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LOCAL CHERN CLASSES

BY BIRGER IVERSEN

The purpose of this paper is to give a construction of local Chern classes as conjectured by Grothendieck [6] (XIV 7.2).

The construction is given in the framework of complex vector bundles on topological spaces where it appears as a generalization of the relative Chern classes obtained from the "difference construction" in K-theory notably used by Atiyah ([1]-[4]).

It will be clear that the constructions performed work equally well in other theories, especially the etale cohomology of algebraic geometry.

CONTENTS

1. **Introduction**

By a complex $K^{\cdot}$ of vector bundles on a topological space X we understand a finite complex of **C**-vector bundles each having constant rank. By the support of $K^{\cdot}$ we understand the complement to the set of points $x \in X$ for which $K^{\cdot}_x$ is an exact complex of vector spaces.

For a space X, $H^{\cdot}(X; \mathbf{Z})$ denotes integral cohomology in the sense of sheaf theory, $\hat{H}^{\cdot}(X; \mathbf{Z}) = \prod_i H^i(X; \mathbf{Z})$. For a closed subset we use interchangably

$$H^{\cdot}_Z(X; \mathbf{Z}) = H^{\cdot}(X, X-Z; \mathbf{Z})$$

for cohomology with support in Z.

A theory of local Chern classes consists in assigning to a complex $K^{\cdot}$ on X with support in Z a cohomology class

$$c_{\cdot}^{Z}(K^{\cdot}) \in \hat{H}_{Z}^{\cdot}(X; \mathbf{Z})$$

with the following two properties

(1.1) For a continuous map f: $X \to Y$, closed subsets $Z \subseteq X$, $V \subseteq Y$ with $f(X-Z) \subseteq Y-V$ and a complex $L^{\cdot}$ on Y with support in V:

$$c_{\cdot}^{Z}(f^{*}L^{\cdot}) = f^{*}c_{\cdot}^{V}(L^{\cdot}).$$

(1.2) Let r: $\hat{H}_{Z}^{\cdot}(X; \mathbf{Z}) \to \hat{H}^{\cdot}(X; \mathbf{Z})$ denote the canonical map.

Then

$$r(c_{\cdot}^{Z}(K^{\cdot}))+1 = \prod_{i} c_{.}(K^{2i})\, c_{.}(K^{2i-1})^{-1}.$$

The main result of this paper is

THEOREM 1.3. — *A theory of local Chern classes exists and is unique.*

As usual we introduce a local Chern character

$$ch^{Z}(K^{\cdot}) \in \hat{H}_{Z}^{\cdot}(X; \mathbf{Q})$$

with the following properties:

(1.4) FUNCTORIALITY. — $f^{*}ch^{V}(L^{\cdot}) = ch^{Z}(f^{*}L^{\cdot})$.

(1.5) $r(ch^{Z}(K^{\cdot})) = \sum_{i}(-1)^{i}\, ch(K^{i})$.

(1.6) DECALAGE. — $ch^{Z}(K^{\cdot}[1]) = -ch^{Z}(K^{\cdot})$.

(1.7) ADDITIVITY. — For complexes $K^{\cdot}$ and $L^{\cdot}$ on X with support in Z:

$$ch^{Z}(K^{\cdot} \oplus L^{\cdot}) = ch^{Z}(K^{\cdot}) + ch^{Z}(L^{\cdot}).$$

(1.8) MULTIPLICATIVITY. — Let $K^{\cdot}$ and $L^{\cdot}$ be complexes on X with support in Z and V, respectively. Then

$$ch^{Z \cap V}(K^{\cdot} \otimes L^{\cdot}) = ch^{Z}(K^{\cdot})\, ch^{V}(L^{\cdot}).$$

The proof of 1.3 is given in paragraphs 2 and 3 while paragraphs 4 and 5 derives multiplicative and additive properties of $c_{\cdot}^{Z}$ and ch^{Z}.

In paragraph 6 we derive Riemann-Roch formulas for the Thom class and paragraph 7 initiates applications to algebraic geometry.

In cases where X is an oriented topological manifold of dimension n, Poincaré duality

$$H^i_Z(X; \mathbf{Z}) \xrightarrow{\sim} H_{n-i}(Z; \mathbf{Z})$$

transforms our local cohomology classes into homology classes. In cases where X is a smooth algebraic variety/$\mathbf{C}$, this should be compared with the homology classes constructed by means of MacPherson's graph construction [5] compare [10], [14], [16].

It should also be mentioned that Illusie ([13] V.6) has constructed local Chern classes "à la Atiyah" in Hodge cohomology.

I should like to thank K. Suominen for stimulating my interest in these matters.

2. The canonical complex

Throughout this paragraph we shall work with the following data.

A topological space X, a sequence of vector bundles $(K^i)_{i \in \mathbf{Z}}$ on X with $K^i = 0$ except for finitely many $i \in \mathbf{Z}$.

$$\nu_i = \operatorname{rank} K^i.$$

We shall assume that there exists a sequence $(\lambda_i)_{i \in \mathbf{Z}}$ of integers with

$$\lambda_i + \lambda_{i+1} = \nu_i, \qquad i \in \mathbf{Z},$$
$$\lambda_i \geqq 0, \qquad i \in \mathbf{Z}.$$

Put $K = \bigoplus_{i \in \mathbf{Z}} K^i$. The flag manifold whose sections are flags in K of nationality $\nu_.$ will be denoted Fl_ν. The fixed flag defined by

$$F_i = \bigoplus_{t \leq i} K^t$$

is denoted $F_.$.

DEFINITION 2.1. — $T \subseteq \mathrm{Fl}_\nu$ denote the closed subspace whose sections are flags $D_.$ with the property that

$$F_{i-1} \subseteq D_i \subseteq F_{i+1}, \qquad i \in \mathbf{Z}.$$

The canonical projection is denoted $p: T \to X$. The restriction to T pf the canonical flag on Fl_ν will be denoted $E_.$. On T we have a canonical complex $C^\cdot$ given by

$$C^i = E_i / p^* F_{i-1},$$
$$\partial^i : \ E_i / p^* F_{i-1} \to E_{i+1} / p^* F_i$$

is induced by the inclusion $E_i \subseteq E_{i+1}$. $\partial^{i+1}\partial^i = 0$ since

$$p^* F_{i-1} \subseteq E_i \subseteq p^* F_{i+1}, \qquad i \in \mathbf{Z}.$$

Finally T_Ψ is the complement in T of the support of $C^\cdot$, and $p_\Psi: T_\Psi \to X$ denotes the restriction of p to T_Ψ.

LEMMA 2.2. — *A section of T over X represented by a flag* $D_\cdot$ *is a section of* T_Ψ *if and only if for all* $x \in X$:

$$\operatorname{rank}(D_{i,x} \cap F_{i,x}/F_{i-1,x}) = \lambda_i.$$

Proof. — By definition $D_\cdot$ represents a section of T_Ψ if and only if the complex

$$\to D_{i-1}/F_{i-2} \to D_i/F_{i-1} \to D_{i+1}/F_i \to$$

has exact fibres for all $x \in X$. Note that D_i/F_{i-1} has rank ν_i, and the lemma follows from the definition of $(\lambda_i)_{i \in \mathbf{Z}}$.

THEOREM 2.3. — *Let* $i_\Psi: T_\Psi \to T$ *denote the inclusion. Then*

$$i_\Psi^* : \quad H^\cdot(T; \mathbf{Z}) \to H^\cdot(T_\Psi; \mathbf{Z})$$

is surjective.

Proof. — Define

$$G_\lambda = \prod_i \operatorname{Grass}_{\lambda_i}(K^i) \to X,$$

where $p_\cdot$: $\operatorname{Grass}_{\lambda_i}(K^i) \to X$ is the fibre space whose sections are rank λ_i-subbundles of K^i.

$$f_\lambda : \quad T_\Psi \to G_\lambda$$

denotes the map which on the level of sections (compare 2.2) transforms

$$D_\cdot \mapsto (D_i \cap F_i/F_{i-1})_{i \in \mathbf{Z}}.$$

We shall first prove

$$f_\lambda^* : \quad H^\cdot(G_\lambda; \mathbf{Z}) \to H^\cdot(T_\Psi; \mathbf{Z}) \tag{2.4}$$

is an isomorphism.

We shall prove that f_λ is a fibration with fibres of type $\mathbf{A}^d$ ($\mathbf{A}^d$: affine space of dimension $d = \sum \lambda_i^2$). For this assume $X = P^t$. The fibre of f_λ above $B^\cdot \in G_\lambda$ consists of sequences $(G^i)_{i \in \mathbf{Z}}$, where G^i is a λ_{i+1}-plane in $2\lambda_{i+1}$-space B^{i+1}/B^i intersection the λ_{i+1}-plane F_i/B^i in zero.

Next define

$$G_\nu = \prod_i \mathrm{Grass}_{\nu_i}(K^i \oplus K^{i+1})$$

and maps

$$f_\nu : \quad T \to G_\nu, \qquad D_\cdot \mapsto (D_i/F_{i-1})_{i \in \mathbf{Z}};$$
$$g : \quad G_\lambda \to G_\nu, \qquad B_\cdot \mapsto (B^i \oplus B^{i+1})_{i \in \mathbf{Z}};$$
$$s_\lambda : \quad G_\lambda \to T_\Psi;$$
$$B^\cdot \mapsto (\bigoplus_{t<i} K^t \oplus B^i \oplus B^{i+1})_{i \in \mathbf{Z}},$$

where in each case the transformation on the level of sections is given.

We have the following diagram

$$\begin{array}{ccc} T & \overset{i_\Psi}{\longleftarrow} & T_\Psi \\ {\scriptstyle f_\nu}\downarrow & & {\scriptstyle f_\lambda}\downarrow\uparrow{\scriptstyle s_\lambda} \\ G_\nu & \overset{g}{\longleftarrow} & G_\lambda \end{array}$$

with

$$f_\nu\, i_\Psi\, s_\lambda = g, \qquad f_\lambda\, s_\lambda = 1$$

$(f_\nu\, i_\Psi \neq g f_\lambda)$.

Let us grant (2.5 below) that g^* is surjective.

$s_\lambda^* f_\lambda^* = 1$ and whence by 2.4;

$f_\lambda^* s_\lambda^* = 1$, on the other hand;

$s_\lambda^* i_\Psi^* f_\nu^* = g^*$ and whence;

$i_\Psi^* f_\nu^* = f_\lambda^* g^*$. Thus i_Ψ^* surjective.

Q. E. D.

LEMMA 2.5. — *The map*

$$g : \quad \prod_i \mathrm{Grass}_{\lambda_i} K^i \to \prod_i \mathrm{Grass}_{\nu_i} K^i \oplus K^{i+1},$$
$$B^\cdot \mapsto (B^i \oplus B^{i+1})_{i \in \mathbf{Z}}$$

induces a surjective map g^* on integral cohomology.

Proof. — Let P^i denote the canonical λ_i-bundle on $\mathrm{Grass}_{\lambda_i}(K^i)$. Consider

$$H^\cdot(\prod_i \mathrm{Grass}_{\lambda_i} K^i; \mathbf{Z})$$

as a $H^\cdot(X; \mathbf{Z})$-algebra. As is well known this algebra is generated by the homogeneous components of

$$\mathrm{pr}_i{}^* c_\cdot(P^i), \qquad i \in \mathbf{Z}.$$

Consider the composite of g and the i'th projection

$$\prod_i \mathrm{Grass}_{\lambda_i} \mathrm{K}^i \to \mathrm{Grass}_{\nu_i} \mathrm{K}^i \oplus \mathrm{K}^{i+1}$$

to see that

$$\mathrm{pr}_i{}^* c.(\mathrm{P}^i)\, \mathrm{pr}_{i+1}{}^* c.(\mathrm{P}^{i+1})$$

and the inverse to that element belongs to the image of g^*. It is now clear by decreasing induction that $\mathrm{pr}_i{}^* c.(\mathrm{P}_i)$ and $\mathrm{pr}_i{}^* c.(\mathrm{P}_i)^{-1}$ belong to the image of g^*.

Q. E. D.

PROPOSITION 2.6. — *The* $\mathrm{H}^{\bullet}(\mathrm{X};\mathbf{Z})$*-module* $\mathrm{H}^{\bullet}(\mathrm{T}_{\Psi};\mathbf{Z})$ *is finitely generated free and for any map* $\mathrm{X}' \to \mathrm{X}$.

$$\mathrm{H}^{\bullet}(\mathrm{T}_{\Psi};\mathbf{Z}) \underset{\mathrm{H}^{\bullet}(\mathrm{X};\mathbf{Z})}{\otimes} \mathrm{H}^{\bullet}(\mathrm{X}';\mathbf{Z}) \to \mathrm{H}^{\bullet}(\mathrm{T}_{\Psi} \times_{\mathrm{X}} \mathrm{X}';\mathbf{Z})$$

is an isomorphism.

Proof. — By 2.4 we may replace T_{Ψ} by a product of Grassmannian bundles for which this is well known.

Q. E. D.

3. Construction of the local Chern class

With the notation of paragraph 2 let $(\partial^{\bullet})_{i\in\mathbf{Z}}$ be a family of linear maps $\partial^i : \mathrm{K}^i \to \mathrm{K}^{i+1}$ with $\partial^{i+1}\partial^i = 0$, $i \in \mathbf{Z}$. Define a flag $s.(\partial^{\bullet})$ in $\mathrm{K} = \bigoplus_i \mathrm{K}^i$ as follows: $s_i(\partial^{\bullet})$ is the graph of the map

$$\bigoplus_{t\leq i} \mathrm{K}^t \to \bigoplus_{t>i} \mathrm{K}^t,$$

$$(\ldots, k_{i-1}, k_i) \mapsto (\partial^i k_i, 0, \ldots).$$

Clearly,

$$\mathrm{F}_{i-1} \subseteq s_i(\partial^{\bullet}) \subseteq \mathrm{F}_{i+1}, \qquad i \in \mathbf{Z}.$$

Thus we may interprete $s.(\partial^{\bullet})$ as a section of $p: \mathrm{T} \to \mathrm{X}$

$$s.(\partial^{\bullet}) : \mathrm{X} \to \mathrm{T}.$$

Clearly

(3.1) $$s.(\partial^{\bullet})^* \mathrm{C}^{\bullet} = (\mathrm{K}^{\bullet}, \partial^{\bullet}).$$

Let now $Z \subseteq X$ denote a closed subset such that $\operatorname{Supp}(K^{\cdot}, \partial^{\cdot}) \subseteq Z$ then

$$s_{\cdot}(\partial^{\cdot})(X - Z) \subseteq T_{\Psi^{\cdot}}.$$

Consider the exact sequence, (2.3):

$$0 \to \hat{H}^{\cdot}(T, T_{\Psi}; \mathbf{Z}) \xrightarrow{r_{\Psi}} \hat{H}^{\cdot}(T; \mathbf{Z}) \xrightarrow{i_{\Psi}^{*}} \hat{H}^{\cdot}(T_{\Psi}; \mathbf{Z}) \to 0.$$

The image by i_{Ψ}^{*} of the cohomology class

$$c_{\cdot}(C^{\cdot}) - 1 = \prod_{i} c_{\cdot}(C^{2i}) c_{\cdot}(C^{2i-1})^{-1} - 1$$

is zero since $C^{\cdot}$ is exact on T_{Ψ}. Let

$$\gamma_{T} \in \hat{H}^{\cdot}(T, T_{\Psi}; \mathbf{Z})$$

denote the cohomology class characterized by

$$r_{\Psi}(\gamma_{T}) + 1 = c_{\cdot}(C^{\cdot}). \tag{3.2}$$

DEFINITION 3.3. — Consider the map induced by $s_{\cdot}(\partial^{\cdot})$

$$s_{\cdot}(\partial^{\cdot})^{*} : \quad H^{\cdot}(T, T_{\Psi}; \mathbf{Z}) \to H_{Z}^{\cdot}(X; \mathbf{Z})$$

and define the local Chern class of $(K^{\cdot}, \partial^{\cdot})$ supported in Z by

$$c_{\cdot}^{Z}(K^{\cdot}, \partial^{\cdot}) = s_{\cdot}(\partial^{\cdot})^{*} \gamma_{T}.$$

Proof of 1.3. — Follows from 3.1 and 3.2.

Q. E. D.

As above we consider the cohomology class

$$\gamma\varkappa_{T} \in \hat{H}^{\cdot}(T, T_{\Psi}; \mathbf{Q})$$

characterized by

$$r_{\Psi}(\gamma\varkappa_{T}) = \sum_{i} (-1)^{i} \operatorname{ch}(C^{i}). \tag{3.4}$$

DEFINITION 3.5:

$$\operatorname{ch}^{Z}(K^{\cdot}, \partial^{\cdot}) = s_{\cdot}(\partial^{\cdot})^{*} \gamma\varkappa_{T}.$$

The local Chern character thus defined satisfies clearly 1.4-6. Let us remark that ch^{Z} can be derived directy from $c_{\cdot}^{Z}$ by means of the theory of λ-rings, compare paragraph 5.

4. Properties of the local Chern character

In this paragraph we shall prove the multiplication property 1.8 of ch^Z. The proof of the additive property 1.7 is similar but simpler and will not be given. Finally, we give some variants of the additive property.

Proof of 1.8. — Let us first note that 1.8 is true if the canonical map

$$H^{\cdot}_{Z \cap V}(X; \mathbf{Z}) \to H^{\cdot}(X; \mathbf{Z})$$

is injective. We are going to reduce the problem to this case. Let $T = T(K^{\cdot})$ and $S = T(L^{\cdot})$ with a slight abuse of notation. It will now suffice to prove that

$$H^{\cdot}(T \times S; \mathbf{Z}) \to H^{\cdot}(S \times T_{\Psi} \cup T \times S_{\Psi}; \mathbf{Z})$$

is surjective. Here and in the following all products are formed in the category of spaces/X. $H^{\cdot}(-)$ denotes integral cohomology. Let us first recall that if $Z \subseteq Y$ is a closed subset of the space Y and if $U \subseteq Y$ is an open subset, then there is a canonical exact sequence

$$\to H^i_{Z-U}(X) \to H^i_Z(X) \to H^i_{Z \cap U}(U) \to H^{i+1}_{Z-U}(X) \to .$$

Put $X = S - S_{\Psi}$ and $Y = T - T_{\Psi}$. It follows from 2.6 that the following commutative diagram is exact [$\otimes$ is formed in the category of H (X)-modules]:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & H^{\cdot}(S_{\Psi}) \otimes H^{\cdot}_Y(T) \\
 & & & & & & \downarrow \\
0 \longrightarrow & H^{\cdot}_{X \times T}(S \times T) & \longrightarrow & H^{\cdot}(S \times T) & \longrightarrow & & H^{\cdot}(S_{\Psi} \times T) \\
 & \downarrow & & \downarrow & & & \downarrow \\
0 \to & H^{\cdot}_{X \times T_{\Psi}}(S \times T_{\Psi}) & \to & H^{\cdot}(S \times T_{\Psi}) & \to & & H^{\cdot}(S_{\Psi} \times T_{\Psi}) \\
 & & & \downarrow & & & \downarrow \\
 & & & 0 & & & 0
\end{array}$$

From this follows that

$$H^{\cdot}_{X \times T}(S \times T) \to H^{\cdot}_{X \times T_{\Psi}}(S \times T_{\Psi})$$

is surjective by remarking that $H^{\cdot}(S) \otimes H^{\cdot}_Y(T) \to H^{\cdot}(S_{\Psi}) \otimes H^{\cdot}_Y(T)$ is surjective, taking into account the map from $H^{\cdot}(S) \otimes H^{\cdot}_Y(T)$ into the kernel of $H^{\cdot}(S \times T) \to H^{\cdot}(S \times T_{\Psi})$. Next, apply the above long exact sequence to $(S \times T, S \times T_{\Psi}, X \times T)$ to get the exact sequence

$$\to H^{\cdot}_{X \times Y}(S \times T) \to H^{\cdot}_{X \times T}(S \times T) \to H^{\cdot}_{X \times T_{\Psi}}(S \times T_{\Psi}) \to$$

from which we conclude that

$$H^{\cdot}_{X \times Y}(S \times T) \to H_{X \times T}(S \times T)$$

is injective. From the following exact sequence and 2.6

$$\to H^{i}_{X \times T}(S \times T) \to H^{i}(S \times T) \to H^{i}(S_{\Psi} \times T) \to$$

follows that

$$H^{\cdot}_{X \times T}(S \times T) \to H^{\cdot}(S \times T)$$

is injective. Compose the last two results and write still another long exact sequence to derive the result.

Q. E. D.

PROPOSITION 4.1. — *Let* $K^{\cdot\cdot}$ *denote a finite double complex on the topological space* X. *Suppose* Z *is a closed subset of* X *such that* $K^{p\cdot}$, *has support in* Z *for all* $p \in \mathbf{Z}$. *Then*

$$ch^{Z}(\text{tot}\, K^{\cdot\cdot}) = \sum (-1)^{i} ch^{Z}(K^{p\cdot}),$$

where tot $K^{\cdot\cdot}$ *denotes the total single complex associated to* $K^{\cdot\cdot}$.

Proof. — We shall first change notation and let $K^{\cdot\cdot}$ denote the double indexed family of vector bundles on X underlying the above double complex. Let $C(K^{\cdot\cdot})$ denote the fibre space over X whose sections are pairs (∂', ∂'') of endomorphisms of $K^{\cdot\cdot}$ such that $(K^{\cdot\cdot}, \partial', \partial'')$ form a double complex. Let $E^{\cdot\cdot}$ denote the canonical double complex on $C(K^{\cdot\cdot})$ and C_{Ψ} the complement of the support of tot $E^{\cdot\cdot}$.

Consider now a fixed pair (∂', ∂'') as above and assume that $(K^{\cdot\cdot}, 0, \partial'')$ has support in Z. Consider the map of spaces/X:

$$\theta : \ X \times \mathbf{A}^{1} \to C(K^{\cdot\cdot})$$

which on the section level is given by

$$t \mapsto (K^{\cdot\cdot}, t\,\partial', \partial'').$$

Clearly

$$\theta(X - Z) \subseteq C_{\Psi}$$

and

$$\theta_{t}^{*}(\text{tot}\, E^{\cdot\cdot}) = \text{tot}(K^{\cdot\cdot}, t\,\partial', \partial'').$$

Conclusion by (1.6), (1.7) and a simple homotopy argument.

Q. E. D.

COROLLARY 4.2. — *Consider an exact sequence of complexes of vector bundles on* X:

$$0 \to K^{\bullet} \to L^{\bullet} \to M^{\bullet} \to 0$$

and suppose all three complexes have support in the closed subset Z *of* X. *Then*

$$\mathrm{ch}^{Z}(L^{\bullet}) = \mathrm{ch}^{Z}(K^{\bullet}) + \mathrm{ch}^{Z}(M^{\bullet}).$$

Proof. — Consider an appropriate double complex and apply 4.1 twice.

Q. E. D.

COROLLARY 4.3. — *Let* $f: K^{\bullet} \to L^{\bullet}$ *be a linear map of complexes on* X *and let* $K^{\bullet}$ *and* $L^{\bullet}$ *have support in* Z. *If for all* $x \in X$:

$$H^{\bullet}(f_x) : \ H^{\bullet}(K^{\bullet}_x) \to H^{\bullet}(L^{\bullet}_x)$$

is an isomorphism, then

$$\mathrm{ch}^{Z}(K^{\bullet}) = \mathrm{ch}^{Z}(L^{\bullet}).$$

Proof. — Construct the mapping cone and apply 4.2.

Q. E. D.

5. Formulas without denominators

Let Z be a closed subspace of the space X and consider the commutative graded ring with 1:

$$\mathbf{Z} \oplus H_Z^{ev}(X; \mathbf{Z}^{+}).$$

To this we associate

$$1 + \hat{H}_Z^{ev}(X; \mathbf{Z})^{+} = 1 + \prod_{i \geq 1} H_Z^{2i}(X; \mathbf{Z})$$

which is an abelian group under cup product. Recall that $1 + \hat{H}_Z^{ev}(X; \mathbf{Z})^{+}$ comes equipped with a product $\bigstar$ with the property

$$(1 + x_m + \text{higher terms}) \bigstar (1 + y_n + \text{higher terms}) =$$

$$1 - \frac{(n+m-1)!}{(m-1)!\,(n-1)!} x_m y_n + \text{higher terms} \tag{5.1}$$

[6] (0, App. § 3).

If $K^\cdot$ is a complex on X with support in Z, we put

$$\tilde{c}^Z(K^\cdot) = 1 + c_\cdot^Z(K^\cdot),$$

$$\tilde{c}^Z(K^\cdot) \in 1 + \hat{H}_Z^{ev}(X; \mathbf{Z})^+.$$

With the notation of the corresponding formulas for ch^Z, 1.4-8, we have

(5.2) $$\tilde{c}^Z(f^* L^\cdot) = f^* \tilde{c}^V(L^\cdot),$$

(5.3) $$r(\tilde{c}^Z(K^\cdot)) = \prod_i c_\cdot(K^{2i})\, c_\cdot(K^{2i-1})^{-1},$$

(5.4) $$\tilde{c}^Z(K^\cdot[1]) = \tilde{c}^Z(K^\cdot)^{-1},$$

(5.5) $$\tilde{c}^Z(K^\cdot \oplus L^\cdot) = \tilde{c}^Z(K^\cdot)\,\tilde{c}^Z(L^\cdot),$$

(5.6) $$\tilde{c}^{Z\cap V}(K^\cdot \otimes' L^\cdot) = \tilde{c}^Z(K^\cdot) \bigstar \tilde{c}^V(L^\cdot).$$

These formulas are easily derived by the method developed in paragraph 4. From *loc. cit.* follows

(5.7) Suppose $c_\cdot^Z(K^\cdot) = a_n +$ higher terms, then $ch^Z(K^\cdot) = 1/(-1)^{n-1}(n-1)!\, a_n +$ higher terms

6. Riemann-Roch formula for the Thom class

Let $\pi: E \to X$ denote a rank n vector bundle, and let λ_E denote the canonical complex on E. Recall that $(\lambda_E)^i = \Lambda^i \pi^* E$. The Koszul complex, i. e. the complex dual to λ_E will be denoted $\lambda_E^\vee$.

PROPOSITION 6.1. — *With the above notation*

$$(-1)^n \operatorname{Todd}(E^\vee)\, ch^X(\lambda_E) = \operatorname{Todd}(E)\, ch^X(\lambda_E^\vee) = \text{Thom class of E}.$$

Proof. — Let $\tilde{E} = \operatorname{Proj}(E \oplus 1)$ and let H denote the canonical line bundle on $\tilde{E}$. From the canonical imbedding ([1], p. 100):

$$H^\vee \subseteq E \oplus 1$$

we derive the canonical section

$$s \in \Gamma(\tilde{E}, E \otimes H \oplus H).$$

The projection of s onto $E \otimes H$ will be denoted

$$t \in \Gamma(\tilde{E}, E \otimes H).$$

The zero's of t all lie on the canonical section $X \to \tilde{E}$. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{\tau} \in \hat{H}^{\cdot}_{X}(\tilde{E}; \mathbf{Q}) & \xrightarrow{\tilde{r}} & \hat{H}^{\cdot}(\tilde{E}; \mathbf{Q}) \\ \downarrow \quad \downarrow \wr & & \downarrow \\ \tau \in \hat{H}^{\cdot}_{X}(E; \mathbf{Q}) & \xrightarrow{r} & \hat{H}^{\cdot}(E; \mathbf{Q}) \end{array}$$

where τ denotes the Thom class. Let us first prove that

$$\tilde{r}(\tilde{\tau}) = c_n(E \otimes H).$$

For this let us note that $\tilde{r}$ is injective. Namely, $H^*(\tilde{E}; \mathbf{Q}) \to H^*(\tilde{E}-X; \mathbf{Q})$ is surjective since the restriction to $\tilde{E}-X$ of

$$1, c_1(H), \ldots, c_1(H)^{n-1}$$

form a basis for the $H^{\cdot}(X; \mathbf{Q})$-module $H^{\cdot}(E-X; \mathbf{Q})$. Note that the restriction of $c_n(E \otimes H)$ to $\tilde{E}-X$ is zero because of the section t. Let $\sigma \in H_X(\tilde{E})$ be such that

$$\tilde{r}(\sigma) = c_n(E \otimes H).$$

We shall show that σ is the Thom class. For this it suffices to treat the case $X = P^t$. In this case $c_n(E \otimes H) = c_1(H)^n$ and the statement is clear.

We shall now prove the first formula. Let $\lambda^{\sim}$ denote the Koszul complex associated with the section t of $E \otimes H$. The restriction of $\lambda^{\sim}$ to E is $\lambda_{\check{E}}$. Let us recall [8], Lemma 18 that for a rank n bundle N we have

(6.2) $$\operatorname{ch}(\lambda_{-1} N^{\vee}) = c_n(N)\operatorname{Todd}(N)^{-1}.$$

The formula will now follow by applying (1.5) to $\lambda^{\sim}$

$$\tilde{r}(\operatorname{ch}^X \lambda^{\sim}) = \operatorname{ch}(\lambda_{-1} \check{E} \otimes \check{H}) = c_n(E \otimes H)\operatorname{Todd}(E \otimes H)^{-1},$$

$$\operatorname{Todd}(E \otimes H)^{-1} \equiv \operatorname{Todd}(E)^{-1} \bmod c_1(H),$$

$$c_n(E \otimes H)\, c_1(H) = 0$$

as it follows from the fact that $t \in \Gamma(\tilde{E}, E \otimes H \oplus H)$ has no zeros. Whence

$$\tilde{r}(\operatorname{ch}^X \lambda^{\sim}) = c_n(E \otimes H)\operatorname{Todd}(E)^{-1}.$$

Q. E. D.

Remark. — The above formula should be considered as generalizations of formulas used in [2], [3], [4].

7. Multiplicity in algebraic geometry

In this paragraph we shall work in the framework of [7] and prove a fundamental relation 7.1 between local Chern classes and the multiplicity of local algebra [15], compare [2], 6.2.

Let V denote a smooth (connected) algebraic variety/**C** and $X \subseteq V$ a closed subvariety of codimension d. The local fundamental class will be denoted

$$\mathrm{cl}^X \in H_X^{2d}(V; \mathbf{Z}).$$

The fundamental class of X, i. e. the image of cl^X in $H^{2d}(V; \mathbf{Z})$ will be denoted

$$\mathrm{cl}(X) \in H^{2d}(V; \mathbf{Z}).$$

For a coherent (algebraic) sheaf M on V with support in X, $l(M)$ denotes the length of the stalk of M at the generic point of X.

THEOREM 7.1. — *Let* $E^{\cdot}$ *denote a complex of locally free coherent* (*algebraic*) *sheaves on* V *with* $\mathrm{Supp}(E^{\cdot}) \subseteq X$. *Then*

$$\mathrm{ch}^X(E^{\cdot}) = \sum (-1)^i\, l(H^i E^{\cdot})\, \mathrm{cl}^X + \text{higher terms}.$$

Proof. — Let O denote the local ring of V at the generic point of X, m denotes the maximal ideal of O. Let $K_m(O)$ denote the Grothendieck group of the category of finite complexes of finitely generated free O-modules with homology of finite length (modulo exact complexes). We are going to define a topological character

$$l: \quad K_m(O) \to \mathbf{Z}.$$

Recall first that if U is a Zariski open subset of V with $X \cap U \neq \varnothing$, then the restriction map

$$H_X^{2d}(V; \mathbf{Z}) \to H_{X \cap U}^{2d}(U; \mathbf{Z})$$

is an isomorphism which carries cl^X to $\mathrm{cl}^{X \cap U}$. From this follows that there is a character l as above such that for any complex $E^{\cdot}$ as in the theorem

$$\mathrm{ch}^X(E^{\cdot}) = l(E^{\cdot})\, \mathrm{cl}^X + \text{higher terms}.$$

As is well known $K_m(O) \simeq \mathbf{Z}$ since O is a regular local ring [6]. Thus it will suffice to find a resolution $E^{\cdot}$ of O/m by finitely generated free sheaves with $l(E^{\cdot}) = 1$. Let us first consider the case $V = \mathbf{A}^d$, $X = \{0\}$. In this case we can take for $E^{\cdot}$ the standard Koszul complex. That $l(E^{\cdot}) = 1$ follows from 6.1.

In the general case choose a Zariski open set U of V and $f_1, \ldots, f_d \in \Gamma(U, O_V)$ which defines $X \cap U$. This defines a map

$$f: \quad U \to \mathbf{A}^d$$

with $f^{-1}(\{0\}) = U \cap X$. It follows that

$$f^*: \quad H^{2d}_{\{0\}}(\mathbf{A}^d; \mathbf{Z}) \to H^{2d}_{X \cap U}(U; \mathbf{Z})$$

is an isomorphism. The pull-back of the complex considered before will now do the job.

Q. E. D.

Remark 7.2. — Taking in particular a resolution of the structure sheaf O_X of X we obtain by means of (5.7):

$$c_d(O_X) = (-1)^{d-1}(d-1)!\,\mathrm{cl}(X)$$

due to Grothendieck [11] formula 17, compare [12] (p. 53, Lemma 2).

Remark 7.3. Combining 7.1 and 1.8 we obtain Serre's "alternating Tor-formula" [15] for the topological intersection number.

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