

ANNALES SCIENTIFIQUES DE L'É.N.S.

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Induction theorems for Grothendieck groups and Whitehead groups of finite groups

Annales scientifiques de l'É.N.S. 4^e série, tome 1, n° 1 (1968), p. 91-148

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INDUCTION THEOREMS FOR GROTHENDIECK GROUPS AND WHITEHEAD GROUPS OF FINITE GROUPS ⁽¹⁾

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⁽¹⁾ Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, in the Faculty of Pure Science, Columbia University, 1967.

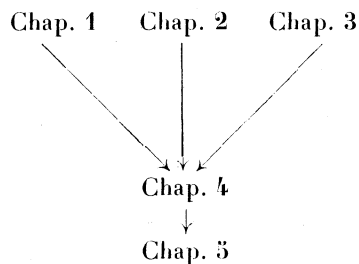
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INTRODUCTION.

For an abelian category \mathcal{C} , the Grothendieck group $K^0(\mathcal{C})$ of \mathcal{C} is an abelian group which solves the universal problem of finding “additive” maps of $\text{obj } \mathcal{C}$ into abelian groups. The Whitehead group $K^1(\mathcal{C})$ is the abelian group which furnishes a “universal determinant theory” for \mathcal{C} . In this paper, we conduct a survey of certain techniques useful in the study of these groups. While we treat them in some generality in the first chapter, we very shortly restrict ourselves to group rings, i. e. to the case when \mathcal{C} is the category of $\mathbf{Z}\pi$ -modules or else the category of projective $\mathbf{Z}\pi$ -modules, for finite groups π . In this setting, the crucial (and most subtle) question seems to be that of determining the torsion of the Whitehead group, and this is chiefly what engages our interest throughout the last two chapters.

Before we give the glossary of results, the following diagram will hopefully give the casual reader a quick glance at the structure of this paper :



In the first chapter, we recapitulate some generalities about Grothendieck groups, part of which has more or less become folklore. We then carry out a little spade work on orders in semi-simple algebras, and state for future reference the finiteness theorems (th. 3.1, 3.2, 3.3) for these orders. In the last section we deal with abelian orders and compute the Whitehead group G^1 for them (th. 4.1).

Chapter 2 is mainly a summary of [17], in which the author introduced the notion of the Artin exponent for finite groups. Main results in [17] are stated to path way for applications to chapter 4. Proofs of these

results require techniques of a different vintage, and will thus be omitted from the present treatise. For full details we refer the reader to [17].

Chapter 3, axiomatic in nature, is again independent of the preceding chapters. We define a Frobenius functor of rings on a category \mathcal{G} as a presheaf of rings on \mathcal{G} which carries an induction structure satisfying a Frobenius reciprocity law (§ 1). *A priori* we impose no condition on the nature of the category \mathcal{G} , but in practice we let \mathcal{G} be the category of subgroups of a fixed group or else the category of finite coverings of a fixed space. Examples abound in nature, and, to fix ideas, we record a handful of them, from group cohomology, cohomology theory and topological K-theory. Every Frobenius functor gives rise, in a natural way, to a category of “modules”. We then show, à la Swan ([21]), that an induction theorem for a Frobenius functor G will automatically imply induction and restriction theorems for the module category over G (th. 3.4). The Sylow subgroup theorems in group cohomology, for example, are seen to follow immediately from this abstract set-up.

The first three chapters then find simultaneous expression in chapter 4, in which we study the Grothendieck groups and Whitehead groups of a finite group π , “by induction”. We show that the functor G^0 is a Frobenius functor over which all other familiar functors behave as modules. Interpreting the classical induction theorems appropriately on G^0 (th. 2.1), we can apply the machinery of chapter 3 and immediately write down induction theorems for K^i , Wh, In this context, the Artin exponent shows up as the requisite exponent for induction from cyclic subgroups. On the other hand, the Whitehead group of a *cyclic* group has been completely determined, and known to be torsion free (Bass-Milnor-Serre, [10]). Consequently our induction theorem yields estimates on torsion exponents of the Whitehead group in general (th. 4.1, 4.2), and the calculation of the Artin exponent in chapter 2 enables us to write down some sufficient conditions for the Whitehead group to be p -torsion free, for a fixed prime p (th. 4.1). As a by-product of the architecture we also give easy “inductive” proofs of three theorems of Brauer on modular representations (th. 3.1, 3.2, 3.3).

In the last chapter, we exploit some techniques developed in a recent work of Bass-Milnor-Serre ([10]), and show how one can make explicit computations via these, in some circumstances. The Whitehead group of an abelian group of type (p, p^n) is p -torsion free (th. 1.1) and the same is true of an abelian group whose p -part has cardinality $\leq p^2$, provided that p does not divide the Euler function of the other prime divisors of the order of the group (th. 2.1). Finally we handle a small non-abelian group, S_3 , and show that it has Whitehead group equal to zero (th. 3.8).

While it may surprise the reader that about six pages of tedious calculations were needed to treat a group as innocent as S_3 , there is apparently no other non-abelian group to which any existing technique has access.

Lastly, a word on notations. π always means a *finite* group, and π' , π'' denote subgroups of π . For $x \in \pi$, $\langle x \rangle$ means the cyclic subgroup generated by x . N and C (mainly in chapter 2) indicate "normalizer" and "centralizer" in π .

All rings have identity and all modules are finitely generated and unitary. For a ring A , a module P is said to be a *generator* of the category of A -modules if A is a direct summand of a direct sum of copies of P . A "*faithfully projective*" module means a (finitely generated) projective generator in the module category. For a commutative ring A , $\mathfrak{p}_1, \mathfrak{p}_2, \dots$ usually refer to prime ideals in A and $\mathfrak{M}_1, \mathfrak{M}_2, \dots$ to maximal ideals in A . For an A -algebra Γ and left Γ -module M , $M_{\mathfrak{p}}$ denotes localization of M at a prime ideal \mathfrak{p} of A , i. e. $M_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A M$. This will be canonically regarded as a $\Gamma_{\mathfrak{p}}$ -module.

A *global field* K means a number field of finite degree over the rationals or else a function field in one variable over a finite field. We write K^* or sometimes $U(K)$ for the group of units in K . $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}, \dots$ will have their usual meanings.

To conclude this introduction, I would like to express my gratitude to Professor H. Bass, my thesis adviser, who led me to this research and made many valuable suggestions and contributions during the preparation of this work.

CHAPTER 1.

FINITENESS THEOREMS.

1. GENERALITIES AND DEFINITIONS. — Let \mathfrak{A} be an abelian category and \mathfrak{C} a full additive subcategory of \mathfrak{A} closed under finite direct sum formations. Suppose further that for any exact sequence

$$(\star) \quad 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$$

of \mathfrak{A} with $X, X'' \in \text{obj } \mathfrak{C}$, we have $X' \in \text{obj } \mathfrak{C}$. We define the *Grothendieck group* of \mathfrak{C} to be an abelian group $K^0(\mathfrak{C})$ with generators $[X]$ for $X \in \text{obj } \mathfrak{C}$ and relations $[X] = [X'] + [X'']$, for exact sequences (\star) , with $X, X'' \in \text{obj } \mathfrak{C}$. The map

$$[\] : \text{obj } \mathfrak{C} \rightarrow K^0(\mathfrak{C}).$$

then solves the universal problem of finding additive maps of $\text{obj } \mathfrak{C}$ into abelian groups.

The *Whitehead group* $K^1(\mathcal{C})$ of the category \mathcal{C} can also be defined, as an abelian group, by generators and relations. Its generators are $[X, \alpha]$, where $X \in \text{obj } \mathcal{C}$ and $\alpha: X \rightarrow X$ is an automorphism of X . The relations are of the following types :

(1) Additive relations : For any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & & \\ 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \end{array}$$

in \mathcal{C} , where horizontal sequences are exact and vertical maps are automorphisms ($X, X'' \in \text{obj } \mathcal{C}$), we have relation

$$[X, \alpha] = [X', \alpha'] + [X'', \alpha''].$$

(2) Multiplicative relations : If $\alpha: X \rightarrow X$ and $\beta: X \rightarrow X$ are automorphisms in \mathcal{C} , we have relation

$$[X, \alpha\beta] = [X, \alpha] + [X, \beta].$$

Let A be a (not necessarily commutative) ring with identity. Define $\mathfrak{M}(A)$ to be the category of (finitely generated) left A -modules, and $\mathfrak{P}(A)$ to be the category of (finitely generated) projective left A -modules. We write

$$\begin{aligned} G^i(A) &= K^i(\mathfrak{M}(A)) & (i = 0, 1); \\ K^i(A) &= K^i(\mathfrak{P}(A)) & (i = 0, 1). \end{aligned}$$

We shall now state, for future reference, some well-known facts about these groups. (See [4], [5].)

PROPOSITION 1.1. — *If \mathcal{C} is an artinian category (i. e. every object has finite length), and \mathcal{C}_0 is the full subcategory of semi-simple objects, then the inclusion functor induces isomorphisms $K^i(\mathcal{C}_0) \approx K^i(\mathcal{C})$, $i = 0, 1$. Suppose $\{S_i : i \in I\}$ is a set of representatives of the isomorphism classes of simple objects, then $K^0(\mathcal{C}_0)$ is the free abelian group generated by $\{[S_i] : i \in I\}$. If D_i is the division ring (Schur's lemma!) of endomorphisms of D_i , we have $K^1 \mathcal{C}_0 \approx \bigoplus D_i^* / [D_i^*, D_i^*]$.*

This enables us to compute, among other things, the K^0 and K^1 of artinian rings, and hence the K^0 and K^1 of finite dimensional algebras over fields.

We also have

PROPOSITION 1.2. — *Let \mathcal{C} be an abelian category and \mathfrak{P} the full subcategory of projectives. If every object of \mathcal{C} has a finite resolution by objects of \mathfrak{P} , then the inclusion functor induces isomorphisms $K^i \mathfrak{P} \approx K^i \mathcal{C}$ ($i = 0, 1$).*

For proof, see ([5]), th. 4 and 5).

In the present chapter, we seek to establish that $G'(A)$ and $K'(A)$ are finitely generated abelian groups for a class of rings A , notably for orders in a “nice” finite dimensional algebra over a field. We recall the definition of an order :

DEFINITION 1.3. — *Let R be a noetherian domain, with quotient field K . Let Σ be a finite dimensional K -algebra, with unit element 1 . An R -order Γ in Σ is a subset of Σ :*

- (1) Γ is a finitely generated R -module;
- (2) Γ is a subring of Σ containing 1 ;
- (3) $K \otimes_R \Gamma = \Sigma$.

In the situation of definition 1.1 we can consider several interesting subcategories of $\mathfrak{P}(\Gamma)$:

- (a) $\mathfrak{P}_0(\Gamma)$ denotes the subcategory of “special projectives”, i. e. objects M of $\mathfrak{P}(\Gamma)$ with the property that $K \otimes_R M$ is a free Σ -module;
- (b) $\mathfrak{P}_l(\Gamma)$ denotes the subcategory of “locally free projectives”, i. e. objects M of $\mathfrak{P}(\Gamma)$ with the property that $M_{\mathfrak{M}}$ is a free $\Gamma_{\mathfrak{M}}$ -module for every $\mathfrak{M} \in \max R$;
- (c) $\mathfrak{P}_f(\Gamma)$ denotes the subcategory of faithfully projective Γ -modules;
- (d) $\mathfrak{P}(\Gamma)$ denotes the subcategory of free Γ -modules.

Now let \mathfrak{P}^* be any one of these. The elements $[P]$, $P \in \text{obj } \mathfrak{P}^*$ generate an additive subgroup in $K^0(\Gamma)$, which we denote by $[\mathfrak{P}^*]$. We can now form the quotient groups

$$C_0(\Gamma) = \frac{[\mathfrak{P}_0(\Gamma)]}{[\mathfrak{P}(\Gamma)]},$$

$$C_l(\Gamma) = \frac{[\mathfrak{P}_l(\Gamma)]}{[\mathfrak{P}(\Gamma)]}.$$

They are loosely referred to as the *projective class groups* (reduced projective class groups, ...) of Γ .

Recall that a ring Γ is (left) *hereditary* if every left ideal is projective; it is (left) *regular* if every module has a finite resolution by projective modules. If Γ is an order over a Dedekind ring R in the sense of definition 1.3, then Γ is hereditary if and only if it is regular.

In general, if Γ is any ring, we can define a map $\alpha(\Gamma) : K^0(\Gamma) \rightarrow G^0(\Gamma)$ by $[P] \rightarrow [P]$. This is called the Cartan map of Γ . In case Γ is regular, $\alpha(\Gamma)$ is an isomorphism, by proposition 1.2.

2. CLEAN ORDERS AND STROOKER THEOREMS. — In this section we recapitulate some theorems of J. R. Strooker [24] and set up some convenient terminology.

THEOREM 2.1 (Strooker [24], th. 3.6). — *Suppose, in definition 1.3 that R is local with maximal ideal \mathfrak{M} and Σ is semi-simple K -algebra. Then for any R -order $\Gamma \subset \Sigma$, $C_0(\Gamma)$ is a free abelian group of rank equal to the rank of the kernel of the Cartan map $\alpha(\Gamma/\mathfrak{M}\Gamma) : K^0(\Gamma/\mathfrak{M}\Gamma) \rightarrow G^0(\Gamma/\mathfrak{M}\Gamma)$.*

This theorem computes the reduced projective class group $C_0(\Gamma)$ in the local setting. Globally, we have another theorem of Strooker ([24], th. 3.11).

THEOREM 2.2. — *Suppose R is a Dedekind domain, and Σ a separable K -algebra, and $\Gamma \subset \Sigma$ an R -order. Then*

$$C_0(\Gamma) \cong C_t(\Gamma) \oplus \bigoplus_{\mathfrak{M} \in \text{Max } R} C_0(\Gamma_{\mathfrak{M}}),$$

where the direct sum is taken over $\mathfrak{M} \in \text{Max } R$. $C_0(\Gamma_{\mathfrak{M}})$ is zero for almost all $\mathfrak{M} \in \text{Max } R$. If further K is a global field, $C_t(\Gamma)$ is finite, and is therefore the torsion subgroup of $C_0(\Gamma)$.

Thus $C_0(\Gamma)$ is a finitely generated abelian group whose rank can be computed by theorem 2.1. We can now define.

DEFINITION 2.3. — Γ is a clean order if $\mathfrak{X}_0(\Gamma) \subset \mathfrak{X}_f(\Gamma)$, i. e. if every special projective is faithfully projective.

The following theorem of Strooker gives various alternative descriptions of cleanness. We record without proof

THEOREM 2.4 ([24], th. 3.4, 3.10). — *If R is Dedekind, and Σ is a separable algebra over the global field K , then the following conditions are equivalent :*

- (a) Γ is clean;
- (b) $\Gamma_{\mathfrak{M}}$ is clean for every maximal ideal $\mathfrak{M} \subset R$;
- (c) $C_0(\Gamma_{\mathfrak{M}}) = 0$ for every maximal ideal $\mathfrak{M} \subset R$;
- (d) The Cartan map $\alpha(\Gamma/\mathfrak{M}\Gamma) : K^0(\Gamma/\mathfrak{M}\Gamma) \rightarrow G^0(\Gamma/\mathfrak{M}\Gamma)$ is injective for every maximal ideal $\mathfrak{M} \subset R$;
- (e) The reduced projective class group $C_0(\Gamma)$ is finite.

Most implications are apparent from theorems 2.1 and 2.2. An immediate consequence of (a) \Leftrightarrow (d) is that group rings $R\pi$ are clean orders. Here we use a theorem of Brauer on the injectivity of the Cartan map, of which we shall give a new proof later in chapter 3.

We finally remark that Strooker has also characterized maximal orders in Σ as those which are both hereditary and clean.

3. FINITENESS THEOREMS FOR G^i, K^i ($i = 0, 1$). — In this section, R denotes the ring of algebraic integers in a number field K , and Σ denotes

a separable K -algebra. Γ , Γ' and Γ'' will refer to R -orders in Σ . We shall prove that $G^i(\Gamma)$, $K^i(\Gamma)$ are all finitely generated abelian groups, and we compute their ranks in terms of two integers, r and q , defined as follows. Write C for the center of Σ . Then q denotes the number of simple factors of C , and r denotes the number of simple factors of $C_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} C$.

To handle G^i and K^i , we first consider the case $i = 0$. The following theorem is implicit in Strooker's thesis :

THEOREM 3.1. — *Let Γ be an R -order in Σ . Then :*

- (a) $G^0(\Sigma) = K^0(\Sigma)$ is a free abelian group of rank q ;
- (b) $G^0(\Gamma)$ is finitely generated, of rank $\geq q$. If Γ is clean, the rank is precisely q . If Γ is hereditary, then the rank equals q if and only if Γ is maximal;
- (c) $K^0(\Gamma)$ is finitely generated. If Γ is clean, the rank of $K^0(\Gamma)$ is $\leq q$. If Γ is hereditary, then the rank equals q if and only if Γ is maximal;
- (d) For a clean Γ , $G^0(\Gamma) \rightarrow G^0(\Sigma)$ and $K^0(\Gamma) \rightarrow K^0(\Sigma)$ have finite kernels.

Proof :

(a) Is a straightforward consequence of proposition 1.1.

(b) Choose a maximal order $\Gamma' \supset \Gamma$. Then we have an exact sequence

$$0 \rightarrow C_0(\Gamma') \rightarrow G^0(\Gamma') \rightarrow G^0(\Sigma) \rightarrow 0$$

by ([22], propr. 5.1), where the first map is defined by $[M] \mapsto [M] - [\Gamma'^m]$ ($m = \text{rank of } K \otimes_{\mathbf{R}} M \text{ over } \Sigma$) for $M \in \mathcal{O}_0(\Gamma')$. Since Γ' is clean, $C_0(\Gamma')$ is finite by theorem 2.4, so $G^0(\Gamma')$ is finitely generated of rank = rank $G^0(\Sigma) = q$ by (a). Now consider the following commutative diagram

$$\begin{array}{ccccccc} & & \ker f & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & C_0(\Gamma') & \longrightarrow & G^0(\Gamma') & \longrightarrow & G^0(\Sigma) \longrightarrow 0 \\ & & & & \downarrow f & & \parallel \\ & & (C_0(\Gamma)) & \xrightarrow{k} & G^0(\Gamma) & \xrightarrow{i} & G^0(\Sigma) \longrightarrow 0 \end{array}$$

where f is induced from the forgetful functor $\mathfrak{M}(\Gamma') \rightarrow \mathfrak{M}(\Gamma)$. $\text{Ker } f$ is a subgroup of $C_0(\Gamma')$, by commutativity, hence is finite. Pick $0 \neq r \in \mathbf{R}$, such that $r\Gamma' \subset \Gamma$, then $r\Gamma'$ is an ideal in Γ' as well as in Γ . Another forgetful functor induces $g : G^0(\Gamma/r\Gamma') \rightarrow G^0(\Gamma)$. We claim that $G^0(\Gamma) = \text{im } f + \text{im } g$. Indeed, since for any $M \in \mathfrak{M}(\Gamma)$, $[M] = [P] - [F]$

for some P and F , both R -torsion free, one need only catch $[M]$ in $\text{im } f + \text{im } g$ for a torsion free $M \subset K \otimes_R M$. With this inclusion in mind, we can talk about $\Gamma' M$. Now the exact sequence

$$0 \rightarrow (r\Gamma') M \rightarrow M \rightarrow M/(r\Gamma') M \rightarrow 0$$

yields $[M] = [(r\Gamma') M] + [M/(r\Gamma') M]$ in $G^0(\Gamma)$. Here $(r\Gamma') M$ is an Γ' -module, since $\Gamma'(r\Gamma') M \subset r\Gamma'\Gamma' M \subset (r\Gamma') M$; and $M/(r\Gamma') M$ is an $\Gamma/r\Gamma'$ -module, hence $[(r\Gamma') M] \in \text{im } f$ and $[M/(r\Gamma') M] \in \text{im } g$. Finally $\Gamma/r\Gamma'$ is an Artin ring, therefore $G^0(\Gamma/r\Gamma')$ is free of finite rank. $\text{Ker } f$ being finite, we conclude that $G^0(\Gamma)$ is finitely generated of rank $\geq q$. Now suppose that Γ is clean, so that $C_0(\Gamma)$ is finite. If we show that $\ker j/\text{im } k$ is a torsion abelian group, then $\ker j$ is finite, and $\text{rank } G^0(\Gamma) = \text{rank } G^0(\Sigma) = q$. Now according to a theorem of Swan ([21], prop. 1.1), the sequence

$$\sum_{\mathfrak{p}} G^0(\Gamma/\mathfrak{p}\Gamma) \rightarrow G^0(\Gamma) \xrightarrow{j} G^0(\Sigma) \rightarrow 0$$

is exact, where \mathfrak{p} runs through all $\mathfrak{p} \in \text{Spec } R$, $\mathfrak{p} \neq 0$. Take any $x \in \ker j$. We could clearly assume that $x = [M]$ where $M \in \mathcal{M}(\Gamma/\mathfrak{p}\Gamma)$ for some $\mathfrak{p} \in \text{Spec } R$, $\mathfrak{p} \neq 0$. Since Γ is clean, the Cartan map $\alpha(\Gamma/\mathfrak{p}\Gamma)$ is injective, and hence has a finite cokernel. By multiplying x with an integer if necessary, we can assume that M itself is $\Gamma/\mathfrak{p}\Gamma$ -projective. Now $\Gamma/\mathfrak{p}\Gamma$ considered as left Γ -module has projective dimension ≤ 1 (by exact sequence $0 \rightarrow \mathfrak{p} \otimes_R \Gamma \rightarrow \Gamma \rightarrow \Gamma/\mathfrak{p}\Gamma \rightarrow 0$), hence M considered as Γ -module also has projective dimension ≤ 1 . This enables us to take a resolution $0 \rightarrow P \rightarrow F \rightarrow M \rightarrow 0$ in which P and F are Γ -projective and Γ -free, respectively. Now, clearly, $K \otimes_R P \approx K \otimes_R F$ is Σ -free, so starting with

$$y = -|P| \in C_0(\Gamma), \quad k(y) = -(|P| - |F|) = |F| - |P| = |M| = x.$$

Finally, suppose Γ is hereditary. We assume that $\text{rank } G^0(\Gamma) = q$ and want to conclude that Γ is maximal [this means that for an order Γ which is hereditary but not maximal, $\text{rank } G^0(\Gamma) > q$]. But now $G^0(\Gamma) \approx K^0(\Gamma)$, (since Γ is regular), so our contention is equivalent to the last statement of (c), which we shall prove shortly.

(c) Consider the exact sequence

$$0 \rightarrow C_0(\Gamma) \rightarrow K^0(\Gamma) \rightarrow K^0(\Sigma).$$

By theorem 2.2, $C_0(\Gamma)$ is a finitely generated group, and by (a) of the present theorem, so is $K^0(\Sigma)$. Hence $K^0(\Gamma)$ is finitely generated. If Γ is clean, $C_0(\Gamma)$ is finite, so $\text{rank } K^0(\Gamma) \leq \text{rank } K^0(\Sigma) = q$. Finally, suppose Γ is hereditary, but non-maximal. By the last remark of paragraph 2, Γ is not clean, so $C_0(\Gamma)$ has rank ≥ 1 . But now $K^0(\Gamma) \approx G^0(\Gamma)$, hence $K^0(\Gamma) \rightarrow K^0(\Sigma)$ is onto. Evidently $\text{rank } K^0(\Gamma) = \text{rank } C_0(\Gamma) + q > q$.

(d) Follows readily from (b) and (c).

Q. E. D.

We now come to G^i and K^i for $i=1$. To handle K^1 , we have the following theorem of Bass ([6], lemma 3.6, and [4], §19).

THEOREM 3.2 (Bass). — *Let Γ be an R -order in Σ . Then :*

- (a) $K^1(\Gamma)$ is a finitely generated abelian group of order $r-q$;
- (b) For R -orders $\Gamma \subset \Gamma'$, the map $K^1(\Gamma) \rightarrow K^1(\Gamma')$ has finite kernel and cokernel;
- (c) $K^1(\Gamma) \rightarrow K^1(\Sigma)$ has finite kernel.

Whereas we won't repeat the proof of this theorem, we shall state and prove its analogue for G^1 , as follows :

THEOREM 3.3. — *Let Γ be an R -order in Σ . Then :*

- (a) $G^1(\Gamma)$ is a finitely generated abelian group of rank $r-q$;
- (b) For R -orders $\Gamma \subset \Gamma'$, the map $G^1(\Gamma') \rightarrow G^1(\Gamma)$ has finite kernel and cokernel;
- (c) $G^1(\Gamma) \rightarrow G^1(\Sigma)$ has finite kernel;
- (d) $K^1(\Gamma) \rightarrow G^1(\Gamma)$ has finite kernel and cokernel.

Proof:

(a) Choose a maximal order $\Gamma' \supset \Gamma$ and $0 \neq r \in R$, such that $r\Gamma' \subset \Gamma$. We observe that

$$\ker(G^1(\Gamma') \rightarrow G^1(\Gamma)) \subseteq \ker(G^1(\Gamma') \rightarrow G^1(\Sigma)) \cong \ker(K^1(\Gamma') \rightarrow K^1(\Sigma)),$$

hence is finite, by (c) of the Bass theorem. Here we have been able to identify G^1 and K^1 because Γ' is hereditary. We then have to handle the cokernel of $f: G^1(\Gamma') \rightarrow G^1(\Gamma)$. Writing g for the map $G^1(\Gamma/r\Gamma') \rightarrow G^1(\Gamma)$, we claim [as in (b) of theorem 3.1], that $G^1(\Gamma) = \text{im } f + \text{im } g$. Indeed the former arguments carry over *verbatim*. We only have to add the observation that $(r\Gamma')M$ is stable under any Γ -endomorphisms of M . Note that given $x \in G^1(\Gamma)$, we can always suppose that $x = [M, \alpha]$, where $M \in \mathcal{M}(\Gamma)$ is R -torsion free (by theorem of Bass-Heller-Swan, [5], th. 5) so $(r\Gamma')M$ make sense. Now the category $\mathcal{M}(\Gamma/r\Gamma')$ is artinian, so we can apply proposition 1.1 to conclude that

$$G^1(\bar{\Gamma}) \cong G^1(\bar{\Gamma}/\text{Jac } \bar{\Gamma})$$

where $\bar{\Gamma} = \Gamma/r\Gamma'$ and Jac denotes Jacobson radical. Here, $\bar{\Gamma}/\text{Jac } \bar{\Gamma}$ is finite and semi-simple (\Rightarrow regular), so

$$G^1(\bar{\Gamma}/\text{Jac } \bar{\Gamma}) \cong K^1(\bar{\Gamma}/\text{Jac } \bar{\Gamma}) \cong U(\bar{\Gamma}/\text{Jac } \bar{\Gamma})$$

is finite. It is clear, then, that $G^1(\Gamma)$ is finitely generated of

$$\text{rank} = \text{rank } G^1(\Gamma') = \text{rank } K^1(\Gamma') = r - q,$$

by the Bass theorem.

(b) Follows immediately by taking a maximal order $\Gamma'' \supset \Gamma' \supset \Gamma$ and looking at the commutative triangle induced for G^1 .

(c) We only have to prove that $\ker k$ is torsion in the commutative diagram. In fact take $x \in \ker k$; assuming that we have proved (d), we can choose $o \neq d \in \mathbf{Z}$, such that $dx \in \text{im } g$, say $dx = g(y)$, $y \in K^1(\Gamma)$.

$$\begin{array}{ccc} K^1(\Gamma) & & \\ \downarrow g & \searrow h & \\ G^1(\Gamma) & \xrightarrow{k} & G^1(\Sigma) \end{array}$$

But then $y \in \ker h$, and so $d'y = o$ for some $o \neq d' \in \mathbf{Z}$, by Bass. Finally $d'dx = g(d'y) = o$. We now prove (d).

(d) $\ker (K^1(\Gamma) \rightarrow G^1(\Gamma)) \subset \ker (K^1(\Gamma) \rightarrow K^1(\Sigma))$, hence is finite, by (c) of theorem 3.2. Since $K^1(\Gamma)$ and $G^1(\Gamma)$ have the same rank ($= r - q$), $\text{coker} (K^1(\Gamma) \rightarrow G^1(\Gamma))$ must be finite too.

Q. E. D.

In chapter 4 (th. 5.1, 5.2, 5.4), we shall give sharper versions of the statements in the last two theorems, in the case when $\Gamma = \mathbf{Z}\pi$, the integral group ring of a finite group.

4. COMPUTATION OF G^1 FOR ABELIAN ORDERS. — Let R be the ring of integers in a number field K , and Σ be a semi-simple commutative K -algebra. Then $\Sigma \cong K_1 \oplus K_2 \oplus \dots \oplus K_q$ where K_j ($1 \leq j \leq q$) are finite extensions of K . Write R_j for the ring of integers in K_j . The subring $\Gamma^* = R_1 \oplus \dots \oplus R_q \subset \Sigma$ is then the unique maximal R -order in Σ . Other orders in Σ are denoted by Γ, Γ', \dots etc. By the theorem of Bass-Milnor on unimodular matrices over algebraic rings of integers ([8], [10]), one knows that $K^1(R_j) \cong U(R_j)$ where the isomorphism is defined via the determinant map. We now state the main theorem of this section, which refines theorem 3.3.

THEOREM 4.1. — *Let Γ be any order in Σ . Suppose for every j , the projection $\Gamma \rightarrow R_j$ is onto. Then $G^1(\Gamma^*) \rightarrow G^1(\Gamma)$ is an isomorphism. In particular*

$$G^1(\Gamma) \cong U(R_1) \times \dots \times U(R_q).$$

We break up the proof into several lemmas.

LEMMA 4.2. — *Let A be a commutative noetherian ring. For any $M \in \mathfrak{M}(A)$, there exists a fully invariant filtration $o = M_0 \subset M_1 \subset \dots \subset M_r = M$ of M such that, for each j , M_j/M_{j-1} is a torsion free A/\mathfrak{p} -module for some prime \mathfrak{p} .*

Any two such filtrations have a common refinement of the same type. (Here, “fully invariant” means: stable under all A -endomorphisms of M .)

Proof. — We need only show that any fully invariant filtration of M has a refinement of the above type. Indeed, once this is done, the first assertion follows on refining the trivial filtration $0 \subset M$, and the second statement follows from the observation that the filtration obtained by “intersecting” two fully invariant filtrations is a fully invariant refinement. Now to refine an arbitrary fully invariant filtration to the requisite type, it clearly suffices to refine the composition factors, hence we are back again to the first assertion of the lemma, which we shall now prove, by “Noetherian induction” on M . Among all prime ideals $\mathfrak{p} \in \text{Spec } A$ for which \mathfrak{p} annihilates some $x \neq 0$ in M , choose a maximal member \mathfrak{p} ($A =$ noetherian!). Set

$$M_1 = \{x \in M, \mathfrak{p}x = 0\}.$$

Evidently M_1 is a fully invariant submodule, and M_1 can be viewed as an A/\mathfrak{p} -module. If M_1 is not torsion free over A/\mathfrak{p} , then its torsion submodule contains elements $\neq 0$ annihilated by primes bigger than \mathfrak{p} , contradicting our choice of \mathfrak{p} . Hence M_1 is A/\mathfrak{p} -torsion-free. Now continue on M/M_1 by noetherian induction.

Q. E. D.

LEMMA 4.3. — Let A be a commutative noetherian ring, with minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$. Let $B = \prod_j^n A/\mathfrak{p}_j$. Then the natural ring homomorphism $A \rightarrow B$ induces epimorphisms $G^i(B) \rightarrow G^i(A) \rightarrow 0$ for $i = 0, 1$.

Proof. — First work with $i = 0$. Given $M \in \mathfrak{M}(A)$, choose a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_r = M$ as in the preceding lemma. Then $[M] = \sum_j [M_j/M_{j-1}]$ in $G^0(A)$. Suppose \mathfrak{p} is a prime for which $N = M_j/M_{j-1}$ is a torsion free A/\mathfrak{p} -module. Choose a minimal prime $\mathfrak{p}_i \subset \mathfrak{p}$ and view N as a B -module N' by “pulling back” along the projection $B = \prod_j^n A/\mathfrak{p}_j \rightarrow A/\mathfrak{p}_i$.

Evidently the image of N' under $\mathfrak{M}(B) \rightarrow \mathfrak{M}(A)$ is isomorphic to N , whence $[N] = [M_j/M_{j-1}] \in G^0(A)$ is captured in the image of $G^0(B) \rightarrow G^0(A)$. Therefore so is $[M]$.

Next consider G^1 . Given $[M, \alpha] \in G^1(A)$, $\alpha \in \text{Aut}_A M$, choose a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_r = M$ as in the preceding lemma. Then $\alpha M_j \subset M_j$ and $\alpha^{-1} M_j \subset M_j$, so $\alpha M_j = M_j$ for every j . Let α_j denote the automorphism induced by α on $N_j = M_j/M_{j-1}$. Then in $G^1(A)$, we have $[M, \alpha] = \sum_j [N_j, \alpha_j]$ by (generalized) additivity. To show surjectivity of $G^1(B) \rightarrow G^1(A)$ therefore, we only have to catch element $[M, \alpha] \in G^1(A)$

for which M is a torsion free A/\mathfrak{p} -module for some prime \mathfrak{p} . Pick a minimal prime $\mathfrak{p}_i \subset \mathfrak{p}$ and let M' denote the corresponding module over A/\mathfrak{p}_i , and hence over B , via $B \rightarrow A/\mathfrak{p}_i$. Since α is an A -automorphism, it corresponds to an A/\mathfrak{p}_i -automorphism and hence a B -automorphism, α' , of M' . Clearly $[M, \alpha]$ is the image of $[M', \alpha'] \in G^1(B)$.

Q. E. D.

We can now resume the proof of the theorem. Since Γ is a finite R -algebra in commutative Σ , Γ is a commutative noetherian ring. Also, being an integral extension of R , Γ has dimension 1, by the Cohen-Seidenberg theorem; consequently the minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ of Γ are no other than the kernels of the "coordinate projections" $\Gamma \rightarrow R_j$. Since all these projections are assumed to be surjective, $\Gamma/\mathfrak{p}_j \approx R_j$ for every j , hence $B = \prod_i \Gamma/\mathfrak{p}_i \approx \prod_i R_j \approx \Gamma^*$, the maximal order in Σ .

Now the diagram of categories and functors on the left induces a diagram of G^1 -groups on the right :

$$\begin{array}{ccc}
 \mathfrak{N}(\Sigma) & & G^1(\Sigma) \\
 \otimes \nearrow & & \mu \nearrow \\
 \mathfrak{N}(\Gamma^*) & \longrightarrow & \mathfrak{N}(\Gamma) & & G^1(\Gamma^*) & \longrightarrow & G^1(\Gamma) \\
 \otimes \searrow & & \theta \searrow & & & & \\
 & & & & & &
 \end{array}$$

In the right diagram, $G^1(\Gamma^*) \approx U(R_1) \times \dots \times U(R_g)$ (by a remark we made at the beginning of paragraph 4), and $G^1(\Sigma) = U(\Sigma)$, hence μ is a monomorphism, hence also θ . But $\Gamma^* \approx B$, and θ was an epimorphism by lemma 4.3. Consequently

$$G^1(\Gamma) \approx G^1(\Gamma^*) \approx U(R_1) \times \dots \times U(R_g).$$

Q. E. D.

Remark. — Theorem 4.1 applies notably to orders which are group rings over cyclotomic integers. In fact if R denotes the ring of integers in a cyclotomic field K , and π a (finite) abelian group, then a typical component K_j of $\Sigma = K\pi$ is a cyclotomic field $K(\zeta_j)$, in which the ring of integers is $R_j = R[\zeta_j]$, by a familiar theorem in number theory. If $\Gamma = R\pi$, the projections $\Gamma \rightarrow R_j$ are manifestly surjective.

CHAPTER 2.

ARTIN EXPONENT OF FINITE GROUPS.

This chapter is a digression into the theory of rational representations of finite groups. Theorems will be stated without proof, and the interested reader is referred to the work [17].

Let us first recall the classical induction theorem of Artin :

THEOREM 1. — *Let π be a finite group of order n , and let π_1, \dots, π_q , be a maximal non-conjugating set of cyclic subgroups of π . Let μ_1, \dots, μ_q be the characters of π induced from the trivial representations of π_1, \dots, π_q . Then for any rational character χ of π , there exist integers a_1, \dots, a_q such that*

$$(\star) \quad n\chi = \sum_{k=1}^q a_k \mu_k.$$

In practice, however, we find that in most cases smaller multiples of χ already have the same property. One can then naturally ask, for a given group π , what is the smallest choice of $m \in \mathbf{Z}^+$ such that the equation

$$(\star\star) \quad m\chi = \sum_{k=1}^q a_k \mu_k$$

is solvable for integral unknowns a_k , for any given rational character χ . This question motivates the following.

Definition. — Let π be a finite group. An integer $m \in \mathbf{Z}$ is said to be an *Artin exponent* for π if, given any rational character χ on π , the equation $(\star\star)$ is solvable for $a_k \in \mathbf{Z}$. All Artin exponents clearly form an ideal in the integers, and, by theorem 1 $[\pi : 1]$ is in this ideal. We pick the (unique) positive generator $A(\pi)$ for this ideal and shall call it the *Artin exponent* of π . Our last remark shows that $A(\pi)$ divides $[\pi : 1]$.

Example. — Let us compute the Artin exponent for S_3 , the (full) symmetric group on three elements. Choose, in notation of theorem 1,

$$\pi_1 = \langle 1 \rangle, \quad \pi_2 = \langle (12) \rangle, \quad \pi_3 = \langle (123) \rangle.$$

S_3 has three (absolutely) irreducible representations, namely, the principal representation χ^1 , the sign representation χ^2 , and the unique two-dimensional irreducible representation χ^3 . We easily verify that

$$\begin{aligned} 2\chi^1 &= -\mu_1 + 2\mu_2 + \mu_3, \\ 2\chi^2 &= \mu_1 - 2\mu_2 + \mu_3, \\ 2\chi^3 &= \mu_1 - \mu_3. \end{aligned}$$

Thus $A(S_3) = 2$.

In [17] we furnish a complete determination of the invariant $A(\pi)$ in terms of the inner structure of the group π . The main results are as follows :

THEOREM 2. — *$A(\pi) = 1$ if and only if π is a cyclic group.*

THEOREM 3. — *If π' is a subgroup of π , then $A(\pi')$ divides $A(\pi)$. If π'' is a quotient group of π , $A(\pi'')$ also divides $A(\pi)$.*

THEOREM 4 (Brauer). — *If π_1, π_2 are two finite groups of relatively prime orders, then $A(\pi_1 \times \pi_2) = A(\pi_1) A(\pi_2)$.*

THEOREM 5 (Witt Induction Theorem) :

$$A(\pi) = \text{l. c. m. } A(\pi')$$

where the l.c.m. is taken over all hyperelementary subgroups π' of π . In particular, a group is cyclic if and only if all hyperelementary subgroups are cyclic.

(Recall that a hyperelementary group is a group π which has a cyclic normal subgroup π' such that $\frac{\pi}{\pi'}$ is a p -group for some prime p .)

THEOREM 6 :

- (a) *Let p be an odd prime, and π be a p -group of order p^n , which is not cyclic. Then $A(\pi) = p^{n-1}$.*
- (b) *Let π be a 2-group which is either "quaternion", or "dihedral", or "semi-dihedral", then $A(\pi) = 2$. If π is a non-cyclic 2-group of order 2^n which is not of any of the above types, then $A(\pi) = 2^{n-1}$.*

This theorem therefore provides the complete computation of $A(\pi)$ for p -groups. It is now possible to use theorem 5 to obtain global theorems :

THEOREM 7. — *Let p be an odd prime and π be a finite group, a p -Sylow subgroup $\pi^{(p)}$ of which is of order p^n and not cyclic. Write $A_p(\pi)$ for the p -part of $A(\pi)$. Then we have :*

- (a) $A_p(\pi) = p^n$ if there exists a cyclic p -free group $D \subset \pi$ normalised by $\pi^{(p)}$ on which the conjugation action of $\pi^{(p)}$ is faithful;
- (b) $A_p(\pi) = p^{n-1}$ if otherwise.

COROLLARY 8. — $A_p(\pi) = p^{n-1}$ if any of the following conditions holds :

- (a) $\pi^{(p)}$ is normal;
- (b) $\pi^{(p)}$ is abelian;
- (c) p^n does not divide the product of all numbers $q - 1$, for q running over all prime divisors of $[\pi : 1]$ distinct from p .

THEOREM 9. — *Suppose p is any prime and a p -Sylow subgroup $\pi^{(p)}$ of π is cyclic. Then $A_p(\pi) = p^s$, where s is the smallest positive integer satisfying the following property ;*

(*) *For any $x \in \pi^{(p)}$ and any p -free cyclic subgroup D of π ,*

$$x \in N(D) \Rightarrow x^{p^s} \in C(D).$$

We get immediately

COROLLARY 10. — Let $\pi^{(p)}$ denote a p -Sylow subgroup of π ; then $A(\pi)$ does not involve the prime p if and only if the following two conditions holds :

- (a) $\pi^{(p)}$ is cyclic;
- (b) For any subgroup $P \subset \pi^{(p)}$ and any p -free cyclic subgroup D of π ,

$$P \subset N(D) \Rightarrow P \subset C(D).$$

COROLLARY 11. — Suppose $\pi^{(p)}$ is cyclic. Then $A(\pi)$ does not involve p under any one of the following conditions :

- (a) $\pi^{(p)}$ is normal;
- (b) p does not divide $q - 1$, for any prime divisor q of $[\pi : 1]$;
- (c) p is the largest prime dividing $[\pi : 1]$.

COROLLARY 12. — Let $\pi = S_n (n \geq 3)$ be the symmetric group of permutations of n letters. Then $A(\pi)$ is free of primes $p > \frac{n}{2}$.

CHAPTER 3.

FROBENIUS FUNCTORS AND INDUCTION THEOREMS.

1. FROBENIUS FUNCTORS G AND G -MODULES. — Let \mathcal{G} be a category and \mathcal{A} the category of commutative rings with identity (with identity preserving ring homomorphisms). A presheaf of (commutative) rings on \mathcal{G} is then a contravariant functor $G : \mathcal{G} \rightarrow \mathcal{A}$. For a morphism $i : \pi' \rightarrow \pi$ in \mathcal{G} , we write i^* for $G(i)$. The presheaf of rings G is called a *Frobenius functor* if it carries the following additional structure of *induction* :

There is a functorial association of group homomorphism $i_* : G(\pi') \rightarrow G(\pi)$ to morphisms $i : \pi' \rightarrow \pi$ in \mathcal{G} [i. e. we have a rule $i \rightarrow i_*$ such that $(ij)_* = i_* j_*$ whenever ij makes sense in \mathcal{G} , and such that $Id_* = Id$], satisfying the following *Frobenius reciprocity formula*

$$(F1) \quad i_*(i^*a \cdot b) = a \cdot i_*b$$

for $a \in G(\pi)$, $b \in G(\pi')$ and $i : \pi' \rightarrow \pi$.

We easily see that such a G can be regarded as a “genuine” functor on \mathcal{G} in the usual sense of category theory, if we manufacture an appropriate recipient category. In detail, we define the *universal Frobenius category* FROB to be a category with $\text{obj FROB} = \text{obj } \mathcal{A}$ and morphisms

$$\text{FROB}(R, R') = \{ (i^*, i_*) : i^* \in \mathcal{A}(R, R'), \\ i_* : R' \rightarrow R \text{ a group homomorphism, and } i^*, i_* \text{ satisfying (F1)} \}.$$

Composition of morphisms in FROB is defined by

$$(j^*, j_*) \circ (i^*, i_*) = (j^* i^*, i_* j_*).$$

It is then an easy exercise to check that if the pairs (j^*, j_*) and (i^*, i_*) satisfy Frobenius reciprocity (F 1), so does the pair $(j^* i^*, i_* j_*)$. With FROB thus defined, a Frobenius functor is just a (usual) functor G from \mathcal{G}^0 (dual category) to FROB .

We now come to the concept of "modules" over a Frobenius functor G . A G -module will mean a module $K : \mathcal{G} \rightarrow (\text{abelian groups})$ over G regarded as presheaf of rings, together with the following additional structure of induction :

There is a functorial association of group homomorphisms $I_* : K(\pi') \rightarrow K(\pi)$ to morphisms $i : \pi' \rightarrow \pi$ in \mathcal{G} , satisfying the following two reciprocity formulae :

$$(F 2) \quad I_*(y \cdot I^*(a)) = i_*(y) \cdot a$$

$$(F 3) \quad I_*(i^*(x) \cdot b) = x \cdot I_*(b)$$

where $x \in G(\pi)$, $y \in G(\pi')$, $a \in K(\pi)$, $b \in K(\pi')$ and $I^* = K(i)$.

REMARK 1.1 :

(a) The formula $I^*(x \cdot a) = i^*(x) \cdot I^*(a)$ ($x \in G(\pi)$, $a \in K(\pi)$) expressing the semi-linearity of I^* with respect to i^* is built into the requirement that K be a module over the presheaf of rings G .

(b) G itself, e. g. is a G -module. In this case (F 2) and (F 3) amalgamate, by commutativity, into a single formula (F 1).

(c) To elucidate the reciprocity formulae, the following orientation may help. Regard $G(\pi')$ as $G(\pi)$ -module by action

$$a \circ b = i^*(a) \cdot b, \quad a \in G(\pi), \quad b \in G(\pi').$$

Then the fact that i^* is a ring homomorphism translates into the statement that i^* is a morphism of $G(\pi)$ -modules. Similarly, (F 1) translates into the statement that i^* is (also) a $G(\pi)$ -module homomorphism. Similar interpretations hold for (F 3).

Let K and K' be G -modules. A *morphism of G -modules* will mean a natural transformation $f : K \rightarrow K'$ which is a morphism of modules over the presheaf of rings, satisfying the following commutativity condition :

$$\begin{array}{ccc} K(\pi) & \xrightarrow{f(\pi)} & K'(\pi) \\ \uparrow i_* & & \uparrow i_* \\ K(\pi') & \xrightarrow{f(\pi')} & K'(\pi') \end{array}$$

for each morphism $i : \pi' \rightarrow \pi$ in \mathcal{G} .

We can define, in a natural manner, the kernel and the image of such a morphism $f: K \rightarrow K'$, and these again carry natural G -module structures. All G -modules with morphisms as above therefore form an abelian category.

Suppose G and G' are both Frobenius functors of rings over a fixed category \mathcal{G} . A *homomorphism* f from G to G' will mean a natural transformation of functors $f: G \rightarrow G'$. The presence of such a homomorphism converts G' into a G -module in the obvious manner, and with this structure on G' , f becomes a morphism of G -modules. In this way all the usual constructions in ring theory and module theory carry over; in particular we have the concept of "ideals" in a Frobenius functor of rings, and we can "complete" the latter in " \mathfrak{A} -adic topologies" for various ideals \mathfrak{A} .

The usefulness of these concepts, as well as many examples of Frobenius functors and modules over them, will appear in the coming section.

2. EXAMPLES :

EXAMPLE 2.1. — Let π be a fixed finite group. Write \mathcal{G}_π for the category of all subgroups $\pi' \subset \pi$ and all monomorphisms $i: \pi'' \rightarrow \pi'$. We shall refer to \mathcal{G}_π briefly as *the category of π* . Take a fixed π -ring R , i. e. a ring on which π acts as a group of ring automorphisms. There exists a unique natural family of maps $\zeta^{p,q}$:

$$\begin{array}{ccc}
 \hat{H}^p(\pi, R) \otimes_{\mathbf{Z}} \hat{H}^q(\pi, R) & \xrightarrow{\zeta^{p,q}} & \hat{H}^{p+q}(\pi, R \otimes_{\mathbf{Z}} R) \\
 \searrow \mu^{p+q} \zeta^{p,q} & & \downarrow \mu^{p+q} \\
 & & \hat{H}^{p+q}(\pi, R)
 \end{array}$$

extending the zero dimensional (unreduced) map

$$\begin{aligned}
 R^\pi \otimes_{\mathbf{Z}} R^\pi &\rightarrow R^\pi, \\
 x \otimes y &\rightarrow xy.
 \end{aligned}$$

(See [11], p. 242.) Here \hat{H} denotes taking "complete" group cohomology, in the sense of Artin-Tate. If we now write

$$\hat{H}(\pi, R) = \bigoplus_n \hat{H}^n(\pi, R),$$

$\hat{H}(\pi, R)$ has the structure of a graded ring, via $\mu^{p+q} \zeta^{p,q}$. For a morphism $i: \pi'' \subset \pi'$ in \mathcal{G}_π , we have the following familiar maps in group

cohomology ([20], p. 124-130).

$$(2.2) \quad \hat{H}(\pi', R) \xrightarrow{\text{res}} \hat{H}(\pi'', R),$$

$$(2.3) \quad \hat{H}(\pi', R) \xleftarrow{\text{ver}} \hat{H}(\pi'', R),$$

Here *res* is a morphism of graded rings and *ver* is a morphism of graded abelian groups. Writing i^* for *res* and i_* for *ver*, we immediately see that $\pi' \rightarrow \hat{H}(\pi', R)$ is a Frobenius functor of commutative graded rings on \mathcal{G}_π , the requisite reciprocity formulate (F 1) being given by equations (12) (13) of page 256 in [11]. Call this functor *G* (suppressing the role of *R*, which is fixed).

By a π -*R*-module we shall mean an *R*-module *K* with π -action satisfying $\sigma(ax) = \sigma a \cdot \sigma x$, for $a \in R$, $x \in K$ and $\sigma \in \pi$. We can construct a functor from the category of all such π -*R*-modules into the category of *G*-modules, as follows. Given a fixed π -*R*-module *K*, we have a diagram similar to (★) with the *R*'s there “partially” replaced by *K*. This furnishes $\hat{H}(\pi', K)$ with a natural structure of a $G(\pi') (= \hat{H}(\pi', R))$ -module. We also have maps as in (2.2) and (2.3) with *K* replacing *R*. Writing I^* and I_* for these maps, we easily verify that the rule

$$\pi' \mapsto \{K(\pi') = \hat{H}(\pi', K), I^*, I_*\}$$

for $\pi' \in \mathcal{G}_\pi$, defines a *G*-module. In fact, semi-linearity (Remark 1.1 *a*) and Frobenius reciprocities are replica of formulas (11), (12), (13) on page 256 of [11]. Further, to a morphism $f: K \rightarrow K'$ of π -*R*-modules we can associate maps $\{\hat{f}(\pi'): \hat{H}(\pi', K) \rightarrow \hat{H}(\pi', K')\}$ induced on cohomologies. This way we get a (covariant) functor from π -*R*-modules to *G*-modules.

REMARK 2.4. — Taking $R = \mathbf{Z}$, a trivial π -ring, we have the basic Frobenius functor $G: \pi' \rightarrow \hat{H}(\pi', \mathbf{Z})$. π - \mathbf{Z} -modules then mean just π -modules and any such defines a *G*-module.

EXAMPLE 2.5. — Let $\mathcal{C}\mathfrak{R}\mathfrak{W}$ be the category in which objects are CW-complexes and morphisms are (regular) finite coverings. For $X \in \mathcal{C}\mathfrak{R}\mathfrak{W}$ write $C_q(X)$ for the group of (say, integral) *q*-chains and $C^q(X)$ for the group of *q*-cochains. If $i: X' \rightarrow X$ is a morphism in $\mathcal{C}\mathfrak{R}\mathfrak{W}$, we can define $i^*: C_q(X') \rightarrow C_q(X)$ and $i_*: C_q(X) \rightarrow C_q(X')$ as follows. For a *q*-cell e' in X' , $i^*(e') = i(e')$, and for a *q*-cell e' in X , $i_*(e') = \sum e''$ where e'' ranges over all *q*-cells of X' which cover e' . These maps induce $I^*: C^q(X) \rightarrow C^q(X')$ and $I_*: C^q(X') \rightarrow C^q(X)$ for cochains, which, in turn, induce $I^*: H^q(X) \rightarrow H^q(X')$, $I_*: H^q(X') \rightarrow H^q(X)$ for (ordinary) cohomology.

We easily check that if $H(X) = \sum H^r(X)$ is given the ring structure defined by the cup-product of cochains, I^* and I_* together satisfy the Frobenius reciprocity (F 1). In this way $X \mapsto H(X)$ and $i \mapsto (I^*, I_*)$ define a Frobenius functor of rings on $\mathcal{C}\mathcal{R}\mathcal{C}$.

EXAMPLE 2.6. — For $X \in \mathcal{C}\mathcal{R}\mathcal{C}$ as in the above example and X finite, let $K^0(X)$ denote the Grothendieck ring of (complex, say) vector bundles on X . If X is infinite CW-complex we define $K^0(X)$ by a limit process. Let $i: X' \rightarrow X$ be a morphism in $\mathcal{C}\mathcal{R}\mathcal{C}$. Given a vector bundle F on X , we can define its inverse image i^*F , a bundle on X' ; and, given a bundle E on X' , we can define a direct image i_*E , a bundle on X . In topology, we have the familiar fact that $i_*(E \otimes i^*F) \cong i_*(E) \otimes F$, so $X \mapsto K^0(X)$, $i \mapsto (i^*, i_*)$ define a Frobenius functor of rings on $\mathcal{C}\mathcal{R}\mathcal{C}$. Exactly the same remarks hold if we replace $K^0(X)$ by $K^*(X) = K^0(X) \oplus K^1(X)$, where $K^1(X)$ is defined to be the kernel of $K^0(X \times S^1) \rightarrow K^0(X)$.

Finally sheaves on topological spaces also lead to an example of a Frobenius functor. We just adhere to the same procedure of taking inverse and direct images for the construction of I^* and I_* .

EXAMPLE 2.7. — In $\mathcal{C}\mathcal{R}\mathcal{C}$, consider a fixed object $X \in \mathcal{C}\mathcal{R}\mathcal{C}$ with a finite fundamental group π . Let \tilde{X} be a contractible universal covering of X , and we consider \tilde{X} as defining a principal π -bundle on X , in the fashion familiar to topologists. Recall that \mathcal{G}_π denotes the category of subgroups and inclusion homomorphisms in π . We can define a similar category topologically, namely, we agree that $\mathcal{C}\mathcal{R}\mathcal{C}_X$ is the subcategory of $\mathcal{C}\mathcal{R}\mathcal{C}$ “over X ”: this means: objects of $\mathcal{C}\mathcal{R}\mathcal{C}_X$ are taken as maps $X' \rightarrow X$ in $\mathcal{C}\mathcal{R}\mathcal{C}$ and morphisms of $\mathcal{C}\mathcal{R}\mathcal{C}_X$ are taken as maps $f: X' \rightarrow X''$ in $\mathcal{C}\mathcal{R}\mathcal{C}$ which make the following diagram commutative

$$\begin{array}{ccc} & X' & \\ & \swarrow & \searrow \\ X & & X'' \\ & \nwarrow & \nearrow \end{array}$$

The homotopy operator $X' \mapsto \pi_1(X')$ then defines a functor $\pi_1: \mathcal{C}\mathcal{R}\mathcal{C}_X \rightarrow \mathcal{G}_\pi$, which is easily seen to be an equivalence of categories, by the theory of covering spaces. Note that under this equivalence $X \leftrightarrow \pi$ and $\tilde{X} \leftrightarrow \{1\}$.

Consider now any $\pi' \in \mathcal{G}_\pi$, and a complex representation $\rho: \pi' \rightarrow GL(n, \mathbf{C})$ of π' . Under the equivalence of categories defined by π_1 , π' corresponds to some X' with $\pi_1(X') = \pi'$. The induced epimorphism $\tilde{X} \rightarrow X'$ defines a principal π' -bundle ξ on X' , so given ρ , we can canonically construct

a complex vector bundle $\rho(\xi)$ on X' . It is easy to check that the process $\xi \mapsto \rho(\xi)$ induces a natural ring homomorphism $G(\mathbf{C}\pi') \rightarrow K^0(X')$, where $G(\mathbf{C}\pi')$ stands for the (complex) character ring of π' . If we identify the categories \mathcal{G}_π and $\mathcal{C}\mathcal{R}_X$ via π_1 , this defines a natural transformation of the Frobenius functor G to the Frobenius functor K^0 , and this, in particular, converts K^0 into a module over G . Since we have also a transformation $K^0 \rightarrow K^*$, K^* can likewise be regarded as a G -module.

REMARK 2.8. — For $X \in \mathcal{C}\mathcal{R}$ with finite fundamental group π , let us identify $\mathcal{C}\mathcal{R}_X \approx \mathcal{G}_\pi$. Then examples 2.1, 2.5 and 2.7 give three different Frobenius functors on \mathcal{G}_π . Take a morphism $i: \pi' \rightarrow \pi''$ in \mathcal{G}_π . This gives rise to a pair (I^*, I_*) . We have $I_* I^* =$ multiplication by $[\pi'' : \pi']$ in examples 2.1 and 2.5, but this is not the case with example 2.7.

EXAMPLE 2.9. — Trivial modules :

Let π be a fixed finite group and K a number field. Then $\pi' \rightarrow G^0(K\pi')$ together with restriction and induction maps in representation theory define a Frobenius functor. (For details, see § 1 of chapter 4.) Denote this briefly by G and write $\mathfrak{M}(G)$ for the category of G -modules. Write $((ab))$ for the category of abelian groups. We shall construct two full imbeddings $\Phi_0, \Phi_1: ((ab)) \rightarrow \mathfrak{M}(G)$ which will be useful later. To define Φ_0 , pick any abelian group F , and take $\Phi_0(F)(\pi') = F$ for any $\pi' \in \mathcal{G}_\pi$. Given $i: \pi' \rightarrow \pi''$ define the restriction map $I^*: \Phi_0(F)(\pi'') \rightarrow \Phi_0(F)(\pi')$ to be the identity map $F \rightarrow F$, and define the induction map $I_*: \Phi_0(F)(\pi') \rightarrow \Phi_0(F)(\pi'')$ to be the multiplication by $[\pi'': \pi']$. Finally we let $G(\pi') = G^0(K\pi')$ act on $\Phi_0(F)(\pi')$ by $[M].x = [M:K]x$ where $[M] \in G^0(K\pi')$ and $x \in F$. Given a morphism $\varphi: F_1 \rightarrow F_2$ in $((ab))$, we define $\Phi_0(\varphi)$ to be the constant family of maps $\{\varphi: F_1 \rightarrow F_2\}$. To get Φ_1 , we proceed in exactly the same way but interchange the definition of restriction and induction maps.

To show that these definitions really work, let us at least check one reciprocity formula, say F 2) for Φ_0 :

$$I_*(y \cdot I^*(a)) = i_*(y) \cdot a \quad | y \in G(\pi'), a \in \Phi_0(F)(\pi'') = F.$$

Write $y = [M]$:

$$\begin{aligned} I_*(y \cdot I^*(a)) &= I_*([M:K]a) = [\pi'': \pi'] \cdot [M:K]a, \\ i_*(y) \cdot a &= [K\pi'' \otimes_{K\pi'} M] \cdot a = [K\pi'' \otimes_{K\pi'} M:K]a = [\pi'': \pi'] \cdot [M:K]a. \end{aligned}$$

Both imbeddings are clearly full and exact. They even have (exact) left inverses. To show that the concepts of trivial modules is convenient, let us look into one example. Consider the trivial module $\Phi_0(\mathbf{Z})$. The natural multiplication on \mathbf{Z} makes $\Phi_0(\mathbf{Z})$ a “trivial” Frobenius functor.

For $\pi' \in \mathcal{G}_\pi$, define $G(\pi') \rightarrow \Phi_0(\mathbf{Z})(\pi')$ by \dim_K , i. e. $[M] \mapsto [M : K]$. Then \dim can be regarded both as a (Frobenius) ring homomorphism and as a module homomorphism. Following Atiyah [3], we write $I(\pi') = I(K\pi')$ for the kernel and call it the augmentation ideal. We can complete $G(\pi')$ in the $I(\pi')$ -adic topology, to get $\widehat{G}(\pi')$, and $\pi' \mapsto \widehat{G}(\pi')$ defines a new Frobenius functor.

Over the trivial Frobenius module $\Phi_0(\mathbf{Z})$ on \mathcal{G}_π what are the modules? Unwinding the two formulas (F 2) and (F 3), we see that a $\Phi_0(\mathbf{Z})$ -module is just a presheaf of abelian groups $\{\pi' \mapsto K(\pi'), i \mapsto I^*\}$ carrying a functorial induction structure I_* which satisfies $I_* I^* = [\pi'' : \pi']$ for every morphism $i : \pi' \rightarrow \pi''$ in \mathcal{G}_π . Now recall example 2.1, in which we established $\{\pi' \mapsto H(\pi', M)\}$ as a module over $\{\pi' \mapsto H(\pi', \mathbf{Z})\}$. In particular the former is a presheaf of abelian groups on \mathcal{G}_π with an induction structure. By a familiar property in group cohomology, we have $I_* I^* = \text{ver} \circ \text{res} = [\pi'' : \pi']$ for every $i : \pi' \rightarrow \pi''$ in \mathcal{G}_π . Consequently we can view $\{\pi' \mapsto \widehat{H}(\pi', M)\}$ as a $\Phi_0(\mathbf{Z})$ -module, and therefore a G -module by pulling back along $G \rightarrow \Phi_0(\mathbf{Z})$. This point of view will prove to be convenient in paragraph 1 of chapter 4, where we construct the Whitehead functor on \mathcal{G}_π .

Similarly, in the terminology of example 2.5, the functor $X' \rightarrow H(X')$ can be viewed as defining a module over $\Phi_0(\mathbf{Z})$, thanks to remark 2.8.

We shall also have examples in the future of trivial modules of the type Φ_1 .

3. INDUCTION AND RESTRICTION THEOREMS. — In this section we shall develop some axiomatic techniques for a Frobenius functor G and a G -module K , defined on some category \mathcal{G} . These techniques will find applications in chapter 4 as well as in the following section of the present chapter.

Let M be a collection of objects in \mathcal{G} . We define, for $\pi \in \mathcal{G}$:

$$K_M(\pi) = \sum_i \{ \text{im}(I_* : K(\pi') \rightarrow K(\pi)) \mid i : \pi' \rightarrow \pi, \pi' \in M \}$$

$$K^M(\pi) = \bigcap_i \{ \ker(I^* : K(\pi) \rightarrow K(\pi')) \mid i : \pi' \rightarrow \pi, \pi' \in M \}$$

with similar definitions for G_M and G^M , using i_* and i^* .

PROPOSITION 3.1. — *On all $\pi \in \text{obj } \mathcal{G}$, we have:*

- (1) $G^M G_M = 0$;
- (2) $G^M K_M = G_M K^M = 0$;
- (3) $GK_M + G_M K \subset K_M$;
- (4) $GK^M + G^M K \subset K^M$.

Proof :

- (1) is a special case of (2);
- (2) $G^M K_M = 0$ follows from (F 3),
 $G_M K^M = 0$ follows from (F 2);
- (3) $G K_M \subset K_M$ follows from (F 3),
 $G_M K \subset K_M$ follows from (F 2).
- (4) Both $G K^M \subset K^M$ and $G^M K \subset K^M$ follow from semi-linearity.

COROLLARY 3.2. — For any π , $G_M(\pi)$ and $G^M(\pi)$ are ideals of $G(\pi)$; $K_M(\pi)$ and $K^M(\pi)$ are $G(\pi)$ -submodules of $K(\pi)$.

COROLLARY 3.3. — Let \mathfrak{A} be an ideal in $G(\pi)$. If \mathfrak{A} annihilates $K_M(\pi)$ then $\mathfrak{A}G_M(\pi)$ annihilates $K(\pi)$. In particular if $d \in \mathbf{Z}$ and $dK(\pi') = 0$ for every $\pi' \in M$, then $dG_M(\pi)$ annihilates $K(\pi)$.

Proof :

$$\mathfrak{A} \cdot G_M(\pi) \cdot K(\pi) \subset \mathfrak{A} \cdot K_M(\pi) = 0.$$

If $dK(\pi') = 0$ for every $\pi' \in M$, then

$$\begin{aligned} dK_M(\pi) &= d \sum_i \text{im} (I_* : K(\pi') \rightarrow K(\pi)) \\ &\subset \sum_i \text{im} (dK(\pi') \rightarrow K(\pi)) = 0. \end{aligned}$$

Q. E. D.

Convention. — Suppose $M \subset N$ is an inclusion of abelian groups. We shall say M has exponent d in N if $dN \subset M$. We say N has exponent d if 0 has exponent d in N .

THEOREM 3.4. — Suppose for a fixed $\pi \in \text{obj } \mathcal{G}$, that, $G_M(\pi)$ has exponent δ in $G(\pi)$. Then we have :

- (I) (Principle of Restriction) : $K^M(\pi)$ is of exponent δ ;
- (II) (Principle of Induction) : $K(\pi)$ is of exponent $d\delta$, provided $K(\pi')$ is of exponent d for every $\pi' \in M$;
- (III) $K_M(\pi)$ has exponent δ in $K(\pi)$.

Proof :

- (I) Follows from proposition 3.1 : $G_M K^M = 0$, since $\delta = \delta \cdot 1 \in G_M(\pi)$;
- (II) By corollary 3.3, $dG_M(\pi)$ annihilates $K(\pi)$. Since now $\delta \in G_M(\pi)$,
 $d\delta K(\pi) = 0$;
- (III) $\delta K(\pi) \subset G_M(\pi) K(\pi) \subset K_M(\pi)$.

COROLLARY 3.5. — Suppose, for a fixed $\pi \in \text{obj } \mathcal{G}$, $K(\pi)$ is a free abelian group. Then, given any collection M of objects of \mathcal{G} with $G_M(\pi)$ having finite index in $G(\pi)$, we have $K^M(\pi) = 0$. Also, if $\varphi : K \rightarrow K'$ is a morphism of G -modules, to prove that $\varphi(\pi) : K(\pi) \rightarrow K'(\pi)$ is one to one, it suffices to prove that $\varphi(\pi') : K(\pi') \rightarrow K'(\pi')$ is one to one for every $\pi' \in M$ with $i : \pi' \rightarrow \pi$.

COROLLARY 3.6. — Suppose, for a fixed $\pi \in \text{obj } \mathcal{G}$, $G(\pi)$ is free abelian, and that $G_M(\pi)$ has finite exponent δ in $G(\pi)$. If, for every $\pi' \in M$, $G(\pi')$ is free of nilpotent elements, then $G(\pi)$ is also free of nilpotent elements.

Proof. — Say $x^n = 0$, $x \in G(\pi)$. Then x restricts to zero in $G(\pi')$ for every $\pi' \in M$. This means that $x \in G^M(\pi)$. By the theorem we get $\delta x = 0$ and thus $x = 0$.

4. APPLICATIONS TO GROUP COHOMOLOGY AND TOPOLOGY. — In this section we give several “trivial” applications of the induction theorems in paragraph 3. We say “trivial” because, firstly, these applications are gotten by applying paragraph 3 only to modules of trivial types (ex. 2.9), and secondly, the results could be obtained equally well without appeal to paragraph 3. Nevertheless, we choose to call them “applications” of paragraph 3. Aside from being somewhat circuitous, the present approach seems to place certain facts in better perspective.

Consider the trivial Frobenius functor $\Phi_0(\mathbf{Z})$ on \mathcal{G}_π , defined in example 2.9. Write S for the collection of Sylow subgroups of π . We then have :

PROPOSITION 4.1 :

$$\Phi_0(\mathbf{Z})_S(\pi) = \Phi_0(\mathbf{Z})(\pi).$$

Proof. — Let π' be a Sylow subgroup and $i : \pi' \rightarrow \pi$ be the inclusion. Then by definition $I_*(1) = [\pi : \pi']$. Therefore $\Phi_0(\mathbf{Z})_S(\pi)$ is the principal ideal of \mathbf{Z} generated by GCD $([\pi : \pi'] : \pi' \in S)$. This GCD is clearly 1, hence the conclusion.

In example 2.9, we see that cohomology of groups defines a module over $\Phi_0(\mathbf{Z})$. Now by theorem 3.4 (with $\delta = 1$) we immediately deduce the following familiar facts in group cohomology.

COROLLARY 4.2 :

(I) If M is any π -module,

$$\hat{H}(\pi, M) = \sum_{\pi' \in S} \text{im}(\text{ver} : \hat{H}(\pi', M) \rightarrow \hat{H}(\pi, M));$$

(2) If $\hat{H}(\pi', M)$ has exponent d for every $\pi' \in S$, then $d\hat{H}(\pi, M) = 0$;

$$(3) \quad \bigcap_{\pi' \in S} \ker \{ \hat{H}(\pi, M) \rightarrow \hat{H}(\pi', M) \} = 0.$$

(Compare [19], prop. 4.7.)

Applying the same principles to a module defined by a kernel we can record :

COROLLARY 4.3. — *Let $f: P \rightarrow N$ be a morphism of π -modules. If the kernel of $\hat{f}: \hat{H}(\pi', P) \rightarrow \hat{H}(\pi', N)$ is of exponent d for every $\pi' \in S$, then $\ker(\hat{f}: \hat{H}(\pi, P) \rightarrow \hat{H}(\pi, N))$ also has exponent d .*

Noting that cohomology theory H of spaces also gives trivial module over $\Phi_0(\mathbf{Z})$ (concluding remark of example 2.9), we can state the analog of corollary 4.2 for H . Finally, viewing K^* as module over G^0 as in example 2.7 and anticipating the Brauer induction theorem for G^0 (th. 2.1, chap. 4), we can state :

COROLLARY 4.4:

(I) For any $X \in \mathcal{C}\mathcal{R}\mathcal{V}$:

$$K^*(X) = \sum_{X'} \text{im} (I_* : K^*(X') \rightarrow K^*(X));$$

$$(2) \quad \bigcap_{X'} \ker (f^* : K^*(X) \rightarrow K^*(X')) = 0.$$

In these statements, X' ranges over all finite covers $(i: X' \rightarrow X)$ of X whose fundamental groups $\pi_1(X')$ correspond to elementary subgroups of π .

When X is a finite CW-complex, (2) can be paraphrased into the following statement on bundles : a bundle ξ on X is stably trivial iff the inverse images $i^*\xi$ are stably trivial for any $i: X' \rightarrow X$ where $\pi_1(X')$ corresponds to an elementary subgroup of π .

In example 2.9 we have also defined the Frobenius functor \hat{G} , which was the completion of G with respect to the augmentation ideal of $\text{dim}: G \rightarrow \Phi_0(\mathbf{Z})$. Atiyah has shown ([3]) that if X is a classifying space for π , $\alpha: G \rightarrow K^*$ factors through $G \rightarrow \hat{G}$, so we have a homomorphism $\hat{\alpha}: \hat{G} \rightarrow K^*$. The main theorem in [3] is that $\hat{\alpha}$ is an equivalence of Frobenius functors. By virtue of (II) of theorem 3.4 (applied to $\ker \hat{\alpha}$ and $\text{coker } \hat{\alpha}$ as modules over G) it suffices to prove isomorphism for $\hat{\alpha}: \hat{G}(\pi') \rightarrow K^*(X')$ where $\pi_1(X') = \pi' = \text{elementary}$. In particular we can assume that the fundamental group $\pi_1(X')$ is solvable, which is what

Atiyah did (§ 10, solvable groups, in [3]), except that, without the present machinery, he had to set aside one section setting up “ the completion of Brauer’s theorem ” (§ 11 in [3]) to achieve the induction step.

CHAPTER 4.

GROTHENDIECK GROUPS AND WHITEHEAD GROUPS.

1. GROTHENDIECK GROUPS AND WHITEHEAD GROUPS AS FROBENIUS FUNCTORS AND MODULES. — Throughout this chapter, π will denote a fixed finite group, and we investigate Frobenius functors and modules on the category \mathcal{G}_π . To set up the machinery, we will show in the present section that there exists a basic functor G^0 , which is a Frobenius functor, and over which all other interesting functors behave as modules.

Let R be a Dedekind domain with field of quotients K . The groups $G^i(R\pi)$ have been defined in chapter 1 as the K^i of $\mathcal{M}(R\pi)$, the category of all (left) $R\pi$ -modules. Taking advantage of the presence of R , we can also consider $\mathcal{M}_{tf}(R\pi)$, the category of all R -torsion free ($= R$ projective) $R\pi$ -modules, and compute the K^i of this category. The resulting groups will be denoted by $G_{tf}^i(R\pi)$. According to Swan ([21], prop. 1.1), the maps $\theta^i : G_{tf}^i(R\pi) \rightarrow G^i(R\pi)$ defined by $[X] \rightarrow [X]$ for $i=0$ and $[X, \alpha] \rightarrow [X, \alpha]$ for $i=1$ are isomorphisms, so we can identify G^i with G_{tf}^i via θ^i . For $i=0$, write $[M] \cdot [N] = [M \otimes_R N]$ for $M, N \in \mathcal{M}_{tf}(R\pi)$, where $M \otimes_R N$ is regarded as a π -module by the diagonal action. It is easy to verify that this induces on $G_{tf}^0(R\pi)$ the structure of a commutative (associative) ring with unity ($= [R]$, R regarded as trivial π -module). Identifying $G_0(R\pi)$ with $G_{tf}^0(R\pi)$ via θ^0 , we see that $G^0(R\pi)$ carries the same structure. We shall speak of $G^0(R\pi)$ as the *Grothendieck ring of π* .

Consider a subgroup π' of π . The restriction operation in representation theory induces a map of Grothendieck groups : $G^0(R\pi) \rightarrow G^0(R\pi')$. If we write $i : \pi' \rightarrow \pi$ for the inclusion (a morphism in \mathcal{G}_π), we denote this induced map by i^* . On the other hand, the same morphism i in \mathcal{G}_π induces a map the other way $i_* : G^0(R\pi') \rightarrow G^0(R\pi)$, via the procedure of induced representation in group theory. Explicitly $i_*[N] = [R\pi \otimes_{R\pi'} N]$. With this set-up we can consider the functor $\pi' \mapsto G^0(R\pi')$ on \mathcal{G}_π . One easily checks ([21], § 1) that i^* is a ring homomorphism (functorially) and that the pair (i^*, i_*) satisfies the “ Frobenius reciprocity ” as set forth in (F1) of paragraph 1, chapter 3. Consequently G^0 is a Frobenius functor.

Consider, now, $K^0(R\pi)$, as defined in chapter 1. We let $G_{tf}^0(R\pi)$ operate on it by defining $[M] \cdot [P] = [M \otimes_R P]$, where $M \in \mathcal{M}_{tf}(R\pi)$, $P \in \mathcal{X}(R\pi)$. It is a lemma of Swan (prop. 5.1 of [21]) that our conditions on M and P imply $M \otimes_R P \in \mathcal{X}(R\pi)$. In this way $G^0(R\pi)$ also acts on $K^0(R\pi)$. Now

for $i : \pi' \rightarrow \pi$ in \mathcal{G}_π , we can define I^* and I_* for K^0 exactly as we did for G^0 and again we can check (F 2) and (F 3) of paragraph 1 in chapter 3. Consequently K^0 is a G^0 -module.

Next we turn to G^1 and K^1 and expect them to turn out as G^0 -modules as well. This is indeed the case. We briefly sketch the construction for K^1 as follows :

For $M \in G_{lf}(\mathbb{R}\pi)$ and $P \in \mathcal{P}(\mathbb{R}\pi)$, write

$$[M].[P, \alpha] = [M \otimes_{\mathbb{R}} P, {}_1M \otimes_{\mathbb{R}} \alpha] \in K^1(\mathbb{R}\pi).$$

Note that the right hand expression makes sense by the lemma of Swan quoted above. To show that the above equation gives a well-defined action, we must check two things :

(1) If we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \alpha' \downarrow & & \alpha \downarrow & & \alpha'' \downarrow & & \\ 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \end{array}$$

where rows are exact, vertical maps are automorphisms and $P', P, P'' \in \mathcal{P}(\mathbb{R}\pi)$, then

$$[M].[P, \alpha] = [M].[P', \alpha'] + [M].[P'', \alpha''].$$

(2) If α, β are automorphisms of P , then

$$[M].[P, \alpha\beta] = [M].[P, \alpha] + [M].[P, \beta].$$

We shall omit this routine verification. Via identification $\theta^0 : G_{lf}^0(\mathbb{R}\pi) \rightarrow G^0(\mathbb{R}\pi)$, we can therefore let $G^0(\mathbb{R}\pi)$ act on $K^1(\mathbb{R}\pi)$ [or $G^1(\mathbb{R}\pi)$]. Consider now a morphism $i : \pi' \rightarrow \pi$ in \mathcal{G}_π . We can define I^* and I_* for K^1 (and G^1) again exactly as we did for G^0, K^0 (now "with automorphisms"), and so K^1 (resp. G^1) becomes a candidate for G^0 -module, facing only the testing of (F 2) and (F 3) in chapter 3. We shall now illustrate the process by checking (F 2) (which is not as trivial as one would think) :

$$I_*(y.I^*(a)) = i_*(y).a \quad (a \in K^1(\mathbb{R}\pi), y \in G_{lf}^0(\mathbb{R}\pi)).$$

Without loss of generality, take $y = [Y], a = [A, \alpha]$ with $Y \in \mathcal{M}_{lf}(\mathbb{R}\pi), A \in \mathcal{P}(\mathbb{R}\pi)$, and $\alpha =$ automorphism of A :

$$\begin{aligned} I_*(y.I^*(a)) &= I_*([Y \otimes_{\mathbb{R}} A, {}_1Y \otimes_{\mathbb{R}} \alpha]) \\ &= [R\pi \otimes_{R\pi'} (Y \otimes_{\mathbb{R}} A), {}_1R\pi \otimes_{R\pi'} ({}_1Y \otimes_{\mathbb{R}} \alpha)]. \end{aligned}$$

We shall now make the identification [in $\mathcal{M}(\mathbb{R}\pi)$] :

$$\varphi : R\pi \otimes_{R\pi'} (Y \otimes_{\mathbb{R}} A) \xrightarrow{\cong} (R\pi \otimes_{R\pi'} Y) \otimes_{\mathbb{R}} A$$

defined by $\varphi(g \otimes (y \otimes a)) = (g \otimes y) \otimes g^{-1}a$, after Swan, ([21], lemma 1.1) where $g \in \pi$, $y \in Y$, and $a \in A$. We claim that the following diagram is commutative

$$\begin{array}{ccc} R\pi \otimes_{R\pi'} (Y \otimes_{R} A) & \xrightarrow{1 \otimes (1 \otimes \alpha)} & R\pi \otimes_{R\pi'} (Y \otimes_{R} A) \\ \varphi \downarrow & & \downarrow \varphi \\ (R\pi \otimes_{R\pi'} Y) \otimes_{R} A & \xrightarrow{(1 \otimes 1) \otimes \alpha} & (R\pi \otimes_{R\pi'} Y) \otimes_{R} A \end{array}$$

Indeed, operating on $g \otimes (y \otimes a)$ in the left upper corner

$$\begin{aligned} [(1 \otimes 1) \otimes \alpha] \varphi(g \otimes (y \otimes a)) &= ((1 \otimes 1) \otimes \alpha) ((g \otimes y) \otimes g^{-1}a) \\ &= (g \otimes y) \otimes \alpha(g^{-1}a) \\ &= (g \otimes y) \otimes g^{-1}(\alpha a) \\ &= \varphi(g \otimes (y \otimes \alpha a)) \\ &= \varphi(1 \otimes (1 \otimes \alpha))(g \otimes (y \otimes a)). \end{aligned}$$

Thus, under identification by φ ,

$$\begin{aligned} I_*(y \cdot I^*(a)) &= [(R\pi \otimes_{R\pi'} Y) \otimes_{R} A, (1 \otimes 1) \otimes \alpha] \\ &= i_*(y) \cdot a, \end{aligned}$$

as desired.

Replacing R by K , its quotient field, we get another Frobenius functor $\{\pi' \mapsto G^0(K\pi') : \pi' \in \mathcal{G}_\pi\}$. The family $\{G^0(R\pi') \rightarrow G^0(K\pi') : \pi' \in \mathcal{G}_\pi\}$ is easily seen to define a "homomorphism" of Frobenius functors in the sense of paragraph 1 of chapter 3. Thus any module over $\{G^0(K\pi')\}$ can be "pulled back" along this homomorphism, to become a module over $\{G^0(R\pi')\}$. This remark is in particular true of the modules $\{G^i(K\pi')\}$, $\{K^i(K\pi')\}$. It is easy to identify the new action of $G^0(R\pi')$ on these groups. For $K^0(K\pi')$, for example, we have

$$[M] \cdot [P] = [K \otimes_R M] \cdot [P] = [(K \otimes_R M) \otimes_R P] = [M \otimes_R P]$$

[where $M \in \mathcal{M}_{lf}(R\pi')$ and $P \in \mathcal{P}(K\pi')$].

From this point of view, $\{G^i(R\pi') \rightarrow G^i(K\pi')\}$, $\{K^i(R\pi') \rightarrow K^i(K\pi')\}$, $\{K^i(R\pi') \rightarrow G^i(R\pi')\}$ are morphisms of G^0 -modules, so their kernel and cokernel define new G^0 -modules. Let us identify some of these.

(1) For $K^0(R\pi') \rightarrow K^0(K\pi')$, the kernel is $C_0(R\pi')$, the reduced projective class group defined in paragraph 1 of chapter 1. $\{C_0(R\pi')\}$ therefore also defines a G^0 -module.

(2) For $K^1(R\pi') \rightarrow K^1(K\pi')$ in the case when K is a number field, the kernel is customarily denoted by $SK^1(R\pi')$. This therefore defines a new G^0 -module.

(3) $\alpha : K^0(R\pi') \rightarrow G^0(R\pi')$ is the Cartan map of $R\pi'$. The study of this map is most interesting when R is a "modular field", \bar{K} , in which case we will show (§ 3) that $\ker \alpha$ is the \mathfrak{o} -module, and $\text{coker } \alpha$ is a G^0 -module whose underlying groups are all p -groups ($p = \text{char } \bar{K}$), under appropriate assumptions.

(4) For $K^1(R\pi') \rightarrow G^1(R\pi')$, $G^1(R\pi') \rightarrow G^1(K\pi')$ and $G^0(R\pi') \rightarrow G^0(K\pi')$, we will be able to give estimates on the kernels and cokernels, in paragraphs 4 and 5.

At this point, it is appropriate to recall some general results. Suppose K has characteristic \mathfrak{o} . Then $K\pi$ is a semi-simple K -algebra in which $\Gamma = R\pi$ is an R -order. In particular, the finiteness theorems apply to $R\pi$, so $G^i(R\pi)$, $K^i(R\pi)$, $C_0(R\pi)$ and $SK^1(R\pi)$ are all finitely generated abelian groups. If we further assume that K is a number field, then we know $C_0(R\pi)$ is finite ([24], prop. 9.1) and $K^0(R\pi)$ has rank one ([22], th. A). The ranks of $G^i(R\pi)$ ($i = 0, 1$) and $K^1(R\pi)$ are as computed in chapter 1, theorem 3.2, $SK^1(R\pi)$ will be proved to be finite.

Finally we come to the definition of the Whitehead group of a finite group. For a fixed group π , we can define a homomorphism $\pm \pi \rightarrow K^1(\mathbf{Z}\pi)$ by bringing $\pm g (g \in \pi)$ to $[\mathbf{Z}\pi, \pm g]$, where the latter means the rank one free module $\mathbf{Z}\pi$ together with the automorphism defined by right multiplication of $\pm g$. The cokernel of this map is known as the Whitehead group of π , denoted by $\text{Wh}(\pi)$. In this way, however, it is not apparent that Wh can be set up as a module over the Frobenius functor G^0 . We will show, in the following, that nevertheless this could be done.

Recall, first of all, from example 2.7, of chapter 3, that group cohomology can be regarded as a module over $\{\pi' \mapsto G^0(K\pi')\}$, and hence over $\{\pi' \mapsto G^0(R\pi')\}$ by pullback. Let's now take this point of view and restrict attention to the submodule H given by first (integral) homology (or \hat{H}^{-2}). It is well known that on this submodule,

$$\{\hat{H}^{-2}(\pi', \mathbf{Z}) = \pi'_{ab} = \pi'/[\pi', \pi']\},$$

the induction map is induced by the identity and the restriction coincides with an operation which is classically known as the "transfer map." For $M \in \mathcal{M}_f(R\pi')$ and $x \in \pi'_{ab}$, the operation of $[M]$ on x is clearly given by $[M].x = x^{[K \otimes_{R^M} : K]}$. Finally, recall that $\Phi_1(\pm 1)$ denotes the trivial module over \mathcal{G}_π with constant values $\{\pm 1\}$, identity induction maps, and restriction maps = multiplication by an index. We can now state the "definition" of Whitehead group in the following fancy way :

THEOREM AND DEFINITION 1.1. — *We can define $\Phi_1(\pm 1) \rightarrow \{K^1(\mathbf{Z}\pi')\}$, and this is a monomorphism of modules over $\{G^0(\mathbf{Z}\pi')\}$, the cokernel of which*

is denoted by $\{\tilde{K}^1(\mathbf{Z}\pi')\}$. We can then define $\{H_1(\pi', \mathbf{Z})\} \rightarrow \{\tilde{K}^1(\mathbf{Z}\pi')\}$, and this is a monomorphism of modules over $\{G^0(\mathbf{Z}\pi')\}$, the cokernel of which is denoted by $\{Wh(\pi')\}$, called the G^0 -module defined by "Whitehead groups." The following is therefore an exact sequence of G^0 -modules

$$0 \rightarrow \{H_1(\pi', \mathbf{Z})\} \rightarrow \{\tilde{K}^1(\mathbf{Z}\pi')\} \rightarrow \{Wh(\pi')\} \rightarrow 0.$$

Proof. — To define $\Phi_1(\pm 1) \rightarrow K^1(\mathbf{Z}\pi')$, we just send $1 \rightarrow [\mathbf{Z}\pi', 1]$ and $-1 \rightarrow [\mathbf{Z}\pi, -1]$. This clearly gives a monomorphism of G^0 -modules. To define $\alpha: \pi'_{ab} \rightarrow \tilde{K}^1(\mathbf{Z}\pi')$, we take any element of π'_{ab} represented by $g \in \pi'$ and send it first to $[\mathbf{Z}\pi', g] \in K^1(\mathbf{Z}\pi')$ and then project the latter to $\tilde{K}^1(\mathbf{Z}\pi')$. It is not hard to see that this gives a monomorphism, but the point being made is that this definition respects: (1) module structure, (2) induction and (3) restriction, on the modules involved.

To start with, (2) presents no difficulty, because in both cases the induction maps are essentially "induced from the identity." To verify (3), take morphism $i: \pi' \rightarrow \pi''$ and consider $g \in \pi''$. We shall compute the restriction of its image $[\mathbf{Z}\pi'', g] \in K^1(\mathbf{Z}\pi'')$. Decompose π'' into coset spaces $\pi'' = \pi'g_1 \cup \pi'g_2 \cup \dots \cup \pi'g_m$. Then

$$I^*(\alpha(g)) = I^*[\mathbf{Z}\pi'', g] = [\mathbf{Z}\pi'', g],$$

where the last bracket is to be considered in \tilde{K} and $\mathbf{Z}\pi''$ is considered as a left $\mathbf{Z}\pi'$ -module. As such, $\mathbf{Z}\pi''$ has a free basis g_1, \dots, g_m and we can write the automorphism g by a matrix with entries from $\mathbf{Z}\pi'$. To find this matrix, write

$$g_j g = x_j g_{\varphi(j)}, \quad x_j \in \pi', \quad 1 \leq \varphi(j) \leq m.$$

φ is clearly a permutation of $1, 2, \dots, m$. The matrix of g with respect to the basis g_1, \dots, g_m is, up to permutation of rows (or columns), of the form $\text{diag}(x_1, x_2, \dots, x_m)$. Since we are computing the restriction in \tilde{K}^1 , permutation of rows does not affect $I^*(\alpha(g))$. But in $K^1(\mathbf{Z}\pi')$ already, $\text{diag}(x_1, \dots, x_m)$ and $x_1 x_2 \dots x_m$, as automorphisms on $(\mathbf{Z}\pi')^m$ and $\mathbf{Z}\pi'$, represent the same element, by a lemma of Whitehead ([4], lemma 1.7). Hence

$$I^*(\alpha(g)) = [\mathbf{Z}\pi', x_1 x_2 \dots x_m] \quad [\text{in } \tilde{K}^1(\mathbf{Z}\pi')].$$

However, on recalling the classical definition of transfer ([14], p. 201), we have $x_1 x_2 \dots x_m = \text{trans. } g$, so $I^*(\alpha(g)) = \alpha(I^* g)$, i. e. α respects induction.

We now check the module actions by $G^0(\mathbf{Z}\pi')$. Pick $[M] \in G_{lf}^0(\mathbf{Z}\pi')$, and $g \in \pi'$. First compute $[M] \cdot \alpha(g) = [M] \cdot [\mathbf{Z}\pi', g]$, everything done in \tilde{K}^1 . Let $\{e_\beta\}$ be a \mathbf{Z} -free-basis of M ; then $\{e_\beta \otimes 1\}$ is a free $\mathbf{Z}\pi'$ -basis

for $M \otimes_{\mathbf{Z}} \mathbf{Z}\pi'$ ([21], lemma 5.1). $[M].[\mathbf{Z}\pi', g] = [M \otimes_{\mathbf{Z}} \mathbf{Z}\pi', M \otimes_{\mathbf{Z}} g]$ so we must compute the matrix for $M \otimes_{\mathbf{Z}} g$ with respect to $\{e_{\beta} \otimes 1\}$. Now write

$$\begin{aligned} g^{-1}e_{\beta} &= \sum_{j=1}^n a_{j\beta} e_j, \quad a_{j\beta} \in \mathbf{Z}; \\ (M \otimes g)(e_{\beta} \otimes 1) &= e_{\beta} \otimes g = g \cdot (g^{-1}e_{\beta} \otimes 1) \\ &= g \cdot \left(\sum_{j=1}^n a_{j\beta} e_j \otimes 1 \right) \\ &= \sum_{j=1}^n a_{j\beta} g \cdot (e_j \otimes 1). \end{aligned}$$

The representing matrix for $M \otimes_{\mathbf{Z}} g$ on $M \otimes_{\mathbf{Z}} \mathbf{Z}\pi'$ is therefore

$$\begin{pmatrix} a_{11}g & \dots & a_{1n}g \\ a_{21}g & \dots & a_{2n}g \\ \vdots & \vdots & \vdots \\ a_{n1}g & \dots & a_{nn}g \end{pmatrix} = \begin{pmatrix} g & & \\ & g & \\ & & \ddots \\ & & & g \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Now $(a_{ij}) \in \text{GL}(n, \mathbf{Z}) = \pm \text{SL}(n, \mathbf{Z})$, so (a_{ij}) represents the trivial element of $\tilde{K}^1(\mathbf{Z}\pi')$. The first matrix, by the Whitehead lemma again, represents the element $[\mathbf{Z}\pi', g \dots g] = [\mathbf{Z}\pi', g^n]$. Therefore $[M].\alpha(g) = [\mathbf{Z}\pi', g^n] = \alpha(g^n)$. But $n = [M : \mathbf{Z}] = [M \otimes_{\mathbf{Z}} \mathbf{Q} : \mathbf{Q}]$ so $\alpha(g^n) = \alpha([M].g)$, by definition of action of $G^0(\mathbf{Z}\pi')$ on $\pi'_{ab} = \hat{H}^{-2}(\pi', \mathbf{Z})$.

Q. E. D.

We close the present section by the following remarks :

REMARKS 1.2 :

- (1) The composition $\text{SK}^1(\mathbf{Z}\pi') \rightarrow \text{K}^1(\mathbf{Z}\pi') \rightarrow \text{Wh}(\pi')$ defines a morphism of G^0 -modules $\{\text{SK}^1(\mathbf{Z}\pi') \rightarrow \text{Wh}(\pi')\}$, which is clearly a monomorphism;
- (2) Let K be any G^0 -module. Write $\text{Tor } K$ for the "torsion submodule" of K , i. e. the module $\{(\text{Tor } K)(\pi')\} = \{\text{Tor}(K(\pi'))\}$, where $\text{Tor}(K(\pi'))$ denotes elements of finite (additive) order in the additive group $K(\pi')$. We can therefore speak of $\text{Tor } K^1$, $\text{Tor } \text{Wh}$, and they are G^0 -modules. Since $\text{SK}^1(\mathbf{Z}\pi')$ is finite (to be proved), the morphism in (1) factors through

$$\{\text{SK}^1(\mathbf{Z}\pi')\} \rightarrow \{\text{Tor } \text{Wh}(\pi')\} \rightarrow \{\text{Wh}(\pi')\}.$$

An interesting conjecture ([6], § 11) is, that the first morphism is an isomorphism. In case π is abelian, this conjecture can easily be seen to be equivalent to the statement that units of finite

order in $\mathbf{Z}\pi$ are all of the form $\pm g$, $g \in \pi$. But this is a theorem of Higman ([16]). So the conjecture holds for abelian groups. The author can prove that the conjecture is true for the symmetric group S_3 and the quaternion group Q . But nothing beyond seems to be manageable.

- (3) Let $\mathcal{A}_i(\mathbf{R}\pi)$ denote the full subcategory of $\mathcal{M}(\mathbf{R}\pi)$ consisting of all (left) $\mathbf{R}\pi$ -modules of projective dimension ≤ 1 , which are \mathbf{R} -torsion. Denote the Grothendieck group of this category by $K_i^0(\mathbf{R}\pi)$. Meanwhile write $\mathcal{M}_i(\mathbf{R}\pi)$ for the subcategory of $\mathcal{M}(\mathbf{R}\pi)$ consisting of \mathbf{R} -torsion modules, with Grothendieck group $G_i^0(\mathbf{R}\pi)$. It is immediate that K_i^0 and G_i^0 are G^0 -modules and natural morphisms into these (or out from these) are G^0 -morphisms. The following diagram is therefore a commutative diagram of G^0 -modules, with exact rows :

$$\begin{array}{ccccccccc} \{K^1(\mathbf{R}\pi')\} & \longrightarrow & \{K^1(\mathbf{K}\pi')\} & \longrightarrow & \{K_i^0(\mathbf{R}\pi')\} & \longrightarrow & \{K^0(\mathbf{R}\pi')\} & \longrightarrow & \{K^0(\mathbf{K}\pi')\} \\ & & \cong \downarrow & & \downarrow & & \downarrow & & \downarrow \cong \\ & & \{G^1(\mathbf{K}\pi')\} & \longrightarrow & \{G_i^0(\mathbf{R}\pi')\} & \longrightarrow & \{G^0(\mathbf{R}\pi')\} & \longrightarrow & \{G^0(\mathbf{K}\pi')\} \longrightarrow 0 \end{array}$$

(for the top sequence, see [7], [9]; the bottom sequence was first observed by Heller and Reiner [15]).

2. INDUCTION THEOREMS FOR G^i AND K^i . — A group is an *elementary group* if it is a direct product of a cyclic group and a p -group (some prime p). It is a *hyerelementary group* if it has a normal cyclic subgroup with respect to which the quotient is a p -group (some prime p). For a fixed group π , we write C , E , and H respectively for the families of cyclic, elementary, and hyerelementary subgroups. In the terminology of paragraph 3 in chapter 3, we can talk of G_c^i , G^{ic} , K_{ii}^i , ... etc. ($i = 0, 1$). By identifying $G^0(\mathbf{K}\pi)$ with the character ring of π ($\text{char } \mathbf{K} = 0$), the classical induction theorems of Artin, Brauer and Witt can be translated into statements about the exponent of the quotient groups $G^0(\mathbf{K}\pi)/G_c^0(\mathbf{K}\pi)$, ... etc. These statements have been generalized by Swan to the case of integral representation, and we shall presently record his result. Recall that, by definition, the Artin exponent $A(\pi)$ of a finite group π is the (smallest) exponent of $G(\mathbf{Q}\pi)/G_c(\mathbf{Q}\pi)$, or the characteristic of this quotient ring.

THEOREM 2.1 (Swan [21], cor. 4.2). — *Let π be a fixed group with order n , and \mathbf{R} a commutative ring with unit. We write $d = (A(\pi), \Phi(n))$ where Φ is the Euler function. Then, in the notation of paragraph 3, chapter 3,*

- (a) $G_C^0(\mathbf{R}\pi)$ has exponent $A(\pi)^2$ in $G^0(\mathbf{R}\pi)$;
 (b) $G_E^0(\mathbf{R}\pi)$ has exponent d^2 in $G^0(\mathbf{R}\pi)$;
 (c) $G_H^0(\mathbf{R}\pi) = G^0(\mathbf{R}\pi)$.

If \mathbf{R} is a field we can replace $A(\pi)^2$ and d^2 in (a) and (b) by $A(\pi)$ and d . If \mathbf{R} contains a primitive n -th root of 1, we can replace d^2 in (b) by 1.

Using theorem 3.4 of chapter 3, we immediately see that the above theorem carries over verbatim for all G^0 -modules G^i , K^i , C_0 , SK^1 , Wh , ...! This, for example, covers theorem 2 of [6], theorem 6.8 of [18], etc. Also, we can write down an analogous theorem on the exponents of G^{iC} , K^{iH} , ... etc., covering corollary 9.4 of [21], ... and so forth. This and the following sections will be spent in exploiting these induction theorems in circumstances where we can analyze the induction parts G_C^i , K_E^i , Let's now record some corollaries of theorem 2.1.

COROLLARY 2.2. — Let K be a field (any characteristic). Then

$$K^{0C}(K\pi) = K^{0E}(K\pi) = K^{0H}(K\pi) = 0$$

and

$$G^{0C}(K\pi) = G^{0E}(K\pi) = G^{0H}(K\pi) = 0.$$

Proof. — $K^0(K\pi)$ is a free abelian group by the Krull-Schmidt theorem, and $G^0(K\pi)$ is free by the Jordan-Hölder theorem. We can then apply theorem 2.1 and appeal to corollary 3.5 of chapter 3. [The statement of the corollary is trivial, of course, when we can make an identification of $G^0(K\pi)$ with the K -character ring of π . But this could be achieved when and only when K has zero characteristic, cf. [12], p. 214.]

COROLLARY 2.3. — If π is a (not necessarily abelian) group having exponent 4 or 6, then $K^1(\mathbf{Z}\pi)$ is finite. Any irreducible \mathbf{Q} -representation of π must remain irreducible over \mathbf{R} .

Proof. — Use induction from, say, abelian subgroups. For the latter, since the exponent is either 4 or 6, the numbers of irreducible real and rational representations coincide. But the K^1 group has rank equal to the difference of these numbers, by theorem 3.2 of chapter 1. Hence $K^1(\mathbf{Z}\pi')$ is finite for all abelian $\pi' \subset \pi$. But $K_C^1(\mathbf{Z}\pi)$ is of finite index in $K^1(\mathbf{Z}\pi)$ by theorem 2.1 (for K^1), and the latter is finitely generated. Hence $K^1(\mathbf{Z}\pi)$ is finite. But then $\text{rank } K^1(\mathbf{Z}\pi) = 0$, so r , the number of real irreducible representations, equals q , the number of rational irreducible representations, by theorem 3.2 of chapter 1, again.

COROLLARY 2.4. — Let π be an abelian group. We say that a unit $u \in U(\mathbf{Z}\pi)$ "belongs" to a subgroup $\pi' \subset \pi$ if $u \in U(\mathbf{Z}\pi')$. Given any $u \in U(\mathbf{Z}\pi)$, $u^{A(\pi)^2}$ is a product of units belonging to cyclic subgroups.

Proof. — Since π is abelian, we can construct $K^1(\mathbf{Z}\pi) \rightarrow U(\mathbf{Z}\pi)$ by the determinant map (a split epimorphism). The kernel of this map can easily be seen to be $SK^1(\mathbf{Z}\pi)$. Thus, for any subgroup π' , we can think of $U(\mathbf{Z}\pi')$ as the quotient $K^1(\mathbf{Z}\pi')/SK^1(\mathbf{Z}\pi')$. This sets up $\{U(\mathbf{Z}\pi')\}$ as a G^0 -module, and the statement of the corollary then becomes the Induction Theorem 2.1 (a) for this G^0 -module.

3. APPLICATIONS TO MODULAR REPRESENTATIONS. — In this section we shall show how one can prove some well known theorems in modular representations by using the induction of paragraph 2. π will be a finite group, with order n , exponent e . R will denote the integers in a number field K , and \mathfrak{p} a fixed prime ideal of R , sitting over the (unique) rational prime p , which presumably divides $[\pi : 1]$. Write $\bar{K} = R/\mathfrak{p}$ for the residue field. We shall consider the Frobenius functor $\{G^0(\bar{K}\pi')\}$ on \mathcal{G}_π , and modules over it. Recall the theorem of Swan (th. 2.1) which says that $G_c^0(\bar{K}\pi)$ has exponent $A(\pi)$ in $G^0(\bar{K}\pi)$, and $G_E^0(\bar{K}\pi) = G^0(\bar{K}\pi)$ if K has a primitive n -th root of unity, ζ . We first recapture two theorems of Brauer (th. 3.1 and 3.2) :

THEOREM 3.1 (Brauer's theorem on injectivity of Cartan map). — *The Cartan map $\chi(\bar{K}\pi) : K^0(\bar{K}\pi) \rightarrow G^0(\bar{K}\pi)$ is injective.*

Proof. — By corollary 3.5 of chapter 3, we can assume that π is cyclic. But then $\bar{K}\pi$ is a direct product of commutative artinian local rings. If we write $\chi(\bar{K}\pi)$ as a matrix, in terms of the natural bases of the free abelian groups $K^0(\bar{K}\pi)$, $G^0(\bar{K}\pi)$, the matrix is diagonal, so χ is injective.

THEOREM 3.2 (Brauer's theorem on minors of decomposition matrix). — *Suppose $\zeta \in K$. Let $\delta(\pi)$ be the (unique) map $G^0(K\pi) \rightarrow G^0(\bar{K}\pi)$ which renders the following diagram commutative (Brauer-Swan) :*

$$\begin{array}{ccc} G^0(R\pi) & \longrightarrow & G^0(K\pi) \\ & \searrow & \swarrow \delta(\pi) \\ & & G^0(\bar{K}\pi) \end{array}$$

Write D for the matrix of $\delta(\pi)$ with respect to the canonical bases of the free abelian groups $G^0(K\pi)$ and $G^0(\bar{K}\pi)$. Let d_k be the GCD of the k by k minors of D . Then $d_k = 1$ for every k .

Proof. — By the theory of elementary divisors of matrices over \mathbf{Z} our statement on GCD of minors is tantamount to the fact that $\delta(\pi)$ is surjective. View $\{\delta(\pi') : \pi' \in \mathcal{G}_\pi\}$ as a morphism of modules over, say $\{G^0(R\pi')\}$, and consider the module $\{\text{coker } \delta(\pi')\}$. Since $\zeta = \sqrt[n]{1} \in K$,

to prove $\text{coker } \delta(\pi) = 0$, it suffices to prove that $\text{coker } \delta(\pi') = 0$ for every elementary subgroup π' of π , by theorem 2.1, so we may assume that π is elementary, and we can write $\pi = \pi_p \times \pi_{p'}$, a direct product, in which π_p is a p -group and $\pi_{p'}$ is p -free. But then $\delta(\pi) = \delta(\pi_p) \otimes \delta(\pi_{p'}) : G^0(K\pi_p) \otimes G^0(K\pi_{p'}) \xrightarrow{\sim} G^0(\overline{K}\pi_p) \otimes G^0(\overline{K}\pi_{p'})$ because K and \overline{K} are both splitting fields of π . It therefore suffices to deal with $\delta(\pi_p)$ and $\delta(\pi_{p'})$ individually. For $\pi_{p'}$, the decomposition map $\delta(\pi_{p'})$ is clearly an isomorphism, so there is no difficulty. For π_p , $G^0(\overline{K}\pi_p)$ is a free abelian group of rank one, with basis given by the trivial representation $\mathbf{1}$. Let $[M_1], \dots, [M_k] \in G^0(R\pi_p)$ be such that $[K \otimes_R M_1], \dots, [K \otimes_R M_k]$ form the natural basis of $G^0(K\pi_p)$, and $M_j \in \mathcal{M}_f(R\pi)$. We then have

$$\begin{aligned} \delta(\pi) [K \otimes_R M_j] &= \left[\frac{M_j}{\mathfrak{p}M_j} : \frac{R}{\mathfrak{p}} \right] \cdot \mathbf{1} \\ &= [M_j : K] \cdot \mathbf{1}. \end{aligned}$$

Therefore, the decomposition matrix D becomes a one-rowed matrix $([M_1 : K], \dots, [M_k : K])$ and the GCD of $\mathbf{1}$ by $\mathbf{1}$ minors is $\text{GCD}([M_1 : K], \dots, [M_k : K]) = \mathbf{1}$, because at least one $[M_j : K] = \mathbf{1}$ (for M_j corresponding to trivial representation of $R\pi_p$).

Q. E. D.

Remark. — The possibility of giving a proof in this fashion has also been noted by Giuritti in his thesis ([13]).

THEOREM 3.3 (*Brauer's theorem on cokernel of the Cartan map*). — Suppose $\chi(\overline{K}\pi) : K^0(\overline{K}\pi) \rightarrow G^0(\overline{K}\pi)$ is the Cartan map of the modular algebra $\overline{K}\pi$. Then $\text{coker } \chi(\overline{K}\pi)$ is an (abelian) p -group with exponent $[\pi^{(p)} : \mathbf{1}]$, where $\pi^{(p)}$ denotes a Sylow p -subgroup of π .

Proof. — We first handle the case when $\zeta \in K$. With this assumption we have $K_E(\overline{K}\pi) = K(\overline{K}\pi)$ for any module K over $\{G^0(\overline{K}\pi')\}$, by theorem 2.1. Applying this to $\{\text{coker } \chi(\overline{K}\pi')\}$, we see that we can work with elementary subgroups. So suppose π is elementary itself and write $\pi = \pi_p \times \pi_{p'}$ as in the preceding theorem, where π_p is a p -group and $\pi_{p'}$ is a p -free group. Then

$$\overline{K}\pi = \overline{K}[\pi_p \times \pi_{p'}] = \overline{K}[\pi_{p'}][\pi_p].$$

Since \overline{K} has characteristic prime to $[\pi_{p'} : \mathbf{1}]$, $\overline{K}[\pi_{p'}]$ is a (finite) semi-simple ring, and therefore a direct product of simple rings $R_1 \oplus \dots \oplus R_s$, where $R_j \cong \mathbf{M}_{m_j}(\overline{K}_j)$ is a full matrix algebra over a finite extension \overline{K}_j

of \bar{K} . We can further decompose

$$\begin{aligned}\bar{K}\pi &= (R_1 \oplus \dots \oplus R_s) [\pi_p] \\ &= R_1 [\pi_p] \oplus \dots \oplus R_s [\pi_p] \\ &\cong \mathbf{M}_{m_1}(\bar{K}_1) [\pi_p] \oplus \dots \oplus \mathbf{M}_{m_s}(\bar{K}_s) [\pi_p] \\ &\cong \mathbf{M}_{m_1}(\bar{K}_1 [\pi_p]) \oplus \dots \oplus \mathbf{M}_{m_s}(\bar{K}_s [\pi_p]).\end{aligned}$$

Since this is a direct product of rings, it suffices to handle the Cartan map of individual components. Now the categories $\mathfrak{M}(\mathbf{M}_{m_i}(\bar{K}_j[\pi_p]))$ and $\mathfrak{M}(\bar{K}_j[\pi_p])$ are equivalent by [7], so the Cartan maps of the two rings are identical and we are led back to the ring $\bar{K}_j[\pi_p]$, i. e. back to the case of p -groups. For the p -group π_p , the Cartan map $K^0(\bar{K}_j\pi_p) \rightarrow G^0(\bar{K}_j\pi_p)$ is a mapping between rank 1 free abelian groups. Picking natural bases for domain and range, the Cartan map is given by $\mathbf{1} \mapsto \text{length of } \bar{K}_j[\pi_p]$ as a $\bar{K}_j[\pi_p]$ -module. But this length = $\dim(\bar{K}_j[\pi_p] : \bar{K}_j) = [\pi_p : \mathbf{1}]$. Consequently the cokernel of $\chi(\bar{K}_j[\pi_p])$ is a p -group of exponent $[\pi_p : \mathbf{1}]$. [It is also easy to give a proof by noting, again, that $G^0(\bar{K}\pi) = G^0(\bar{K}\pi_p) \otimes G^0(\bar{K}\pi_{p'})$] ⁽³⁾.

To treat the case when $\zeta \notin K$, write $K' = \bar{K}(\zeta)$ and make an extension of scalars from \bar{K} to K' . We can prove (see footnote) that $K^0(\bar{K}\pi) \rightarrow K^0(K'\pi)$ and $G^0(\bar{K}\pi) \rightarrow G^0(K'\pi)$ are both splitting monomorphisms. Therefore $\text{coker } \chi(\bar{K}\pi)$ is isomorphic to a subgroup of $\text{coker } (\chi K'\pi)$, and we are back to the splitting case again.

To conclude this section, we should mention that the induction method can also be applied to some advantage to the study of indecomposable $\bar{K}\pi$ -modules. Let us write $\text{Gr}(\bar{K}\pi)$ for the modular representation ring of J. A. Green, i. e. $\text{Gr}(\bar{K}\pi)$ denotes the free abelian group generated by the indecomposable $\bar{K}\pi$ -modules, equipped with a multiplication induced by the tensor product. This Gr is, of course, no longer a G^0 -module, but one can still prove mild forms of induction theorems for it. Roughly speaking, the class M required in this case to exhaust a considerable part of $\text{Gr}(\bar{K}\pi)$ is the class of subgroups of π which have a "cyclic p -component". Now it has been conjectured by Green that the ring $\text{Gr}(\bar{K}\pi)$ is always free of nilpotent elements. Using an appropriate induction theorem for $\text{Gr}(\bar{K}\pi)$, and applying corollary 3.6 in chapter 3, we see that, in trying to prove Green's conjecture in the affirmative, we can

⁽¹⁾ Cf. SERRE, *Introduction à la théorie de Brauer*, in *Seminaire I. H. E. S.*, 1966, which has the same approach.

first make a reduction of the non- p -part of π . To be precise, we can prove the following theorem :

THEOREM 3.4. — *Let π be any finite group. Suppose, for any $\pi' \subset \pi$ which has a normal p -Sylow subgroup $\pi'^{(p)}$ such that $\frac{\pi'}{\pi'^{(p)}}$ is cyclic, we know that $\text{Gr}(\overline{\mathbf{K}}\pi')$ is free of nilpotent elements. Then $\text{Gr}(\overline{\mathbf{K}}\pi)$ has no nilpotent elements.*

For the proof, we first apply the induction technique to render the “ non- p -part ” of π cyclic; then the “ transfer theorem ” of Green enables us to assume that $\pi^{(p)}$ is actually normal. Details of this proof, and exposition of related ideas will be published elsewhere.

4. APPLICATIONS TO $\text{SK}^1(\mathbf{Z}\pi)$, $\text{Tor Wh}(\pi)$, ETC. — In this section we come back to the set-up of paragraph 1 and establish some estimate on exponents of certain groups. First of all we have

THEOREM 4.1. — *$\text{SK}^1(\mathbf{Z}\pi)$ is a finite abelian group of exponent $A(\pi)^2$. It is p -torsion free if the following conditions on the prime p are satisfied :*

- (1) *A Sylow p -subgroup $\pi^{(p)}$ of π is cyclic;*
- (2) *For any subgroup $P \subset \pi^{(p)}$ and any p -free cyclic subgroup D of π ;
 $P \subset N(D) \Rightarrow P \subset C(D)$.*

In particular $\text{SK}^1(\mathbf{Z}\pi)$ has no p -torsion under any of the following provisions :

- (a) *$\pi^{(p)}$ is normal cyclic;*
- (b) *$\pi^{(p)}$ is cyclic, and $(p, q - 1) = 1$ for any prime q dividing $[\pi : 1]$;*
- (c) *$\pi^{(p)}$ is cyclic and p is the largest prime divisor of $[\pi : 1]$.*

Proof. — By a theorem of Bass-Milnor (prop. 4.12, [10]), $\text{SK}^1(\mathbf{Z}\pi) = 0$ for any cyclic subgroup π' of π . Using (a) of theorem 2.1, we immediately get $A(\pi)^2$ as an exponent for $\text{SK}^1(\mathbf{Z}\pi)$. Since $\text{SK}^1(\mathbf{Z}\pi) \subset K^1(\mathbf{Z}\pi)$ and $K^1(\mathbf{Z}\pi)$ is finitely generated, we conclude that $\text{SK}^1(\mathbf{Z}\pi)$ is finite. Under (1) and (2), we know that $A(\pi)$ is free of p , by corollary 10 of chapter 2, hence $\text{SK}^1(\mathbf{Z}\pi)$ is void of p -torsion. By corollary 11 of chapter 2, any of (a), (b), (c) implies (1) and (2), which proves the second part of the theorem.

The same technique applies to prove.

THEOREM 4.2 :

- (1) *$\text{Tor Wh}(\pi)$ is finite of exponent $A(\pi)^2$;*
- (2) *$\text{Tor } K^1(\mathbf{Z}\pi)$ is finite of exponent $2e \cdot A(\pi)^2$ where e is an exponent of π .*

We only have to note that for a cyclic π , $\text{Tor Wh}(\pi) \cong \text{SK}^1(\mathbf{Z}\pi) = 0$, and $\text{Tor K}^1(\mathbf{Z}\pi) \cong \{\pm \pi\}$ is of exponent $2e$. From (1) and corollary 12 of chapter 2 we conclude :

COROLLARY 4.3. — $\text{Wh}(S_n)$ is a (finite) group with no p -torsion for $p > \frac{n}{2}$. [But we are unable to answer fully Milnor's question asking whether or not $\text{Wh}(S_n) = 0$ ([18], § 6).]

THEOREM 4.4. — Let π be a finite group, for which e is an exponent. Let ζ_e be a primitive e -th root of unity, and δ be an exponent for the abelian group $U(\mathbf{Z}[\zeta_e]/e)$. Then :

- (1) $\ker(K^1(\mathbf{Z}\pi) \rightarrow G^1(\mathbf{Z}\pi))$ has exponent $A(\pi)^2$;
- (2) $\text{coker}(K^1(\mathbf{Z}\pi) \rightarrow G^1(\mathbf{Z}\pi))$ has exponent $\delta A(\pi)^2$.

Proof. — Since $K^1(\mathbf{Z}\pi) \rightarrow K^1(\mathbf{Q}\pi)$ factors through $K^1(\mathbf{Z}\pi) \rightarrow G^1(\mathbf{Z}\pi)$, we have $\ker(K^1(\mathbf{Z}\pi) \rightarrow G^1(\mathbf{Z}\pi)) \subset \text{SK}^1(\mathbf{Z}\pi)$, so (1) follows from theorem 5.1. For (2), we refer to (a) of theorem 2.1, which says that it suffices to see that δ is an exponent for $\text{coker}(K^1(\mathbf{Z}\pi') \rightarrow G^1(\mathbf{Z}\pi'))$, for any cyclic subgroup π' of π . Write $B = \mathbf{Z}\pi'$ and write \bar{B} for the unique maximal order of $\mathbf{Q}\pi'$. Since e is an exponent for π' , e is a multiple of $[\pi' : 1]$, so $e\bar{B}$ is a common ideal of B and \bar{B} (cf. e. g. [22], lemma 5.1). By theorem 4.1 of chapter 1, $G^1(\mathbf{Z}\pi') \cong U(\bar{B}) = K^1(\bar{B})$. Now according to Bass-Murthy ([9], th. 7.2) there is a commutative diagram (associated to $f: B \rightarrow \bar{B}$ and $f' : \frac{B}{eB} \rightarrow \frac{\bar{B}}{e\bar{B}}$) :

$$\begin{array}{ccccc} K^1(B) & \xrightarrow{a} & K^1(\bar{B}) & \longrightarrow & K^1(\Phi f) \\ \downarrow & & \downarrow & & \downarrow \varphi \\ K^1\left(\frac{B}{eB}\right) & \xrightarrow{b} & K^1\left(\frac{\bar{B}}{e\bar{B}}\right) & \longrightarrow & K^1(\Phi f') \end{array}$$

in which the rows are exact, $K^1(\Phi f)$, $K^1(\Phi f')$ are some relative groups, and φ is an isomorphism. It follows in particular that the cokernel of a is isomorphic to a subgroup of the cokernel of b . We are then reduced to showing that δ is an exponent for $\text{coker}\left(K^1\left(\frac{B}{eB}\right) \rightarrow K^1\left(\frac{\bar{B}}{e\bar{B}}\right)\right)$. Since $\frac{\bar{B}}{e\bar{B}}$ is artinian, $K^1\left(\frac{\bar{B}}{e\bar{B}}\right) = U\left(\frac{\bar{B}}{e\bar{B}}\right)$, and we are done if we show that the latter has exponent δ . Now \bar{B} is a direct product of rings of cyclotomic integers of the form $\mathbf{Z}[\zeta_f]$ where $f|e$, so it suffices to examine $U\left(\frac{\mathbf{Z}[\zeta_f]}{e}\right)$. But $\mathbf{Z}[\zeta_f]$

is a pure subgroup of $\mathbf{Z}[\zeta_e]$, so the ring homomorphism $\frac{\mathbf{Z}[\zeta_f]}{e} \rightarrow \frac{\mathbf{Z}[\zeta_e]}{e}$ is a monomorphism, inducing a monomorphism for the units. Since δ is defined to be an exponent for $U\left(\frac{\mathbf{Z}[\zeta_e]}{e}\right)$, it is likewise an exponent for $U\left(\frac{\mathbf{Z}[\zeta_f]}{e}\right)$.

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Remark. — The exponent given in the theorem is not very satisfactory, and sometimes *ad hoc* arguments provide more refined estimates. Take, for example $\pi = \frac{\mathbf{Z}}{p\mathbf{Z}}$, where p is a prime. Then the cokernel of $K^1(\mathbf{Z}\pi) \rightarrow G^1(\mathbf{Z}\pi)$ is finite of exponent $\Phi(p) = p - 1$. We briefly sketch a proof (for the non-trivial case $p > 2$) as follows. $\mathbf{Q}\pi$ is a direct sum of two fields $\mathbf{Q} \oplus \mathbf{Q}(\zeta_p)$, therefore the maximal order is $\bar{B} = \mathbf{Z} \oplus \mathbf{Z}[\zeta_p]$. An element $(a, b) \in \mathbf{Z} \oplus \mathbf{Z}[\zeta_p]$ is caught in the image of the map $\mathbf{Z}\pi \rightarrow B$ if and only if $a \equiv b \pmod{(1 - \zeta_p)}$ in $\mathbf{Z}[\zeta_p]$. Now $(1 - \zeta_p)$ is the unique prime of $\mathbf{Z}[\zeta_p]$ lying over p and the prime $p \in \mathbf{Z}$ totally ramifies, so $\frac{\mathbf{Z}[\zeta_p]}{1 - \zeta_p}$ is a finite field of p elements. If $(a, b) \in B$ is a unit, we have $a = \pm 1$ and $b \in U(\mathbf{Z}[\zeta_p])$. In either case $a^{p-1} = 1 \equiv b^{p-1} \pmod{(1 - \zeta_p)}$ since p is odd. Hence $(a, b)^{p-1} = (a^{p-1}, b^{p-1})$ is caught in the image of $K^1(\mathbf{Z}\pi) \rightarrow G^1(\mathbf{Z}\pi)$, establishing our claim.

THEOREM 4.5. — *Let π be a finite group, for which e is an exponent. Let δ be defined as in the preceding theorem. Let λ be the LCM of the class numbers of all cyclotomic subrings $\mathbf{Z}[\zeta_f]$ of $\mathbf{Z}[\zeta_e]$. Then $\delta\lambda A(\pi)^2$ is an exponent for $C_0(\mathbf{Z}\pi)$ and $\lambda A(\pi)^2$ is an exponent for $\ker(G^0(\mathbf{Z}\pi) \rightarrow G^0(\mathbf{Q}\pi))$.*

Proof. — By the induction theorem, it suffices to prove that $\delta\lambda$ is an exponent for $C_0(\mathbf{Z}\pi')$, for any cyclic subgroup $\pi' \subset \pi$. Now for abelian groups π' , $C_0(\mathbf{Z}\pi')$ can be identified with $\text{Pic}(\mathbf{Z}\pi')$, the group of rank one projectives ([9], cor. 3.5) over $\mathbf{Z}\pi'$. Writing $B = \mathbf{Z}\pi'$ and $\bar{B} =$ maximal order in $\mathbf{Q}\pi'$, we know further from ([9], cor. 7.5) that the kernel of $\text{Pic } B \rightarrow \text{Pic } \bar{B}$ is isomorphic to $U\left(\frac{B}{eB}\right) \Big| U(\bar{B}) \cdot U\left(\frac{B}{eB}\right)$. By definition of λ , it follows easily that λ is an exponent for $\text{Pic } \bar{B}$. On the other hand the proof of the preceding theorem shows that δ is an exponent of $U\left(\frac{B}{eB}\right)$. Consequently $\delta\lambda$ is an exponent for $\text{Pic } B \approx C_0(\mathbf{Z}\pi)$. To prove the last statement in the theorem, we must prove that λ is an exponent for $\ker(G^0(\mathbf{Z}\pi') \rightarrow G^0(\mathbf{Q}\pi'))$, for π' cyclic. According to Swan ([22], cor. 2) there is an exact sequence

$$C_0(B) \rightarrow G^0(\mathbf{Z}\pi') \rightarrow G^0(\mathbf{Q}\pi') \rightarrow 0.$$

Since $C_0(\bar{B}) \cong \text{Pic } \bar{B}$, our conclusion follows immediately.

CHAPTER 5.

SOME EXPLICIT COMPUTATIONS.

1. ABELIAN GROUPS OF TYPE (p, p^n) . — Throughout this section, π denotes an abelian group of type (p, p^n) , i. e. $\pi \cong \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p^n\mathbf{Z}$, where p is a fixed prime, and $n \geq 0$. We aim to prove the following.

THEOREM 1.1 : $\mathrm{SK}^1(\mathbf{Z}\pi) = 0$.

For the proof, we shall rely heavily on the techniques of treating SK^1 developed by Bass, Milnor and Serre in [10], and especially on two of their main results, which we shall state as theorems 1.2 and 1.5.

A few remarks on notation will set the stage. K always denotes a number field and \mathfrak{p} an (integral) prime ideal of K . We write $K_{\mathfrak{p}}$ for the completion of K at \mathfrak{p} , and $U_{\mathfrak{p}}$ for the group of local units in $K_{\mathfrak{p}}$. We filter the latter by

$$U_{\mathfrak{p}}(n) = \{u \in U_{\mathfrak{p}} : \mathrm{ord}_{\mathfrak{p}}(1-u) \geq n\} = 1 + \mathfrak{p}^n.$$

Now suppose that K contains the group of m -th roots of unity, μ_m . The local m -th reciprocity symbol $\left(\frac{\cdot}{\mathfrak{p}}\right)_m$ is then defined ([2], chap. 11) and gives a non-degenerate anti-symmetric bilinear pairing of $\frac{K_{\mathfrak{p}}^*}{K_{\mathfrak{p}}^{*m}}$ with itself into μ_m . We can now state the following theorem in ([10], A. 17).

THEOREM 1.2 (Bass-Milnor-Serre). — *Suppose $m = p^n$, $\mu_m \subset K$, and that \mathfrak{p} lies over the rational prime p with ramification index e . Then, for any non-negative integer h ,*

$$\left(\frac{U_{\mathfrak{p}}(h), U_{\mathfrak{p}}}{\mathfrak{p}}\right)_{\rho^n} = \left(\frac{U_{\mathfrak{p}}(h+1), K_{\mathfrak{p}}^n}{\mathfrak{p}}\right)_{\rho^n} = \mu_{p^{n-j}},$$

where

$$j = j(h) = \left[\frac{h}{e} - \frac{1}{p-1} \right]_{[0, n]}.$$

Here, for $x \in \mathbf{R}$, $[x]$ denotes the largest integer $\leq x$, and for $a \in \mathbf{Z}$, $a_{[0, n]}$ denotes the nearest integer to a in the interval $[0, n]$.

We can record an immediate corollary.

COROLLARY 1.3. — *Let ζ_{p^n} be a primitive p^n -th root of unity, $K = \mathbf{Q}(\zeta_{p^n})$ and $\mathfrak{p} = (1 - \zeta_{p^n}) = (\lambda)$. Then, for any $1 \leq m \leq n$ and $h = p^{n-1}[(p-1)m + 1]$ we have*

$$\left(\frac{U_{\mathfrak{p}}(h), \lambda}{\mathfrak{p}}\right)_{\rho^n} = \mu_{p^{n-m+1}}.$$

Proof. — In fact $e = \text{ord}_p p = p^{n-1}(p-1)$, and

$$j(h) = \left[\frac{(p-1)m+1}{p-1} - \frac{1}{p-1} \right]_{[0, n]} = m.$$

Similarly

$$j(h-1) = \left[m - \frac{1}{p^{n-1}(p-1)} \right]_{[0, n]} = m-1.$$

First we claim that $\left(\frac{U_p(h), \lambda}{p} \right)_{p^n} \supset \mu_{p^{n-m+1}}$. Indeed, if otherwise, we would have $\left(\frac{U_p(h), \lambda}{p} \right)_{p^n} \subset \mu_{p^{n-m}}$. Since $\left(\frac{U_p(h), U_p}{p} \right)_{p^n} = \mu_{p^{n-j(h)}} = \mu_{p^{n-m}}$ by theorem 1.2, we have, by multiplicativity in the second variable : $\left(\frac{U_p(h), K_p^*}{p} \right)_{p^n} \subset \mu_{p^{n-m}}$. (Observe that λ is a local uniformizer.) However, again by theorem 1.2,

$$\left(\frac{U_p(h), K_p^*}{p^n} \right)_{p^n} = \mu_{p^{n-j(h-1)}} = \mu_{p^{n-m+1}},$$

a contradiction. To finish the proof, we suppose that $\left(\frac{U_p(h), \lambda}{p} \right)_{p^n} = \mu_{p^r}$, and proceed to show that $r \leq n-m+1$. But the chain of inclusions

$$\mu_{p^{n-j(h-1)}} = \left(\frac{U_p(h), K_p^*}{p} \right)_{p^n} \supset \left(\frac{U_p(h), \lambda}{p} \right)_{p^n} = \mu_{p^r}$$

implies that $r \leq n-j(h-1) = n-m+1$.

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COROLLARY 1.4. — *There exists an element a of the form $1 + \alpha\lambda^{p^n}$, where $\alpha \in \mathbf{Z}[\zeta_{p^n}]$, such that $\left(\frac{a, \lambda}{p} \right)_{p^n}$ is a primitive p^n -th root of unity.*

Proof. — In the preceding corollary, put $m=1$, so $h=p^n$. The conclusion of this corollary is then $\left(\frac{U_p(p^n), \lambda}{p} \right)_{p^n} = \mu_{p^n}$, which is the present corollary.

Write $A = \mathbf{Z}\pi$, where π is, for the moment, any finite abelian group. Let \bar{A} be the maximal order of $\mathbf{Q}\pi$, and \mathfrak{C} be the conductor from \bar{A} to A , i. e. $\mathfrak{C} = \{x \in \bar{A} : x\bar{A} \subset A\}$. Since \bar{A} is a direct product of rings $\bar{A} = \prod_{\rho} A_{\rho}$,

we have a corresponding decomposition $\mathfrak{C} = \prod_{\rho} \mathfrak{C}_{\rho}$ for the ideal \mathfrak{C} . Here

we can think of the indexing set $\{\rho\}$ as the set of irreducible rational representations of π . We shall need the following result from [10] (prop. 4.10 there) :

THEOREM 1.5 (Bass-Milnor-Serre). — *For a rational irreducible representation ρ , write k_{ρ} for the order of the kernel of the representation, and m_{ρ}*

for the degree of the representation. Let p be an odd prime; then the p -primary part of $\text{SK}^1(A_\rho, \mathfrak{C}_\rho)$ is cyclic of order p^j , where

$$j = \begin{cases} \min(\text{ord}_p(k_\rho), \text{ord}_p(m_\rho)) & \text{if } m_\rho > 2, \\ 0 & \text{if } m_\rho \leq 2. \end{cases}$$

The same result holds for $p = 2$, provided that m_ρ is even.

From here on π will denote again the abelian group of type (p, p^n) . We are now ready to begin the proof of theorem 1.1, which is by induction on n . For $n = 0$, the group π is cyclic, so we can appeal to theorem 4.1 in chapter 4. Suppose now $n > 0$ and $\pi = \langle x, y \rangle$, $x^p = y^{p^n} = 1$. Let $\pi' = y^{p^{n-1}}$ be the unique minimal subgroup of $\langle y \rangle$, and \mathfrak{B} be the kernel of the surjection $\mathbf{Z}\pi \rightarrow \mathbf{Z}\frac{\pi}{\pi'}$. Further, write \mathfrak{A} for the sum of components \mathfrak{C}_ρ of \mathfrak{C} which are contained in \mathfrak{A} . According to ([10], 4.6, 4.7) we have an exact sequence

$$(1.6) \quad \text{SK}^1(\mathbf{Z}\pi, \mathfrak{A}) \rightarrow \text{SK}^1(\mathbf{Z}\pi) \rightarrow \text{SK}^1\left(\mathbf{Z}\frac{\pi}{\pi'}\right)$$

where the first term stands for the “relative” SK^1 -group (see [4]). Since $\frac{\pi}{\pi'}$ is an abelian group of type (p, p^{n-1}) , we can suppose that $\text{SK}^1\left(\mathbf{Z}\frac{\pi}{\pi'}\right) = 0$ by an inductive hypothesis. The relative group $\text{SK}^1(\mathbf{Z}\pi, \mathfrak{A})$ can be identified with $\text{SK}^1(\overline{\mathbf{A}}, \mathfrak{A})$ by ([9], lemma 10.5), and hence breaks up into a direct sum $\bigoplus_\rho \text{SK}^1(A_\rho, \mathfrak{C}_\rho)$ where ρ ranges over all irreducible rational representations of π which are not trivial on π' . If we can verify that for any such ρ , the map $\text{SK}^1(A_\rho, \mathfrak{C}_\rho) \rightarrow \text{SK}^1(\mathbf{Z}\pi)$ is trivial, then we are finished, by exactness of (1.6). Since ρ is not trivial on π' , it must be one of the following representations

$$\rho_i : \begin{cases} x \rightarrow \zeta^{p^{n-1}i} \\ y \rightarrow \zeta \end{cases} \quad (\zeta = \zeta_{p^n}), \quad 1 \leq i \leq p.$$

The kernel of this character is therefore a subgroup of order $k_\rho = p$. Discarding the possibility that π is of type (2.2), we can apply theorem 1.5 to conclude that $\text{SK}^1(A_\rho, \mathfrak{C}_\rho)$ is cyclic of order p . The explicit isomorphism $\text{SK}^1(A_\rho, \mathfrak{C}_\rho) \cong \mu_p \subset A_\rho$ can be given by the “power residue symbol” ([10], Appendix), as follows. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents an element of $\text{SK}^1(A_\rho, \mathfrak{C}_\rho)$, the isomorphism sends this element to $\left(\frac{b}{a}\right)_p$. Now let's pick $a = 1 + \alpha\lambda^{p^n}$ as in corollary 1.4, so that $\left(\frac{a, \lambda}{p}\right)_{p^n}$ is a primitive p^n -th root of unity. Choose $b = \lambda^{p^{n+1}}$. Then, using formulas in [10],

$$\left(\frac{b}{a}\right)_p = \left(\left(\frac{b}{a}\right)_{p^n}\right)^{p^{n-1}} = \left(\left(\frac{\lambda}{a}\right)_{p^n}\right)^{p^{n-1}} = \left(\left(\frac{a, \lambda}{p}\right)_{p^n}\right)^{p^{n-1}} \neq 1,$$

where the passage from power residue symbol to p^n -reciprocity symbol is achieved by the reciprocity law, as follows :

$$\begin{aligned} \left(\frac{\lambda}{a}\right)_{p^n} &= \prod_{\mathfrak{p}|a} \left(\frac{\lambda}{\mathfrak{p}}\right)_{p^n}^{\text{ord}_{\mathfrak{p}} a} = \prod_{\mathfrak{p}|a} \left(\frac{\lambda, a}{\mathfrak{p}}\right)_{p^n} \\ &= \prod_{\mathfrak{p}|a} \left(\frac{a, \lambda}{\mathfrak{p}}\right)_{p^n} = \left(\frac{a, \lambda}{\mathfrak{p}}\right)_{p^n}. \end{aligned}$$

The last product degenerates into a single term because (1) there are no real primes \mathfrak{p} , (2) at the complex primes \mathfrak{p} , the symbols are trivial and (3) at primes $\mathfrak{p} \neq \lambda$, both a, λ are units so the symbols are again trivial.

Since $\left(\frac{b}{a}\right)_p \neq 1$, we see that any matrix (of $\det = 1$) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $(c, d) \equiv (0, 1) \pmod{\mathfrak{C}_\rho}$, represents a cyclic generator for $\text{SK}^1(\mathbb{A}_\rho, \mathfrak{C}_\rho)$. We suppose that $\rho = \rho_i$ and compute the image of this generator under the homomorphisms $\text{SK}^1(\mathbb{A}_{\rho_i}, \mathfrak{C}_{\rho_i}) \rightarrow \text{SK}^1(\mathbb{Z}\pi)$. Set

$$z = (y^{\rho^n-1} - 1) (x - y^{\rho^{n-1}}) (x - y^{2\rho^{n-1}}) \dots (x - y^{(\rho^{n-1})}) \dots (x - y^{\rho^n})$$

and consider the images of z under the coordinate projections associated with the various rational irreducible representations. We easily see that all these images are 0, except the one under the projection associated with ρ_i , which is the following element of \mathbb{A}_{ρ_i} : $\lambda^{\rho^{n-1}} \cdot \lambda^{(\rho-1)\rho^{n-1}} = \lambda^{\rho^n}$, up to a unit. We can then find $\theta \in \mathbb{Z}\pi$, so that the ρ_i -projection of θz is λ^{ρ^n} . The other projections of θz are clearly still zero. Find also $c \in \mathbb{Z}\pi$ so that the ρ_i -projection of c equals α (in corollary 1.4). Finally set $A = 1 + c\theta z$ and $B = (1 - y)\theta z$. Then, under ρ_i , A projects to the coordinate $1 + \alpha\lambda^{\rho^n} = a$, and B projects to the coordinate $(1 - \zeta)\lambda^{\rho^n} = \lambda^{\rho^{n+1}} = b$. Under any other projections A clearly goes to 1 and B to zero. Therefore, under the homomorphism $\text{SK}^1(\mathbb{A}_{\rho_i}, \mathfrak{C}_{\rho_i}) \rightarrow \text{SK}^1(\mathbb{Z}\pi)$ the element represented by $\begin{pmatrix} a & b \\ \star & \star \end{pmatrix}$ goes to the element represented by $\begin{pmatrix} A & B \\ \star & \star \end{pmatrix}$ in $\text{SK}^1(\mathbb{Z}\pi)$.

Suppose we designate the mapping $\text{SL}_2(\mathbb{Z}\pi) \rightarrow \text{SK}^1(\mathbb{Z}\pi)$ by $\begin{pmatrix} x & y \\ v & w \end{pmatrix} \rightarrow \begin{bmatrix} y \\ x \end{bmatrix}$ (well-defined!). Then

$$\begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} (1-y)\theta z \\ 1+c\theta z \end{bmatrix} = \begin{bmatrix} 1-y \\ 1+c\theta z \end{bmatrix} \times \begin{bmatrix} \theta z \\ 1+c\theta z \end{bmatrix} = \begin{bmatrix} 1-y \\ 1+c\theta z \end{bmatrix}$$

(by "multiplicativity of symbols", see [10]).

But z has a "factor" $1 - y$, therefore $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1-y \\ 1 \end{bmatrix} = 1$.

Q. E. D.

Remark. — Theorem 4.1 is also proved independently by Professor M. Kervaire.

2. ABELIAN GROUPS WHOSE p -PARTS HAVE CARDINALITY p^2 . — In this section we investigate the p -torsion of $\text{SK}^1(\mathbf{Z}\pi)$ for abelian groups π whose orders involve at most a second power of p . We show, using methods analogous to those in paragraph 1, that under a mild condition on $[\pi : \mathbf{1}]$, the group $\text{SK}^1(\mathbf{Z}\pi)$ is void of p -torsion. To be precise, we have.

THEOREM 2.1. — *Suppose π is an abelian group for which $\pi^{(p)}$ is of order p^2 . Suppose further that $(p, q - 1) = 1$ for any prime divisor q of $[\pi : \mathbf{1}]$. Then $\text{SK}^1(\mathbf{Z}\pi)$ has no p -torsion.*

[If $\pi^{(p)}$ is cyclic, then the theorem is a special case of theorem 4.1 in chapter 4, and so is true without any assumption on p . The case in point is, however, when $\pi^{(p)}$ has type (p, p) . So we suppose, in the following, that $\pi^{(p)} \approx \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z}$.]

Proof. — The whole proof of the theorem relies on the following lemma on power residue symbols.

LEMMA 2.2. — *Let $m = pm'$, where $(m', p) = 1 = (\Phi(m'), p)$. Suppose ζ is a (complex) primitive p -th root of unity, and ζ' a primitive m' -th root of unity. Write $\lambda = 1 - \zeta$ and write $\mathfrak{p} = (\lambda)$ for the ideal generated by λ in $\mathbf{Z}[\zeta]$. Then, for any given $c \in \mathbf{Z}[\zeta, \zeta']$ prime to $\mathfrak{p}\mathbf{Z}[\zeta, \zeta'] = \lambda\mathbf{Z}[\zeta, \zeta']$, there exists a natural number n such that, for $d = c^n$ and $a = 1 - \lambda^p d$, we have $\left(\frac{\lambda}{a}\right)_p = a$ primitive p -th root of unity, where $\left(\frac{\lambda}{a}\right)_p$ stands for the power residue symbol of $\mathbf{Z}[\zeta, \zeta']$ with respect to p .*

We shall postpone the proof of this lemma (which is unfortunately quite technical) and show first how we can use it to finish the proof of theorem 2.1.

According to ([10], 4.6, 4.7) we have a surjection

$$\psi : \bigoplus_{\rho \neq 1} \text{SK}^1(A_\rho, \mathfrak{C}_\rho) \rightarrow \text{SK}^1(\mathbf{Z}\pi) \rightarrow 0$$

where A_ρ are the components of the maximal order \overline{A} , \mathfrak{C}_ρ are the components of the conductor \mathfrak{C} from \overline{A} to A , and ρ 's stand for the various irreducible rational representations, which correspond in a one to one fashion to the components of \overline{A} . The surjection ψ restricts to a surjection of the p -parts of the two groups involved, so, to establish the theorem, it suffices to see that the restriction of ψ to $\text{SK}^1(A_\rho, \mathfrak{C}_\rho)^{(p)}$ is zero for each $\rho \neq 1$. Write $k = k_\rho$ for the order of the kernel of ρ , and $m = m_\rho$ for the degree of ρ . If one of the numbers $\text{ord}_p(k_\rho)$, $\text{ord}_p(m_\rho)$ is zero, then

$SK^1(A_\rho, \mathfrak{C}_\rho)^{(p)} = 0$ by theorem 1.5, so the issue is trivial. This enables us to deal exclusively with those ρ for which $\text{ord}_p(k_\rho) = \text{ord}_p(m_\rho) = 1$, and hence $SK^1(A_\rho, \mathfrak{C}_\rho)^{(p)} \cong \mu_p$. As remarked already in the proof of theorem 1.5, this isomorphism is established by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \left(\frac{b}{a}\right)_p$. Now write $m = m_\rho = pm'$, and $k = k_\rho = pk'$, fixing ρ ; and consider the cyclotomic ring $\mathbf{Z}[\zeta, \zeta']$, where ζ and ζ' are defined as in the lemma. Here $\Phi(m')$ is a product of factors of the form q or $q - 1$, where $q \mid [\pi : 1]$, $q \neq p$, so by hypothesis of theorem 2.1, $(\Phi(m'), p) = 1$. The lemma is therefore applicable. Write $m' = q_1^{a_1} \dots q_s^{a_s}$, and let ζ_i be a primitive $q_i^{a_i}$ -th root of unity in $\mathbf{Z}[\zeta, \zeta']$. Set $\theta_i = 1 - \zeta_i^{q_i^{a_i-1}}$ and $\lambda_i = 1 - \zeta_i$. We easily see that θ_i and $\lambda_i^{q_i^{a_i-1}}$ differ only by a unit in $\mathbf{Z}[\zeta, \zeta']$. Finally, set $\theta = \theta_1 \theta_2 \dots \theta_s$, $h = \theta k'$, and $c = h^p$. Since θ and k' are both prime to $\lambda \mathbf{Z}[\zeta, \zeta']$, the condition of lemma 2.2 on c is satisfied. We can therefore discover a a as described in the lemma, such that $\left(\frac{\lambda}{a}\right)_p = a$ primitive p -th root of unity. Write $b = \lambda^{p+1} d \in \mathbf{Z}[\zeta, \zeta'] = A_\rho$. Modulo the verification that $(a, b) \equiv (1, 0) \pmod{\mathfrak{C}_\rho}$, we see that there exists a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant one, with $(c, d) \equiv (0, 1) \pmod{\mathfrak{C}_\rho}$, such that α represents a generator of $SK^1(A_\rho, \mathfrak{C}_\rho)^{(p)}$. The reason is that

$$\left(\frac{b}{a}\right)_p = \left(\frac{\lambda^{p+1} d}{a}\right)_p = \left(\frac{\lambda^{p+1} h^{np}}{a}\right)_p = \left(\frac{\lambda}{a}\right)_p \left(\frac{\lambda h^n}{a}\right)_p^p = \left(\frac{\lambda}{a}\right)_p \neq 1.$$

Now pick y, y_1, y_2, \dots, y_s in π , such that their ρ coordinate projections are respectively $\zeta, \zeta_1, \zeta_2, \dots, \zeta_s$ in A_ρ ; and set

$$z = \frac{1}{p} (1 - y)^p (k')^{pn-1} \left(\sum_{x \in \pi_0} x \right) \left(\prod_{i=1}^s (1 - y_i^{q_i^{a_i-1}}) \right)^{pn},$$

where π_0 is the "kernel" of the representation ρ , consisting of elements of π having coordinate 1 under the projection associated with ρ . Note that $\frac{(1-y)^p}{p} \in \mathbf{Z}\pi$, so $z \in \mathbf{Z}\pi$. If $\rho' \neq \rho$ is any other irreducible rational representation, the coordinate projection of z under ρ' will be zero. Under the projection associated to ρ itself, the coordinate of z is

$$\begin{aligned} & \frac{1}{p} (1 - \zeta)^p (k')^{pn-1} \cdot (k) \cdot \left(\prod_{i=1}^s (1 - \zeta_i^{q_i^{a_i-1}}) \right)^{pn} \\ &= \lambda^p k'^{pn} \prod_{i=1}^s \theta_i^{pn} \\ &= \lambda^p (\theta k')^{pn} = \lambda^p (h^p)^n = \lambda^p c^n = \lambda^p d. \end{aligned}$$

From this it is clear that $\lambda^p d \in \mathfrak{C}_\rho$, so $(a, b) = (1 - \lambda^p d, \lambda^{p+1} d) \equiv (1, 0) \pmod{\mathfrak{C}_\rho}$. Now we come to the last stage of the argument. Set $A = 1 - z$ and $B = (1 - y)z$. Then (A, B) projects to $(1, 0)$ under any $\rho' \neq \rho$ and to (a, b) under ρ . Therefore a generator for $\psi(\text{SK}^1(A_\rho, \mathfrak{C}_\rho^{(p)}))$ could be taken as the element of $\text{SK}^1(\mathbf{Z}\pi)$ represented by a special linear matrix with first row (A, B) . Writing $\begin{bmatrix} B \\ A \end{bmatrix}$ for this element, we have

$$\begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} (1-y)z \\ 1-z \end{bmatrix} = \begin{bmatrix} 1-y \\ 1-z \end{bmatrix} \times \begin{bmatrix} z \\ 1-z \end{bmatrix} = \begin{bmatrix} 1-y \\ 1-z \end{bmatrix}$$

by "multiplicativity" of the symbols.

The last bracket is equal to 1 because z has a factor

$$\begin{aligned} \frac{1}{p} (1-y)^p &= \frac{1}{p} \left[1 - \binom{p}{1} y + \dots - \binom{p}{p-1} y^{p-1} + 1 \right] \\ &= [(-1)^1 y + (-1)^{p-1} y^{p-1}] \frac{\binom{p}{1}}{p} + \dots \\ &\quad + [(-1)^2 y + (-1)^{p-2} y^{p-2}] \frac{\binom{p}{2}}{p} + \dots \end{aligned}$$

and hence has a factor $1 - y$.

Q. E. D.

We now come to the proof of lemma 2.2, and the notations will again refer there. Take any prime ideal \mathfrak{p}_0 of $\mathbf{Z}[\zeta\zeta']$ which extends p , and consider the following diagram

$$\begin{array}{ccc} \mathbf{Z}[\zeta]/\mathfrak{p} & \rightarrow & \mathbf{Z}[\zeta\zeta']/\mathfrak{p}_0 \\ \parallel & & \parallel \\ \mathbf{F}_p & \longrightarrow & \mathbf{F}_{p^f} \end{array}$$

where f is the residue class field extension degree. Write a bar for passage to residue classes, and write Tr (resp. Tr^*) for the trace function from \mathbf{F}_{p^f} [resp. $\mathbf{F}_p(\bar{c})$] to \mathbf{F}_p . By assumption, we have $\bar{c} \neq 0$. We claim that there exists a natural number n such that $\text{Tr}^*(\bar{c}^n) \neq 0$. In fact, if $\text{Tr}^*(\bar{c}^j) = 0$ for all $j = 1, 2, \dots$, then the discriminant of the field extension $\mathbf{F}_p \subset \mathbf{F}_p(\bar{c})$, computed via a basis consisting of powers of \bar{c} , will be $\det \text{Tr}^*(\bar{c}^i \cdot \bar{c}^j) = 0$. This implies that $[\mathbf{F}_p(\bar{c}) : \mathbf{F}_p]$ is a multiple of p ([23], p. 93). Therefore f is also a multiple of p . But $fr = \Phi(m')$, where r is the number of distinct primes of $\mathbf{Z}[\zeta\zeta']$ which extend \mathfrak{p} . Hence $\Phi(m')$ is a multiple of p , a contradiction to our hypothesis. Say $d = c^n$, $\text{Tr}^*(\bar{d}) \neq 0$. We then have $\text{Tr}(\bar{d}) = [\mathbf{F}_{p^f} : \mathbf{F}_p(\bar{d})] \cdot \text{Tr}^*(\bar{d}) \neq 0$, since the square bracket is prime to p . Observe that with the

present choice of d , $d + d^p + d^{p^2} + \dots + d^{p^{f-1}} \in \mathbf{Z}[\zeta] - \varphi$, because $0 \neq \text{Tr}(\bar{d}) = \bar{d} + \sigma \bar{d} + \dots + \sigma^{f-1} \bar{d} \in \mathbf{F}_p$ for $\sigma =$ the Frobenius substitution for the Galois group $\text{Gal}(\mathbf{F}_{p^f}/\mathbf{F}_p)$. We claim now that this d just constructed meets the requirement of lemma 2.2. The argument leading up to this is however quite complicated.

Let $K = \mathbf{Q}(\zeta\zeta')$ and consider the completion $K_{\mathfrak{p}}$ of the number field K at any prime \mathfrak{p} of $\mathbf{Z}[\zeta\zeta']$ above \mathfrak{p} . We adjoin a p -th root of $a = 1 - \lambda^p d$ to $K_{\mathfrak{p}}$, say $A = a^{1/p}$, to get an extension $K_{\mathfrak{p}}(A)$ of the local field $K_{\mathfrak{p}}$. We first establish :

LEMMA 2.3. — $K_{\mathfrak{p}}(A)$ is an unramified extension of $K_{\mathfrak{p}}$.

Proof. — By definition of A , it satisfies the following equation

$$A^p - 1 + \lambda^p d = 0.$$

Set $B = \frac{A-1}{\lambda} \in K_{\mathfrak{p}}(A)$. Then B satisfies the following equation

$$\begin{aligned} 0 &= (\lambda B + 1)^p - 1 + \lambda^p d \\ &= \lambda^p B^p + \binom{p}{1} \lambda^{p-1} B^{p-1} + \dots + p \lambda B + \lambda^p d. \end{aligned}$$

Dividing by λ^p , we get :

$$(*) \quad 0 = B^p + \frac{\binom{p}{1} \lambda^{p-1} B^{p-1}}{\lambda^p} + \dots + \frac{\binom{p}{p-2} \lambda^2 B^2}{\lambda^p} + \frac{p \lambda B}{\lambda^p} + d.$$

Since p divides $\binom{p}{r}$ for $1 \leq r \leq p-1$, and $(p) = (\lambda)^{p-1}$ in $\mathbf{Z}[\zeta]$, we see that the above equation forces B to be integral over $\mathbf{Z}[\zeta\zeta']_{\mathfrak{p}}$; and also that

$$B^p + \left(\frac{p\lambda}{\lambda^p}\right) B + d \equiv 0 \pmod{\lambda}.$$

Now, in $\mathbf{Z}[\zeta]$, we have the congruence (cf. [2], p. 160) :

$$\frac{p\lambda}{\lambda^p} \equiv -1 \pmod{\lambda}$$

so we can rewrite the congruence in B as

$$B^p - B + d \equiv 0 \pmod{\lambda}.$$

Let $f(X) \in \mathbf{Z}[\zeta\zeta']_{\mathfrak{p}}[X]$ be the polynomial obtained from the LHS of (\star) by replacement of B by X . Then

$$\bar{f}(X) = X^p - X + \bar{d} \in (\mathbf{Z}[\zeta\zeta']_{\mathfrak{p}}/\hat{\mathfrak{p}})[X] = \mathbf{F}_{p^f}[X], \quad \text{where } \hat{\mathfrak{p}} = \mathfrak{p}\mathbf{Z}[\zeta\zeta']_{\mathfrak{p}}.$$

Since, by choice of d , $\bar{d} + \bar{d}_p + \dots + \bar{d}^{p^{f-1}} \neq 0$ we know that $f(X)$ is an irreducible polynomial in $\mathbf{F}_{p^f}[X]$, by a familiar theorem in finite field

theory ([1], th. 22). Now $f(B) = 0$ and \bar{f} is irreducible over the residue class field. These two facts together will imply ([20], chap. 3, § 6) that $K_{\mathfrak{p}}(B)$ is unramified over $K_{\mathfrak{p}}$. Since $K_{\mathfrak{p}}(A) = K_{\mathfrak{p}}(B)$, the lemma follows.

LEMMA 2.4. — Let \mathfrak{P} be any prime of $\mathbf{Z}[\zeta, \zeta']$ which lies over \mathfrak{p} and let $\left(\frac{a, \lambda}{\mathfrak{p}}\right)_p$ be the local reciprocity symbol for $K_{\mathfrak{p}}$ with respect to \mathfrak{p} . Then $\left(\frac{a, \lambda}{\mathfrak{p}}\right)_p$ is a primitive p -th root of unity. Furthermore this symbol is independent of the choice of \mathfrak{P} .

Proof. — To compute the symbol, we adjoin $A = a^{\frac{1}{p}}$ to $K_{\mathfrak{p}}$, and consider the automorphism ψ of $K_{\mathfrak{p}}(A)$ over $K_{\mathfrak{p}}$ that corresponds to the element λ under reciprocity. Since $K_{\mathfrak{p}}(A)$ is unramified over $K_{\mathfrak{p}}$ by lemma 2.3 and λ is a local uniformizer for $\mathbf{Z}[\zeta, \zeta']_{\mathfrak{p}}$, the automorphism ψ is precisely given by the Frobenius automorphism of $K_{\mathfrak{p}}(A)$ over $K_{\mathfrak{p}}$. By definition $\left(\frac{a, \lambda}{\mathfrak{p}}\right)_p = A^{\psi-1}$, so we must compute the latter. We can first compute B^{ψ} . To start with, I claim that, for any $n \geq 1$,

$$B^{p^n} \equiv B - (d + d^p + \dots + d^{p^{n-1}}) \pmod{\mathfrak{P}}.$$

Indeed, for $n = 1$, this is trivial. Assuming the congruence for $n = k - 1$, we get

$$\begin{aligned} B^{p^k} &= (B^{p^{k-1}})^p \equiv [B - (d + d^p + \dots + d^{p^{k-2}})]^p \\ &\equiv B^p - (d + d^p + \dots + d^{p^{k-2}})^p \\ &\equiv B - (d + d^p + \dots + d^{p^{k-1}}) \pmod{\mathfrak{P}}. \end{aligned}$$

Now by definition of a Frobenius automorphism, we have $B^{\psi} \equiv B^{p^f} \pmod{\mathfrak{P}}$, so by what we just said,

$$B^{\psi} \equiv B - (d + d^p + \dots + d^{p^{f-1}}) \pmod{\mathfrak{P}}$$

then

$$\begin{aligned} (2.5) \quad A^{\psi-1} &= \frac{(1 + \lambda B)^{\psi}}{1 + \lambda B} \\ &= (1 + \lambda B^{\psi})(1 - \lambda B + \lambda^2 B^2 \dots) \\ &\equiv 1 + \lambda(B^{\psi} - B) \pmod{\mathfrak{P}^2} \\ &\equiv 1 - \lambda(d + d^p + \dots + d^{p^{f-1}}) \pmod{\mathfrak{P}^2}. \end{aligned}$$

The contention of the lemma is that $A^{\psi-1} \neq 1$. Suppose the contrary. We get, from the above congruence: $\lambda(d + d^p + \dots + d^{p^{f-1}}) \in \mathfrak{P}^2$, hence $d + d^p + \dots + d^{p^{f-1}} \in \mathfrak{P}$. But then $d + d^p + \dots + d^{p^{f-1}} \in \mathfrak{P} \cap \mathbf{Z}[\zeta] = \mathfrak{p}$, a contradiction. Since λ and d are both independent of the choice of \mathfrak{P} , the congruence (2.5) clearly shows that $A^{\psi-1} \in \mu_p$ is independent of \mathfrak{P} .

Q. E. D.

The informed reader would not have failed to notice that the above argument is adapted from the proof of a theorem of Artin-Tate ([2], chap. 12, th. 8), of which lemma 2.4 is a mild generalization.

Finally we verify that $a = 1 - \lambda' d$ fulfills the conclusion of lemma 2.2, i. e.

COROLLARY 2.6. — *The power residue symbol $\left(\frac{\lambda}{a}\right)_p$ is a primitive p -th root of unity.*

Proof. — Repeating the arguments in the proof of theorem 1.5 for handling power residue symbols via the reciprocity law, we get

$$\left(\frac{\lambda}{a}\right)_p = \prod_{\mathfrak{p} | \mathfrak{p}} \left(\frac{a, \lambda}{\mathfrak{p}}\right)_p.$$

The number of factors involved in the product is r , the number of distinct primes of $\mathbf{Z}[\zeta']$ which extend \mathfrak{p} . All the factors are equal by lemma 2.4, and $rf = \Phi(m')$ implies that r is prime to p , so the product is not one.

Q. E. D.

3. THE SYMMETRIC GROUP S_3 . — In this section, π denotes the symmetric group S_3 , with elements (1) , (12) , etc. We shall conduct a very explicit computation with the various groups associated with the group ring $\mathbf{Z}\pi$, the upshot of which will be that $\text{SK}'(\mathbf{Z}\pi) = \text{Wh}(\pi) = \mathfrak{o}$ (theorem 3.9). The following notations will be fixed throughout this section.

$A = \mathbf{Z}\pi$, \bar{A} = the maximal order of $\mathbf{Q}\pi$ containing A (unique because \mathbf{Q} is a splitting field for π);

$\bar{\mathfrak{C}}$ = the conductor from \bar{A} to A ;

p_1, p_{-1}, p_2 will denote interchangeably the three irreducible representations of S_3 and the three projections of $\mathbf{Q}\pi$ into the simple constituents. Here p_1 = trivial representation, p_{-1} = the sign representation, and p_2 = the unique irreducible two dimensional representation.

Notice that for p_j , $|j|$ is its dimension;

$$\tilde{A} = \prod_{j=\pm 1, 2} \text{im}(A \rightarrow p_j \mathbf{Q}\pi) = \tilde{A}_1 \times \tilde{A}_{-1} \times \tilde{A}_2, \quad \tilde{\mathfrak{C}} = \text{conductor from } \tilde{A} \text{ to } A;$$

$$\bar{A} = \bar{A}_1 \times \bar{A}_{-1} \times \bar{A}_2;$$

$$\bar{\mathfrak{C}} = \bar{\mathfrak{C}}_1 \times \bar{\mathfrak{C}}_{-1} \times \bar{\mathfrak{C}}_2 \text{ and } \tilde{\mathfrak{C}} = \tilde{\mathfrak{C}}_1 \times \tilde{\mathfrak{C}}_{-1} \times \tilde{\mathfrak{C}}_2.$$

To start the computation, we must first determine A_j , \tilde{A}_j , \mathfrak{C}_j and $\tilde{\mathfrak{C}}_j$. We have $p_1 \mathbf{Q}\pi \cong \mathbf{Q}_1$, $p_{-1} \mathbf{Q}\pi \cong \mathbf{Q}_{-1}$, $p_2 \mathbf{Q}\pi \cong \mathbf{M}_2(\mathbf{Q})$ and $\bar{A}_1 \cong \mathbf{Z}_1$, $\bar{A}_{-1} \cong \mathbf{Z}_{-1}$, $\bar{A}_2 \cong \mathbf{M}_2(\mathbf{Z})$, where \mathbf{Q}_j (resp. \mathbf{Z}_j) denote copies of the rationals

(resp. integers). The identification $p_2 \mathbf{Q} \pi \cong \mathbf{M}_2(\mathbf{Q})$ can be set up by the following "choice of coordinates"

$$\begin{aligned} p_2(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & p_2(13) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & p_2(12) &= \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \\ p_2(23) &= \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, & p_2(123) &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, & p_2(132) &= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Now, by an easy computation which we omit, one gets

$$\bar{\mathbf{c}}_1 = 6\mathbf{Z}_1, \quad \bar{\mathbf{c}}_{-1} = 6\mathbf{Z}_{-1}, \quad \bar{\mathbf{c}}_2 = 3\mathbf{M}_2(\mathbf{Z}) = \mathbf{M}_2(3\mathbf{Z}).$$

We next determine $\tilde{\mathbf{A}}$. Since clearly $\tilde{\mathbf{A}}_1 = \mathbf{Z}_1$, $\tilde{\mathbf{A}}_{-1} = \mathbf{Z}_{-1}$, we need only compute $\tilde{\mathbf{A}}_2$.

LEMMA 3.1. — *The underlying abelian group of the subring $\tilde{\mathbf{A}}_2 \subset \mathbf{M}_2(\mathbf{Z})$ is free on the following basis :*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Also $[\mathbf{M}_2(\mathbf{Z}) : \tilde{\mathbf{A}}_2] = 3$.

Proof. — Say

$$\alpha = p_2(132) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \beta = p_2(13) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$p_2(12) = \alpha\beta, \quad p_2(23) = \beta\alpha \quad \text{and} \quad p_2(123) = \alpha^2.$$

It is easy, to begin with, to check that the four matrices enumerated in the lemma are \mathbf{Z} -independent. Among these,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \beta, \quad \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \alpha + \beta, \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \beta\alpha - \alpha;$$

so the free abelian groups generated by the four matrices is contained in $\tilde{\mathbf{A}}_2$. On the other hand,

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta\alpha = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \alpha,$$

and

$$\alpha\beta = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} - \beta - \alpha,$$

so we prove the first statement in the lemma. To determine the index of $\tilde{\mathbf{A}}_2$, we use the method of functionals. Define a \mathbf{Z} -linear map $g : \mathbf{M}_2(\mathbf{Z}) \rightarrow \mathbf{Z} \oplus \mathbf{Z}$ by sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $(a-d, b-c)$. This is an epimorphism whose

$$\text{kernel} = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbf{Z} \right\} = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \subset \tilde{\mathbf{A}}_2.$$

It thus induces an isomorphism $\mathbf{M}_2(\mathbf{Z})/\tilde{\mathbf{A}}_2 \cong (\mathbf{Z} \oplus \mathbf{Z})/g(\tilde{\mathbf{A}}_2)$. Now $g(\tilde{\mathbf{A}}_2)$ has free basis consisting of

$$g\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = (-1, 2) = \nu_1 \quad \text{and} \quad g\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = (1, 1) = \nu_2.$$

Hence

$$\begin{aligned} g(\tilde{\mathbf{A}}_2) &= \mathbf{Z}\nu_1 \oplus \mathbf{Z}\nu_2 = \mathbf{Z}(\nu_1 + \nu_2) \oplus \mathbf{Z}\nu_2 \\ &= \{ (b, b + 3a) : a, b \in \mathbf{Z} \}. \end{aligned}$$

We therefore get $[\mathbf{M}_2(\mathbf{Z}) : \tilde{\mathbf{A}}_2] = [\mathbf{Z}^2 : g(\tilde{\mathbf{A}}_2)] = 3$.

Next we have to study the reduced norm on $\mathbf{K}^1(\mathbf{A})$ and determine its image. Recall, from ([6], § 1) that the map N_{rd} is defined on $\mathbf{K}^1(\mathbf{A})$, with values in the unit group of the maximal order in Center $\mathbf{Q}\pi$. The latter is easily identified as

$$\mathbf{U}(\mathbf{Z}_1 \times \mathbf{Z}_{-1} \times \mathbf{Z}) = \mathbf{Z}_{1/2} \mathbf{Z}_1 \oplus \mathbf{Z}_{-1/2} \mathbf{Z}_{-1} \oplus \mathbf{Z}/2\mathbf{Z} \cong (\mathbf{Z}/2\mathbf{Z})^3.$$

We have :

LEMMA 3.2 : $N_{rd}(\mathbf{U}(\mathbf{Z}\pi)) = N_{rd}(\mathbf{K}^1(\mathbf{A}))$ and they coincide with the subgroup of $(\mathbf{Z}/2\mathbf{Z})^3$ consisting of the following four elements :

$$\begin{aligned} N_{rd}(1) &= (1, 1, 1), & N_{rd}(-1) &= (-1, -1, 1), & N_{rd}(12) &= (1, -1, -1), \\ N_{rd}(-(12)) &= (-1, 1, -1). \end{aligned}$$

Proof. — In the terminology of Bass ([4], chap. 1), the ring $\mathbf{Z}\pi$ has “ stable range ” 2, hence, to compute $N_{rd}(\mathbf{K}^1(\mathbf{A}))$, it suffices to consider only elements of $\mathbf{K}^1(\mathbf{A})$ represented by 2 by 2 invertible matrices over \mathbf{A} [which, in fact, exhaust $\mathbf{K}^1(\mathbf{A})$]. To make the computation possible, we must produce more expedient bases for $\mathbf{M}_2(\mathbf{Z}\pi)$. Think of $\mathbf{M}_2(\mathbf{Z}\pi)$ as contained in the maximal order $\mathbf{M}_2(\mathbf{Z}_1) \oplus \mathbf{M}_2(\mathbf{Z}_{-1}) \oplus \mathbf{M}_2(\mathbf{Z})$ of $\mathbf{M}_2(\mathbf{Q}\pi)$. We first perform a sequence of change of basis in $\mathbf{Z}\pi$:

$$\begin{aligned} (1) = e_1 &= \left(1, 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), & (1\bar{3}) = e_2 &= \left(1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \\ (1\bar{3}2) = e_3 &= \left(1, 1, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \right), & (12) = e_4 &= \left(1, -1, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right), \\ (2\bar{3}) = e_5 &= \left(1, -1, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \right), & (12\bar{3}) = e_6 &= \left(1, 1, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right). \end{aligned}$$

We then vary this basis to

$$\begin{aligned} f_1 = e_1 &= \left(1, 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), & f_2 = e_1 - e_3 &= \left(0, 0, \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \right), \\ f_3 = e_1 - e_6 &= \left(0, 0, \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \right), & f_4 = e_2 - e_4 &= \left(0, 0, \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \right), \\ f_5 = e_2 - e_5 &= \left(0, 0, \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \right), & f_6 = e_1 - e_2 &= \left(0, 2, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) \end{aligned}$$

then to

$$\begin{aligned} g_1 = f_1 &= \left(1, 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), & g_2 = f_6 &= \left(0, 2, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right), \\ g_3 = f_2 + f_5 &= \left(0, 0, \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}\right), & g_4 = f_3 + f_4 &= \left(0, 0, \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}\right), \\ g_5 = f_2 &= \left(0, 0, \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\right), & g_6 = f_2 + f_3 &= \left(0, 0, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right). \end{aligned}$$

Using the last set of basis elements for $\mathbf{Z}\pi$ we can write down a basis for $\mathbf{M}_2(\mathbf{Z}\pi)$, thus :

$$\begin{aligned} h_1 &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array} \right] \right], \\ h_2 &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{array}{c|c} 1 & -1 \\ -1 & 1 \\ \hline 0 & 0 \end{array} \right] \right], \\ \dots & \\ h_6 &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{array}{c|c} 3 & 0 \\ 0 & 3 \\ \hline 0 & 0 \end{array} \right] \right], \\ h_7 &= \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \left[\begin{array}{c|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array} \right] \right], \\ \dots & \\ h_{24} &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \left[\begin{array}{c|c} 0 & 0 \\ 0 & 3 & 0 \\ \hline 0 & 0 & 3 \end{array} \right] \right]. \end{aligned}$$

Take any unit in $\mathbf{M}_2(\Lambda)$, say $x = \sum_{j=1}^{24} a_j h_j \in \mathrm{GL}_2(\Lambda)$. We want to compute

its reduced norm. We know that the answer is of the form $(\pm 1, \pm 1, \pm 1)$ therefore it is harmless to carry out the computation mod 3. In terms of its component matrices, x is

$$\left[\begin{array}{cc} \begin{pmatrix} a_1 & a_7 \\ a_{13} & a_{19} \end{pmatrix}, & \begin{pmatrix} a_1 + 2a_2 & a_7 + 2a_8 \\ a_{13} + 2a_{14} & a_{19} + 2a_{20} \end{pmatrix}, \\ \left(\begin{array}{cccc} a_1 + a_2 + 2a_5 & -a_2 - a_5 & a_7 + a_8 + 2a_{11} & -a_8 - a_{11} \\ -a_2 + a_5 & a_1 + a_2 + a_5 & -a_8 + a_{11} & a_7 + a_8 + a_{11} \\ a_{13} + a_{14} + 2a_{17} & -a_{14} - a_{17} & a_{19} + a_{20} + a_{23} & -a_{20} - a_{23} \\ -a_{14} + a_{17} & a_{13} + a_{14} + a_{17} & -a_{20} + a_{23} & a_{19} + a_{20} + a_{23} \end{array} \right) \end{array} \right].$$

We get the third component of $N_{rd}(x)$ by evaluating the determinant of the 4 by 4 matrix, (mod 3) :

$$\begin{aligned}
 \det &\equiv \begin{vmatrix} a_1 & a_1 & a_7 & a_7 \\ -a_2 + a_5 & a_1 + a_2 + a_5 & -a_8 + a_{11} & a_7 + a_8 + a_{11} \\ a_{13} & a_{13} & a_{19} & a_{19} \\ a_{13} + a_{17} & a_{13} + a_{13} + a_{17} & -a_{20} + a_{23} & a_{19} + a_{20} + a_{23} \end{vmatrix} \\
 &\equiv \begin{vmatrix} a_1 & 0 & a_7 & 0 \\ -a_2 + a_5 & a_1 - a_2 & -a_8 + a_{11} & a_7 - a_8 \\ a_{13} & 0 & a_{19} & 0 \\ -a_{13} + a_{17} & a_{13} - a_{13} & -a_{20} + a_{23} & a_{19} - a_{20} \end{vmatrix} \\
 &\equiv a_1 \begin{vmatrix} a_1 - a_2 & -a_8 + a_{11} & a_7 - a_8 \\ 0 & a_{19} & 0 \\ a_{13} - a_{13} & -a_{20} + a_{23} & a_{19} - a_{20} \end{vmatrix} + a_7 \begin{vmatrix} -a_2 + a_5 & a_1 - a_2 & a_7 - a_8 \\ a_{13} & 0 & 0 \\ -a_{13} + a_{17} & a_{13} - a_{13} & a_{19} - a_{20} \end{vmatrix} \\
 &\equiv a_1 a_{19} \begin{vmatrix} a_1 - a_2 & a_7 - a_8 \\ a_{13} - a_{13} & a_{19} - a_{20} \end{vmatrix} - a_7 a_{13} \begin{vmatrix} a_1 - a_2 & a_7 - a_8 \\ a_{13} - a_{13} & a_{19} - a_{20} \end{vmatrix} \\
 &\equiv \begin{vmatrix} a_1 & a_7 \\ a_{13} & a_{19} \end{vmatrix} \times \begin{vmatrix} a_1 + 2a_2 & a_7 + 2a_8 \\ a_{13} + 2a_{13} & a_{19} + 2a_{20} \end{vmatrix} \pmod{3},
 \end{aligned}$$

i. e. the third coordinate is the product of the first and the second. It is then easy to conclude that $N_{rd}(x)$ is one of the four elements of $(\mathbf{Z}/2\mathbf{Z})^3$ enumerated in the lemma. An immediate consequence of the lemma is

COROLLARY 3.3 : $\text{Wh}(\pi) \approx \text{SK}^1(\mathbf{Z}\pi)$ and they are groups of exponent 4.

Proof. — The lemma shows $\text{K}^1(\mathbf{Z}\pi) = \text{SK}^1(\mathbf{Z}\pi) \oplus \text{im}(\pm \pi)$ where im refers to image of the natural map $\pm \pi \rightarrow \text{K}^1(\mathbf{Z}\pi)$. Thus

$$\text{Wh}(\pi) = \text{K}^1(\mathbf{Z}\pi) / (\text{im } \pm \pi) \approx \text{SK}^1(\mathbf{Z}\pi).$$

The last statement follows from corollary 4.3 of chapter 4.

LEMMA 3.4 : $\text{SK}^1(\bar{\mathbf{A}}, \bar{\mathbf{C}}) = \text{SK}^1(\bar{\mathbf{A}}) = 0$ and $\text{SK}^1(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$ is a 3-group.

Proof. — The first statement is the “congruence subgroup theorem” for the rational integers ([10], corollary 4.3). For the other statement, we need only handle the third component $\tilde{\mathbf{A}}_2$. Pick a big integer n and set

$$\mathbf{G} = \text{SL}_n(\tilde{\mathbf{A}}_2, \tilde{\mathbf{C}}_2) = \text{SL}_{2n}(\mathbf{Z}, 3\mathbf{Z}), \quad \tilde{\mathbf{G}} = \text{SL}_n(\tilde{\mathbf{A}}_2, \tilde{\mathbf{C}}_2).$$

Now $[\mathbf{G}, \mathbf{G}] \subset \text{SL}_{2n}(\mathbf{Z}, 9\mathbf{Z})$ and $[\text{E}_{2n}(\mathbf{Z}, 3\mathbf{Z}), \text{E}_{2n}(\mathbf{Z}, 3\mathbf{Z})] \supset \text{E}_{2n}(\mathbf{Z}, 9\mathbf{Z})$, by the formulas of ([4], § 1), so $\text{SL}_{2n}(\mathbf{Z}, 9\mathbf{Z}) \supset [\mathbf{G}, \mathbf{G}] \supset \text{E}_{2n}(\mathbf{Z}, 9\mathbf{Z})$. By the “congruence subgroup theorem” for \mathbf{Z} , the extreme ends are

equal, so they both equal $[G, G]$. Now look at the following sequence of Serre-Hochschild for the group extension $G \triangleleft \tilde{G}$:

$$(3.5) \quad \frac{G}{[G, \tilde{G}]} \rightarrow \frac{\tilde{G}}{[\tilde{G}, \tilde{G}]} \rightarrow \frac{\tilde{G}/G}{[\tilde{G}/G, \tilde{G}/G]} \rightarrow 1.$$

Since $\text{SK}'(\tilde{\mathcal{A}}_2, \tilde{\mathcal{C}}_2)$ is a quotient of $\frac{\tilde{G}}{[\tilde{G}, \tilde{G}]}$, it suffices to see that the latter is a 3-group. This in turn would follow from the statements :

- (a) $\frac{G}{[G, \tilde{G}]}$ is a 3-group;
 (b) $\frac{\tilde{G}/G}{[\tilde{G}/G, \tilde{G}/G]}$ is a 3-group

and the exactness of (3.5).

(a) and (b) are proved as follows.

(a) $\frac{G}{[G, \tilde{G}]}$ is a quotient of $\frac{G}{[G, G]}$, so we proceed to show that the latter is a 3-group. Now, using corollary 5.2, of [4], for the semi-local ring $\frac{\mathbf{Z}}{9\mathbf{Z}}$:

$$\frac{G}{[G, G]} = \frac{\text{SL}_{2n}(\mathbf{Z}, 3\mathbf{Z})}{\text{SL}_{2n}(\mathbf{Z}, 9\mathbf{Z})} \cong \text{SL}_{2n}\left(\frac{\mathbf{Z}}{9\mathbf{Z}}, \frac{3\mathbf{Z}}{9\mathbf{Z}}\right).$$

For simplicity, put $R = \frac{\mathbf{Z}}{9\mathbf{Z}}$ and $\mathfrak{A} = \frac{3\mathbf{Z}}{9\mathbf{Z}}$. Notice that $\mathfrak{A}^2 = 0$. If $\mathfrak{SL}_{2n}(\mathfrak{A})$ denotes the additive subgroup of $\mathbf{M}_{2n}(R)$ consisting of matrices with entries from \mathfrak{A} and trace 0, the map $M \mapsto I + M$ sets up an isomorphism between $\mathfrak{SL}_{2n}(\mathfrak{A})$ and $\text{SL}_{2n}(R, \mathfrak{A})$. Since the former has exponent 3, our claim follows.

(b) We have to handle the commutator quotient group of \tilde{G}/G . Since $\tilde{\mathcal{A}}_2/\bar{\mathcal{C}}_2$ is clearly semi-local, we can apply corollary 5.2 of [4] to obtain the following exact sequence :

$$1 \rightarrow \text{SL}_n(\tilde{\mathcal{A}}_2, \bar{\mathcal{C}}_2) \rightarrow \text{SL}_n(\tilde{\mathcal{A}}_2, \tilde{\mathcal{C}}_2) \rightarrow \text{SL}_n(\tilde{\mathcal{A}}_2/\bar{\mathcal{C}}_2, \tilde{\mathcal{C}}_2/\bar{\mathcal{C}}_2) \rightarrow 1.$$

Since $\text{SL}_n(\tilde{\mathcal{A}}_2, \bar{\mathcal{C}}_2) = \text{SL}_n(\bar{\mathcal{A}}_2, \bar{\mathcal{C}}_2) = \text{SL}_{2n}(\mathbf{Z}, 3\mathbf{Z})$ we immediately deduce that

$$\tilde{G}/G = \text{SL}_n(\tilde{\mathcal{A}}_2, \tilde{\mathcal{C}}_2) / \text{SL}_n(\tilde{\mathcal{A}}_2, \bar{\mathcal{C}}_2) \approx \text{SL}_n(\tilde{\mathcal{A}}_2/\bar{\mathcal{C}}_2, \tilde{\mathcal{C}}_2/\bar{\mathcal{C}}_2).$$

Now $\tilde{\mathcal{A}}_2/\bar{\mathcal{C}}_2$ is an \mathbf{F}_3 -algebra, in which the ideal $\tilde{\mathcal{C}}_2/\bar{\mathcal{C}}_2$ has square zero (this would be made clear in a later lemma), so we can repeat the argument

in (a) to conclude that \tilde{G}/G is a 3-group, hence so is its commutator quotient group.

COROLLARY 3.5. — $SK^1(A, \tilde{\mathcal{C}})$ is a 3-group.

Proof. — By lemma 10.5 of [9], we have $SL(A, \tilde{\mathcal{C}}) = SL(\tilde{A}, \tilde{\mathcal{C}})$ and $E(A, \tilde{\mathcal{C}}) = E(\tilde{A}, \tilde{\mathcal{C}})$. Observe that this result that we quoted from [9] depends only on the fact that the projections $p_j: A \rightarrow \tilde{A}_j$ are surjective, but not on the commutativity of the ring A . We thus have

$$\begin{aligned} SK^1(A, \tilde{\mathcal{C}}) &= SL(A, \tilde{\mathcal{C}})/E(A, \tilde{\mathcal{C}}) \\ &= SL(\tilde{A}, \tilde{\mathcal{C}})/E(\tilde{A}, \tilde{\mathcal{C}}) \\ &= SK^1(\tilde{A}, \tilde{\mathcal{C}}) \end{aligned}$$

and this is a 3-group by our lemma.

We next claim

LEMMA 3.6. — $k: K^1(A/\tilde{\mathcal{C}}) \rightarrow K^1(\tilde{A}/\tilde{\mathcal{C}})$ is a monomorphism.

Proof. — Since both rings are artinian, our conclusion follows if we show that $\tilde{A}/\tilde{\mathcal{C}}$ is commutative, because then k is identifiable with the inclusion map $U(A/\tilde{\mathcal{C}}) \rightarrow U(\tilde{A}/\tilde{\mathcal{C}})$ (by §1, chap. 1). Now $\tilde{A}_1/\tilde{\mathcal{C}}_1$ and $\tilde{A}_{-1}/\tilde{\mathcal{C}}_{-1}$ are both isomorphic to $\frac{\mathbf{Z}}{6\mathbf{Z}}$, so it suffices to examine

$$\tilde{A}_2/\tilde{\mathcal{C}}_2 \approx (\tilde{A}_2/\overline{\mathcal{C}}_2)/(\tilde{\mathcal{C}}_2/\overline{\mathcal{C}}_2).$$

Since $\overline{\mathcal{C}}_2 = \mathbf{M}_2(3\mathbf{Z})$, everything in sight is a vector space over \mathbf{F}_3 . By lemma 3.1, $\tilde{A}_2/\overline{\mathcal{C}}_2$ is 3-dimensional over \mathbf{F}_3 . If we can show that $\tilde{\mathcal{C}}_2$ is bigger than $\overline{\mathcal{C}}_2$, then $\tilde{A}_2/\tilde{\mathcal{C}}_2$ is two-dimensional algebra over \mathbf{F}_3 and we are done. It suffices to show that $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \in \tilde{\mathcal{C}}_2$. Let's use the basis $g_j (1 \leq j \leq 6)$ for A . Then a matrix α belongs to $\tilde{\mathcal{C}}_2$ if and only if $\alpha\beta \in A$ and $\beta\alpha \in A$ for any β which is the last component of some g_j . Since we can compute mod 3, there are only two non-trivial choices for β , namely,

$$\beta_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (\text{for } g_2) \quad \text{and} \quad \beta_2 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \quad (\text{for } g_6).$$

For $\alpha = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ we have

$$\begin{aligned} \beta_1\alpha &\equiv \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \equiv 2 \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \pmod{3}, \\ \alpha\beta_1 &\equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{3} \end{aligned}$$

and also

$$\beta_2 \alpha \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \equiv \alpha \beta_2 \pmod{3},$$

so indeed $\alpha \in \tilde{\mathcal{C}}_2$.

LEMMA 3.7. — *The composite map in the following commutative diagram is zero*

$$\begin{array}{ccc} \mathrm{SK}^1(\mathbf{A}) & \xrightarrow{j} & \mathrm{K}^1(\mathbf{A}/\tilde{\mathcal{C}}) \\ \downarrow & & \downarrow k \\ \mathrm{SK}^1(\tilde{\mathbf{A}}) & \longrightarrow & \mathrm{K}^1(\tilde{\mathbf{A}}/\tilde{\mathcal{C}}) \end{array}$$

Proof. — Since

$$\mathrm{K}^1(\tilde{\mathbf{A}}/\tilde{\mathcal{C}}) = \mathrm{K}^1(\tilde{\mathbf{A}}_1/\tilde{\mathcal{C}}_1) \oplus \mathrm{K}^1(\tilde{\mathbf{A}}_{-1}/\tilde{\mathcal{C}}_{-1}) \oplus \mathrm{K}^1(\tilde{\mathbf{A}}_2/\tilde{\mathcal{C}}_2)$$

and $\mathrm{SK}^1(\mathbf{A}) \rightarrow \mathrm{K}^1(\tilde{\mathbf{A}}_j/\tilde{\mathcal{C}}_j)$ are clearly trivial for $j = 1, -1$, we need only handle the third component of $\mathrm{K}^1(\tilde{\mathbf{A}}/\tilde{\mathcal{C}})$. We take the basis g_j ($1 \leq j \leq 6$) for \mathbf{A} and perform another unimodular transformation :

$$\begin{aligned} g'_1 = g_1 &= \left(1, 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), & g'_2 = g_1 - g_2 &= \left(1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right), \\ g'_3 = g_3 &= \left(0, 0, \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix} \right), & g'_4 = g_4 &= \left(0, 0, \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix} \right), \\ g'_5 = g_5 &= \left(0, 0, \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \right), & g'_6 = g_6 &= \left(0, 0, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right). \end{aligned}$$

Take a 2 by 2 matrix x over \mathbf{A} representing (any) element of $\mathrm{SK}^1(\mathbf{A})$. Following the proof of lemma 3.2 we can write down a \mathbf{Z} -basis h'_j ($1 \leq j \leq 24$) for $\mathbf{M}_2(\mathbf{A})$, via the basis g'_j for \mathbf{A} . The x in question is therefore $x = \sum_{i=1}^{24} a_i h'_i$.

Since its reduced norm is 1, we have the following two equations

$$\begin{vmatrix} a_1 + a_2 & a_7 + a_8 \\ a_{13} + a_{14} & a_{19} + a_{20} \end{vmatrix} = 1, \quad \begin{vmatrix} a_1 - a_2 & a_7 - a_8 \\ a_{13} - a_{14} & a_{19} - a_{20} \end{vmatrix} = 1.$$

From these we get

$$\begin{aligned} \begin{vmatrix} a_1 & a_7 \\ a_{13} & a_{19} \end{vmatrix} + \begin{vmatrix} a_2 & a_8 \\ a_{14} & a_{20} \end{vmatrix} &= 1, \\ \begin{vmatrix} a_1 & a_7 \\ a_{14} & a_{20} \end{vmatrix} + \begin{vmatrix} a_2 & a_8 \\ a_{13} & a_{19} \end{vmatrix} &= 0. \end{aligned}$$

To compute the image of x in $\mathrm{K}^1(\tilde{\mathbf{A}}_2/\tilde{\mathcal{C}}_2)$, we can reduce mod 3 ($\tilde{\mathcal{C}}_2 \subset \tilde{\mathcal{C}}_2!$) and ignore $\alpha = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \in \tilde{\mathcal{C}}_2$. This image is thus the following :

$$\begin{aligned} & \left[a_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \cdot \left[a_{19} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_{20} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \\ & - \left[a_7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_8 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \cdot \left[a_{13} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_{14} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]. \end{aligned}$$

Writing $I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ and $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ ($J^2 = 0$), this becomes

$$\begin{aligned} & (a_1 I + a_2 J) (a_{19} I + a_{20} J) - (a_7 I + a_8 J) (a_{13} I + a_{14} J) \\ &= [(a_1 a_{19} + a_2 a_{20}) I + (a_2 a_{19} + a_1 a_{20}) J] \\ &\quad - [(a_7 a_{13} + a_8 a_{14}) I + (a_8 a_{13} + a_7 a_{14}) J] \\ &= \left(\begin{vmatrix} a_1 & a_7 \\ a_{13} & a_{19} \end{vmatrix} + \begin{vmatrix} a_2 & a_8 \\ a_{14} & a_{20} \end{vmatrix} \right) I + \left(\begin{vmatrix} a_1 & a_7 \\ a_{14} & a_{20} \end{vmatrix} + \begin{vmatrix} a_2 & a_8 \\ a_{13} & a_{19} \end{vmatrix} \right) J = I. \end{aligned}$$

So the image is zero, as claimed.

We just need one more lemma to finish :

LEMMA 3.8. — *The sequence $SK^1(A, \tilde{\mathfrak{C}}) \xrightarrow{i} SK^1(A) \xrightarrow{j} K^1(A/\tilde{\mathfrak{C}})$ is exact with $\text{im } i = \ker j = 0$. In particular j is a monomorphism.*

Proof. — Consider the commutative diagram

$$\begin{array}{ccccc} K^1(A, \tilde{\mathfrak{C}}) & \xrightarrow{i} & K^1(A) & \longrightarrow & K^1(A/\tilde{\mathfrak{C}}) \\ \cup & & \cup & & \nearrow j \\ SK^1(A, \tilde{\mathfrak{C}}) & \xrightarrow{i} & SK^1(A) & & \end{array}$$

where the top sequence is a section of the “ K-theory exact sequence ” for the map $A \rightarrow A/\tilde{\mathfrak{C}}$. It follows immediately that $\xrightarrow{i} \xrightarrow{j}$ is exact, since $l^{-1}(SK^1(A)) = SK^1(A, \tilde{\mathfrak{C}})$. Now $SK^1(A, \tilde{\mathfrak{C}})$ is a 3-group by corollary 3.5 and $SK^1(A)$ is a 2-group by corollary 3.3. Hence $\text{im } i = 0$.

We finally get

THEOREM 3.9 : $SK^1(\mathbf{Z}\pi) \approx \text{Wh}(\pi) = 0$.

Proof. — We refer to the diagram in lemma 3.7. There, we have $kj = 0$. Since k is a monomorphism (lemma 3.6) we have $j = 0$. But by lemma 3.8, j is a monomorphism, therefore $SK^1(\mathbf{Z}\pi) = 0$. Appealing now to corollary 3.3, $\text{Wh}(\pi) = 0$, which finishes the proof of theorem 3.9.

BIBLIOGRAPHY.

- [1] A. A. ALBERT, *Fundamental concepts of higher algebra*, University of Chicago Press, 1956.
- [2] E. ARTIN and J. TATE, *Class Field Theory*, Harvard University Notes, 1961.
- [3] M. ATIYAH, *Characters and cohomology of finite groups (Publications mathématiques, I. H. E. S., No. 9)*.
- [4] H. BASS, *K-theory and stable algebra (Publications mathématiques, I. H. E. S., No. 22, p. 4-59)*.
- [5] H. BASS, A. HELLER and R. SWAN, *The Whitehead group of a polynomial extension (Publications mathématiques, I. H. E. S., No. 22, p. 61-79)*.
- [6] H. BASS, *The Dirichlet unit theorem, induced characters, and Whitehead groups of finite groups (Topology, vol. 4, 1966, p. 391-410)*.

- [7] H. BASS, *Topics in algebraic K-theory*, Tata Institute Notes, 1966.
- [8] H. BASS and J. MILNOR, *On the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$)* (mimeographed notes of Institute for Advanced Studies, Princeton, 1966).
- [9] H. BASS and M. P. MURTHY, *Grothendieck groups and Picard groups of abelian group rings* (to appear in *Annals of Math.*).
- [10] H. BASS, J. MILNOR and J.-P. SERRE, *Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$)*, to appear in *Publications mathématiques*, I. H. E. S.
- [11] H. CARTAN and S. EILENBERG, *Homological algebra*, Princeton University Press, 1956.
- [12] C. CURTIS and I. REINER, *Representation theory of finite groups and associative algebras*, J. Wiley, New York, 1962.
- [13] M. I. GIORGIUTTI, *Thèse de l'Université de Paris*, 1963.
- [14] M. HALL, *Theory of groups*, Mc Millan, New York, 1959.
- [15] A. HELLER and I. REINER, *Grothendieck groups of orders in semi-simple algebras* (*Trans. Amer. Math. Soc.*, vol. 112, 1964, p. 344-355).
- [16] G. HIGMAN, *The units of group rings* (*Proc. London Math. Soc.*, vol. 46, 1940, p. 231-248).
- [17] T. Y. LAM, *Artin exponent of finite groups* (*Journal of Algebra*) (to appear).
- [18] J. MILNOR, *Whitehead torsion* (*Bull. Amer. Math. Soc.*, vol. 72, No. 3, 1966, p. 358-426).
- [19] D. S. RIM, *Modules over finite groups* (*Ann. of Math.*, vol. 69, 1960, p. 700-712).
- [20] J.-P. SERRE, *Corps locaux*, Herman, Paris, 1962.
- [21] R. G. SWAN, *Induced representations and projective modules* (*Ann. of Math.*, vol. 71, 1960, p. 552-578).
- [22] R. G. SWAN, *The Grothendieck ring of a finite group* (*Topology*, vol. 2, 1963, p. 85-110).
- [23] O. ZARISKI and P. SAMUEL, *Commutative algebra*, vol. 1, Van Nostrand, Princeton, 1958.
- [24] J. STROOKER, *Thesis at Rijkuniversiteit te Utrecht*, 1965.

(Manuscript reçu le 25 Août 1967).

