ANNALES SCIENTIFIQUES DE L'É.N.S.

CALVIN C. MOORE
JONATHAN ROSENBERG

Comments on a paper of Brown and Guivarc'h : "Espaces de Poisson des groupes de Lie"

Annales scientifiques de l'É.N.S. 4e série, tome 8, nº 3 (1975), p. 379-381
<http://www.numdam.org/item?id=ASENS_1975_4_8_3_379_0>

Ann. Scient. Éc. Norm. Sup.,
4e série, t. 5, 1975, p. 379 à 382.

COMMENTS ON A PAPER OF BROWN AND GUIVARC'H

BY CALVIN C. MOORE (1) AND JONATHAN ROSENBERG (2)

In a recent paper [2], Brown and Guivarc'h announce a proof of the following conjecture from [1]: Let G be a connected Lie group with radical R such that G/R has finite center; then G is type T in the sense of [1] if and only if the eigenvalues of ad (X) restricted to the Lie algebra $\mathscr{L}$ (R) of R are purely imaginary for every X in the Lie algebra $\mathscr{L}$ (G) of G. The proof given, however, has a gap in it, and in particular the crucial Proposition 4 is clearly false as stated. The difficulty occurs in the next to last sentence of the proof of this Proposition for it is surely possible for $G \cap V$ to leave invariant a compact set in $\mathscr{G}_p$ (V), for instance a one point set consisting of an affine subspace containing $G \cap V$. We shall show how the difficulty can be repaired by modifying both Propositions 4 and 5; in the end, the modified version is a bit more direct than the original version. We also show that the condition in the theorem that G/R have finite center is necessary; in fact, we show that the universal covering group of SL_2 (**R**) fails to have property T.

Specifically, Proposition 4 should be modified to read as follows:

PROPOSITION 4'. — *Let* G *be a connected Lie group contained in the affine group of a vector space* V. *If* $G \supset V$, *and if* G *is type* T, *then* G *is type* R.

Proof. — The given proof applies directly except that the affine Grassmann manifold $\mathscr{G}_r$ (V) (use some letter other than *p*) must be chosen so that $0 < r < \dim V$ which is possible by the proof of Proposition 3. The next to last sentence of the proof must be changed; the point is that if a compact subset C of $\mathscr{G}_r$ (V) is invariant under a subspace V' of V, then C must consist of affine subspaces parallel to V'. In particular, if $V' = V$, we have an impossibility since $r < \dim V$. This completes the proof.

Now Proposition 5 has to be strenghtened as follows:

PROPOSITION 5'. — *Let* G *be a connected Lie group with radical* R (*which is non-compact*) *and nil-radical* N. *Then there exists a homomorphism h of* G *onto a group h* (G) *such that*

(1) Supported in part by NSF Grant MPS-74-19876.

(2) Supported by an NSF Graduate Fellowship. MOS (1970) Subject Classification. 22A10, 60J45.

the kernel of h operates unipotently on $\mathscr{L}$ (R) *and such that either:* (i) $h(G) \subset GA(V)$ *for some vector space* V *and* $h(N) \supset V$, *or else,* (ii) $h(G)$ *is a solvable group.*

Proof. – Exactly as in the paper, one reduces to the case when N is a vector group. Let $\mathscr{S}$ be a Levi factor for $\mathscr{L}(G)$ so that $\mathscr{L}(G) = \mathscr{L}(R) + \mathscr{S}$. Now let $\mathscr{L}(N_0)$ be the subspace of $\mathscr{L}(N)$ where $\mathscr{S}$ acts trivially and let N_0 be the corresponding vector subgroup of N. If $N = N_0$, then as $\mathscr{S}$ acts trivially on $\mathscr{L}(R)/\mathscr{L}(N)$ and as $\mathscr{S}$ is semisimple, $\mathscr{S}$ acts trivially on $\mathscr{L}(R)$ so that $\mathscr{L}(G)$ is the Lie algebra direct sum of $\mathscr{L}(R)$ and $\mathscr{S}$. The commutator subalgebra of $\mathscr{L}(G)$ is $[\mathscr{L}(R), \mathscr{L}(R)] + \mathscr{S} \subset \mathscr{L}(N) + \mathscr{S}$ which acts nilpotently on the radical $\mathscr{L}(R)$. Hence, the commutator subgroup $[G, G]$ of G acts unipotently on $\mathscr{L}(R)$ and hence so does its closure G_1. In this case, we choose h to be the projection of G onto G/G_1 and (ii) holds.

Now if $N_0 \neq N$, we note that N_0 is a normal subgroup of G since N is abelian and since $\mathscr{L}(R/N)$ is central in $\mathscr{L}(G/N)$. Since $\mathscr{S}$ is semisimple and acts trivially on $\mathscr{L}(R)/\mathscr{L}(N)$, we may find a subspace $\mathscr{A}$ of $\mathscr{L}(R)$ complementary to $\mathscr{L}(N)$ which is centralized by $\mathscr{S}$. Since $[\mathscr{A}, \mathscr{A}]$ is also centralized by $\mathscr{S}$, it is contained in $\mathscr{L}(N_0)$. Dividing out by N_0, let $G' = G/N_0$, $R' = R/N_0$, $N' = N/N_0$, and let $\mathscr{A}' \simeq \mathscr{A}$, $\mathscr{S}' \simeq \mathscr{S}$ be the images of $\mathscr{A}$ and $\mathscr{L}$ in $\mathscr{L}(G')$. (Note that R' is the radical of G', but that N' may be smaller than the nil-radical of G'.) Then $\mathscr{A}'$ is an abelian subalgebra and $\mathscr{S}' + \mathscr{A}'$ is a complement to $\mathscr{L}(N')$ so that $\mathscr{L}(G')$ is the semi-direct product of $\mathscr{L}(N')$ and $\mathscr{S}' + \mathscr{A}'$. Now let H be the connected subgroup of G with Lie algebra $\mathscr{S}' + \mathscr{A}'$, and let $\overline{H}$ be its closure. Then $\overline{H} \cap N'$ consists of elements n such that Ad (n) is trivial on $\mathscr{L}(N')$ and on $\mathscr{L}(G')/\mathscr{L}(N')$ and which stablize $\mathscr{S}' + \mathscr{A}'$. That implies that Ad (n) is the identity, or in other words, that n is in the center of G'. However, by the construction of N_0, and semi-simplicity of $\mathscr{S}'$, this implies that $n = e$. Thus, $\overline{H} \cap N' = \{e\}$ so $H = \overline{H}$ is closed and G' is the semi-direct product of N' and H.

We choose our vector space V to be N'; for $g \in G$, let g' be its image in G' and write $g' = \tau(g)\rho(g)$ with $\tau(g) \in V$, and $\rho(g) \in H$. Now we let $h(g)v = g' v g'^{-1} + \tau(g)$ for $v \in V$; then h is a homomorphism of G into GA (V). Moreover $h(N) = V$ and the kernel of h consists of elements $g \in G$ whose projection in G' lies in H and which act trivially on V. Thus the kernel surely acts unipotently on $\mathscr{L}(R)$ and (i) holds. Proposition 5' is proved.

The proof of the main theorem now proceeds as in [2] if h satisfies (i) and is trivial if h satisfies (ii).

We turn now to the second point about necessity of the condition that G/R have finite center. Let G be semisimple with center Z. By Proposition V.1 of [1] G will fail to have property T if and only if there is an open semigroup S in G such that $S S^{-1} \cap Z$ has infinite index in Z. Now let G be universal covering group of $G_0 = SL_2(\mathbf{R})$, so that $Z = \mathbf{Z}$, the integers, and let S_0 be the open semigroup of $SL_2(\mathbf{R})$ consisting of matrices with all entries strictly positive. It is known that $S_0 S_0^{-1}$ meets the center of G_0 in only one point. Now on page 46 of [3], there is constructed a very explicit cross section $s: G_0 \to G$ for the group extension so that the corresponding cocycle b from $G_0 \times G_0$ into Z defined by $s(g)\,s(h) = b(g, h)\,s(gh)$ is explicitly computable. The cross section s is continuous and hence a homeomorphism on a dense open set D, specifically, the dense double coset of the triangular subgroup of G_0. It is clear that $S_0 \subset D$, and a direct calculation using the formulas on page 46 of [3] shows that the cocycle is trivial on $S_0 \times S_0$ and that $s(g^{-1}) = s(g)^{-1}$ for $g \in S_0$. It follows that s is a homomorphism on S_0 and that $S = s(S_0)$ is an open semigroup in G, and that $SS^{-1} \cap Z = \{ e \}$. Thus G fails to have property T.

REFERENCES

[1] R. AZENCOTT, *Espaces de Poisson des groupes localement compacts* (*Lecture Notes in Mathematics*, vol. 148, Springer Verlag, 1970).

[2] I. D. BROWN et Y. GUIVARC'H, *Espaces de Poisson des groupes de Lie* (*Ann. scient. Éc. Norm. Sup.*, t. 7, 1974, p. 175-180).

[3] C. C. MOORE, *Group Extensions of p-Adic and Adelic Linear Groups* (*Publ. Math. I.H.E.S.*, n° 35, 1968, p. 1-70).

(Manuscrit reçu le 17 mai 1975.)

Calvin C. MOORE
and
Jonathan ROSENBERG,
Department of Mathematics,
University of California,
Berkeley, California 94720,
U.S.A.