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Effective resistances for supercritical percolation clusters in boxes[1]

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Abstract. Let $\mathcal{C}^n$ be the largest open cluster for supercritical Bernoulli bond percolation in $[-n,n]^d \cap \mathbb{Z}^d$ with $d \geq 2$. We obtain a sharp estimate for the effective resistance on $\mathcal{C}^n$. As an application we show that the cover time for the simple random walk on $\mathcal{C}^n$ is comparable to $n^d(\log n)^2$. Noting that the cover time for the simple random walk on $[-n,n]^d \cap \mathbb{Z}^d$ is of order $n^d \log n$ for $d \geq 3$ (and of order $n^2(\log n)^2$ for $d = 2$), this gives a quantitative difference between the two random walks for $d \geq 3$.

Résumé. On considère la percolation de Bernoulli par arêtes dans le régime surcritique. Soit $\mathcal{C}^n$ le plus grand amas de percolation dans $[-n,n]^d \cap \mathbb{Z}^d$ avec $d \geq 2$. Nous obtenons une estimation précise de la résistance effective sur $\mathcal{C}^n$. Comme application, nous montrons que le temps de recouvrement d'une marche simple sur $\mathcal{C}^n$ est de l'ordre de $n^d(\log n)^2$. En remarquant que le temps de recouvrement d'une marche simple sur $[-n,n]^d \cap \mathbb{Z}^d$ est de l'ordre de $n^d \log n$ quand $d \geq 3$ (et de $n^2(\log n)^2$ quand $d = 2$), ceci montre une différence quantitative entre les deux marches si $d \geq 3$.

MSC: Primary 60J45; secondary 60K37

Keywords: Effective resistances; Simple random walks; Cover times; Gaussian free fields; Supercritical percolation

1. Introduction

The effective resistance is a fundamental measurement of the conductivity for the electrical network. It has close connections with many subjects of reversible Markov chains such as transience/recurrence, heat kernels, mixing times and cover times. We refer to [17,25,26,29] for the introduction of the theory of reversible Markov chains.

Effective resistances for percolation clusters in $\mathbb{Z}^d$ have been studied for a long time. Grimmett, Kesten and Zhang [19] showed almost-sure finiteness of the effective resistance from a fixed point to infinity on the infinite supercritical percolation cluster in $\mathbb{Z}^d$ for $d \geq 3$. (This is equivalent to almost-sure transience of the simple random walk on the cluster.) The result was extended to a series of works on more general energy of flows on percolation clusters [2,9,20–22,28]. The study of effective resistances for boxes on $\mathbb{Z}^d$ under percolation goes back to 1980's [11,18,23]. These results showed that critical phenomena occur at the critical percolation probability for effective resistances between opposing faces of boxes (see Remark 1.3 below). In [10], Boivin and Rau estimated the effective resistances from a fixed point to boundaries of boxes in the infinite supercritical percolation cluster on $\mathbb{Z}^2$ (see Remark 1.4 below). In [7], general upper bounds for effective resistances on general graphs are given by using isoperimetric inequalities. The estimates are used to obtain upper bounds of effective point-to-point resistances on supercritical percolation clusters in boxes on $\mathbb{Z}^2$; however, there seems to be a gap in this part (see Remark 1.2 below).

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Many properties of simple random walks on supercritical percolation clusters in $\mathbb{Z}^d$ have been investigated such as transience [2,9,19,20], mixing times [8,13], heat kernel decays [4,31], invariance principles [6,30,35], collisions of two independent simple random walks [5,12], and the existence of the harmonic measure [10]. These properties are very similar to those of simple random walks on $\mathbb{Z}^d$. A famous folk conjecture is that most important properties of simple random walks survive for simple random walks on supercritical percolation clusters (see, for example, [33]).

In this paper, we consider the largest supercritical percolation cluster in $[-n, n]^d \cap \mathbb{Z}^d$. We obtain the correct order of the maximum of the effective resistances between vertices in the giant cluster. Applying the result, we obtain a sharp estimate of the cover time for the simple random walk on the largest cluster; the result shows that it is much larger than the cover time for the simple random walk on $[-n, n]^d \cap \mathbb{Z}^d$ when $d \geq 3$. The author should mention that the referee pointed out that this result was a conjecture.

In order to describe our results more precisely, we begin with some notation. Let $|\cdot|_1$ be the ℓ^1-distance on $\mathbb{Z}^d$. We define the set of edges between all nearest-neighbour pairs on $\mathbb{Z}^d$ by $E(\mathbb{Z}^d) := \{\{x, y\}: x, y \in \mathbb{Z}^d, |x-y|_1 = 1\}$. For $p \in [0, 1]$, let $\mathbb{P}_p$ be the product Bernoulli measure on $\{0, 1\}^{E(\mathbb{Z}^d)}$ with $\mathbb{P}_p(\omega(e) = 1) = 1 - \mathbb{P}_p(\omega(e) = 0) = p$ for each $e \in E(\mathbb{Z}^d)$. We say that an edge e is open for $\omega \in \{0, 1\}^{E(\mathbb{Z}^d)}$ if $\omega(e) = 1$. For $A \subseteq \mathbb{Z}^d$, we define a random set of edges by

$$\mathcal{O}_A = \mathcal{O}_A(\omega) := \bigl\{\{x, y\} \in E\bigl(\mathbb{Z}^d\bigr): x, y \in A, \omega\bigl(\{x, y\}\bigr) = 1\bigr\}. \tag{1.1}$$

An open cluster in A is a connected component of the graph $(A, \mathcal{O}_A)$. We define the critical probability $p_c(\mathbb{Z}^d)$ by

$$\inf\bigl\{p \in [0, 1]: \mathbb{P}_p\bigl(\text{the cluster in } \mathbb{Z}^d \text{ containing the origin is infinite}\bigr) > 0\bigr\}. \tag{1.2}$$

We focus our attention to clusters in the box

$$B(n) := [-n, n]^d \cap \mathbb{Z}^d. \tag{1.3}$$

It is known that for $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$, we have the unique largest open cluster in $B(n)$ whose size is proportional to n^d, $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$. For $d \geq 3$, see [34], Theorem 1.2. The $d = 2$ case is well-known, and [14] will serve as a convenient reference. We write $\mathcal{C}^n$ to denote the largest open cluster in $B(n)$.

Let $G = (V(G), E(G))$ be a finite connected graph; the set $V(G)$ is the vertex set and $E(G)$ is the edge set. We define the Dirichlet energy by

$$\mathcal{E}(f) := \frac{1}{2} \sum_{\substack{u,v \in V(G) \\ \{u,v\} \in E(G)}} \bigl(f(u) - f(v)\bigr)^2, \quad f \in \mathbb{R}^{V(G)}.$$

For $A, B \subset V(G)$ with $A \cap B = \varnothing$, the effective resistance between A and B for G is defined by

$$R_{\mathrm{eff}}^G(A, B)^{-1} := \inf\bigl\{\mathcal{E}(f): f \in \mathbb{R}^{V(G)}, \mathcal{E}(f) < \infty, f|_A = 1, f|_B = 0\bigr\}.$$

We write $R_{\mathrm{eff}}^G(x, y)$ to denote $R_{\mathrm{eff}}^G(\{x\}, \{y\})$. We now state our result on the effective resistance for $\mathcal{C}^n$.

Theorem 1.1. *For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$,*

$$c_1 \cdot \log n \leq \max_{x, y \in \mathcal{C}^n} R_{\mathrm{eff}}^{\mathcal{C}^n}(x, y) \leq c_2 \cdot \log n.$$

Remark 1.2. *Corollary* 3.1 *of* [7] *says that for $d = 2$ and $p > p_c(\mathbb{Z}^2)$, there exists $c_1 > 0$ such that $\lim_{n\to\infty} \mathbb{P}_p(\max_{x,y\in\mathcal{C}^n} R_{\mathrm{eff}}^{\mathcal{C}^n}(x, y) \leq c_1 \cdot \log n) = 1$. The proof is based on general upper bounds for effective resistances given in Theorem* 2.1 *of* [7] *and an isoperimetric profile for $\mathcal{C}^n$ studied in* [8]. *However, according to the latest arXiv version of their paper* [7] *and* [24], *the proof only implies the following*: *there exists $c_1 > 0$ such that $\lim_{n\to\infty} \mathbb{P}_p(\max_{x,y\in\mathcal{C}^n} R_{\mathrm{eff}}^{\mathcal{C}^n}(x, y) \leq c_1 \cdot (\log n)^2) = 1$. This is due to bad isoperimetry of some small subsets of $\mathcal{C}^n$.*

Remark 1.3. *Fix $d \geq 2$. Let us consider a random graph G_n which has the vertex set $[0,n]^d \cap \mathbb{Z}^d$ and the edge set $\mathcal{O}_{[0,n]^d \cap \mathbb{Z}^d}$. We write R_n to denote the effective resistance*

$$R_{\mathrm{eff}}^{G_n}\big([0,n]^{d-1} \times \{0\}, [0,n]^{d-1} \times \{n\}\big).$$

In [11,18], *it was shown that the following holds: there exist $c_1, c_2 > 0$ such that if $p < p_c(\mathbb{Z}^d)$,*

$$R_n = \infty \quad \text{for large } n \in \mathbb{N}, \mathbb{P}_p\text{-a.s.},$$

and if $p_c(\mathbb{Z}^d) < p \leq 1$,

$$c_1 \leq \liminf_{n\to\infty} n^{d-2} R_n \leq \limsup_{n\to\infty} n^{d-2} R_n \leq c_2, \quad \mathbb{P}_p\text{-a.s.}$$

Remark 1.4. *Let $\mathcal{C}_\infty$ be the infinite Bernoulli bond percolation cluster in $\mathbb{Z}^2$. Define spheres in $\mathcal{C}_\infty$ by*

$$\partial B_{\mathcal{C}_\infty}(x,r) = \big\{y \in \mathcal{C}_\infty \colon d_{\mathcal{C}_\infty}(x,y) = r\big\}, \quad x \in \mathcal{C}_\infty, r \in \mathbb{N},$$

where $d_{\mathcal{C}_\infty}$ is the graph distance of $\mathcal{C}_\infty$. In Proposition 4.3 of the published version of [10], *Boivin and Rau proved the following: for $p \in (p_c(\mathbb{Z}^2), 1]$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for all $x_0 \in \mathcal{C}_\infty$, for large $n \in \mathbb{N}$,*

$$c_1 \log n \leq R_{\mathrm{eff}}^{\mathcal{C}_\infty}\big(x_0, \partial B_{\mathcal{C}_\infty}(x_0,n)\big) \leq c_2 \log n.$$

For a finite connected graph $G = (V(G), E(G))$ and $x \in V(G)$, let $\tau_x(G)$ be the hitting time of x by the simple random walk on G. We define the cover time by

$$t_{\mathrm{cov}}(G) := \max_{x \in V(G)} E_x\Big(\max_{y \in V(G)} \tau_y(G)\Big).$$

Applying Theorem 1.1, we can obtain an improvement of Proposition 3.3 of the arXiv version of [1].

Theorem 1.5. *For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$,*

$$c_1 \cdot n^d (\log n)^2 \leq t_{\mathrm{cov}}\big(\mathcal{C}^n\big) \leq c_2 \cdot n^d (\log n)^2.$$

Remark 1.6. *It is known that $t_{\mathrm{cov}}(B(n))$ is comparable to $n^2(\log n)^2$ when $d = 2$ and to $n^d \log n$ when $d \geq 3$ (see, for example, Section 11.3.2 of* [26]). *Therefore, Theorem 1.5 implies that the simple random walk on $\mathcal{C}^n$ exhibits anomalous behavior when $d \geq 3$. This is due to irregularity of $\mathcal{C}^n$; the cluster $\mathcal{C}^n$ contains a lot of one-dimensional objects (see Lemma 2.6 below).*

Let $G = (V(G), E(G))$ be any finite connected graph. Let us define the Gaussian free field on G. This is a centered Gaussian process $\{\eta_x^G\}_{x \in V(G)}$ with $\eta_{x_0}^G = 0$ for some fixed vertex $x_0 \in V(G)$ and the covariances are given by

$$\mathbf{E}\big(\eta_x^G \eta_y^G\big) = \frac{1}{2}\big(R_{\mathrm{eff}}^G(x,x_0) + R_{\mathrm{eff}}^G(y,x_0) - R_{\mathrm{eff}}^G(x,y)\big), \quad x,y \in V(G).$$

We define the expected maximum of the Gaussian free field by

$$M_G := \mathbf{E}\Big(\max_{x \in V(G)} \eta_x^G\Big).$$

Note that M_G does not depend on the choice of x_0. For a set S, we will write $|S|$ to denote the cardinality of S. In [16], Ding, Lee and Peres proved the following: there exist universal constants $c_1, c_2 > 0$ such that

$$c_1 \cdot \big|E(G)\big| \cdot (M_G)^2 \leq t_{\mathrm{cov}}(G) \leq c_2 \cdot \big|E(G)\big| \cdot (M_G)^2. \tag{1.4}$$

By (1.4) and Theorem 1.5, we obtain the following estimate of $M_{\mathcal{C}^n}$ immediately.

Corollary 1.7. *For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$,*

$$c_1 \cdot \log n \leq M_{\mathcal{C}^n} \leq c_2 \cdot \log n.$$

Remark 1.8. *By Remark 1.6 and (1.4), $M_{B(n)}$ is comparable to $\log n$ when $d = 2$ and to $\sqrt{\log n}$ when $d \geq 3$. Thus there is a marked quantitative difference between $M_{\mathcal{C}^n}$ and $M_{B(n)}$ when $d \geq 3$.*

Let us describe the outline of the paper. Section 2 gives preliminary lemmas about percolation estimates. In Section 3.1, we give a proof of Theorem 1.5 via Theorem 1.1. The upper bound is due to Theorem 1.1 and the Matthews bound [32]. The lower bound is based on the general bound on the cover time via the Gaussian free field given in [16]. The key of the proof of the lower bound is counting the number of one-way open paths of length of order $\log n$. In Section 3.2, we construct the so-called "Kesten grid" (the terminology comes from Mathieu and Remy [31]). This is an analogue of the square lattice and consists of "white" sites on a renormalized lattice; the concept of "white" sites is from the renormalization argument of Antal and Pisztora [3]. In Section 3.3, we prove Theorem 1.1. The lower bound is immediately followed by the fact that $\mathcal{C}^n$ has a one-way open path of length of order $\log n$. In the proof of the upper bound, we construct unit flows between vertices of $\mathcal{C}^n$ with energy of order $\log n$. Thanks to Kesten grids, we can let the flows run almost on some "sheets" and adapt an argument of estimating upper bounds of effective resistances for $\mathbb{Z}^2$. We use a result on the chemical distance for $\mathcal{C}^n$ based on [13]; this guarantees that every vertex of $\mathcal{C}^n$ can be connected with a Kesten grid decorated with small boxes by an open path in $\mathcal{C}^n$ of length of order $\log n$.

Throughout the paper, we will write c, c', c'' to denote positive constants depending only on the dimension of the lattice and the percolation parameter. Values of c, c' and c'' will change from line to line. We use $c_1, c_2, \ldots$ to denote constants whose values are fixed within each argument. If we cite elsewhere the constant c_1 in Lemma 2.4, we write it as $c_{2.4.1}$, for example.

2. Percolation estimates

In this section, we collect some useful results on percolation estimates. We need the following facts to prove Theorem 1.1 and Theorem 1.5.

(1) *Size and connectivity pattern of $\mathcal{C}^n$*

A path is a sequence $(x_0, x_1, \ldots, x_\ell)$ satisfying that $|x_{i-1} - x_i|_1 = 1$ for all $1 \leq i \leq \ell$. An open path is a path all of whose edges are open. We write $|\cdot|_\infty$ to denote the ℓ^∞-distance on $\mathbb{Z}^d$. The diameter of an open path $(x_0, x_1, \ldots, x_\ell)$ is defined by $\max_{0 \leq i, j \leq \ell} |x_i - x_j|_\infty$. For $\kappa > 0$, let A_κ^n be the event that every open path in $B(n)$ with diameter larger than $\kappa \log n$ is contained in $\mathcal{C}^n$. The following fact follows easily from [34], Theorem 1.2 and Theorem 3.1, for $d \geq 3$ and [14], Theorem 1 and Theorem 9, for $d = 2$.

Lemma 2.1. *Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. There exist $c_1, c_2 > 0$ and $\kappa_0, n_0 \in \mathbb{N}$ such that for all $\kappa \geq \kappa_0$ and $n \geq n_0$,*

$$\mathbb{P}_p\left(A_\kappa^n \cap \left\{\left|\mathcal{C}^n\right| \geq c_1 n^d\right\}\right) \geq 1 - 2\exp(-c_2 \kappa \log n).$$

Remark 2.2. *By Lemma 2.1 and the Borel–Cantelli lemma, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$, the event $A_{c_1}^n \cap \{|\mathcal{C}^n| \geq c_2 n^d\}$ holds.*

(2) *Chemical distance of $\mathcal{C}^n$*

Let $d_{\mathcal{C}^n}$ be the graph distance of the graph $(\mathcal{C}^n, \mathcal{O}_{\mathcal{C}^n})$. The following result is immediately followed by Lemma 3.2 and (4.1) in [13].

Lemma 2.3. *Fix $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. There exist $c_1 > 0$ and $\kappa_0 \in \mathbb{N}$ such that the following holds for all $\kappa \geq \kappa_0$, $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$: for all $x, y \in \mathcal{C}^n$ with $|x - y|_1 \leq \kappa \log n$,*

$$d_{\mathcal{C}^n}(x, y) \leq c_1 \kappa \log n.$$

(3) *Crossing probabilities*
In (3), we restrict our attention to Bernoulli site percolation on the square lattice. Let $\mathbb{Q}_q$ be the product Bernoulli measure on $\{0,1\}^{\mathbb{Z}^2}$ with $\mathbb{Q}_q(\omega(x)=1)=1-\mathbb{Q}_q(\omega(x)=0)=q$ for each $x\in\mathbb{Z}^2$. For $\omega\in\{0,1\}^{\mathbb{Z}^2}$ and $x\in\mathbb{Z}^2$, we will say that x is occupied if $\omega(x)=1$. The critical probability $q_c(\mathbb{Z}^2)$ is defined similarly to (1.2). We write $x\cdot y$ to denote the inner product of x and y. A self-avoiding path is a path all of whose vertices are distinct. Let e_1,e_2 be the standard basis for $\mathbb{Z}^2$. Fix $m,n\in\mathbb{N}$. A horizontal crossing in $([0,m]\times[0,n])\cap\mathbb{Z}^2$ is a self-avoiding path $(x_0,x_1,\dots,x_\ell)$ with $x_0\cdot e_1=0$ and $x_\ell\cdot e_1=m$.

Lemma 2.4 ([23], Theorem 11.1). *For $d=2$ and $q>q_c(\mathbb{Z}^2)$, there exist $c_1,c_2,c_3>0$ such that*

$$\begin{aligned}&\mathbb{Q}_q\big(\textit{there exist at least } c_1 n \textit{ disjoint horizontal crossings}\\&\qquad \textit{consisting of occupied sites in } \big([0,m]\times[0,n]\big)\cap\mathbb{Z}^2\big)\\&\quad\ge 1-c_2\cdot m\exp(-c_3 n).\end{aligned}$$

(4) *Renormalization argument*
We recall the renormalization argument of [3]. We will write $a,b,\dots$ rather than $x,y,\dots$ to denote vertices of the renormalized lattice. Fix a positive integer K. To $a\in\mathbb{Z}^d$, we associate boxes

$$B_a(K):=(2K+1)a+[-K,K]^d\cap\mathbb{Z}^d, \tag{2.1}$$

$$B'_a(K):=(2K+1)a+\left[-\frac{5}{4}K,\frac{5}{4}K\right]^d\cap\mathbb{Z}^d. \tag{2.2}$$

We will say that an open cluster crosses a box if the cluster intersects all the faces of the box. For $a\in\mathbb{Z}^d$, we define an event R_a^K satisfying the following:

- There exists a unique open cluster $\mathcal{C}$ in $B'_a(K)$ crossing $B'_a(K)$.
- The open cluster $\mathcal{C}$ crosses all the subboxes of $B'_a(K)$ of side length larger than $\frac{K}{10}$.
- Any open paths in $B'_a(K)$ of diameter larger than $\frac{K}{10}$ are contained in $\mathcal{C}$.

We say $a\in\mathbb{Z}^d$ is white for $\omega\in\{0,1\}^{E(\mathbb{Z}^d)}$ if $\mathbf{1}_{R_a^K}(\omega)=1$. By [34], Theorem 3.1, for $d\ge 3$ and [14], Theorem 9, for $d=2$, if $p>p_c(\mathbb{Z}^d)$, we have for all $a\in\mathbb{Z}^d$

$$\lim_{K\to\infty}\mathbb{P}_p\big(R_a^K\big)=1.$$

By [27], Theorem 0.0(ii), the following holds:

Lemma 2.5. *For $d\ge 2$ and $p>p_c(\mathbb{Z}^d)$, there exists a function $q:\mathbb{N}\to[0,1]$ with $\lim_{K\to\infty}q(K)=1$ such that*

the process $(\mathbf{1}_{R_a^K})_{a\in\mathbb{Z}^d}$ stochastically dominates

a Bernoulli site percolation process in $\mathbb{Z}^d$ with parameter $q(K)$.

(5) *Special open paths*
Let $e_1,\dots,e_d$ be the standard basis for $\mathbb{Z}^d$. We define a one-sided boundary of $B(n)$ by

$$\partial_1 B(n):=\big\{x\in B(n):\ x\cdot e_1=n\big\}.$$

We will say that $x\in\partial_1 B(n)$ is m-special if it is a base point of a special open path; that is, the edge $\{x+ie_1,x+(i+1)e_1\}$ is open for each $0\le i\le m-1$ and the vertex $x+ie_1$ does not have any other open edges for each $1\le i\le m$. See Figure 1. The following is a key to prove the lower bounds of Theorem 1.1 and Theorem 1.5.

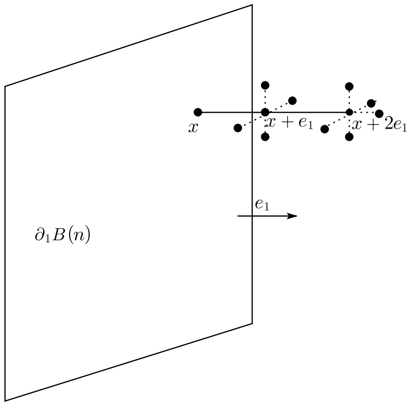

Fig. 1. An illustration of a 2-special vertex x and the corresponding special open path. The dotted line segments at $x+ie_1$ $(i=1,2)$ are closed edges. The solid line segment between x and $x+2e_1$ are composed of open edges.

Lemma 2.6. *For $d \geq 2$ and $p \in (p_c(\mathbb{Z}^d), 1)$, there exist $c_1, c_2 > 0$ such that $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$,*

$$\left|\left\{x \in \partial_1 B(n) \cap \mathcal{C}^n\colon\ x \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\right\}\right| \geq n^{c_2}.$$

Proof. Fix a large constant $K > 0$. We define a subset of $\partial_1 B(n)$ by

$$\partial_{1,2}^K B(n) := \left\{x \in \partial_1 B(n)\colon\ |x \cdot e_i| \leq \frac{5}{4} K \text{ for all } 3 \leq i \leq d\right\}.$$

Let S_n^K be the "square" defined by

$$\left\{a \in \mathbb{Z}^d\colon\ |a \cdot e_1| \leq \left\lceil \frac{n}{2K+1} \right\rceil, |a \cdot e_2| \leq \left\lfloor \frac{n - 5K/4}{2K+1} \right\rfloor, a \cdot e_i = 0 \text{ for } 3 \leq i \leq d\right\}.$$

Note that S_n^K is isomorphic to $[0, 2\lceil \frac{n}{2K+1} \rceil] \times [0, 2\lfloor \frac{n-5K/4}{2K+1} \rfloor] \cap \mathbb{Z}^2$. Let $\varepsilon > 0$ be a small constant. By Lemma 2.4 and Lemma 2.5, the following holds: there exist $c, c' > 0$ such that with $\mathbb{P}_p$-probability at least $1 - c \cdot n \exp(-c'n)$, we have more than εn disjoint crossings of white sites in S_n^K. By this fact and the definition of white sites together with Lemma 2.1, the event $G_n^K := \{|\partial_{1,2}^K B(n) \cap \mathcal{C}^n| < \varepsilon n\}$ satisfies the following: there exist $c, c' > 0$ and $\kappa_0 \in \mathbb{N}$ such that for all $\kappa \geq \kappa_0$ and sufficiently large $n \in \mathbb{N}$,

$$\mathbb{P}_p\big(G_n^K\big) \leq c \exp\big(-c' \kappa \log n\big). \tag{2.3}$$

Take $c_1 > 0$ small enough to make $r := 1 - c_1(\log \frac{1}{p} + 2(d-1) \log \frac{1}{1-p})$ positive. Fix $0 < c_2 < r$. We write $\mathrm{Bin}(m, q)$ to denote a binomial random variable with parameter m and q. Let q_n be the probability of the event that a fixed vertex in $\partial_1 B(n)$ is $\lfloor c_1 \log n \rfloor$-special. Consider the σ-field generated by finite-dimensional cylinders associated with configurations restricted to $B(n)$. Note that conditioned on the σ-field, events $\{x \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\}$ and $\{y \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\}$ are independent for $x, y \in \partial_1 B(n)$ with $|x - y|_\infty \geq 2$. By conditioning on the σ-field, we have for some $c, c' > 0$ and some $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned}
&\mathbb{P}_p\left(\left\{\left|\left\{x \in \partial_1 B(n) \cap \mathcal{C}^n\colon\ x \text{ is } \lfloor c_1 \log n \rfloor\text{-special}\right\}\right| \leq n^{c_2}\right\} \cap \big(G_n^K\big)^c\right) \\
&\quad \leq \mathbb{P}_p\big(\mathrm{Bin}\big(\lfloor \varepsilon' n \rfloor, q_n\big) \leq n^{c_2}\big) \\
&\quad \leq c \exp\big(-c' n^r\big).
\end{aligned} \tag{2.4}$$

In the last inequality, we have used the Chebyshev inequality and the fact that $nq_n \geq cn^r$ for some $c > 0$. By (2.3) and (2.4) together with the Borel–Cantelli lemma, we obtain the conclusion. □

3. Proof of Theorem 1.1 and Theorem 1.5

3.1. *Cover time estimate*

In this subsection we prove Theorem 1.5 via Theorem 1.1. We begin with general bounds on cover times based on [16,32]. The following is immediately followed by [26], Proposition 10.6 and Theorem 11.2 (see also [15,32]) and [16], Theorem 1.1 and Lemma 1.11.

Lemma 3.1. *Let $G = (V(G), E(G))$ be any finite connected graph.*

(1) *There exists a universal constant $c_1 > 0$ such that*

$$t_{\mathrm{cov}}(G) \le c_1 \cdot \big|E(G)\big| \cdot \Big(\max_{x,y \in V(G)} R^G_{\mathrm{eff}}(x,y)\Big) \cdot \log\big|V(G)\big|.$$

(2) *There exists a universal constant $c_2 > 0$ such that for all subset $\tilde{V} \subset V(G)$,*

$$t_{\mathrm{cov}}(G) \ge c_2 \cdot \big|E(G)\big| \cdot \Big(\min_{\substack{x,y \in \tilde{V} \\ x \ne y}} R^G_{\mathrm{eff}}(x,y)\Big) \cdot \log|\tilde{V}|.$$

Proof of Theorem 1.5 via Theorem 1.1. The upper bound is immediately followed by Lemma 3.1(1) and Theorem 1.1. From now, we prove the lower bound. By Remark 2.2, we have $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$,

$$\mathcal{C}^{n-\lceil c_{2.6.1} \log n \rceil} \subseteq \mathcal{C}^n. \tag{3.1}$$

Set $m_n := n - \lceil c_{2.6.1} \log n \rceil$. We define a set V_n of tips of special open paths by

$$\big\{x + \lfloor c_{2.6.1} \log m_n \rfloor e_1 \colon x \in \partial_1 B(m_n) \cap \mathcal{C}^{m_n} \text{ and } x \text{ is } \lfloor c_{2.6.1} \log m_n \rfloor\text{-special}\big\}.$$

By (3.1), Lemma 2.6 and the Nash–Williams inequality (see, for example, [26], Proposition 9.15), we have the following $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$: for all $x, y \in V_n$ with $x \ne y$,

$$V_n \subseteq \mathcal{C}^n, \qquad |V_n| \ge (m_n)^{c_{2.6.2}} \quad \text{and} \quad R^{\mathcal{C}^n}_{\mathrm{eff}}(x,y) \ge \lfloor c_{2.6.1} \log m_n \rfloor.$$

Therefore, by Remark 2.2 and Lemma 3.1 (2) with $\tilde{V} = V_n$, we obtain the lower bound. $\square$

3.2. *Kesten grid*

In this subsection, we construct the so-called "Kesten grids" on the renormalized lattice (recall Section 2(4)) in order to obtain the upper bound of Theorem 1.1. The terminology is due to Mathieu and Remy [31]. We note that the construction of the Kesten grid is based on Theorem 11.1 of [23] (see Lemma 2.4).

Let us define some notations. Recall that $e_1, \dots, e_d$ is the standard basis for $\mathbb{Z}^d$. Fix a sufficiently large positive integer K. Recall the notation (2.2). For a subset S of the renormalized lattice, we define a fattened version of S by

$$W(S) := \bigcup_{a \in S} B'_a(K). \tag{3.2}$$

Let α be a positive constant. We will choose α sufficiently large later. Let ℓ_n be the largest integer ℓ satisfying that

$$(2K+1)\big(\big(2\lceil \alpha \log n \rceil + 1\big)\ell + \lceil \alpha \log n \rceil\big) + \frac{5}{4}K \le n,$$

and set

$$\tilde{\ell}_n := \big(2\lceil \alpha \log n \rceil + 1\big)\ell_n + \lceil \alpha \log n \rceil. \tag{3.3}$$

Note that we have $W(B(\tilde{\ell}_n)) \subset B(n)$ (recall the notation (1.3)). Thus we can regard $B(\tilde{\ell}_n)$ as a box in the renormalized lattice corresponding to the original box $B(n)$. The number of two-dimensional sections in $B(\tilde{\ell}_n)$ is $\frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2}$; each of them is isomorphic to the square $[-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2$. In Section 3.3 below, we will focus our attention on one of them, namely

$$F_1 := \{k_1 e_1 + k_2 e_2 \colon -\tilde{\ell}_n \le k_1, k_2 \le \tilde{\ell}_n\}. \tag{3.4}$$

We write

$$F_i, \quad 2 \le i \le \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2} \tag{3.5}$$

to denote the other two-dimensional sections of $B(\tilde{\ell}_n)$. For $-\ell_n \le m \le \ell_n$, we define the mth horizontal strip of F_1

$$R_1^n(m) := \bigl\{k_1 e_1 + k_2 e_2 \colon -\tilde{\ell}_n \le k_1 \le \tilde{\ell}_n, \bigl|k_2 - \bigl(2\lceil \alpha \log n\rceil + 1\bigr)m\bigr| \le \lceil \alpha \log n\rceil\bigr\} \tag{3.6}$$

and the mth vertical strip of F_1

$$R_2^n(m) := \bigl\{k_1 e_1 + k_2 e_2 \colon -\tilde{\ell}_n \le k_2 \le \tilde{\ell}_n, \bigl|k_1 - \bigl(2\lceil \alpha \log n\rceil + 1\bigr)m\bigr| \le \lceil \alpha \log n\rceil\bigr\}. \tag{3.7}$$

We define a horizontal (respectively, vertical) crossing of F_1 as a self-avoiding path with endvertices a, b satisfying $a \cdot e_1 = -\tilde{\ell}_n$ and $b \cdot e_1 = \tilde{\ell}_n$ (respectively, $a \cdot e_2 = -\tilde{\ell}_n$ and $b \cdot e_2 = \tilde{\ell}_n$). We define strips and crossings for the other two-dimensional sections of $B(\tilde{\ell}_n)$ in a similar fashion. We note that each strip is isomorphic to $([0, 2\tilde{\ell}_n] \times [0, 2\lceil \alpha \log n\rceil]) \cap \mathbb{Z}^2$. Taking α large enough, the following holds immediately by Lemma 2.4 and Lemma 2.5.

Corollary 3.2. *Fix $d \ge 2$ and $p > p_c(\mathbb{Z}^d)$. The following holds $\mathbb{P}_p$-a.s., for large $n \in \mathbb{N}$: for all $1 \le i \le \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2}$ and $-\ell_n \le m \le \ell_n$, there exist at least $2c_{2.4.1}\lceil \alpha \log n\rceil$ disjoint horizontal (respectively, vertical) crossings consisting of white sites in the mth horizontal (respectively, vertical) strip of F_i.*

Set $L_n := \lceil 2c_{2.4.1}\lceil \alpha \log n\rceil\rceil$. Fix $1 \le i \le \frac{d(d-1)}{2}(2\tilde{\ell}_n + 1)^{d-2}$ and a configuration satisfying the event of Corollary 3.2. We can choose L_n disjoint crossings of white sites in each strip of F_i. Since F_i is isomorphic to $[-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2$, the horizontal and the vertical crossings intersect (see Figure 2) and form a grid on F_i. We will call it a Kesten grid. Note that Kesten grids on F_i and F_j do not intersect if $i \ne j$ in general.

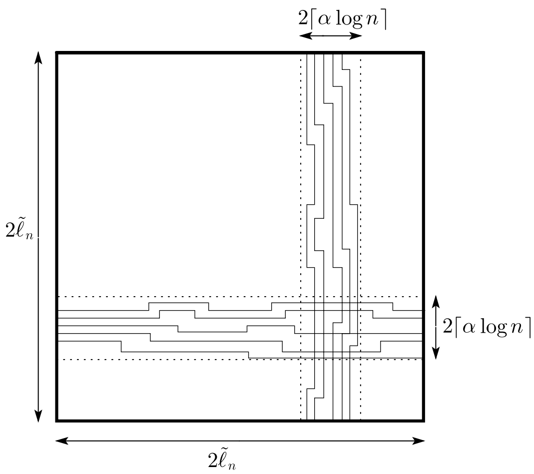

Fig. 2. An illustration of disjoint crossings of white sites (thin solid lines) in strips (rectangles with dotted boundaries) of F_1 (square with thick solid boundary). These horizontal and vertical crossings intersect since F_1 is isomorphic to $[-\tilde{\ell}_n, \tilde{\ell}_n]^2 \cap \mathbb{Z}^2$.

3.3. *Effective resistance estimate*

In this subsection we prove Theorem 1.1. To prove it, we only need to estimate the upper bound of effective resistances for pairs of vertices on $W(F_i)\cap\mathcal{C}^n$, $1\le i\le\frac{d(d-1)}{2}(2\tilde{\ell}_n+1)^{d-2}$ (recall the notations (3.2)–(3.5)). In the following proposition, we will focus our attention to $W(F_1)\cap\mathcal{C}^n$. Recall the constant α in Section 3.2. To $k_1,k_2\in\mathbb{Z}$, we associate the "square" defined by

$$S^n_{k_1,k_2}:=\{s_1e_1+s_2e_2\colon\ |s_i-(2\lceil\alpha\log n\rceil+1)k_i|\le\lceil\alpha\log n\rceil, i=1,2\}.$$

Proposition 3.3. *Fix $d\ge 2$ and $p>p_c(\mathbb{Z}^d)$. Let $m_1,m_2,\tilde{m}_2$ be integers satisfying that $m_2<\tilde{m}_2$, $\tilde{m}_2-m_2$ is even, and $m_1+\frac{\tilde{m}_2-m_2}{2}\le\ell_n$. Then there exists $c_1>0$ (not depending on choices of $m_1,m_2,\tilde{m}_2$) such that $\mathbb{P}_p$-a.s., for large $n\in\mathbb{N}$, the following holds: for all $x\in W(S^n_{m_1,m_2})\cap\mathcal{C}^n$ and $y\in W(S^n_{m_1,\tilde{m}_2})\cap\mathcal{C}^n$,*

$$R^{\mathcal{C}^n}_{\mathrm{eff}}(x,y)\le c_1\cdot\log n.$$

The upper bound of Theorem 1.1 follows from a similar argument of the proof of Proposition 3.3 below together with Lemma 2.3 and the triangle inequality for the effective resistance (see, for instance, [26], Corollary 10.8). We omit the details. The lower bound of Theorem 1.1 holds by Lemma 2.6, (3.1) and the Nash–Williams inequality. In the rest of the article, we focus on proving Proposition 3.3. Fix some integers r,r' with $r<r'$. A left-to-right (respectively, right-to-left) path in $[r,r']\times\mathbb{Z}^{d-1}$ is a path in $[r,r']\times\mathbb{Z}^{d-1}$ whose initial and final vertices a,b satisfy $a\cdot e_1=r$ and $b\cdot e_1=r'$ (respectively, $a\cdot e_1=r'$ and $b\cdot e_1=r$). A bottom-to-top path in $\mathbb{Z}\times[r,r']\times\mathbb{Z}^{d-2}$ is a path in $\mathbb{Z}\times[r,r']\times\mathbb{Z}^{d-2}$ whose initial and final vertices a,b satisfy $a\cdot e_2=r$ and $b\cdot e_2=r'$. For a path Γ and $a,a'\in\Gamma$ (we assume that a appears before a' in Γ), we write $\Gamma[a,a']$ to denote a part of Γ from a to a'. In particular, when a' is the final vertex of Γ, we write $\Gamma[a]$ in the place of $\Gamma[a,a']$.

Proof of Proposition 3.3. We fix a configuration satisfying the events of Remark 2.2, Corollary 3.2, and Lemma 2.3 with κ large enough. We choose vertices x and y as in Proposition 3.3. We will construct a unit flow between x and y. Our argument is based on [29], Proposition 2.15, and Section 4.1 of the arXiv version of [10].

We first construct a random self-avoiding open path from x to y, most of whose parts lie on $W(F_1)\cap\mathcal{C}^n$. Recall the notations (3.6) and (3.7). A Kesten grid guarantees the existence of the following self-avoiding paths:

- disjoint left-to-right (respectively, right-to-left) self-avoiding paths of white sites $H^m_1,\dots,H^m_{L_n}$ contained in $R^n_1(m_2+m)$ for $1\le m\le\frac{\tilde{m}_2-m_2}{2}$ (respectively, $\frac{\tilde{m}_2-m_2}{2}<m<\tilde{m}_2-m_2$),
- disjoint bottom-to-top self-avoiding paths of white sites $V^m_1,\dots,V^m_{L_n}$ contained in $R^n_2(m_1+m)$ for $0\le m\le\frac{\tilde{m}_2-m_2}{2}$,
- left-to-right self-avoiding paths of white sites H_x and H_y of length at most $c\log n$ for some $c>0$, contained in $S^n_{m_1,m_2}$ and in $S^n_{m_1,\tilde{m}_2}$ respectively.

From now, we construct an open path from x to y corresponding to a fixed label (i,j) with $i\in\{0,\dots,\frac{\tilde{m}_2-m_2}{2}\}$ and $j\in\{1,\dots,L_n\}$ by the following three steps.

Step 1: For convenience, we will associate a vertex $(k_1,k_2)\in\mathbb{Z}^2$ to $S^n_{k_1,k_2}$. Let $f:\{0,\dots,2i\}\to\{0,\dots,i\}$ be a function defined by $f(r)=r$ for $0\le r\le i$ and $f(r)=2i-r$ for $i<r\le 2i$. Take a self-avoiding path Γ_i as follows:

- Let $(s_r)_{0\le r\le 2i+1}$ be a sequence with $0=s_0<s_1<\cdots<s_{2i+1}=\tilde{m}_2-m_2$. For $0\le r\le 2i$, set $v_{2r}=(m_1+f(r),m_2+s_r)$ and $v_{2r+1}=(m_1+f(r),m_2+s_{r+1})$. The self-avoiding path Γ_i is obtained from the sequence $(v_r)_{0\le r\le 4i+1}$ by linear interpolation.
- The path Γ_i lies within 1 in ℓ^∞-distance from the piecewise line segments connecting (m_1,m_2), $(m_1+i,\frac{m_2+\tilde{m}_2}{2})$ and $(m_1,\tilde{m}_2)$.

See the left-hand side of Figure 3.

Step 2: Recall the notation before the proof of Proposition 3.3. Let h_x (respectively, h_y) be the initial (respectively, the final) vertex of H_x (respectively, H_y). We define vertices $a_1,\dots,a_{4i+2}$ as follows:

- For $0\le r\le 2i$, a_{2r+1} is the last-visited vertex of $H^{s_r}_j[a_{2r}]$ by $V^{f(r)}_j$, where $a_0:=h_x$ and $H^{s_0}_j:=H_x$.

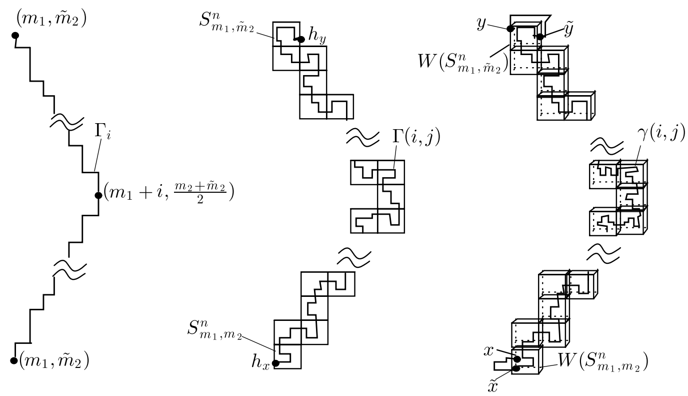

Fig. 3. Illustrations of paths Γ_i, $\Gamma(i,j)$ and $\gamma(i,j)$.

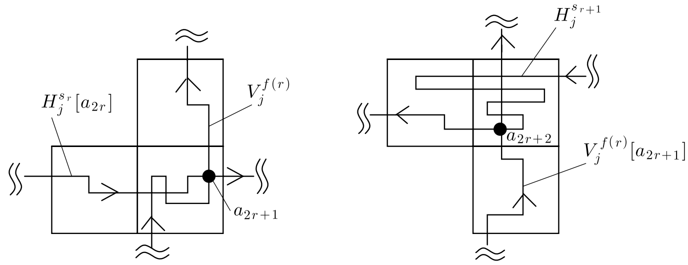

Fig. 4. Illustrations of a_{2r+1} and a_{2r+2} for $0 \le r \le 2i$.

- For $0 \le r \le 2i$, a_{2r+2} is the last-visited vertex of $V^{f(r)}_j[a_{2r+1}]$ by $H^{s_{r+1}}_j$, where $H^{s_{2i+1}}_j := H_y$.

See Figure 4. We can obtain a self-avoiding path $\Gamma(i,j)$ of white sites from h_x to h_y in $\bigcup_{(k_1,k_2)\in\Gamma_i} S^n_{k_1,k_2}$ by connecting segments

$$H^{s_r}_j[a_{2r},a_{2r+1}], \qquad V^{f(r)}_j[a_{2r+1},a_{2r+2}], \quad 0 \le r \le 2i, \quad \text{and} \quad H_y[a_{4i+2},h_y].$$

See the middle of Figure 3.

Step 3: Recall the notation (2.1). Fix vertices $\tilde{x} \in B_{h_x}(K) \cap \mathcal{C}^n$ and $\tilde{y} \in B_{h_y}(K) \cap \mathcal{C}^n$. Indeed, we can take such $\tilde{x}$ and $\tilde{y}$ by Remark 2.2. By the definition of white sites, we can take a self-avoiding open path $\tilde{\gamma}(i,j)$ from $\tilde{x}$ to $\tilde{y}$ contained in $W(\Gamma(i,j))$. By Lemma 2.3, we have self-avoiding open paths γ_x and γ_y in $\mathcal{C}^n$ from x to $\tilde{x}$ and from $\tilde{y}$ to y respectively of length at most $c\log n$ for some $c>0$. We can get a self-avoiding open path $\gamma(i,j)$ from x to y in $\mathcal{C}^n$ via γ_x, $\tilde{\gamma}(i,j)$ and γ_y. See the right-hand side of Figure 3.

Let X and Y be random variables distributed uniformly on $\{0,\dots,\frac{\tilde{m}_2-m_2}{2}\}$ and on $\{1,\dots,L_n\}$ respectively; they are defined on a probability space with probability measure $\mathbf{P}$. From now, we construct a unit flow from x to y via the random open path $\gamma(X,Y)$. Recall the notation (1.1). We define a random function ψ on $\mathcal{C}^n \times \mathcal{C}^n$ by

$$\psi(u,v) := \begin{cases} 1 & \text{if } \{u,v\} \in \mathcal{O}_{\mathcal{C}^n} \text{ and } \gamma(X,Y) \text{ passes } u,v \text{ in this order,} \\ -1 & \text{if } \{u,v\} \in \mathcal{O}_{\mathcal{C}^n} \text{ and } \gamma(X,Y) \text{ passes } v,u \text{ in this order,} \\ 0 & \text{otherwise.} \end{cases}$$

We define a function θ on $\mathcal{C}^n \times \mathcal{C}^n$ by

$$\theta(u,v) := \mathbf{E}\big(\psi(u,v)\big) \quad \text{for } u,v \in \mathcal{C}^n.$$

Since ψ is a unit flow from x to y, θ is a unit flow from x to y. In order to bound θ, let us define the following function on $\mathcal{C}^n \times \mathcal{C}^n$:

$$p(u,v) := \begin{cases} \mathbf{P}\big(\gamma(X,Y) \text{ passes the edge } \{u,v\}\big) & \text{if } \{u,v\} \in \mathcal{O}_{\mathcal{C}^n}, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 \le r \le \tilde{m}_2 - m_2$, we define a set of labels of the rth level by

$$D_r := \begin{cases} \{(m_1+s, m_2+r) \in \mathbb{Z}^2 \colon 0 \le s \le r+1\} & \text{if } r \le \frac{\tilde{m}_2 - m_2}{2}, \\ \{(m_1+s, m_2+r) \in \mathbb{Z}^2 \colon 0 \le s \le \tilde{m}_2 - m_2 - r + 1\} & \text{otherwise.} \end{cases}$$

Let U be the union of γ_x, γ_y, $W(H_x)$ and $W(H_y)$. By Thomson's principle (see, for example, [26], Theorem 9.10) and the construction of $\gamma(X,Y)$, we have

$$\begin{aligned} R_{\mathrm{eff}}^{\mathcal{C}^n}(x,y) &\le \frac{1}{2} \sum_{u,v \in \mathcal{C}^n} \theta(u,v)^2 \\ &\le \sum_{u,v \in U} p(u,v)^2 + \sum_{r=0}^{(\tilde{m}_2 - m_2)/2} \sum_{(k_1,k_2) \in D_r} \sum_{\substack{u,v \in W(S^n_{k_1,k_2}) \\ v \notin U}} p(u,v)^2 \\ &\quad + \sum_{r=(\tilde{m}_2-m_2)/2+1}^{\tilde{m}_2 - m_2} \sum_{(k_1,k_2) \in D_r} \sum_{\substack{u,v \in W(S^n_{k_1,k_2}) \\ v \notin U}} p(u,v)^2. \end{aligned} \tag{3.8}$$

Since γ_x, γ_y, H_x and H_y are self-avoiding paths of length of order $\log n$, the first term on the right-hand side of (3.8) is bounded by $c \log n$ for some $c > 0$. Fix $1 \le r \le \frac{\tilde{m}_2 - m_2}{2}$, $(k_1,k_2) \in D_r$ and $u, v \in W(S^n_{k_1,k_2})$ with $v \notin U$ and $\{u,v\} \in \mathcal{O}_{\mathcal{C}^n}$. By the construction of the random open path $\gamma(X,Y)$, we have for some $c > 0$,

$$p(u,v) \le \frac{c}{r \cdot L_n}.$$

So we have for some $c, c' > 0$,

$$\sum_{r=0}^{(\tilde{m}_2 - m_2)/2} \sum_{(k_1,k_2) \in D_r} \sum_{\substack{u,v \in W(S^n_{k_1,k_2}) \\ v \notin U}} p(u,v)^2 \le c \sum_{r=1}^{(\tilde{m}_2 - m_2)/2} \frac{1}{r} \le c' \log n.$$

By a similar argument, the last term on the right-hand side of (3.8) is bounded by $c \log n$ for some $c > 0$. Therefore by (3.8), we obtain the conclusion. □

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