A. JOSEPH

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Annales scientifiques de l'É.N.S. 4e série, tome 10, nº 4 (1977), p. 419-439

<http://www.numdam.org/item?id=ASENS_1977_4_10_4_419_0>

Ann. scient. Éc. Norm. Sup.
4 série, t. 10, 1977, p. 419 à 440.

ON THE ANNIHILATORS OF THE SIMPLE SUBQUOTIENTS OF THE PRINCIPAL SERIES (1)

BY A. JOSEPH

ABSTRACT. — Let $\mathfrak{g}$ be a complex semisimple Lie algebra and denote by $U(\mathfrak{g})$ its enveloping algebra. The main result of this paper (Th. 5.2) gives a formula for the annihilators of the simple subquotients of the (spherical) principal series in terms of the annihilators of simple quotients of Verma modules. The proof involves a description of the principal series in terms (Th. 5.1) of products of the almost minimal primitive ideals of $U(\mathfrak{g})$. It was motivated by an attempt to find a method for distinguishing primitive ideals of $U(\mathfrak{g})$. In particular for $\mathfrak{g}$ of type A_n (Cartan notation) it is shown (Cor. 6.6) that a conjecture of Jantzen ([1], 5.9) is equivalent to the simple subquotients of the principal series having distinct annihilators.

INDEX OF NOTATION. — Symbols frequently used in the text are given below in order of appearance.

1.1. $\mathfrak{g}$, $\mathfrak{n}^+$, $\mathfrak{h}$, $\mathfrak{n}^-$, R, R^+, B, ρ, W, Σ, s_α, X_α, H_α, $\alpha^\vee$, P (R), Q (R).
1.2. J (A), Spec A, Prim A, $\mathfrak{a}^\wedge$, U ($\mathfrak{a}$), Z ($\mathfrak{a}$), S (V), V*.
1.3. π, Max Z ($\mathfrak{g}$), R_λ, R_λ^+, B_λ, W_λ, Σ_λ, D_λ, w_λ, $\hat{\lambda}$, $\mathfrak{b}$, e_λ, E_λ, M (λ), $I_{\hat{\lambda}}$, $Z_{\hat{\lambda}}$, $\overline{M(\lambda)}$, L (λ), I_λ, $X_{\hat{\lambda}}$, $\varphi_{\hat{\lambda}}$, φ.
1.4. ${}^t u$, $\check{u}$.
1.5. U, j, $\mathfrak{k}$, $F_{\hat{\lambda}}$, L (λ, μ), L^0 (λ, μ), V (λ, μ).
2.1. $S_\lambda(w)$, $l_\lambda(w)$, $\tau_\lambda(w)$, $\leqq$.
2.2. S_λ, $\subseteq$.
3.3. L (M (μ), M (λ)).
3.4. P, P_λ, $\mathscr{V}$ (I), $\langle\ ,\ \rangle$, ψ_T, ψ.
3.6. I_α, θ_w.
3.9. LAnn V ($-w\lambda$, $-\lambda$), RAnn V ($-w\lambda$, $-\lambda$).
4.0. I_α, I_α^*, I_B.
4.1. τ.
4.9. $J_{w,r}$.
5.0. $I_{B'}^*$.
5.1. J_w.
5.4. $J_w^{w'}$, $\bar{J}_w^{w'}$.
6.1. St (ξ), Yg (ξ), T^i, T_i, m (T).
6.2. $\vee$, $\cup$.
6.4. Φ.

(1) Work supported by the C.N.R.S.

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ ([8], 1.10.14). Let $R \subset \mathfrak{h}^*$ denote the set of non-zero roots, $R^+ \subset R$ a system of positive roots, $B \subset R^+$ a $\mathbf{Z}$ basis for R, ρ the half sum of the positive roots, W the Weyl group for the pair $\mathfrak{g}$, $\mathfrak{h}$, Σ the subset of involutions of W, s_α the reflection corresponding to the root α. Fix a Chevalley base for $\mathfrak{g}$, let X_α denote an element of weight $\alpha \in R$ of this base, set $H_\alpha = [X_\alpha, X_{-\alpha}]$ and $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Let $Q(R)$ [resp. $P(R)$] $\subset \mathfrak{h}^*$ denote the lattice of radicial (resp. integral) weights.

1.2. For each Noetherian $\mathbf{C}$-algebra A, let $\mathbf{J}(A)$ (resp. Spec A, Prim A) denote the set of two-sided (resp. prime, primitive) ideals of A. For each $\mathbf{C}$-Lie algebra $\mathfrak{a}$, let $\mathfrak{a}^\wedge$ denote the set of classes of finite dimensional irreducible representations of $\mathfrak{a}$, $U(\mathfrak{a})$ the enveloping algebra of $\mathfrak{a}$, $Z(\mathfrak{a})$ the centre of $U(\mathfrak{a})$. For each $\mathbf{C}$-vector space V, let $S(V)$ denote the symmetric algebra over V and V^* the dual of V.

1.3. The principal aim of this paper is the study of Prim $U(\mathfrak{g})$. In this recall [3], (3.2) that $\pi : I \mapsto I \cap Z(\mathfrak{g})$ is a surjection of Prim $U(\mathfrak{g})$ onto Max $Z(\mathfrak{g})$. For each $\lambda \in \mathfrak{h}^*$, set $R_\lambda = \{ \alpha \in R : (\lambda, \alpha^\vee) \in \mathbf{Z} \}$, $R_\lambda^+ = R_\lambda \cap R^+$, $B_\lambda \subset R_\lambda^+$ a $\mathbf{Z}$ basis for R_λ, W_λ the subgroup of W generated by the s_α: $\alpha \in B_\lambda$. Set

$$\Sigma_\lambda = \Sigma \cap W_\lambda, \qquad D_\lambda = \{ w \in W : \ w R_\lambda^+ \subset R^+ \}$$

and w_λ the unique element of W_λ taking B_λ to $-B_\lambda$. Call λ *dominant* if $(\lambda, \alpha^\vee) \notin \mathbf{N}^-$, for all $\alpha \in R^+$ and *regular* if $(\lambda, \alpha) \neq 0$, for all $\alpha \in R$. Let $\hat{\lambda}$ denote the orbit of λ under W. With $\mathfrak{b} := \mathfrak{n}^+ \oplus \mathfrak{h}$, let $E_\lambda := \mathbf{C}\, e_\lambda$ denote the one-dimensional $\mathfrak{b}$ module defined through $X e_\lambda = 0 : X \in \mathfrak{n}^+$, $H e_\lambda = (H, \lambda) e_\lambda : H \in \mathfrak{h}$, and set $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} E_{\lambda-\rho}$, considered as a left $U(\mathfrak{g})$ module (*cf.* [8], Chaps. 5, 7). Recalling [8] (8.4.4), set $I_{\hat{\lambda}} = \operatorname{Ann} M(\lambda)$, $Z_{\hat{\lambda}} = \pi(I_{\hat{\lambda}})$. Recalling [8] (7.1.11), let $\overline{M(\lambda)}$ denote the unique maximal submodule of $M(\lambda)$ and set $L(\lambda) = M(\lambda)/\overline{M(\lambda)}$, $I_\lambda = \operatorname{Ann} L(\lambda)$ and $\mathbf{X}_{\hat{\lambda}} = \{ I_\mu : \mu \in \hat{\lambda} \}$ considered as an ordered set (by inclusion of elements). (After Duflo [7], II, Thm. 1.)

THEOREM. — *For each* $\hat{\lambda} \in \mathfrak{h}^*/W$, *one has* $\mathbf{X}_{\hat{\lambda}} = \pi^{-1}(Z_{\hat{\lambda}})$.

This reduces the study of Prim $U(\mathfrak{g})$ to that of finite sets $\mathbf{X}_{\hat{\lambda}} : \lambda \in \mathfrak{h}^*$. Now the Borho-Jantzen translation principle ([3], 2.12), shows that it suffices to determine $\mathbf{X}_{\hat{\lambda}}$ for λ regular and then fixing $-\lambda \in \mathfrak{h}^*$ dominant and regular, the map $\varphi_{\hat{\lambda}} : w \mapsto I_{w\lambda}$ is a surjection of W onto $\pi^{-1}(Z_{\hat{\lambda}})$. Furthermore if we write $w = w_1 w_2 : w_1 \in D_\lambda$, $w_2 \in W_\lambda$, then by [10] (4.2), we have $\varphi_{\hat{\lambda}}(w) = \varphi_{\hat{\lambda}}(w_2)$. That is $\varphi_{\hat{\lambda}}$ factors through W_λ giving a map φ of W_λ onto $\pi^{-1}(Z_{\hat{\lambda}})$. The Borho-Jantzen translation principle for say $\lambda \in P(R)$ shows that φ is in a natural sense independent of $\hat{\lambda}$ and suggests that in general φ should only depend on W_λ. In [10], we indicated what this dependence might be by exhibiting a partition of W_λ into cells so that each point in a given cell defines the same ideal. The main question that remains is to show that points in different cells define distinct ideals. Now this and the calculations of Borho-Jantzen on the low rank cases ([3], [4]), indicate that

Duflo's upper bound, namely card $\mathbf{X}_{\hat{\lambda}} \leqq$ card Σ_λ ([7], II, 2) should be very nearly saturated, that is one should expect to have card $\mathbf{X}_{\hat{\lambda}} \geqq \sqrt{\text{card } W_\lambda}$. Let us see how such a bound might arise.

1.4. Let $u \mapsto {}^t u$ (resp. $u \mapsto \check{u}$) denote the involutory antiautomorphism of $U(\mathfrak{g})$ defined by ${}^t X_\alpha = X_{-\alpha}$, for all $\alpha \in R$ and ${}^t H = H$, for all $H \in \mathfrak{h}$ (resp. $\check{X} = -X$, for all $X \in \mathfrak{g}$). As noted by Duflo ([7], I, Modules de Verma), one has:

LEMMA. — ${}^t I_\lambda = I_\lambda$, *for all* $\lambda \in \mathfrak{h}^*$.

1.5. Identity $U := U(\mathfrak{g}) \otimes U(\mathfrak{g})$ canonically with $U(\mathfrak{g} \oplus \mathfrak{g})$. Define the embedding $j : \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ through $j(X) = (X, -{}^t X)$, for all $X \in \mathfrak{g}$ and set $\mathfrak{k} = j(\mathfrak{g})$ which is naturally isomorphic to $\mathfrak{g}$. In this $\mathfrak{k}^\wedge$ identifies canonically with $P(R)/W$ and we let $F_{\hat{\nu}} : \nu \in P(R)$ denote the unique simple finite dimensional $\mathfrak{k}$ module with extreme weight ν. For each $\mu \in \mathfrak{h}^*$, let $F_{\hat{\nu}}(\mu)$ denote the subspace of $F_{\hat{\nu}}$ spanned by vectors of weight μ. Given λ $\mu \in \mathfrak{h}^*$, consider $(M(-\lambda) \otimes M(-\mu))^*$ as a U module by transposition and let $L(\lambda, \mu)$ denote the subspace spanned by all $\mathfrak{k}$ finite elements (which is a U submodule). As noted in say [6] (3.2), Frobenius reciprocity ([8], 5.5.7, 5.5.8) gives:

LEMMA. — *For all* λ, $\mu \in \mathfrak{h}^*$, $\hat{\nu} \in P(R)/W$, *one has*

$$mtp(\hat{\nu}, L(\lambda, \mu)) = \dim F_{\hat{\nu}}(\lambda - \mu).$$

In particular $L(\lambda, \mu) = 0$, unless $\lambda - \mu \in P(R)$. Again if $\lambda - \mu \in \hat{\nu}$, then $\hat{\nu}$ occurs with multiplicity one in $L(\lambda, \mu)$ and we denote this component by $L^0(\lambda, \mu)$. Let $V(\lambda, \mu)$ denote the unique simple quotient of $UL^0(\lambda, \mu)$ admitting a $\mathfrak{k}$ submodule of type $\hat{\nu}$. These modules which are said to belong to the principal series have been systematically studied. The results are reviewed in [6].

1.6. After Duflo ([7], Prop. 7), one has:

PROPOSITION. — *For all* λ, $\mu \in \mathfrak{h}^* : \lambda - \mu \in P(R)$, *there exist* $\lambda' \in \hat{\lambda}$, $\mu' \in \hat{\mu}$ *such that*

$$\operatorname{Ann} V(-\mu, -\lambda) = \check{I}_{\mu'} \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \check{I}_{\lambda'}.$$

Consider the special case when λ is regular and $\mu \in \hat{\lambda}$. Then $\mu = w\lambda$, for some $w \in W_\lambda$ and through the isomorphisms of the $V(-w\lambda, -\lambda)$ ([6], 4.1), we can assume $-\lambda$ fixed and say dominant. It follows that if the (non-isomorphic) U modules $V(-w\lambda, -\lambda) : w \in W_\lambda$, have distinct annihilators, then $(\text{card } \mathbf{X}_{\hat{\lambda}})^2 \geqq \text{card } W_\lambda$. Unfortunately we shall see that the former assertion is generally false; yet it is obviously of interest to determine a precise formula for $\operatorname{Ann} V(-w\lambda, -\lambda)$. Our main result (Th. 5.2) shows that under the above hypotheses we can take $\mu' = w_\lambda w \lambda$, $\lambda' = w_\lambda w^{-1} \lambda$ (recall that the $I_{w'\lambda} : w' \in W_\lambda$ are not all distinct). For W_λ simple of type A_n (Cartan notation) it is further shown (Sect. 6) that the $\operatorname{Ann} V(-w\lambda, -\lambda) : w \in W_\lambda$, are pairwise distinct if and only if card $\mathbf{X}_{\hat{\lambda}}$=card Σ_λ. In this we recall that if $\lambda \in P(R)$, then Borho and Jantzen ([3], [4]) have shown that the former equality holds up to $n = 5$. Perhaps the most interesting results are those of Section 4 which give remarkable sum and product formulae for the "almost minimal" primitive ideals which generalize [8] (7.8.12) and [7] (Prop. 12).

1.7. The proofs we give are entirely algebraic; but depend on results on complex Lie groups, so we have preferred to simply assume $\mathfrak{g}$ defined over $\mathbf{C}$. The use of the principal antiautomorphism U is not strictly necessary; but it seemed preferable to stick to the notational conventions of ([5], [6], [7]) where logically possible. I should like to thank M. Duflo for many discussions concerning these papers. Part of this work was done during a stay at the Sonderforschungsbereich, Bonn and I should like to thank W. Borho for a preview of his recent results with Jantzen concerning $\mathrm{X}_{\hat{\lambda}}$.

2. Two order relations on the Weyl group

2.1. For each $\lambda \in \mathfrak{h}^*$, $w \in \mathrm{W}_\lambda$ set

$$\mathrm{S}_\lambda(w) = w^{-1}\mathrm{R}_\lambda^- \cap \mathrm{R}_\lambda^+, \qquad l_\lambda(w) = \operatorname{card} \mathrm{S}_\lambda(w), \qquad \tau_\lambda(w) = \mathrm{S}_\lambda(w) \cap \mathrm{B}_\lambda.$$

Recall that $l_\lambda(w)$ is just the least number of ways of writing w as a product of the generating reflections $\{ s_\alpha : \alpha \in \mathrm{B}_\lambda \}$ and such a product is called a reduced decomposition for w. The group W_λ admits an order relation $\leqq$ defined as follows. Let

$$w = s_1 s_2 \ldots s_n, \qquad s_i = s_{\alpha_i}, \quad \alpha_i \in \mathrm{B}_\lambda$$

be a reduced decomposition for w. Then $w' \leqq w$ iff we can write $w' = s_{i_1} s_{i_2} \ldots s_{i_m}$, where $1 \leqq i_1 < i_2 < \ldots < i_m \leqq n$. It is easy to show that the expression for w' can be assumed reduced and then by [8] (7.7.4), this is the same order relation as that defined in [8] (7.7.3).

LEMMA. — *For all w, $w' \in \mathrm{W}_\lambda$,*

(i) $w \leqq w' \Leftrightarrow w^{-1} \leqq w'^{-1}$;

(ii) $w \leqq w' \Leftrightarrow w_\lambda w \geqq w_\lambda w'$.

(i) is clear. (ii) follows from [8] (7.7.3) and the relation $l_\lambda(w_\lambda) = l_\lambda(w_\lambda w) + l_\lambda(w)$ (*cf.* [10], 3.1).

2.2. Recall that the map $\mathrm{S}_\lambda : w \mapsto \mathrm{S}_\lambda(w)$ of W_λ into $\mathbf{P}(\mathrm{R}_\lambda^+)$ is injective ([10], 3.9). The group W_λ admits an order relation $\subseteq$, defined through $w' \subseteq w$ iff $\mathrm{S}_\lambda(w') \subseteq \mathrm{S}_\lambda(w)$. By say [10] (3.1), we have:

LEMMA. — *For each $w \in \mathrm{W}_\lambda$, $\alpha \in \tau_\lambda(w^{-1})$ one has $s_\alpha w \subseteq w$. Moreover $\{ s_\alpha w : \alpha \in \tau_\lambda(w^{-1}) \}$ is the set of all maximal elements of W_λ strictly less than w (for $\subseteq$).*

In particular $w \subseteq w'$ implies $w \leqq w'$.

2.3. Let α, β be distinct elements of B_λ and suppose that $(\alpha, \alpha) \leqq (\beta, \beta)$. Then $(\alpha^\vee, \beta) = -k$, with $k = 0, 1, 2$, or 3. One has

$$s_\alpha s_\beta = s_\beta s_\alpha : \ k = 0, \qquad s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta : \ k = 1, \qquad (s_\alpha s_\beta)^k = (s_\beta s_\alpha)^k : \ k = 2, 3.$$

For the appropriate k we call this a *pair relation* (for the pair α, β). Recall that W_λ is generated by the involutions $s_\alpha : \alpha \in \mathrm{B}_\lambda$ satisfying all possible pair relations.

LEMMA. — *Let $w \in \mathrm{W}_\lambda$. Any two reduced decompositions of w can be transformed into one another through just the pair relations* (i. e. *without using the identities* $s_\alpha^2 = 1 : \alpha \in \mathrm{B}_\lambda$).

The proof is by induction on $l_\lambda(w)$. If $l_\lambda(w) = 0$, then $w = 1$ and the assertion is trivial. Suppose in the respective reduced decompositions we have $w = s_\alpha w_1$, $w = s_\beta w_2 : \alpha, \beta \in B_\lambda$. We can assume $\alpha \neq \beta$ and then by say the first part of [10] (3.6), we have $\alpha \in \tau_\lambda(w_2^{-1})$, $\beta \in \tau_\lambda(w_1^{-1})$. By the induction hypothesis and 2.2, we can write $w_1 = s_\beta w_3$, $w_2 = s_\alpha w_4$ up to pair relations. If $k = 0$, then $w_3 = w_4$ and the assertion holds in this case. Otherwise by [10] (3.6), we have as above $w_3 = s_\alpha w_5$, $w_4 = s_\beta w_6$, up to pair relations. This process eventually gives the required assertion.

Remark. — This elementary (but for us important) fact is for example noted in [17] (Lemma 83 *a*) where its proof is left as an exercise.

3. The principal series

3.0. Fix $\lambda, \mu \in \mathfrak{h}^*$, with $\lambda - \mu \in P(R)$. We start by summarizing some classical results on the modules $M(\lambda)$, $L(\lambda, \mu)$, $V(\lambda, \mu)$.

3.1. THEOREM. — (*cf.* [6], I, 4):

(i) $V(\lambda, \mu)$ *is isomorphic to* $V(\lambda', \mu')$ *iff* $\lambda' = w\lambda$, $\mu' = w\mu$, *for some* $w \in W$;

(ii) $UL^0(\lambda, \mu) = L(\lambda, \mu)$, *if* λ *or* μ *is dominant;*

(iii) $UL^0(\lambda, \mu) = V(\lambda, \mu)$, *if* $-\lambda$ *or* $-\mu$ *is dominant;*

(iv) $L(\lambda, \mu)$ *has finite length as a* U *module, its simple factors are amongst the* $V(\lambda', \mu') : \lambda' \in \hat{\lambda}$, $\mu' \in \hat{\mu}$, *with* $V(\lambda, \mu)$ *occurring exactly once.*

3.2. THEOREM. — (*cf.* [7], [8], 7.6.23. *Suppose* $-\lambda \in \mathfrak{h}^*$ *dominant and regular. For each pair* $w, w' \in W_\lambda$, $M(w'\lambda)$ [*resp.* $L(w'\lambda)$] *is a submodule* (*resp. subquotient*) *of* $M(w\lambda)$ *iff* $w \geqq w'$.

3.3. Consider $\mathrm{Hom}_{\mathbf{C}}(M(\mu), M(\lambda))$ as a U module through $((a \otimes b).T)\, m = ({}^t\hat{a}\, T\, \check{b})\, m$, for all $T \in \mathrm{Hom}_{\mathbf{C}}(M(\mu), M(\lambda))$, $a, b \in U(\mathfrak{g})$, $m \in M(\mu)$. Let $L(M(\mu), M(\lambda))$ denote the subspace of $\mathrm{Hom}_{\mathbf{C}}(M(\mu), M(\lambda))$ spanned by all $\mathfrak{k}$ finite elements (which is a U submodule).

Suppose $M(\lambda)$ is a submodule of $M(\mu)$ and suppose given $I \in J(U(\mathfrak{g})/I_{\hat{\mu}})$ satisfying $IM(\mu) \subset M(\lambda)$. Then the representation of $U(\mathfrak{g})$ in $M(\mu)$ defines an embedding of I in $L(M(\mu), M(\lambda))$.

3.4. Let P denote the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ [which identifies with $S(\mathfrak{h})$] defined by the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+)$. For each $\lambda \in \mathfrak{h}^*$, define $P_\lambda : U(\mathfrak{g}) \to \mathbf{C}$, through $P_\lambda(a) = (P(a), \lambda - \rho)$. Given $I \in J(U(\mathfrak{g}))$, set

$$\mathscr{V}(I) = \{\lambda \in \mathfrak{h}^* :\ P_{\lambda+\rho}(a) = 0, \text{ for all } a \in I\}.$$

Define a bilinear form on $M(\lambda)$ through $\langle a\, e_{\mu-\rho}, b e_{\alpha-\rho} \rangle = P_\lambda({}^t ab)$ (which we recall is $\mathfrak{k}$ invariant and determined up to a scalar by this latter property). Identify $E_{\mu-\rho}$ (resp. $E_{\lambda-\rho}$) with the corresponding weight space in $M(\mu)$ [resp. $M(\lambda)$]. Given $T \in L(M(\mu), M(\lambda))$ define

$$\psi_T \in \mathrm{Hom}_{\mathbf{C}}(U, \mathrm{Hom}_{\mathbf{C}}(E_{\mu-\rho}, E_{\lambda-\rho}))$$

through

$$\langle e_{\lambda-\rho}, \psi_T(a \otimes b)\, e_{\mu-\rho} \rangle = \langle a e_{\lambda-\rho}, T b e_{\mu-\rho} \rangle,$$

for all $a, b \in U(\mathfrak{g})$. After Conze-Berline, Duflo ([5], 5.3, 5.5), we have

PROPOSITION. — *The map* $\psi : T \mapsto \psi_T$ *is a* U *module homomorphism of* $L(M(\mu), M(\lambda))$ *into* $L(-\lambda, -\mu)$. *Furthermore:*

(i) $\ker \psi = \{ T \in L(M(\mu), M(\lambda))$ *such that* $TM(\mu) \subset \overline{M(\lambda)} \}$;

(ii) *given* $M(\lambda)$ *simple, then* ψ *is an isomorphism.*

Remark. — Assume $-\lambda \in \mathfrak{h}^*$ dominant. Then as noted in [5] (6.3), it follows from 3.1 (ii), 3.2, 3.4 (ii) that ψ induces a U module isomorphism of $U(\mathfrak{g})/I_{\hat{\lambda}}$ onto $L(-\lambda, -\lambda)$ and *we identify these modules.*

3.5. We require the following refinement of 3.4. Take $-\lambda \in \mathfrak{h}^*$ dominant and $w \in W_\lambda$. Suppose we have $J \in \mathbf{J}(U(\mathfrak{g})/I_{\hat{\lambda}})$ satisfying *mtp* $((\lambda - w\lambda)^\wedge, J) = 1$, $JM(w_\lambda \lambda) = M(w_\lambda w \lambda)$ and generated as a U module by its component of type $(\lambda - w\lambda)^\wedge$.

THEOREM:

(i) *given* $J \supsetneq K \in \mathbf{J}(U(\mathfrak{g})/I_{\hat{\lambda}})$, *then* $KM(w_\lambda \lambda) \subset \overline{M(w_\lambda w \lambda)}$;

(ii) *if* $J \not\subset I_{w_\lambda w \lambda}$, *then* $(J + I_{w_\lambda w\lambda})/I_{w_\lambda w \lambda}$ *is isomorphic to* $V(-w\lambda, -\lambda)$ *as a* U *module.*

Consider J as a submodule of $L(M(w_\lambda \lambda), M(w_\lambda w \lambda))$ and restrict ψ defined in 3.4 to J. Then

$$\begin{aligned}\ker \psi &= \{ a \in J : \ a M(w_\lambda \lambda) \subset \overline{M(w_\lambda w \lambda)} \} \\ &\subset \{ a \in J : \ a M(w_\lambda w \lambda) \subset \overline{M(w_\lambda w \lambda)} \} \subset I_{w_\lambda w \lambda}.\end{aligned}$$

Through the hypothesis $JM(w_\lambda \lambda) = M(w_\lambda w \lambda)$, we have $\operatorname{Im} \psi \neq 0$ and since J is generated as a U module by a $\mathfrak{k}$ submodule of type $(w_\lambda \lambda - w_\lambda w \lambda)^\wedge$ it follows that $\operatorname{Im} \psi = UL^0(-w_\lambda w \lambda, -w_\lambda \lambda)$. Yet $w_\lambda \lambda$ is dominant and so by 3.1 (i) and 3.1 (iii), $\operatorname{Im} \psi$ is isomorphic to the simple U module $V(-w\lambda, -\lambda)$.

If $K \subsetneq J$, then $\psi(K)$ can have no component of type $(\lambda - w\lambda)^\wedge$ and so is a strict submodule of $V(-w\lambda, -\lambda)$. By 3.4 (i), this gives (i).

If $J \subsetneq I_{w_\lambda w \lambda}$, then $\ker \psi = J \cap I_{w_\lambda w \lambda}$ by the simplicity of $\operatorname{Im} \psi$. This gives (ii).

3.6. Fix $-\lambda \in \mathfrak{h}^*$ dominant and regular. By [7] (Cor. 2 to Prop. 10), $\{ I_{s_\alpha \lambda} : \alpha \in B_\lambda \}$, is the set of smallest primitive ideals of $U(\mathfrak{g})$ strictly containing the minimal primitive ideal $I_\lambda = I_{\hat{\lambda}}$. We call then the almost minimal primitive ideals. Set $I_\alpha := I_{s_\alpha \lambda}/I_\lambda$. Take $w \in W_\lambda$ and recall 3.2. The injection $M(\lambda) \subsetneq M(w\lambda)$ defines by transposition a U module homomorphism θ_w of $L(-\lambda, -w\lambda)$ into $L(-\lambda, -\lambda)$ and by restriction a U module homomorphism Θ_w of $L(M(w\lambda), M(\lambda))$ into $L(M(\lambda), M(\lambda))$. Define ψ (resp. ψ') as in 3.4 with $\mu = w\lambda$ (resp. $\mu = \lambda$). This gives the commutative diagram

$$\begin{array}{ccc} L(M(w\lambda), M(\lambda)) & \xrightarrow{\Theta_w} & L(M(\lambda), M(\lambda)) \\ \psi \downarrow & & \downarrow \psi' \\ L(-\lambda, -w\lambda) & \xrightarrow{\theta_w} & L(-\lambda, -\lambda). \end{array}$$

Since $-\lambda$ is dominant, $M(\lambda)$ is simple and so by 3.4 (ii) ψ, ψ' are isomorphisms.

Set $I = \operatorname{Ann} M(w\lambda)/M(\lambda)$ [computed in $U(\mathfrak{g})$] and define Dim as in [15] (2.1). It follows exactly as in 4.7 that $\operatorname{Dim} U(\mathfrak{g})/I = \operatorname{card} R - 2 = \operatorname{Dim} U(\mathfrak{g})/I_\lambda - 2$ and so $I/I_\lambda \neq 0$. Recalling that $\operatorname{Ann} M(w\lambda) = \operatorname{Ann} M(\lambda)$ and 3.3, it follows that the representation of $U(\mathfrak{g})$ in $M(w\lambda)$ defines an embedding of I/I_λ in $L(M(w\lambda), M(\lambda))$ and the restriction of Θ_w to I/I_λ is injective. Hence $\theta_w \neq 0$.

THEOREM. — *For all* $w \in W_\lambda$, $\alpha \in B_\lambda$:

(i) *there exists a* U *module monomorphism* θ'_w (*resp.* θ''_w) *of* $L(-w\lambda, -\lambda)$ [*resp.* $L(-\lambda, -w\lambda)$] *into* $L(-\lambda, -\lambda)$;

(ii) *mtp* $(V(-w_\lambda\lambda, -\lambda), L(-w\lambda, -\lambda)) =$ *mtp* $(V(-w_\lambda\lambda, -\lambda), L(-\lambda, -w\lambda)) = 1$;

(iii) *any non-zero* U *module homomorphism of* $L(-w\lambda, -\lambda)$ [*resp.* $L(-\lambda, -w\lambda)$] *into* $L(-\lambda, -\lambda)$ *is injective and coincides up to a scalar with* θ'_w (*resp.* θ''_w). *In particular we can take* θ''_w *to be* θ_w;

(iv) $L(-\lambda, -s_\alpha\lambda) = L(-s_\alpha\lambda, -\lambda) = I_\alpha$, *considered as submodules of* $L(-\lambda, -\lambda)$.

The first part of (i) follows on taking a reduced decomposition of w and repeated application of the first part of [7], Lemma 5. Consider $L(-w\lambda, -\lambda)$ as a submodule of $U(\mathfrak{g})/I_\lambda$. Then as noted in [7] (Prop. 9), ${}^tL(-w\lambda, -\lambda)$ is isomorphic as a U module to $L(-\lambda, -w\lambda)$. This proves the second part of (i).

The proofs of the two parts of (ii) and (iii) are similar and we consider only $L(-w\lambda, -\lambda)$. By [7] (Prop. 4), *mtp* $(V(-w_\lambda\lambda, -\lambda), L(-w\lambda, -\lambda)) \geqq 1$. By (i) it suffices to reverse this inequality in the case when $w = 1$. By (i) and 3.1 (iii), $V(-w_\lambda\lambda, -\lambda)$ identifies with a submodule of $L(-\lambda, -\lambda)$ and so by 3.4, there exists $I \in J(U(\mathfrak{g}))$ such that $I/I_\lambda = V(-w_\lambda\lambda, -\lambda)$ up to isomorphism. Yet I_λ is prime (in fact completely prime) and so by [2] (3.6), one has $\operatorname{Dim} U(\mathfrak{g})/I < \operatorname{Dim} U(\mathfrak{g})/I_\lambda$. It follows from say [2] (5.5), that $U(\mathfrak{g})/I$ is too small to admit a subquotient isomorphic to I/I_λ. This proves (ii).

We have seen that $L(-\lambda, -\lambda)$ admits a submodule V isomorphic to $V(-w_\lambda\lambda, -\lambda)$. By [7], Remark preceeding Proposition 12, $L(-\lambda, -\lambda)$ admits a unique simple submodule which must hence coincide with V. By (i), $L(-w\lambda, -\lambda)$ admits just one simple submodule and this is necessarily isomorphic to V. Now let θ be a U module homomorphism of $L(-w\lambda, -\lambda)$ into $L(-\lambda, -\lambda)$. If $\operatorname{Im}\theta \neq 0$, then it contains V. If $\ker\theta \neq 0$, it contains a submodule isomorphic to V. Then (iii) follows from (ii). (iv) follows from [7] (Lemme 5 and Proposition 10).

Remarks. — The assertions corresponding to (ii) and (iii) for Verma modules are well-known [8] (7.6.6), and the proof of (ii) was inspired by the improved Borho-Jantzen proof of [8] (7.6.6). The way to obtain (iii) from (i) and (ii) was pointed out to me by Duflo.

3.7. HYPOTHESES 3.6

COROLLARY. — *For each* $w \in W_\lambda$, *one has* ${}^tL(-w\lambda, -\lambda) = L(-\lambda, -w\lambda)$, *considered as submodules of* $U(\mathfrak{g})/I_\lambda$.

As remarked in [7] (Prop. 9), the above are isomorphic as U submodules of $U(\mathfrak{g})/I_\lambda$. Hence the assertion follows from 3.4 and 3.6 (iii).

3.8. NOTATION and HYPOTHESES 3.6. Consider $L(-\lambda, -w\lambda)$ as a two-sided ideal of $U(\mathfrak{g})/I_\lambda$.

PROPOSITION. — *For all* $w, w' \in W_\lambda$:

(i) $\ker \Theta_w = 0$;

(ii) $L(-\lambda, -w\lambda) \supset \operatorname{Ann} M(w\lambda)/M(\lambda)$ [*computed in* $U(\mathfrak{g})/I_\lambda$];

(iii) $L(-\lambda, -w\lambda) \supset L(-\lambda, -w'\lambda)$, *if* $w \leqq w'$.

Through the commutative diagram defined in 3.6, we obtain (i) from 3.6 (iii) and (ii) from 3.3, 3.4 and 3.6 (iii). Given $w' \geqq w$, we obtain from 3.2 the injections $M(\lambda) \subsetneq M(w\lambda) \subsetneq M(w'\lambda)$ and hence by restriction the homomorphisms

$$L(M(w'\lambda), M(\lambda)) \xrightarrow{\Theta} L(M(w\lambda), M(\lambda))$$

and

$$L(M(w\lambda), M(\lambda)) \xrightarrow{\Theta_w} L(M(\lambda), M(\lambda)).$$

Clearly $\Theta_w \Theta = \Theta_{w'}$ and so Θ is injective by (i). This gives (iii).

3.9. Fix $-\lambda \in \mathfrak{h}^*$ dominant and regular. For each $w \in W_\lambda$, set

$$\operatorname{LAnn} V(-w\lambda, -\lambda) = \{a \in U(\mathfrak{g}) : (\check{a} \otimes 1) V(-w\lambda, -\lambda) = 0,$$
$$\operatorname{RAnn} V(-w\lambda, -\lambda) = \{a \in U(\mathfrak{g}) : (1 \otimes \check{a}) V(-w\lambda, -\lambda) = 0\}.$$

In the notation of 1.6 taking $\mu = w\lambda$, we have $\operatorname{LAnn} V(-w\lambda, -\lambda) = I_{\mu'}$ and $\operatorname{RAnn} V(-w\lambda, -\lambda) = I_{\lambda'}$. We set $\mu' = w_1 \lambda$, $\lambda' = w_2 \lambda : w_1, w_2 \in W_\lambda$.

PROPOSITION. — *For all* $w \in W_\lambda$:

(i) $w_1 \leqq w_\lambda w$;

(ii) $\operatorname{LAnn} V(-w\lambda, -\lambda) = \operatorname{RAnn} V(-w^{-1}\lambda, -\lambda)$;

(iii) $\{\operatorname{LAnn} V(-\sigma\lambda, -\lambda) : \sigma \in \Sigma_\lambda\} = X_{\hat{\lambda}}$ (Duflo [7]).

Recall the argument of [7] (Prop. 7). By 3.1 (i), (iii), $V(-w\lambda, -\lambda)$ identifies with a submodule of $L(-w_\lambda w\lambda, -w_\lambda \lambda)$ and then its orthogonal M in $M(w_\lambda w\lambda) \otimes M(w_\lambda \lambda)$ is a proper submodule of the latter. Let M′ be a submodule (not necessarily unique) of $M(w_\lambda w\lambda) \otimes M(w_\lambda \lambda)$ containing M such that M′/M is simple. By 3.2, M′/M is isomorphic to $L(w_1\lambda) \otimes L(w_2\lambda)$, for some $w_1, w_2 \in W_\lambda$ with $w_1 \leqq w_\lambda w$. By duality this gives (i). (ii) follows from 1.4, 3.1 (i), 3.1 (ii) and 3.7. (iii) is just [7] (Prop. 9).

Remark. — By (ii), (iii) one has card $X_{\hat{\lambda}}$ = card Σ_λ, iff the $\operatorname{Ann} V(-\sigma\lambda, -\lambda) : \sigma \in \Sigma_\lambda$ are pairwise distinct (*cf.* 1.6 and 6.6).

4. The almost minimal primitive ideals

4.0. In this section we fix $-\lambda \in \mathfrak{h}^*$ dominant and regular. For all $\alpha \in B_\lambda$, we set $I_\alpha := I_{s_\alpha \lambda}/I_\lambda$ and $I_\alpha^* := I_{w_\lambda s_\alpha \lambda}/I_\lambda$ (this latter notation is motivated by a conjecture of Borho-Jantzen [3], 2.19). For all $B' \subset B_\lambda$, we set $I_{B'} = I_{w_{B'}\lambda}/I_\lambda$.

4.1. Define a map $\tau : X_{\hat{\lambda}} \to P(B_\lambda)$, through $\tau(I_{w\lambda}) = \{\alpha \in B_\lambda : I_{w\lambda} \supset I_{s_\alpha \lambda}\}$. Borho-Jantzen and Duflo established independantly [*cf.* [10], 4.4 (ii)] that

THEOREM. — $\tau(I_{w\lambda}) = \tau_\lambda(w)$, *for all* $w \in W_\lambda$.

4.2. COROLLARY. — *For all* $w \in W_\lambda$, $\alpha \in B_\lambda$ *one has* $I_\alpha M(w\lambda) = M(w\lambda)$ *iff* $\alpha \notin \tau_\lambda(w)$:

$$I_\alpha M(w\lambda) \subset \overline{M(w\lambda)} \Leftrightarrow I_\alpha \subset I_{w\lambda}/I_\lambda \Leftrightarrow I_{s_\alpha\lambda} \subset I_{w\lambda} \Leftrightarrow \alpha \in \tau_\lambda(w),$$

by 4.1.

4.3. LEMMA. — *Suppose* $w \in W_\lambda$. *Then* $\mathrm{LAnn}\, V(-w\lambda, -\lambda) = I_{w_\lambda\lambda}$, *iff* $w = 1$.

In the identification $U(\mathfrak{g})/I_\lambda = L(-\lambda, -\lambda)$, the image of 1 is the unique trivial $\mathfrak{k}$ submodule of $L(-\lambda, -\lambda)$. Then recalling 3.1 (ii) we have $V(-\lambda, -\lambda) = U(\mathfrak{g})/I_{w_\lambda\lambda}$, which gives sufficiency. Necessity follows from 2.9 (i).

4.4. LEMMA. — *For each* $B' \subset B_\lambda$, $\alpha \in B'$, *one has:*

(i) $I^2_{B_\lambda} = I_{B_\lambda}$;

(ii) $I_\alpha + I^*_\alpha = I_{B_\lambda}$;

(iii) $(I^*_\alpha + I_{B'})/I^*_\alpha$ [*and hence* $I_{B'}/(I^*_\alpha \cap I_{B'})$] *is isomorphic to the simple* U *module* $V(-s_\alpha\lambda, -\lambda)$;

(iv) $I^2_\alpha = I_\alpha$.

Suppose (i) is false. Then by 3.1 (iv) and 3.4, there exists $J \in \mathbf{J}(U(\mathfrak{g})/I_\lambda)$ containing $I^2_{B_\lambda}$ such that I_{B_λ}/J is isomorphic to $V(-w\lambda, -\lambda)$, for some $w \in W_\lambda$. Then clearly $\mathrm{LAnn}\, V(-w\lambda, -\lambda) = I_{w_\lambda\lambda}$ and so $w = 1$, by 4.3. Yet this is impossible since I_{B_λ} does not contain the trivial $\mathfrak{k}$ submodule. Hence (i). Now suppose $J \in \mathbf{J}(I_{B_\lambda})$ strictly contains some I^*_α. Then since $I_{w_\lambda s_\alpha\lambda}$ is almost maximal (*cf.* [3], 2.19) it follows that $\sqrt{J} = I_{B_\lambda}$. Hence $J = I_{B_\lambda}$ by (i). Combined with 4.1, this gives (ii) and (iii). Suppose (iv) is false. By 3.6 (ii) and 3.1 (ii), I_α admits a unique maximal submodule and by (iii) taking $B' = \{\alpha\}$, it follows that this coincides with $I^*_\alpha \cap I_\alpha$. Then $I^2_\alpha \subset I^*_\alpha \cap I_\alpha$, which contradicts 4.1. Hence (iv).

4.5. The best we could do to prove that $I^2_{B'} = I_{B'}$ for all $B' \subset B_{\lambda'}$ is the following:

PROPOSITION. — *For all* $B' \subset B_\lambda$, *the following statements are equivalent:*

(i) $I^2_{B'} = I_{B'}$;

(ii) $I_{B'}$ *admits exactly* card B' *distinct simple quotients;*

(iii) $I_{B'} = \sum_{\alpha \in B'} I_\alpha$.

Recalling 3.4, we have

$$\mathscr{V}\Big(\sum_{\alpha \in B'} I_{s_\alpha\lambda}\Big) = \bigcap_{\alpha \in B'} \mathscr{V}(I_{s_\alpha\lambda}) = \mathscr{V}(I_{w_{B'}\lambda}),$$

by [10] [4.5 (ii)]. Then $I_{w_{B'}} = \sqrt{\sum_{\alpha \in B'} I_{s_\alpha\lambda}}$, by [10] (2.1) and so (i) $\Rightarrow$ (iii). Through 4.4 (iii), it follows that (ii) $\Rightarrow$ (i) as in the proof of 4.4 (iv). Now let K be a maximal submodule of $I_{B'}$. If (iii) holds, then $K \cap I_\alpha \subsetneq I_\alpha$ for some $\alpha \in B'$ and so $I_{B'}/K = I_\alpha/(I_\alpha \cap K) = I_\alpha/(I_\alpha \cap I^*_\alpha)$ where the last equality follows by 3.6 (iv), 3.1 (ii) and 4.4 (iii). Hence (iii) $\Rightarrow$ (ii).

Remarks. — By [7] (Prop. 12), (iii) holds if $B' \subset B$ and by [10] (4.2), it is sufficient that there exists $w \in W$ such that $w B' \subset B$. Conversely by 4.4 (i) and 4.5, we have

$$I_{w_\lambda\lambda} = \sum_{\alpha \in B_\lambda} I_{s_\alpha\lambda}.$$

This shows that for all $\lambda \in \mathfrak{h}^*$ regular the maximal ideal in the fibre $\pi^{-1}(Z_{\hat{\lambda}})$ (i. e. $I_{w_\lambda \lambda}$) though generally not itself induced (*cf.* [3], 4.2) is nevertheless a sum of the induced ideals $I_{s_\alpha \lambda} : \alpha \in B_\lambda$.

4.6. PROPOSITION. — *For all* $w \in W_\lambda$, $\alpha \in \tau_\lambda(w)$, *one has* $I_\alpha M(w\lambda) = M(ws_\alpha \lambda)$.

Since $\alpha \in \tau_\lambda(w)$, we have $w \geqq ws_\alpha$ by 2.2 and so by 3.2, $M(ws_\alpha \lambda)$ is a submodule of $M(w\lambda)$. Then $I_\alpha M(w\lambda) \supset I_\alpha M(ws_\alpha \lambda) = M(ws_\alpha \lambda)$, by 4.2.

For the opposite inclusion, let ε be a real positive number and set

$$C_{\alpha,\lambda,\varepsilon} = \{ \nu \in \mathfrak{h}^* : (\nu, \alpha) = 0,\ B_{\lambda+\nu} \subset B_\lambda,\ (\nu, \nu) < \varepsilon \}.$$

Assume ε sufficiently small so that $\lambda + C_{\alpha,\lambda,\varepsilon}$ lies in a fixed Weyl chamber (and hence $-(\lambda+\nu) : \nu \in C_{\alpha,\lambda,\varepsilon}$ is dominant). Observe that $\alpha \in B_{\lambda+\nu}$ and set

$$C^0_{\alpha,\lambda,\varepsilon} = \{ \nu \in C_{\alpha,\lambda,\varepsilon} : B_{\lambda+\nu} = \{ \alpha \} \}.$$

Set $\mu := w(\lambda+\nu) - ws_\alpha(\lambda+\nu) = w\lambda - ws_\alpha \lambda = (\alpha^\vee, \lambda) w\alpha \in \mathbf{N}R^+ \setminus \{0\}$ and set $\beta = w\alpha$ Then $s_\beta w = ws_\alpha$, so by [8] (7.6.23), $M(ws_\alpha(\lambda+\nu))$ is a submodule of $M(w(\lambda+\nu))$ and (*cf.* [8], 7.5):

$$(\bigstar) \qquad \operatorname{ch}\frac{M(w(\lambda+\nu))}{M(ws_\alpha(\lambda+\nu))} = e^{w\nu} \operatorname{ch}\frac{M(w\lambda)}{M(ws_\alpha\lambda)}.$$

for all $\nu \in C_{\alpha,\lambda,\varepsilon}$. Identify (*cf.* [8], 7.1.5) $M(w(\lambda+\nu))$ canonically with $U(\mathfrak{n}^-)$. Then by [11] (Lemma 1), there exists a polynomial map $\nu \mapsto a_\nu$ of $C_{\alpha,\lambda,\varepsilon}$ into $U(\mathfrak{n}^-)$ such that $M(ws_\alpha(\lambda+\nu))$ identifies with $U(\mathfrak{n}^-) a_\nu$. By ($\bigstar$) the dimension of each weight space of $U(\mathfrak{n}^-)/U(\mathfrak{n}^-) a_\nu$ is independent of ν. Hence the representation of $U(\mathfrak{g})$ in $M(w(\lambda+\nu))/M(ws_\alpha(\lambda+\nu))$ depends rationally on ν about $\nu = 0$.

By [7] (Prop. 1), there exists a U module homomorphism

$$B(s_\alpha, -\lambda-\nu, -\lambda-\nu) : \quad L(-\lambda-\nu, -\lambda-\nu) \to L(-s_\alpha(\lambda+\nu), -s_\alpha(\lambda+\nu)),$$

with $\ker B(s_\alpha, -\lambda-\nu, -\lambda-\nu) = I_{s_\alpha(\lambda+\nu)}/I_{\lambda+\nu}$, [7] (Prop. 10). By [7] (Lemma 5), and 3.1 (ii), $I_{s_\alpha(\lambda+\nu)}/I_{\lambda+\nu}$ is generated by a simple $\mathfrak{k}$ submodule of type $(\lambda - s_\alpha \lambda)^\wedge$ in $L(-\lambda-\nu, -\lambda-\nu)$ and hence by the lowest weight vector f_ν of this submodule. The restriction b_ν of $B(s_\alpha, -\lambda-\nu, -\lambda-\nu)$ to the lowest weight space of the isotypical component of type $(\lambda - s_\alpha \lambda)^\wedge$ in $L(-\lambda-\nu, -\lambda-\nu)$ has for image the lowest weight space in the isotypical component of type $(\lambda - s_\alpha \lambda)^\wedge$ in $L(-s_\alpha(\lambda+\nu), -s_\alpha(\lambda+\nu))$ and for suitable n (*cf.* 1.5) is an $n \times n$ matrix with entries depending rationally on ν, [7] (Prop. 1). Since $-\lambda-\nu$ is always dominant the singularities in b_ν lie outside $C_{\alpha,\lambda,\varepsilon}$. (This is made explicit in [6], III, 3.8, 4.7.) Evidently rank $b_\nu = n-1$, for all $\nu \in C_{\alpha,\lambda,\varepsilon}$. Choose a cofactor b^{kj}_ν which is non-zero at $\nu = 0$. Then we may write $f_\nu = (b^{j1}_\nu, b^{j2}_\nu, \ldots, b^{jn}_\nu)$ and so it follows that the map $\nu \mapsto f_\nu$ is rational in ν about $\nu = 0$.

By 4.2, we have $I_{s_\alpha(\lambda+\nu)} M(w(\lambda+\nu)) \subset \overline{M(w(\lambda+\nu))}$, and for all $\nu \in C^0_{\alpha,\lambda,\varepsilon}$ one has $\overline{M(w(\lambda+\nu))} = M(ws_\alpha(\lambda+\nu))$ by [9] (Satz 3). Let $\bar{e}_{w(\lambda+\nu)-\rho}$ be the representative of $e_{w(\lambda+\nu)-\rho}$ in $M(w(\lambda+\nu))/M(ws_\alpha(\lambda+\nu))$. We have shown that $f_\nu \bar{e}_{w(\lambda+\nu)-\rho}$ depends rationally on ν about $\nu = 0$ and vanishes in the Zariski dense set $C^0_{\alpha,\lambda,\varepsilon}$. Hence it vanishes at $\nu = 0$ and so $I_\alpha M(w\lambda) \subset M(ws_\alpha \lambda)$, as required.

Remarks. — The special case when $\mathfrak{g}$ is simple of type A_2 and $\lambda = \rho$ was given by Dixmier [8] (7.8.12). It is not known if an arbitrary induced ideal depends rationally on the available parameters (but it does depend continuously [3], 3.9 *b*), and in fact the crucial point in the above argument is the description of I_α through the Kunze-Stein intertwining operators. The equality $I_\alpha M(w\lambda) = M(ws_\alpha\lambda)$ would not have followed from the second part of the proof and only follows from the deep fact noted in 4.1. (In this connection see [10], 4.6.)

4.7. We note the following fact which finds application in 7.2.

COROLLARY. — *For each* $\alpha \in B_\lambda$, *and each* $w \in W_\lambda$ *satisfying* $\alpha \in \tau_\lambda(w)$ *one has* $I_{s_\alpha\lambda} = \operatorname{Ann} M(w\lambda)/M(ws_\alpha\lambda)$.

Set $M_w = M(w\lambda)/M(ws_\alpha\lambda)$ and $K_w = \operatorname{Ann} M_w$. By 4.6, one has $K_w \supset I_{s_\alpha\lambda}$ with equality if $w = s_\alpha$. Define Dim as in [15] (2.1). Since each M_w identifies with $U(\mathfrak{n}^-)/U(\mathfrak{n}^-)a_w$, for suitable $a_w \in U(\mathfrak{n}^-)$, it follows that $\operatorname{Dim} M_w = \dim \mathfrak{n}^- - 1$. Again (*cf.* [2], 3.1) one has

$$\operatorname{Dim} M_w = \sup \{ \operatorname{Dim} L : L \in \mathscr{JH}\, M_w \}$$

and

$$\operatorname{Dim} U(\mathfrak{g})/K_w = \sup \{ \operatorname{Dim} U(\mathfrak{g})/\operatorname{Ann} L : L \in \mathscr{JH}\, M_w \}.$$

Hence by 3.2 and [15], 2.7 it follows that $\operatorname{Dim} U(\mathfrak{g})/K_w = \operatorname{card} R - 2 = \operatorname{Dim} U(\mathfrak{g})/I_{s_\alpha\lambda}$. Yet $I_{s_\alpha\lambda}$ is a prime ideal and so by [2] (3.6), one has $K_w = I_{s_\alpha\lambda}$, as required.

4.8. For each $w \in W_\lambda$, consider $L(-w\lambda, -\lambda)$ as a U submodule of $U(\mathfrak{g})/I_\lambda$ (*cf.* 3.7). Recalling 3.9, choose $w_1 \in W_\lambda$ (not necessarily unique) such that $L \operatorname{Ann} V(-w\lambda, -\lambda) = I_{w_1\lambda}$.

LEMMA:

(i) $\tau_\lambda(w_1) = B_\lambda$, *iff* $w = 1$;

(ii) $B_\lambda \setminus \tau_\lambda(w_1) = \{ \alpha \in B_\lambda : I_\alpha L(-w\lambda, -\lambda) = L(-w\lambda, -\lambda) \}$.

(i) follows from 4.3. (ii) follows from 3.1 (ii) which implies that $L(-w\lambda, -\lambda)$ has a *unique* maximal submodule and the quotient is isomorphic to $V(-w\lambda, -\lambda)$.

4.9. Given $w \in W_\lambda \setminus \{1\}$, let $w = s_1 s_2 \ldots s_n : s_i = s_{\alpha_i} : \alpha_i \in B_\lambda$, be a reduced decomposition r of w and set $J_{w,r} := I_{\alpha_n} I_{\alpha_{n-1}} \ldots I_{\alpha_1}$. (We shall eventually see that $J_{w,r}$ is independent of r.)

PROPOSITION. — *For all* $w, w' \in W_\lambda$ *with reduced decompositions* r, r' *one has:*

(i) $J_{w,r} M(w_\lambda\lambda) = M(w_\lambda w\lambda)$;

(ii) $J_{w,r} \subset {}^t(\operatorname{Ann} M(w\lambda)/M(\lambda)) \subset L(-w\lambda, -\lambda)$;

(iii) $J_{w,r} \subset J_{w',r}$ *implies that* $w \geqq w'$.

(i) obtains on successive application of 4.6. Combined with 2.1 (ii) and 3.2, this gives (iii). Again successive application of 4.6 gives ${}^tJ_{w,r} M(w\lambda) = M(\lambda)$. Combined with 3.7 and 3.8 (ii) this gives (ii).

4.10. Recalling 2.3, let α, β be distinct elements of B_λ and suppose $(\alpha, \alpha) \leqq (\beta, \beta)$. Set $k = -(\alpha\check{}, \beta)$.

LEMMA:

(i) $I_\alpha I_\beta = I_\beta I_\alpha : k = 0$;

(ii) $I_\alpha I_\beta I_\alpha = I_\beta I_\alpha I_\beta : k = 1$;

(iii) $(I_\alpha I_\beta)^k = (I_\beta I_\alpha)^k : k = 2, 3$.

By 3.6 (iv), 3.7, 3.8 (iii) and 4.9 (ii) we have $I_\beta I_\alpha \subset L(-s_\alpha s_\beta \lambda, -\lambda) \subset I_\alpha \cap I_\beta$. Choose $\gamma \in B_\lambda \setminus \{\alpha, \beta\}$. If $\gamma \notin \tau_\lambda((s_\alpha s_\beta)_1)$ (notation 4.8), then by 4.8 (ii), we obtain $I_\beta I_\alpha \subset L(-s_\alpha s_\beta \lambda, -\lambda) \subset I_\lambda I_\alpha$, which contradicts 4.9 (iii). Then by 4.8 (i) we must either have $\alpha \notin \tau_\lambda((s_\alpha s_\beta)_1)$ or $\beta \notin \tau_\lambda((s_\alpha s_\beta)_1)$. Suppose the first holds. Then by 4.8 (ii), we obtain $I_\beta I_\alpha \subset I_\alpha I_\beta$. By 4.9 (iii) this can only hold if $k = 0$ and then by 1.4 we obtain (i). If $k \neq 0$, then we must have $\beta \notin \tau_\lambda((s_\alpha s_\beta)_1)$ and so from 4.8 (ii) we obtain $I_\beta I_\alpha = L(-s_\alpha s_\beta \lambda, -\lambda)$. A similar argument with α, β interchanged gives

$$I_\alpha I_\beta = L(-s_\beta s_\alpha \lambda, -\lambda).$$

Substitution from 3.7, 3.8 (iii) and 4.9 (ii) gives

$$I_\alpha I_\beta I_\alpha \subset L(-s_\alpha s_\beta s_\alpha \lambda, -\lambda) \subset I_\alpha I_\beta \cap I_\beta I_\alpha.$$

Then by 4.8 and 4.9 (iii) either $\alpha \notin \tau_\lambda((s_\alpha s_\beta s_\alpha)_1)$ or $\beta \notin \tau_\lambda((s_\alpha s_\beta s_\alpha)_1)$. Suppose $\beta \notin \tau_\lambda((s_\alpha s_\beta s_\alpha)_1)$. Then by 4.8 (ii), $I_\alpha I_\beta I_\alpha \subset I_\beta I_\alpha I_\beta$ and so $k = 1$ by 4.9 (iii). Yet as above:

$$I_\beta I_\alpha I_\beta \subset L(-s_\beta s_\alpha s_\beta \lambda, -\lambda) = L(-s_\alpha s_\beta s_\alpha \lambda, -\lambda) = I_\alpha I_\beta \cap I_\beta I_\alpha,$$

and since $\beta \notin \tau_\lambda((s_\alpha s_\beta s_\alpha)_1)$, this gives $I_\beta I_\alpha I_\beta = L(-s_\beta s_\alpha s_\beta \lambda, -\lambda)$. Recalling 1.5, it follows from 3.5 (ii) that either $I_\beta I_\alpha I_\beta = I_\alpha I_\beta I_\alpha$ or $I_\alpha I_\beta I_\alpha M(w_\lambda \lambda) \subsetneq M(w_\lambda s_\alpha s_\beta s_\alpha \lambda)$. The latter contradicts 4.9 (i) and so we obtain (ii). The remaining cases follow similarly.

4.11. COROLLARY. — *For each* $w \in W_\lambda \setminus \{1\}$, $J_{w,r}$ *is independent of the reduced decomposition* r *of* w.

Apply 4.10 and 2.3.

4.12. PROPOSITION. — *Choose* $B' \subset B_\lambda$ *for which* 4.5 (iii) *holds. Then*

$$\overline{M(w_{B'} \lambda)} = \sum_{\alpha \in B'} M(w_{B'} s_\alpha \lambda).$$

Set $M = \sum_{\alpha \in B'} M(w_{B'} s_\alpha \lambda)$. Certainly $M \subsetneq M(w_\lambda \lambda)$. Recalling [8] [7.6.1 (i)], let M' be a submodule of $M(w_{B'} \lambda)$ strictly containing M such that M'/M is simple and hence isomorphic to $L(w\lambda)$, for some $w \leqq w_{B'}$ (by 3.2). From the hypothesis and 4.6 we obtain $I_{w_{B'},\lambda} \subset \operatorname{Ann} M'/M = I_{w\lambda}$. By 4.1, this gives $\tau(w) \supset \tau(w_{B'}) = B'$ and so $w = w_{B'}$. Hence $M' = M(w_{B'} \lambda)$ and so $M = \overline{M(w_{B'} \lambda)}$.

Remark. — In particular by 4.4 and 4.5, it follows that $\overline{M(w_\lambda \lambda)} = \sum_{\alpha \in B_\lambda} M(w_\lambda s_\alpha \lambda)$ and so is generated by the Verma submodules it contains. This is well-known if $\lambda \in P(R)$ ([8], 7.2.5).

5. Main theorems

5.0. In this section we retain the conventions of 4.0 and in addition set $I^*_{B'} = I_{w_\lambda w_{B'}}/I_\lambda$, for all $B' \subset B_\lambda$. Identify $L(-w\lambda, -\lambda)$ and $L(-\lambda, -w\lambda)$ with U submodules of $U(\mathfrak{g})/I_\lambda$ (*cf.* 3.6).

5.1. Set $J_1 = U(\mathfrak{g})/I_\lambda = L(-\lambda, -\lambda)$ (*cf.* 3.4). Given $w \in W_\lambda \setminus \{1\}$, let

$$w = s_1 s_2 \ldots s_n, \qquad s_i = s_{\alpha_i}, \quad \alpha_i \in B_\lambda,$$

be a reduced decomposition for w and recalling 4.11 set $J_w := I_{s_n} I_{s_{n-1}} \ldots I_{s_1}$.

THEOREM. — *For all* $w, w' \in W_\lambda$:

(i) $J_w = L(-w\lambda, -\lambda)$;

(ii) $L(-w\lambda, -\lambda) = L(-\lambda, -w^{-1}\lambda)$;

(iii) $L(-\lambda, -w\lambda) = \operatorname{Ann} M(w\lambda)/M(\lambda)$;

(iv) $L(-\lambda, -w\lambda) \supset L(-\lambda, -w'\lambda)$, *iff* $w \leqq w'$.

The proof of (i) is by induction of $l_\lambda(w)$. It has already been established for $l_\lambda(w) = 0,1$ [*cf.* 3.6 (ii) and 3.4]. Take w as above and set $w' = s_1 w$. Then $l_\lambda(w') = l_\lambda(w) - 1$, so by 3.7, 3.8 (iii), 4.9 (ii) and the induction hypothesis we obtain

$$(\bigstar) \qquad J_w \subset L(-w\lambda, -\lambda) \subset J_{w'}.$$

We show that $\tau_\lambda(w_1) \supset B_\lambda \setminus \tau_\lambda(w)$ (notation 4.8). If this is false choose

$$\alpha \in B_\lambda \setminus (\tau_\lambda(w) \cup \tau_\lambda(w_1)).$$

Since $\alpha \notin \tau_\lambda(w_1)$, we obtain from 4.8 (ii) and ($\bigstar$) that $J_w \subset I_\alpha J_{w'} = J_{w' s_\alpha}$ and so by 4.9 (iii) that $w \geqq w' s_\alpha$. Yet $\alpha \notin \tau_\lambda(w)$ and so by [10], 3.1 (iii), we obtain

$$l_\lambda(ws_\alpha) = l_\lambda(w) + 1 = l_\lambda(w') + 2.$$

Further application of [10], 3.1 (iii) and 3.1 (iv) gives $l_\lambda(w' s_\alpha) = l_\lambda(w)$ and so $w = w' s_\alpha$ This contradicts $\alpha \notin \tau_\lambda(w)$.

Through the above inclusion and 4.8 (i) it follows that there exists $\alpha \notin \tau_\lambda(w)$ with $\alpha \notin \tau_\lambda(w_1)$. Set $w'' = ws_\alpha$. Then $l_\lambda(w'') = l_\lambda(w) - 1$, so by 3.7, 3.8 (iii) and the induction hypothesis we obtain $L(-w\lambda, -\lambda) \subset J_{w''}$. Then by 4.8 (ii) and 4.11, $L(-w\lambda, -\lambda) \subset I_\alpha J_{w''} = J_w$, which combined with ($\bigstar$) proves the required assertion.

(ii) follows from (i), 1.4 and 3.7. (iii) follows from (i), 3.7, 3.8, 4.9 (ii). (iv) follows from (i), 3.7, 3.8 (iii) and 4.9 (iii).

Remarks. — Necessity in (iv) also follows from 3.1 (iv) and [7] (Prop. 4). By 3.1 (ii) the embedding defined in 5.1 (iv) is unique.

5.2. THEOREM. — *For all* $w \in W_\lambda$, *one has*

$$\operatorname{Ann} V(-w\lambda, -\lambda) = \check{I}_{w_\lambda w\lambda} \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \check{I}_{w_\lambda w^{-1}\lambda}.$$

By 1.6 and 3.9 (ii) it is enough to show that $\operatorname{LAnn} V(-w\lambda, -\lambda) = I_{w_\lambda w\lambda}$. By 3.1 (ii), $L(-w\lambda, -\lambda)$ admits a unique maximal submodule and so $I_{w_1\lambda} := \operatorname{LAnn} V(-w\lambda, -\lambda)$

is just the largest element of $\mathbf{J}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda)$ such that $\mathrm{I}_{w_1\lambda}\,\mathrm{L}(-w\lambda, -\lambda) \subsetneq \overline{\mathrm{L}(-w\lambda, -\lambda)}$. By 5.1, 4.9 (i) and 3.5 (i) this is equivalent to $\mathrm{I}_{w_1\lambda}\,\mathrm{M}(ww_\lambda\lambda) \subset \overline{\mathrm{M}(ww_\lambda\lambda)}$, which gives the required assertion.

5.3. In [7] (II), Duflo notes that for each $\mathrm{I} \in \operatorname{Spec}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda)$, there exists a unique smallest $\mathrm{J} \in \mathbf{J}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda)$ strictly containing I. When $\mathrm{I} = \mathrm{I}^*_{\mathrm{B}'}$ we compute J below.

THEOREM. — *For all* $\mathrm{B}' \subset \mathrm{B}_\lambda$:

(i) $\mathrm{J}_{w_{\mathrm{B}'}}$ *is the smallest element of* $\mathbf{J}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda)$ *with radical equal to* $\bigcap_{\alpha\in\mathrm{B}'} \mathrm{I}_\alpha$. *In particular* $\mathrm{J}_{w_{\mathrm{B}'}} = (\bigcap_{\alpha\in\mathrm{B}'} \mathrm{I}_\alpha)^l$, *for all integer l sufficiently large;*

(ii) $\mathrm{J}_{w_{\mathrm{B}'}}+\mathrm{I}^*_{\mathrm{B}'}$ *is the unique smallest element of* $\mathbf{J}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda)$ *strictly containing* $\mathrm{I}^*_{\mathrm{B}'}$. *Furthermore* $(\mathrm{J}_{w_{\mathrm{B}'}}+\mathrm{I}^*_{\mathrm{B}'})/\mathrm{I}^*_{\mathrm{B}'} = \mathrm{V}(-w_{\mathrm{B}'}\lambda, -\lambda)$, *up to a* U *module isomorphism.*

Choose $\mathrm{J} \in \mathbf{J}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda)$ such that $\sqrt{\mathrm{J}} = \bigcap_{\alpha\in\mathrm{B}'} \mathrm{I}_\alpha$, and set $\mathrm{K} = \mathrm{J}+\mathrm{I}_\lambda$, considered as an element of $\mathbf{J}(\mathrm{U}(\mathfrak{g}))$. By [10] [2.1 (v) (notation 3.4)] we have $\mathscr{V}(\mathrm{K}) = \bigcup_{\alpha\in\mathrm{B}'} \mathscr{V}(\mathrm{I}_{s_\alpha\lambda})$., Then by [10] [2.1 (i), 2.1 (ii)], the inclusion $\mathrm{K} \subset \mathrm{I}_{w_\lambda w_{\mathrm{B}'}\lambda}$ implies $w_\lambda w_{\mathrm{B}'}\lambda \in \mathscr{V}(\mathrm{I}_{s_\beta\lambda})$, for some $\beta \in \mathrm{B}'$ and so by [10], 2.1 (i) that $\mathrm{I}_{w_\lambda w_{\mathrm{B}'}\lambda} \supset \mathrm{I}_{s_\beta\lambda}$. This contradicts 4.1. Hence $(\mathrm{J}+\mathrm{I}^*_{\mathrm{B}'})/\mathrm{I}^*_{\mathrm{B}'} \neq 0$. In particular we may take $\mathrm{J} = \mathrm{J}_{w_{\mathrm{B}'}}$, and then by 5.1 (i) and 3.5 (ii) this gives (ii). Again if $\mathrm{J} \subsetneq \mathrm{J}_{w_{\mathrm{B}'}}$, then by 3.5 (i), $\mathrm{KM}(w_\lambda w_{\mathrm{B}'}\lambda) \subset \overline{\mathrm{M}(w_\lambda w_{\mathrm{B}'}\lambda)}$ which implies $\mathrm{K} \subset \mathrm{I}_{w_\lambda w_{\mathrm{B}'}\lambda}$. This contradiction gives (i).

5.4. Fix $w, w' \in \mathrm{W}_\lambda$ satisfying $l_\lambda(w'w) = l_\lambda(w')+l_\lambda(w)$. Then $\mathrm{M}(w'\lambda)$ is a submodule of $\mathrm{M}(w'w\lambda)$ and we set $\mathrm{J}^{w'}_w = \operatorname{Ann}\mathrm{M}(w'w\lambda)/\mathrm{M}(w'\lambda)$, $\bar{\mathrm{J}}^{w'}_w = \operatorname{Ann}\overline{\mathrm{M}(w'w\lambda)/\mathrm{M}(w'\lambda)}$ computed in $\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda$. Certainly $\bar{\mathrm{J}}^{w'}_w \subset \mathrm{J}^{w'}_w$. Let $\mathrm{B}(w', -\lambda, -w\lambda)$ be the U module homomorphism of $\mathrm{L}(-\lambda, -w\lambda)$ into $\mathrm{L}(-w'\lambda, -w'w\lambda)$ defined in [7] (Prop. 1), and let $\psi_{w'w\lambda, w'\lambda}$ (or simply, ψ) be the U module homomorphism of $\mathrm{L}(\mathrm{M}(w'w\lambda), \mathrm{M}(w'\lambda))$ into $\mathrm{L}(-w'\lambda, -w'w\lambda)$ defined in 3.4.

THEOREM:

(i) $\mathrm{L}(-\lambda, -w\lambda) \subset \mathrm{J}^{w'}_w$, *with equality if* $w' = 1$ *or* $w = s_\alpha : \alpha \in \mathrm{B}_\lambda$;

(ii) $\mathrm{UL}^0(-w'\lambda, -w'w\lambda) = \psi(\mathrm{L}(-\lambda, -w\lambda)) \subset (\overline{\mathrm{M}(w'\lambda)} \otimes \mathrm{M}(w'w\lambda))^\perp$;

(iii) *up to a non-zero scalar (depending on* $w'w\lambda$ *and* $w'\lambda$*) the restriction of* $\psi_{w'w\lambda, w'\lambda}$ *to* $\mathrm{L}(-\lambda, -w\lambda)$ *coincides with* $\mathrm{B}(w', -\lambda, -w\lambda)$;

(iv) $\ker\mathrm{B}(w', -\lambda, -w\lambda) = \bar{\mathrm{J}}^{w'}_w \cap \mathrm{L}(-\lambda, -w\lambda)$.

(i) obtains from 4.6, 4.7 and 5.1. Then by 3.4 (i), 4.9, 5.1,

$$\psi(\mathrm{L}^0(-\lambda, -w\lambda)) = \mathrm{L}^0(-w'\lambda, -w'w\lambda),$$

which by 3.1 (ii) gives the first part of (ii). By [11] (Sect. 2), the bilinear form $\langle\, , \rangle$ defined on $\mathrm{M}(w'\lambda)$ has kernel $\overline{\mathrm{M}(w'\lambda)}$. Hence $\operatorname{Im}\psi \subset (\overline{\mathrm{M}(w'\lambda)} \otimes \mathrm{M}(w'w\lambda))^\perp$ which gives the second part of (ii). By 3.1 (ii), any U module homomorphism of $\mathrm{L}(-\lambda, -w\lambda)$ into $\mathrm{L}(-w'\lambda, -w'w\lambda)$ is determined by its restriction to the lowest weight vector of the $\mathfrak{k}$-submodule $\mathrm{L}^0(-\lambda, -w\lambda)$. Hence (iii) and (iv).

Remarks. — The importance of (iii) is that it gives a new way of representing the Kunze-Stein intertwining operators $B(w', -\lambda, -w\lambda)$. Taking $w' w = w_\lambda$ and recalling the argument of 3.9, we see that (ii) implies 5.2 and is indeed a stronger result. Taking $w = 1$ in (iv), we recover [7] (Prop. 10) as a special case. It would be rather useful to establish equality in (i). For example this would give $\ker B(w', -\lambda, -w\lambda) = \bar{J}_w^{w'}$ and a further application is noted in 7.1. It is part of a general question raised in [8] (Prob. 30). By 4.7 and the definition of J_w, it follows that $L(-\lambda, -w\lambda)$ and $J_w^{w'}$ have the same radical. Again we note that

$$\bar{J}_w^{s_\alpha} = \operatorname{Ann} M(s_\alpha w\lambda)/M(\lambda) = L(-\lambda, -s_\alpha w\lambda) \subset L(-\lambda, -w\lambda),$$

by 5.1 and so $\ker B(s_\alpha, -\lambda, -w\lambda) = L(-\lambda, -s_\alpha w\lambda)$, which is essentially [7] (Lemma 5). Finally does one have $\ker B(w', -\lambda, -w\lambda) = (I_{w'\lambda}/I_\lambda) L(-\lambda, -w\lambda)$? By (iv) they have the same radical.

5.5 Take $w \in W_\lambda$. Then $\overline{\sum_{w'<w} M(w'\lambda)}$ is generally a strict submodule of $\overline{M(w\lambda)}$. Recalling 3.1 (ii) let $\overline{L(-\lambda, -w\lambda)}$ denote the unique maximal submodule of $L(-\lambda, -w\lambda)$. By 1.5 and 5.1 (iv), $\sum_{w'<w} L(-\lambda, -w'\lambda)$ is contained in $\overline{L(-\lambda, -w\lambda)}$ and we show that this inclusion is generally strict.

Lemma. — *Take* $\alpha \in B_\lambda$ *and set* $B' = B_\lambda \setminus \{\alpha\}$. *If* $I_\alpha^* \supsetneqq I_{B'}$, *then* $J := \sum_{w'>s_\alpha} L(-\lambda, -s_\alpha\lambda)$ *is a strict submodule of* $\overline{L(-\lambda, -s_\alpha\lambda)}$.

By 4.1 and 5.1, $J = \sum_{\beta \in B'} (I_\alpha I_\beta + I_\beta I_\alpha) \subset I_\alpha \cap I_{B'}$. By 4.4 and 4.5,

$$I_\alpha/(I_\alpha \cap I_{B'}) \cong (I_\alpha + I_{B'})/I_{B'} = I_{B_\lambda}/I_{B'},$$

which by 4.4 (iii) is a simple U module *iff* $I_{B'} = I_\alpha^*$. This establishes the required assertion.

Example. — Take R of type A_3 with $\lambda \in P(R)$ and $\alpha = \alpha_2$. By [3] (4.4, 4.17), one has $I^* \supsetneqq I_{B'}$. Also $\overline{M(w_\lambda s_\alpha \lambda)}$ is not generated by the Verma modules it contains.

5.6. Fix $-\lambda \in \mathfrak{h}^*$ dominant and let $L(L(w\lambda), L(w\lambda)) : w \in W_\lambda$ denote the subspace of all $\mathfrak{k}$ finite elements of $\operatorname{Hom}(L(w\lambda), L(w\lambda))$ (which is a U submodule). Recalling 3.4, let $\langle\, ,\rangle$ denote the non-degenerate bilinear $\mathfrak{k}$ invariant form on $L(w\lambda)$. Given $T \in L(L(w\lambda), L(w\lambda))$ define $\psi_T \in (L(w\lambda) \otimes L(w\lambda))^*$ through $(\psi_T, m \otimes n) = \langle m, Tn\rangle$, for all $m, n \in L(w\lambda)$. Let $\overline{(M(w\lambda) \otimes M(w\lambda))}$ denote the unique maximal submodule of $M(w\lambda) \otimes M(w\lambda)$ and let $\overline{(M(w\lambda) \otimes M(w\lambda))}^\perp$ denote its orthogonal complement in $(M(w\lambda) \otimes M(w\lambda))^*$.

Lemma. — *The map* $\psi : T \mapsto \psi_T$ *induces a* U *module isomorphism of* $L(L(w\lambda), L(w\lambda))$ *onto* $L(-w\lambda, -w\lambda) \cap \overline{(M(w\lambda) \otimes M(w\lambda))}^\perp$.

It follows exactly as in [5], 5.5 that ψ is a U module isomorphism of $L(L(w\lambda), L(w\lambda))$ onto the subspace of all $\mathfrak{k}$ finite elements of $(L(w\lambda) \otimes L(w\lambda))^*$ which further identifies with the subspace of all $\mathfrak{k}$ finite elements of $\overline{(M(w\lambda) \otimes M(w\lambda))}^\perp$. This gives the required assertion.

5.7. From say 3.3 we obtain an embedding $U(\mathfrak{g})/I_{w\lambda} \subsetneq L(L(w\lambda), L(w\lambda))$. This is generally strict (*cf.* [5], 6.5). Yet

THEOREM. — *For all—* $\lambda \in \mathfrak{h}^*$ *dominant and regular one has*

$$U(\mathfrak{g})/I_{w_\lambda \lambda} = L(L(w_\lambda \lambda), L(w_\lambda \lambda)).$$

Set $L = L(L(w_\lambda \lambda), L(w_\lambda \lambda))$. By 3.1 (iv) and 5.6, L has finite length as a U module. Let V be one of its non-zero simple U subquotients. Since ${}^t\check{I}_{w_\lambda \lambda} L = 0$, we obtain from 1.4 and 1.6 that LAnn $V \supset I_{w_\lambda \lambda}$. This by 4.3 and the maximality of $I_{w_\lambda \lambda}$ gives $V = V(-\lambda, -\lambda)$ up to isomorphism. By 3.1 (iv) and 5.6, $V(-\lambda, -\lambda)$ occurs with multiplicity at most once in L which is therefore itself a simple U module. Hence $L = U(\mathfrak{g})/I_{w_\lambda \lambda}$ as required.

Remark.—In the special case for which $B_\lambda \subset B$ this result is due to Conze-Berline and Duflo (combine 2.12, 6.2, 6.3 of [5]). More generally they show that

$$L(L(w_{B'} \lambda), L(w_{B'} \lambda)) = U(\mathfrak{g})/I_{w_{B'}\lambda},$$

for all $B' \subset (B_\lambda \cap B)$ and $-\lambda$ dominant. For λ regular we sketch an alternative proof based on 5.7. Let ε be a real positive number and set $C_{B', \lambda, \varepsilon} = \{ \nu \in \mathfrak{h}^* : (\nu, \alpha) = 0 : \alpha \in B', B_{\lambda+\nu} \subset B_\lambda, (\nu, \nu) < \varepsilon \}$. Given $B' \subset B$ and taking ε sufficiently small it follows from [6], 3.9 and 4.3.3 (as pointed out to me by Duflo) that $B(w_{B'}, -(\lambda+\nu), -(\lambda+\nu))$ is independent of $\nu \in C_{B', \lambda, \varepsilon}$. Hence by [7], Prop. 10, $I_{w_{B'}(\lambda+\nu)}/I_{\lambda+\nu}$ is independent of ν. By [10], 4.3, this also holds when $B' = \{ \alpha \} \subset B$ and hence it is true for any subset B' of B satisfying 4.5 (iii). Conversely since we can always choose $\nu \in C_{B', \lambda, \varepsilon}$ such that $B' = B_{\lambda+\nu}$, the independence of $I_{w_{B'}(\lambda+\nu)}/I_{\lambda+\nu}$ on ν implies 4.5 (iii). This gives an independent proof of [7], Proposition 12. By 4.5 (iii) and 4.12 it follows that $L(w_{B'}(\lambda+\nu))$ identifies with an induced module. Then by 5.6, $L(L(w_{B'}(\lambda+\nu), w_{B'}(\lambda+\nu))$ identifies with a principle series module and so as a $\mathfrak{k}$ module is independent of ν. Taking ν so that $B' = B_{\lambda+\nu}$ the required assertion follows from 5.7.

6. The symmetric group

6.0. Theorem 5.2 is slightly unsatisfactory in the sense that the $I_{w\lambda} : w \in W_\lambda$ are not pairwise distinct. Here we recast this formula into a better form when W_λ is of type A_{n-1} (that is when it is isomorphic to the symmetric group S_n). We follow the notation of [10] (Sect. 7), briefly outlined below.

6.1. Let n be an integer > 0, ξ a partition of n and set $|\xi| = n$. Let $St(\xi)$ [resp. $Yg(\xi)$] denote the set of standard (resp. Young) Tableaux of type ξ. Given $T \in Yg(\xi)$, let T^i (resp. T_i) : $i = 1, 2, \ldots, n$, denote the columns (resp. rows) of T and $m(T)$ the set of positive integers (assumed pairwise distinct) occurring in T. We recall that by definition $m(T) = \{ 1, 2, \ldots, n \}$ iff $T \in St(\xi)$.

6.2. By an ordinal we mean an element of $\mathbf{N}^+ \cup \{ \infty \}$ given its natural order. Let T be a Young Tableau and k, l positive integers not occurring in $m(T)$. We define a new Table $T \vee k$ (resp. $T \cup l$) by the following rule. First complete T to an infinite square array by putting ∞ into the empty places. Then define ordinals $k_0 \leqq k_1 \leqq k_2 \leqq \ldots$ (resp. $l_0 \leqq l_1 \leqq l_2 \leqq \ldots$) inductively as follows. Set $k_0 = k$ (resp. $l_0 = l$) and for each $i \in \mathrm{N}^+$, let k_i (resp. l_i) be the smallest ordinal $\geqq k_{i-1}$ (resp. l_{i-1}) in T^i (resp. T_i). Finally set $(\mathrm{T} \vee k)^i = (\mathrm{T}^i \setminus \{ k_i \}) \cup \{ k_{i-1} \}$ [resp. $(\mathrm{T} \cup l)_i = (\mathrm{T}_i \setminus \{ l_i \}) \cup \{ l_{i-1} \}$]. The following result is due to Schensted [16] (Lemma 6).

LEMMA. — *Let* T *be a Young Tableau (possibly the trivial empty Tableau). Given* $k,\ l \in \mathrm{N}^+ \setminus m(\mathrm{T})$ *distinct, then* $(\mathrm{T} \vee k) \cup l = (\mathrm{T} \cup l) \vee k$.

Define $k_1, k_2, \ldots$ (resp. $l_1, l_2, \ldots$) as above and call it the k (resp. l)-sequence. Observe, for example, that l_{i-1} takes the place of l_i in $(\mathrm{T} \cup l)$ and that the l-sequence moves downwards and to the left in T. Both sequences are increasing and hence have either exactly one common element which is finite, or (possibly) several infinite ones. Choose $r, s \in \mathrm{N}^+$ such that $l_r = k_s$. We can assume without loss of generality that there are no further common elements and that $l_{r-1}, k_{s-1} < \infty$. One has $l_r \in \mathrm{T}^s$ and so either $r = 1$ or $l_{r-1} \in \mathrm{T}^v$, for some $v \geqq s$. Again $s = 1$ or $k_{s-1} \in \mathrm{T}_u$, for some $u \geqq r$. Suppose $u = r$. Then $l_{r-1} > k_{s-1}$ by definition of the l-sequence and so $v > s$ (for otherwise $k_{s-1} > l_{r-1}$ by definition of the k-sequence). Now

$$(\mathrm{T} \cup l)_s = (\mathrm{T}_s \setminus \{ l_s \}) \cup \{ l_{s-1} \} \qquad \text{and} \qquad (\mathrm{T} \vee k)_s = (\mathrm{T}_s \setminus \{ k_t \}) \cup \{ k_{t-1} \},$$

for some ordinal $t \geqq v > s$. Hence $((\mathrm{T} \vee k) \cup l)_s = (\mathrm{T}_s \setminus \{ l_s, k_t \}) \cup \{ k_{t-1}, l_{s-1} \}$. Again since $l_r = k_s$, it follows from the definition of the k-sequence that l_{r-1} is the smallest integer $> k_{s-1}$ in $(\mathrm{T} \cup l)^s$. Hence $((\mathrm{T} \cup l) \vee k)_s = (\mathrm{T}_s \setminus \{ l_s, k_t \}) \cup \{ k_{t-1}, l_{s-1} \}$, as required. The remaining rows coincide because they do not contain the intersection point of the sequences. The case $u > r$, $v > s$, $l_{r-1} > k_{s-1}$ is exactly the same and the remaining cases follow by interchanging rows and columns.

Remark. — The above proof is different and shorter than Schensted's which uses induction on n.

6.3. Given $\mathrm{T} \in \mathrm{Yg}(\xi)$, then after Robinson (*cf.* [10], 7.5 (i)), we can always write $\mathrm{T} = ((\ldots (l_1 \cup l_2) \cup l_3) \cup \ldots \cup l_n$, for some (pairwise distinct) $l_i \in \mathrm{N}^+$. Given $l \in \mathrm{N}^+ \setminus m(\mathrm{T})$, set $l \cup \mathrm{T} := ((\ldots (l \cup l_1) \cup l_2) \cup \ldots \cup l_n$. Let T^* denote the Young Table obtained by rotating T about its main diagonal.

COROLLARY (Schensted [16], Lemma 7):

(i) $l \cup \mathrm{T} = \mathrm{T} \vee l$;

(ii) $((\ldots (l_1 \cup l_2) \cup l_3) \cup \ldots \cup l_n)^* = ((\ldots (l_n \cup l_{n-1}) \cup l_{n-2}) \cup \ldots \cup l_1$;

(i) follows easily from 6.2. For (ii), observe that $(\mathrm{T} \cup l) = (\mathrm{T}^* \vee l)^* = (l \cup \mathrm{T}^*)^*$, by (i). Hence $(l \cup \mathrm{T}^*) = (\mathrm{T} \cup l)^*$ and (ii) follows by induction on n.

6.4. Assume that R is of type A_{n-1}. Then W_λ is isomorphic to the symmetric group S_n which we consider as the permutation group of $\{ 1, 2, \ldots, n \}$. For each $w \in \mathrm{S}_n$, set $k_i = w^{-1}\ i : i = 1, 2, \ldots, n$ and $\mathrm{A}(w) := ((\ldots (k_1 \cup k_2) \cup k_3) \cup \ldots \cup k_n$, $\mathrm{B}(w) := \mathrm{A}(w^{-1})$. Then after Robinson, Schensted and Schützenberger (*cf.* [10], 7.5), the map $\Phi : w \mapsto (\mathrm{A}(w), \mathrm{B}(w))$ is a bijection of S_n onto : $\bigcup \{ \mathrm{Yg}(\xi) \times \mathrm{Yg}(\xi) : | \xi | = n \}$.

LEMMA:

(i) $A(w_\lambda w) = A(w)^*$;

(ii) $A(w_\lambda w^{-1}) = B(w)^*$.

One has $(w_\lambda w)^{-1} i = w^{-1}(n+1-i) = k_{n+1-i}$ and so (i) follows from 6.3 (ii). By (i), $B(w)^* = A(w^{-1})^* = A(w_\lambda w^{-1})$, which is (ii).

6.5. HYPOTHESES 6.4. — For each $w \in W_\lambda$ define the involutions

$$\sigma_1(w) := \Phi^{-1}(A(w)^*, A(w)^*) \qquad \sigma_2(w) := \Phi^{-1}(B(w)^*, B(w)^*).$$

Through the injectivity of Φ, the map $w \mapsto (\sigma_1(w), \sigma_2(w))$ of W_λ into $(\Sigma_\lambda, \Sigma_\lambda)$ is injective.

THEOREM (R_λ *of type* A_{n-1}). — *For all* $w \in W_\lambda$, *one has*

$$\operatorname{Ann} V(-w\lambda, -\lambda) = \check{I}_{\sigma_1(w)\lambda} \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \check{I}_{\sigma_2(w)\lambda}.$$

This follows 5.2, 6.4, and [10] (5.1, 6.1 and 7.9).

6.6. COROLLARY (R_λ *of type* A_{n-1}). — *The following two statements are equivalent :*

(i) card $X_{\hat{\lambda}}$ = card Σ_λ;

(ii) card $\{ \operatorname{Ann} V(-w\lambda, -\lambda) : w \in W_\lambda \}$ = card W_λ.

Remark. — If $\lambda \in P(R)$, then after Borho-Jantzen ([3], [4]), (i) holds up to $n = 6$.

6.7. If B_λ admits roots α, β which span a subsystem of type B_2 or G_2 one has $I_{s_\alpha \lambda} = I_{s_\alpha s_\beta s_\alpha \lambda}$ by [10] (5.1). Then by 5.2

$$\operatorname{Ann} V(-w_\lambda s_\alpha \lambda, -\lambda) = \operatorname{Ann} V(-w_\lambda s_\alpha s_\beta s_\alpha \lambda, -\lambda)$$

and since $w_\lambda = -1$ (under the above hypothesis) this gives card $X_{\hat{\lambda}}$ < card Σ_λ by 3.9 (iii). Consequently card $\{ V(-w\lambda, -\lambda) : w \in W_\lambda \}$ < card W_λ and one can expect this to also hold if R_λ admits a subsystem of type D_n or E_n. Yet it is plausible that

$$\operatorname{Ann} V(-w\lambda, -\lambda) \neq \operatorname{Ann} V(-w^{-1}\lambda, -\lambda) \qquad \text{if} \quad w \neq w^{-1},$$

holds in general. In case A_{n-1} such a result would distinguish the $\{ I_{\sigma\lambda} : \sigma \in \Sigma_\lambda \}$ associated through Φ with standard Tableaux of the same form (i. e. defined by the same partition ξ). Furthermore if $\sigma = \Phi^{-1}(A, A)$ with $A \in St(\xi)$, then one expects that the zero variety of the graded ideal gr $I_{\sigma\lambda}$ will admit a dense nilpotent orbit corresponding to ξ (*cf.* [1], 5.9) and together these results would distinguish the $\{ I_{\sigma\lambda} : \sigma \in \Sigma_\lambda \}$. In [15] (4.2), when $\mathfrak{g}$ itself is of type A_{n-1} we have already shown that the zero variety of gr $I_{\sigma\lambda}$ has the expected dimension. This is an important and rather non-trivial application of our main result 5.2. It further allows us to classify $X_{\hat{\lambda}}$ when card $B_\lambda = 3$, [15] (Sect. 5). Finally we remark that Spaltenstein [12] has pointed out in case A_{n-1} that the Robinson map Φ can be viewed as the inverse of a map recently introduced by Steinberg in connection with the unipotent variety. The Steinberg map is defined without restriction on type; but in the general case there is a tantalizing distinction between this map and what would be required to generalize 6.5 for arbitrary W_λ.

7. The rank 2 case

7.1. Retain the notation and conventions of 4.0. It is well-known and follows easily from 5.3 (i) that card $\mathrm{J}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda) = 1+\mathrm{card}\, \mathrm{B}_\lambda$, if card $\mathrm{B}_\lambda \leqq 1$. Here we consider the case when card $\mathrm{B}_\lambda = 2$. In this situation Jantzen [14], has recently shown that $\mathcal{JH}$ $\mathrm{M}(w_\lambda \lambda)$ is multiplicity free. It is natural to then ask if $\mathcal{JH}$ $\mathrm{L}(-\lambda, -\lambda)$ is also multiplicity free and we remark that such a result leads easily to a complete description of $\mathrm{J}(\mathrm{U}(\mathfrak{g})/\mathrm{I}_\lambda)$. We can show that this would result from equality in 5.4 (iii) when $w' w = w_\lambda$. Unfortunately we were not quite able to establish the latter; but to illustrate our method we give a new proof of [3] (Folg. 2.20). First we note an easy and well-known consequence of the fact that $\mathcal{JH}$ $\mathrm{M}(w_\lambda \lambda)$ is multiplicity free.

LEMMA. – *Set* $\mathrm{B}_\lambda = \{\alpha, \beta\}$. *Then*

$$\overline{\mathrm{M}(w_\lambda s_\alpha \lambda)} = \overline{\mathrm{M}(w_\lambda s_\beta \lambda)} = \mathrm{M}(w_\lambda s_\alpha \lambda) \cap \mathrm{M}(w_\lambda s_\beta \lambda).$$

Remark. – When B_λ is of type $\mathrm{A}_1 \times \mathrm{A}_1$ or A_2, this also follows from [13] (Lemma 1).

7.2. COROLLARY. – $\mathrm{I}_\alpha^* = \mathrm{I}_\beta$, $\mathrm{I}_\beta^* = \mathrm{I}_\alpha$.

By definition $\mathrm{I}_\alpha \subset \mathrm{I}_\beta^*$. For the opposite inclusion, note that $\mathrm{I}_\beta^* \subset \mathrm{I}_\alpha + \mathrm{I}_\beta$ by 4.4 (i) and 4.5 and so $\mathrm{I}_\beta^* = \mathrm{I}_\beta^* \cap \mathrm{I}_\beta + \mathrm{I}_\alpha$. By 4.4 (iii) taking $\mathrm{B}' = \{\beta\}$, we have $\mathrm{I}_\beta^* \cap \mathrm{I}_\beta \subsetneq \mathrm{I}_\beta$ and so by 3.5 (i) and 5.1 (i) it follows that $(\mathrm{I}_\beta^* \cap \mathrm{I}_\beta)\, \mathrm{M}(w_\lambda \lambda) \subset \overline{\mathrm{M}(w_\lambda s_\beta \lambda)} \subset \mathrm{M}(w_\lambda s_\alpha \lambda)$, by 7.1. Then by 4.6, $\mathrm{I}_\beta^*\, \mathrm{M}(w_\lambda \lambda) \subset \mathrm{M}(w_\lambda s_\alpha \lambda)$ and so $\mathrm{I}_\beta^* \subset \mathrm{I}_\alpha$, by 4.7.

Remark. – It is clear that this result is equivalent to [3] (Folg. 2.20).

7.3. Set $\mathrm{B}_\lambda = \{\alpha, \beta\}$ and choose $l \in \{2, 3, \ldots, (1/2)\ \mathrm{card}\, \mathrm{W}_\lambda - 1\}$. Then there are exactly two distinct elements $w, w' \in \mathrm{W}_\lambda$ satisfying $l_\lambda(w) = l_\lambda(w') = l$. Furthermore

LEMMA (notation 5.1) :

(i) $\mathrm{J}_w + \mathrm{J}_{w'} = (\mathrm{I}_\alpha \cap \mathrm{I}_\beta)^{l-1}$;

(ii) $\mathrm{J}_w \cap \mathrm{J}_{w'} = (\mathrm{I}_\alpha \cap \mathrm{I}_\beta)^{l}$;

Clearly $(\mathrm{I}_\alpha \cap \mathrm{I}_\beta) \supset \mathrm{I}_\alpha \mathrm{I}_\beta + \mathrm{I}_\beta \mathrm{I}_\alpha$ and any non-zero simple subquotient V of

$$(\mathrm{I}_\alpha \cap \mathrm{I}_\beta)/(\mathrm{I}_\alpha \mathrm{I}_\beta + \mathrm{I}_\beta \mathrm{I}_\alpha)$$

must satisfy LAnn $\mathrm{V} = \mathrm{I}_\alpha + \mathrm{I}_\beta$. Then by 4.3, 4.4 (i) and 4.5, $\mathrm{V} = \mathrm{V}(-\lambda, -\lambda)$ up to isomorphism which contradicts the fact that $(\mathrm{I}_\alpha \cap \mathrm{I}_\beta)$ does not admit the trivial $\mathfrak{k}$ submodule. This gives (i) for $l = 2$ and the general case obtains by taking powers. Again $\mathrm{J}_w \cap \mathrm{J}_{w'} \supset (\mathrm{I}_\alpha \cap \mathrm{I}_\beta)^l$ and $\mathrm{I}_\alpha (\mathrm{J}_w \cap \mathrm{J}_{w'}) \subset \mathrm{J}_{ws_\alpha} \cap \mathrm{J}_{w's_\alpha} \subset (\mathrm{I}_\alpha \cap \mathrm{I}_\beta)^l$ by (i). Thus a similar argument gives (ii).

7.4. Set $\mathrm{B}_\lambda = \{\alpha, \beta\}$ and define k as in 2.3.

PROPOSITION. – *Suppose* $k = 0, 1, 2$. *Then* $\mathcal{JH}$ $\mathrm{L}(-\lambda, -\lambda)$ *is multiplicity free.*

Suppose B_λ is of type B_2. We show (see *Fig.*) that

$$\{\mathrm{J}_w : w \in \mathrm{W}_\lambda,\ \mathrm{I}_\alpha + \mathrm{I}_\beta,\ (\mathrm{I}_\alpha \cap \mathrm{I}_\beta)^l : l = 1, 2, 3\}$$

is the set of non-zero U submodules of $U(\mathfrak{g})/I_\lambda$. By 5.1 (i), (iv) and 7.3, these are pairwise distinct and satisfy the given inclusion relations. We show that each arrow defines a simple quotient. By 4.4 (i) and 4.5, $I_\alpha+I_\beta$ is the unique maximal submodule. By [7] (II), there exists a unique minimal submodule which by 5.3 (i) is $L(-w_\lambda \lambda, -\lambda)$ and this by the argument of 7.3 (i) and 4.10 (iii) equals $(I_\alpha \cap I_\beta)^3$. By 7.3 and α, β interchange it suffices to prove simplicity for the arrows labelled 1, 2, 3. For 1, this follows from 7.2 and 4.4 (iii). Consider 2. By 3.1 (iv) and 3.4 any simple subquotient of $I_\alpha I_\beta/(I_\alpha \cap I_\beta)^2$ is isomorphic to $V(-w\lambda, -\lambda)$ for some $w \in W_\lambda$. Taking 4.3 into account it follows from 7.2 that

$$\operatorname{Ann} V(-w\lambda, -\lambda) = \check{I}_\beta \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \check{I}_\alpha .$$

Substitution in 5.2 gives $w = s_\beta s_\alpha$. Yet $I_\alpha I_\beta = L(-s_\beta s_\alpha \lambda, -\lambda)$ by 5.1 (i) and so by 3.1 (iv), $V(-s_\beta s_\alpha \lambda, -\lambda)$ can only occur once in $I_\alpha I_\beta/(I_\alpha \cap I_\beta)^2$ which is hence simple. Consider 3. Let $V(-w\lambda, -\lambda) : w \in W_\lambda$ be a simple subquotient of $I_\alpha I_\beta I_\alpha/(I_\alpha \cap I_\beta)^3$. Then $\operatorname{Ann} V(-w\lambda, -\lambda) = \check{I}_\beta \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \check{I}_\beta$ and so by 5.2, $w = s_\alpha$ or $w = s_\alpha s_\beta s_\alpha$. The former choice contradicts [7] (Prop. 4) and the latter implies the simplicity of $I_\alpha I_\beta I_\alpha/(I_\alpha \cap I_\beta)^3$ as above.

$$\begin{tikzcd}
 & U(\mathfrak{g})/I_\lambda \arrow[d] & \\
 & I_\alpha+I_\beta \arrow[dl] \arrow[dr] & \\
I_\alpha \arrow[dr, "1"'] & & I_\beta \arrow[dl] \\
 & I_\alpha \cap I_\beta \arrow[dl] \arrow[dr] & \\
I_\alpha I_\beta \arrow[dr, "2"'] & & I_\beta I_\alpha \arrow[dl] \\
 & (I_\alpha \cap I_\beta)^2 \arrow[dl] \arrow[dr] & \\
I_\alpha I_\beta I_\alpha \arrow[dr, "3"'] & & I_\beta I_\alpha I_\beta \arrow[dl] \\
 & (I_\alpha \cap I_\beta)^3 \arrow[d] & \\
 & 0 &
\end{tikzcd}$$

The submodules of $U(\mathfrak{g})/I_\lambda$ and their inclusion relations for B_λ of type B_2.
The notation $M \longrightarrow N$ denotes $M \supset N$ with M/N a simple U module.

By 4.4 (i) and 4.5, $I_\alpha + I_\beta$ admits exactly two simple quotients. By 7.3 (i) and 3.1 (ii) the same is true of the $(I_\alpha \cap I_\beta)^l : l = 1, 2$. By 3.1 (ii) and 5.1 (i) the $J_w : w \in W_\lambda$ admit exactly one simple quotient. Since each arrow defines a simple subquotient it follows that there can be no other submodules of $U(\mathfrak{g})/I_\lambda$ than those given in the Figure.

Then by [7] (Prop. 4), or by direct computation it follows that $\mathcal{J}\mathcal{H}\, L(-\lambda, -\lambda)$ is mulitplicity free for B_λ of type B_2 (i. e. when $k = 2$). The remaining cases follow similarly.

Remarks. The cases $k = 0,1$ are unpublished results of Duflo. In genera[l] $\mathcal{J}\mathcal{H}\, L(-\lambda, -\lambda)$ is not multiplicity free if card $B_\lambda \geqq 3$ (*cf.* [5], 7.1 and [7], Cor. 1 to Prop. 11). Recalling 4.6 and 5.1 one can also easily verify that the conclusion of the proposition implies that the map $I \mapsto IM(w_\lambda \lambda)$ is a bijection of $\mathbf{J}(U(\mathfrak{g})/I_\lambda)$ onto the set of submodules of $M(w_\lambda \lambda)$ (*cf.* [8], Prob. 30). It would be important to show that this holds in general and we remark that the counterexample to surjectivity given in [13], Ex. 1, is for λ non-regular. More generally if B_λ is of type A_2 (resp. B_2) with λ on exactly one wall (i.e. subregular) then similar calculations show that card $\mathbf{J}(U(\mathfrak{g})/I_\lambda) = 2$ (resp. 3) whereas after Jantzen [14], $M(w_\lambda \lambda)$ admits 3 (resp. 4) distinct proper submodules.

REFERENCES

[1] W. Borho, *Recent Advances in the Enveloping Algebras of Semisimple Lie Algebras* (*Séminaire Bourbaki*, No. 489, November 1976).

[2] W. Borho und H. Kraft, *Über die Gelfand-Kirillov-Dimension* (*Math. Ann.*, Vol. 220, 1976, pp. 1-24).

[3] W. Borho und J. C. Jantzen, *Über primitive Ideale in der Einhüllenden einer halbeinfacher Lie-Algebra* (*Invent. Math.*, Vol. 39, 1977, pp. 1-53).

[4] W. Borho und J. C. Jantzen (to appear).

[5] N. Conze-Berline et M. Duflo, *Sur les représentations des groupes semi-simple complexes*, *Compos. Math.*, Vol. 34, 1977, pp. 307-336).

[6] M. Duflo, *Représentations irréductibles des groupes semi-simple complexes* (*Lecture Notes*, No. 497, 1975, pp. 26-88).

[7] M. Duflo, *Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple* (*Ann. Math.*, Vol. 105, 1977, pp. 107-120).

[8] J. Dixmier, *Algèbres enveloppantes* (*Cahiers scientifiques*, Vol. XXXVII, Gauthier-Villars, Paris, 1974).

[9] J. C. Jantzen, *Kontravariante formen auf induzierte Darstellungen halbeinfacher Lie-Algebren* (*Math. Ann.*, Vol. 226, 1977, pp. 53-65).

[10] A. Joseph, *A Characteristic Variety for the Primitive Spectrum of a Semisimple Lie Algebra*, preprint, *Bonn*, 1976. Short version in *Lecture Notes*, No. 587. 1977, pp. 102-118.

[11] N. N. Shapovalov, *On a Bilinear Form on the Universal Enveloping Algebra of a Complex Semisimple Lie Algebra* (*Funkt. Analiz i Ego Pril.*, Vol. 6, 1972, pp. 65-70).

[12] N. Spaltenstein, *Remarques sur certaines relations d'équivalence dans les groupes de Weyl*, preprint, Warwick, 1977.

[13] N. Conze et J. Dixmier, *Idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple* (*Bull. Sc. math.*, 2e série, Vol. 96, 1972, pp. 339-351).

[14] J. C. Jantzen (to appear).

[15] A. Joseph, *Gelfand-Kirillov Dimension for the Annihilators of Simple Quotients of Verma Modules* [*J. Lond. Math. Soc.* (to appear)].

[16] C. Schensted, *Longest Increasing and Decreasing Subsequences* (*Canad. J. Math.*, Vol. 13, 1961, pp. 179-191).

[17] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, 1967.

(Manuscrit reçu le 25 avril 1977.)

A. Joseph,
Paris XI,
Centre de Mathématiques,
Bât. 425,
91405 Orsay.