ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 6, nº 1 (1973), p. 95-101
<http://www.numdam.org/item?id=ASENS_1973_4_6_1_95_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 6, 1973, p. 95 à 101.

STABLE VECTOR BUNDLES AND THE FROBENIUS MORPHISM

By David GIESEKER

1. Let X be a curve of genus g, proper and smooth over an algebraically closed field, and let E be a vector bundle over X. Mumford defines E to be semi-stable if whenever F is a quotient bundle of E, then

$$\frac{\deg \mathrm{F}}{\operatorname{rank} \mathrm{F}} \geq \frac{\deg \mathrm{E}}{\operatorname{rank} \mathrm{E}},$$

where deg E is the degree of the line bundle $\overset{r}{\wedge} \mathrm{E}$, r the rank of E. If the characteristic of X is $p > 0$, $\mathrm{E}^{(p)}$ will denote Frobenius pullback of E.

Theorem 1. — *For each prime p and integer* $g > 1$, *there is a curve* X *of genus g in characteristic p and a semi-stable bundle* E *of rank two on* X *so that* $\mathrm{E}^{(p)}$ *is not semi-stable.*

Examples of non-ample semi-stable bundles of positive degree constructed by Serre for $p = g = 3$ and later by Tango for $p(p-1) = 2g$ incidentally proved Theorem 1 when $p(p-1) = 2g$.

We prove Theorem 1 by constructing a sequence of bundles E_n so that $\mathrm{E}_n^{(p)}$ is isomorphic to E_{n-1}, and E_1 is not semi-stable. In such a sequence, we must have E_n semi-stable for $n \gg 0$, and then we obtain the E of Theorem 1 as the first semi-stable E_n.

The bundles E_n will be constructed in the following situation : Let $\mathrm{A} = k[[t]]$, where k is a field of characteristic $p > 0$, and let X be a stable curve over A with k-split degenerate fiber in the sense of Mumford [5]. Thus by definition X is proper and flat over A, and its geometric fibers are reduced, connected and one dimensional. Further, all the normalizations of the components of the special fiber X_0 of X are isomorphic to $\mathbf{P}_k^1$, and the singularities of X_0 are double points with 2 k-rational branches. Further each component X_0 meets at least three other components counting itself. We also assume the generic fiber is smooth over K, the quotient field of A.

Let Y_0 be the universal covering scheme of X_0, i. e. there is an etale map p_0 from Y_0 to X_0 with the usual universal mapping property. Y_0 is not of finite type over A. Mumford shows that the group G of covering transformations of Y_0 over X_0 is a free group on g generators, g the genus of X_K. G operates freely and discontinuously in the Zariski topology of Y_0.

Section two is devoted to associating to each representation ρ of G on K^m a sequence of bundles E_n on $X_K^{(p^n)}$ so that $F^* E_n$ is isomorphic to E_{n-1}, where $X^{(p^n)}$ is the fiber product of X with the n^{th} iterate of the Frobenius map on Spec K, and F is relative Frobenius. The construction of E_n from ρ is analogous to the construction of a bundle E′ on a smooth, compact complex variety X′ from a representation ρ' of the fundamental group of X′ on $\mathbf{C}^m$. Further, the sequence $\{ E_n \}$ defines a stratification on E_1, which is analogous to the stratification on E′ whose monodromy is ρ' [2].

Section three is devoted to the study of the bundle associated to a particular representation ρ of G on K^2 which arises in Mumford's work. We show that the E_1 associated to ρ is not semi-stable. This ρ is analogous to the following ρ' associated to a compact Riemann surface X′. Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be the usual generators of $\pi_1(X')$, and let U be an open subset of $\mathbf{P}^1_{\mathbf{C}}$, and let π be a covering map from U to X′. Assume that the group G of covering transformations acts on U by linear fractional transformations, and that G is freely generated by the images of $b_1, \ldots, b_g$. Such a π is called a Schottky uniformization. Thus we have a homomorphism from G to PGL $(2, \mathbf{C})$, and this lifts to a homomorphism ρ' of $\pi_1(X)$ to SL $(2, \mathbf{C})$. Following Gunning, one may show the bundle E associated to ρ' is an extension

$$0 \to L \to E \to L^{-1} \to 0$$

where $L^{\otimes 2}$ is isomorphic to $\Omega^1_{X/\mathbf{C}}$. In particular, E is not semi-stable. The representation ρ of G on K^2 is the rigid analytic analogue of ρ', and the bundle E_1 associated to ρ is an extension of the above type.

We conclude by noting that semi-stable bundles are not closed under symmetric product and with some examples of semi-stable bundles of positive degree which are not ample.

2. X will continue to denote a stable curve over A with smooth generic fiber and k-split degenerate fiber, Y_0 the universal covering space of X_0, and G the group of covering transformations of Y_0. There is a unique structure of a formal scheme Y with underlying space Y_0 and an etale

map p of Y to $\hat{X}$ which reduces to p_0, $\hat{X}$ being the completion of X along X_0.

DEFINITION. — Meromorphic descent data on a coherent sheaf F over Y is a collection of elements $h_g \in \Gamma\left(Y, \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(F, g^* f) \otimes_A K\right)$ for each $g \in G$ so that

$$h_g \circ g^* (h_{g'}) = h_{gg'}$$

and h_e is the identity. If $\{h_g\}$ and $\{k_g\}$ are sets of meromorphic descent data on F and G respectively, a map from $\{h_g\}$ to $\{k_g\}$ is an element $f \in \Gamma\left(Y, \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(F, G) \otimes_A K\right)$ so that

$$k_g \circ f = g^* (f) \circ h_g.$$

We will show the category of coherent sheaves on Y with meromorphic descent data is equivalent to the category of coherent sheaves on X_K.

LEMMA 1. — *Given meromorphic descent data on a coherent sheaf* F *on* Y, *there is a coherent* F′ *with descent data* $h'_g \in \mathrm{Hom}_{\mathcal{O}_Y}(F', g^* F')$ *so that* $\{h_g\}$ *and* $\{h'_g\}$ *are isomorphic.* F′ *may be taken to have no* A *torsion.*

Proof. — We may assume F has no A torsion by replacing it by its image in $F \otimes_A K$. We will construct a coherent subsheaf F′ of $F \otimes_A K$ so that the map of $F' \otimes_A K$ to $F \otimes_A K$ is an isomorphism and so that

$$h_g (F') = g^* F'$$

where we are regarding h_g as a map of $F \otimes_A K$ to $g^* (F \otimes_A K)$. Suppose such an F′ has been constructed over a G invariant open set U of Y, and let V be a quasi compact open not contained in U so that $V \cap g V \subseteq U$ if $g \neq e$. V exists, since G acts discontinuously and has no torsion.

F′ may be extended to a coherent subsheaf of $F \otimes_A K$ over $V \cup U$ using the following idea of Raynaud. On $V \cap U$, we may find an N so that

$$t^N F \subseteq F' \subseteq t^{-N} F,$$

where t is a uniformizing parameter of A. Let $\overline{F}'$ be the image of F′ in $t^{-N} F/t^N F$. $\overline{F}'$ is a coherent sheaf on a scheme whose sheaf of local rings is $\mathcal{O}_Y/t^{-2N} \mathcal{O}_Y$. Thus $\overline{F}'$ extends to a coherent subsheaf $\overline{F}''$ of $t^{-N} F/t^N F$ over $V \cup U$. The inverse image F″ of $\overline{F}''$ in $t^{-N} F$ extends F′. Finally, F′ may be extended to the G invariant open set consisting of the union of the translates of $V \cup U$ by taking the subsheaf of $F \otimes_A K$ generated by $h_g^{-1} (g^* F'')$ over $U \cap g^{-1} V$.

Given a coherent F on $\hat{X}$, the natural map

$$h_g^F : \quad p^* F \to g^* p^* F$$

gives meromorphic descent data on p^* F.

LEMMA 2. — *Let* $\{ h_g \}$ *be meromorphic descent data on a coherent* F. *There is a coherent* H *on* $\hat{X}$ *so that* $\{ h_g \}$ *is isomorphic to* $\{ h_g^H \}$. *Further the natural map* α,

$$\mathrm{Hom}_{\mathcal{O}_{\hat{X}}}(H, H') \otimes_A K \xrightarrow{\alpha} \mathrm{Hom}(\{ h_g^H \}, \{ h_g^{H'} \})$$

is an isomorphism.

Proof. — By Lemma 1, we may assume h_g maps F to g^* (F). There is a quasi-compact open U of Y so that the translates of U by G cover Y. $\{ h_g \}$ gives descent data for the morphism $U \to \hat{X}$ and so F descends to a coherent H on $\hat{X}$, and $\{ h_g \}$ is isomorphic to $\{ h_g^H \}$.

If $f \in \mathrm{Hom}(\{ h_g^H \}, \{ h_g^{H'} \})$, then $t^N f$ gives a morphism from p^* H to p^* H' compatible with descent data, and so a morphism from H to H'. Thus α is surjective. On the other hand, if $f \in \mathrm{Hom}_{\mathcal{O}_{\hat{X}}}(H, H')$ and if $\alpha(f) = 0$, then $t^N p^*(f) = 0$, and so $t^N f = 0$ for some integer N. So f is zero in $\mathrm{Hom}_{\mathcal{O}_{\hat{X}}}(H, H') \otimes_A K$, and α is injective.

PROPOSITION 1. — *There is an equivalence of categories* α *from the category of coherent sheaves on* X_K *to the category of coherent sheaves on* Y *with meromorphic descent data. If* F *is a coherent sheaf on* X, $\alpha(F_K)$ *is the descent data* $\{ h_g^{\hat{F}} \}$ *on* $p^*(\hat{F})$.

Proof. — Consider the category C whose objects are coherent sheaves on X, with $\mathrm{Hom}'(F, G) = \mathrm{Hom}_{\mathcal{O}_X}(F, G) \otimes_A K$. C maps isomorphically to the category of coherent sheaves on X_K. On the other hand, Grothendieck's existence theorem and lemma 2 show that it maps isomorphically to the category of coherent sheaves with descent data on Y.

Any representation ρ of G on K^n gives meromorphic descent data $\{ h_g \}$ on $\mathcal{O}_Y^n$, and so a bundle $F_{\rho, X} = \alpha^{-1} \{ h_g \}$. When char $k = p > 0$, we let $F : X \to X^{(p)}$ be the relative Frobenius morphism. $X^{(p)}$ is a stable curve with k split degenerate fiber, and the fundamental group of its special fiber is G. Further we have

$$F_{\rho, X} = F^*(F_{\rho, X^{(p)}}).$$

Thus we have proven :

PROPOSITION 2. — *If* ρ *is a representation of* G *on* K^n *and* F_1 *is pullback of* F_ρ *to* $X_{\overline{K}}$, *then there is a sequence of bundles* $F_1, F_2, \ldots$ *so that* $F_{k+1}^{(p)}$ *is isomorphic to* F_k.

Remark. — There is in fact a unique stratification on F_1 associated to the sequence $\{F_i\}$ [2]. This stratification may be defined directly, and exists even when char $K = 0$.

3. Mumford's theory [5] gives us a natural representation of G on K^2 in the following way : Let D be a positive Cartier divisor on Y so that D meets only one component of Y_0, and let L be the quotient field of

$$\bigcup_{n=0}^{\infty} \Gamma(Y, \mathcal{O}_Y(nD)).$$

L does not depend on D, and since G acts on Y, we get a homomorphism from G to the K-linear automorphisms PGL (2, K) of L. This homomorphism may be lifted to a homomorphism ρ of G to SL (2, K) since G is free. We will show that $F_{\rho, X}$ is not semi-stable.

LEMMA 3. — *There is a transcendance basis* $\{z\}$ *of* L *over* K *so that* z *and* $\frac{1}{z}$ *are sections of* $\mathcal{O}_Y \otimes_A K$. *Further, multiplication by* dz *gives an isomorphism of* $\mathcal{O}_Y \otimes_A K$ with $\Omega^1_{Y/A} \otimes_A K$.

Proof. — Let $\gamma \in \mathrm{PGL}(2, K)$ be a non-identity element in the image of G. γ is known to be hyperbolic, so let P_1 and P_2 be its two fixed points in $\mathbf{P}^1_K$, and let z be a function on $\mathbf{P}^1_K$ having a pole at P_1, a zero at P_2 and no other poles or zeros Identifying L with the functions on $\mathbf{P}^1_K$, we get an element z of L. Any quasi-compact open V of Y may be embedded via an open immersion in the formal completion of an A-scheme whose generic fiber is $\mathbf{P}^1_K$ so that L is identified with the rational functions on $\mathbf{P}^1_K$ as above and so that the closures of P_1 and P_2 do not meet $V \cap Y_0$ ([5], Prop. 2.5, 4.20). The lemma follows using this z.

LEMMA 4. — *There is an exact sequence*

$$0 \to L \to F_{\rho, x} \to L^{-1} \to 0$$

where $F_{\rho, x}$ *is the bundle associated to the representation* ρ *of* G *on* K^2 *considered above, and* $L^{\otimes 2} \cong \Omega^1_{X_K/K}$.

Proof. — Let $\{h_g\}$ be the meromorphic descent data on $\mathcal{O}_Y^2$ defined by ρ. If

$$\rho_g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as a matrix, define descent data $\{h'_g\}$ on $\mathcal{O}_Y$,

$$h'_g : \mathcal{O}_Y \to \mathcal{O}_Y \otimes_A K$$

by

$$h'_g(f) = \frac{f}{cz+d}.$$

Let φ be the section of $\underline{\mathrm{Hom}}(\mathcal{O}_Y, \mathcal{O}_Y^2) \otimes_A K$ defined by sending f to the vector (zf, f). φ is a map of the descent data $\{h'_g\}$ to $\{h_g\}$ since

$$g^*(z) = \frac{az+b}{cz+d}.$$

The cokernel of φ is $\mathcal{O}_Y$ with descent data $\{h''_g\}$,

$$h''_g(f) = (cz+d)f.$$

Letting L denote the line bundle on X_K obtained from $\{h'_g\}$, we have an exact sequence

$$0 \to L \to F_\rho \to L^{-1} \to 0.$$

It remains to identify $L^{\otimes 2}$ with $\Omega^1_{X_K/K}$. $L^{\otimes 2}$ is the bundle associated to the meromorphic descent data

$$h'''_g = \frac{1}{(cz+d)^2}.$$

Since $g^*(dz) = \dfrac{dz}{(cz+d)^2}$ and since multiplication by dz gives an isomorphism of $\Omega^1_{\hat{X}/A} \otimes_A K$ with $\mathcal{O}_Y \otimes_A K$, we see $L^{\otimes 2}$ is $\Omega^1_{X_K/K}$.

Let F_1 denote the pullback of $F_{\rho, x}$ to $X_{\bar{K}}$. Proposition 2 shows there is a sequence of bundles F_k on $X_{\bar{K}}$ so that $F_k^{(p)} \simeq F_{k-1}$.

LEMMA 5. — *If $g \leq p^{k-1}$, then the F_k above is semi-stable.*

Proof. — Suppose F_k were not semi-stable. Then F_k would have a quotient bundle of negative degree. Thus $F_1 = F_k^{(p^{k-1})}$ would have a quotient bundle L′ of degree at most $-p^{k-1}$. Then there is a non-zero map φ from either L or L^{-1} to L′. The degree of L is $g-1$, and so φ cannot exist if $g-1 < p^{k-1}$.

Proof of Theorem 1. — It suffices to show that for each $g > 1$ and each algebraically closed field k of characteristic p, there is a stable curve of genus g over $k[[t]]$ whose generic fiber is smooth and geometrically connected, and whose special fiber is k-split degenerate. Let X_0 be a rational curve over k with g nodes. There is a complete regular local ring B of characteristic p with residue field k and a lifting X of X_0 to Spec B so that the generic fiber of X is smooth and connected [1]. Pulling back by a suitably generic map of Spec $k[[t]]$ to Spec B gives the desired curve.

Finally, we give two consequences of Theorem 1.

COROLLARY 1. — *For each prime $p > 0$ and integer $g > 1$, there is a smooth curve* X *of genus g over an algebraically closed field k of characteristic p and a semi-stable bundle* E *so that* $S^p(E)$ *is not semi-stable.*

Proof. — $E^{(p)}$ is a subbundle of $S^p(E)$, and the degree of $S^p(E)$ is zero, where E is the bundle of Theorem 1.

Remark. — Hartshorne has shown that in characteristic zero, every symmetric power of a semi-stable bundle is semi-stable [3].

COROLLARY 2. — *For each prime $p > 0$ and integer $g > 1$, and each positive integer $n < \frac{g-1}{p}$, there is a semi-stable bundle of rank* 2 *and degree* $2n$ *on a curve of genus g which is not ample.*

Proof. — If E is the bundle of Theorem 1, consider $E \otimes L$, where L is a line bundle of degree n. $(E \otimes L)^{(p)}$ has a quotient of non-positive degree, so $E \otimes L$ is not ample.

It is known that if $\deg E > \frac{2g-2}{p}$, and E is semi-stable of rank two, then E is ample [4].

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(Manuscrit reçu le 6 novembre 1972.)

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