ORLANDO E. VILLAMAYOR

An extension to fields of positive characteristic of Mather's construction of the Thom-Boardman sequence

Annales scientifiques de l'É.N.S. 4e série, tome 11, nº 1 (1978), p. 1-28
<http://www.numdam.org/item?id=ASENS_1978_4_11_1_1_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 11, 1978, p. 1 à 28.

AN EXTENSION TO FIELDS OF POSITIVE CHARACTERISTIC OF MATHER'S CONSTRUCTION OF THE THOM-BOARDMAN SEQUENCE (1)

BY ORLANDO E. VILLAMAYOR (h)

0. Introduction

In [3] J. Mather gives the relation between the numbers introduced by Thom in [7] and certain numbers that he obtains for an ideal in the power series ring on n indeterminates over a field k of characteristic zero.

The main tool in this direction is the concept of Jacobian extension of ideals.

Also Mount and Villamayor have introduced this concept in [6] making use of the Fitting invariant theory ([2], [4]).

The object of this work is to extend the numbers associated by Mather for a given ideal $I \subset k[[x_1, \ldots, x_n]]$ where k is now a field of positive characteristic.

So the first concept to extend was the one of Jacobian extension of ideals and this was possible making use of the Fitting ideals [6] corresponding to the "higher order differentials", and certain operators introduced by Dieudonné in [1].

1. Modules of differentials [8]

In this work ring or k-algebra will mean unitary and commutative.

1.1. Given a k-algebra A we define $\overline{\Phi} : A \times A \rightarrow A$ $\overline{\Phi}(a, b) = a.b$ which is k-bilinear so there is a well defined linear morphism Φ such that the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{\overline{\Phi}} & A \\ {\scriptstyle p}\downarrow & \nearrow_{\Phi} & \\ A \otimes_k A & & \end{array}$$

commutes.

(1) This work was partially supported by a fellowship of the Consejo Nacional de Investigaciones Cientificas Técnicas (Argentina).

Let $I(A/k)$ be the kernel of Φ. If we give to $A \underset{k}{\otimes} A$ the natural structure of a left A-module the ideal $I(A/k)$ is generated (as a submodule) by $\{1 \otimes a - a \otimes 1 / a \in A\}$.

In fact given $x \in I(A/k)$:

$$x = \sum_{i=1}^{n} a_i \otimes b_i \qquad \text{and} \qquad \Phi(x) = \sum_{i=1}^{n} a_i b_i = 0,$$

$$x = x - 0 = \sum_i (a_i \otimes b_i) - (\sum_i a_i b_i) \otimes 1$$

$$= \sum_i a_i \otimes b_i - a_i b_i \otimes 1 = \sum_i a_i (1 \otimes b_i - b_i \otimes 1).$$

Q.E.D.

We define now $T_k : A \to I(A/k)$ by $T_k(a) = 1 \otimes a - a \otimes 1$ which has the following properties:

(i) $T_k(1) = 0$;

(ii) T_k is k-linear;

(iii) $T_k(a.b) = a\, T_k(b) + b\, T_k(a) + T_k(a)\, T_k(b)$.

The application T_k will be called the universal Taylor k-map. If B is an A-algebra a map $L: A \to B$ which has properties (i), (ii) and (iii) will be called a Taylor k-map.

PROPERTY 1.1. — Given A, B k-algebras and $L: A \to B$ a Taylor k-map, then there is one and only one A-algebra morphism $F: I(A/k) \to B$ such that $F \circ T_k = L$ ([5]).

LEMMA 1.2. — *If* $\Phi: A \to M$ *is a k-linear morphism from a k-algebra* A *to an* A-*module* M *such that* $\Phi(1) = 0$, *then there is one and only one* A-*morphism* $\theta: I(A/k) \to M$ *such that* $\theta \circ T_k = \Phi$.

Proof. — First of all let us show that $A \underset{k}{\otimes} A = A(1 \otimes 1) \underset{A}{\oplus} I(A/k)$ direct sum of left A-modules.

The map $T_k: A \to I(A/k)$ can be extended to an A-linear map $1_A \otimes T_k: A \otimes A \to I(A/k)$ where $(1_A \otimes T_k)(a \otimes b) = a\, T_k(b)$. And $1_A \otimes T_k$ is a natural projection of A-modules, in fact $I(A/k)$ is generated as an A-module by the set $\{1 \otimes a - a \otimes 1 / a \in A\}$ and

$$(1_A \otimes T_k)(1 \otimes b - b \otimes 1) = 1\, T_k(b) - b\, T_k(1) = T_k(b).$$

On the other hand whenever $y \in A \otimes A$:

$$y = \sum_{i=1}^{n} a_i \otimes b_i = \sum_i a_i (1 \otimes b_i - b_i \otimes 1) + \sum_i a_i b_i \otimes 1$$

$$= \sum_i a_i T_k(b_i) + (\sum_i a_i b_i)(1 \otimes 1)$$

as it was to be shown.

Given $\Phi: A \to M$ k-linear we extend to $1_A \otimes \Phi: A \otimes A \to M$

$$(1_A \otimes \Phi)(a \otimes b) = a. \quad \Phi(b).$$

The condition $\Phi(1) = 0$ assures that $(1 \otimes \Phi)(1 \otimes 1) = 0$ then $1 \otimes \Phi$ is A-linear and factorizes through $I(A/k)$.

Q.E.D.

Let R be a ring, $\{a_1, \ldots, a_n\}$ a set of elements of R we will denote

$$a_1 \ldots \hat{a}_{i_1} \ldots \hat{a}_{i_r} \ldots a_n = \prod_{k \neq i_1 \ldots i_r} a_k.$$

DEFINITION 1.3. — Given R and k rings, R a k-algebra and M an R-module. An n-derivation or derivation of order n, k-linear from R to M will be a k-linear L_n which verifies:

(i) for any set $\{\alpha_0, \ldots, \alpha_n\} \subset R$:

$$L_n(\alpha_0 \ldots \alpha_n) = \sum_{i=1}^{n} (-1)^{i+1} \Big(\sum_{j_1 < \ldots < j_i} \alpha_{j_1} \ldots \alpha_{j_i} L_n(\alpha_0 \ldots \hat{\alpha}_{j_1} \ldots \hat{\alpha}_{j_i} \ldots \alpha_n) \Big);$$

(ii) $L_n(1) = 0$.

Given the map $T_k: R \to I(R/k)$ defined in 1.1 we will denote

$$D^n(R/k) = I(R/k)/I(R/k)^{n+1}$$

and by T_k^n or simply T^n the map $p \circ T_k$, p the natural projection from $I(R/k)$ to $D^n(R/k)$.

THEOREM 1.4. — *Let R, k be rings,* M *a* R*-module* R *a k-algebra and* $L: R \to M$ *a k-linear derivation of order n. The k-linear map* $T^n: R \to D^n(R/k)$ (def. 1.3) *is a k-linear derivation of order n and there is a unique* R*-linear morphism* $h: D^n(R/k) \to M$ *such that* $h \circ T^n = L$.

Conversely, if $h: D^n(R/k) \to M$ *is an* R*-linear morphism then* $h \circ T^n: R \to M$ *is a k-linear derivation of order n.*

Proof. — First of all let us show by induction on n that given a set $\{x_0, \ldots, x_n\}$ in R and $\{T_k(x_0), \ldots, T_k(x_n)\}$ in $I(R/k)$ we have

$$T_k(x_0) \ldots T_k(x_n) = \sum_{i=0}^{n} (-1)^i \sum_{j_1 < \ldots < j_i} x_{j_1} \ldots x_{j_i} T_k(x_0 \ldots \hat{x}_{j_1} \ldots \hat{x}_{j_i} \ldots x_n)$$

if $n = 1$; $T_k(x_0 x_1) - x_0 T_k(x_1) - x_1 T_k(x_0) = T_k(x_1). \; T_k(x_0)$ by definition.

If the formula is valid for n:

$$\begin{aligned}
&T_k(x_0)\dots T_k(x_n).T_k(x_{n+1})\\
&=\sum_{i=0}^{n}(-1)^i\sum_{j_1<\dots<j_i} x_{j_1}\dots x_{j_i}T(x_0,\dots,\hat{x}_{j_1}\dots x_{j_i}\dots x_n)T(x_{n+1})\\
&=\sum_{i=0}^{n}(-1)^i\sum_{j_1<\dots<j_i} x_{j_1}\dots x_{j_i}[T_k(x_0\dots\hat{x}_{j_1}\dots\hat{x}_{j_i}\dots x_n x_{n+1})\\
&\quad-(x_0\dots\hat{x}_{j_1}\dots\hat{x}_{j_i}\dots x_n)T(x_{n+1})-x_{n+1}T_k(x_0\dots\hat{x}_{j_1}\dots\hat{x}_{j_i}\dots x_n)]\\
&=\sum_{i=0}^{n+1}(-1)^i\sum_{j_1<\dots<j_i} x_{j_1}\dots x_{j_i}T_k(x_0\dots\hat{x}_{j_1}\dots\hat{x}_{j_i}\dots x_{n+1})\\
&\quad-\sum_{i=0}^{n}(-1)^i\sum_{j_1<\dots<j_i} x_0\dots x_n T_k(x_{n+1})\\
&=\sum_{i=0}^{n+1}(-1)^i\sum_{j_1<\dots<j_i} x_{j_1}\dots x_{j_i}T(x_0\dots\hat{x}_{j_1}\dots\hat{x}_{j_i}\dots x_{n+1})
\end{aligned}$$

since:

$$\sum_{i=0}^{n}(-1)^i\sum_{j_1<\dots<j_i}1=\sum_{i=0}^{n}(-1)^i\binom{n}{i}=(1-1)^n=0$$

and $T(x_0)\dots T(x_n)=0$ in $D^n(R/k)$ so

$$T_k^n(x_0\dots x_n)=\sum_{i=1}^{n}(-1)^{i+1}\sum_{j_1<\dots<j_i} x_{j_1}\dots x_{j_i}T_k^n(x_0\dots\hat{x}_{j_1}\dots\hat{x}_{j_i}\dots x_n).$$

Let $L: R\to M$ be a k-linear derivation of order n. By Lemma 1.2 there is one and only one morphism $h^*: I(R/k)\to M$ of R-modules such that $h^*\circ T_k = L$. To complete the proof we note that h^* is zero on $I(R/k)^{n+1}$:

$$\begin{aligned}
&h^*(T(x_0)\dots T(x_n))\\
&=h^*\left(\sum_{i=0}^{n}(-1)^i\sum_{j_1<\dots<j_i} x_{j_1}\dots x_{j_i}T_k(x_0\dots\hat{x}_{j_1}\dots x_{j_i}\dots x_n)\right)\\
&=\sum_{i=0}^{n}(-1)^i\sum_{j_1<\dots<j_i} x_{j_1}\dots x_{j_i}L(x_0\dots\hat{x}_{j_1}\dots\hat{x}_{j_i}\dots x_n)=0,
\end{aligned}$$

because L is a k-linear derivation of order n (Def. 1.3).

COROLLARY 1.4. — *The pair* $(T_k^n, D^n(R/k))$ *is well defined* (*up to isomorphisms*) *with the properties of Theorem* 1.4.

1.5. If R is a local ring with radical M then the R-module

$$D^n(R/k)/\bigcap_{n\in N} M^n D^n(R/k)=\hat{D}^n(R/k)$$

is separated in the M-adic topology.

Let $\theta: D^n(R/k) \to \hat{D}^n(R/k)$ be the natural projection $\theta\, T_k^n = \hat{T}_k^n$ is obviously a k-linear derivation of order n and a pair $(\hat{T}_k^n, \hat{D}^n(R/k))$ is universal with the properties of Theorem 1.4 if we restrict ourselves to the subcategory of separated modules in the M-adic topology [8].

NOTE 1.6. — Let A, B be k-algebras, a k-algebra morphism $\lambda: A \to B$ gives B a structure of A-algebra and $D^n(B/k)$ becomes an A-module.

Since T_k^n is a k-linear derivation of order n there is a unique A-module morphism $d(\lambda)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ {\scriptstyle T_k^n}\downarrow & & \downarrow{\scriptstyle T_k^n} \\ D^n(A/k) & \xrightarrow{d(\lambda)} & D^n(B/k) \end{array}$$

commutes.

An analogous proof will show that given A, B local k-algebras and $\lambda: A \to B$ a local morphism of k-algebras there will be a morphism $\hat{d}(\lambda): \hat{D}^n(A/k) \to \hat{D}^n(B/k)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & B \\ {\scriptstyle \hat{T}_k^n}\downarrow & & \downarrow{\scriptstyle \hat{T}_k^n} \\ \hat{D}^n(A/k) & \xrightarrow{\hat{d}(\lambda)} & \hat{D}^n(B/k) \end{array}$$

commutes.

PROPOSITION 1.7. — *In the conditions of Note* 1.6, *given the diagram*

$$\begin{array}{ccccc} A & \xrightarrow{\lambda} & B & & \\ {\scriptstyle T_k^n}\downarrow & & \downarrow{\scriptstyle T_k^n} & & \\ B \otimes_A D^n(A/k) & \to & D^n(B/k) & \xrightarrow{p} & C \to 0 \end{array}$$

with a commutative square and a lower exact row, then $(p \circ T_k^n, C) \simeq (T_A^n, D^n(B/A))$ *in the sense of Corollary* 1.4.

Proof. — Let $\Delta: B \to M$ an A-linear derivation of order n in a B-module M, since λ is a k-algebra morphism Δ becomes k-linear because it is A-linear, so there is one and only one morphism of B-modules γ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\Delta} & M \\ {\scriptstyle T_k^n}\downarrow & \nearrow{\scriptstyle \gamma} & \\ D^n(B/k) & & \end{array}$$

commutes.

By hypothesis

$$\Delta(\lambda(a)) = 0, \qquad \forall a \in A, \qquad \gamma(d(\lambda) T_k^n(a)) = \gamma(T_k^n(\lambda(a))) = \Delta(\lambda(a)) = 0, \qquad \forall a \in A,$$

so Image $d(\lambda) \subset$ kernel γ and γ factorizes by C.

The unicity becomes because p is an epimorphism, in fact if γ and γ' are B-module morphisms form C to M and:

$\gamma \circ p \circ T_k^n = \gamma' \circ p \circ T_k^n = \Delta$ and by the universal property of $D^n(B/k)$;

$\gamma \circ p = \gamma' \circ p$ so $\gamma = \gamma'$ because p is an epimorphism.

PROPOSITION 1.8. — *Given a multiplicative system* S *of a k-algebra* R, *then:*

$$D^n(R_s/k) \simeq R_s \underset{R}{\otimes} D^n(R/k).$$

2. Modules of higher order differentials for the ring of power series in n-variables over a field k

2.1. Dieudonné has pointed out in [1] that given the ring $k[[x]]$ of series on one indeterminate over a field k and $f(x) \in k[[x]]$ then: $f(x+Y) = Tf(x)$ where $Tf(x)$ is the Taylor expansion on the variable Y. Let us say that if we develop $f(x+Y)$ we obtain

$$f(x+Y) = \sum_{i \geqq 0} \Delta_i'(f(x)) Y^i.$$

If the characteristic of k is zero then it is well known that

$$\Delta_i'(f(x)) = \frac{1}{i!} \frac{\partial^i f(x)}{\partial^i x}.$$

But whenever the characteristic of $k = p \geqq 0$ then $i! = 0$ for any $i \geqq p$ and the operator $\partial^i/\partial^i x$ is also trivial.

However these operators Δ_i' are always well defined and if we take $\Delta_e = \Delta_t'$ for $t = p^e \, e \geqq 0$, given $n \in N$:

$$n = \alpha_0 + \alpha_1 p + \ldots + \alpha_r p^r, \qquad 0 \leqq \alpha_i < p,$$

for some r, we have

$$\Delta_n' = \Delta_r^{\alpha_r} \ldots \Delta_1^{\alpha_1} \Delta_0^{\alpha_0},$$

the product denoting the composition of operators [1].

The operator Δ_e has the following properties ($e \geqq 0$):

(i) In the restriction to the subring $k[[F^e(x)]]$ of formal series it acts as $\partial/\partial F^e(x)$;

(ii) If $f \in k[[F^e(x)]]$ and $g \in k[[x]]$:

$$\Delta_e(f \cdot g) = f \Delta_e(g) + g \Delta_e(f).$$

F denotes here the Frobenious morphism $F(x) = x^p$ and F^e means the composition of the operator e-times.

Given a local regular k-algebra R with maximal ideal M we will denote R^* the completion of R in the M-adic topology.

Suppose $\Delta: R \to N$ is a k-linear derivation of order n (1.3) on a complete separated R-module N.

PROPOSITION 2.2. — *Under the above conditions the derivation Δ of order n can be extended to a k-linear derivation of order n $\Delta: R^* \to N$.*

Proof. — The k-linear derivation Δ of order n is continuous in the M-adic topology, in fact given $\{ m_0, \ldots, m_n \} \subset M$:

$$\Delta(m_0 \ldots m_n) = \sum_{i=1}^{n} (-1)^{i+1} \sum_{j_1 < \ldots < j_i} m_{j_1} \ldots m_{j_i} \Delta(m_0 \ldots \hat{m}_{j_1} \ldots \hat{m}_{j_i} \ldots m_n)$$

so $\Delta(m_0 \ldots m_n) \subset MN$ and $\Delta(M^{n+1}) \subset MN$.

Let r^* be an element of R^* and $\{ r_n \} \subset R$, $r_n \to r^*$, we will define

$$\Delta(r) = \lim_{n \in N} \Delta(r_n),$$

which is well defined because Δ is continuous and N is a complete separated R-module.

Given a set $\{ r_0^*, \ldots, r_n^* \} \subset R^*$ and $\{ r_k^i / k \geqq 0 \} \subset R$, $i = 0, \ldots, n$ such that $r_k^i \to r_i^*$ then:

$$\begin{aligned} \Delta(r_0^* \ldots r_n^*) &= \Delta(\lim_k r_k^0 \ldots r_k^n) \\ &= \lim_k \sum_{i=1}^{n} (-1)^{i+1} \sum_{j_1 < \ldots < j_i} r_k^{j_1} \ldots r_k^{j_i} \Delta(r_k^0 \ldots \hat{r}_k^{j_1} \ldots \hat{r}_k^{j_i} \ldots r_k^n) \\ &= \sum_{i=1}^{n} (-1)^{i+1} \sum_{j_1 < \ldots < j_i} r_{j_1}^* \ldots r_{j_i}^* \Delta(r_0^* \ldots \hat{r}_{j_1}^* \ldots \hat{r}_{j_i}^* \ldots r_n^*), \end{aligned}$$

so $\Delta: R^* \to N$ becomes obviously a k-linear derivation of order n.

PROPOSITION 2.3. — *The natural inclusion i: $R \to R^*$ gives the following commutative diagram* (Note 1.6):

$$\begin{array}{ccc} R & \xrightarrow{\quad i \quad} & R^* \\ {\scriptstyle T^n}\downarrow & & \downarrow{\scriptstyle T^n_*} \\ R^* \otimes \hat{D}^n(R/k) & \xrightarrow{1 \otimes d(i)} & \hat{D}^n(R^*/k). \end{array}$$

If $\hat{D}^n(R/k)$ is a finitely generated R-module then $1 \otimes d(i)$ splits.

Proof. — Since $\hat{D}^n(R/k)$ is a finitely generated R-module then $R^* \underset{R}{\otimes} \hat{D}^n(R/k)$ will be a completely separated R-module so there is D: $R^* \to R^* \underset{R}{\otimes} \hat{D}^n(R/k)$ such that $D \circ i = T^n$. Now by the universal property of $\hat{D}^n(R^*/k)$ there is a R^*-linear morphism

$$\gamma: \quad D^n(R^*/k) \to R^* \otimes D^n(R/k),$$

such that $D = \gamma\, T^n_*$.

We will show that $\gamma\,(1 \otimes d(i))$ = identity of $R^* \underset{R}{\otimes} D(R/k)$.

γ and $1 \otimes d(i)$ are R^*-linear and $R^* \otimes D^n(R/k)$ is generated over R^* by the set $\{ 1 \otimes T^n(r) \in R \}$. We can show that $[\gamma\,(1 \otimes d(i))]\,(1 \otimes T^n(r)) = 1 \otimes T^n(r)$ in fact:

$$(1 \otimes d(i)).T^n = T^n_* i \quad \gamma(1 \otimes d(i))(1 \otimes T^n r) = \gamma\, T^n_*(i(r)) = D(i(r)) = 1 \otimes T^n(r).$$

Q.E.D.

2.4. Let us take $A = k[x_1, \ldots, x_n]$, a polynomial ring with n indeterminates over a ring k and go back to the definition of $I(A/k)$ and T_k: $A \to I(A/k)$ of 1.1:

$$A \underset{k}{\otimes} A \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

where $x_i \otimes 1$ corresponds to x_i and $1 \otimes x_i$ to y_i so $T_k(x_i) = x_i - y_i$.

PROPOSITION 2.5. — (i) *If x belongs to* A, *a k-algebra and* T_k: $A \to I(A/k)$ *is the universal Taylor map* (1.1) *then:* $T_k(x^n) = (x + T_k(x))^n - x^n$ *in* $A \underset{R}{\otimes} A$ (*where x means* $x \otimes 1$).

Proof. — In fact $a \to a + T(a) = 1 \otimes a$ is a ring homomorphism, so

$$a^n + T(a^n) = (a + T(a))^n \qquad \text{and} \qquad T(a^n) = (a + T(a))^n - a^n$$

(ii) On the conditions of the last proposition if $\{ x_1, \ldots, x_r \}$ are r elements of A then for nonnegative integers $\alpha_1, \ldots, \alpha_r$:

$$T(x_1^{\alpha_1} \ldots x_r^{\alpha_r}) = (x_1 + T x_1)^{\alpha_1} \ldots (x_r + T x_r)^{\alpha_r} - x_1^{\alpha_1} \ldots x_r^{\alpha_r}.$$

Proof. — Again, since $a \to a + T(a)$ is a ring homomorphism

$$T(x_1^{\alpha_1} \ldots x_r^{\alpha_r}) + x_1^{\alpha_1} \ldots x_r^{\alpha_r} = (x_1 + T x_1)^{\alpha_1} \ldots (x_r + T x_r)^{\alpha_r}$$

as was to be shown.

COROLLARY 2.6. — *Taking* $A = k[x_1, \ldots, x_n]$ *the ring of polynomials in n-indeterminates over a field k then the universal Taylor map:*

$$T_k: \quad A \to k[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

satisfies

$$T_k(f(x_1, \ldots, x_n)) = f(x_1 + T x_1, \ldots, x_n + T x_n) - f(x_1, \ldots, x_n)$$

in

$$A \underset{k}{\otimes} A \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

2.7. Since $T(x_i) = x_i - y_i$ $i = 1, \ldots, n$ is an algebraically independent set over the subring $k[x_1, \ldots, x_n]$ of $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ then by the last corollary and 1.1 we can assure that the module $I(A/k)$ is freely generated by the monomials in $\{T x_1, \ldots, T x_n\}$ and if $N^* = N \cup \{0\}$.

$$T_k(f(x_1, \ldots, x_n))$$
$$= \sum_{(\alpha(1), \ldots, \alpha(n)) \in (N^*)^n} \Delta(\alpha(1), \ldots, \alpha(n)).(f).(T x_1)^{\alpha(1)} \ldots (T x_n)^{\alpha(n)},$$

where $\Delta(\alpha(1), \ldots, \alpha(n))(f)$ is obviously zero for almost all $(\alpha(1), \ldots, \alpha(n)) \in (N^*)^n$. (This was introduced in 2.1 [1].)

COROLLARY 2.7. — *Given* A *in the above conditions then* $D^r(A/k) = I(A/k)/I(A/k)^{r+1}$ *is the* A*-module freely generated by the image of the set*

$$\{T x_1^{\alpha(1)} \ldots T x_n^{\alpha(n)} / \alpha(1) + \ldots + \alpha(n) \leqq r\}$$

with dual base

$$\{\gamma(\alpha(1) \ldots \alpha(n)) / \alpha(1) + \ldots + \alpha(n) \leqq r\}$$

and

$$\gamma(\alpha(1), \ldots, \alpha(n)) T_k^n = \Delta(\alpha(1), \ldots, \alpha(n)).$$

If we take $R = k[x_1, \ldots, x_n]_M$ $M = (x_1, \ldots, x_n)$ *the localization of the ring of polynomials in n variables over k on the complement of* M, *the completion of* R *in the* M*-adic topology will be*

$$R^* = k[[x_1, \ldots, x_n]]$$

the formal power series in n variables over k.

PROPOSITION 2.8 ([9] Lemma 4.7). — *Under the above conditions*

$$\hat{D}^n(R^*/k) \simeq R^* \otimes_R \hat{D}^n(R/k).$$

Proof. — $D^n(R/k)$ is finitely generated by Corollary 2.7 and Proposition 1.8 so $D^n(R/k) = \hat{D}^n(R/k)$.

Applying now Proposition 2.3: $\hat{D}^n(R^*/k) \simeq R^* \otimes_R D^n(R/k) \oplus N$ for some R^*-submodule N.

If $\gamma: \hat{D}^n(R^*/k) \to P$ is a R^*-linear morphism of separated modules and if $R^* \otimes_R D^n(R/k) \subset \ker \gamma$ then γ corresponds to a k-linear derivation of order n, Δ: $R^* \to P$ for $\Delta = \gamma \circ T_k^n$ so $\Delta(i(r)) = 0$ if $r \in R$, $i: R \to R^*$ the natural inclusion.

Since Δ is continuous then Δ is the zero operator and so is γ. Let $d(i)$:

$$R^* \otimes D^n(R/k) \to \hat{D}^n(R^*/k)$$

be the natural inclusion and

$$p: \quad \hat{D}^n(R^*/k) \to N,$$

the natural projection.

We showed above that given any separated R^*-module P and a R^*-linear map

$$\gamma: \quad \hat{D}^n(R^*/k) \to P$$

such that $\gamma \circ d(i) = 0$, then $\gamma = 0$.

Since $p \circ d(i) = 0$, then $p = 0$, so $N = 0$ as was to be shown.

3. Jacobian extensions

3.1. Let us consider a finitely generated A-module M and the following exact sequence $0 \to R \to A^n \xrightarrow{\varphi} M \to 0$ where R is the set of n-tuples such that their image by φ is zero. We can form a matrix whose rows are vectors that generate R as A-module, and for any natural number s; $1 \leqq s \leqq n$ we define $f_s(M) = \langle \det(M_\alpha) \rangle$ ideal generated by determinants of M_α, where M_α runs over all $(n-s+1) \times (n-s+1)$ sub-matrices we can obtain from that matrix. And $f_t(M) = A$ if $t > n$.

Fitting [2] shows that these ideals are independent of the solution given before.

3.1.1. Let $\{v_1, \ldots, v_n\} \subset A^n$ such that $\sum_{i=1}^{n} Av_i = A^n$ and $\{v_1, \ldots, v_r\} \subset R$.

If

$$p: \quad A^n \to \sum_{i=r+1}^{n} A v_i \simeq A^{n-r}$$

is the natural projection then $0 \to p(R) \to A^{n-r} \to M \to 0$ is also an exact sequence.

Given a prime ideal $P \subset A$ the rank of M_P is s if and only if $f_s(M) \subset P$ and $f_{s+1}(M) \not\subset P$, it can be immediately proved that

$$f_s(M) \subset f_t(M) \quad \text{whenever } s \leqq t.$$

The ideals $f_s(M)$ will be called Fitting ideals.

If A is a local ring we will denote by $f(M)$ the biggest proper Fitting ideal.

3.1.2. If A is a local ring $I = \operatorname{rad}(A)$ and $R \subset IA^n$ then $f(M)$ is the ideal generated by the coefficients of the n-tuples that belong to R, i. e. $f(M) = f_n(M)$.

In what follows $A = k[[x_1, \ldots, x_n]]$ will be the formal power series in n independent variables over a perfect field k of characteristic $p > 0$, F as before will be the Frobenious morphism, $F(a) = a^p$.

$M = \operatorname{rad}(A)$ and R.S.P. will mean a regular system of parameters.

An ideal will always mean a proper ideal and rank of an ideal I will mean $\dim_k (I+M^2)/M^2$.

LEMMA 3.2. — *Given an ideal* $I \subset k[[y_1, \ldots, y_n]] = A$ *generated by a set*

$$\{y_1, \ldots, y_s\} \cup B, \qquad 0 \leqq s \leqq n, \qquad B \subset k[[y_j]]_{j>s}(k[[y_{s+1}, \ldots, y_n]]),$$

then:

$$I \cap k[[y_j]]_{j>s} = B\, k[[y_j]]_{j>s}.$$

Proof. — If we consider the isomorphism $\alpha = \theta\, i$

$$k[[y_j]]_{j>s} \xrightarrow{i} A \xrightarrow{\theta} k[[y_1, \ldots, y_n]]/\langle y_1, \ldots, y_s \rangle.$$

Since $\langle y_1, \ldots, y_s \rangle \subset I$ we can identify $I \cap k[[y_j]]_{j>s}$ with $\theta(I) = B.k[[y_j]]_{j>s}$ as was to be shown.

LEMMA 3.3. — *If an ideal* $I \subset A$ *admits a set of generators* $B \subset k[[F(x_1), \ldots, F(x_n)]]$ *then:*

$$I \cap k[[F(x_1), \ldots, F(x_n)]] = B.k[[F(x_1), \ldots, F(x_n)]].$$

Proof. — Suppose $\sum_{i=1}^{r} h_i f_i \in k[[F(x_1), \ldots, F(x_n)]]$, $h_i \in B$, $f_j \in A$. Since A is a free finitely generated $k[[F(x_1), \ldots, F(x_n)]]$-module with basis:

$$\{x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \alpha = (\alpha_1, \ldots, \alpha_n)/0 \leqq \alpha_i < p\}$$

let

$$f_i = \sum_\alpha a_\alpha^i x^\alpha, a_\alpha^i \in k[[F(x_1), \ldots, F(x_n)]], \qquad \sum_i h_i f_i = \sum_\alpha (\sum_i h_i a_\alpha^i) x^\alpha$$

so

$$\sum h_i a_\alpha^i = 0 \qquad \text{if } \alpha \neq (0, \ldots, 0) = 0 \qquad \text{and} \qquad \sum h_i f_i = \sum h_i a_0^i.$$

Q.E.D.

COROLLARY 3.4. — *Let* $A = k[[y_1, \ldots, y_n]]$ *and an ideal*

$$I = \langle y_1, \ldots, y_{s(0)} \rangle + \langle F(y_1), \ldots, F(y_{s(1)}) \rangle + \ldots + \langle F^e(y_1), \ldots, F^e(y_{s(e)}) \rangle$$
$$+ \langle B \rangle, s(0) \leqq s(1) \leqq \ldots \leqq s(e) \qquad \text{and} \qquad B \subset k[[F^e(y_j)]]_{j>s(e)}$$

then:

$$I \cap k[[F^e(y_j)]]_{j>s(e)} = B.k[[F^e(y_j)]]_{j>s(e)}.$$

Proof. — By induction on e.

For $e = 0$ it was proved in Lemma 3.2. $k \Rightarrow k+1$.

Let

$$I = \langle y_1, \ldots, y_{s(0)} \rangle + \ldots + \langle F^k(y_1), \ldots, F^k(y_{s(k)}) \rangle$$
$$+ \langle F^{k+1}(y_1), \ldots, F^{k+1}(y_{s(k+1)}) \rangle + \langle B \rangle,$$
$$s(0) \leqq s(1) \leqq \ldots \leqq s(k) \leqq s(k+1) \qquad \text{and} \qquad B \subset k[[F^{k+1}(y_j)]]_{j>s(k+1)}.$$

By hypothesis

$$\begin{aligned}I \cap k[[F^k(y_j)]]_{j>s(k)}\\ = \{\{F^{k+1}(y_{s(k)+1}), \ldots, F^{k+1}(y_{s(k+1)})\} \cup B\}\, k[[F^k(y_j)]]_{j>s(k)}\end{aligned}$$

by Lemma 3.3:

$$\begin{aligned}(I \cap k[[F^k(y_j)]]_{j>s(k)}) \cap k[[F^{k+1}(y_j)]]_{j>s(k)}\\ = \{\{F^{k+1}(y_{s(k)+1}), \ldots, F^{k+1}(y_{s(k+1)})\} \cup B\}.k[[F^{k+1}(y_j)]]_{j>s(k)}.\end{aligned}$$

Now by Lemma 3.2

$$\begin{aligned}[\{\{F^{k+1}(y_{s(k)+1}), \ldots, F^{k+1}(y_{s(k+1)})\} \cup B\}\, k[[F^{k+1}(y_j)]]_{j>s(k)}]\\ \cap\, k[[F^{k+1}(y_j)]]_{j>s(k+1)} = B.k[[F^{k+1}(y_j)]]_{j>s(k+1)}\end{aligned}$$

as it was to be shown.

LEMMA 3.5. — *If* $I \subset A$ *is an ideal in the conditions of Corollary* 3.4 *then*

$$I \cap k[[F^e(y_1), \ldots, F^e(y_n)]] = \langle F^e(y_1), \ldots, F^e(y_{s(e)})\rangle + \langle B\rangle$$

(*the ideals generated in the subring* $k[[F^e(y_1), \ldots, F^e(y_n)]]$).

Proof. — Clearly

$$\langle F^e(y_1), \ldots, F^e(y_{s(e)})\rangle \subset I \cap k[[F^e(y_1), \ldots, F^e(y_n)]]$$

if

$$f \in I \cap k[[F^e(y_1), \ldots, F^e(y_n)]]$$

then

$f = f' + f''$, $f' \in \langle F^e(y_1), \ldots, F^e(y_{s(e)})\rangle$, $f'' \in I \cap k[[F^e y_j]]_{j>s(e)} = B\,k[[F^e y_j]]_{j>s(e)}$ by Corollary 3.4.

We will say that an ideal $I \subset A = k[[x_1, \ldots, x_q]]$ is closed by the action of the derivations if it has the following property: $\partial f/\partial x_i \in I\ \forall f \in I$, $i = 1, \ldots, n$.

LEMMA 3.6. — *An ideal* $I \subset A$ *is closed by the action of the derivations if and only if it admits a family of generators in the subring* $k[[F(x_1), \ldots, F(x_n)]]$.

Proof. — Since the sufficient condition is trivial we will show the necessity.

Let $P \subset Z^n$, $P = \{\alpha = (\alpha_1, \ldots, \alpha_n)/0 \leqq \alpha_i < p, i = 1, \ldots, n\}$ we have already pointed out that A is a free $k[[F(x_1), \ldots, F(x_n)]]$-module with basis

$$\{x^\alpha = x_1^{\alpha_1}.x_2^{\alpha_2} \ldots x_n^{\alpha_n},\ \alpha = (\alpha_1, \ldots, \alpha_n) \in P\}$$

if $f \in I$, $f = \sum_{\alpha \in F} a_\alpha x^\alpha$, $a_\alpha \in k[[F(x_1), \ldots, F(x_n)]]$, there is $\alpha_0 \in F$ such that

(i) $|\alpha| = \Sigma \alpha_i \leqq |\alpha_0|$ if $a_\alpha \neq 0$;

(ii) $a_{\alpha_0} \neq 0$,

if $\alpha_0 = (\beta_1, \ldots, \beta_n)$ it can be shown that

$$\left[\frac{\partial}{\partial x_1}\right]^{\beta_1} \cdots \left[\frac{\partial}{\partial x_n}\right]^{\beta_n} f = {}_{\beta 1}! \ldots {}_{\beta n}!\, a_{\alpha_0} \qquad \text{so} \quad a_{\alpha_0} \in \mathrm{I},$$

and since F is finite we can assure that $a_\alpha \in \mathrm{I}\ \forall\ \alpha \in \mathrm{F}$.

PROPOSITION 3.7. — *Given any ideal* $\mathrm{I} \subset \mathrm{A}$ *there is a regular system of parameters* (R.S.P.) $\{y_1, \ldots, y_n\}$ *such that*

$$\mathrm{I} = \langle y_1, \ldots, y_{s(0)} \rangle + \langle \mathrm{F}(y_1), \ldots, \mathrm{F}(y_{s(1)}) \rangle + \ldots$$
$$+ \langle \mathrm{F}^e(y_1), \ldots, \mathrm{F}^e(y_{s(e)}) \rangle + \langle \mathrm{B} \rangle\, s(0) \leqq s(\mathrm{i}) \leqq \ldots \leqq s(e),$$
$$\mathrm{B} \subset \mathrm{rad}\,(k[[\mathrm{F}^e(y_j)]]_{j>s(e)})^2.$$

Proof. — It is enough to show that for any ideal the proposition is true taking $e = 0$.

Let $\{y_1, \ldots, y_{s(0)}\} \subset \mathrm{I}$ such that $\{\bar{y}_1, \ldots, \bar{y}_{s(0)}\}$ is a base of the k-vector space $(\mathrm{I}+\mathrm{M}^2)/\mathrm{M}^2$, $\mathrm{M} = \mathrm{rad}\,(\mathrm{A})$. $\{y_1, \ldots, y_{s(0)}\}$ can now be extended to a set of generators of I taking a set $\mathrm{B} \subset (k[[y_j]]_{j>s(0)})$. Since rank $\mathrm{I} = s_0$, we can take

$$\mathrm{B} \subset \mathrm{rad}\,(k[[y_j]]_{j>s(0)})^2.$$

Given an ideal I in the conditions of Proposition 3.7 we will denote

$$\mathrm{Y} = \{\{y_1, \ldots, y_n\}; \{s(0), \ldots, s(e)\}; \mathrm{B}\}.$$

DÉFINITION. — Given an ideal I and Y in the above conditions

$$\delta_e^y(\mathrm{I}) = \left\langle \mathrm{I}, \frac{\partial g}{\partial \mathrm{F}^e(y_j)}, g \in \mathrm{B}, j > s(e) \right\rangle.$$

PROPOSITION 3.8. — *In the above conditions if* $\mathrm{I} = \delta_e^y(\mathrm{I})$ *then* B *can be chosen in* $\mathrm{rad}\,(k[[\mathrm{F}^{e+1}(y_j)]]_{j>s(e)})$.

Proof. — If $\mathrm{I} = \delta_e^y(\mathrm{I})$ then: for any $g \in \mathrm{B}$, $r > s(e)$:

$$\frac{\partial g}{\partial \mathrm{F}^e(y_r)} \in \mathrm{I} \cap k[[\mathrm{F}^e(y_j)]]_{j>s(e)} = \mathrm{B}\, k[[\mathrm{F}^e(y_j)]]_{j>s(e)} \quad \text{(Cor. 3.4)}$$

but $\mathrm{B}.k[[\mathrm{F}^e(y_j)]]_{j>s(e)}$ closed by the derivations means that B′ can be taken in $\mathrm{rad}\,(k[[\mathrm{F}^{e+1}(y_j)]]_{j>s(e)})$ such that

$$\mathrm{B}'.k[[\mathrm{F}^e(y_j)]]_{j>s(e)} = \mathrm{B}\, k[[\mathrm{F}^e(y_j)]]_{j>s(e)} = \mathrm{I} \cap k[[\mathrm{F}^e(y_j)]]_{j>s(e)}.$$

(Lemma 3.6).

COROLLARY 3.9. — *If* $I = \delta_e^y(I)$ *then there is a new set* $Y' = \{\{y'_1, \ldots, y'_n\}; \{s'(0), \ldots, s'(e+1)\}; B'\}$, $\{y'_1, \ldots, y'_n\}$ *an R.S.P.;*

$$s'(0) \leqq \ldots \leqq s'(e+1),\ B' \subset \operatorname{rad}(k[[F^{e+1}(y_j)]])_{j>s(e+1)})^2$$

such that

$$I = \langle y'_1, \ldots, y'_{s(0)} \rangle + \langle F(y'_1), \ldots, F(y'_{s(1)}) \rangle + \ldots$$
$$+ \langle F^e(y'_1), \ldots, F^e(y'_{s(e)}) \rangle + \langle F^{e+1}(y'_1), \ldots, F^{e+1}(y'_{s(e+1)}) \rangle + \langle B' \rangle.$$

Proof. — In fact since B can be chosen in $\operatorname{rad}(k[[F^{e+1}(y_j)]]_{j>s(e)})$ (Prop. 3.8) then there is a number $s(e+1) \geqq s(e)$ such that

$$B\,k[[F^{e+1}(y_j)]]_{j>s(e)} = \{F^{e+1}(y_{s(e)+1}), \ldots, F^{e+1}(y_{s(e+1)})\}\,k[[F^{e+1}(y_j)]]_{j>s(e)}$$
$$+B'.k[[F^{e+1}(y_j)]]_{j>s(e+1)}$$

and $B' \subset \operatorname{rad}(k[[F^{e+1}(y_j)]]_{j>s(e+1)})^2$. (Prop, 3.7 applied to

$$B\,k[[F^{e+1}(y_j)]]_{j>s(e)} \subset k[[F^{e+1}(y_j)]]_{j>s(e)}.$$

NOTATION. — Let $\Omega(e)$ be $\hat{D}^n(A/k)$ if $n = p^e$ $(e \geq 0)$ (1.5),

THEOREM 3.10. — *Given* $I \subset A$ *an ideal and a system of parameters* $\{y_1, \ldots, y_n\}$ such that

$$I = \langle y_1, \ldots, y_{s(0)} \rangle + \langle F(y_1), \ldots, F(y_{s(1)}) \rangle + \ldots + \langle F^e(y_1), \ldots, F^e(y_{s(e)}) \rangle$$
$$+ \langle B \rangle,\ s(0) \leqq s(1) \leqq \ldots \leqq s(e)\, B \subset \operatorname{rad}(k[[F^e(y_j)]]_{j>s(e)})^2$$

then:

(3.1) (i) $$f(A/I \otimes \Omega(e)/\Delta I) = \left\langle I, \frac{\partial f}{\partial F^e(y_j)}, f \in I,\ j > s(e) \right\rangle;$$

(ii) $$f(A/I \otimes \Omega(e)/\Delta I) = \left\langle I, \frac{\partial g}{\partial F^e(y_j)}, g \in B,\ j > s(e) \right\rangle.$$

Where ΔI is the submodule generated by the elements $\{1 \otimes Tf / f \in I\}$ and $T: A \to \hat{D}^n(A/k)$ is the natural derivation.

Proof, — By induction on $e \in \mathbb{Z}$, $e = 0$,

Given an ideal $a \subset A$ and a regular system of parameters $\{y_1, \ldots, y_n\}$ such that $a = \langle y_1, \ldots, y_{s(0)} \rangle + \langle B \rangle$ $B \subset \operatorname{rad}(k[[y_j]])^2_{j>s(0)}$ then

$$\{Ty_1, \ldots, Ty_{s(0)}\} \subset \Delta a \subset \hat{D}^1(A/k) = \Omega(0)$$

the hypothesis assures that $(\partial f/\partial y_j)(0, \ldots, 0) = 0$ for any $f \in a$, $j > s(0)$, So we know that

$$f(A/a \otimes \Omega(0)/\Delta a) = \left\langle a, \frac{\partial f}{\partial y_j}, f \in a\, j > s(0) \right\rangle \quad (3.1.1,\ 3.1.2).$$

On the other hand, given $g \in a$, $f \in \mathrm{A}$, $\mathrm{T}(f \circ g) = f\mathrm{T}g + g\mathrm{T}f$ where $\mathrm{T}: \mathrm{A} \to \Omega(0)$ is the natural derivation, so given any family G of generators for a then

$$\bar{\mathrm{G}} = \{ 1 \otimes \mathrm{T}g, g \in \mathrm{G} \}$$

is a family of generators for the submodule Δa in $\mathrm{A}/a \otimes \Omega(0)$ and using Fitting theory (3.1):

$$f(\mathrm{A}/a \otimes \Omega(0)/\Delta a) = \left\langle a, \frac{\partial g}{\partial y_j} g \in \mathrm{B}\, j > s(0) \right\rangle.$$

$k \Rightarrow k+1$.

Since the natural derivation $\mathrm{T}: \mathrm{A} \to \hat{\mathrm{D}}^n(\mathrm{A}/k)$ satisfies

$$\mathrm{T}(f.g) = f\mathrm{T}g + g\mathrm{T}f + \mathrm{T}f.\mathrm{T}g \text{ if } n \geqq 2,$$

then given an ideal $\mathrm{I} \subset \mathrm{A}$ the A-submodule of $\mathrm{A}/\mathrm{I} \otimes \hat{\mathrm{D}}^n(\mathrm{A}/k)$ generated by the family $\{ 1 \otimes \mathrm{Th}/h \in \mathrm{I} \}$ is also an ideal in the n-truncated algebra $\hat{\mathrm{D}}^n(\mathrm{A}/k)$. In fact given $g \in \mathrm{I}$ and $f \in \mathrm{A}$, $\mathrm{T}(g).\mathrm{T}(f) = -g\mathrm{T}f. - f.\mathrm{T}g + \mathrm{T}(f.g)$ so

$$(1 \otimes \mathrm{T}g).(1 \otimes \mathrm{T}f) = -f \otimes \mathrm{T}g + 1 \otimes \mathrm{T}(f.g) \quad \text{in } \mathrm{A}/\mathrm{I} \otimes \hat{\mathrm{D}}^n(\mathrm{A}/k),$$

where both g and $g.f$ belong to I,

Now let $\mathrm{I} \subset \mathrm{A}$ be an ideal such that

$$\mathrm{I} = \langle y_1, \ldots, y_{s(0)} \rangle + \langle \mathrm{F}(y_1), \ldots, \mathrm{F}(y_{s(1)}) \rangle + \ldots + \langle \mathrm{F}^{k+1}(y_1), \ldots, \mathrm{F}^{k+1}(y_{s(k+1)}) \rangle$$
$$+ \langle \mathrm{B} \rangle, s(0) \leqq s(1) \leqq \ldots \leqq s(k) \leqq s(k+1), \mathrm{B} \subset \operatorname{rad}(k[[\mathrm{F}^{k+1}(y_j)]]_{j > s(k+1)})^2.$$

For every t, $0 \leqq t < k+1$ we have

$$\mathrm{I} = \langle y_1, \ldots, y_{s(0)} \rangle + \langle \mathrm{F}(y_1), \ldots, \mathrm{F}(y_{s(1)}) \rangle + \ldots + \langle \mathrm{F}^t(y_1), \ldots, \mathrm{F}^t(y_{s(t)}) \rangle + \langle \mathrm{B}_t \rangle$$

where $\mathrm{B}_t \subset \operatorname{rad}(k[[\mathrm{F}^{t+1}(y_j)]]_{j > s(t)})$ so combining (i) and (ii) of the inductive hypothesis we have

$$\text{(A)} \qquad \frac{\partial f}{\partial \mathrm{F}^t y_j} \in \mathrm{I}, \quad \forall f \in \mathrm{I}, \quad j > s_t, \quad t = 0, \ldots, k.$$

On the other hand we have an ideal, E of the $p^{k+1}+1$-truncated algebra $\Omega(k+1)$,

$$\mathrm{E} = \langle \mathrm{T}y_1, \ldots, \mathrm{T}y_{s(0)} \rangle + \langle \mathrm{TF}(y_1), \ldots, \mathrm{TF}(y_{s(1)}) \rangle$$
$$+ \ldots + \langle \mathrm{TF}^{k+1}(y_1), \ldots, \mathrm{TF}^{k+1}(y_{s(k+1)}) \rangle \subset \Delta\mathrm{I}.$$

We will consider as a base of $\Omega(e)$ the monomials on $\{ \mathrm{T}y_1, \ldots, \mathrm{T}y_n \}$ of degree at most p^e, since

$$\mathrm{TF}^i(y_j) = \mathrm{F}^i(\mathrm{T}y_j)$$

for the Fitting theory we will restrict our attention to the coordinates of the elements of ΔI which do not belong to the ideal E, let us say to the coordinates on the monomials of the form

$$(\mathrm{T}\, y_{j\,(0.1)} . \mathrm{T}\, y_{j\,(0.2)} \ldots \mathrm{T}\, y_{j\,(0.\,i\,(0))}).(\mathrm{TF}\, y_{j\,(1.1)} \ldots \mathrm{TF}\, y_{j\,(1.\,i\,(1))}) \times \ldots$$
$$\times (\mathrm{TF}^{k+1}\, y_{j\,(k+1.\,1)} \ldots \mathrm{TF}^{k+1}\, y_{j\,(k+1.\,i\,(k+1))}); \qquad j(s, h) \leqq j(s, i),$$

if $h \leqq i$, $s = 0, \ldots, k+1, j(m, 1) > s(m)\ m = 0, \ldots, k+1$ and where none of the $\mathrm{TF}^t(y_{j\,(t,\,i)})$ is reepeated p-times (3.1.1),

By the result (A) we know that the coordinates of an element $\mathrm{T}f$ when $f \in \mathrm{I}$ on this coordinates are again elements of I [zero on the module $\mathrm{A}/\mathrm{I} \otimes \Omega(e)$] except, may be, the coordinates on the elements $\mathrm{TF}^{k+1} y_{j,\,j} > s(k+1)$.

If we can show then that $(\partial f/\partial \mathrm{F}^{k+1} y_j)\ (0, \ldots, 0) = 0$ whenever $f \in \mathrm{I}\ j > s(k+1)$ then by Fitting theory (3.1.2):

$$f(\mathrm{A}/\mathrm{I} \otimes \Omega(k+1)/\Delta\mathrm{I}) = \left\langle \mathrm{I}, \frac{\partial f}{\partial \mathrm{F}^{k+1} y_j} / f \in \mathrm{I}, j > s(k+1) \right\rangle.$$

In fact suppose $f \in \mathrm{I}$ such that $(\partial f/\partial \mathrm{F}^{k+1} y_j)(0, \ldots, 0) \neq 0$ for some fixed $j > s(k+1)$, if $n < p^{k+1}$, $n = \alpha(0) + \alpha(1) + \ldots + \alpha(k) p^k\ \ 0 < \alpha(i) < p$, using once again the result (A):

$$f' = \left[\frac{\partial}{\partial y_j}\right]^{\alpha(0)} \ldots \left[\frac{\partial}{\partial \mathrm{F}^k y_j}\right]^{\alpha(k)} f \in \mathrm{I}, \qquad \text{if } f \in \mathrm{I},$$

then $f'(0, \ldots, 0) = 0$. Since this can be done for any $n < p^{k+1}$, the order of the series $f(0, \ldots, 0, y_j, 0, \ldots, 0) \in k[[y_j]]$ is p^{k+1}.

By Weierstrass preparation theorem there is $u \in \mathrm{A}$ and

$$\{g_t, t = 0, \ldots, p^{k+1}-1\} \subset k[[y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n]]$$

such that

$$uf = \mathrm{F}^{k+1} y_j + \sum_{t=0}^{p^{k+1}-1} g_t y_j^t$$

and since I is closed by the action of $(\partial/\partial \mathrm{F}^t y_j)$, $t = 0, \ldots, k$ (A) we have

$$\{g_t/t = 0, \ldots, p^{k+1}-1\} \subset \mathrm{I}$$

so $\mathrm{F}^{k+1} y_j \in \mathrm{I}$ which can not be since:

$$\mathrm{I} \cap k[[\mathrm{F}^{k+1} y_r]]_{r > s(k+1)} = \mathrm{B}\, k[[\mathrm{F}^{k+1} y_r]]_{r > s(k+1)}$$

(Cor. 3.4) and $\mathrm{B} \subset \mathrm{rad}\,(k[[\mathrm{F}^{k+1} y_r]]_{r > s(k+1)})^2$.

If

$$f \in \mathrm{I}, f = \sum_{t=0}^{k+1} \sum_{j=1}^{s(t)} a_j^t \mathrm{F}^t y_j + \sum_{i=1}^{n} b_i h_i, \{a_j^t\} \cup \{b_i\} \subset \mathrm{A}, \{h_i\} \subset \mathrm{B}.$$

Hence only the last summand will affect the coordinates of Tf on the monomials $TF^{k+1} y_j, j > s(k+1)$.

Now, $T(\Sigma b_i h_i) = \Sigma b_i T h_i + \Sigma h_i T(b_i) + \Sigma T b_i T h_i$ since:

$$h_i \in B \subset k[[F^{k+1} y_1, \ldots, F^{k+1} y_n]] T(h_i) \in (\Omega(k+1)) p^{k+i} \quad \text{then } T h_i T(b_i) = 0$$

in the $p^{k+1}+1$ truncated algebra $\Omega(k+1)$ so in $A/I \otimes \Omega(k+1)$ we have

$$1 \otimes T(\Sigma b_i h_i) = \Sigma \bar{b}_i \otimes T h_i$$

and using once again Fitting theory (3.1):

$$f(A/I) \otimes \Omega(k+1)/\Delta I) = \left\langle I, \frac{\partial h}{\partial F^{k+1} y_j} h \in B, \ j > s(k+1) \right\rangle.$$

COROLLARY 3.11. — *Given an ideal* $I \subset A$ *as in Proposition* 3.7 *the ideal* $\delta_e^y(I)$ *does not depend on the system of parameters but only on e. And* $I = \delta_e^y(I)$ *if and onlv if there is a family* $B' \subset \text{rad}(k[[F^{e+1} y_j]]_{j>s(e)})$ *such that*

$$I \cap k[[F^e y_j]]_{j>s(e)} = B k[[F^e y_j]]_{j>s(e)}$$
$$= B' k[[F^e y_j]]_{j>s(e)}$$

and in this case we can find a number $i(e, 1) \geqq s(e)$ *and a family*

$$B'' \subset \text{rad}(k[[F^{e+1} y_j]])^2_{j>i(e,1)}$$

such that

$$I = \langle y_1, \ldots, y_{s(0)} \rangle + \ldots + \langle F^e(y_1), \ldots, F^e(y_{s(e)}) \rangle$$
$$+ \langle F^{e+1}(y_1), \ldots, F^{e+1}(y_{i(e,1)}) \rangle + \langle B'' \rangle.$$

Proof. — This is a consequence of Theorem 3.10 (ii) and Lemma 3.6.

NOTATION. — Given an ideal I as is Proposition 3.7 let $\delta_e(I) = \delta_e^y(I)$.

COROLLARY 3.12. — *The numbers* $s(t)$ $0 \leqq t \leqq e$ *of Proposition* 3.7 *are well defined as:* $s_t = \text{rank}(I \cap k[[F^t(y_1), \ldots, F^t(y_n)]])$ *as an ideal of* $k[[F^t(y_1), \ldots, F^t(y_n)]]$.

Proof. — *See* Lemma 3.5.

COROLLARY 3.13. — *Given* $I \subset I'$ *ideals of* A *such that*

$$\text{rank}(I \cap k[[F^s(y_1), \ldots, F^s(y_n)]]) = \text{rank}(I' \cap k[[F^s y_1, \ldots, F^s y_n]]),$$

$$s = 0, \ldots, e \quad \text{and} \quad I = \delta_s(I), I' = \delta_s(I') \quad \text{for } 0 \leqq s \leqq e-1,$$

then:

(i) *there is a system of parameters* $\{y_1, \ldots, y_n\}$ $s(0) \leqq \ldots \leqq s(e)$ *and a set* $B \subset \text{rad}(k[[F^e y_j]])^2_{j>s(e)}$ *such that*

$$I = \langle y_1, \ldots, y_{s(0)} \rangle + \ldots + \langle F^e y_1, \ldots, F^e y_{s(e)} \rangle + \langle B \rangle.$$

and there is a set $B' \subset \operatorname{rad}(k[[F^e y_j]]_{j>s(e)})^2$ *such that*

(ii) $I' = \langle y_1, \ldots, y_{s(0)} \rangle + \ldots + \langle F^e y_1, \ldots, F^e y_{s(e)} \rangle + \langle B' \rangle$ *and* $B \subset B'$,

Proof. – (i) by successive applications of Theorem 3.10 (ii), Corollary 3.4 and Lemma 3.6.

(ii) This is a consequence of Corollary 3.12 and Corollary 3.4, in fact B′ must be such that

$$\begin{aligned} &B' k[[F^e(y_j)]]_{j>s(e)} \\ &\quad = I' \cap k[[F^e(y_j)]]_{j>s(e)} \supset I \cap k[[F^e(y_j)]]_{j>s(e)} = B\, k[[F^e(y_j)]]_{j>s(e)}, \end{aligned}$$

so we can take $B' \supset B$.

4. Thom-Boardman singularities

4.1, Let us make some remarks on Mather's construction of the Thom-Boardman sequence [3],

Given an ideal $I \subset \mathbf{C}[[x_1, \ldots, x_n]]$ a set $\{ y_1, \ldots, y_s \} \subset I$ can be found such that $\{ \bar{y}_1, \ldots, \bar{y}_s \}$ is a base of

$$\frac{I+M^2}{M^2}, \qquad M = \operatorname{rad}(\mathbf{C}[[x_1, \ldots, x_n]]).$$

Extending the set $\{ y_1, \ldots, y_s \}$ to a regular system of parameters $\{ y_1, \ldots, y_n \}$ he shows that the Jacobian extension of I is

$$\delta_0(I) = \left\langle I, \frac{\partial f}{\partial y_j} f \in I j > s \right\rangle.$$

What we do in Proposition 3.7 and the definition that follows is to extend the concept in such a way to obtain a good definition in series over fields of positive characteristic of the operator β also introduced in [3]

$$\beta(I) = I + (\delta_0(I))^2 + \ldots + (\delta_0^k(I))^{k+1} + \ldots$$

For which there is a R.S.P. $\{ y_1, \ldots, y_n \}$ and a sequence of non-negative numbers $0 \leqq s(0) \leqq s(1) \leqq \ldots \leqq s(k) \leqq \ldots \leqq n$ such that

$$\beta(I) = \sum_{j \geqq 0} (y_1, \ldots, y_{s(j)})^{j+1}, \{ y_1, \ldots, y_{s(j)} \} \subset \delta_0^j(I),$$

$$s(j) = \dim_k \frac{(\delta_0^j(I) + M^2)}{M^2} \text{ i.e. } s(j) = \text{rank of } \delta_0^j(I).$$

This is not true in general when the field k is of positive characteristic $p > 0$, take $I = \langle x_1^p, \ldots, x_n^p \rangle$, $\delta_0(I) = I$ and there will be no R.S.P. such that $\beta(I) = I$ has the form described above. If we take a principal ideal $I = \langle F \rangle F \in M^2$, $F = F^1 + F^{11}$ such that $F^{11} \in (x_1^p, \ldots, x_n^p)$:

$$\delta_0(I) = \left\langle I, \frac{\partial F}{\partial x_j} j = 1, \ldots, n \right\rangle$$

since $(x_1^p, \ldots, x_n^p)$ is closed by the action of the partial derivations (it is also the biggest ideal with this property as shown in Lemma 3.6), then F'' and his partial derivations will always be in $(x_1^p, \ldots, x_n^p) \subset M^2$ so will never affect the numbers $s(k)$ obtained in [3].

Another important difference of the operator δ_0 in positive characteristic is the following, If characteristic of k is zero, let $s(k) = \operatorname{rank}(\delta_0^k(I))$ if m is such that

$$s(m) = s(j)\ \forall j \geqq m \text{ then } \delta_0^m(I) = \delta_0^j(I).$$

It is enough to prove that $\delta_0(\delta_0^m(I)) = \delta_0^m(I)$ in fact

$$\delta_0^m(I) = \langle y_1, \ldots, y_{s(m)} \rangle + \langle B \rangle, B \subset \operatorname{rad}(k[[y_j]]_{j > s(m)})^2$$

(Prop. 3.7 for charac $k = 0$) $s(m) = s(m+r)\ \forall r \geqq 0$ means that

$$\left\{ B, \frac{\partial^s}{\partial y_{j(1)} \partial y_{j(s)}} g,\ g \in B,\ j(i) > s(m),\ s \leqq r \right\} \subset \operatorname{rad}(k[[y_j]]_{j > s(m)})^2,$$

$$\forall r \geqq 0 \text{ fixed.}$$

If charac $k = 0$ this assures that $B = 0$. Again this is not true in general if characteristic is $p > 0$. Take the ideal

$$I = \langle x_1^{p+1} \rangle \subset \langle x_1^p, \ldots, x_n^p \rangle \subset M^2, \qquad \delta^k(I) \subset (x_1^p, \ldots, x_n^p) \subset M^2, \qquad \forall k \geqq 0,$$

so $s(k) = 0$, $\forall k \geqq 1$ but $\delta_0(I) = \langle x_1^p \rangle \neq I$.

We have to define the operators δ, β and the Thom-Boardman numbers in order to solve these problems when characteristic of k is not zero.

NOTE 4.1. – Given an ideal $D \subset A$ such that $D = \delta_0(D) = \ldots = \delta_{e-1}(D)$ there will be a R.S.P. $\{y_1, \ldots, y_n\}$ and nonnegative numbers $s(0) \leqq s(1) \leqq \ldots \leqq s(e-1)$ such that

$$D = \sum_{r=0}^{e-1} \langle F^r y_1, \ldots, F^r y_{s(r)} \rangle + \langle B \rangle B \subset \operatorname{rad}(k[[F^e y_j]]_{j > s(e-1)})$$

(applying Prop. 3.8 several times). Now modifying the set $\{y_j\} j > s(e-1)$ if necessary we can take

$$B = \{F^e y_{s(e-1)+1}, \ldots, F^e y_{s(e)}\} \cup B', \qquad B' \subset \operatorname{rad}(k[[F^e y_j]]_{j > s(e)})^2$$

$$\delta_e(D) = D + \left\langle \frac{\partial g}{\partial F^e y_j}, g \in B', j > s(e) \right\rangle.$$

Since

$$\frac{\partial g}{\partial F^e y_v} \in k[[F^e y_j]]_{j>s(e)} \qquad \text{if} \quad g \in B', v > s(e)$$

then:

$$\delta_e(D) = \delta(\delta_e(D)) = \ldots = \delta_{e-1}(\delta_e(D)).$$

Even if we have to modify the subset $\{y_j\}_{j>s(e-1)}$ since the chains

$$\delta_e^k(D) \subset \delta_e^{k+1}(D) \subset \ldots$$

are stationary we can define $D_e = \delta_e^k(D)$ for k big enough, now $D_e = \delta_e(D_e)$ so we are in the conditions of Corollary 3.11 and we can define $\delta_{e+1}(D_e)$ and obtain an increasing chain:

$$D_e \subset D_{e+1} \subset \ldots,$$

a R.S.P. can be taken so we can define:

DEFINITION 4.1. – If $\delta^k = \delta . \delta^{k-1}$ let:

(i) $I_{-1} = I$ and given $e \in N$ $e \geqq 0$:

$$I_e = \delta_e^k(I_{e-1}) \qquad \text{for} \quad k \text{ big enough.}$$

(ii) $s(I, e)$: $Z \geqq 0 \to Z \geqq 0$ non decreasing applications $s(I, e)(k) = p(e) \leqq n$ for k big enough and

$$\delta_e^t(I_{e-1}) = \sum_{v=0}^{e-1} \langle F^v y_1, \ldots, F^v y_{p(v)} \rangle + \langle F^e y_1, \ldots, F^e y_w \rangle + \langle B \rangle,$$

$$B \subset \operatorname{rad}(k[[F^e y_j]]_{j>w})^2, \qquad w = s(I, e)(t).$$

For some R.S.P. $\{y_1, \ldots, y_n\}$ (Note 4.1). So $s(I, e)(t)$ is the rank of

$$\delta_e^t(I_{e-1}) \cap k[[F^e y_1, \ldots, F^e y_n]]$$

as an ideal of $k[[F^e y_1, \ldots, F^e y_n]]$ (Lemma 3.5). If the ideal I is fixed we will write: $i(e, k) = s(I, e)(k)$.

NOTE 4.2. – By successive application of result (i) of Theorem 3.10 we have

$$\delta_e^t(I_{e-1}) = \left\langle I, \left[\frac{\partial}{\partial y_{j(0,0)}} \cdots \frac{\partial}{\partial y_{j(0,n(0))}}\right] \cdots \left[\frac{\partial}{\partial F^e y_{j(e,0)}} \cdots \frac{\partial}{\partial F^e y_{j(e,n(e))}}\right] f / f \in I \right\rangle$$

$$j(s, h) \leqq j(s, i) \quad \text{if} \quad h \leqq i, s = 0, \ldots, e \quad \text{and} \quad j(u, v) > s(I, u)(v).$$

NOTE 4.3. – If $I = I_0 = \ldots = I_{e-1}$ then $s(I, t) = s(\delta_e(I), t)$ $t = 0, \ldots, e-1$ and $s(\delta_e(I), e)(k) = s(I, e)(k+1)$. In fact by hypothesis $I = \delta(I) = \ldots = \delta_{e-1}(I)$ and as we noted out before (Def. 4.1) there is a R.S.P. $\{y_1, \ldots, y_n\}$ of A and $0 \leqq p(0) \leqq \ldots \leqq p(e-1) \leqq s(e) \leqq n$ such that

$$I = \sum_{r=0}^{e-1} \langle F^r y_1, \ldots, F^r y_{p(r)} \rangle + \langle F^e y_1, \ldots, F^e y_{s(e)} \rangle + \langle B \rangle,$$

$$B \subset \operatorname{rad}(k[[F^e y_j]]_{j>s(e)})^2 \quad \text{and} \quad \delta_e(I) = I + \left\langle \frac{\partial g}{\partial F^e y_j} g \in B,\ j > s(e) \right\rangle,$$

since

$$\frac{\partial g}{\partial F^e y_j} \in \operatorname{rad}(k[[F^e y_j]]_{j>s(e)})$$

then:

$$\operatorname{rank}(I \cap k[[F^t y_1, \ldots, F^t y_n]])$$
$$= \operatorname{rank}(\delta_e(I) \cap k[[F^t y_1, \ldots, F^t y_n]]), \qquad 0 \leqq t \leqq e-1.$$

$I = \delta_t(I)$ and $\delta_e(I) = \delta_t(\delta_e(I))$ $t = 0, \ldots, e-1$ so $s(I, t)(k) = p(t)$, $\forall k$ and

$$s(\delta_e(I), t)(k) = p(t), \forall k.$$

If $t = e$:

$$s(\delta_e(I), e)(k)$$
$$= \operatorname{rank}(\delta_e^k(\delta_e(I)) \cap k[[F^e y_1, \ldots, F^e y_n]])$$
$$= \operatorname{rank}(\delta_e^{k+1}(I) \cap k[[F^e y_1, \ldots, F^e y_n]] = s(I, e)(k+1).$$

PROPOSITION 4.4. — *Suppose* $I \subset I'$, $I = I_0 = \ldots = I_{e-1}$, $I' = I'_0 = \ldots = I'_{e-1}$ $s(I, t) = s(I', t)$ $0 \leqq t \leqq e-1$ *and* $s(I', e)(0) = s(I, e)(0)$ *then:* $\delta_e(I_{e-1}) \subset \delta_e(I'_{e-1})$.

Proof. — Since we are in the conditions of Corollary 3.13, then there is a R.S.P. $\{y_1, \ldots, y_n\}$, $s(0) \leqq \ldots \leqq s(e)$ and $B \subset B' \subset \operatorname{rad}(k[[F^e y_j]]_{j>s(e)})^2$ such that

$$I = \sum_{r=0}^{e} \langle F^r y_1, \ldots, F^r y_{s(r)} \rangle + \langle B \rangle; \qquad I' = \sum_{r=0}^{e} \langle F^r y_1, \ldots, F^r y_{s(r)} \rangle + \langle B' \rangle,$$

$$s(r) = p(r) \text{ (Def. 4.1) } r = 0, \ldots, e-1,\ s(e) = s(I', e)(0) = s(I, e)(0)$$

and

$$\delta_e(I) = I + \left\langle \frac{\partial g}{\partial F^e y_j} g \in B,\ j > s(e) \right\rangle \subset I' + \left\langle \frac{\partial g'}{\partial F^e y_j} g' \in B' j > s(e) \right\rangle = \delta_e(I)$$

(Th. 3.10). If characteristic of k is zero only $s(I, 0)$ will have sense. Mather in [3] assigns to an ideal I a non increasing sequence of natural numbers M (I):

$$M(I)(r) = i_r = n - s(I, 0)(r-1)$$

then it is found that $M(\delta_0(I))(r) = i_{r+1}$, which we generalize in Note 4.3.

This concept together with Proposition 4.4 assures us that if I and I′ are as in Proposition 4.4 and $s(I, e) = s(I', e)$ then:

$$I_e \subset I'_e$$

in fact $I_e = \delta_k^e(I)$ for k big enough and so is I'_e. Applying once more Proposition 4.4 we have:

COROLLARY 4.5. — *Suppose* $I \subset I'$ *ideals of* A *such that*

$$s(I, t) = s(I', t)\, 0 \leqq t \leqq e-1 \quad \text{and} \quad s(I, e)(k) = s(I', e)(k), \quad 0 \leqq k \leqq k_0 - 1,$$

then:

$$\delta_e^{ko}(I_{e-1}) \subset \delta_e^{ko}(I'_{e-1}).$$

NOTE 4.6. — Let $\{y_1, \ldots, y_n\}$ be a R.S.P. of A,

$$s(0) \leqq s(I) \leqq \ldots \leqq s(r) \leqq \ldots \leqq n \quad \text{and} \quad \mathscr{A} = \sum_{r=0}^{\infty} \langle y_1, \ldots, y_{s(r)} \rangle^r \subset A.$$

$\mathscr{A}$ is an ideal generated by monomials then given $f \in k[[y_1, \ldots, y_n]] = A, f \notin \mathscr{A}$:

$$f = \sum_{\alpha \in Z^n} k_\alpha M_\alpha, \qquad \alpha = (\alpha_1, \ldots, \alpha_n)\, \alpha_i \geqq 0, \qquad M_\alpha = y_1^{\alpha_1}, \ldots, y_n^{\alpha_n}.$$

There must be $\alpha \in Z^n$ such that $k_\alpha \neq 0$ and $M_\alpha \notin \mathscr{A}$:

$$M_\alpha = y_{j(1)} y_{j(2)} \ldots y_{j(r)} j(1) \leqq j(2) \leqq \ldots \leqq j(r)$$

by direct computation if $M_\alpha \notin \mathscr{A} \Rightarrow j(1) > s(1), j(2) > s(2), \ldots, j(r) > s(r)$.

THEOREM 4.6. — *Given an ideal* $I \subset A$ *and a regular system of parameters (R.S.P.)* $\{y_1, \ldots, y_n\}$ *in the conditions of Definition* 4.1 *then:*

(i) $$I \subset \mathscr{A} = \sum_{e \geqq 0} \left(\sum_{h \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e,k)} \rangle^{k+1} \right), \qquad i(e, k) = s(I, e)(k).$$

(ii) *For each* $e \geqq 0\; s(I, e) = s(\mathscr{A}, e)$.

(iii) $\mathscr{A}$ *is maximal among the ideals* B *such that* $s(B, e) = s(\mathscr{A}, e)\ \forall e \geqq 0$.

Proof. — (i) Every $f \in A$ may be written

$$f = \sum_{\alpha \in Z^n} k_\alpha, M_\alpha, M = y_1^{\alpha(1)}, \ldots, y_n^{\alpha(n)}; \qquad k_\alpha \in k$$

and

$$\alpha(i) = \sum_{t=0}^{N} \alpha(i, t) p^t, 0 \leqq \alpha(i, t) < p$$

(p-adic notation). Let $f \in I$ and

$$f' = \left[\left[\frac{\partial}{\partial y_1}\right]^{\alpha(1,0)} \ldots \left[\frac{\partial}{\partial y_n}\right]^{\alpha(n,0)} \ldots \left[\frac{\partial}{\partial F^N y_1}\right]^{\alpha(1,N)} \ldots \left[\frac{\partial}{\partial F^N y_n}\right]^{\alpha(n,N)}\right] f,$$

then: $f'(0, 0, \ldots, 0) = (\prod_{i,j} \alpha(i,j)!)\, k_\alpha$ and

$$M_\alpha = \prod_{t=0}^{N} M_\alpha^t, \qquad M_\alpha^t = (F^t y_{j(t,1)})^{\alpha(j(t,1),t)} \ldots (F^t y_{j(t,h)})^{\alpha(j(t,h),t)},$$

$$1 \leqq j(t, i) < j(t, k) \leqq n \quad \text{if} \quad 0 \leqq i < k \leqq n-1.$$

Now

$$M_\alpha \notin \mathscr{A} \Rightarrow M_\alpha^t \notin \sum_{k>0} \langle F^t y_1, \ldots, F^t y_{i(t,k)} \rangle^{k+1}; \qquad t = 0, 1, \ldots, N.$$

So $j(t, h) > i(t, h) = s(I, t)(h)$. for every h (Note 4.6). But then going back to Note 4.2 we have

$$f' \in I_e \subset \text{rad}(A), \quad \text{then } f'(0, \ldots, 0) = 0 \quad \text{so } k_\alpha = 0 \quad \text{and} \quad f \in \mathscr{A}.$$

(ii) Mather shows in [3] that given

$$B = \sum_{t=0}^{\infty} (y_1, \ldots, y_{s(t)})^{t+1}, \qquad s(0) \leqq s(1) \leqq \ldots \leqq s(t) \leqq \ldots \leqq n,$$

$$\delta_0^k(B) = \sum_{r=k}^{\infty} (y_1, \ldots, y_{s(r)})^{r-k+1}$$

if we make use of this fact together with the definition of the operators δ_e, since

$$\frac{\partial F^r y_j}{\partial F^e y_i} = 0 \quad \text{if} \quad r > e,$$

we have

$$\delta_0^k(\mathscr{A}) = \sum_{t \geqq 0}^{\infty} (y_1, \ldots, y_{i(0, t+k)})^{t+1} + \sum_{e \geqq 1} \Big(\sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e,r)} \rangle^{r+1}\Big)$$

so

$$\mathscr{A}_0 = \langle y_1, \ldots, y_{p(0)} \rangle + \sum_{e \geqq 1} \Big(\sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e,r)} \rangle^{r+1}\Big).$$

Applying now the operator δ_1 we have

$$\delta_1^k(\mathscr{A}_0) = \langle y_1, \ldots, y_{p(0)} \rangle + \sum_{t \geqq 0} \langle F y_1, \ldots, F y_{i(1, t+k)} \rangle^{t+1}$$
$$+ \sum_{e \geqq 2} \sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e,r)} \rangle^{r+1}.$$

In general

$$\mathscr{A}_{e-1} = \langle y_1, \ldots, y_{r(0)} \rangle + \langle F y_1, \ldots, F y_{p(1)} \rangle + \ldots + \ldots$$
$$+ \langle F^{e-1} y_1, \ldots, F^{e-1} y_{p(e-1)} \rangle + \sum_{h \geqq e} \sum_{r \geqq 0} \langle F^h y_1, \ldots, F^h y_{i(h,r)} \rangle^{r+1}$$

and

$$\delta_e^k(\mathscr{A}_{e-1}) = \sum_{i=0}^{e-1} \langle F^i y_1, \ldots, F^i y_{p(i)} \rangle + \sum_{r\geq 0} \langle F^e y_1, \ldots, F^e y_{i(e,r+k)} \rangle^{r+1}$$
$$+ \sum_{h\geq e+1} \sum_{r\geq 0} \langle F^h y_1, \ldots, F^h_{i(h,r)} \rangle^{1+r},$$

then rank $(\delta_e^k(\mathscr{A}_{e-1}) \cap k[[F^e y_1, \ldots, F^e y_n]]) = i(e, k)$ in fact it will be given by: $\langle F^e y_1, \ldots, F^e y_{i(e, k)} \rangle$.

(iii) Suppose an ideal $B \supset \mathscr{A}$ such that $s(\mathscr{A}, e) = s(B, e)$ $e \geqq 0$, then by Corollary 4.5:

$$\delta_e^k(B_{e-1}) \supset \delta_e^k(\mathscr{A}_{e-1})$$

and by Corollary 3.13:

$$\delta_e^k(B_{e-1}) = \langle y_1, \ldots, y_{p(0)} \rangle + \ldots + \langle F^{e-1}(y_1), \ldots, F^{e-1}(y_{p(e-1)} \rangle$$
$$+ \langle F^e y_1, \ldots, F^e y_{i(e,k)} \rangle + \langle B' \rangle$$
$$B' \subset \operatorname{rad}(k[[F^e y_j]]_{j>i(e,k)})^2; \qquad i(e, k) = s(I, e)(k)$$

so $\{ y_1, \ldots, y_n \}$ is also a R.S.P. in the conditions of Definition 4.1 for the ideal B. Then using (i) of this theorem

$$B \subset \mathscr{A}$$

as it was to be shown.

PROPOSITION 4.7. — *Let* $\{ y_1, \ldots, y_n \}$ *be a R.S.P. of* A, $I_r = \langle F^e y_1, \ldots, F^e y_r \rangle$ $0 \leqq e$ *fixed:*

$$0 \leqq s(0) \leqq s(1) \leqq \ldots \leqq s(k) \leqq \ldots \leqq n,$$

then:

$$(\sum_{i\geq k} I_{s(i)}^{i+1-k})^{k+1} \subset \sum_{i\geq 0} I_{s(i)}^{i+1}.$$

In fact

$$(\sum_{i\geq k} I_{s(i)}^{i+1-k})^{k+1}$$
$$= \sum_{k\leq j(1)\leq \ldots \leq j(k+1)} \Pi I_{s(j(l))}^{j(l)+1-k} \quad \prod_{l=1}^{k+1} I_{s(j(l))}^{j(l)+1-k} = (\prod_{l<k+1} I_{s(j(l))}^{j(l)+1-k})(I_{s(j(k+1))}^{j(k+1)+1-k})$$

since $I_{s(i)} \subset I_{s(j)}$ *if* $i \leqq j$; *and* $j(l) \geqq k$:

$$j(l)+1-k \geqq 1 \quad \textit{and} \quad \prod_{1\leq l\leq k} I_{s(j(l))}^{j(l)+1-k} \subset I_{s(j(k+1))}^k,$$

so

$$\prod_{l=1}^{k+1} I_{s(j(l))}^{j(l)+1-k} \subset I_{s(j(k+1))}^{j(k+1)+1}$$

and this proves the proposition.

NOTE 4.7. — We will now extend what Mather defines in [3] as the ideal $\beta(I)$, if characteristic of k is zero $\beta(I) = \sum_{k \geqq 0} (\delta_0^k(I))^{k+1}$ and the ideal $\beta(I)$ is what we called $\mathscr{A}$ in Theorem 4.6 (taking $p = \text{charac. } k = 0$).

We will show that the ideal $\mathscr{A}$ depends only on I and not on the R.S.P. $\{y_1, \ldots, y_n\}$ in the conditions of Definition 4.1.

PROPOSITION 4.8. — *Given* I *and* $\mathscr{A}$ *ideals of* A *as in Theorem* 4.6.

$$I_{e,k} = \delta_e^k(I_{e-1}) \cap k[[F^e x_1, \ldots, F^e x_n]],$$

then:

$$\mathscr{A} = \sum_{e \geqq 0} \sum_{k \geqq 0} \langle I_{e,k} \rangle^{k+1}.$$

Proof. — Since the R.S.P. $\{y_1, \ldots, y_n\}$ was taken such that

$$\{F^e y_1, \ldots, F^e y_{s(i,e)(k)}\} \subset I_{e,k},$$

then obviously

$$\mathscr{A} \subset \sum_{e \geqq 0} \sum_{k \geqq 0} \langle I_{e,k} \rangle^{k+1}.$$

We proved in Theorem 4.6 that $I \subset \mathscr{A}$ and $s(I, e) = s(\mathscr{A}, e)\ \forall e \geqq 0$ then by Corollary 4.5:

$$\delta_e^k(I_{e-1}) \subset \delta_e^k(\mathscr{A}_{e-1}), \qquad \forall e, k \geqq 0,$$

so $I_{e,k} \subset \mathscr{A}_{e,k}\ \forall e, k \geqq 0$ ($\mathscr{A}_{e,k}$ defined as $I_{e,k}$):

$$\sum_{e \geqq 0} \sum_{k \geqq 0} \langle I_{e,k} \rangle^{k+1} \subset \sum_{e \geqq 0} \sum_{k \geqq 0} \langle \mathscr{A}_{e,k} \rangle^{k+1},$$

it will be enough to prove that

$$\sum_{e \geqq 0} \sum_{k \geqq 0} \langle \mathscr{A}_{e,k} \rangle^{k+1} \subset \mathscr{A}, \qquad \delta_e^k(\mathscr{A}_{e-1}) =$$

$$= \sum_{i=0}^{e-1} \langle F^i y_1, \ldots, F^i y_{p(i)} \rangle + \sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e, r+k)} \rangle^{r+1}$$

$$+ \sum_{h \geqq e+1} \sum_{r \geqq 0} \langle F^h y_1, \ldots, F^h y_{i(h,r)} \rangle^{r+1}, \qquad i(h, r) = s(\mathscr{A}, h)(r) \quad [\text{Th. } 4.6 \text{ (ii)}],$$

so

$$\mathscr{A}_{e,k} = \sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e, r+k)} \rangle^{r+1}$$

$$+ \sum_{h \geqq e+1} \sum_{r \geqq 0} \langle F^h y_1, \ldots, F^h y_{i(n,r)} \rangle^{r+1} \quad \text{(Lemma 3.5)}.$$

Let us show that $\langle \mathscr{A}_{e,k} \rangle^{k+1} \subset \mathscr{A}$ since:

$$\sum_{h \geqq e+1} \sum_{r \geqq 0} \langle F^h y_1, \ldots, F^h y_{i(h,r)} \rangle^{r+1} \subset \mathscr{A}$$

it is enough to verify:

$$(\sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e, r+k)} \rangle^{r+1})^{k+1} \subset \mathscr{A}$$

in fact

$$(\sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e, r+k)} \rangle^{r+1})^{k+1} \subset \sum_{r \geqq 0} \langle F^e y_1, \ldots, F^e y_{i(e, r)} \rangle^{r+1} \subset \mathscr{A}$$

by Proposition 4.7.

DEFINITION 4.8. — Given I and $\mathscr{A}$ ideal of A as in Theorem 4.6 we will define:

$$\beta(I) = \mathscr{A}.$$

DEFINITION 4.9. — For a given ideal $I \subset A$ we have defined the ideals $\{ I_e \} e \geqq -1$ (Def. 4.1), let: $h(e)$ be the smallest k such that $\delta_e^k(I_{e-1}) = \delta_e^{k+1}(I_{e-1})$. We will define non-increasing applications.

$$TB(I, e): \{0, 1, \ldots, h(e)\} \to N \cup \{0\},$$

$$TB(I, e)(k) = n - s(I, e)(k) e \geqq 0$$

that we will call the Thom-Boardman numbers associated to the ideal I. Since $I_e \subset I_{e+1} \subset \ldots$ then for e big enough $I_e = I_{e+1} = \ldots$ and

$$I_e = \delta_{e+1}(I_e); \; I_{e+k} = \delta_{e+k+1}(I_{e+k})$$

so $h(e) = 0$ for e big enough.

Example 1. — Let $A = k[[t]]$, k of characteristic p, the ideals $I_1 = \langle t^{p+1} \rangle$ and $\langle t^{p+2} \rangle = I_2$ (p = charac k) are such that $s(e, I_1) = s(e, I_2)$ $\forall e \geqq 0$ but there Thom-Boardman numbers are different, in fact

$$\delta_0(I_1) = \langle t^p \rangle = \delta_0^n(I_1), \qquad \forall n \geqq 1,$$

$$\delta_0(I_2) = \langle t^{p+1} \rangle, \qquad \delta_0^2(I_2) = \langle t^p \rangle = \delta_0^n(I_2), \qquad \forall n \geqq 2,$$

also $\delta_e(\langle t^p \rangle) = \langle t^p \rangle$ for $e \geqq 1$ so:

$$s(I_1, 0)(k) = s(I_2, 0)(k) = 0, \qquad \forall k \geqq 0,$$

$$s(I_1, e)(k) = s(I_2, e)(k) = 1, \qquad \forall k \geqq 0, \quad e \geqq 1,$$

but TB $(I_1, 0) = (1, 1)$; TB $(I_1, 1) = (0)$; TB $(I_1, e) = (0)$, $e \geqq 2$ and TB $(I_2, 0) = (1, 1, 1)$; TB $(I_2, I) = (0)$; TB $(I_2, e) = (0)$ $e \geqq 2$, so the monomials t^{p+1} and t^{p+2} will have the same sequences $s(e, I)$, but different Thom-Boardman numbers.

$$\beta(I_1) = \beta(I_2) = \langle t^p \rangle.$$

Example 2. — $I = \langle xy + z^p \rangle \subset k[[x, y, z]]$ characteristic of $k = p$:

$$\delta_0(I) = \langle x, y, z^p \rangle = \delta_0^k(I) = I_0, k \geqq 2 \quad \text{(Def. 4.1)}$$

$$\delta_1(I_0) = I_0 \quad \text{and} \quad \delta_e(I_0) = I_0, \quad e \geqq 1,$$

$$s(I, 0)(0) = 0; \quad s(I, 0)(k) = 2 \,\forall k \geqq 1; \quad s(I, e)(k) = 3, \quad \forall k \geqq 0, \ e \geqq 1,$$

$$TB(I, 0) = (3, 1); \quad TB(I, 1) = (0) = TB(I, e), \quad e \geqq 2,$$

$$\beta(I) = \langle x, y \rangle^2 + \langle x^p, y^p, z^p \rangle.$$

Example 3. — $k[[x, y, z]]$ as before $I = \langle x^p, y^p, z^p \rangle$:

$$I = I_e \,\forall e \geqq 0; \quad s(I, 0)(k) = 0, \quad \forall k \geqq 0; \quad s(I, e)(k) = 3, \quad \forall k, \ e \geqq 1,$$

$$TB(I, 0) = (3); \quad TB(I, 1) = (0) = TB(I, e), \quad e \geqq 2,$$

$$\beta(I) = I.$$

Note. — The only information that we have of these 3 examples in characteristic $p \neq 0$ using the same method that in characteristic zero is the one given by TB (I, 0) with the last integer repeated infinite times.

In examples 2 and 3 if we define the ideal β (I) as in characteristic zero:

$$\beta(I) = \sum_{i=0} (\delta_0^i(I))^{i+1}$$

there will be no R.S.P. $\{y_1, y_2, y_3\}$ of $k[[x, y, z]]$ such that

$$\beta(I) = \sum_{i=0} (y_1, \ldots, y_{s(i)})_{i+1}$$

for any non decreasing sequence $0 \leqq s(0) \leqq s(1) \leqq \ldots \leqq 3$ as in [3].

REFERENCES

[1] J. Dieudonné, *Le calcul différentiel dans les corps de caractéristique* $p = 0$ (*Proc. International Congress of Math.*, Amsterdam, 1954).

[2] R. Fitting, *Die determinanten ideale eines modules* (*Jahresbericht der Deut. Math. Vereinigung*, V, 1936, pp. 195-221).

[3] J. Mather, *On Thom-Boardman Singularities* (*Dynamical Systems*, ed. M. M. Peixoto, Academic Press, New York, 1973, pp. 233-247).

[4] K. R. Mount, *Some remarks on Fitting's invariants* (*Pacific Journal of Math.*, Vol. 13, 1963, pp. 1353-1357).

[5] K. R. Mount and O. E. Villamayor, *Taylor Series and Higher Derivations* (*Publ. Dept. Matematicas*, Facultad Ciencias Exactas, Univ. Buenos Aires).

[6] K. R. Mount and O. E. Villamayor, *An Algebraic Construction of the Generic Singularities of Boardman-Thom* (Publ. Math. I.H.E.S., Vol. 43, 1974, pp. 205-244).
[7] R. Thom, *Les singularités des applications différentiables* (Ann. Inst. Fourier, Vol. 6, 1956, pp. 17-86).
[8] Y. Nakai and S. Suzuki, *On M-adic differentials*, (J. Sci. Hiroshima Univ., Serie a, 24, 1960).
[9] J. Roberts, *Singularity subschemes and generic projections* (*Trans. Amer. Math. Soc.*, Vol. 212, 1975, pp. 229-268).

(Manuscrit reçu le 10 février 1977,
révisé le 20 octobre 1977.)

Orlando E. Villamayor (h)
Department of Mathematics,
Northwestern University,
Evanston, Ill. 60 201,
U.S.A.