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EQUATIONS DEFINING RATIONAL SINGULARITIES

PAR JONATHAN M. WAHL

INTRODUCTION

Suppose $R = P/I$ is a complete two-dimensional rational singularity (e. g., [2]) of embedding dimension e, where P is a formal power series ring in e variables over an algebraically closed field k. The tangent cone $\overline{R} = \text{gr } R$ is a quotient of the polynomial ring $\overline{P} = \text{gr } P$.

THEOREM 1. (*see* 2.1). — *A minimal projective resolution for* $P/I = R$ *is*

$$0 \to P^{b_{e-2}} \xrightarrow{\varphi_{e-2}} \ldots \to P^{b_2} \xrightarrow{\varphi_2} P^{b_1} \xrightarrow{\varphi_1} P \to P/I \to 0,$$

where

(*a*) *the Betti numbers are* $b_i = \binom{e-1}{i+1}$, $i \geqq 1$,

(*b*) *the associated graded sequence:*

$$0 \to \overline{P}^{b_{e-2}} \xrightarrow{\overline{\varphi}_{e-2}} \ldots \to \overline{P}^{b_2} \xrightarrow{\overline{\varphi}_2} \overline{P}^{b_1} \xrightarrow{\overline{\varphi}_1} \overline{P} \to \overline{P/I} \to 0,$$

is a minimal projective resolution for $\overline{R}$, *and* $\overline{\varphi}_1$ *has degree* 2, $\overline{\varphi}_i$ *has degree* 1 ($i > 1$).

We therefore may say that R is defined by quadratic equations, and all the higher syzygies are linear. The proof of this result is cohomological, but not difficult; one uses Castelnuovo's lemma on the projectivized tangent cone, showing it admits a 2-regular resolution, and then uses a variant of the Artin-Rees theorem to lift the equations for $\overline{R}$ to those for R (§ 1). Apparently, more elementary algebraic proofs are available using only that the multiplicity is one less than the embedding dimension (2.6).

The same techniques yield an analogous result for the "minimally elliptic" singularities of Laufer [12]; these are Gorenstein singularities (hence have self-dual resolutions), and include cones over elliptic curves and the cusp singularities of the two-dimensional Hilbert modular group.

THEOREM 2. (*see* 2.8). — *A minimally elliptic singularity* (*over* **C**) *with* $e \geqq 4$ *has a minimal resolution as in Theorem* 1, *except that*

(*a*) $$b_{e-2} = 1,\ b_i = \frac{i(e-i-2)}{e-1}\binom{e}{i+1}, \qquad i = 1, \ldots, e-3,$$

(*b*) $\bar{\varphi}_i$ *has degree* 2 *for* $i = 1$ *and* $e-2$, *and degree* 1 *otherwise.*

If a rational R happens to be defined determinantally (*see* § 3) by the 2×2 minors of a $2 \times (e-1)$ matrix, then the Theorem follows from the Eagon-Northcott complex [7], which gives a concrete projective resolution. However, once $e \geqq 5$, very few rational singularities are determinantal.

THEOREM 3. (*see* 3.4). — *The graph of a determinantal rational singularity of embedding dimension e consists of one* $-(e-1)$ *curve and* (*possibly*) *some* -2 *curves.*

We conjecture that the converse is true; we have verified it for the quotient singularities over **C** (3.7) and singularities with reduced tangent cone (3.6). The proof of this theorem uses a partial desingularization of Spec R with only rational double points as singularities; this should be contrasted with the Seifert construction of Orlik-Wagreich for singularities with $\mathbf{G}_m$-action, where a space with only cyclic quotient singularities is constructed. It is amusing (and perhaps significant) that many non-determinantal rational singularities can be written as the 2×2 minors of a big matrix (3.8).

An elementary but useful observation is that every rational singularity in characteristic 0 is a cyclic quotient, étale off the singular point, of a Gorenstein singularity (which is never rational, unless R is a quotient singularity). In many cases, one can construct an exceptional configuration for this "canonical cover", and use it to write the equations of rational singularities, given their graph (§ 4). Some needed facts about cyclic covers are gathered in an Appendix.

These results suggest a number of conjectures (*see* § 5).

CONJECTURE 1. — Every rational singularity is a normally flat (i. e., "equimultiple") specialization of a cone (over a rational curve).

A much more general result should be true, e. g., replacing "rational" by "minimally elliptic"; in fact, this has been proved recently by Karras and Kulikov for the cusp singularities, using Kodaira's work on exceptional elliptic configurations. Their method can be used in the rational case for certain taut singularities. In any case, we verify the result for determinantal rational singularities (Proposition 5.4).

CONJECTURE 2. — A normal two-dimensional singularity with resolution as in Theorem 1 is rational if and only if it is absolutely isolated.

This would be a generalization of Kirby's result [10] that the absolutely isolated double points are the rational double points; again, we verify the conjecture in the determinantal case (5.11). Of course, that rational singularities are absolutely isolated is well-known (e. g., [14]).

Our restriction that k be algebraically closed is not essential—see, for instance, a forthcoming paper by J. Lipman on two-dimensional resolution of singularities.

Our interest in these questions arose from a paper of O. Riemenschneider [17], who computed b_1 and b_2 for the cyclic quotient singularities by writing down generators and relations for I. (Recently, using our result, he and Behnke have done the same for the "dihedral" quotients.) We are grateful to David Eisenbud, David Mumford, and

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1. Projective resolutions of local rings and their tangent cones

1.1. Let $V^d \subset \mathbf{P}_k^r = \mathbf{P}$ be a connected purely d-dimensional projective subscheme contained in no hyperplane, with $r > d > 0$. The cone over V is

$$\overline{R} = \bigoplus_{n=0}^{\infty} \Gamma(\mathcal{O}_{\mathbf{P}}(n))/\Gamma(\mathcal{I}(n)) = \overline{P}/\overline{I},$$

where $\mathcal{I} \subset \mathcal{O}_{\mathbf{P}}$ is the ideal sheaf of V,

$$\overline{I} = \oplus \Gamma(\mathcal{I}(n)) \subset \overline{P} = \oplus \Gamma(\mathcal{O}_{\mathbf{P}}(n)) = k[z_0, \ldots, z_r];$$

recall also that V = Proj $(\overline{P}/\overline{I})$. $\overline{R}$ is Cohen-Macaulay iff it is Cohen-Macaulay at the vertex.

PROPOSITION 1.2 (Serre-Grothendieck e. g., [11], 2.24). — $\overline{R}$ *is Cohen-Macaulay* (*i. e.*, $V \subset \mathbf{P}$ *is arithmetically Cohen-Macaulay*) *iff*:

(i) $\overline{R} \xrightarrow{\sim} \bigoplus_{0}^{\infty} \Gamma(\mathcal{O}_V(n))$ (i.e., $\Gamma(\mathcal{O}_{\mathbf{P}}(n)) \twoheadrightarrow \Gamma(\mathcal{O}_V(n))$, or $H^1(\mathcal{I}(n)) = 0$, *all* n).

(ii) $H^i(\mathcal{O}_V(n)) = 0$, $i \neq 0, d$, *all* n.

(1.3) A coherent sheaf F on $\mathbf{P}$ is said to be m-regular in the sense of Castelnuovo ([15], § 14) if $H^i(F(m-i)) = 0$, all $i > 0$. This condition implies:

(1.3.1). $F(m)$ is generated by its global sections;

(1.3.2). $\Gamma(\mathcal{O}_{\mathbf{P}}(i)) \otimes \Gamma(F(m)) \twoheadrightarrow \Gamma(F(m+i))$, all $i \geqq 0$;

(1.3.3). $H^i(F(j)) = 0$, all $j \geqq m-i$.

If $V \subset \mathbf{P}$ is arithmetically Cohen-Macaulay, then $\mathcal{I}$ is 2-regular iff $H^d(\mathcal{O}_V(1-d)) = 0$, since (1.2. (ii)) already yields the other vanishing.

PROPOSITION 1.4. — *Suppose* $V \subset \mathbf{P}$ *is arithmetically Cohen-Macaulay and* $H^d(\mathcal{O}_V(1-d)) = 0$. *Then the cone* $\overline{R}$ *has a minimal graded projective resolution*

$$0 \to \overline{P}^{b_{r-d}} \xrightarrow{\varphi_{r-d}} \ldots \to \overline{P}^{b_2} \xrightarrow{\varphi_2} \overline{P}^{b_1} \xrightarrow{\varphi_1} \overline{P} \to \overline{P}/\overline{I} = \overline{R} \to 0,$$

where φ_i *is homogeneous on each summand of degree* 1 (*if* $i > 1$) *or* 2 ($i = 1$). *Further, the Betti numbers* b_i *are inductively computable from the Hilbert function* $H(n) = \dim \Gamma(\mathcal{O}_V(n))$, *viz*,

$$(1.4.1) \qquad (1-t)^{r+1} \sum_{0}^{\infty} H(n) t^n = 1 + \sum_{1}^{r-d} (-1)^i b_i t^{i+1}.$$

Proof. — As above, $\mathscr{I}$ is 2-regular, so there is a surjection (1.3.1):

$$\mathscr{O}_{\mathbf{P}}^{b_1} \twoheadrightarrow \mathscr{I}(2),$$

where $b_1 = h^0(\mathscr{I}(2))$. Since $\bar{\mathrm{I}} = \bigoplus_{2}^{\infty} \Gamma(\mathscr{I}(n))$ (V is in no hyperplane), (1.3.2) implies $\bar{\mathrm{I}}$ is generated as $\bar{\mathrm{P}}$-module by $\mathrm{H}^0(\mathscr{I}(2))$. Thus, from $\mathscr{O}_{\mathbf{P}}(-2)^{b_1} \to \mathscr{I} \subset \mathscr{O}_{\mathbf{P}}$, we form the map:

$$\varphi_1 : \underset{\displaystyle \bar{\mathrm{P}}^{b_1}}{\underset{\|}{\bigoplus_{0}^{\infty} \Gamma(\mathscr{O}_{\mathbf{P}}(n-2))^{b_1}}} \to \underset{\displaystyle \bar{\mathrm{P}}}{\underset{\|}{\bigoplus_{0}^{\infty} \Gamma(\mathscr{O}_{\mathbf{P}}(n))}},$$

homogeneous of degree 2, with image $\bar{\mathrm{I}}$. This is the beginning of a minimal resolution.

Let $\mathscr{R}_1$ be the kernel of $\mathscr{O}_{\mathbf{P}}^{b_1} \to \mathscr{I}(2)$. By construction, $\mathrm{H}^0(\mathscr{R}_1) = \mathrm{H}^1(\mathscr{R}_1) = 0$, and $\mathscr{R}_1$ is easily checked to be 1-regular. By (1.3), $\bar{\mathrm{R}}_1 = \bigoplus_{0}^{\infty} \Gamma(\mathscr{R}_1(n))$ is generated as $\bar{\mathrm{P}}$-module by $\Gamma(\mathscr{R}_1(1))$, and we have a surjection (with $b_2 = h^0(\mathscr{R}_1(1))$):

$$\mathscr{O}_{\mathbf{P}}^{b_2} \to \mathscr{R}_1(1) \subset \mathscr{O}_{\mathbf{P}}(1)^{b_1}.$$

The kernel of φ_1 is $\bar{\mathrm{R}}_1$, and the map:

$$\varphi_2 : \underset{\displaystyle \bar{\mathrm{P}}^{b_2}}{\underset{\|}{\bigoplus_{0}^{\infty} \Gamma(\mathscr{O}_{\mathbf{P}}(n-3))^{b_2}}} \to \underset{\displaystyle \bar{\mathrm{P}}^{b_1}}{\underset{\|}{\bigoplus_{0}^{\infty} \Gamma(\mathscr{O}_{\mathbf{P}}(n-2))^{b_1}}},$$

has image exactly $\bar{\mathrm{R}}_1$. Note that $b_1 = \#$ of quadratic generators of $\bar{\mathrm{I}}$, and $b_2 = \#$ of "linear" relations on the quadratic generators.

Suppose inductively we have found exact sequences

(1.4.2)
$$\mathscr{O}_{\mathbf{P}}(-i)^{b_{i-1}} \to \ldots \to \mathscr{O}_{\mathbf{P}}(-2)^{b_1} \to \mathscr{O}_{\mathbf{P}} \to \mathscr{O}_{\mathrm{V}} \to 0,$$

and

(1.4.3)
$$\begin{array}{ccccccccc} \bigoplus \Gamma(\mathscr{O}_{\mathbf{P}}(n-1))^{b_{i-1}} & \to & \ldots & \to & \bigoplus \Gamma(\mathscr{O}_{\mathbf{P}}(n-2))^{b_1} & \to & \bigoplus \Gamma(\mathscr{O}_{\mathbf{P}}(n)) & \to & \bigoplus \Gamma(\mathscr{O}_{\mathrm{V}}(n)) \to 0 \\ \| & & & & \| & & \| & & \| \\ \bar{\mathrm{P}}^{b_{i-1}} & \xrightarrow{\varphi_{i-1}} & \ldots & \longrightarrow & \bar{\mathrm{P}}^{b_1} & \xrightarrow{\varphi_1} & \bar{\mathrm{P}} & \longrightarrow & \bar{\mathrm{P}}/\bar{\mathrm{I}} \longrightarrow 0, \end{array}$$

such that the kernel-image short exact sequences arising from (1.4.2):

(1.4.4)
$$0 \to \mathscr{R}_j \to \mathscr{O}_{\mathbf{P}}^{b_j} \to \mathscr{R}_{j-1}(1) \to 0,$$

have $\mathscr{R}_{j-1}$ 1-regular and $b_j = h^0(\mathscr{R}_{j-1}(1))$ (here $1 \leqq j \leqq i-1$, with $\mathscr{R}_0 = \mathscr{I}(1)$). By definition of b_{i-1}, $\mathrm{H}^0(\mathscr{R}_{i-1}) = \mathrm{H}^1(\mathscr{R}_{i-1}) = 0$, and a check (using the 1-regularity of $\mathscr{R}_{i-2}$) shows $\mathscr{R}_{i-1}$ is 1-regular. Thus, $\overline{\mathrm{R}}_{i-1} = \bigoplus_0^\infty \mathrm{H}^0(\mathscr{R}_{i-1}(n))$ is generated as $\overline{\mathrm{P}}$-module by $\mathrm{H}^0(\mathscr{R}_{i-1}(1))$. Letting $b_i = h^0(\mathscr{R}_{i-1}(1))$, one forms the surjection

$$\mathcal{O}_{\mathbf{P}}^{b_i} \to \mathscr{R}_{i-1}(1),$$

and continues as before. A well-known property of Cohen-Macaulay ideals says that the sequence terminates after $r-d$ steps.

The identity (1.4.1), pointed out to us by R. Stanley, is apparently a special case of a more general result of Hilbert. That says that if V is arithmetically Cohen-Macaulay with "pure" resolution, i. e.,

$$(1.4.5)\qquad \mathcal{O}(-a_i)^{b_i} \to \ldots \to \mathcal{O}(-a_2)^{b_2} \to \mathcal{O}(-a_1)^{b_1} \to \mathcal{O} \to \mathcal{O}_{\mathrm{V}} \to 0,$$

where $a_1 < a_2 <, \ldots,$ then:

$$(1.4.6)\qquad \sum_{n=0}^{\infty} \mathrm{H}(n)t^n = (1-t)^{-(r+1)} . \sum_{0}^{r-d} (-1)^i b_i t^{a_i}.$$

This is proved by comparing coefficients on each side; use the Cohen-Macauley property (as in 1.2) to show that if $a_i \leqq a_i + l < a_{i+1}$, then:

$$b_i(h^0(\mathcal{O}(l)) = \sum_{j=1}^{i} (-1)^{j+1} b_{i-j} h^0(\mathcal{O}(l+a_i-a_{j-i})) + (-1)^i \mathrm{H}(a_i+l).$$

1.5. Let R be a Cohen-Macaulay local ring of residue field k, of dimension $d+1 \geqq 2$ and embedding dimension e; assume $\mathrm{R} = \mathrm{P}/\mathrm{I}$, where P is a regular local k-algebra of dimension e. The tangent cone is

$$\overline{\mathrm{R}} = \mathrm{gr}\,\mathrm{R} = \bigoplus_0^\infty m^n/m^{n+1} = \overline{\mathrm{P}}/\overline{\mathrm{I}}, \qquad \text{where} \quad \overline{\mathrm{P}} = k[x_1, \ldots, x_e],$$

and $\overline{\mathrm{I}}$ is the ideal of leading forms of I. The projectivized tangent cone is

$$\mathrm{V} = \mathrm{Proj}\,\overline{\mathrm{R}} \subset \mathbf{P}^{e-1} = \mathbf{P},$$

a d-dimensional scheme contained in no hyperplane, to which we can try to apply the preceding theory (if $\overline{\mathrm{R}}$ is Cohen-Macaulay). Viewing P as filtered by the m-adic filtration (by abuse of notation, we let m denote the maximal ideal of P or R), we wish to compare projective resolutions of P/I and $\overline{\mathrm{P}}/\overline{\mathrm{I}}$. We need first a precise form of the Artin-Rees Theorem.

LEMMA 1.6. — *Suppose* $\mathrm{M} \subset \mathrm{P}^b$ *is a submodule, with the induced filtration* $\mathrm{M}_n = \mathrm{M} \cap m^n \mathrm{P}^b$. *If* $\mathrm{gr}\,\mathrm{M} \subset \mathrm{gr}(\mathrm{P}^b)$ *is generated by homogeneous elements of degree* s, *then:*

$$\mathrm{M} \cap m^{n+s} p^b = m^n \mathrm{M}, \quad \text{all } n > 0,$$

and M *is generated by any set of elements whose images generate* gr M.

Proof. — By assumption, the inclusion $m\,M_i \to M_{i+1}$ is surjective mod M_{i+2}, for all $i \geqq s$. On the other hand, the Artin-Rees theorem (*e. g.*, [19], II-9) guarantees the existence of an integer j so that

$$m\,M_i = M_{i+1}, \quad \text{all } i \geqq j.$$

An easy descending induction starting at $\max(j, s)$ shows

$$m\,M_i = M_{i+1}, \quad \text{all } i \geqq s;$$

thus,

$$m^n M_s = M_{n+s}, \quad \text{all } n \geqq 0,$$

as desired (since $M \subset m^s P^b$).

Next, suppose $\{ x_i \}$ in M have images generating gr M, and let $M' \subset M$ be the submodule they generate. If $x \in M$, by assumption there are $a_i \in P$ with

$$x - \sum a_i x_i \in M_{s+1} = m\,M_s \subset m\,M.$$

Applying Nakayama's Lemma, $M' = M$.

THEOREM 1.7. — *With notations as in* (1.5), *suppose* $\overline{R}$ *is Cohen-Macaulay and* $H^d(\mathcal{O}_V(1-d)) = 0$. *Then there exist minimal projective resolutions:*

$$(1.7.1) \qquad 0 \to P^{b_{e-d-1}} \xrightarrow{\varphi_{e-d-1}} \ldots \to P^{b_2} \xrightarrow{\varphi_2} P^{b_1} \xrightarrow{\varphi_1} P \to P/I \to 0,$$

$$(1.7.2) \qquad 0 \to \overline{P}^{b_{e-d-1}} \xrightarrow{\overline{\varphi}_{e-d-1}} \ldots \to \overline{P}^{b_2} \xrightarrow{\overline{\varphi}_2} \overline{P}^{b_1} \xrightarrow{\overline{\varphi}_1} \overline{P} \to \overline{P}/\overline{I} \to 0,$$

such that

(*a*) (1.7.2) *is the associated graded complex attached to* (1.7.1).

(*b*) $\overline{\varphi}_i$ *is homogeneous of degree* 1 $(i > 1)$ *or* 2 $(i = 1)$.

(*c*) *The* b_i*'s are inductively computable from* H(n), *the Hilbert function of* R.

Proof. — Proposition 1.4 gives the sequence (1.7.2) satisfying (*b*) and (*c*). It remains to derive (1.7.1).

Let $\overline{K}_i = \operatorname{Ker} \overline{\varphi}_i$; it is generated by degree 1 elements of $\overline{P}^{b_i}$ $(1 \leqq i < e-d-1)$.

By Lemma 1.6 and the construction of $\overline{I}$,

$$(1.7.3) \qquad I \cap m^{n+2} = m^n I, \quad \text{all } n \geqq 0,$$

and I is minimally generated by b_1 elements, whose leading forms (all of degree 2) are independent over k. We can therefore construct $\varphi_1 : P^{b_1} \to P$, with $\varphi_1(m^n P^{b_1}) \subset m^{n+2}$ and with image I, whose associated graded is $\overline{\varphi}_1$.

Let $K_1 = \operatorname{Ker} \varphi_1$. We claim $\operatorname{gr} K_1 \xrightarrow{\sim} \overline{K}_1$, where the filtration on K_1 is induced by that on P^{b_1}. The injectivity is obvious. It suffices to lift an $\overline{a} \in \overline{K}_1$ of degree 1 to K_1. Lifting first to $a \in m\,P^{b_1}$, we have $\varphi_1(a) \in m^3 \cap I$ (since φ_1 increases the filtration by 2); in fact, since $\overline{\varphi}_1(\overline{a}) = 0$, $\varphi_1(a) \in m^4 \cap I = m^2\,I$ (by 1.7.3). Thus,

$$\varphi_1(a) = \sum n_i \varphi_1(a_i), \qquad \text{for } n_i \in m^2, \quad a_i \in P^{b_1}.$$

So,

$$a-\sum n_i a_i \in K_1$$

has image $\bar{a}$, and the claim is established. By Lemma 1.6,

$$K_1 \cap m^{n+1} P^{b_1} = m^n K_1, \quad \text{all } n \geqq 1, \tag{1.7.4}$$

and K_1 is generated (necessarily minimally) by b_2 elements whose leading forms are independent over k. Consequently, we can construct $\varphi_2 : P^{b_2} \to P^{b_1}$ which has image K_1, increases the filtration by 1, and whose associated graded is $\bar{\varphi}_2$.

The same argument works successively for each $K_i = \operatorname{Ker} \varphi_i$, except that if $a \in m P^{b_i}$ is a lifting of $\bar{a} \in \bar{K}_i = \operatorname{Ker} \bar{\varphi}_i$, then $\varphi_i(a) \in m^2 P^{b_{i-1}} \cap K_{i-1}$ (since φ_i will increase the filtration by 1). But again, since $\bar{\varphi}_i(\bar{a}) = 0$, we get:

$$\varphi_i(a) \in m^3 P^{b_{i-1}} \cap K_{i-1} = m^2 K_{i-1},$$

and we proceed as above.

Remarks 1.8. — The method of proof above allows one to go quite generally from a minimal projective resolution of $\bar{I}$ to one for I, if the resolution of $\bar{I}$ is "pure" [as in (1.4.5)].

1.9. A particular example of a resolution as in Theorem 1.7 arises from the Eagon-Northcott complex. Here, $R = \bar{R}$ is the generic determinantal singularity, defined by the 2×2 minors of a $2 \times n$ matrix, *viz.*

$$rk \begin{bmatrix} x_{11} & \dots & x_{1n} \\ x_{21} & \dots & x_{2n} \end{bmatrix} \leqq 1,$$

in $k[x_{11}, \dots, x_{2n}]$. The minimal resolution is given in [7]. However, the Veronese embedding $\mathbf{P}^2 \subsetneq \mathbf{P}^5$ is 2-regular but not determinantal (*see* § 3).

2. Syzygies for rational and minimally elliptic singularities

THEOREM 2.1. — *Let* $R = P/I$ *be a rational surface singularity of embedding dimension e, with* P *a regular local k-algebra of dimension e. Then there exist minimal projective resolutions:*

$$0 \to P^{b_{e-2}} \xrightarrow{\varphi_{e-2}} \dots \to P^{b_2} \xrightarrow{\varphi_2} P^{b_1} \xrightarrow{\varphi_1} P \to P/I \to 0,$$

$$0 \to \bar{P}^{b_{e-2}} \xrightarrow{\bar{\varphi}_{e-2}} \dots \to \bar{P}^{b_2} \xrightarrow{\bar{\varphi}_2} \bar{P}^{b_1} \xrightarrow{\bar{\varphi}_1} \bar{P} \to \bar{P}/\bar{I} \to 0;$$

so that:

(*a*) *the second resolution is the associated graded complex attached to the first;*

(*b*) $\bar{\varphi}_i$ *is homogeneous, of degree* $1\,(i > 1)$ *or* $2\,(i = 1)$;

(*c*) $b_i = i\binom{e-1}{i+1}$.

Proof. — By (1.4) and (1.7), we can prove (*a*) and (*b*) once we know that the tangent cone $\overline{R}$ is Cohen-Macaulay, and that $H^1(\mathcal{O}_V) = 0$ for the projectivized tangent cone $V \subset \mathbf{P}^{e-1}$.

Let $X \to \operatorname{Spec} R$ be the minimal desingularization, factoring $X \xrightarrow{g} B \to \operatorname{Spec} R$ *via* the first blow-up. Now, $m\,\mathcal{O}_B = \mathcal{O}_B(-V)$, where $V \subset B$ is the exceptional divisor (= projectivized tangent cone, with very ample line bundle $\mathcal{O}_V(-V)$). Further, $m\,\mathcal{O}_X = \mathcal{O}_X(-Z)$, where Z is the fundamental cycle of M. Artin ([2]). We claim:

$$\mathcal{O}_V(-nV) \xrightarrow{\sim} g_*\mathcal{O}_Z(-nZ), \quad \text{all } n \geqq 0. \tag{2.1.1}$$

For, apply g_* to the exact sequence

$$0 \to \mathcal{O}_X(-(n+1)Z) \to \mathcal{O}_X(-nZ) \to \mathcal{O}_Z(-nZ) \to 0;$$

and use that

$$g_*\mathcal{O}_X(-nZ) \cong g_*(g^*\mathcal{O}_B(-nV)) \cong \mathcal{O}_B(-nV) \otimes g_*\mathcal{O}_X \cong \mathcal{O}_B(-nV),$$

(projection formula) and $R^1 g_*\mathcal{O}_X(-nZ) = 0$ for $n \geqq 0$ (by rationality of the singularities of B—*see* [14], 12.1). This gives:

$$0 \to \mathcal{O}_B(-(n+1)V) \to \mathcal{O}_B(-nV) \to g_*\mathcal{O}_Z(-nZ) \to 0,$$

whence the claim.

Artin has proved ([2]) that

$$m^n/m^{n+1} \xrightarrow{\sim} H^0(Z, \mathcal{O}_Z(-nZ)), \quad \text{all } n \geqq 0. \tag{2.1.2}$$

Therefore, by (2.1.1), we have

$$m^n/m^{n+1} \xrightarrow{\sim} H^0(V, \mathcal{O}_V(-nV)), \quad \text{all } n \geqq 0. \tag{2.1.3}$$

By (1.2), $\overline{R}$ is Cohen-Macaulay. Finally, we have a surjection $\mathcal{O}_B \twoheadrightarrow \mathcal{O}_V$; $H^1(\mathcal{O}_B) = 0$ by rationality, while all H^2's are 0 on B, so $H^1(\mathcal{O}_V) = 0$. This completes the proof of (*a*) and (*b*).

For (*c*), use (1.4.1) and the fact that the Hilbert function of a rational singularity is $H(n) = n(e-1)+1$ ([2]). Or, note that the b_i's depend only on *e*. Since the cone over $\mathbf{P}^1 \subsetneq \mathbf{P}^{e-1}$, defined determinantally by

$$rk\begin{bmatrix} x_1 & x_2 & \dots & x_{e-1} \\ x_2 & x_3 & \dots & x_e \end{bmatrix} \leqq 1,$$

is such a singularity, merely read off its Betti numbers from the (explicit) projective resolution of Eagon-Northcott [7].

Remarks 2.2.1. — Therefore, rational singularities of embedding dimension *e* are defined formally by $[(e-1)(e-2)/2]$ power series of multiplicity 2, with independent leading forms; and there are $[(e-1)(e-2)(e-3)/3]$ relations with independent linear terms. This much was proved by O. Riemenschneider [17] for the cyclic quotient singularities; but he wrote down actual equations and relations.

2.2.2. Not every tangent cone as in Theorem 2.1 can arise from a rational singularity; the absolute isolatedness [14] implies the following:

PROPOSITION 2.3. — *Let* $R = k[[x_1, \ldots, x_e]]/I$ *be a complete rational singularity, with* $e \geqq 4$. *Then the strict tangent cone of* R *has dimension* $\leqq 1$, i. e., *the space of linear derivations* $\sum a_i(\partial/\partial x_i)$ *vanishing on the tangent cone has dimension* $\leqq 1$.

Proof. — We must show that one cannot have two variables, say x_1 and x_2, missing from the leading forms of $r = [(e-1)(e-2)/2]$ generators of I. Suppose in fact we have $I = (f_1, \ldots, f_r)$, with

$$f_1 = Q_1(x_3, \ldots, x_e) + C_1(x_1, \ldots, x_e) + \ldots$$
$$\vdots$$
$$f_r = Q_r(x_3, \ldots, x_e) + C_r(x_1, \ldots, x_r) + \ldots,$$

where the Q_i are quadrics, the C_i are cubics, etc. Since $e > 3$, generators for the linear relations on the Q_i are of the form:

$$\sum L_{ij} Q_i = 0,$$

where $L_{ij} = L_{ij}(x_3, \ldots, x_e)$. By the Theorem, these relations extend to relations on the f_i; thus,

$$\sum L_{ij} C_i = \sum M_{ij} Q_i,$$

where the M_{ij} are quadrics. We claim that C_i has no terms in $(x_1, x_2)^3$. For instance, suppose $C_i = a_i x_1^2 x_2 + \ldots$ Since Q_i has no (x_1, x_2) terms, $\sum M_{ij} Q_i$ has no $x_1^2 x_2$ term, so $\sum a_i L_{ij} = 0$, all j. If some $a_i \neq 0$, we may perform a linear transformation of the f_i's, and assume $L_{ij} = 0$, all j. But the relation $(Q_2, -Q_1, 0, \ldots, 0)$ on the Q_i's could not then be in the span of the relations $L_{\cdot j}$ (since $Q_2 \neq 0$). This establishes the claim.

Now, an easy computation shows that blowing up the origin *via* $x_i = x'_i x_1$ $(i > 1)$ yields $f'_1, \ldots, f'_r$ containing no linear terms and no terms of the form $x_2'^n$ or $x_2'^n$.(linear form in $x_1, x'_3, \ldots, x'_e$). Therefore, by the Jacobian criterion, the first blow-up contains singularities along $x_1 = x'_3 = \ldots = x'_e = 0$, x'_2 arbitrary, hence is not normal. This contradicts absolute isolatedness.

Remark 2.3.1. — For $e = 3$, only the A_n-singularities satisfy the Proposition.

2.4. The highest Betti number b_{e-2}, called "the type" of a Cohen-Macaulay singularity, may be computed as either:

(2.4.1) dim $\mathrm{Ext}^2_R(k, R)$;

(2.4.2) length of the socle, i. e., the length of the annihilator of the maximal ideal in $R/(u, v)$, where u, v is an appropriate R-sequence.

(*See* [9].) Theorem 2.1 says this is $e-2$ for a rational singularity, but we outline another proof.

First, $\mathrm{Ext}^1_R(m, R) \xrightarrow{\sim} \mathrm{Ext}^2_R(k, R)$. Choose e generators for m, and write:

$$0 \to J \to R^e \to m \to 0.$$

Then $\text{Ext}^1_R(m, R) = \text{Coker}((R^e)^{\vee} \to J^{\vee})$. Pulling this sequence back to the minimal desingularization $f: X \to \text{Spec } R$ and dualizing, one gets:

$$0 \to \mathcal{O}_X(Z) \to \mathcal{O}_X^e \to \mathbf{Hom}(f^* J, \mathcal{O}_X) \to 0.$$

Taking cohomology (i. e., f_*) yields:

$$\dim \text{Ext}^1_R(m, R) = h^1(\mathcal{O}_X(Z)).$$

But $h^1(\mathcal{O}_X(Z)) = h^1(\mathcal{O}_Z(Z))$, and one may use Riemann-Roch to show this equals $e-2$.

2.5. Since the type of a Gorenstein singularity is 1, we have that the only Gorenstein rational singularities are the double points. However, this could also be proved using Serre duality on X, and in any case is generally known. More generally, since a result of Serre implies Gorenstein resolutions are self-dual [3], the only 2-regular arithmetically Gorenstein subschemes are degree 2 hypersurfaces [in the notation of (1.4), the degree of φ_{r-d} must be equal to 2, the degree of φ_1].

2.6. As Eisenbud has pointed out, there is a purely algebraic way to obtain some of the above results, since a rational singularity R is Cohen-Macaulay of multiplicity $e-1$. If u, v is an appropriate R-sequence, then the artinian ring $R/(u, v)$ has length $e-1$ and embedding dimension $e-2$, hence:

$$R/(u, v) \cong k[z_1, \ldots, z_{e-2}]/(z_i z_j).$$

Thus, the type of R = length of annihilator of maximal ideal in $R/(u, v) = e-2$, and $R/(u, v)$ is determinantal:

$$rk\begin{bmatrix} z_1 & z_2 & \ldots & z_{e-2} & 0 \\ 0 & z_1 & \ldots & z_{e-3} & z_{e-2} \end{bmatrix} \leqq 1.$$

But a recent result of J. Sally [18] implies that the tangent cone of R is also Cohen-Macaulay (because emb dim = mult + dim − 1), hence the same analysis applies.

2.7. Our methods can be modified to determine the syzygies of the minimally elliptic singularities of Laufer [12]. These are Gorenstein singularities with $h^1(\mathcal{O}_X) = 1$ for a desingularization, and are absolutely isolated for $e \geqq 4$; included are cones over elliptic curves, and the cusps of the 2-dimensional Hilbert modular group.

THEOREM 2.8. — *Let* $R = \mathbf{C}[[x_1, \ldots, x_e]]/I = P/I$ *be a minimally elliptic singularity with* $e \geqq 4$, *with* $\bar{R} = \bar{P}/\bar{I}$ *the tangent cone. Then there exist minimal projective resolutions:*

$$0 \to P \xrightarrow{\varphi_{e-2}} P^{b_{e-3}} \to \ldots \to P^{b_2} \xrightarrow{\varphi_2} P^{b_1} \xrightarrow{\varphi_1} P \to P/I \to 0$$

$$0 \to \bar{P} \xrightarrow{\bar{\varphi}_{e-2}} \bar{P}^{b_{e-3}} \to \ldots \to \bar{P}^{b_2} \xrightarrow{\bar{\varphi}_2} \bar{P}^{b_1} \xrightarrow{\bar{\varphi}_1} \bar{P} \to \bar{P}/\bar{I} \to 0$$

so that

(*a*) *the second resolution is the associated graded complex attached to the first;*

(*b*) φ_i *is homogeneous, of degree* 2 ($i = 1$ *or* $e-2$) *or* 1 ($1 < i < e-2$);

(*c*) $$b_i = \frac{i(e-i-2)}{e-1}\binom{e}{i+1}, \qquad i = 1, \ldots, e-3.$$

Proof. – Suppose we know $\bar{R}$ is Gorenstein. Letting $V = \operatorname{Proj} \bar{R} \subset \mathbf{P}^{e-1}$, write a minimal graded resolution:

$$0 \to F_{e-2} \to \ldots \to F_1 \to \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_V \to 0;$$

here,

$$F_i = \mathcal{O}(-d_i) \oplus \ldots \oplus \mathcal{O}(-d_i - r_i),$$

with $r_i \geqq 0$ (summands may appear many times), and all intermediary summands are $\mathcal{O}(-d_i - l)$, $0 \leqq l \leqq r_i$. The resolution has length $e-2$, by the Cohen-Macaulay property.

We claim $d_{i+1} > d_i$. For, if $d_{i+1} < d_i$, then $\operatorname{Hom}(\mathcal{O}(-d_{i+1}), F_i) = 0$, contradicting the minimality of the resolution. If $d_{i+1} = d_i$, then an $\mathcal{O}(-d_{i+1})$ summand in F_{i+1} would map either isomorphically or as the zero map to the $\mathcal{O}(-d_i)$ summands in F_i; this again contradicts minimality. This proves the claim. Note also that $d_1 = 2$; since the Hilbert function is ne [12], an easy check shows $\bar{I}$ contains some quadrics.

Since the Gorenstein resolution is self-dual, we have $F_{e-2} = \mathcal{O}(-d_{e-2})$; in fact, $F_{e-2} = \mathcal{O}(-e)$ (since $\dim H^1(\mathcal{O}_V) = 1$). Further, $F^{\vee}_{e-2-i}(-e) \cong F_i$, or

$$(2.8.1) \qquad -d_i - r_i = d_{e-2-i} - e.$$

Now, $2 = d_1 < d_2 < \ldots < d_{e-3} < d_{e-2} = e$. But $d_{e-3} = e - d_1 - r_1 \leqq e-2$, so $d_i = i+1$, $2 \leqq i \leqq e-3$. From (2.8.1), each $r_i = 0$. Thus,

$$F_i = \mathcal{O}(-(i+1))^{b_i}, \qquad 1 \leqq i \leqq e-3.$$

This proves (*b*). To compute (*c*), one uses (1.4.6); $H(n) = ne$, so:

$$\sum H(n) t^n = 1 + \frac{et}{(1-t)^2} = (1-t)^{-e}\Big(1 + \sum_{i=1}^{e-3} (-1)^i b_i t^{i+1} + (-1)^{e-2} t^e\Big).$$

A calculation now gives the result.

To prove (*a*), use the Artin-Rees technique of Theorem 1.7. Thus, it remains to show that $\bar{R}$ is Gorenstein.

Now, $\bar{R}$ is Cohen-Macaulay, by the same argument as in the proof of Theorem 2.1. Here $\bar{R} \cong \oplus H^0(Z, \mathcal{O}_Z(-nZ))$ (by [12], 3.13, since $e \geqq 4$), and B has only rational (double point) singularities (*loc. cit.*, 3.15); thus (2.11) is still valid.

Let u, v be an R-sequence such that $\bar{u}$, $\bar{v}$ is an $\bar{R}$-sequence (by taking leading forms). Now, $R/(u, v)$ is artinian, of length e and embedding dimension $e-2$, and the annihilator of the maximal ideal has length 1 (the type is 1 for Gorenstein ring). Choose a $\mathbf{C}$-basis 1, $z_1, \ldots, z_{e-2}, g$, where z_i's generate the maximal ideal m, and $g \in m^2$. Then $z_i z_j = a_{ij} g$, $z_i g = 0$, and $g^2 = 0$; by the annihilator condition, the symmetric matrix (a_{ij}) is invertible. So, after linearly changing the z_i's, we may assume $z^2 = g$, $z_i z_j = 0$ $(i \neq j)$. It is then an easy matter to check that:

$$R/(u, v) \cong \mathbf{C}[z_1, \ldots, z_{e-2}]/J,$$

where $J = (z_i z_j, z_i^2 - z_1^2)$ $(i \neq j)$; in particular $(z_1, \ldots, z_{e-2})^3 \subset J$. Thus, $R/(u, v)$ is graded, defined by quadrics. It follows that $\overline{R}/(\overline{u}, \overline{v})$ is a quotient of $R/(u, v)$, whence the length of the first is $\leqq e$; but it couldn't be $e-1$ [i. e., there can be no new quadratic relation in $R/(u, v)$], so it's e, and $R/(u, v) \xrightarrow{\sim} \overline{R}/(\overline{u}, \overline{v})$. Since each is Gorenstein, R is Gorenstein.

Remarks 2.9.1. — Thus, minimally, elliptic singularities with $e \geqq 4$ are defined by $[e(e-3)/2]$ quadratic equations, and the higher syzygies are linear, except for the last one. Note that for $e = 3$ (a hypersurface), the singularity may be cubic (e. g., the cone over a plane cubic).

2.9.2. Theorem 2.8 predicts the nature of higher syzygies of theta functions, since one can use theta functions to embed elliptic curves into projective space.

2.9.3. Another recent result of J. Sally yields directly that the tangent cone of R in Theorem 2.8 is Gorenstein, because emb dim = mult+dim−2.

3. Determinantal rational singularities

3.1. Suppose $R = k[[x_1, \ldots, x_e]]/I = P/I$ is a normal surface singularity of embedding dimension e. R is said to be *determinantal* if I is defined by the $t \times t$ minors of an $r \times s$ matrix of P $(t \leqq r \leqq s)$, for which $e-2 = (r-t+1)(s-t+1)$ (= codimension of I) — see [6]. If $e = 3$ or 4, then R is automatically determinantal, being (respectively) a hypersurface or Cohen-Macaulay of codimension 2 (a result of Hilbert). We will see below (Theorem 3.4) that "few" rational singularities with $e \geqq 5$ are determinantal.

PROPOSITION 3.2. — *Suppose* $R = P/I$ *as above is a determinantal rational singularity,* $e \geqq 4$. *Then* I *is defined by the* 2×2 *minors of a* $2 \times (e-1)$ *matrix.*

Proof. — One may assume that all the matrix entries are in the maximal ideal of P. Since I is generated by elements with independent quadratic terms (Theorem 2.1), necessarily $t = 2$. Thus $(r-1)(s-1) = e-2$. There are $\binom{r}{2} \cdot \binom{s}{2}$ 2×2 minors of a $t \times s$ matrix; since I is generated by $\binom{e-1}{2}$ elements (2.1):

$$\frac{r(r-1)}{2} \cdot \frac{s(s-1)}{2} \geqq \frac{(e-1)(e-2)}{2} = \frac{[(r-1)(s-1)+1](r-1)(s-1)}{2}.$$

A simple computation reduces this to

$$(r-2)(s-2) \leqq 0.$$

We may suppose $r \geqq 2$, $s \geqq 3$ (since $e \geqq 4$). Thus, $r = 2$ and $s = e-1$.

3.3. Thus, a determinantal rational singularity is defined by the maximal minors of a $2 \times (e-1)$ matrix. As mentioned earlier, the Eagon-Northcott complex provides an explicit resolution, and any rational singularity has the Betti numbers of a determinantal singularity.

THEOREM 3.4. — *Let* R = P/I *be a determinantal rational singularity of multiplicity* $d = e-1$. *Then the graph of* R *consists of one* $-d$ *curve and (possibly) some* -2 *curves.*

Proof. — Write the equations of R formally via $\begin{bmatrix} f_1 \dots f_d \\ g_1 \dots g_d \end{bmatrix}$. Let $V \subset \operatorname{Spec} R \times \mathbf{P}^1$ be defined by $sf_1 = tg_1, \dots, sf_d = tg_d$, where (s, t) are homogeneous coordinates on $\mathbf{P}^1$. (V should be thought of as the closure of the graph of the rational map Spec R ---> $\mathbf{P}^1$ given by the columns of the matrix).

It is easy to see that V $\longrightarrow$ Spec R is proper and surjective, and an isomorphism off the singular point of Spec R. Also, the ideal (f_i, g_j) defines in V a reduced $\mathbf{P}^1$, a section of the map V $\longrightarrow \mathbf{P}^1$. We will be done if we show V is normal with only hypersurface singularities. For, letting X $\longrightarrow$ V be the minimal resolution, X $\longrightarrow$ Spec R is a resolution, so those hypersurface singularities were rational. Then, X $\longrightarrow$ Spec R is the minimal resolution, with graph consisting of one $-d'$ curve and some -2's. But the d' has multiplicity 1 in the fundamental cycle Z, since it has multiplicity 1 in V. But $Z.K = -Z^2-2 = d-2$, so $d' = d$.

Next, it suffices to check at $t = 0$ on $f_i = tg_i$, since more general points can be put in this form by adding a multiple of one row of the matrix to the other. This done, we can add linear combinations of columns to each other.

The leading forms $\bar{f}_1, \dots, \bar{f}_d$ span a space of dimension $\geqq d-1$; otherwise, the tangent cone

$$\begin{bmatrix} \bar{f}_1 & \dots & \bar{f}_d \\ \bar{g}_1 & \dots & \bar{g}_d \end{bmatrix}$$

can be assumed to have 2 zeros in the top row, which would mean fewer than $[d(d-1)/2]$ quadratic equations for the tangent cone, contradicting Theorem 2.1. If $\bar{f}_1, \dots, \bar{f}_d$ are linearly independent, then clearly $f_i = tg_i$ is nonsingular at $t = 0$.

So, suppose $\bar{f}_2, \dots, \bar{f}_d$ are linearly independent and $\bar{f}_1 = 0$ (performing operations on the rows). Choose coordinates so that $f_2 = x_1, \dots, f_d = x_{d-1}$. Then $\bar{g}_1 \neq 0$, or again we'd have too few quadratic equations. If $\bar{g}_1 \notin (\bar{f}_2, \dots, \bar{f}_d)$, we may assume $g_1 = x_d$. Then V is given by $f_1 = tx_d$, where $f_1 \in (x_1, \dots, x_{d+1})^2$. But $(\partial/\partial x_d)(f_1 - tx_d)$ at $x_1 = \dots = x_{d+1} = 0$ is $-t$; so, V has an isolated hypersurface singularity at $t = 0$, (hence is normal).

We are left with the case $\bar{g}_1 \in (x_1, \dots, x_{d-1})$; we may suppose $\bar{g}_1 = x_1$. In fact, assume the tangent cone is of the form:

$$\begin{bmatrix} 0 & x_1 & \dots & x_{n-1} & x_n & x_{n+1} & \dots \\ x_1 & x_2 & & x_n & \bar{g} & . & \dots \end{bmatrix},$$

for some $n \geqq 1$. Suppose $\bar{g} \in (x_1, \dots, x_n)$. Then the tangent cone contains the subscheme defined by $x_1 = \dots = x_n = 0$ and a (possibly empty) set of equations

$$\begin{bmatrix} x_{n+1} & \dots & x_{e-2} \\ . & \dots & . \end{bmatrix}.$$

If $n \leqq e-4$, then every component has dimension $\geqq 3$, by [7], a contradiction.

If $n = e-3$, then $x_1 = \ldots = x_n = 0$ is a component of dimension $= 3$ (x_{e-2}, x_{e-1}, x are arbitrary), also impossible. Finally, if $n = e-2$, then 2 variables are missing from the tangent cone, contrary to Proposition 2.3. Therefore, $\bar{g} \notin (x_1, \ldots, x_n)$.

If $\bar{g} \in (x_1, \ldots, x_{d-1})-(x_1, \ldots, x_n)$, we write $\bar{g} = \mathrm{A}\,(x_{n+1}, \ldots, x_{e-2})+\mathrm{B}\,(x_1, \ldots, x_n)$, where $\mathrm{A} \neq 0$. Changing the last few coordinates, we may suppose $\mathrm{A} = x_{n+1}$. Adding appropriate linear combinations of columns 2 through $n+1$ to the $n+2$ column, and changing the last few coordinates again, we may suppose $\bar{g} = x_{n+1}$.

We eventually reach a stage where $\bar{g} \notin (x_1, \ldots, x_{d-1})$; changing coordinates again, we write the tangent cone as

$$\begin{bmatrix} 0 & x_1 & \ldots & x_{n-1} & x_n & x_{n+1} & \ldots & x_{d-1} \\ x_1 & x_2 & & x_n & x_d & . & \ldots & . \end{bmatrix}.$$

Therefore, final (non-linear) coordinate changes, after adding multiples of certain columns to the first column, lets us write R as

$$\begin{bmatrix} \mathrm{Q}(x_d, x_{d+1}) & x_1 & \ldots & x_{n-1} & x_n & x_{n+1} & \ldots & x_{d-1} \\ x_1+h_1 & x_2+h_2 & & x_n+h_n & x_d & . & \ldots & . \end{bmatrix},$$

where multiplicities of Q, $h_1, \ldots, h_n$ are $\geqq 2$. Now, V at $t = 0$ has coordinates t, x_d, x_{d+1}, with equations

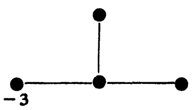
$$\begin{aligned} \mathrm{Q} &= t(x_1+h_1), \\ x_1 &= t(x_2+h_2), \\ &\vdots \\ x_{n-1} &= t(x_n+h_n), \\ x_n &= tx_d, \\ &\vdots \end{aligned}$$

These reduce to $\mathrm{Q}\,(x_d, x_{d+1}) = th_1+t^2 h_2+ \ldots +t^n h_n+t^{n+1} x_d = t^{n+1} x_d+\mathrm{B}$, where $\mathrm{B} \in t\,(x_d, x_{d+1})^2$. To be singular at $x_d = x_{d+1} = 0$ requires:

$$\frac{\partial}{\partial x_d}(\mathrm{Q}-t^{n+1} x_d-\mathrm{B})\big|_{x_d=x_{d+1}=0} = -t^{n+1}$$

to be zero, whence $t = 0$, and the singularity is isolated (hence normal). This completes the proof.

Remarks 3.5.1. – The construction of the partial resolution V, suggested by Mumford, should be thought of as analogous to the partial Seifert resolution of Orlik-Wagreich [16] for singularities with $\mathbf{G}_m$-action. However, these spaces are in general different for determinantal rational singularities with $\mathbf{G}_m$-action. For example, the triple point



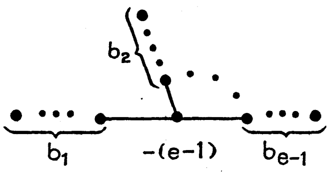
has on V an A_3-singularity, while the Seifert space has 3 cyclic quotient singularities. Showing the deformations of V inject into the deformations of Spec R is equivalent to the standard conjecture on Res $\rightarrow$ Def for determinantal singularities ([22]).

3.5.2. For the Theorem to be true, it is not sufficient that there be exactly one curve not a -2 (e. g., 5.7.3); but it is (necessary and) sufficient that the curve have multiplicity 1 in the fundamental cycle.

3.5.3. The converse of Theorem 3.4 is presumably true, but appears to require much more work. However, we know it in some cases. We use the standard dual graph notation (see [13]); $\bullet$ means a -2 curve.

COROLLARY 3.6. — *Suppose* R *is a rational singularity of embedding dimension e in char.* 0, *with reduced tangent cone (i. e., the fundamental cycle is reduced). Then* R *is determinantal iff its graph contains a* $-(e-1)$ *curve. In this case, the graph is:*

(3.6.1)



where the $b_i \geqq 0$ *are the lengths of strings of* -2 *curves. The equations are:*

(3.6.2)
$$rk\begin{bmatrix} x_1 & x_2 & \cdots & x_{e-2} & x_e^{b_{e-1}+1} \\ x_e^{b_1+1} & a_2 x_2 + x_e^{b_2+1} & \cdots & a_{e-2}x_{e-2}+x_e^{b_{e-2}+1} & x_{e-1} \end{bmatrix} \leqq 1,$$

for appropriate $a_2, \ldots, a_{e-2}$ *in* k, $a_i \neq a_j \neq 0$ $(i \neq j)$.

Proof. — First of all, by the discussion in paragraph 2 and a result of Tjurina [20], the projectivized tangent cone is reduced iff the fundamental cycle $Z = \sum r_i E_i$ has the property that $Z.E_i < 0$ implies $r_i = 1$. Assuming this property of Z, we show all $r_i = 1$. Write $Z = \sum_1^t L_i + F$, where the L_i are reduced and connected cycles, $L_i \cap L_j = \emptyset$ $(i \neq j)$, and F is the sum of those E_i with $r_i > 1$. For every $E_i \subset |F|$, we have $Z.E_i = 0$, hence $Z.F = 0$. One computes

$$-2 = Z(Z+K) = \sum L_i(L_i+K)+L_i.F+F.K.$$

But L_i is the fundamental cycle of its support (an easy check), so $L_i(L_i+K) = -2$. If $F \neq 0$, then $L_i.F \geqq 2$, all i, since $L_i \cap |F| \neq \emptyset$ and every component in F is counted at least twice. So,

$$-2 \geqq -2t+2t+F.K,$$

or $F.K \leqq -2$. Since $K.E_i \geqq 0$, all i, this is absurd. So, $F = 0$, and $Z = E$ is reduced. Since the tangent cone has no embedded components (§ 2), it is reduced iff the projectivized tangent cone is reduced, hence this is true iff Z is reduced.

So, since $Z = E$ is reduced, we have $E.E_i \leqq 0$, all i; hence each -2 curve intersects at most 2 other curves. Thus, the graph of R is as in (3.6.1). Such singularities are formally determined by the graph and the PGL (1)-equivalence class of the intersection points of the $-(e-1)$ curve with the non-empty strings (by [13], 4.1). It remains to check, therefore, that (3.6.2) gives such a singularity, and the choice of the a_i's allows one to fix the isomorphism class of the intersection points. This is straight-forward, except in the case that all $b_i > 0$; here, the projectivized tangent cone consists of $e-1$ lines through a point. It suffices to show that in $\mathbf{P}^{e-1}$, the intersections of $x_e = 0$ with $\mathbf{P}^1 \subset \mathbf{P}^{e-1}$ defined by

$$rk\begin{bmatrix} x_1 & x_2 & \dots & x_{e-2} & x_e \\ x_e & a_2 x_2 + x_e & & a_{e-2} x_{e-2} + x_e & x_{e-1} \end{bmatrix} \leqq 1,$$

yield, for variable a_i's, all equivalence classes of $e-1$ points on $\mathbf{P}^1$. This is done by a concrete "uniformization", *via* (assuming $a_2 = 1$):

$$x_e = uv(u-v)\prod_3^{e-2}\left(u+\frac{a_i}{1-a_i}v\right),$$

$$x_{e-1} = u^2 v\prod_3^{e-2}\left(u+\frac{a_i}{1-a_i}v\right).$$

(It is now easy to figure out the other x_i's).

COROLLARY 3.7. — *A quotient singularity* (*over* **C**) *is determinantal iff the graph contains a* $-(e-1)$ *curve.*

(A) The cyclic quotient singularities $R_{n,q}$ $(0 < q < n,\ (n, q) = 1)$ are determinantal if and only if the terms $a_2, \dots, a_{e-1}$ of the continued fraction expansion of $(n/n-q)$ satisfy $a_i = 2$ $(i \neq 2,\ e-1)$. (*See* [17] for notation.)

(B) The quotient singularity $\langle b;\ n_1, q_1;\ n_2, q_2;\ n_3, q_3 \rangle$ (*see* Brieskorn's list [5]) is determinantal iff it is of the form

(i)	$\langle b; 2, 1; 2, 1; n, n-1 \rangle$	$b \geqq 2$
	$\langle b; 2, 1; 3, 2; 3, 2 \rangle$	$b \geqq 2$
	$\langle b; 2, 1; 3, 2; 4, 3 \rangle$	$b \geqq 2$
	$\langle b; 2, 1; 3, 2; 5, 4 \rangle$	$b \geqq 2$
(ii)	$\langle 2; 2, 1; 2, 1; n, q \rangle$	n, q as in (a).
	$\langle 2; 2, 1; 3, 2; 4, 1 \rangle$	
	$\langle 2; 2, 1; 3, 2; 5, 1 \rangle$	
(iii)	$\langle 2; 2, 1; 3, 1; 3, 2 \rangle$	
	$\langle 2; 2, 1; 3, 1; 4, 3 \rangle$	
	$\langle 2; 2, 1; 3, 2; 5, 3 \rangle$	
	$\langle 2; 2, 1; 3, 1; 5, 4 \rangle$	
	$\langle 2; 2, 1; 3, 2; 5, 2 \rangle$	

The equations in (A) and (B) (i) with $b > 2$ can be read off (3.6.2); if $b = 2$, one has rational double points. The equations in (B) (ii) are given respectively, by the matrices:

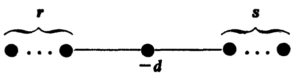
$$\begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_{e-2} & x_{e-1}^{a_{e-1}-1} \\ x_2 & x_1^{a_2}+x_3^2 & x_4 & \dots & x_{e-1} & x_e \end{bmatrix},$$

$$\begin{bmatrix} x_1 & x_2 & x_4 & x_5+x_1^2 \\ x_2 & x_3 & x_5 & x_1 x_4 \end{bmatrix},$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_5 & x_6 \\ x_2 & x_3 & x_4 & x_6+x_1^2 & x_1 x_5 \end{bmatrix}.$$

The equations for the triple points of (B) (iii) are found in [20].

Proof. – Since the quotient singularities are taut [5], it is only a matter of checking that the equations give graphs of the appropriate type. As for (A), use induction and [17] to check that the singularity with graph:



$r, s \geqq 0, \quad d \geqq 3;$

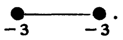
corresponds to

$$n = (r+1)(s+1)d-(2r+1)s-r,$$

$$q = r(s+1)d-(2r-1)s-(r-1),$$

so $a_2 = r+2, \quad a_d = s+2, \quad a_i = 2 \quad (2 < i < d)$.

3.8. The simplest non-determinantal quotient singularity is the cyclic (8,3) quotient, with graph



However, the equations defining R may be expressed as

$$rk \begin{bmatrix} x_1 & x_2 & x_3^2 \\ x_2 & x_3 & x_4 \\ x_3^2 & x_4 & x_5 \end{bmatrix} \leqq 1.$$

Similarly, one can use Riemenschneider's equations [17] to write the equations of any cyclic quotient of embedding dimension e as the 2×2 minors of an $[(e+1)/2]$ by $[(e+2)/2]$ matrix, where [ ] means "greatest integer in". Of course, this is not what determinantal means.

3.9. Note that the equations (3.6.2) are weighted homogeneous, where x_e has weight 1, x_i has weight b_i+1 $(1 \leqq i \leqq e-1)$. In particular, one has equations for an algebraic variety with $\mathbf{G}_m$-action, not just equations giving the analytic type.

3.10. We outline another proof of Theorem 3.4 in the case of reduced tangent cone. If R is determinantal, so is its projectivized tangent cone $C \subset \mathbf{P}^{e-1}$. If C is determinantal and is not a cone, then

$$C = \bigcup E_i = \mathbf{P}^{e-1} \bigcap (\mathbf{P}^1 \times \mathbf{P}^{e-2}) \subset \mathbf{P}^{2e-3}.$$

An easy check shows that r_i, the degree of E_i in $\mathbf{P}^{e-1}$ (or $\mathbf{P}^{2e-3}$) is the sum of the degrees of the compositions $E_i \to \mathbf{P}^1$ and $E_i \to \mathbf{P}^{e-2}$. In particular, the image of E_i in $\mathbf{P}^{e-2}$ spans an s_i-dimensional linear space, with $s_i \leqq r_i$, and $s_i = r_i$ iff $E_i \to \mathbf{P}^1$ lands in a point. Also, $\sum r_i = e-1$.

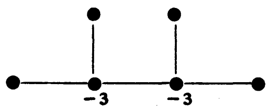
But the image of C in both $\mathbf{P}^1$ and $\mathbf{P}^{e-2}$ must span the whole spaces, since otherwise C would be defined by too few quadratic equations. Thus (since C is connected):

$$e-2 \leqq \sum s_i \leqq e-1 = \sum r_i, \qquad \text{and some} \quad s_i < r_i.$$

Therefore, $s_i = r_i$ for all but one $i = j$, whence $s_j = r_j - 1$. Let E_1 be a boundary curve for which $r_1 = s_1$ (we may assume C is not one curve). One then shows that if $r_1 > 1$, then one could choose coordinates for the matrix of C with two zeros in one row, an impossibility. Thus, $r_1 = s_1 = 1$ (E_1 is a line). If $E_1 \cap E_2 \neq \emptyset$, and $r_2 = s_2$, one derives a similar contradiction. The net result is that C consists of $E_1, \ldots, E_t, E'$, where the E_i are non-intersecting lines, and where $s' = r' - 1$. From this configuration of C, one can show E is as in (3.6.1).

If C is a cone, one blows up, obtaining another determinantal singularity of the same embedding dimension. Now continue as before.

3.11. The following example is a non-determinantal singularity with determinantal projectivized tangent cone:



Its equations can be computed using the techniques of paragraph 4, and is defined as in (3.8) by:

$$rk \begin{bmatrix} x_0 & x_1^2 & x_3 \\ x_1^2 & x_2 & x_4 \\ x_3 & x_4 & x_0 + x_2 - 2x_1^2 + x_0^2 \end{bmatrix} \leqq 1.$$

3.12. Suppose R is a rational singularity whose projectivized tangent cone C is not a cone. Then if R is determinantal, C is a local complete intersection (since C is cut out on $\mathbf{P}^1 \times \mathbf{P}^{e-2}$ by $e-2$ hyperplanes). Occasionally one may use this fact to show an R is not determinantal (if one knows the equations). Furthermore, the singularities of the first blow-up of a determinantal R have either the same multiplicity, or are rational double points (clear from the Theorem or from the proof of 5.11 below).

4. The canonical cover of a rational singularity

4.1. Let R be a two-dimensional normal local ring, essentially of finite type over k algebraically closed, with a rational singularity. Let $X \to \operatorname{Spec} R$ be a desingularization, with exceptional fibre $E = \bigcup E_i$. The canonical line bundle $K = \Omega^2_{X/k}$ satisfies

$$K.E_i = d_i - 2, \tag{4.1.1}$$

where $d_i = E_i.E_i$.

4.2. There is a smallest integer $n \geqq 1$ and an integral cycle $Z_1 = \sum r_i E_i$ so that numerically $n K \equiv -Z_1$. One has but to solve in rational numbers the linear equations:

$$-(\sum r_i E_i).E_j = d_j - 2,$$

using the definiteness of the intersection matrix; then, clear denominators minimally. If the desingularization is minimal, each $d_i \geqq 2$, so $Z_1.E_i \leqq 0$, all i, hence Z_1 is 0 or an effective cycle ([2]). Since R is rational, numerical equivalence of line bundles implies isomorphism ([14], § 14), so $K^n \simeq \mathcal{O}(-Z_1)$. Letting $U = X - E = \operatorname{Spec} R - \{ m \}$, we have $K' = K|_U$ is torsion, of order n.

4.3. Assuming $(n, p) = 1$ (p = char. k), choose an isomorphism $\alpha : K'^n \xrightarrow{\sim} \mathcal{O}_U$. One gets an n-cyclic étale cover:

$$V_\alpha = \operatorname{Spec}(\mathcal{O}_U \oplus K' \oplus \ldots \oplus K'^{n-1}) \to U,$$

where multiplication is given *via* α. (*See* the Appendix for this and other facts on cyclic étale covers.)

LEMMA 4.4. – *The canonical line bundle of* V_α *is trivial* (*i. e.*, $\simeq \mathcal{O}_{V_\alpha}$).

Proof. – Let $\pi : V_\alpha \to U$. By étaleness,

$$\Omega^1_{V_\alpha/U} = 0 \quad \text{and} \quad \Omega^1_{V_\alpha/k} \xrightarrow{\sim} \pi^* \Omega^1_{U/k}.$$

Taking second exterior powers gives $K_{V_\alpha} \xrightarrow{\sim} \pi^* K_U$; but by construction, $\pi^* K_U \cong \mathcal{O}_{V_\alpha}$.

4.5. V_α is smooth and connected (since n = order of K_U – Appendix A.4), hence $S_\alpha = \Gamma(\mathcal{O}_{V_\alpha})$ is a normal local domain, finite over $R = \Gamma(\mathcal{O}_U)$ and étale off the singular point. R is the ring of invariants for an action of Z_n on S_α. Since $V_\alpha = \operatorname{Spec} S_\alpha - \{ \text{singular point} \}$ and $K_{V_\alpha} \cong \mathcal{O}_{V_\alpha}$, it follows that S_α is Gorenstein ([8], 1.6). If we change α, or use K^i instead of K (where $(i, n) = (i, p) = 1$), we get covers that are isomorphic over the henselization of R (Appendix, A.1 *b*). We will therefore speak of the *canonical cover* $R \to S$. Note that if S is rational, it is a double point (2.5), hence a quotient singularity, so R is a quotient singularity.

PROPOSITION 4.6. – *Every rational singularity* R *in characteristic zero is an n-cyclic quotient, étale off the singular point, of a Gorenstein singularity; n, the order of* $K|_U$, *may be computed from the exceptional configuration of* R. *The canonical cover is unique over the henselization of* R. *In case* R *is a quotient singularity* $\mathbf{C}[x, y]^G$, $G \subset GL(2)$ (as in [5]), then $S = \mathbf{C}[x, y]^{G'}$, where $G' = G \cap SL(2)$.

Proof. – Only the last assertion requires proof. We have exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & Z_n & \to & 0, \\ & & \cap & & \cap & & \cap & & \\ 0 & \to & SL(2) & \to & GL(2) & \to & \mathbf{C}^* & \to & 0, \end{array}$$

where the last vertical map is by the subgroup of nth roots of 1. Then $S = \mathbf{C}[x, y]^{G'}$ is a rational double point (hence Gorenstein), and S is an n-cyclic cover of R, étale off the singular point. The canonical cover, being Galois, is defined by a subgroup $G' \subset G$; being Gorenstein, $G'' \subset SL(2) \cap G = G'$ (the rational double points are the only Gorenstein rational singularities). Letting n_1 be the order of K_U ($= [G : G'']$), we have $n \mid n_1$. But $K|_U$ becomes trivial in Spec $S - \{$singular point$\}$, by (3.4), so, by (A.5) in the Appendix, $n_1 \mid n$. Thus, $n = n_1$, so $G' = G''$.

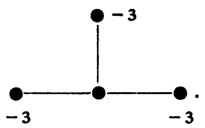
4.7. If $X \to$ Spec R is a desingularization, we construct a proper $\overline{Y} \to X$, inducing $V \to U$ off E, with $\overline{Y}$ non-singular; then $S = \Gamma(\mathcal{O}_{\overline{Y}})$, and we wish to determine the exceptional configuration on $\overline{Y}$. So, let $Y' \to X$ be the normalization of X in the quotient field of V. Y′ may be obtained as follows. Choose an isomorphism $K^n \xrightarrow{\sim} \mathcal{O}_X(-Z_1) \subset \mathcal{O}_X$. If $(i, n) = (i, p) = 1$ and D is a divisor for which $i Z_1 - n D$ is effective, one has an isomorphism

$$(K^i(D))^n \xrightarrow{\sim} \mathcal{O}_X(-(iZ_1 - nD)) \subset \mathcal{O}_X. \tag{4.7.1}$$

Consider the finite flat X-scheme $Y = \text{Spec}(\mathcal{O} \oplus K^i(D) \oplus \ldots \oplus (K^i(D))^{n-1})$, where multiplication is given by the inclusion (4.7.1). Then Y′ is the normalization of Y, $Y' \to X$ is a brached cover, and Y′ has only rational (even cyclic) singularities (obtained from $z^n = x^a y^b$, where $x^a y^b$ is a local equation for $i Z_1 - nD$). Now, let $\overline{Y} \to Y'$ be the minimal desingularization of the singularities.

4.8. If $i Z_1 - nD = F$ is a reduced, disjoint union of some of the E_i, then $\overline{Y} \xrightarrow{\sim} Y' \xrightarrow{\sim} Y$, and one can easily compute the exceptional configuration of $\overline{Y}$. Let $\pi : \overline{Y} \to X$. If $E_i \cap |F| = \emptyset$, then $\pi^{-1}(E_i) \to E_i$ is étale, of degree n: thus, $\pi^{-1}(E_i)$ is the disjoint union of n copies of $\mathbf{P}^1$, and each has self-intersection $E_i . E_i$ in $\overline{Y}$. If $E_i \cap |F|$ is a finite number of points, then $\pi^{-1}(E_i) \to E_i$ is a branched cover of degree n, étale off $E_i \cap |F|$, with one point lying over each point of $E_i \cap |F|$. Thus, $\pi^{-1}(E_i)$ is connected and smooth, of genus $1/2\,(n-1)\,(s_i - 2)$ ($s_i = E_i \cap |F|$) by Hurwitz's formula, and with self-intersection $n\,E_i . E_i$ in $\overline{Y}$. Finally, if $E_i \subset |D|$, then $\pi^{-1}(E_i)$ is one smooth curve ($a\ \mathbf{P}^1$) of self-intersection $1/n\ E_i . E_i$.

Example 4.8.1.

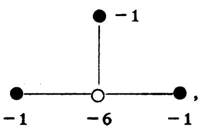


. Then Z_1 is $\begin{matrix} & 2 & \\ 2 & 3 & 2 \end{matrix}$, $n = 3$,

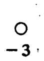
and one has $(K_X^2(Z))^3 = \mathcal{O}(-F)$, where Z is the fundamental cycle and F is

$$\begin{matrix} & 1 & \\ 1 & 0 & 1. \end{matrix}$$

By (4.8), the exceptional configuration of $\overline{Y}$ is



, or



(○ = elliptic curve).

By construction, this elliptic curve admits a 3-fold cyclic cover of $\mathbf{P}^1$; there is one such, and S is the cone over it. In fact, $S = k[x, y, z]/z^3 = x^3+y^3$, and R is the ring of invariants for the action $(x, y, z) \mapsto (\omega x, \omega^2 y, \omega z)$, where $\omega^3 = 1$. The equations for R are easily written down, and may be expressed in the form:

$$rk \begin{bmatrix} x_0 & x_1 & x_2 & x_4 \\ x_1 & x_2 & x_3 & x_5 \\ x_4^2 & x_4 x_5 & x_5^2 & x_0+x_3 \end{bmatrix} \leqq 1,$$

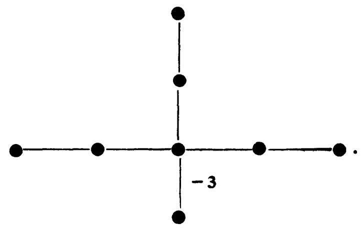
where $x_0 = x^3$, $x_1 = x^2 y$, $x_2 = xy^2$, $x_3 = y^3$, $x_4 = xz$, $x_5 = yz$.

4.9. Blowing up points of a given resolution X will frequently simplify matters. Let $Z_1 = \sum r_i E_i$ on X, and suppose E_1 and E_2 intersect. Blowing up that point yields $X' \to X$, with proper transforms E'_i, and a new -1 curve E'. One then checks that:

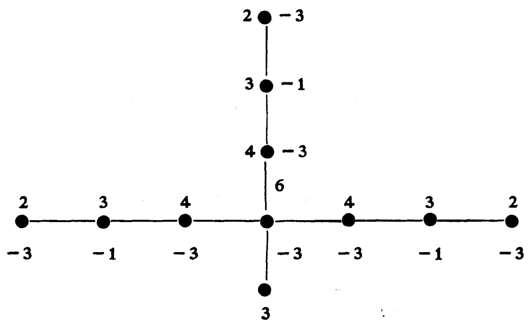
$$K^n_{X'} \xrightarrow{\sim} \mathcal{O}(-\sum r_i E'_i - (r_1+r_2-n) E'),$$

by dotting both sides with E'_1, E'_2, E'. Also, note that one can compute $\overline{Y}$ locally in a neighborhood of each E_i; so, one can apply (4.8) when $| i Z_1 - n D |$ is a disjoint union of non-singular curves.

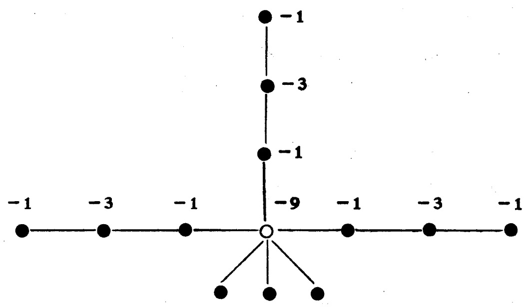
Example 4.9.1.



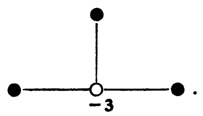
Here, $n = 3$, $e = 6$. Blowing up, and writing the multiplicities in the new Z_1, we have



Thus, $\overline{Y}$ is



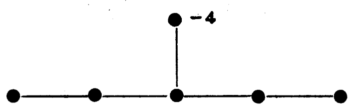
or



An example of such a Gorenstein S is $k[x, y, z]/z^3 = x^3+y^6$, and the others are weighted homogeneous deformations of this. The action $(x, y, z) \mapsto (\omega x, \omega y, \omega^2 z)$, with $\omega^3 = 1$, will define an R with the desired graph.

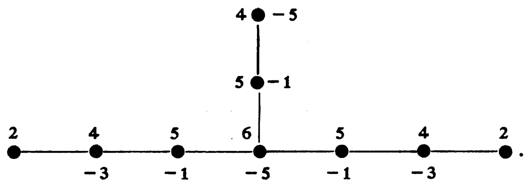
4.10. In the general case, the preceding methods are not sufficient. It may still be possible to reconstruct the configuration on $\overline{Y}$ by recognizing "known" canonical covers on the connected components of some $|i Z-n D|$.

Example 4.11.



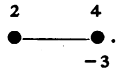
$n = 5, \quad e = 5.$

Blowing up, and writing the multiplicities of the new Z_1, yields:

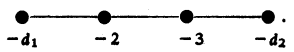


Over each -1 curve is a rational -5 curve. One must find the configuration on $\overline{Y}$ over each end, i. e., over

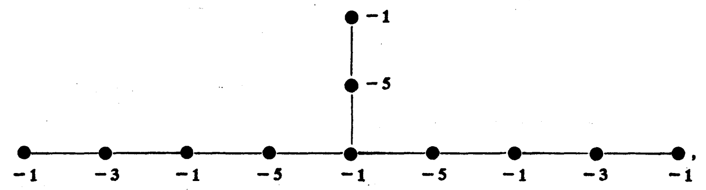
(4.11.1)



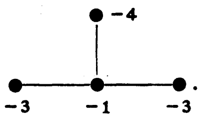
But $\overline{Y}$ over these components is equivalent to the canonical cover of the (5, 3) cyclic quotient singularity (recall $2-1/3 = 5/3$). There is one rational curve in $\overline{Y}$ over each component, and a desingularization for $z^5 = x^2 y^4$ (equivalently, $z^5 = xy^2$) connecting them. But $z^5 = x^2 y$ normalized and resolved (use $x = s^5$, $y = t^5$, $z = s^2 t$) gives a (5, 2) singularity. Thus, lying above (4.11.1) on $\overline{Y}$ is



(One must check that it is -2, -3, not -3, -2). But since this corresponds to the canonical cover of the (5, 3) singularity, and since that cover is smooth, we must have $d_1 = d_2 = 1$. Therefore, the configuration on $\overline{Y}$ is



or



This is recognized to be a minimally elliptic singularity; it is listed as Tr (2, 2, 3) in Laufer's tables, and one such S is $k[x, y, z]/z^2 = x^3+y^8$. R is the quotient by the action $(x, y, z) \mapsto (\omega x, \omega y, \omega^4 z)$, where $\omega^5 = 1$. R is determinantal (*cf.* Conjecture 3.5.3), *viz.*

$$rk\begin{bmatrix} x_0 & x_1 & x_3 & x_4^2-x_2^3 \\ x_1 & x_2 & x_4 & x_0 \end{bmatrix} \leq 1,$$

where $x_0 = x^2 y^3$, $x_1 = xy^4$, $x_2 = y^5$, $x_3 = xz$, $x_4 = yz$.

4.12. In the previous examples, one was able to recognize S from its graph (at least up to "equisingular deformation"). In the case of (4.9.1), one needed to know that S was Gorenstein; for, a theorem of Laufer ([12], 4.3) states that only minimally elliptic and rational double point graphs force the corresponding singularity to be Gorenstein.

4.13. One can start with simple singularities S (say, hypersurfaces) and try to construct linear actions by groups of roots of unity, in hopes of producing rational singularities R. It is easy to see, for instance, that if S is strongly elliptic ($h^1(\mathcal{O}_X) = 1$ for a resolution), a cyclic quotient is either strongly elliptic or rational. For instance, let

$$S = k[x, y, z]/z^2 = f(x, y),$$

where all monomials of f have even degree. There is a natural $\mathbf{Z}_2$-action

$$(x, y, z) \mapsto (-x, -y, -z).$$

The invariants for this action on $k[x, y, z]$, i. e. $x_0 = x_2$, $x_1 = xy$, $x_2 = xz$, $x_3 = y^2$, $y_4 = yz$, $x_5 = z^2$, define the cone of the Veronese embedding $\mathbf{P}^2 \subsetneq \mathbf{P}^5$, and the equations among them are given by

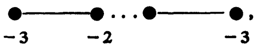
$$(4.13.1) \qquad rk\begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix} \leq 1.$$

One could easily write down an explicit projective resolution for (4.13.1), a Cohen-Macaulay singularity of codimension 3 (it is not determinantal). One gets equations for R, the ring of invariants of S, by specialization; since $f(x, y) = g(x_0, x_1, x_3)$ by assumption, R is

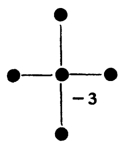
(4.13.2) $$rk \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & g(x_0, x_1, x_3) \end{bmatrix} \leqq 1.$$

Of course, R is rarely rational (it is not absolutely isolated if $\deg g \geqq 3$). One can probably check that the only possibilities are those covered in the following proposition.

PROPOSITION 4.14. — *Let* R *be a rational singularity* (char $k = 0$) *for which* $K^2 \cong \mathcal{O}(-Z)$ *on the minimal resolution* ($Z =$ *fundamental cycle*). *Then* R *is either a cyclic quotient with graph:*

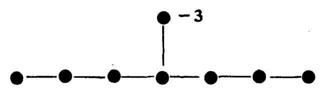
or


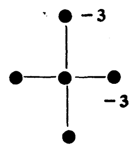
or R *is of type* (4.13.2), *where we list graphs for* R *and* S *and an equation for* S.



$z^2 = x^4 + y^4$,

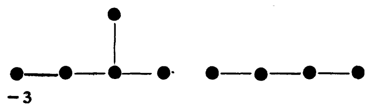
(a)

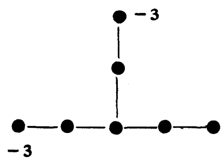




$z^2 = x^4 + y^6$,

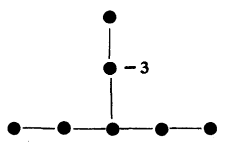
(b)

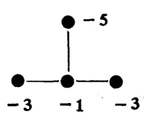




$z^2 = x(x^3 + y^5)$,

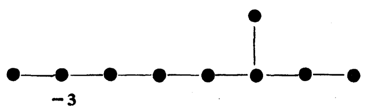
(c)

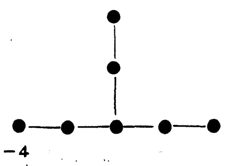




$z^2 = y(x^3 + y^5)$,

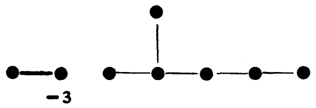
(d)

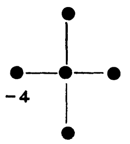




$z^2 = y(x^3 + y^7)$,

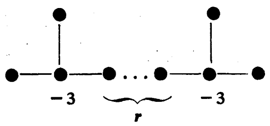
(e)

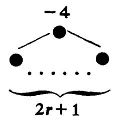




$$z^2 = xy(x^2+y^4),$$

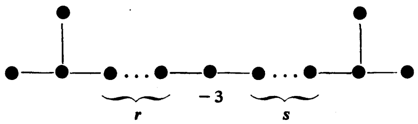
(*f*)

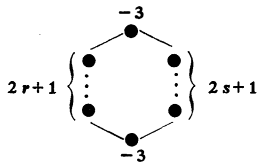




$$z^2 = x(y+x) \times (y^2+x^{2r+4})$$

(*g*)





$$z^2 = (y^2+x^{2r+4}) \times (x^2+y^{2s+4}).$$

(*h*)

Proof. – Since $2K = -Z$ and $Z(Z+K) = -2$, we have $Z^2 = -4$, whence $e = 5$. Writing $Z = \sum r_i E_i$, we deduce:

$$Z.E_i = 2(2-d_i),$$

$$\sum r_i(d_i-2) = 2.$$

It is then a direct matter to verify that only the graphs above can occur. The equations for S are from [12], Table 5.2.

Remark 4.15. The equations on the right correspond to only one analytic type of singularity with the given (for S or R). In fact, types (*a*)–(*f*) are not taut [13]. For instance, more general S of type (*f*) may be written $z^2 = xy(x^2+y^4+axy^3)$, $a \in k$, and the corresponding R accordingly. Also note all types except (*g*) and (*h*) have $\mathbf{G}_m$-action.

5. Problems and Conjectures

5.1. The fact that the Betti numbers of a rational singularity depend only on the multiplicity suggests the following:

CONJECTURE 5.2. – Every rational singularity R is a normally flat specialization of a cone.

That is, there is a one-parameter family $\{\operatorname{Spec} R_t\}$ of rational singularities which is normally flat ("equimultiple") along some section, $R_0 \simeq R$, and R_t is (for $t \neq 0$) a cone over a rational curve.

5.3. The Conjecture should be thought of as having two parts. Let $C \subset \mathbf{P}^{e-1} = \mathbf{P}$ be the projectivized tangent cone of R; let Hilb be the (infinitesimal or formal) Hilbert scheme functor for C; and let E be the functor of isomorphism classes of deformations of R plus normally flat sections ([4] or [21]). (Recall that one must consider the sections as part of the data of E in order that E have a good deformation theory.) There is a natural morphism

$$\alpha: \quad E \to \text{Hilb},$$

(*see* Bennett [4]) because one can trivialize the section to get a well-defined projective deformation of C (1). It is a problem for general singularities to determine whether α is smooth; one knows this for strict (i. e., tangential) complete intersections. Note that (5.2) would follow if one could prove:

5.3.1. α: $E \to \text{Hilb}$ is surjective;

5.3.2. C is a smoothable scheme.

For, if C is smoothable, then it is smoothable in **P**; for, $H^1(\mathcal{O}_C(1)) = 0$, so $H^1(C, \Theta_{\mathbf{P}} \otimes \mathcal{O}_C) = 0$, whence all deformations of C can be realized in **P**. But the only smooth curve which can specialize to C is (for Hilbert polynomial reasons) a non-singular rational curve of degree $(e-1)$. Thus (5.3.1) and (5.3.2) together imply the Conjecture; of course, the Conjecture also implies (5.3.2). It appears that the surjectivity of α is quite subtle. Our partial results are:

PROPOSITION 5.4. — *Conjecture* 5.2 *is true for determinantal rational singularities* R. *If, furthermore,* C *is not a cone, then* $\alpha : E \to \text{Hilb}$ *is surjective.*

Proof. — The key point is that one can deform arbitrarily the entries of the $2 \times (e-1)$ matrix defining R and still have a flat deformation; this is because all relations can be read off the matrix by doubling a row, and noting all 3×3 subdeterminants are zero.

If C is not a cone, then we can view $C \subset X = \mathbf{P}^1 \times \mathbf{P}^{e-2} \subset \mathbf{P}^{2e-1}$, and C is a complete intersection on X defined by $(e-2)$ linear forms on $\mathbf{P}^{2e-1}$ (*cf.* 3.10). Thus, one can write an explicit projective resolution for $\mathcal{O}_X \to \mathcal{O}_C$ (a Koszul complex), and compute that $H^1(\Theta_X \otimes \mathcal{O}_C) = 0$ (recalling $X = \mathbf{P}^1 \times \mathbf{P}^{e-2}$). Therefore, every small abstract deformation of C lives on $\mathbf{P}^1 \times \mathbf{P}^{e-2}$; we conclude that nearby deformations of $C \subset \mathbf{P}^{e-1}$ are also determinantal. Combining this with our first remark, $\alpha : E \to \text{Hilb}$ is surjective. Also, the smoothability of C follows from repeated use of Bertini's Theorem, since C is cut from $\mathbf{P}^1 \times \mathbf{P}^{e-2}$ by $e-2$ linear hyperplanes in $\mathbf{P}^{2e-1}$.

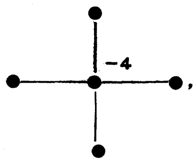
If C is a cone, then we may assume the matrix defining C is missing some homogeneous coordinate, say x_1, of $\mathbf{P}^{e-1}$. But it is easy to see that adding tx_1 to an appropriate entry of the matrix gives a determinantal deformation of C for which the general fibre is not a cone. This lifts to a normally flat deformation of R, by the first remark. Now one can use the preceding argument to find a cone specializing in a normally flat way to R.

Example 5.5. — There exist determinantal C which are cones and are specializations of non-determinantal curves. For instance, the family:

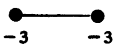
$$rk \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2+tx_4 & x_3 \\ x_2 & x_3 & x_0+x_4^2 \end{bmatrix} \leqq 1,$$

(1) α maps to Hilb/PGL, not to Hilb.

consists of a determinantal singularity



for $t = 0$, and a non-determinantal singularity



for $t \neq 0$. Of course, this does not imply that α is not surjective; it appears that both E and Hilb are obstructed. Note also that this normally flat deformation is *not* on the Artin component ([1] or [22]) of the moduli space of R.

PROPOSITION 5.6. — *If the projectivized tangent cone* C *is reduced, then* C *is smoothable.*

Proof. — Since C is a reduced curve, local deformations can be patched globally, so it suffices to show each singularity is smoothable. However, it follows by construction of C from the fundamental cycle Z ([20]) that the only singularities are analytically equivalent to r lines (in general position) through the origin t-space ($t \geqq r$), e. g.,

$$rk\begin{bmatrix} x_1 & \dots & x_r \\ a_1 x_1 & \dots & a_r x_r \end{bmatrix} \leqq 1,$$

where $a_i \neq a_j$ ($i \neq j$). It is easy to check (e. g., using the Jacobian criterion) that

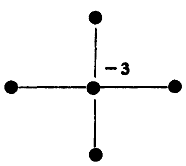
$$rk\begin{bmatrix} x_1+t & \dots & x_r+t \\ a_1 x_1 & \dots & a_r x_r \end{bmatrix} \leqq 1$$

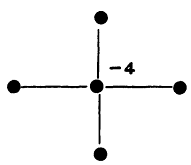
is a smoothing deformation.

Remarks 5.7.1. — It follows from Hartshorne's Theorem that any C as above lies on the same connected component of Hilb as a non-singular rational curve; the question is whether they lie on the same irreducible component, since the non-singular curves are dense there.

5.7.2. Riemenschneider has shown explicitly [17] that Conjecture 5.2 is correct for cyclic quotient singularities.

5.7.3. The following are both weighted dual graphs of rational singularities with $e = 5$:





Perhaps surprisingly, both are specializations of $\overset{-4}{\bullet}$ singularities. This is obvious in the second (determinantal) case. Writing an example of the first as a $\mathbf{Z}_2$-quotient of

$z^2 = x^4+y^4$ (4.14 *a*), we get a deformation *via* the $\mathbf{Z}_2$-invariants of the family $z^2 = x^4+y^4+txy$. This is easily checked to be of the desired type.

5.7.4. What is really wanted to prove (5.3.1) is a representation of the equations of R in a "weakly determinantal" form, so that all relations may be read off the matrix, and there is sufficient freedom in deformation of the entries to be able to generalize to a cone.

5.8. D. Kirby ([10]) originally identified the rational double points as those double points in $\mathbf{C}^3$ that are absolutely isolated, i. e., resolve by successively blowing up points, without normalization. Now, all rational singularities are absolutely isolated, and Theorem 2.1 gives a generalization of double point in 3-space. One is therefore tempted to make the:

CONJECTURE 5.9. — Suppose R is a complete two-dimensional normal domain over k, of embedding dimension e, and suppose the tangent cone $\overline{\mathrm{R}}$ admits a projective resolution as in (2.1) (in particular, $\overline{\mathrm{R}}$ is Cohen-Macaulay, with Hilbert function $n(e-1)+1$). If Spec R is absolutely isolated, then R is a rational singularity.

5.10. If R as in (5.9) resolves in one blow-up $\mathrm{X} \to \mathrm{Spec}\ \mathrm{R}$, then the result is true. For, $m_{\mathrm{R}}\, \mathcal{O}_{\mathrm{X}} = \mathcal{O}_{\mathrm{X}}(-\mathrm{V})$ is invertible, where V is isomorphic to the projectivized tangent cone; as divisors, $\mathrm{V} \geqq \mathrm{Z}$ (the fundamental cycle), since $\mathrm{V}.\mathrm{E}_i \leqq 0$ for all i (e. g., [14], § 18). Since $\mathrm{H}^1(\mathcal{O}_{\mathrm{V}}) = 0$ (follows from the projective resolution for $\overline{\mathrm{R}}$), $\mathrm{H}^1(\mathcal{O}_{\mathrm{Z}}) = 0$, so R is rational (Artin [2], Theorem 3). The same proof works if the first blow-up of R has only rational singularities. Finally, it is not hard to show that if the singularities of the first blow-up have the same multiplicity as R, then these singularities have the appropriate kind of projective resolution (and of course are absolutely isolated). Our only general result is

THEOREM 5.11. — *Suppose* R *is a complete two-dimensional determinantal domain over* k, *of embedding dimension* $e \geqq 4$ *defined by*

$$rk\begin{bmatrix} f_1 & \cdots & f_{e-1} \\ g_1 & \cdots & g_{e-1} \end{bmatrix} \leqq 1,$$

and suppose the projectivized tangent cone is 2-*regular as in paragraph* 2. *If* R *is absolutely isolated, then* R *is rational.*

Proof. — We will show that the singularities of the first blow-up $g : \mathrm{B} \to \mathrm{Spec}\ \mathrm{R}$ are either rational double points, or determinantal singularities of the same embedding dimension with 2-regular projectivized tangent cones. Thus, by induction, if $f : \mathrm{X} \to \mathrm{B}$ is a resolution, then $\mathrm{R}^1 f_* \mathcal{O}_{\mathrm{X}} = 0$ (B has only rational singularities); of course, $\mathcal{O}_{\mathrm{B}} \xrightarrow{\sim} f_* \mathcal{O}_{\mathrm{X}}$. On the other hand, writing $m_{\mathrm{R}}\, \mathcal{O}_{\mathrm{B}} = \mathcal{O}_{\mathrm{B}}(-\mathrm{V})$, we have that V is the projectivized tangent cone, with very ample line bundle $\mathcal{O}_{\mathrm{V}}(-\mathrm{V})$. By 2-regularity, $\mathrm{H}^1(\mathcal{O}_{\mathrm{V}}(-n\,\mathrm{V})) = 0$ for all $n \geqq 0$; the usual exact sequences give $\mathrm{H}^1(\mathcal{O}_{n\mathrm{V}}) = 0$, $n > 0$, hence

$$\mathrm{H}^1(\mathcal{O}_{\mathrm{B}}) = \mathrm{R}^1 g_* \mathcal{O}_{\mathrm{B}} = 0.$$

The Leray spectral sequence then yields $\mathrm{H}^1(\mathcal{O}_{\mathrm{X}}) = 0$, whence R is rational.

Write the tangent cone of R as

$$rk\begin{bmatrix} L_1 & \dots & L_{e-1} \\ M_1 & \dots & M_{e-1} \end{bmatrix} \leqq 1,$$

where the L_i, M_j are linear forms. The spaces of linear forms (L_i) and (M_j) have dimension $\geqq e-2$, by the usual remark on the number of equations defining $\overline{R}$. The dimension of (L_i, M_j) is $\geqq e-1$ by the absolute isolatedness of R (proof of Proposition 2.3), and is $e-1$ if Proj $\overline{R}$ is a cone, e otherwise.

Blow up by inverting the coordinate $L_1 = x_1$, say; we may assume in fact that $f_1 = x_1$ (if $L_1 = 0$, try another one). Assume a singularity exists in the first blow-up at $x'_i = x_i/x_1 = a_i$, $(i > 1)$ and $x_1 = 0$ (otherwise, there is nothing to prove). Make a linear change of variables with new x_i equal to old $x_i - a_i x_1$ $(i > 1)$; thus, we may assume all $a_i = 0$. Next, adding multiples of the first column of

$$\begin{bmatrix} x_1 & f_2 & \dots & f_{e-1} \\ g_1 & g_2 & \dots & g_{e-1} \end{bmatrix},$$

to other columns, and the same with the first row, we may suppose the linear terms of $f_2, \dots, f_{e-1}$, and g_1 contain no x_1 terms. But since $x_1 = 0$, $x'_i = 0$ $(i > 1)$ is a point on the first blow-up, it follows that $g_2, \dots, g_{e-1}$ contain no linear terms in x_1. By the span conditions above, we may assume $g_2 = x_2, \dots, g_{e-2} = x_{e-2}$, without affecting any of the previous choices. Thus, R is given by

$$rk\begin{bmatrix} x_1 & f_2 & \dots & f_{e-2} & f_{e-1} \\ g_1 & x_2 & \dots & x_{e-2} & g_{e-1} \end{bmatrix} \leqq 1.$$

Again, set $x'_i = x_i/x_1$ $(i > 1)$; we obtain a singularity R′ defined by

$$rk\begin{bmatrix} 1 & f'_2 & \dots & f'_{e-2} & f'_{e-1} \\ g'_1 & x'_2 & \dots & x'_{e-2} & g'_{e-1} \end{bmatrix} \leqq 1.$$

Since $x'_i = f'_i g'_i$, $2 \leqq i \leqq e-2$, is in the maximal ideal squared, we may eliminate these equations; so, R′ is the hypersurface singularity $g'_{e-1} = f'_{e-1} g'_1$, where we have now expressed everything in terms of x_1, x'_{e-1}, and x'_e. It suffices to show R′ is a double point; being absolutely isolated, by Kirby's Theorem, it will be a rational double point.

If $M_{e-1} \notin (x_2, \dots, x_{e-2})$, then g_{e-1} and g'_{e-1} are regular parameters, hence R′ would be non-singular. Otherwise, adding combinations of the columns to the last column, we may suppose $M_{e-1} = 0$. By the span condition, $M_1 \notin (x_1, \dots, x_{e-2})$ (remember it has no x_1 term); we may suppose in fact $g_1 = x_{e-1}$. Since $M_{e-1} = 0$, we have $L_{e-1} \neq 0$ (by the number of equations defining $\overline{R}$), and there is no x_1 term in L_{e-1}. Thus, f'_{e-1} has a non-0 linear term involving $x'_2, \dots, x'_e$. This means $f'_{e-1} g'$ contains a term $x'_i x'_{e-1}$, some $i > 1$. Since $x_1 \mid g'_{e-1}$, it follows that $g'_{e-1} - f'_{e-1} g'_1$ is a double point.

There remains to consider the case that some variable x_e does not occur in any L_i or M_j. Blowing-up x_e gives a new determinantal singularity of the same size at the origin; one has the same equations as before in $x'_1, \dots, x'_{e-1}, x_e$, except that some $x_e \cdot g(x'_1, \dots, x_e)$

terms are added. Note that since x_e does not occur in the equations for $\overline{R}$, there is no x_e^3 term in the equations for R (look at the matrix); thus, the embedding dimension stays the same after blowing up. Therefore, R′ (and its tangent cone) satisfy the conditions of the Theorem, and we may induct.

This completes the proof.

COROLLARY 5.12. — *Let* R *be an absolutely isolated* (*normal*) *triple point in* 4-*space. Then* R *is a rational triple point.*

Proof. — We prove directly that R is determinantal. We assume R is complete. Let u, v be an R-sequence. Then (2.6):

$$R/(u, v) \cong k[y, z]/(y^2, yz, z^2).$$

Thus, R is the quotient of $k[[u, v, y, z]]$ by

$$y^2+l_1 y+l_2 z+q_1 = 0,$$

$$yz+m_1 y+m_2 z+q_2 = 0,$$

$$z^2+n_1 y+n_2 z+q_3 = 0;$$

where $l_i, m_i, n_i, q_i \in k[[u, v]]$ (R is a free $k[[u, v]]$ module, with basis $1, y, z$). Replacing y by $y-m_2$ and z by $z-m_1$, we may suppose $m_1 = m_2 = 0$. The 2 relations arising from the associative law (e. g., $z.y^2 = y.yz$) give:

$$q_1 = l_2 n_2, \qquad q_2 = -n_1 l_2, \qquad q_3 = n_1 l_1.$$

We may therefore write the 3 equations as

$$rk\begin{bmatrix} -y & n_1 & z+n_2 \\ l_2 & -z & y+l_1 \end{bmatrix} \leqq 1;$$

one can now apply Theorem 5.11.

5.13. Presumably one can replace "2-regular resolution" in (5.9) by "multiplicity is one less than embedding dimension".

APPENDIX

A.1. We recall some facts about n-cyclic étale covers. Let X be a scheme over k, where k contains the nth roots of 1, and $(n, p) = 1$ (p = char. k). Let L be a torsion invertible sheaf, and $\alpha : L^n \xrightarrow{\sim} \mathcal{O}_X$ an isomorphism (write $L^j = L^{\otimes j}$). One may use α to define multiplication in $\mathcal{O}_X \oplus L \oplus \ldots \oplus L^{n-1}$, forming a finite flat morphism:

$$\begin{array}{l} Y_\alpha = \operatorname{Spec}(\mathcal{O}_X \oplus L \oplus \ldots \oplus L^{n-1}) \\ \Big\downarrow \pi_\alpha \\ X \end{array}$$

Straightforward arguments show

(*a*) π_α is an n-cyclic étale cover [*recall* $(n, p) = 1$].

(*b*) If $\alpha' : L^n \xrightarrow{\sim} \mathcal{O}_X$ is another isomorphism, then Y_α and $Y_{\alpha'}$ are isomorphic covers iff $\alpha' \circ \alpha^{-1} \in \Gamma(\mathcal{O}_X)^*$ has an nth root.

(*c*) Using L^i and $\alpha^{\otimes i} : (L^i)^n \xrightarrow{\sim} \mathcal{O}_X$, where $(i, n) = (i, p) = 1$, yields a cover isomorphic to Y_α.

(*d*) $\pi_\alpha^*(L) \cong \mathcal{O}_Y$.

A.2. Conversely, every n-cyclic étale cover arises in this way. Given one, $\pi : Y \to X$, choose ζ, a primitive nth root of 1 in k. Define a sheaf L on X by

$$L(U) = \{ f \in \Gamma(\pi^{-1}(U), \mathcal{O}_Y) \,|\, \sigma^* f = \zeta f \},$$

where σ^* is the induced action of a fixed generator σ of $\mathbf{Z}/n$. Then L is invertible, comes with an isomorphism $\alpha : L^n \xrightarrow{\sim} \mathcal{O}_X$, and $Y \to X$ is isomorphic to $Y_\alpha \to X$. This is checked locally and on geometric points. One could also examine the cohomology sequence for

$$\{1\} \to \mu_n \to \mathbf{G}_m \to \mathbf{G}_m \to \{1\}.$$

THEOREM A.3 (Krull-Schmidt). — *Suppose X is a k-scheme with $\Gamma(\mathcal{O}_X)$ a local ring, $L_1, \ldots, L_s, M_1, \ldots, M_t$ are indecomposable locally free sheaves, and*

$$\varphi : \ L_1 \oplus \ldots \oplus L_s \xrightarrow{\sim} M_1 \oplus \ldots \oplus M_t.$$

is an isomorphism. Suppose further that either:

(*a*) *the L_i are line bundles; or*

(*b*) *$\Gamma(\mathcal{O}_X)$ is henselian.*

Then $s = t$, and after reordering the M_i's, $L_i \xrightarrow{\sim} M_i$.

Proof. — We copy the argument in Jacobson, *Lectures in Abstract Algebra*, Vol. I, p. 157. Letting $M_i' = \varphi^{-1}(M_i)$, we may as well assume the L_i and M_j are subsheaves of a fixed locally free F. Suppose inductively we have isomorphisms $L_i \xrightarrow{\sim} M_i$, $i \leqq r-1$, so that

$$(\bigstar)_k \qquad F = M_1 \oplus \ldots \oplus M_{k-1} \oplus L_k \oplus \ldots \oplus L_s, \quad \text{all } k \leqq r-1.$$

(Here, $r = 1$ at the start.) Let $\lambda_i : F \to L_i$, $\eta_j : F \to M_j$ be the projection maps. Then:

$$\lambda_r = \lambda_r \left(\sum_1^t \eta_i \right),$$

where $\lambda_r \eta_i : F \to M_i \subset F \to L_r$. But $(\bigstar)_{r-1}$ implies

$$\lambda_r \eta_i = 0, \quad i < r, \qquad \text{so} \quad \lambda_r = \sum_r^t \lambda_r \eta_i.$$

Now restricting λ_r and $\lambda_r \eta_i$ to L_r, we may consider $\lambda_r = 1$ and $\lambda_r \eta_i \in \operatorname{Hom}(L_r, L_r)$. We show below that (*a*) or (*b*) above implies that some $\lambda_r \eta_i$ (say for $i = r$) is an automorphism. Since $\lambda_r \eta_r$: $L_r \to M_r \to L_r$, L_r is a direct summand of M_r; by indecomposability, $\eta_r : L_r \xrightarrow{\sim} M_r$, and $\lambda_r : M_r \xrightarrow{\sim} L_r$. Now, the intersection in F:

$$M_r \cap (M_1 \oplus \ldots \oplus M_{r-1} \oplus L_{r+1} \oplus \ldots \oplus L_s) = (0),$$

since λ_r is an isomorphism on M_r and the 0-map on the components of the second term (use $(\bigstar)_{r-1}$). Therefore, the natural map on

$$M_1 \oplus \ldots \oplus M_{r-1} \oplus L_r \oplus \ldots \oplus L_s,$$

sending all $M_i \to M_i$, $L_j \to L_j$ $(j > r)$, and $L_r \to M_r$ (*via* η_r) is a isomorphism onto $M_1 \oplus \ldots \oplus M_r \oplus L_{r+1} \oplus \ldots \oplus L_s$. This proves the induction step.

Assuming (*a*), we have $\operatorname{Hom}(L_r, L_r) = \Gamma(\mathcal{O}_X)$, a local ring; so, if a sum of two things is a unit, one of them is a unit (hence an isomorphism of L_r). In general, for indecomposable L_r, $\operatorname{Hom}(L_r, L_r)$ is an associative $\Gamma(\mathcal{O}_X)$-algebra with 0 and 1 as only idempotents (since L_r is indecomposable). If $S, T \in \operatorname{Hom}(L_r, L_r)$, and $S+T$ is invertible, we claim S or T is invertible. It suffices to assume $S+T = 1$. The subalgebra R of $\operatorname{Hom}(L_r, L_r)$, generated by $\Gamma(\mathcal{O}_X)$ and T is commutative and $\Gamma(\mathcal{O}_X)$-finite (since a local argument shows T satisfies its characteristic polynomial $p(\lambda) = \Lambda^n (T - \lambda I) \in \Gamma(\mathcal{O}_X)[\lambda]$). But since $\Gamma(\mathcal{O}_X)$ is henselian, R is a direct sum of local rings. Since R has only 0 and 1 as idempotents, R must be local; thus, T or $1-T$ is invertible.

COROLLARY A.4. — *Suppose* L *is a torsion invertible sheaf on* X *of order* n, $(n, p) = 1$, *and* $\Gamma(\mathcal{O}_X)$ *is local. Then any* $Y_\alpha = \operatorname{Spec}(\mathcal{O} \oplus L \oplus \ldots \oplus L^{n-1})$ *is connected.*

Proof. — Suppose $Y_\alpha = Y = Y_1 \coprod Y_2$, $Y_i \neq \emptyset$. Then $\pi_i : Y_i \to X$ is finite, étale, and cyclic, of order dividing n; hence, *via* the trace map, $\mathcal{O}_X$ is a direct summand of $\pi_{i*} \mathcal{O}_{Y_i}$. Since:

$$\pi_{1*} \mathcal{O}_{Y_1} \oplus \pi_{2*}, \mathcal{O}_{Y_2} \cong \pi_* \mathcal{O}_Y = \mathcal{O} \oplus L \oplus \ldots \oplus L^{n-1},$$

and since there are two $\mathcal{O}_X$'s on the left side, by A.3 (*a*) we have $L^i \cong \mathcal{O}_X$, some $i < n$; this contradicts the fact that L had order n.

COROLLARY A.5. — *Suppose* $\Gamma(\mathcal{O}_X)$ *is local,* $\alpha : L^n \xrightarrow{\sim} \mathcal{O}_X$, $\pi = \pi_\alpha : Y \to X$ *the associated cover, and* $(n, p) = 1$. *If* M *is an invertible sheaf on* X *such that* $\pi^* M \cong \mathcal{O}_X$, *then* $M = L^i$, *some* i. *In particular, we have an exact sequence:*

$$1 \to \{L^i\} \to \operatorname{Pic} X \xrightarrow{\pi^*} \operatorname{Pic} Y.$$

Proof. — $\pi^* M \simeq \mathcal{O}_Y$ implies $\pi_* \pi^* M \simeq \pi_* \mathcal{O}_Y$. But by the projection formula, $\pi_* \pi^* M \cong M \otimes \pi_* \mathcal{O}_Y$. Thus,

$$M \otimes \bigoplus_{i=0}^{n-1} L^i \cong \bigoplus_{i=0}^{n-1} L^i.$$

Since an $\mathcal{O}_X$ occurs on the right side, $M \otimes L^i = \mathcal{O}_X$, some i, by A.3 *a*), so $M \simeq L^{n-i}$.

Remark A.6. — Note π^* in (A.5) is rarely surjective. In fact, rational singularities are characterized by having finite Pic ([14], § 17), yet the canonical covers of paragraph 4 are rational iff the original singularity is a quotient singularity.

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