ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 8, nº 2 (1975), p. 157-178

<http://www.numdam.org/item?id=ASENS_1975_4_8_2_157_0>

Ann. scient. Éc. Norm. Sup.,
4[e], série, t. 8, 1975, p. 157 à 178.

REPRESENTATIONS OF SOLVABLE LIE ALGEBRAS. II. TWISTED GROUP RINGS

BY J. C. MC CONNELL

Introduction

This paper may be regarded as either an introduction to or a continuation of [4]. In [4] it was shown that the simple algebras which arise in the representation theory of solvable Lie algebras could be viewed as algebras of differential operators in which the multiplication is altered by a Lie 2-cocycle. In this paper we show that these cocycle twisted algebras of differential operators may also be viewed as (cocycle free) twisted group rings.

We first recall the main result of [4]. Let $\mathbf{g}$ be a completely solvable Lie algebra over a field k of characteristic zero and P be a prime ideal of $\mathrm{U} = \mathrm{U}(\mathbf{g})$, the universal enveloping algebra of $\mathbf{g}$. Let

$$\mathrm{E} = \mathrm{E}(\mathrm{U}/\mathrm{P}) = \{u+\mathrm{P} : u+\mathrm{P} \neq \mathrm{P} \text{ and } [g, u]+\mathrm{P} \in ku+\mathrm{P} \text{ for all } g \in \mathbf{g}\}.$$

U/P has a simple quotient ring $(\mathrm{U}/\mathrm{P})_{\mathrm{E}}$ with respect to E. It was shown in [4] that

$$(\mathrm{U}/\mathrm{P})_{\mathrm{E}} \cong (\mathrm{KS} \otimes_{\mathrm{K}} \mathrm{KG}) \#_{\sigma} \mathrm{U}(\mathbf{a}),$$

where K is the centre of $(\mathrm{U}/\mathrm{P})_{\mathrm{E}}$, KS is a commutative polynomial algebra, $\mathrm{KS} = \mathrm{K}[y_1, \ldots, y_n]$, $n \geqq 0$; KG is the group algebra of a free abelian group of finite rank m, $m \geqq 0$,

$$\mathrm{KG} = \mathrm{K}[g_1, g_1^{-1}, \ldots, g_m, g_m^{-1}];$$

$\mathbf{a}$ is a subalgebra of the abelian Lie algebra $\sum_{i,j} k\partial/\partial y_i + kg_j\,\partial/\partial g_j$ such that the ring of formal differential operators $(\mathrm{KS} \otimes_{\mathrm{K}} \mathrm{KG}) \# \mathrm{U}(\mathbf{a})$ is simple, (*see* Theorem 2.2); σ is a Lie 2-cocycle, $\sigma \in \mathrm{Z}^2(\mathbf{a}, \mathrm{KS} \otimes \mathrm{KG})$ and $(\mathrm{KS} \otimes_{\mathrm{K}} \mathrm{KG}) \#_{\sigma} \mathrm{U}(\mathbf{a})$ is the corresponding "twisted ring of differential operators." (*See* Theorem 2.1 and Remark.) The key result of this paper is that, under the conditions stated, the Lie cohomology group $\mathrm{H}^2(\mathbf{a}, \mathrm{KS} \otimes \mathrm{KG})$ is isomorphic to a subgroup of $\mathrm{H}^2(\mathbf{a}, \mathrm{K})$. From this it follows readily that such a simple algebra $(\mathrm{KS} \otimes \mathrm{KG}) \#_{\sigma} \mathrm{U}(\mathbf{a})$ may be "turned upside down" and viewed as a twisted group ring, $(\mathrm{A}_n \otimes_{\mathrm{K}} \mathrm{U}(\mathrm{W})) \# \mathrm{KG}$, where A_n is a Weyl algebra with centre K, U (W) is the enveloping algebra of an abelian Lie algebra W and G is a finitely generated free abelian group of automorphisms of the coefficient ring $A_n \otimes U(W)$. Each element of G preserves the filtration on $A_n \otimes U(W)$ and induces the identity automorphism on the associated graded algebra and $(A_n \otimes U(W)) \# KG$ may be regarded as an algebra $\mathscr{A}(V, \delta, \mathbf{G})$ constructed from a finite dimensional vector space V, an alternating bilinear form δ on V and a finitely generated subgroup $\mathbf{G}$ of the dual space V^*.

It is shown that each such simple algebra $\mathscr{A}(V, \delta, \mathbf{G})$ may be expressed as a $(KS \otimes KG) \#_\sigma U(\mathbf{a})$ and as a $(U(\mathbf{g})/P)_E$. Thus there are bijections between the isomorphism classes of simple algebras,

$$\{(U(\mathbf{g})/P)_E\}, \qquad \{(KS \otimes KG) \#_\sigma U(\mathbf{a})\} \qquad \text{and} \qquad \{\mathscr{A}(V, \delta, \mathbf{G})\}.$$

It is also shown that a simple algebra $\mathscr{A}(V, \delta, \mathbf{G})$ is a subalgebra of $A_n \otimes A'_m$, where $n = \operatorname{rank} \delta/2$, $m = \operatorname{rank} \mathbf{G}$ and A'_m is the localisation of A_m at the powers of the element $x_1 x_2 \ldots x_m$.

A necessary and sufficient condition for $\mathscr{A}(V, \delta, \mathbf{G})$ to be simple is that $V^\delta \cap V^G = 0$, where V^δ is the orthogonal complement of V with respect to δ and $V^G = \cap \operatorname{Ker} \lambda$, $(\lambda \in \mathbf{G})$. The proof of this is elementary and is much easier than the proof of the corresponding theorem of [4] which gave sufficient conditions for $(KS \otimes KG) \#_\sigma U(\mathbf{a})$ to be simple. (*See* Theorems 2.2 and 4.7.)

Finally we consider when two simple algebras

$$\mathscr{A}_1 = \mathscr{A}(V, \delta, \mathbf{G}) \qquad \text{and} \qquad \mathscr{A}_2 = \mathscr{A}(W, \gamma, \mathbf{H})$$

are isomorphic. We conjecture that these algebras are isomorphic if and only if there exists a vector space isomorphism $\varphi : V \to W$ such that (i) φ is compatible with δ and γ and (ii) the dual map φ^* induces an isomorphism $\mathbf{H} \to \mathbf{G}$. If such a φ exists then $\mathscr{A}_1 \cong \mathscr{A}_2$. Conversely, if $\mathscr{A}_1 \cong \mathscr{A}_2$ then we show that there exists an isomorphism φ such that (ii) holds but we are unable to show that φ also satisfies (i) except in a special case. However we do have four integer valued isomorphism invariants associated with with $\mathscr{A}(V, \delta, \mathbf{G})$, (or with a primitive ideal P of the universal enveloping algebra of a completely solvable Lie algebra $\mathbf{g}$) and if the conjecture is true then there is yet another integer valued isomorphism invariant. It is well known that if $\mathbf{g}$ is nilpotent then P determines a single integer n, where $U/P \cong A_n$ and $2n$ is the dimension of the orbit of $\mathbf{g}^*$ associated with P, [6] (4.2.1).

We are indebted to L. Avramov and A. Rosenberg, (participants in the Leeds Ring Theory Year), for their help with the cohomological questions considered in Sections 3.

1. Notation

k is a field of characteristic zero. An algebra is an associative k-algebra with 1 and homomorphisms preserve 1. If A is an algebra, Der A is the Lie algebra of k-derivations of A and Aut A the group of k-algebra automorphisms of A. End A = End_k A and if A is commutative, Diff A is the subalgebra of End A generated by Der A and the multiplications by elements of A.

If $\mathbf{g}$ is a Lie algebra and M a $\mathbf{g}$-module, then $C^p(\mathbf{g}, M)$ is the k-space of p-cochains, $Z^p(\mathbf{g}, M)$ the space of p-cocycles and $H^p(\mathbf{g}, M)$ the corresponding cohomology group. In particular $C^2(\mathbf{g}, k)$ is the space of alternating bilinear forms on $\mathbf{g}$. $M \times \mathbf{g}$ denotes the split extension of M by $\mathbf{g}$ and if $\sigma \in Z^2(\mathbf{g}, M)$, $M \times_\sigma \mathbf{g}$ denotes the extension corresponding to σ. (*See* [1], Chapter 13 Section 8 and Chapter 14 Section 5.)

If A is an algebra and G a subgroup of Aut A then $A \# k G$ denotes the corresponding twisted group ring which as a k-space is $A \otimes_k k G$, where $k G$ is the group algebra of G over k, and multiplication is defined by

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = a_1 g_1(a_2) \otimes g_1 g_2.$$

We will denote $a \otimes g$ by ag as usual. $A \# k G$ is also a free right A-module with the elements of G as a free basis. $A \# k G$ has the universal property that if B is an algebra and φ is a homomorphism of A to B and ψ a group homomorphism of G into the group of units of B such that

$$\psi(g)\varphi(a) = \varphi(g(a))\psi(g),$$

for all $a \in A$, $g \in G$, then there exists a unique (k-algebra) homomorphism of $A \# k G$ to B which extends φ and ψ.

If V is a vector space and δ an alternating bilinear form on V then a basis for V adapted to δ is a basis

$$x_1, y_1, \ldots, x_l, y_l, s_1, \ldots, s_t$$

such that $\delta(x_i, y_j) = \Delta_{ij}$ (where Δ_{ij} is the Kronecker delta) and

$$\delta(x_i, x_j) = \delta(x_i, s_j) = \delta(y_i, y_j) = \delta(y_i, s_j) = \delta(s_i, s_j) = 0.$$

If A_n is a Weyl algebra with its usual filtration and $U(W)$ the enveloping algebra of an abelian Lie algebra W with its usual filtration then $A_n \otimes U(W)$ is a filtered algebra by

$$F_p(A_n \otimes U(W)) = \sum_{s=0}^{p} F_s(A_n) \otimes F_{p-s}(U(W)).$$

Thus if $A_n = k[x_1, y_1, \ldots, x_n, y_n]$ (with $[x_i, y_j] = \Delta_{ij}$), $U(W) = k[s_1, \ldots, s_t]$ and V denotes the subspace of $A_n \otimes U(W)$ spanned by $\{x_1, y_1, \ldots, x_n, y_n, s_1, \ldots, s_t\}$ then

$$F_0(A_n \otimes U(W)) = k1 \qquad \text{and} \qquad F_1(A_n \otimes U(W)) = k1 \oplus V.$$

If θ is an automorphism of $A_n \otimes U(W)$ such that θ preserves the filtration and induces the identity automorphism on the associated graded algebra then there exists a unique $\lambda \in V^*$ such that for $v \in V$, $\theta(v) = v + \lambda(v)$. Conversely, if $\lambda \in V^*$ then there exists a unique automorphism θ of $A_n \otimes U(W)$ defined by $\theta(v) = v + \lambda(v)$, $v \in V$, and θ preserves the filtration and induces the identity automorphism on the associated graded algebra.

The following notation is standard throughout the paper. kS will denote a commutative polynomial algebra over k with $n \geqq 0$ generators, $k S = k[y_1, \ldots, y_n]$, and $k G$ will denote the group algebra over k of a free abelian group of finite rank m, $m \geqq 0$,

$k\,G = k\,[g_1, g_1^{-1}, \ldots, g_m, g_m^{-1}]$. $\mathbf{c}$ denotes the abelian Lie subalgebra of Der $(k\,G \otimes k\,S)$,

$$\mathbf{c} = \sum_{i=1}^{n} k\partial/\partial y_i + \sum_{j=1}^{m} k g_j\,\partial/\partial g_j.$$

This situation may be described more intrinsically as follows. The units in $k\,S \otimes k\,G$ are just the units in $k\,G$, i. e. are scalar multiples of the elements of G. Consider $k\,S$ with its usual graded algebra structure, (induced by the degree in $y_1, \ldots, y_n$) and let $\mathrm{Gr}_p(kS)$ denote the pth subspace in this grading. Then $\mathbf{c}$ is the subspace of Der $(k\,S \otimes k\,G)$ such that,

(i) if $c \in \mathbf{c}$ and h is a unit in $k\,S \otimes k\,G$, then, $c(h) \in kh$

and

(ii) $$c(\mathrm{Gr}_p(k\,S)) \subseteq \mathrm{Gr}_{p-1}(k\,S) \qquad \text{for} \quad p \geqq 1.$$

$\mathbf{c} = \mathbf{c}_1 \oplus \mathbf{c}_2$, where $\mathbf{c}_1 = \sum_i k\partial/\partial y_i$ and $\mathbf{c}_2 = \sum_j k g_j\,\partial/\partial g_j$. For $f = 1, 2$, π_f denotes the projection of $\mathbf{c} = \mathbf{c}_1 \oplus \mathbf{c}_2$ onto $\mathbf{c}_f$. If $g \in G$, $\lambda_g \in \mathbf{c}^*$ is defined by $c(g) = \lambda_g(c)\,g$ for all $c \in \mathbf{c}$.

A subspace $\mathbf{a}$ of $\mathbf{c}$ is said to satisfy condition (Sim) when the following two conditions hold :

(i) $\pi_1(\mathbf{a}) = \mathbf{c}_1$,

(ii) if $g, h \in G$ with $g \neq h$ then $\lambda_g \mid \mathbf{a} \neq \lambda_h \mid \mathbf{a}$. (*See* Theorems 2.2 and 3.2.)

(Note that $\mathbf{c}_1$ (respectively $\mathbf{c}_2$) as defined above corresponds to $\mathbf{c}_2$ (respectively $\mathbf{c}_1$) as defined in [4] Section 5.)

$\mathbf{a} \cap \mathbf{c}_2$ will be denoted by $\mathbf{a}^G$, i. e. $\mathbf{a}^G = \mathrm{Ker}(\pi_1 \mid \mathbf{a})$.

2. A Simplicity Theorem

The following theorem is [4] (Theorem 2.8).

THEOREM 2.1. — *Let* A *be a commutative algebra,* $\mathbf{g}$ *a Lie algebra,* θ *a Lie algebra homomorphism of* $\mathbf{g}$ *into* Der A *and* σ *a Lie 2-cocycle,* $\sigma \in Z^2(\mathbf{g}, A)$. *Let* $I = I_\sigma$ *be the ideal of* $U = U(A \times_\sigma \mathbf{g})$ *generated by*

$$\mathcal{S} = \{1_U - 1_A,\ a.b - ab : a, b \in A\},$$

where $a.b$ *denotes the product of* a *and* b *in* U *and* ab *their product in* A.

(1) *If* χ *is a homomorphism of* U *into an algebra* B *then* $\chi \mid A$ *is an algebra homomorphism from* A *to* B *if and only if* $I \subset \mathrm{Ker}\,\chi$.

(2) *If* ψ *is the canonical homomorphism from* U *to* U/I *then* $\psi \mid A \times_\sigma \mathbf{g}$ *is a monomorphism and if* $g_1, \ldots, g_n$ *is a basis for* $\mathbf{g}$ *and we identify* $A \times_\sigma \mathbf{g}$ *with its image under* ψ *then* U/I *is a free left* A*-module with the standard monomials in* $g_1, \ldots, g_n$ *as a free basis.* ■

Remark. — $A \#_\sigma U(\mathbf{g})$ has the universal property that if B is an algebra, and θ is a Lie algebra homomorphism of $A \times_\sigma \mathbf{g}$ into B such that $\theta \mid A$ is an algebra homomorphism then θ may be extended to an algebra homomorphism of $A \#_\sigma U(\mathbf{g})$ into B.

THEOREM 2.2. — *Let* $\mathbf{a}$ *be a subspace of* $\mathbf{c}$ *and* $\sigma \in Z^2(\mathbf{a}, k S \otimes k G)$. *Then* $(k S \otimes k G) \#_\sigma U(\mathbf{a})$ *is a simple algebra if and only if* $\mathbf{a}$ *satisfies condition* (Sim).

[Thus the simplicity of $(k S \otimes k G) \#_\sigma U(\mathbf{a})$ depends on the action of $\mathbf{a}$ and not on the choice of σ.]

Proof. — If $\pi_1(\mathbf{a}) \neq \mathbf{c}_1$ then there exists $0 \neq y \in \sum_i k y_i$ such that $a(y) = 0$ for all $a \in \mathbf{a}$. Thus y is a central non unit in $(k S \otimes k G) \#_\sigma U(\mathbf{a})$. If there exists $g, h \in G$ with $g \neq h$ and $\lambda_g \mid \mathbf{a} = \lambda_h \mid \mathbf{a}$ then $1 - gh^{-1}$ is a central nonunit in $(k S \otimes k G) \#_\sigma U(\mathbf{a})$. Thus (Sim) is a necessary condition for $(k S \otimes k G) \#_\sigma U(\mathbf{a})$ to be simple. That (Sim) is also sufficient was proved in [4], Section 5, Theorems 5.2 and 5.3.

However in Theorem 4.7 we will give another proof of this, which is entirely elementary for the case $\sigma = 0$ and the same proof can be used when $\sigma \neq 0$ modulo Theorems 3.1 and 3.2. ■

DEFINITION 2.3. — Let A be an algebra and $\mathbf{g}$ a Lie subalgebra of Der A. If, for all $a_1, a_2 \in A$, $g(a_1) = g(a_2)$ for all $g \in \mathbf{g}$ implies that $a_1 - a_2 \in k$ then we say that $\mathbf{g}$ separates the points of A modulo k.

THEOREM 2.4. — *Let* $\mathbf{a}$ *be a subspace of* $\mathbf{c}$. *Then* $\mathbf{a}$ *satisfies* (*Sim*) *if and only if* $\mathbf{a}$ *separates the points of* $k S \otimes k G$ *modulo* k.

Proof. — If $\mathbf{a}$ does not separate points modulo k then, as in the proof of Theorem 2.2, $(k S \otimes k G) \# U(\mathbf{a})$ contains central nonunits and so $\mathbf{a}$ does not satisfy condition (Sim), by Theorem 2.2. Conversely, if $\mathbf{a}$ separates points modulo k then $\mathbf{a}$ must satisfy conditions (i) and (ii) of (Sim). ■

3. Cohomology

THEOREM 3.1. — *Let* A *be a commutative algebra,* $\mathbf{g}$ *a Lie algebra and* θ *a Lie algebra homomorphism of* $\mathbf{g}$ *into* Der A. *Let* $\sigma, \tau \in Z^2(\mathbf{g}, A)$. *If* σ, τ *are cohomologous cocycles then there exists an isomorphism*

$$\varphi' : A \#_\sigma U(\mathbf{g}) \to A \#_\tau U(\mathbf{g})$$

which extends the identity map of A *to* A.

Proof. — Since σ, τ are cohomologous there is a Lie algebra isomorphism $\varphi : A \times_\sigma \mathbf{g} \to A \times_\tau \mathbf{g}$ which extends the identity map on A. φ may be extended uniquely to an algebra isomorphism (again denoted by φ) of $U(A \times_\sigma \mathbf{g}) \to U(A \times_\tau \mathbf{g})$. Consider the following diagram where I, ψ are as in Theorem 2.1,

$$\begin{array}{ccccccccc} 0 & \to & I_\sigma & \to & U(A \times_\sigma \mathbf{g}) & \xrightarrow{\psi_\sigma} & A \#_\sigma U(\mathbf{g}) & \to & 0 \\ & & \downarrow & & \downarrow \varphi & \searrow \psi_\tau & \downarrow \varphi' & & \\ 0 & \to & I_\tau & \to & U(A \times_\tau \mathbf{g}) & \to & A \#_\tau U(\mathbf{g}) & \to & 0 \end{array}$$

Since $\varphi : I_\sigma \to I_\tau$, φ induces an isomorphism of $A \#_\sigma U(\mathbf{g}) \to A \#_\tau U(\mathbf{g})$ which we denote by φ'. Since ψ_σ, ψ_τ and φ induce the identity map on A, φ' also has this property. ■

The rest of this section is devoted to the proof of the following

THEOREM 3.2. — *Let* $\mathbf{a}$ *be a subspace of* $\mathbf{c}$ *which satisfies condition* (*Sim*). *Let* $\mathbf{a}^G = \mathbf{a} \cap \mathbf{c}_2$. *Then for* $p > 0$, *there is an isomorphism of Lie cohomology groups,*

$$H^p(\mathbf{a}, k S \otimes k G) \cong H^p(\mathbf{a}^G, k).$$

LEMMA 3.3 (Rees). — *Let* R *be a commutative ring and* $x \in R$ *a non zerodivisor on* R. *Let* M *be an* R*-module with* $x M = 0$ *and* N *an* R*-module such that* x *is a non zerodivisor on* N.

$$(1) \qquad \operatorname{Ext}^p_R(M, N) \cong \operatorname{Ext}^{p-1}_{R/Rx}(M, N/xN) \qquad \textit{for} \quad p \geqq 1,$$
$$= 0 \qquad \textit{for} \quad p = 0.$$

(2) *If the map* $N \to N$ *given by* $n \mapsto xn$ *is surjective* (*as well as injective*) *then* $\operatorname{Ext}^p_R(M, N) = 0$ *for* $p \geqq 0$.

Proof (1) is the Rees Reduction Theorem [7] and (1) implies (2). ■

The following lemma is a dual version of the Rees Reduction Theorem for the case when the map $N \to N$ given by $n \mapsto xn$ is surjective instead of injective. (Compare the concept of a cosequence as introduced by Matlis [3].)

LEMMA 3.4. — *Let* R *be a commutative ring and* $x \in R$ *a non zerodivisor on* R. *Let* M *be an* R*-module such that* $x M = 0$ *and* N *an* R*-module such that* $x N = N$. *Then for* $p \geqq 0$,

$$\operatorname{Ext}^p_R(M, N) \cong \operatorname{Ext}^p_{R/Rx}(M, \operatorname{Ann}_N x),$$

where $\operatorname{Ann}_N x = \{ n \in N : xn = 0 \}$.

Proof. — By ([1], p. 348, Case 4 or p. 118, Case 4) with $\Gamma = R/Rx$ and $\Lambda = R$, there is an edge homomorphism θ in a spectral sequence,

$$\theta : \operatorname{Ext}^p_{R/Rx}(M, \operatorname{Hom}_R(R/Rx, N)) \to \operatorname{Ext}^p_R(M, N)$$

and by [1] (p. 349 or p. 118, Proposition 4.1.4) θ is an isomorphism if

$$\operatorname{Ext}^q_R(R/Rx, N) = 0 \qquad \text{for} \quad q > 0.$$

Since x is a non zerodivisor on R,

$$0 \to Rx \to R \to R/Rx \to 0$$

is an R-projective resolution of ${}_R(R/Rx)$ and so $\operatorname{Ext}^q_R(R/Rx, N) = 0$ for $q > 1$. Applying $\operatorname{Hom}_R(-, N)$ to this exact sequence, we obtain

$$\begin{tikzcd}
\operatorname{Hom}_R(R, N) \arrow[r] \arrow[d, phantom, "\wr\|"] & \operatorname{Hom}_R(Rx, N) \arrow[r] \arrow[d, phantom, "\wr\|"] & \operatorname{Ext}^1_R(R/Rx, N) \arrow[r] & 0, \\
N \arrow[r, "\Delta"] & N & &
\end{tikzcd}$$

where Δ is defined by $n \mapsto xn$. Since $x\,\mathrm{N} = \mathrm{N}$, Δ is surjective and $\mathrm{Ext}^1_{\mathrm{R}}(\mathrm{R}/\mathrm{R}\,x, \mathrm{N}) = 0$. Thus θ is an isomorphism. Since $\mathrm{R}/\mathrm{R}\,x$ is generated by $1+\mathrm{R}\,x$ and $x.(1+\mathrm{R}\,x) = 0+\mathrm{R}\,x$, $\mathrm{Hom}_{\mathrm{R}}(\mathrm{R}/\mathrm{R}\,x, \mathrm{N}) \cong \mathrm{Ann}_{\mathrm{N}}\,x$. ∎

Proof of Theorem 3.2. – Let $\mathbf{a}'$ be a subspace of $\mathbf{a}$ such that $\mathbf{a} = \mathbf{a}' \oplus \mathbf{a}^{\mathrm{G}}$. Let $\sigma \in \mathrm{C}^p(\mathbf{a}^{\mathrm{G}}, k)$. σ may be extended to an element of $\mathrm{C}^p(\mathbf{a}, k)$ by requiring that $\sigma(x_1, \ldots, x_p) = 0$ if there exists an i, $1 \leqq i \leqq p$, such that $x_i \in \mathbf{a}'$. The canonical embedding $k \to k\,1 \subset k\,\mathrm{S} \otimes k\,\mathrm{G}$ enables us to regard $\sigma \in \mathrm{C}^p(\mathbf{a}, k)$ as an element of $\mathrm{C}^p(\mathbf{a}, k\,\mathrm{S} \otimes k\,\mathrm{G})$ and clearly $\sigma \in \mathrm{Z}^p(\mathbf{a}, k\,\mathrm{S} \otimes k\,\mathrm{G})$. We now show that such a σ cannot be a coboundary. For each $x \in \mathbf{a}^{\mathrm{G}}$, the image of x (viewed as an operator on $k\,\mathrm{S} \otimes k\,\mathrm{G}$) is contained in $\sum_{g \neq 1} k\,\mathrm{S} \otimes kg$. For each $f \in \mathrm{C}^{p-1}(\mathbf{a}, k\,\mathrm{S} \otimes k\,\mathrm{G})$ and $x_1, \ldots, x_p \in \mathbf{a}^{\mathrm{G}}$,

$$\delta f(x_1, \ldots, x_p) = \sum_i (-1)^{i-1} x_i(f(x_1, \ldots, \hat{x}_i, \ldots, x_p))$$

and so $\delta f(x_1, \ldots, x_p) \in \sum_{g \neq 1} k\,\mathrm{S} \otimes kg$. Thus $\sigma \neq \delta f$. Hence the induced homomorphism of

$$\mathrm{H}^p(\mathbf{a}^{\mathrm{G}}, k) \cong \mathrm{C}^p(\mathbf{a}^{\mathrm{G}}, k) \to \mathrm{H}^p(\mathbf{a}, k\,\mathrm{S} \otimes \mathrm{K}\mathrm{G})$$

is injective. It remains to show that this homomorphism is surjective.

Let U be the universal enveloping algebra of $\mathbf{a}$, so U is a commutative polynomial algebra. By [1], Chapter 13, Section 8,

$$\mathrm{H}^p(\mathbf{a}, k\,\mathrm{S} \otimes k\,\mathrm{G}) \cong \mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S} \otimes k\,\mathrm{G}).$$

For $g \in \mathrm{G}$, $k\,\mathrm{S} \otimes kg$ is an $\mathbf{a}$-submodule of $k\,\mathrm{S} \otimes k\,\mathrm{G}$ and

$$k\,\mathrm{S} \otimes k\,\mathrm{G} = \bigoplus_{g \in \mathrm{G}} k\,\mathrm{S} \otimes kg.$$

Since k is a finitely generated U-module and U is noetherian, k has a projective resolution by finitely generated free U-modules. If F is a finitely generated free module then

$$\mathrm{Hom}_{\mathrm{U}}(\mathrm{F}, k\,\mathrm{S} \otimes k\,\mathrm{G}) = \bigoplus_{g \in \mathrm{G}} \mathrm{Hom}_{\mathrm{U}}(\mathrm{F}, k\,\mathrm{S} \otimes kg).$$

Thus

$$\mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S} \otimes k\,\mathrm{G}) \cong \bigoplus_{g \in \mathrm{G}} \mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S} \otimes kg).$$

The action of U of $k\,\mathrm{S} \otimes kg$ is induced by the action of $\mathbf{a}$ on $k\,\mathrm{S} \otimes kg$ which is given by : for $a \in \mathbf{a}$ and $f \in k\,\mathrm{S}$,

$$a(f \otimes g) = a(f) \otimes g + f \otimes a(g) = (a(f) + \lambda_g(a) f) \otimes g.$$

We show first that if $g \neq 1_{\mathrm{G}}$ then $\mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S} \otimes kg) = 0$ for $p \geqq 0$. If $g \neq 1_{\mathrm{G}}$ then, since $\mathbf{a}$ satisfies (Sim), there exists $a_1 \in \mathbf{a}$ with $\lambda_g(a_1) \neq 0$. Choose a basis $\{x_i\}$ for $\mathbf{a}$ with $x_1 = a_1$. Then there exist $c_1, \ldots, c_n, \mu \in k$ with $\mu = \lambda_g(a_1) \neq 0$ such that

$$x_1(f \otimes g) = \left(\left(\sum_{i=1}^{n} c_i \partial/\partial y_i + \mu\,\mathrm{I}\right) f\right) \otimes g,$$

where $\mathrm{I} = 1_{k\mathrm{S}}$. The map $k\,\mathrm{S} \to k\,\mathrm{S}$ given by $f \to (\sum_i c_i\,\partial/\partial y_i + \mu\mathrm{I})f$ is bijective. (It is clearly injective and is easily seen to be surjective by induction on the degree of f.) Thus, if $g \neq 1_{\mathrm{G}}$, then for $p \geqq 0$,

$$\mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S} \otimes kg) = 0$$

by Lemma 3.3 (ii).

We now consider $\mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S} \otimes k\,1_{\mathrm{G}})$ and we will identify the U-modules $k\,\mathrm{S} \otimes k\,1_{\mathrm{G}}$ and $k\,\mathrm{S}$ by the map $f \otimes 1_{\mathrm{G}} \mapsto f$ for $f \in k\,\mathrm{S}$. Since $\mathbf{a}$ satisfies (Sim), we may choose a basis $x_1, \ldots, x_t$ for $\mathbf{a}$ such that for $f \in k\,\mathrm{S}$:

$$\begin{aligned} x_i.f &= \partial f/\partial y_i && \text{for} \quad 1 \leqq i \leqq n, \\ &= 0 && \text{for} \quad n+1 \leqq i \leqq t. \end{aligned}$$

(Thus $x_{n+1}, \ldots, x_t$ are a basis for $\mathbf{a}^{\mathrm{G}}$.) Since k has characteristic zero, $x_1.k\,\mathrm{S} = k\,\mathrm{S}$ and $\mathrm{Ann}_{k\mathrm{S}}\,x_1 = k\,[y_2, \ldots, y_n]$. So by Lemma 3.4,

$$\mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S}) \cong \mathrm{Ext}^p_{\mathrm{U}_1}(k, k\,[y_2, \ldots, y_n]),$$

where $\mathrm{U}_1 = \mathrm{U}/\mathrm{U}\,x_1 \cong k\,[x_2, \ldots, x_t]$. By induction on n, we obtain

$$\mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S}) \cong \mathrm{Ext}^p_{\mathrm{U}_n}(k, k),$$

where

$$\mathrm{U}_n = \mathrm{U}/\mathrm{U}\,x_1 + \ldots + \mathrm{U}\,x_n \cong k\,[x_{n+1}, \ldots, x_t].$$

Thus

$$\mathrm{H}^p(\mathbf{a}, k\,\mathrm{S} \otimes k\,\mathrm{G}) = \mathrm{Ext}^p_{\mathrm{U}}(k, k\,\mathrm{S} \otimes k\,\mathrm{G}) \cong \mathrm{Ext}^p_{\mathrm{U}_n}(k, k) \cong \mathrm{H}^p(\mathbf{a}^{\mathrm{G}}, k).$$

Since these isomorphisms are all k-vector space isomorphisms the theorem is proved. ∎

Remark. – In the case when rank $\mathrm{G} = 0$ and so $k\,\mathrm{S} \otimes k\,\mathrm{G} = k\,[y_1, \ldots, y_n]$ and $\mathbf{a} = \sum_{i=1}^{n} k\,\partial/\partial y_i$, this theorem is classical, (Lemma of Poincaré). *See* [2] (Theorem 2.2 and following remark).

NOTE 3.5. – Let $\mathbf{a}$ be a subspace of $\mathbf{c}$ with $\dim \mathbf{a} > 0$. In this case the converse of Theorem 3.2 is also true, i. e. the monomorphism $\mathrm{C}^p(\mathbf{a}^{\mathrm{G}}, k) \to \mathrm{H}^p(\mathbf{a}, k\,\mathrm{S} \otimes k\,\mathrm{G})$, $p \geqq 1$, is an isomorphism for all $p \geqq 1$ only if $\mathbf{a}$ satisfies (Sim). (This is easily seen by reversing the steps of the proof of Theorem 3.2.) Thus there are three equivalent conditions to (Sim), namely the simplicity of an algebra, Theorem 2.2; the separating points condition, Theorem 2.4; and, when $\dim \mathbf{a} > 0$, the minimality of some cohomology groups. Yet another condition was given in [4] which briefly is as follows.

$$\mathbf{c}_2 = \sum_{j=1}^{m} kg_j\,\partial/\partial g_j$$

may be viewed as the space of diagonal matrices on $kg_1 + \ldots + kg_m$ by

$$\lambda_1\,g_1\,\partial/\partial g_1 + \ldots + \lambda_m\,g_m\,\partial/\partial g_m \to \text{diagonal matrix}(\lambda_1, \ldots, \lambda_m).$$

It was shown in [4] Section 5, that condition (ii) of (Sim) is equivalent to the condition that the algebraic hull of $\pi_2(\mathbf{a})$ in $\mathbf{c}_2$ is $\mathbf{c}_2$, [where the algebraic hull of $\pi_2(\mathbf{a})$ is the usual algebraic hull of a space of diagonal matrices].

4. Twisted Group Rings

In this section we construct the algebras $\mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$ mentioned in the introduction. Let V be a finite-dimensional vector space and δ an alternating bilinear form on V. Let

$$\mathrm{V}^\delta = \{ v \in \mathrm{V} : \delta(v, v') = 0 \text{ for all } v' \in \mathrm{V} \}.$$

Then rank $\delta = \dim \mathrm{V} - \dim \mathrm{V}^\delta$ is even, equal to $2l$ say where $l \geqq 0$. With this notation we have

Lemma 4.1. — *Consider* V *as an abelian Lie algebra, k as a trivial* V*-module and* δ *as an element of* $\mathrm{Z}^2(\mathrm{V}, \mathrm{K})$. *Then*

$$k \#_\delta \mathrm{U}(\mathrm{V}) \cong \mathrm{A}_l \otimes \mathrm{U}(\mathrm{V}^\delta),$$

where A_l *is a Weyl algebra and* $\mathrm{U}(\mathrm{V}^\delta)$ *is the universal enveloping algebra of the abelian Lie algebra* V^δ.

Proof. — Choose a basis for V adapted to δ and use Theorem 2.1 (2). ∎

$k \#_\delta \mathrm{U}(\mathrm{V})$ will be denoted by $\mathrm{U}_\delta(\mathrm{V})$ and should be regarded as the algebra containing **V** as a subspace and generated by the elements of V subject to the relations : for all $v, v' \in \mathrm{V}$, $vv' - v'v = \delta(v, v')$.

As noted in Section 1, if $\lambda \in \mathrm{V}^*$ then there corresponds a unique automorphism θ_λ of $\mathrm{U}_\delta(\mathrm{V})$ defined by $\theta_\lambda(v) = v + \lambda(v)$, $v \in \mathrm{V}$. Let $\Phi : \mathrm{V}^* \to \operatorname{Aut} \mathrm{U}_\delta(\mathrm{V})$ be given by $\lambda \mapsto \theta_\lambda$. If $\lambda, \mu \in \mathrm{V}^*$ then $\theta_{\lambda+\mu} = \theta_\lambda \theta_\mu$ so Φ is a group monomorphism of the additive group V^* into $\operatorname{Aut} \mathrm{U}_\delta(\mathrm{V})$. If G is a subgroup of $\Phi(\mathrm{V}^*)$ then the inverse image of G under Φ will be denoted by **G** and if **H** is a subgroup of V^* then $\Phi(\mathbf{H})$ will be denoted by H. If **G** is a finitely generated subgroup of V^* then **G** is a torsionfree group (since k has characteristic zero) and hence is a free abelian group of finite rank.

We note in passing that if $\lambda \in \mathrm{V}^*$ then $\theta_\lambda = \exp d_\lambda$, (exponential d_λ), where d_λ is the unique derivation of $\mathrm{U}_\delta(\mathrm{V})$ such that

$$\begin{array}{ccc} \mathrm{U}_\delta(\mathrm{V}) & \xrightarrow{d_\lambda} & \mathrm{U}_\delta(\mathrm{V}) \\ \uparrow & & \uparrow \\ \mathrm{V} & \xrightarrow{\lambda} & k \end{array}$$

is commutative. d_λ is locally nilpotent and $\exp d_\lambda$ is defined, (since k has characteristic zero), and is an automorphism of $\mathrm{U}_\delta(\mathrm{V})$ which coincides with θ_λ since they agree on **V.**

Thus given a finite dimensional vector space V, an alternating bilinear form δ on V and a finitely generated subgroup **G** of V^* we can form the twisted group ring $\mathrm{U}_\delta(\mathrm{V}) \# k\mathrm{G}$ and we will denote this algebra by $\mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$. Conversely an algebra $(\mathrm{A}_l \otimes \mathrm{U}(\mathrm{W})) \# k\mathrm{G}$, where G is finitely generated and each element of G preserves the filtration and induces the identity automorphism on the associated graded algebra, is an $\mathscr{A}$ (V, δ, G), where

$$V = W + \sum_{i=1}^{l} kx_i + ky_i$$

and for $v, v' \in V$, $\delta(v, v') = [v, v']$.

Two specialisations of $\mathscr{A} = \mathscr{A}(V, \delta, \mathbf{G})$ are the following. If $\mathbf{G} = 0$ and δ is non-singular on V then $\mathscr{A} \cong A_p$, where $2p = \dim V$. If $\delta = 0$ and $m = \operatorname{rank} \mathbf{G} = \dim V$ and $\mathbf{G}$ spans V^* then $\mathscr{A} \cong A'_m$. To see this let $g_1, \ldots, g_m$ be a set of generators of G, let $\lambda_1, \ldots, \lambda_m$ be the corresponding elements of V^*, which are a basis for V^*, and let $v_1, \ldots, v_m$ be a dual basis for V with $\lambda_i(v_j) = \Delta_{ij}$. So $[g_i, v_j] = \Delta_{ij} g_i$ and

$$\mathscr{A} = k[g_1, g_1^{-1}, \ldots, g_m, g_m^{-1}, v_1, \ldots, v_m].$$

Set $x_i = g_i$ and $y_i = g_i^{-1} v_i$. With the notation as defined in Section 1 we have

THEOREM 4.2. — *Let* $\mathbf{a}$ *be a subspace of* $\mathbf{c}$ *which satisfies (Sim) and let* $\sigma \in Z^2(\mathbf{a}, k\,S \otimes k\,G)$. *Then there exists a finite dimensional vector space* V, *an alternating bilinear form* δ *on* V *and a finitely generated subgroup* $\mathbf{G}$ *of* V^* *such that*

$$(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a}) \cong \mathscr{A}(V, \delta, \mathbf{G}).$$

Proof. — By Theorems 3.1 and 3.2 we may assume that $\sigma \in C^2(\mathbf{a}, k)$. Also if $\tau \in C^2(\mathbf{a}, k)$ then σ and τ determine isomorphic algebras if $\sigma \mid \mathbf{a}^G = \tau \mid \mathbf{a}^G$. Thus if $\mathbf{a}'$ is a subspace of $\mathbf{a}$ such that $\mathbf{a} = \mathbf{a}' \oplus \mathbf{a}^G$ then we may assume that $\mathbf{a}'$ is σ-orthogonal to $\mathbf{a}$. Let rank $(\sigma \mid \mathbf{a}^G)$ be $2r$ and let W be the subspace of $\mathbf{a}^G$ which is the σ-orthogonal complement of $\mathbf{a}^G$. Let V be the subspace $S \oplus \mathbf{a}$ of $(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a})$, where S denotes the space of homogeneous elements of degree one of $k\,S$. $\delta \in C^2(V, k)$ is defined by

$$\delta(v, v') = [v, v'], \qquad v, v' \in V.$$

By (Sim), $\pi_1(\mathbf{a}') = \mathbf{c}_1$ and so the subalgebra generated by S and $\mathbf{a}'$ is A_n, where n is the Krull dimension of $k\,S$. By Theorem 2.1 (2), the subalgebra generated by $\mathbf{a}^G$ is $A_r \otimes U(W)$ and the subalgebra generated by V is $A_{n+r} \otimes U(W)$ and so is $U_\delta(V)$.

For $a \in \mathbf{a}$ and $g \in G$, $[a, g] = \lambda_g(a) g$ or equivalently $ga = (a - \lambda_g(a)) g$. $\lambda_g \in \mathbf{a}^*$ and λ_g may be extended to an element of V^* by setting $\lambda_g(S) = 0$. Thus for $v \in V$ and $g \in G$,

$$gv = (v - \lambda_g(v)) g \qquad \text{or} \qquad gvg^{-1} = v - \lambda_g(v).$$

Thus each $g \in G$ determines an automorphism of $U_\delta(V)$ given by $u \mapsto gug^{-1}$ and if $g, h \in G$ with $g \neq h$, then $\lambda_g \mid \mathbf{a} \neq \lambda_h \mid \mathbf{a}$ by (Sim) and so g and h determine distinct automorphisms of $U_\delta(V)$. Since

$$(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a}) \cong k\,G \otimes k\,S \otimes U(\mathbf{a}),$$

as vector spaces, $(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a})$ is a free right $U_\delta(V)$-module with the elements of G as a free basis. Thus

$$(k\mathrm{S} \otimes k\mathrm{G}) \#_\sigma \mathrm{U}(\mathbf{a}) \cong \mathscr{A}(\mathrm{V}, \delta, \mathbf{G}),$$

where $\mathbf{G} = \{ -\lambda_g : g \in \mathrm{G} \}$ is a subgroup of V^*.

$\mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$ satisfies the following universal property.

THEOREM 4.3. – *Let $\mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$ be as above. Let* B *be an algebra,* $\chi : \mathrm{V} \to \mathrm{B}$ *be a* k*-module map and* $\psi : \mathbf{G} \to$ (*Units of* B) *be a group homomorphism such that*

(i) $[\chi(v), \chi(v')] = \delta(v, v'),\ v, v' \in \mathrm{V}$,

(ii) $\psi(\lambda)\,\chi(v) = (\chi(v) + \lambda(v))\,\psi(\lambda)$.

Then there exists a unique homomorphism $\chi' : \mathscr{A}(\mathrm{V}, \delta, \mathbf{G}) \to \mathrm{B}$ *such that* $\chi' \mid \mathrm{V} = \chi$ *and for* $\lambda \in \mathbf{G}$, $\chi'(\theta_\lambda) = \psi(\lambda)$.

Proof. – By the remark following Theorem 2.1 and (i), χ can be extended to an algebra homomorphism of $\mathrm{U}_\delta(\mathrm{V})$ to B. By (ii) and the universal property of a twisted group ring, there is an algebra homomorphism χ' as required. ∎

LEMMA 4.4. – *Consider an algebra* $\mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$. *If* $\mathbf{G}$ *spans* V^* *then for each* $r \in \mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$ *with* $r \notin k$ *there exists* $\lambda \in \mathbf{G}$ *such that* $\exp d_\lambda(r) \neq r$ *and there exists* $g \in \mathrm{G}$ *such that* $gr \neq rg$.

Proof. – Let $0 \neq \lambda \in \mathbf{G}$, denote $\exp d_\lambda$ by g, set $\mathrm{V}' = \operatorname{Ker} \lambda$ and choose $v \in \mathrm{V}$ so that $\mathrm{V} = \mathrm{V}' \oplus kv$. As k-modules,

$$\mathrm{U}_\delta(\mathrm{V}) \cong \mathrm{U}_\delta(\mathrm{V}') \otimes k[v].$$

For $r \in \mathrm{U}_\delta(\mathrm{V})$, $grg^{-1} = \exp d_\lambda(r)$. If $r = \sum_{i=0}^{s} r_i \otimes v^i$, where $r_i \in \mathrm{U}_\delta(\mathrm{V}')$, $i = 0, \ldots, s$, and $r_s \neq 0$ then, since

$$\exp d_\lambda(v^i) = (v + \lambda(v))^i,$$

the coefficient of v^{s-1} in $\exp d_\lambda(r) - r$ is $s\,\lambda(v)\,r_s$. So if $\exp d_\lambda(r) = r$ then, $\lambda(v)\,r_s \neq 0$, $s = 0$ and $r \in \mathrm{U}_\delta(\mathrm{V}')$. Thus if r commutes with g for all $g \in \mathrm{G}$, then

$$r \in \bigcap_\lambda \mathrm{U}_\delta(\operatorname{Ker} \lambda) = \mathrm{U}_\delta\Big(\bigcap_\lambda \operatorname{Ker} \lambda\Big),$$

by a Poincaré-Birkhoff-Witt argument. Since $\mathbf{G}$ spans V^*, $\mathrm{U}_\delta\big(\bigcap_\lambda \operatorname{Ker} \lambda\big) = k$. ∎

NOTATION 4.5. – Let V be a finite-dimensional vector space, δ an alternating bilinear form on V and $\mathbf{G}$ a subgroup of V^*. Set $\mathrm{V}^{\mathbf{G}} = \bigcap_\lambda \operatorname{Ker} \lambda$, $(\lambda \in \mathbf{G})$. V^δ denotes the δ-orthogonal complement of V and $(\mathrm{V}^{\mathbf{G}})^\delta$ the subspace of $\mathrm{V}^{\mathbf{G}}$ which is δ-orthogonal to $\mathrm{V}^{\mathbf{G}}$.

THEOREM 4.6. – *Let* V *be a finite-dimensional vector space,* δ *an alternating bilinear form on* V *and* $\mathbf{G}$ *a finitely generated subgroup of* V^*. *The following conditions are equivalent.*

(i) $\mathscr{A}(V, \delta, \mathbf{G}) \equiv U_\delta(V) \# k\,G$ *is a simple algebra.*

(ii) $\{ r \in U(V^\delta) : \exp d_\lambda(r) = r$ *for all* $\lambda \in \mathbf{G} \} = k$.

(iii) $\mathbf{G} \mid V^\delta$ *spans* $(V^\delta)^*$, *i. e.* $V^G \cap V^\delta = 0$.

(iv) $U(V^\delta)$ *is the only non-zero ideal of* $U(V^\delta)$ *which is invariant under* G.

(v) $\mathscr{A}(V^\delta, 0, \mathbf{G} \mid V^\delta) = U(V^\delta) \# k\,H$, *where* $H = \mathbf{G} \mid V^\delta$, *is a simple subalgebra of* $\mathscr{A}(V, \delta, \mathbf{G})$.

Proof. — (i) $\to$ (ii) $\to$ (iii) $\to$ (ii) $\to$ (iv) $\to$ (i), (iv) $\to$ (v) $\to$ (iii), (i) $\to$ (ii) If there exists an $r \in U(V^\delta)$ with $r \notin k$ such that $\exp d_\lambda(r) = r$ for all $\lambda \in \mathbf{G}$ then r is a central non unit in $\mathscr{A}(V, \delta, \mathbf{G})$ and so this algebra is not simple.

(ii) $\to$ (iii). If $r \in V^\delta$ then $\exp d_\lambda(r) = r + \lambda(r)$.

(iii) $\to$ (ii) by Lemma 4.4.

(ii) $\to$ (iv). Let I be a nonzero ideal of $U(V^\delta)$ which is invariant under G. If $0 \neq r \in I$ and r has minimal degree then $I \ni \exp d_\lambda(r) - r$ which is zero or has degree less than r. Thus $\exp d_\lambda(r) = r$ for all $\lambda \in \mathbf{G}$ and so $I \cap k \neq 0$.

(iv) $\to$ (i). We use a variation of a classical "minimal length of a relation" argument. (The particular case when $V^\delta = V$ may be deduced from [5], (Theorem 1.5 and the remark on pp 260-261).) Note first that if $v \in V$, $0 \neq u \in U_\delta(V)$ and $g = \exp d_\lambda \in G$ then

$$[v, ug] = vug - u(v + \lambda(v))g = ([v, u] - \lambda(v)u)g$$

and if $\lambda(v) \neq 0$ then $[v, u] - \lambda(v)u \neq 0$ since degree $[v, u] <$ degree u. We show that if J is a nonzero ideal of $U_\delta(V) \# k\,G$ then $J \cap U(V^\delta)$ is a nonzero ideal of $U(V^\delta)$ which is invariant under G. Among the non-zero elements of J, choose $r = \sum_{i=1}^{s} u_i g_i$ with the property firstly that $s =$ length r is minimal and secondly that degree u_1 is minimal. Without loss of generality we may assume that $g_1 = 1_G$. Firstly $u_1 \in U(V^\delta)$, since otherwise there exists $0 \neq v \in V$ with degree $[v, u_1] <$ degree u_1 and $[v, r] \in J$, which contradicts the minimality of r. If length $r = s > 1$, suppose $g_i = \exp d_{\lambda_i}$ and choose $v \in V$ with $v \notin \operatorname{Ker} \lambda_2$. Then

$$[v, r] = \sum_{i=2}^{s} ([v, u_i] - \lambda_i(v)u_i)g_i \neq 0,$$

since $\lambda_2(v) \neq 0$, and this contradicts the minimality of r. Thus $J \cap U(V^\delta) \neq 0$.

(iv) $\to$ (v). That $\mathscr{A}(V^\delta, 0, \mathbf{G} \mid V^\delta)$ is simple follows from (iv) $\to$ (i) by supposing that $V = V^\delta$. It remains to show that $\mathscr{A}(V^\delta, 0, \mathbf{G} \mid V^\delta)$ may be regarded as a subalgebra of $\mathscr{A}(V, \delta, \mathbf{G})$. Consider the canonical homomorphism $\mathbf{G} \to \mathbf{G} \mid V^\delta$. $\mathbf{G} \mid V^\delta$ is finitely generated and hence free abelian so the exact sequence

$$0 \to \operatorname{Ker} \to \mathbf{G} \to \mathbf{G} \mid V^\delta \to 0$$

splits and we may regard $\mathbf{G} \mid V^\delta$ as a subgroup of $\mathbf{G}$. Thus $\mathscr{A}(V^\delta, 0, \mathbf{G} \mid V^\delta)$ may be identified with a subalgebra of $\mathscr{A}(V, \delta, \mathbf{G})$. (This is the only place in the proof of the Theorem where the assumption that $\mathbf{G}$ is finitely generated is used).

(v) $\to$ (iii) is the particular case of (i) $\to$ (iii) corresponding to $V = V^\delta$. ∎

We now give the alternative proof of part of Theorem 2.2.

THEOREM 4.7. *Let* $\mathbf{a}$ *be a subspace of* $\mathbf{c}$ *which satisfies* (*Sim*) *and* $\sigma \in Z^2(\mathbf{a}, k\,S \otimes k\,G)$. *Then* $(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a})$ *is a simple algebra.*

Proof. – Consider first the case when $\sigma = 0$.

Then by Theorem 4.2, $(k\,S \otimes k\,G) \# U(\mathbf{a}) \cong \mathscr{A}(V, \delta, G)$, where $V^\delta = \mathbf{a}^G$ and so $V^\delta \cap V^G = 0$.

If $\sigma \neq 0$ then by Theorem 4.2 again,

$$(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a}) \cong \mathscr{A}(V, \gamma, G),$$

where $V^\gamma \subseteq V^\delta = \mathbf{a}^G$. So $V^\gamma \cap V^G = 0$. In either case apply Theorem 4.6 (iii). ∎

THEOREM 4.8. – *Let* $\mathscr{A} = \mathscr{A}(V, \delta, G)$ *be a simple algebra. Then*

(i) *There exists an abelian by abelian completely solvable Lie algebra* $\mathbf{g}$ *and a prime ideal* P *of* $U = U(\mathbf{g})$ *such that*

$$\mathscr{A} \cong (U/P)_E.$$

(ii) $\mathscr{A} \cong (k\,S \otimes k\,G) \#_\sigma U(\mathbf{a})$, *for a suitable choice of* S, G, $\mathbf{a} \subset \mathbf{c}$ *satisfying* (*Sim*) *and* $\sigma \in Z^2(\mathbf{a}, k\,S \otimes k\,G)$.

Proof. – (i) Let $\dim (V^G)^\delta = q$ and $\dim V^G = 2p+q$. Since $\mathscr{A}$ is simple, $V^G \cap V^\delta = 0$ and we can choose a basis

$$x_1, y_1, \ldots, x_l, y_l, s_1, \ldots, s_t$$

for V adapted to δ so that $y_{p+1}, \ldots, y_{p+q}$ is a basis for $(V^G)^\delta$ and

$$x_1, y_1, \ldots, x_p, y_p, y_{p+1}, y_{p+2}, \ldots, y_{p+q}$$

is a basis for V^G.

Let $g_1, \ldots, g_m$ be generators for G corresponding to linear forms $\lambda_1, \ldots, \lambda_m \in V^*$. (Recall that G is free abelian of rank m.) Consider the Lie algebra $\mathbf{g}$ of dimension $2l+t+m+1$ defined as follows. $\mathbf{g} = kw + \sum_{i=1}^{m} kg_i + V$, i. e. $\mathbf{g}$ has a basis

$$w, g_1, \ldots, g_m, y_1, \ldots, y_{p+q}; y_{p+q+1}, \ldots, y_l, x_1, \ldots, x_l, s_1, \ldots, s_t,$$

and relations $[x_i, y_j] = \Delta_{ij}\,w$,

$$[g_i, v] = \lambda_i(v)\,g_i \qquad \text{for} \quad v \in V,$$

and all other commutators of basis elements are zero.

$w, g_1, \ldots, g_m, y_1, \ldots, y_{p+q}$ span an abelian ideal $\mathbf{h}$ of $\mathbf{g}$ and $\mathbf{g}/\mathbf{h}$ is abelian. $w \in \text{Centre } \mathbf{g}$ and if E denotes the subsemi-group of U generated by $g_1, \ldots, g_m$ then

$$(U/(w-1))_E \cong \mathscr{A},$$

by Theorem 4.3 and the simplicity of $\mathscr{A}$. (The Heisenberg algebra of dimension $2l+1$ is the subalgebra of $\mathbf{g}$ spanned by x_i, y_i and w, $1 \leqq i \leqq l$ and it is easy to see that $\mathbf{g}$ is a subalgebra of the Lie algebra which is the direct sum of the Heisenberg algebra of dimension $2l+1$ and m copies of the two dimensional solvable Lie algebra. Compare Theorem 4.9 below.)

(ii) In $\mathcal{A}$, set $k\,S = k[y_1, \ldots, y_{p+q}]$, $k\,G = k[g_1, g_1^{-1}, \ldots, g_m, g_m^{-1}]$ and let $\mathbf{a}$ be the space spanned by

$$x_1, x_2, \ldots, x_p; x_{p+1}, \ldots, x_{p+q}, x_{p+q+1}, \ldots, x_l, y_{p+q+1}, \ldots, y_l, s_1, \ldots, s_t.$$

Then $\mathbf{a} \subset \mathbf{c}$ and $x_1, \ldots, x_p$ spans $\mathbf{a} \cap \mathbf{c}_1$ and

$$x_{p+q+1}, \ldots, x_l, y_{p+q+1}, \ldots, y_l, s_1, \ldots, s_t$$

spans $\mathbf{a}^G$. The 2-cocycle $\sigma \in C^2(\mathbf{a}, k)$ is the restriction of δ to $\mathbf{a}$. ■

In the next section we show that the integers p, q, $2l+t$ and m = rank $\mathbf{G}$ are isomorphism invariants for $\mathcal{A} = \mathcal{A}(V, \delta, \mathbf{G})$. $\mathcal{A}$ may be presented as $(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a})$ and conversely, (Theorems 4.2 and 4.8) and these integers appear in the two different presentations as follows :

$$q = \dim(V^G)^\delta, \qquad p = \frac{1}{2}(\dim V^G - \dim(V^G)^\delta), \qquad 2l+t = \dim V \quad (\text{and } 2l = \operatorname{rank}\delta);$$

Krull dimension of $k\,S$ is $p+q$,

$$\dim(\mathbf{a} \cap \mathbf{c}_1) = p, \qquad \dim \mathbf{a} = 2l+t-p-q,$$

$$\dim \mathbf{a}^G = 2l+t-2p-2q \quad [\text{and if } \sigma \in C^2(\mathbf{a}, k) \text{ then } 2l-2p-2q = \operatorname{rank}(\sigma \,|\, \mathbf{a}^G)].$$

THEOREM 4.9. — *Let* $\mathcal{A} = \mathcal{A}(V, \delta, \mathbf{G})$ *be a simple algebra. If* rank $\delta = 2l$ *and* rank $\mathbf{G} = m$ *then* $\mathcal{A}$ *is a subalgebra of* $A_l \otimes A'_m$.

Proof. — Choose a basis $x_1, y_1, \ldots, x_l, y_l, s_1, \ldots, s_t$ for V adapted to δ. Let $g_1, \ldots, g_m$ be generators for G and suppose that

$$[g_j, x_i] = \lambda_{ji} g_i, \qquad [g_j, y_i] = \mu_{ji} g_j \qquad \text{and} \qquad [g_j, s_i] = \nu_{ji} g_j.$$

Let

$$A_l = k[X_1, Y_1, \ldots, X_l, Y_l] \qquad \text{with } [X_i, Y_j] = \Delta_{ij}$$

and

$$A'_m = k[U_1, U_1^{-1}, V_1, \ldots, U_m, U_m^{-1}, V_m] \qquad \text{with } [U_i, V_j] = \Delta_{ij} U_i.$$

Consider the vector space map,

$$V \to \sum_{i,j} k X_i + k Y_i + k V_j \qquad (i = 1, \ldots, l, j = 1, \ldots, m)$$

given by

$$x_i \to X_i + \sum_{j=1}^{m} \lambda_{ji} V_j,$$

$$y_i \to Y_i + \sum_{j=1}^{m} \mu_{ji} V_j,$$

$$s_i \to \sum_{j=1}^{m} \nu_{ji} V_j.$$

By the universal property of $\mathscr{A}$, Theorem 4.3, this map extends to an algebra homomorphism $\mathscr{A} \to A_l \otimes A'_m$ mapping $g_j \to U_j$ for $1 \leqq j \leqq m$. Since $\mathscr{A}$ is simple this homomorphism is a monomorphism. ■

The class of simple subalgebras of $A_l \otimes A'_m$, each of which is the image of a simple algebra $\mathscr{A}$ (V, δ, **G**) as in the proof of Theorem 4.9, may be described intrinsically as follows.

In $A_l \otimes A'_m$, let $W_1 = \sum_{i=1}^{m} k\, X_i + k\, Y_i$, $W_2 = \sum_{j=1}^{m} k\, V_j$ and $W = W_1 + W_2$. Let π_1 and π_2 be the canonical projections of W onto W_1 and W_2 respectively. If $w \in W_2$ then $[w, U_i] = \lambda_i(w)\, U_i$, where $\lambda_i \in W_2^*$. The map

$$w \to \operatorname{Diag}(\lambda_1(w), \ldots, \lambda_m(w))$$

is a linear transformation of W_2 into the space of diagonal matrices acting on $k\, U_1 + \ldots + k\, U_m$ and we identify W_2 with this space of diagonal matrices.

THEOREM 4.10. — *Let* V *be a subspace of* W. *Then the subalgebra of* $A_l \otimes A'_m$ *generated by* V *and the units of* $A_l \otimes A'_m$ *is a simple algebra if* V *satisfies*

(i) $\pi_1(V) = W_1$ *and*

(ii) *the algebraic hull of* $\pi_2(V) = W_2$, [*where* $\pi_2(V)$ *and* W_2 are considered as spaces of diagonal matrices].

Proof. — By [4], Remark before Theorem 5.3, the algebraic hull of $\pi_2(V)$ is W_2 if and only if $\lambda_1 \mid \pi_2(V), \ldots, \lambda_m \mid \pi_2(V)$ are linearly independent over the rational field **Q**. If g is a unit in $A_l \otimes A'_m$ then g is a monomial in $U_1, U_1^{-1}, \ldots, U_m, U_m^{-1}$. Let $\lambda_g \in W_2^*$ be defined by $[w, g] = \lambda_g(w)\, g$ for $w \in W_2$.

If g and h are units in $A_l \otimes A'_m$ with $g \neq h$ then $\lambda_1 \mid \pi_2(V), \ldots, \lambda_m \mid \pi_2(V)$ are linearly independent over **Q** if and only if $\lambda_g \mid \pi_2(V) \neq \lambda_h \mid \pi_2(V)$. If the latter condition holds then the subalgebra generated by V and the units is of the form $\mathscr{A}$ (V, δ, **G**) and this is a simple algebra since $V^\delta \cap V^G = 0$. ■

REMARK 4.11. — $A_l \otimes A'_m \cong \operatorname{Diff}(k\, S \otimes k\, G) \subset \operatorname{End}(k\, S \otimes k\, G)$, where

$$k\, S = k[Y_1, \ldots, Y_l] \quad \text{and} \quad k\, G = k[U_1, U_1^{-1}, \ldots, U_m, U_m^{-1}].$$

The simple algebras of Theorem 4.10 are simple subalgebras of Diff $(k\, S \otimes k\, G)$ which do not necessarily contain the multiplications by elements of $k\, S \otimes k\, G$. Is there a sense in which they are " dense subalgebras " of Diff $k\, S \otimes k\, G$?

5. Isomorphism Theorems

We now consider when two *simple* algebras of the form $\mathscr{A} = \mathscr{A}$ (V, δ, **G**) are isomorphic. If **G** $= 0$ then $\mathscr{A} \cong A_n$, where $2n = \dim V$ and $A_n \cong A_m$ if and only if $n = m$, by [6] (2.6, Proposition). If **G** $\neq 0$ then, unlike A_n, $\mathscr{A}$ (V, δ, **G**) has units which do not belong to k. So it is sufficient to consider the case when **G** $\neq 0$ and we examine how $\mathscr{A}$ (V, δ, **G**) is built around its group of units.

LEMMA 5.1. – *Consider* $\mathcal{A} = \mathcal{A}(\mathrm{V}, \delta, \mathbf{G})$, *where* $\mathbf{G} \neq 0$ *and let* V′ *be a subspace of* V *such that* $\mathrm{V} = \mathrm{V}^{\mathrm{G}} \oplus \mathrm{V}'$.

(i) C, *the centraliser in* $\mathcal{A}$ *of the group of units of* $\mathcal{A}$, *is* $\mathrm{U}_\delta(\mathrm{V}^{\mathrm{G}}) \otimes k\,\mathrm{G}$.

(ii) *Let* L *be the Lie subalgebra of* $\mathcal{A}$ *defined by* $r \in \mathrm{L} \Leftrightarrow [r, g] \in kg$ *for each unit* $g \in \mathcal{A}$. *Then* $\mathrm{L} = \mathrm{C}+\mathrm{V}'$.

(iii) D, *the centre of* C, *is* $\mathrm{U}((\mathrm{V}^{\mathrm{G}})^{\delta}) \otimes k\,\mathrm{G}$.

(iv) *The first two terms of the upper central series of* D *as an* L*-module, are* k *and* $k+(\mathrm{V}^{\mathrm{G}})^{\delta}$.

Proof. – (i) Clearly $\mathrm{U}_\delta(\mathrm{V}^{\mathrm{G}}) \otimes k\,\mathrm{G} \subset \mathrm{C}$ since the units of $\mathcal{A}$ are just the scalar multiples of the elements of G. Now $\mathcal{A} \equiv \mathrm{U}_\delta(\mathrm{V}) \# k\,\mathrm{G} \cong \mathrm{U}_\delta(\mathrm{V}^{\mathrm{G}}) \otimes \mathrm{U}_\delta(\mathrm{V}') \otimes k\,\mathrm{G}$, as k-modules but not as algebras. Since $\mathbf{G} \mid \mathrm{V}'$ spans $(\mathrm{V}')^*$, $\mathrm{C} \cap \mathrm{U}_\delta(\mathrm{V}') = k$ by Lemma 4.4. Thus, by a unique representation of elements argument,

$$\mathrm{C} = \mathrm{U}_\delta(\mathrm{V}^{\mathrm{G}}) \otimes k \otimes k\,\mathrm{G} = \mathrm{U}_\delta(\mathrm{V}^{\mathrm{G}}) \otimes k\,\mathrm{G}.$$

(ii) Choose $g_1, \ldots, g_s \in \mathrm{G}$, where g_i corresponds to $\lambda_i \in \mathrm{V}^*$ such that the corresponding elements of $(\mathrm{V}')^*$, $\lambda_1 \mid \mathrm{V}', \ldots, \lambda_s \mid \mathrm{V}'$, are a basis for $(\mathrm{V}')^*$. Let $v_1, \ldots, v_s$ be a dual basis for V′ with $\lambda_i(v_j) = \Delta_{ij}$. Then $[g_i, v_j] = \Delta_{ij}\, g_i$. We show first that

$$\{r \in \mathrm{A} : [g, r] \in \mathrm{C}\} = \mathrm{C}+\mathrm{C}v_1+\ldots+\mathrm{C}v_s.$$

This follows from

$$[g_l, c v_1^h \ldots v_l^i \ldots v_s^j] = i c g_l v_1^h \ldots v_l^{i-1} \ldots v_s^j$$

modulo elements of lower degree, for $1 \leqq l \leqq s$. Also $[g_l, c_1 v_1+\ldots+c_s v_s] = c_l\, g_l$ which belongs to kg_l if and only if $c_l \in k$. Thus $\mathrm{L} = \mathrm{C}+\mathrm{V}'$.

(iii) Let $\mathrm{V}_1 = (\mathrm{V}^{\mathrm{G}})^{\delta}$ and let V_2 be a subspace of V^{G} such that $\mathrm{V}^{\mathrm{G}} = \mathrm{V}_1 \oplus \mathrm{V}_2$. Then $\delta \mid \mathrm{V}_2$ is non singular and $\mathrm{U}_\delta(\mathrm{V}_2) \cong \mathrm{A}_p$, where $p = (1/2) \dim \mathrm{V}_2$. Thus

$$\mathrm{C} \cong \mathrm{A}_p \otimes \mathrm{U}(\mathrm{V}_1) \otimes k\,\mathrm{G}$$

and so the centre of C is $\mathrm{U}(\mathrm{V}_1) \otimes k\,\mathrm{G}$.

(iv) As an L-module, D is a direct sum of L-submodules, $\mathrm{D} = \sum_{g \in \mathrm{G}} \mathrm{U}(\mathrm{V}_1) \otimes kg$. If $g \neq 1_{\mathrm{G}}$ then no nonzero element of $\mathrm{U}(\mathrm{V}_1) \otimes kg$ is annihilated by ad L. [Compare the proof of (Theorem 4.6 (iv) $\Rightarrow$ (i)).] Hence the upper central series for D, as an L-module, is just the upper central series for $\mathrm{U}(\mathrm{V}_1)$ as an L-module and the assertion follows at once since $\mathrm{V}^{\mathrm{G}} \cap \mathrm{V}^{\delta} = 0$. ∎

LEMMA 5.2. – *Let* $\mathbf{a}$ *be a subspace of* $\mathbf{c}$ *which satisfies* (*Sim*) *and* $\sigma \in \mathrm{Z}^2(\mathbf{a}, k\,\mathrm{S} \otimes k\,\mathrm{G})$. *Let* S_1 *denote the subspace* $k+ky_1+\ldots+ky_n$ *of* $k\,\mathrm{S}$. *Then*

$$\mathrm{M} = \{r \in (k\,\mathrm{S} \otimes k\,\mathrm{G}) \#_\sigma \mathrm{U}(\mathbf{a}) : [s, r] \in k \textit{ for } s \in \mathrm{S}_1 \textit{ and } [g, r] \in kg \textit{ for } g \in \mathrm{G}\}$$

is the Lie algebra $(k\,\mathrm{S} \otimes k\,\mathrm{G}) \times_\sigma \mathbf{a}$ [*considered as a subspace of* $(k\,\mathrm{S} \otimes k\,\mathrm{G}) \#_\sigma \mathrm{U}(\mathbf{a})$].

Proof. – Let L be as in Lemma 5.1. Then

$$\mathrm{M} = \{r \in \mathrm{L} : [s, r] \in k \text{ for } s \in \mathrm{S}_1\} = (k\,\mathrm{S} \otimes k\,\mathrm{G}) \times_\sigma \mathbf{a}$$

by a similar argument to that used in Lemma 5.1 (ii). ∎

The following theorem should be compared with Theorem 3.1.

THEOREM 5.3. — *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be subspaces of* $\mathbf{c}$ *each of which satisfies (Sim). Le* $\sigma \in Z^2(\mathbf{a}, k\,S \otimes k\,G)$ *and* $\tau \in Z^2(\mathbf{b}, k\,S \otimes k\,G)$. *Then there exists an isomorphism* θ *of* $(k\,S \otimes k\,G) \#_{\sigma} U(\mathbf{a})$ *onto* $(k\,S \otimes k\,G) \#_{\tau} U(\mathbf{b})$ *which extends the identity map from* $k\,S \otimes k\,G$ *to* $k\,S \otimes k\,G$ *if and only if* $\mathbf{a} = \mathbf{b}$ *and* σ *and* τ *are cohomologous cocycles in* $Z^2(\mathbf{a}, k\,S \otimes k\,G)$.

Proof. — Suppose θ exists. By Lemma 5.2, θ induces a Lie algebra isomorphism

$$(k\,S \otimes k\,G) \times_{\sigma} \mathbf{a} \quad \rightarrow \quad (k\,S \otimes k\,G) \times_{\tau} \mathbf{b}$$

which is the identity map on $k\,S \otimes k\,G$. For $\xi \in k\,S \otimes k\,G$ and $a \in \mathbf{a}$, $[a, \xi] = [\theta(a), \xi]$. Thus, viewed as elements of $\mathrm{Der}(k\,S \otimes k\,G)$, $a = \theta(a)$. Since $\mathbf{b}$ and $\mathbf{a}$ are subspaces of $\mathbf{c}$, this implies that $\mathbf{b} = \mathbf{a}$ and that θ induces the identity map from $\mathbf{a}$ to $\mathbf{a}$ in the following diagram of Lie algebras, where the horizontal map are the obvious inclusions and projections :

$$\begin{tikzcd}
0 \arrow[r] & k\,S \otimes k\,G \arrow[r] \arrow[d, "1"] & (k\,S \otimes k\,G) \times_{\sigma} \mathbf{a} \arrow[r] \arrow[d, "\theta"] & \mathbf{a} \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & k\,S \otimes k\,G \arrow[r] & (k\,S \otimes k\,G) \times_{\tau} \mathbf{a} \arrow[r] & \mathbf{a} = 0.
\end{tikzcd}$$

Thus σ and τ are cohomologous cocycles. ∎

DEFINITION 5.4. — Let $\mathscr{A}(V, \delta, \mathbf{G})$ and $\mathscr{A}(W, \gamma, \mathbf{H})$ be simple algebras. These algebras are said to be *locally isomorphic* if there exists a vector space isomorphism $\varphi : V \rightarrow W$ such that

(i) $\delta(v_1, v_2) = \gamma(\varphi(v_1), \varphi(v_2))$, for all $v_1, v_2 \in V$ and

(ii) the dual map $\varphi^* : W^* \rightarrow V^*$ restricts to an isomorphism of $\mathbf{H}$ onto $\mathbf{G}$.

THEOREM 5.5. — *If* $\mathscr{A}(V, \delta, \mathbf{G})$ *and* $\mathscr{A}(W, \gamma, \mathbf{H})$ *are simple algebras which are locally isomorphic with respect to* $\varphi : V \rightarrow W$ *then there exists an algebra isomorphism* $\psi : \mathscr{A}(V, \delta, \mathbf{G}) \rightarrow \mathscr{A}(W, \gamma, \mathbf{H})$ *which extends* φ.

Proof. — For $\lambda \in \mathbf{G}$ let $g(\lambda)$ denote the corresponding element of G and similarly for H.

Set $\psi(g(\lambda)) = h((\varphi^*)^{-1}(\lambda))$ and $\psi(v) = \varphi(v)$, $\lambda \in \mathbf{G}$, $v \in V$. Then by the universal property (Theorem 4.3) ψ extends to an algebra homomorphism of $\mathscr{A}(V, \delta, \mathbf{G})$ onto $\mathscr{A}(W, \gamma, \mathbf{H})$ and ψ is injective since $\mathscr{A}(V, \delta, \mathbf{G})$ is a simple algebra. ∎

CONJECTURE 5.6. — If $\mathscr{A}(V, \delta, \mathbf{G})$ and $\mathscr{A}(W, \gamma, \mathbf{H})$ are simple algebras which are isomorphic (as algebras) then they are locally isomorphic.

We will prove this conjecture in the case when $V^G = (V^G)^{\delta}$, [i. e. when the centraliser in $\mathscr{A}(V, \delta, \mathbf{G})$ of its group of units is a commutative algebra] and go some way towards answering it in the general case.

Let $\mathscr{A} = \mathscr{A}(V, \delta, \mathbf{G})$, where $\mathbf{G} \neq 0$. By Lemma 5.1, $C = U_{\delta}(V^G) \otimes k\,G$, $L = U_{\delta}(V^G) \otimes k\,G + V'$, $D = U((V^G)^{\delta}) \otimes k\,G$ and $k + (V^G)^{\delta}$ are subsets of $\mathscr{A}$ which are independent of the presentation of $\mathscr{A}$ as $\mathscr{A}(V, \delta, \mathbf{G})$. Denote $\dim (V^G)^\delta$ by q and then $\dim V^G = 2p+q$ for some integer $p \geqq 0$. So $C \cong A_p \otimes D$. If I is an ideal of $A_p \otimes D$ which is generated by an ideal of codimension one of D then $A_p \cong C/I$ and the Krull dimension of C/I is p by [8]. The subalgebra of $\mathscr{A}$ generated by the units is $k\,G$ and the Krull dimension of $k\,G$ is $m = \operatorname{rank} G$. Also $\dim L/C = \dim V'$ is uniquely determined by $\mathscr{A}$ so we have four integer valued isomorphism invariants for $\mathscr{A}$, viz p, q, m and $\dim V = \dim V' + 2p+q$.

Suppose now that χ is an isomorphism of $\mathscr{A}(V, \delta, \mathbf{G})$ onto $\mathscr{A}(W, \gamma, \mathbf{H})$. We endeavour to define a vector space isomorphism $\varphi : V \to W$ which satisfies the conditions stated in the conjecture. Let $V_1 = (V^G)^\delta$ and V_2 be a subspace of V^G such that $V^G = V_2 \oplus V_1$ and define W_1 and W_2 similarly. The essential difficulty is that the only information that we have on $\chi(U_\delta(V_2))$ is that $\chi(C_V) = C_W$, i. e. :

$$\chi(U_\delta(V_2)) \otimes (U(W_1) \otimes k\,H) = U_\gamma(W_2) \otimes (U(W_1) \otimes k\,H).$$

By Lemma 5.1, $\chi(V_1) \subset k + W_1$. Let π be the projection of $k+W_1$ onto W_1 with $\operatorname{Ker} \pi = k$ and define φ on V_1 by $\varphi = \pi\chi$. Extend φ to V^G by $\varphi(V_2) = W_2$ and φ is compatible with $\delta \mid V_2$ and $\gamma \mid W_2$. Let V^C be the δ-orthogonal complement of V_2 in V. Then $V = V_2 \oplus V^C$. Since no element of V_1 is orthogonal to V^C, we may choose a subspace V_3 of V^C such that $V_1 \cap V_3 = 0$, $\delta \mid V_3 = 0$ and δ is non singular on $V_1 \oplus V_3$. Finally let V_4 be the δ-orthogonal complement of $V_1+V_2+V_3$ in V. So

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4.$$

Thus V_3+V_4 is a complementary subspace V' to $V^G = V_1+V_2$ as in Lemma 5.1. Let L_V, L_W, C_V, C_W be as in Lemma 5.1. $\chi(V_3)$ is a subspace of L_W such that $C_W \cap \chi(V_3) = 0$. Let W' be the subspace of W such that

$$W' + C_W/C_W = \chi(V_3) + C_W/C_W.$$

For $v \in V_3$, choose $w' \in W'$ such that $\chi(v) + C_W = w' + C_W$. Then, for

$$v_1 \in V_1, \qquad \delta(v_1, v) = [\chi(v_1), \chi(v)] = [\chi(v_1), w'] = \gamma(\varphi(v_1), w').$$

Hence $\gamma \mid W'$ is non singular. Hence there exists $W_3 \subset W'$ such that

$$W' = W_1 \oplus W_2 \oplus W_3,$$

W_3 is γ-orthogonal to W_2 and $\gamma \mid W_3 = 0$. We now extend the domain of φ from V_1+V_2 to $V_1+V_2+V_3$ by, for $v_3 \in V_3$, let $w_3 \in W_3$ be the unique element such that $\chi(v_3) + C_W = w_3 + C_W$ and define $\varphi(v_3) = w_3$. So $\varphi : V_1+V_2+V_3 \to W_1+W_2+W_3$ and φ is compatible with δ and γ by construction.

Let W_4 be the γ-orthogonal complement of $W_1+W_2+W_3$ in W. Define

$$\varphi : \quad V_4 \to W_4 \qquad \text{by} \quad \varphi(v_4) = w_4 \quad \text{where } \chi(v_4) + C_W = w_4 + C_W.$$

We now show that $\varphi : V \to W$ *satisfies the second condition in Conjecture* 5.6. Let $\lambda \in \mathbf{G}$ and g_λ be the corresponding element of G. Then $\chi(g_\lambda) = \alpha h_\mu$ where $\alpha \in k$ and $h_\mu \in H$

corresponds to $\mu \in \mathbf{H}$. Then, by the construction, we have that for all $v \in \mathrm{V}$, $\lambda(v) g_\lambda = [g_\lambda, v]$, so after applying χ we have

$$\lambda(v) \alpha h_\mu = [\alpha h_\mu, \chi(v)] = [\alpha h_\mu, \varphi(v)] = \mu(\varphi(v)) \alpha h_\mu,$$

since $\chi(v) + \mathrm{C_W} = \varphi(v) + \mathrm{C_W}$. Thus $\lambda(v) = \mu(\varphi(v))$ for all $v \in \mathrm{V}$, i. e. $\lambda = \mu\varphi$ or $\varphi^* \mu = \lambda$. Thus $\varphi^* : \mathrm{W}^* \to \mathrm{V}^*$ induces an isomorphism $\varphi^* : \mathbf{H} \to \mathbf{G}$. *However we are not able to prove that* $\varphi \mid \mathrm{V}_4$ *is compatible with* $\delta \mid \mathrm{V}_4$ *and* $\gamma \mid \mathrm{W}_4$. This difficulty becomes clearer when we write $\mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$ in the form $(k\mathrm{S} \otimes k\mathrm{G}) \#_\sigma \mathrm{U}(\mathbf{a})$ as in Theorem 4.8 (ii) with $\mathrm{V}_2 = \sum_{i=1}^{p} kx_i + ky_i$. Then

$$\mathbf{a} = \sum_{i=1}^{p} kx_i + \mathrm{V}_3 + \mathrm{V}_4,$$

V_4 is $\mathbf{a}^\mathrm{G}$ and

$$\mathscr{A}(\mathrm{V}, \delta, \mathbf{G}) \cong (k\mathrm{S} \otimes k\mathrm{G}) \#_\sigma \mathrm{U}(\mathbf{a}),$$

where $\sigma = \delta \mid \mathbf{a}$ and so $\sigma \in \mathrm{C}^2(\mathbf{a}, k)$. Similarly, since there exists $\varphi : \mathrm{V} \to \mathrm{W}$ which satisfies condition (ii) of conjecture 5.6,

$$\mathscr{A}(\mathrm{W}, \gamma, \mathbf{H}) \cong (k\mathrm{S} \otimes k\mathrm{G}) \#_\tau \mathrm{U}(\mathbf{a}),$$

where $\tau = \gamma \mid \mathbf{a}$. Thus if $\mathscr{A}(\mathrm{V}, \delta, \mathbf{G})$ and $\mathscr{A}(\mathrm{W}, \gamma, \mathbf{H})$ are isomorphic then they are cocycle twisted variants of the same "untwisted algebra" $(k\mathrm{S} \otimes k\mathrm{G}) \# \mathrm{U}(\mathbf{a})$, (but σ and τ need not be cohomologous). The next theorem shows that the "untwisted algebra" is independent of the presentation.

THEOREM 5.7. — *Let* $\mathbf{a}$ *and* $\mathbf{a}'$ *be subspaces of* $\mathbf{c}$ *and* $\mathbf{c}'$ *respectively each of which satisfies* (*Sim*) *and* $\sigma \in \mathrm{Z}^2(\mathbf{a}, k\mathrm{S} \otimes k\mathrm{G})$ *and* $\tau \in \mathrm{Z}^2(\mathbf{a}', k\mathrm{S}' \otimes k\mathrm{G}')$.

(i) *Consider the following diagram*

$$\begin{array}{ccccccccc} 0 & \to & k\mathrm{S} \otimes k\mathrm{G} & \to & (k\mathrm{S} \otimes k\mathrm{G}) \times_\sigma \mathbf{a} & \to & \mathbf{a} & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \\ 0 & \to & k\mathrm{S}' \otimes k\mathrm{G}' & \to & (k\mathrm{S}' \otimes k\mathrm{G}') \times_\tau \mathbf{a}' & \to & \mathbf{a}' & \to & 0 \end{array}$$

If there exist an algebra isomorphism α *and Lie algebra isomorphisms* β, η *such that this diagram is commutative,* (*where the horizontal map are the canonical inclusions and projections*), *then there exists an algebra isomorphism*

$$\beta' : \ (k\mathrm{S} \otimes k\mathrm{G}) \#_\sigma \mathrm{U}(\mathbf{a}) \to (k\mathrm{S}' \otimes k\mathrm{G}') \#_\tau \mathrm{U}(\mathbf{a}')$$

which extends β.

(ii) *If there is an algebra isomorphism*

$$\theta : \ (k\mathrm{S} \otimes k\mathrm{G}) \#_\sigma \mathrm{U}(\mathbf{a}) \to (k\mathrm{S}' \otimes k\mathrm{G}') \#_\tau \mathrm{U}(\mathbf{a}')$$

then there exists an algebra isomorphism α *and Lie algebra isomorphisms* β,

η *such that*

$$\begin{array}{ccccccccc} 0 \to & kS \otimes kG & \to & (kS \otimes kG) \times \mathbf{a} & \to & \mathbf{a} & \to 0 \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & \\ 0 \to & kS' \otimes kG' & \to & (kS' \otimes kG') \times \mathbf{a}' & \to & \mathbf{a}' & \to 0 \end{array}$$

is commutative and hence there is an isomorphism

$$\beta' : \quad (kS \otimes kG) \# U(\mathbf{a}) \to (kS' \otimes kG') \# U(\mathbf{a}')$$

which extends β.

Proof. — (i) follows immediately from the universal property of $(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a})$, *see* Theorem 2.1 and the remark which follows it. Note also that $\alpha \mid k\,S$ is a filtered algebra isomorphism from $k\,S$ to $k\,S'$ since the terms of the usual filtration of $k\,S$ are precisely the terms of the upper central series of $k\,S \otimes k\,G$ as a $(k\,S \otimes k\,G) \times_\sigma \mathbf{a}$-module [compare Lemma 5.1 (iv)].

(ii) Denote G′ by H where necessary. Consider $(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a})$. Let $\mathbf{b}$ be a subspace of $\mathbf{a}$ such that $\mathbf{a} = \mathbf{b} \oplus \mathbf{a}^G$. Without loss of generality we may suppose that $\sigma \in C^2(\mathbf{a}, k)$ and that $\mathbf{b}$ is σ-orthogonal to $\mathbf{a}$. Similarly, let $\mathbf{a}' = \mathbf{b}' \oplus (\mathbf{a}')^H$ and suppose that $\tau \in C^2(\mathbf{a}', k)$ and $\mathbf{b}'$ is τ-orthogonal to $\mathbf{a}'$. Let S and S′ be the spaces of homogenous elements of degree one of $k\,S$ and $k\,S'$ respectively. $(k\,S \otimes k\,G) \#_\sigma U(\mathbf{a}) \cong \mathscr{A}(V, \delta, \mathbf{G})$, where $V = S \oplus \mathbf{a}$ and δ is defined by $\delta(a, s) = [a, s]$, $a \in \mathbf{a}$ and $s \in S$, $\delta \mid S = 0$ and $\delta \mid \mathbf{a} = \sigma$. Similarly

$$(kS' \otimes kG') \neq_\tau U(\mathbf{a}') \cong \mathscr{A}(W, \gamma, \mathbf{H}),$$

where $W = S' \oplus \mathbf{a}$. Define $\delta_1 \in C^2(V, k)$ by requiring that δ_1 coincides with δ on $S \oplus \mathbf{b}$ and $\mathbf{a}^G$ is δ_1-orthogonal to V. Similarly define $\gamma_1 \in C^2(W, k)$ by requiring that γ_1 coincides with γ on $S' \oplus \mathbf{b}'$ and $(\mathbf{a}')^H$ is γ_1-orthogonal to W. Then

$$(kS \otimes kG) \# U(\mathbf{a}) \cong \mathscr{A}(V, \delta_1, \mathbf{G}) \qquad \text{and} \qquad (kS' \otimes kG') \# U(\mathbf{a}') \cong \mathscr{A}(W, \gamma_1, \mathbf{H}).$$

Since $\mathscr{A}(V, \delta, \mathbf{G}) \cong \mathscr{A}(W, \gamma, \mathbf{H})$, there is a vector space isomorphism $\varphi : V \to W$ which satisfies (ii) of Conjecture 5.6 and φ is compatible with δ and γ except possibly from $V_4 = \mathbf{a}^G$ to $W_4 = (\mathbf{a}')^H$. Thus φ satisfies both (i) and (ii) of Conjecture 5.6 with respect to δ_1 and γ_1. Recall that $\varphi : V_1 \to W_1$ and also that $\varphi : V_2 \to W_2$ may be chosen arbitrarily subject to being compatible with δ and γ. Thus we may suppose that φ restricts to a vector space isomorphism from S to S′. Since $V = S \oplus \mathbf{a}$ and $W = S' \oplus \mathbf{a}'$, φ induces a vector space isomorphism $\xi : \mathbf{a} \to \mathbf{a}'$ by $\xi(a) + S' = \varphi(a) + S'$, $a \in \mathbf{a}$. The required maps α, β and η are now defined by $\alpha \mid S = \varphi$, $\alpha \mid k\,G$ is induced by $(\varphi^*)^{-1}$, β is the extension of α defined by $\beta(a) = \xi(a)$, $a \in \mathbf{a}$, and $\eta = \xi$. ■

Theorem 5.7 is unsatisfactory since one would like to prove that if there is an algebra isomorphism θ as in 5.7 (ii), then there exist α, β and η which make the diagram in 5.7 (i) commutative. However, this is equivalent to an affirmative answer to Conjecture 5.6.

EXAMPLE 5.8. — Let rank $G = 2$ and

$$kG = k[g, g^{-1}, h, h^{-1}] \quad \text{and} \quad \mathbf{a} = kg\,\partial/\partial g + kh\,\partial/\partial h.$$

Set $a_1 = g\,\partial/\partial g$ and $a_2 = h\,\partial/\partial h$. Then

$$H^2(\mathbf{a}, kG) \cong H^2(\mathbf{a}, k) = C^2(\mathbf{a}, k),$$

by Theorem 3.2.

$$C^2(\mathbf{a}, k) \cong k \quad \text{via} \quad \sigma \mapsto \sigma(a_1, a_2), \quad \sigma \in C^2(\mathbf{a}, k).$$

Define $\tau \in C^2(\mathbf{a}, k)$ by $\tau(a_1, a_2) = 1$. It is easy to see that $kG \#_\sigma U(\mathbf{a}) \cong kG \#_\tau U(\mathbf{a})$ if $\sigma \neq 0$. By [4], Example 5.8 or by Theorem 5.9 below, $kG \#_\tau U(\mathbf{a})$ is not isomorphic to $kG \# U(\mathbf{a})$ and so the 2-cocycle twisted variants of $kG \# U(\mathbf{a})$ fall into exactly two isomorphism classes.

THEOREM 5.9. — *If* $\mathscr{A}(V, \delta, \mathbf{G})$ *and* $\mathscr{A}(W, \gamma, \mathbf{H})$ *are simple algebras which are isomorphic and* $\delta \mid V^G = 0$, *or equivalently the centraliser of the group of units of* $\mathscr{A}(V, \delta, \mathbf{G})$ *is a commutative algebra, then* $\mathscr{A}(V, \delta, \mathbf{G})$ *and* $\mathscr{A}(W, \gamma, \mathbf{H})$ *are locally isomorphic.*

Proof. — Let χ be an isomorphism. It is sufficient to show that the map $\varphi : V \to W$ which was constructed earlier is compatible with δ and γ. Let $v, v' \in V_4$.

Then $\chi(v) = \varphi(v) + r$, $\chi(v') = \varphi(v') + r'$ for some $r, r' \in C_W$. Now $\delta(v, v') = [v, v']$, so, after applying χ, we have

$$\begin{aligned}\delta(v, v') &= [\chi(v), \chi(v')] \\ &= [\varphi(v) + r, \varphi(v') + r'] \\ &= [\varphi(v), \varphi(v')] + [r, \varphi(v')] + [\varphi(v), r'],\end{aligned}$$

since C_W is commutative.

Now

$$[\varphi(v), \varphi(v')] = \gamma(\varphi(v), \varphi(v')) \in k$$

and

$$[r, \varphi(v')] + [\varphi(v), r'] \in \sum_{g \neq 1} U(V_1) \otimes kg.$$

So $[r, \varphi(v')] + [\varphi(v), r'] = 0$ and φ satisfies the required property. ∎

EXAMPLE 5.10. — We now consider the algebras $\mathscr{A}(V, 0, \mathbf{G})$ in the special case when $\dim V = 1$ and rank $\mathbf{G} = 2$. Let v be a basis of V. If $\lambda_1, \lambda_2 \in V^*$ then λ_1 and λ_2 are linearly independent over $\mathbf{Q}$ if and only if $\lambda_1(v)$ and $\lambda_2(v)$ are linearly independent over $\mathbf{Q}$. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in V^*$ have the property that λ_1 and λ_2 (respectively μ_1 and μ_2) are linearly independent over $\mathbf{Q}$ and denote the subgroup of V^* generated by $\{\lambda_1, \lambda_2\}$ and $\{\mu_1, \mu_2\}$ by G_1 and G_2 respectively. By Theorem 5.9, $\mathscr{A}(V, 0, G_1) \cong \mathscr{A}(V, 0, G_2)$ if and only if there exists $0 \neq \rho \in k$ and a 2×2 unimodular matrix $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ with integer coefficients such that

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \rho \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

Thus if k is $\mathbf{R}$ or $\mathbf{C}$ then there are infinitely many non isomorphic simple algebras $\mathscr{A}(V, 0, \mathbf{G})$ with $\dim V = 1$ and rank $\mathbf{G} = 2$.

REMARK 5.11. — If Conjecture 5.6 is false then there are simple algebras $\mathscr{A}(V, \delta, \mathbf{G})$ and $\mathscr{A}(W, \gamma, \mathbf{H})$ which are isomorphic but not locally isomorphic. Let

$$V = V_1 + V_2 + V_3 + V_4$$

be a decomposition of V as considered earlier with $\dim V_2 = 2p > 0$. Set $V' = V_1 + V_3 + V_4$. Then

$$\begin{aligned} \mathscr{A}(V, \delta, \mathbf{G}) &\cong U_\delta(V_2) \otimes \mathscr{A}(V', \delta, \mathbf{G} \mid V') \\ &\cong A_p \otimes \mathscr{A}(V', \delta, \mathbf{G} \mid V'). \end{aligned}$$

By a similar decomposition,

$$\mathscr{A}(W, \gamma, \mathbf{H}) \cong A_p \otimes \mathscr{A}(W', \gamma, \mathbf{H} \mid W').$$

Now $\mathscr{A}(V', \delta, \mathbf{G} \mid V')$ and $\mathscr{A}(W', \gamma, \mathbf{H} \mid W')$ cannot be locally isomorphic by the hypothesis, and so are not isomorphic by Theorem 5.9. But

$$A_p \otimes \mathscr{A}(V', \delta, \mathbf{G} \mid V') \cong A_p \otimes \mathscr{A}(W', \gamma, \mathbf{H} \mid W')\,!$$

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(Manuscrit reçu le 25 octobre 1973.)

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