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# MÖBIUS INVERSION FOR THE BRUHAT ORDERING ON A WEYL GROUP

BY DAYA-NAND VERMA.

There is a natural partial-ordering, defined below, on any Coxeter group with given Coxeter generators, under which the identity element is the smallest. The group is finite if, and only if, there is a highest element, which is then unique. With a handful of well-known exceptions finite Coxeter groups arise as Weyl groups of semisimple Lie groups and Lie algebras, and this partial-ordering then coincides with one arising from what is known as the Bruhat decomposition [3] for the corresponding semisimple Lie group.

In connection with an attempt to give an *ad hoc* proof of « Kostant's formula » (for the weight-multiplicities of finite-dimensional representations of semi-simple Lie groups and Lie algebras) — in fact, in an attempt to « explain away » the alternating nature of the summation on the Weyl group — it was conjectured in [3] that the Möbius function, defined and explained below, of this partial-ordering on a Weyl group, has an attractively simple form. In this note it is proved that the conjectured formula for such Möbius function holds not only for Weyl groups, but also for arbitrary Coxeter groups, finite or infinite. We prove this only for the finitely generated groups, but the passage to arbitrary Coxeter groups is trivial.

A group W with generators $R_1, R_2, \ldots, R_l$ is said to be *Coxeter* on these generators if each $R_i$ has order 2 and every relation on these generators is a consequence of the $\frac{1}{2}l(l+1)$ relations

$$(R_i R_j)^{\operatorname{ord} R_i R_j} = \mathrm{id}. \qquad (i \leqq j),$$

where $\operatorname{ord} \sigma$ is the order of an element $\sigma \in W$, being either a positive integer or $\infty$, and where $\sigma^{\infty}$ is taken to mean the identity element $\mathrm{id.} \in W$ for every $\sigma \in W$. Refer to [1] as the standard text on Coxeter groups.

Let $\Phi$ be the set of all conjugates of the (distinguished Coxeter) generators of W, under elements of W. For $\sigma \in W$ let $l(\sigma)$, called the length of $\sigma$ (always relative to the distinguished Coxeter generators $R_1, R_2, \ldots, R_l$), denote the smallest integer $k$ such that $\sigma = R_{i_1} R_{i_2} \ldots R_{i_k}$ for some sequence $i_1, \ldots, i_k$ drawn from the set $\{1, 2, \ldots, l\}$. (We reserve the letters $i$, $j$, possibly subscripted, to range in this set.) Clearly $l(\sigma R_j)$ is either $l(\sigma) + 1$ or $l(\sigma) - 1$.

For $\sigma, \tau \in W$, we say $\sigma \preceq \tau$ (and $\tau \succeq \sigma$) iff there exists a sequence of elements $S_1, S_2, \ldots, S_k \in \Phi$ such that

$$\tau = \sigma S_1 \cdot S_2 \ldots S_k$$

and

$$l(\sigma S_1 \ldots S_q) > l(\sigma S_1 \ldots S_{q-1}) \qquad \text{for} \quad 1 \leq q \leq k.$$

(Of course, for $q = 1$ the product $\sigma S_1 \ldots S_{q-1}$ is interpreted to mean simply $\sigma$.) Clearly $\preceq$ is a partial-ordering, such that the set $\{\sigma \in W \mid \sigma \preceq \tau\}$ is finite for all $\tau \in W$.

By the *Möbius function* of a partial-ordered set $\mathbf{S}$, satisfying the finiteness condition : $\{\sigma \in \mathbf{S} \mid \sigma \preceq\}$ is a finite set for every $\tau \in \mathbf{S}$, one means an integer-valued function $\mu : \mathbf{S} \times \mathbf{S} \to \mathbf{Z}$ with two arguments in S such that for arbitrary $\sigma, \tau \in \mathbf{S}$,

$$\sum_{\sigma \preceq \varkappa \preceq \tau} \mu(\sigma, \varkappa) = \delta(\sigma, \tau),$$

where $\delta(\sigma, \tau)$ equals 1 if $\sigma = \tau$ and 0 otherwise. This function is unique and is equally well defined by the (equivalent) condition that

$$\sum_{\sigma \preceq \varkappa \preceq \tau} \mu(\varkappa, \tau) = \delta(\sigma, \tau) \quad \text{for all } \sigma, \tau \in \mathbf{S}.$$

It has the property that if $f$ is any function on $\mathbf{S}$ with values in any abelian group, and $g$ is the partial-sum function

$$g(\varkappa) = \sum_{\sigma \preceq \varkappa} f(\sigma),$$

then $f$ can be expressed in terms of $g$ by

$$f(\tau) = \sum_{\varkappa \preceq \tau} \mu(\varkappa, \tau)\, g(\varkappa).$$

*See* ([2], p. 344) for further details.

THEOREM. — *The Möbius function* $\mu$ *on a Coxeter group (with the above mentioned partial-ordering in terms of the distinguished Coxeter generators) is given by*

$$\mu(\sigma, \tau) = (-1)^{l(\sigma)+l(\tau)}.$$

*Remark.* — It is clear that this is equivalent to saying that $\sigma \neq \tau$ implies $\sum_{\sigma \preceq \varkappa \preceq \tau} (-1)^{l(\varkappa)} = 0$. (*Cf.* [3], Conjecture 2.)

In other words, calling an element $\varkappa \in W$ *even* or *odd* according as $l(\varkappa)$ is even or odd, we have to show that every non-trivial *interval* in W (i. e. $\{\varkappa \in W \mid \sigma \preceq \varkappa \preceq \tau\}$ with $\sigma \neq \tau$) has as many even elements as odd elements.

LEMMA. — *If* $\alpha, \beta \in W$ *are such that for some* $j$

$$\alpha' = \alpha R_j \preceq \alpha \quad \textit{and} \quad \beta' = \beta R_j \preceq \beta,$$

*then the following three statements are equivalent :*

$$\alpha \preceq \beta, \qquad \alpha' \preceq \beta \qquad \text{and} \qquad \alpha' \preceq \beta'.$$

*Proof.* — That the middle statement $\alpha' \preceq \beta$ is implied by the others is trivial from the transitivity of the partial-ordering.

To prove that $\alpha' \preceq \beta$ implies $\alpha \preceq \beta$, start with $\beta = \alpha' S_1 S_2 \ldots S_k$ satisfying $l(\alpha' S_1 \ldots S_q) > l(\alpha' S_1 \ldots S_{q-1})$ for $1 \leqq q \leqq k$.

Let $t$ be such that $1 \leqq t \leqq k$,

$$l(\alpha' S_1 \ldots S_q R_j) > l(\alpha' S_1 \ldots S_q) \qquad \text{for} \quad 1 \leqq q \leqq t-1,$$

and

$$l(\alpha' S_1 \ldots S_t R_j) < l(\alpha' S_1 \ldots S_t).$$

Clearly such $t$ exists uniquely. We have two cases to consider.

*Case* 1 : $S_t = R_j$. — Denoting the conjugate $R_j S R_j$ of $S \in \Phi$ under the (fixed) generator $R_j$ simply by $\tilde{S}$, we claim that the sequence of elements

$\tilde{S}_1, \ldots, \tilde{S}_{t-1}, S_{t+1}, \ldots, S_k$ satisfies the necessary conditions in the definition of $\alpha \preceq \beta$, i. e.

$$(\star) \quad \begin{cases} \beta = \alpha \tilde{S}_1 \ldots \tilde{S}_{t-1} S_{t+1} \ldots S_k, \\ \text{with } \; l(\alpha \tilde{S}_1 \ldots \tilde{S}_r) > l(\alpha \tilde{S}_1 \ldots \tilde{S}_{r-1}) \quad \text{for } \; 1 \leq r \leq t-1, \\ \text{and } \; l(\alpha \tilde{S}_1 \ldots \tilde{S}_{t-1} S_{t+1} \ldots S_q) > l(\alpha \tilde{S}_1 \ldots \tilde{S}_{t-1} S_{t+1} \ldots S_{q-1}) \quad \text{for } \; t+1 \leq q \leq k. \end{cases}$$

Since

$$\alpha \tilde{S}_1 \ldots \tilde{S}_r = \alpha' S_1 \ldots S_r R_j \qquad \text{and} \qquad \alpha \tilde{S}_1 \ldots \tilde{S}_{t-1} S_{t+1} \ldots S_q = \alpha' S_1 \ldots S_q,$$

the top and bottom assertions of $(\star)$ are clear. To verify the middle assertion, note that

$$l(\alpha' S_1 \ldots S_r R_j) = l(\alpha' S_1 \ldots S_r) + 1$$

and

$$l(\alpha' S_1 \ldots S_{r-1} R_j) = l(\alpha' S_1 \ldots S_{r-1}) + 1,$$

so that the part $l(\alpha' S_1 \ldots S_r) > l(\alpha' S_1 \ldots S_{r-1})$ in the hypothesis $\alpha' \preceq \beta$ gives

$$l(\alpha \tilde{S}_1 \ldots \tilde{S}_r) = l(\alpha' S_1 \ldots S_r R_j) > l(\alpha' S_1 \ldots S_{r-1} R_j) = l(\alpha \tilde{S}_1 \ldots \tilde{S}_{r-1})$$

as desired.

*Case* 2 : $S_t \neq R_j$. — Here we claim that the sequence of elements $\tilde{S}_1, \ldots, \tilde{S}_t, R_j, S_{t+1}, \ldots, S_k \in \Phi$ satisfies the following conditions giving $\alpha \preceq \beta$ :

$$\beta = \alpha \tilde{S}_1 \ldots \tilde{S}_t R_j S_{t+1} \ldots S_k,$$

$$l(\alpha \tilde{S}_1 \ldots \tilde{S}_r) > l(\alpha \tilde{S}_1 \ldots \tilde{S}_{r-1}) \qquad \text{for } \; 1 \leq r \leq t,$$

$$l(\alpha \tilde{S}_1 \ldots \tilde{S}_t R_j) > l(\alpha \tilde{S}_1 \ldots \tilde{S}_t)$$

and

$$l(\alpha \tilde{S}_1 \ldots \tilde{S}_t R_j S_{t+1} \ldots S_q) > l(\alpha \tilde{S}_1 \ldots \tilde{S}_t R_j S_{t+1} \ldots S_{q-1}) \qquad \text{for } \; t+1 \leq q \leq k.$$

The proof of this is exactly analogous to that in Case 1. This completes demonstration of $\alpha' \preceq \beta$ implies $\alpha \preceq \beta$. To show that $\alpha' \preceq \beta$ implies $\alpha' \preceq \beta'$, start with $\beta = \alpha' S_1 S_2 \ldots S_k$ as before, and this time find the unique $s$, $1 \leq s \leq k$, such that

$$l(\alpha' S_1 \ldots S_{s-1} R_j) > l(\alpha' S_1 \ldots S_{s-1})$$

and

$$l(\alpha' S_1 \ldots S_q R_j) < l(\alpha' S_1 \ldots S_q) \qquad \text{for } \; s \leq q \leq k.$$

In case $S_s = R_j$, the sequence of elements $S_1, \ldots, S_{s-1}, \tilde{S}_{s+1}, \ldots, \tilde{S}_k \in \Phi$ does the trick of satisfying

$$\beta' = \alpha' S_1 \ldots S_{s-1} \tilde{S}_{s+1} \ldots \tilde{S}_k,$$
$$l(\alpha' S_1 \ldots S_{s-1} \tilde{S}_{s+1} \ldots \tilde{S}_q) > l(\alpha' S_1 \ldots S_{s-1} \tilde{S}_{s+1} \ldots \tilde{S}_{q-1}) \qquad \text{for} \quad s+1 \leqq q \leqq k,$$

from which we get $\alpha' \preceq \beta'$; and in case $S_s \neq R_j$, the sequence $S_1 \ldots S_{s-1}$, $R_j, \tilde{S}_s, \ldots, \tilde{S}_k$ from $\Phi$ does a similar trick.

Q. E. D.

*Proof of the Theorem.* — Let $\sigma, \tau$ be distinct elements of W. We shall prove by induction on $l(\sigma) + l(\tau)$, that the interval $\{\varkappa \in W \mid \sigma \preceq \varkappa \preceq \tau\}$ has as many even elements as odd elements. Since $\sigma \neq \tau$, the smallest value of $l(\sigma) + l(\tau)$ is 1, and occurs (if and) only if $\sigma = \text{id.}$ and $\tau = R_i$ (for some $i \in \{1, 2, \ldots, l\}$), and the corresponding interval has no other element. This is the starting step of the induction.

Now for the induction step, we fix distinct $\sigma$ and $\tau$ satisfying $\sigma \preceq \tau$. Since $\tau \neq \text{id.}$, there exists $j$ such that $\tau' = \tau R_j \preceq \tau$. We have to consider two possibilities :

*Case* 1 : $\sigma' = \sigma R_j \succeq \sigma$. — In this case we claim that $\varkappa \in W$ satisfies $\sigma \preceq \varkappa \preceq \tau$ if and only if $\varkappa' = \varkappa R_j$ satisfies $\sigma \preceq \varkappa' \preceq \tau$. Because of the symmetry of the situation we need only prove this claim when $\varkappa' \preceq \varkappa$. Suppose $\sigma \preceq \varkappa \preceq \tau$, so that $\varkappa' \preceq \varkappa \preceq \tau$. Applying the Lemma to the situation $\alpha = \sigma'$, $\beta = \varkappa$ (thus $\alpha' = \sigma$, $\beta' = \varkappa'$), we find that our hypothesis $\sigma \preceq \varkappa$ gives the conclusion $\sigma \preceq \varkappa'$, whence $\sigma \preceq \varkappa' \preceq \tau$ is proved. Assume conversely $\sigma \preceq \varkappa' \preceq \tau$, so that $\sigma \preceq \varkappa' \preceq \varkappa$. Taking $\alpha = \varkappa$, $\beta = \tau$ in the Lemma our hypothesis $\varkappa' \preceq \tau$ gives $\varkappa \preceq \tau$, whence $\sigma \preceq \varkappa \preceq \tau$ is obtained. Having proved our claim, it is clear now that the interval between $\sigma$ and $\tau$ has as many even elements as odd. (The parity of $\varkappa$ and $\varkappa'$ are necessarily opposite.)

*Case* 2 : $\sigma' = \sigma R_j \preceq \sigma$. — Since the interval $\{\varkappa \mid \sigma \preceq \varkappa \preceq \tau\}$ is contained in the interval $\{\varkappa \mid \sigma' \preceq \varkappa \preceq \tau\}$, which by our induction hypothesis has as many even elements as odd, it will suffice to show that the complimentary set

$$\mathbf{S} = \{\varkappa \in W \mid \sigma' \preceq \varkappa \preceq \tau, \quad \sigma \npreceq \varkappa\}$$

has as many even elements as odd. We claim that

$$(\star\,\star) \qquad \mathbf{S} = \{\varkappa \in W \mid \sigma' \preceq \varkappa \preceq \tau', \quad \sigma \npreceq \varkappa\}.$$

To see this first note that $\varkappa \in \mathbf{S}$ implies $\varkappa \preceq \varkappa' = \varkappa R_j$. For if $\varkappa \npreceq \varkappa'$ then $\varkappa' \preceq \varkappa$ and we can apply the Lemma with $\alpha = \sigma$ and $\beta = \varkappa$, and then $\sigma' \preceq \varkappa$ would give $\sigma \preceq \varkappa$ contradicting $\varkappa \in \mathbf{S}$. So, having proved that $\varkappa' \succeq \varkappa$ holds for every $\varkappa \in \mathbf{S}$, we apply the Lemma once again with $\alpha = \varkappa'$ and $\beta = \tau$, and obtain $\varkappa \preceq \tau$ implies $\varkappa \preceq \tau'$. This establishes our claim ($\star\star$). Since the right side of ($\star\star$) is the complement of the interval $\{ \varkappa \mid \sigma \preceq \varkappa \preceq \tau' \}$ in the larger interval $\{ \varkappa \mid \sigma' \preceq \varkappa \preceq \tau' \}$, each of which has as many even elements as odd (by the induction hypothesis), we find that the set $\mathbf{S}$ also has this property.

Q. E. D.

This completes, in particular, the proof of Conjecture 2 of [3].

Problem. — Let J be a subset of $\{ 1, 2, \ldots, l \}$, and $W_J$ the subgroup of W generated by $\{ R_j \mid j \in J \}$. Consider the subset

$$W^J = \{ \sigma \in W \mid l(\sigma R_j) > l(\sigma) \quad \text{for} \quad j \in J \}$$

of distinguished coset representatives of $W_J$ in W. Give a suitable (closed) formula for the Möbius function of $W^J$ (with the partial-ordering induced from that of W), in such a way that our Theorem above becomes its particular case for J = empty set.

Just as the partial-ordering of W is related to that of the Bruhat cells in the decomposition of the flag manifold $\mathbf{G}/\mathbf{B}$ (where $\mathbf{G}$ is a complex semisimple Lie group and $\mathbf{B}$ its Borel subgroup), that of $W^J$ is related to the partial-ordering on the cellular decomposition of $\mathbf{G}/\mathbf{P}$ for the parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ corresponding to the set J.

Unfortunately this author has no conjecture to offer on the Möbius function of $W^J$.

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