5}$ and $\mathbf{C}\,P^3$ are Kähler-Einstein and when we consider linear combinations of their curvature tensors we will do this in a way compatible with the complex structure. The manifold $G^{2,5}$ is naturally identified with the space of oriented geodesics of S^4 (*cf.* [1]) and therefore a tangent vector to $G^{2,5}$ is represented by a normal Jacobi field j along a great circle of S^4; the complex structure of $G^{2,5}$ maps j onto its derivative j', which is again a Jacobi field.

To get the curvature tensor we view $G^{2,5}$ as the symmetric space $SO(5)/SO(3)\times SO(2)$ and use the standard formula $R(a, b)\,c = [[a, b], c]$ in terms of the Lie bracket of $\mathfrak{so}(5)$ (*see* [6], p. 180).

Later we prefer to compute with eigenvectors of the curvature operator rather than with matrices; therefore we choose the following basis of the tangent space V^6 of $G^{2,5}$:

$$
\begin{aligned}
a &= e_1 \wedge e_4, & b &= e_2 \wedge e_4, & c &= e_3 \wedge e_4,\\
J\,a &= e_1 \wedge e_5, & J\,b &= e_2 \wedge e_5, & J\,c &= e_3 \wedge e_5,
\end{aligned}
$$

where $(e_1, \ldots, e_5)$ is an orthonormal basis of $\mathbf{R}^5$, $e_i \wedge e_j$ are elements of $\mathfrak{so}(5)$ and the complex structure for $G^{2,5}$ described above is, in this basis, the given J. The subspace span $\{a, b, c\}$ can be thought of as the space of normal Jacobi fields vanishing at a point of a great circle.

7.2. Of course $R_{S^6} = 1/2\, g \wedge g$ (so that $\overline{R}_{S^6} = 1$).

Also on any complex space, we have in terms of the complex structure J

$$R_{\mathbf{C}P^3} = 1/2(g \wedge g + J \wedge J + 4\,J \otimes J) \quad (\text{so that } 1 \leqq \overline{R}_{\mathbf{C}P^3} \leqq 4).$$

We know from section 6 that the following basis of $\Lambda^2 V$ consists of eigenvectors of the curvature operator $\hat{R}_{\mathbf{C}P^3}$ (ε is $+$ or $-$):

$$
\begin{aligned}
2\,\omega_1^\varepsilon &= a \wedge b - \varepsilon\, J\,a \wedge J\,b, & 2\,\omega_4^\varepsilon &= a \wedge J\,b + \varepsilon\, J\,a \wedge b,\\
2\,\omega_2^\varepsilon &= b \wedge c - \varepsilon\, J\,b \wedge J\,c, & 2\,\omega_5^\varepsilon &= b \wedge J\,c + \varepsilon\, J\,b \wedge c,\\
2\,\omega_3^\varepsilon &= c \wedge a - \varepsilon\, J\,c \wedge J\,a, & 2\,\omega_6^\varepsilon &= c \wedge J\,a + \varepsilon\, J\,c \wedge a,
\end{aligned}
$$

$$
\begin{aligned}
2\,\omega_1'' &= a \wedge J\,a - b \wedge J\,b,\\
2\sqrt{3}\,\omega_2'' &= a \wedge J\,a + b \wedge J\,b - 2\,c \wedge J\,c,\\
\sqrt{6}\,\omega_3'' &= a \wedge J\,a + b \wedge J\,b + c \wedge J\,c\,(= J).
\end{aligned}
$$

The eigenvalues are also given in section 6.

The Lie bracket computations left out in 7.1 show that, fortunately, this basis of $\Lambda^2 V$ is also a basis of eigenvectors of the curvature operator of $G^{2,5}$. Since the curvature operator of S^6 is twice the identity, we have already the spectral decomposition of the curvature operator $\hat{R}$, where $R = \lambda R_{S^6} + \mu R_{CP^3} + \nu R_{G^{2,5}}$ with eigenvalues:

$A = 2\lambda$ on ω_i^+ for $i = 1, \ldots, 6$, $\quad C = 2\lambda + 4\mu$ on ω_i^- for $i = 4, 5, 6$,

$B = 2\lambda + 4\mu + 4\nu$ on ω_i^- for $i = 1, 2, 3$, $\quad C = 2\lambda + 4\mu$ on ω_i'' for $i = 1, 2$,

$$D = 2\lambda + 16\mu + 6\nu \quad \text{on} \quad \omega_3''.$$

Notice the large kernel of $\hat{R}_{G^{2,5}}$; the two-forms ω_i^+ ($i = 1, \ldots, 6$) lie in the kernel since $G^{2,5}$ is Kählerian.

7.3. The formula for the Euler integrand (*see* [12] or [14]) is, in dimension $2k$,

$$E_k(R) = \sum_{i \in \mathfrak{S}_k} \hat{R}(\omega_{i_1}) \wedge \ldots \wedge \hat{R}(\omega_{i_k}) \otimes \omega_{i_1} \wedge \ldots \wedge \omega_{i_k},$$

where (ω_ν) is an orthonormal basis for $\Lambda^2 V$. If the basis (ω_ν) is of the form $(v_i \wedge v_j)$ with (v_i) an orthonormal basis of V, then the formula reduces to the wellknown expression in the curvature tensor components; if the $\{\omega_\nu\}$ are eigenvectors of $\hat{R}$, we obtain

$$E_k(R) = \sum_{i \in \mathfrak{S}_k} \hat{r}_{i_1} \ldots \hat{r}_{i_k} \omega_{i_1} \wedge \ldots \wedge \omega_{i_k} \otimes \omega_{i_1} \wedge \ldots \wedge \omega_{i_k},$$

an expression which is much more manageable than the one in the curvature tensor components.

In particular, if a curvature operator has the eigenvectors of 7.2 and the eigenvalues A, B, C, D, then, from a list of triple exterior products of the eigenvectors, one obtains

$$24 E_3(R) = 54 A^2 B + 90 A^2 C + 18 A^2 D + 45 B^2 C + 9 B^2 D + 35 C^3 + 15 C^2 D + 4 D^3.$$

Notice that the expression (3.2) of the Bianchi identity reduces to the linear relation $6A - 3B - 5C + 2D = 0$.

7.4. PROPOSITION. — *The sectional curvature function of* $R = \lambda R_{S^6} + \mu R_{CP^3} + \nu R_{G^{2,5}}$ *achieves its minimum* ($= \min(\lambda+\mu, \lambda+4\mu+\nu)$) *on a holomorphic plane, its maximum* ($= \max(\lambda+\mu+\nu, \lambda+4\mu+2\nu)$) *on a real plane if* $3\mu+\nu \leqq 0$ *and vice versa if* $3\mu+\nu \geqq 0$ (*we suppose* $\nu \leqq 0$).

Proposition 7.4 directly gives the following interesting geometric:

7.5. PROPOSITION. — *The curvature tensor* $R_\alpha = R_{S^6} - \alpha R_{CP^3} + 3\alpha R_{G^{2,5}}$ *has the sectional curvature range* $[1-\alpha, 1+2\alpha]$. *The Euler integrand* $E_3(R_\alpha) = 18(5 + 3\alpha + 12\alpha^2 - 28\alpha^3)$ *is negative for* $\alpha \geqq \alpha_0 = 0.8227$. *The pinching at* α_0 *is* 1/15.

Klembeck's example (cf [10]) *is obtained at* $\alpha = 1$.

(*From Corollary* 3.9 *follows that in dimension six a pinching* > 4/7 *for the sectional curvature implies that* $\hat{R}$ *is positive.*)

Proof. — The sectional curvatures of R_{CP^3} are given in terms of the angle φ of holomorphy [if x and y are orthonormal, then $\cos\varphi = (J x, y)$] by

$$\overline{R}_{CP^3}\{x, y\} = 1+3\cos^2\varphi.$$

The situation is considerably more complicated in $G^{2,5}$; it will be described by the following Lemma. As can be seen from the definition of a, b, c in 7.1, the most general tangent vector (up to isometry) to $G^{2,5}$ is

$$x = \cos\gamma(\cos\beta . a+\sin\beta . b)+\sin\gamma . J a.$$

7.6. LEMMA. — *For the sectional curvature of* $G^{2,5}$ *we have*:

(i) *the holomorphic sectional curvature of* $G^{2,5}$, $h^2(x)$, *is given by*

$$h^2(x) = (R_{G^{2,5}}(x, J x)x, J x) = 1+\sin^2\beta . \sin^2 2\gamma;$$

(ii) *the quadratic form* $y \mapsto (R_{G^{2,5}}(x, y) x, y)$ *on span* $\{x, J x\}^{\perp}$ *can be represented by a symmetric operator which has on span* $\{c, J c\}$ *the eigenvalues* $1/2 \pm 1/2\,(2-h^2(x))^{1/2}$ (*hence* $\leqq 1$), *and has on span* $\{d, J d\}$ (*where* $d = (-\sin\beta . a+\cos\beta . b)\cos\gamma-\sin\gamma . J b$) *the eigenvalues* 0 *and* $2-h^2(x)$ (*hence* $\leqq 1$), *the latter with the eigenvector*

$$e' = \cos 2\gamma . d+\cos\beta . \sin 2\gamma . J d,$$

such that $(e', e') = 2-h^2(x)$;

(iii) $(R_{G^{2,5}}(x, J x) x, y) = \sin\beta . \sin 2\gamma . (e', y)$.

7.7. With this Lemma we complete the proof of Proposition 7.4.

Obviously the sectional curvature of R on real planes has $\lambda+\mu$ as minimum and $\lambda+\mu+\nu$ as maximum; the holomorphic sectional curvature of R has $\lambda+4\mu+\nu$ as minimum and $\lambda+4\mu+2\nu$ as maximum. It remains to estimate the sectional curvature of a plane with φ as angle of holomorphy. We have

$$\begin{aligned}\overline{R}\{x, (\cos\varphi . J x+\sin\varphi . y)\} = {} & \lambda+\mu\sin^2\varphi+4\mu\cos^2\varphi+\nu\cos^2\varphi \\ & +\nu[\overline{R}_{G^{2,5}}\{x, y\}\sin^2\varphi+2(R_{G^{2,5}}(x, J x)x, y)\sin\varphi . \cos\varphi \\ & +(\overline{R}_{G^{2,5}}\{x, J x\}-1)\cos^2\varphi].\end{aligned}$$

Now the expression in the last bracket is nonnegative since from Lemma 7.6 we have

$$(R_{G^{2,5}}(x, J x) x, y)^2 \leqq \overline{R}_{G^{2,5}}\{x, y\}(\overline{R}_{G^{2,5}}\{x, J x\}-1),$$

hence

$$\overline{R}\{x, (\cos\varphi . J x+\sin\varphi . y)\} \geqq \min(\lambda+\mu, \lambda+4\mu+\nu).$$

To derive the maximum, let ψ be the angle between y and span $\{d, J d\}$, so that

$$(e', y)^2 \leqq \cos^2\psi(2-h^2(x)),$$
$$\overline{R}_{G^{2,5}}\{x, y\} \leqq \sin^2\psi(1/2+1/2\,(2-h^2(x))^{1/2})+\cos^2\psi(2-h^2(x)).$$

We clearly have

$$|2\sin\varphi\cos\varphi\,(\mathrm{R}_{\mathrm{G}^{2,5}}(x, \mathrm{J}\,x)\,x, y)| \leqq 2|\sin\varphi\cos\varphi\cos\psi||e'|(h^2(x)-1)^{1/2}$$
$$\leqq \cos^2\varphi\,(2-h^2(x))+\sin^2\varphi\cos^2\psi\,(h^2(x)-1),$$

hence the bracket expression is less than 1, so that finally we have

$$\overline{\mathrm{R}}\{x, (\cos\varphi.\mathrm{J}\,x+\sin\varphi.y)\} \leqq \max(\lambda+\mu+\nu, \lambda+4\mu+2\nu). \quad \blacksquare$$

7.8. *Proof of Lemma* 7.6. – Since

$$\begin{aligned} x\wedge \mathrm{J}\,x &= (\cos^2\beta\cos^2\gamma+\sin^2\gamma)\,a\wedge \mathrm{J}\,a+\sin^2\beta\cos^2\gamma.\,b\wedge \mathrm{J}\,b\\ &\quad+\sin\beta\cos\beta\cos^2\gamma\,(a\wedge \mathrm{J}\,b+b\wedge \mathrm{J}\,a)\\ &\quad+\sin\beta\sin\gamma\cos\gamma\,(a\wedge b+\mathrm{J}\,a\wedge \mathrm{J}\,b),\end{aligned}$$

we get from the spectral decomposition of $\hat{\mathrm{R}}_{\mathrm{G}^{2,5}}$ (*see* 7.2)

$$\hat{\mathrm{R}}_{\mathrm{G}^{2,5}}(x\wedge \mathrm{J}\,x) = 2\,\mathrm{J}+2\sin\beta\sin 2\gamma\,(a\wedge b+\mathrm{J}\,a\wedge \mathrm{J}\,b).$$

From this, a direct computation gives

$$\begin{aligned}\mathrm{R}_{\mathrm{G}^{2,5}}(x, \mathrm{J}\,x)\,x &= 1/2\,\hat{\mathrm{R}}_{\mathrm{G}^{2,5}}(x\wedge \mathrm{J}\,x)(x) \quad \text{(with our convention)}\\ &= \mathrm{J}\,x+\sin\beta\sin 2\gamma.\,e\end{aligned}$$

with $e = \sin\beta\sin 2\gamma.\mathrm{J}\,x+e'$.

This proves (i) and (iii).

Similarly we have

$$\begin{aligned}\mathrm{R}_{\mathrm{G}^{2,5}}(x, c)\,x &= \cos^2\gamma.\,c+\cos\beta\sin\gamma\cos\gamma.\mathrm{J}\,c,\\ \mathrm{R}_{\mathrm{G}^{2,5}}(x, \mathrm{J}\,c)\,x &= \cos\beta\ \sin\gamma\cos\gamma.\,c+\sin^2\gamma.\mathrm{J}\,c,\end{aligned}$$

which proves that span $\{c, \mathrm{J}\,c\}$ is an invariant subspace of the mapping $y\mapsto \mathrm{R}_{\mathrm{G}^{2,5}}(x, y)\,x$ with eigenvalues $1/2 \pm 1/2\,(2-h^2(x))^{1/2}$.

Finally $x\wedge d$ and $x\wedge \mathrm{J}\,d$ have most of their components in the kernel of $\hat{\mathrm{R}}_{\mathrm{G}^{2,5}}$ leaving us with

$$\begin{aligned}\hat{\mathrm{R}}_{\mathrm{G}^{2,5}}(x\wedge d) &= 2\cos 2\gamma\,(a\wedge b+\mathrm{J}\,a\wedge \mathrm{J}\,b),\\ \hat{\mathrm{R}}_{\mathrm{G}^{2,5}}(x\wedge \mathrm{J}\,d) &= 2\cos\beta\sin 2\gamma\,(a\wedge b+\mathrm{J}\,a\wedge \mathrm{J}\,b).\end{aligned}$$

This proves that the operator which represents the quadratic form $y\mapsto(\mathrm{R}_{\mathrm{G}^{2,5}}(x, y)\,x, y)$ on span $\{d, \mathrm{J}\,d\}$ has the eigenvalue 0 on the vector $\cos\beta.\sin 2\gamma.d-\cos 2\gamma.\mathrm{J}\,d$ (which is orthogonal to e'). The vector e' is also an eigenvector for the eigenvalue $\cos^2 2\gamma+\cos^2\beta\sin^2 2\gamma$ (we get it as a trace) completing the proof of (ii). ■

7.9. *Remarks:*

(i) with obvious changes the preceeding proof will also handle $G^{2,k}$;

(ii) another example of a linear combination with small integer coefficients where the Euler integrand vanishes is $R = 4\,R_{S^6} - 3\,R_{CP^3} + 8\,R_{G^{2,5}}$ with $A = 8$, $B = 28$, $C = -4$, $D = 8$ as eigenvalues of $\hat{R}$ (*see* 7.3). The sectional curvature range is $[0, 9]$.

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(Manuscrit, reçu le 16 novembre 1977,
révisé le 28 février 1978.)

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