Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 1, 194–206
DOI: 10.1214/13-AIHP574

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Stable limit laws for the parabolic Anderson model between quenched and annealed behaviour[1]

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Received 4 November 2012; revised 3 June 2013; accepted 11 June 2013

Abstract. We consider the solution to the parabolic Anderson model with homogeneous initial condition in large time-dependent boxes. We derive stable limit theorems, ranging over all possible scaling parameters, for the rescaled sum over the solution depending on the growth rate of the boxes. Furthermore, we give sufficient conditions for a strong law of large numbers.

Résumé. Nous considérons la solution du modèle parabolique d'Anderson avec condition initiale homogène sur de grandes boîtes dépendantes du temps. Nous dérivons des théorèmes limites stables, pour toutes les valeurs possibles des paramètres d'échelle, pour la somme de la solution changée d'échelle en fonction du taux de croissance des boîtes. De plus, nous donnons des conditions suffisantes pour une loi des grands nombres.

MSC: Primary 60K37; 82C44; secondary 60H25; 60F05

Keywords: Parabolic Anderson model; Stable limit laws; Strong law of large numbers

1. Introduction

1.1. *The problem*

The parabolic Anderson model (PAM) is the heat equation on the lattice with a random potential, given by

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \kappa\Delta u(t,x) + \xi(x)u(t,x), & (t,x)\in(0,\infty)\times\mathbb{Z}^d, \\ u(0,x) = u_0(x), & x\in\mathbb{Z}^d, \end{cases} \tag{1}$$

where $\kappa > 0$ denotes a diffusion constant, u_0 a nonnegative function, and Δ the discrete Laplacian, defined by

$$\Delta f(x) := \sum_{y\in\mathbb{Z}^d:|x-y|_1=1}\bigl[f(y)-f(x)\bigr],\quad x\in\mathbb{Z}^d,\ f:\mathbb{Z}^d\to\mathbb{R}.$$

Furthermore, $\xi := \{\xi(x), x\in\mathbb{Z}^d\}$ is an i.i.d. random potential. We will stick in this paper to the homogeneous initial condition $u_0\equiv 1$.

The solution u depends on two effects. The Laplacian tends to make it flat whereas the potential tends to make it uneven. In combination this causes the occurrence of small regions where almost all mass of the system is located. This

[1]The work was supported by the DFG International Research Training Group *Stochastic Models of Complex Processes*.

effect is called *intermittency* and it is present for all potentials that are not almost surely constant, see [9], Theorem 3.2. It turns out that the intermittency effect becomes the more pronounced the more heavy tailed the potential is.

Basically, there are two ways of looking at the solution. On the one hand one can pick one realisation of the potential field and consider the almost sure behaviour of u. This is the so-called *quenched* setting. On the other hand one can take expectation with respect to the potential and consider the averaged behaviour of u. This is the so-called *annealed* setting. Expectation with respect to ξ will be denoted by $\langle\cdot\rangle$, and the corresponding probability measure will be denoted by $\mathbf{P}$. Those sites $x \in \mathbb{Z}^d$ whose peaks $\xi(x)$ govern the quenched behaviour of u differ heavily from those that govern the annealed behaviour, see [9]. Therefore, it is interesting to understand the transition mechanism from quenched to annealed behaviour.

To this end we are interested in expressions such as $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ where Q is a large centred box. If Q has a fixed size then $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ follows quenched behaviour as t tends to infinity, i.e., $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)/u(t,0) \to 1$ a.s. This can be deduced from the Feynman–Kac representation of u given by

$$u(t,x) = \mathbb{E}_x \exp\left\{\int_0^t \xi(X_s)\,\mathrm{d}s\right\} u_0(X_t), \quad (t,x) \in [0,\infty) \times \mathbb{Z}^d,$$

where X is a simple, symmetric, continuous time random walk with generator $\kappa\Delta$ and $\mathbb{P}_x$ ($\mathbb{E}_x$) denotes the corresponding probability measure (expectation) if $X_0 = x$ a.s.

On the other hand, if we fix t and let the size of Q tend to infinity then (due to the homogeneous initial condition) by Birkhoff's ergodic theorem $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)$ displays annealed behaviour almost surely, i.e., $\frac{1}{|Q|}\sum_{x\in Q} u(t,x)/\langle u(t,0)\rangle \to 1$ a.s. Therefore, a natural question is what happens if the box Q is time dependent.

More precisely, we want to find for all $\alpha \in (0,2)$ a large box $Q_{L_\alpha(t)}$, with $Q_{r(t)} = [-r(t), r(t)]^d \cap \mathbb{Z}^d$, for any $r(t) > 0$, and numbers $A(t)$, $B_\alpha(t)$ such that

$$\sum_{x\in Q_{L_\alpha(t)}} \frac{u(t,x) - A(t)}{B_\alpha(t)} \overset{t\to\infty}{\Longrightarrow} \mathcal{F}_\alpha,$$

with $\mathcal{F}_\alpha$ a suitable stable distribution.

In the case $\kappa = 0$, i.e., if the solutions at different sites are independent, the problem has been addressed in [1] under the assumption that the logarithmic tail of the distribution is normalized regularly varying. A wider class of distributions was considered in [5]. In [13] a conceptual treatment of several classes of time-dependent sums is offered, in particular explaining the universality of the limit laws in different cases. In [1] the authors also give sufficient and necessary conditions on the growth rate of Q for a weak law of large numbers (WLLN) and for a central limit theorem (CLT) to hold. Corresponding results for a WLLN and a CLT for the PAM, i.e., $\kappa \neq 0$, were derived in [2] and in [3]. They state that, under appropriate regularity assumptions, there exist $J(t)$ and $\gamma_1 < \gamma_2$, all depending on the tails of ξ, such that:

(i) $\frac{1}{|Q_{\gamma J(t)}|}\sum_{x\in Q_{\gamma J(t)}} u(t,x) \sim \langle u(t,0)\rangle$, as $t\to\infty$ if $\gamma > \gamma_1$, in probability, $\frac{1}{|Q_{\gamma J(t)}|}\sum_{x\in Q_{\gamma J(t)}} u(t,x) = \mathrm{o}(\langle u(t,0)\rangle)$, as $t\to\infty$ if $\gamma < \gamma_1$, in probability.
(ii) $\frac{1}{|Q_{\gamma J(t)}|}\sum_{x\in Q_{\gamma J(t)}} \frac{u(t,x)-\langle u(t,0)\rangle}{\sqrt{\langle u(t,0)^2\rangle}} \Longrightarrow \mathcal{N}(0,1)$, as $t\to\infty$ if $\gamma > \gamma_2$, $\frac{1}{|Q_{\gamma J(t)}|}\sum_{x\in Q_{\gamma J(t)}} \frac{u(t,x)-\langle u(t,0)\rangle}{\sqrt{\langle u(t,0)^2\rangle}} = \mathrm{o}(1)$, as $t\to\infty$ if $\gamma < \gamma_2$, in probability.

Here, $\mathcal{N}(0,1)$ denotes the law of the standard normal distribution with variance 1.

However, α-stable limits for the PAM have not been investigated so far. Furthermore, we give sufficient conditions on the growth rate of Q for a strong law of large numbers to hold. So far this has been done neither for the PAM nor for the $\kappa = 0$ case.

For a general overview of the parabolic Anderson model see, for instance [15] and [8]. A WLLN and a CLT for the PAM with time-dependent white noise potential using rather different techniques can be found in [7]. The critical growth rates in that model are of the same order as for the double-exponential case in our model. Similar questions concerning a version of the random energy model were investigated in [6].

1.2. *Main results*

To state the main results we need to introduce some notation. Let

$$\varphi(h) := -\log \mathbf{P}\big(\xi(0) > h\big)$$

and h_t a solution to

$$\sup_{h\in(0,\infty)} \big(th - \varphi(h)\big) = th_t - \varphi(h_t).$$

If φ is ultimately convex then h_t is unique for any large t. Throughout this paper we will assume that $\xi(0)$ is unbounded from above and has finite exponential moments of all orders. Under these circumstances the left-continuous inverse of φ,

$$\psi(s) := \min\big\{r\colon \varphi(r) \ge s\big\}, \quad s > 0,$$

is well defined. Furthermore, this implies that the cumulant generating function

$$H(t) := \log\big\langle \exp\big\{t\xi(0)\big\}\big\rangle, \quad t \ge 0,$$

is well-defined and that $H(t) < \infty$ for all t with $\lim_{t\to\infty} H(t)/t = \infty$. If $\varphi \in C^2$ is ultimately convex and satisfies some mild regularity assumptions then the Laplace method yields that $H(t) = th_t - \varphi(h_t) + \mathrm{o}(t)$. In the sequel we will frequently need the following regularity assumptions.

Assumption F. *There exists* $\rho \in [0, \infty]$ *such that for all* $c \in (0, 1)$,

$$\lim_{t\to\infty} \big[\psi(ct) - \psi(t)\big] = \rho c \log c.$$

Assumption H. *There exists* $\rho \in [0, \infty]$ *such that for all* $c \in (0, 1)$,

$$\lim_{t\to\infty} \frac{1}{t}\big[H(ct) - cH(t)\big] = \rho c \log c.$$

In [10], Theorems 1.2 and 2.2, the authors prove that there exists $\chi = \chi(\rho) \in [0, 2d\kappa]$ such that

$$\frac{\log u(t,0)}{t} = \xi^{(1)}_{Q_t} - \chi + \mathrm{o}(1), \quad \text{a.s.,} \tag{2}$$

with $\xi^{(1)}_A = \sup\{\xi(x)\colon x \in A\}$, if Assumption F is satisfied, and

$$\frac{\log\langle u(t,0)^p\rangle}{t} = \frac{H(pt)}{t} - p\chi + \mathrm{o}(1), \quad p \in \mathbb{N}, \tag{3}$$

if Assumption H is satisfied. Notice that Assumption F implies Assumption H. Furthermore, it turns out that $\chi = \chi(\rho)$ is strictly increasing in ρ with $\chi(0) = 0$ and $\chi(\infty) = 2d\kappa$. For details see [10].

Prominent examples satisfying Assumption F are the double exponential distribution, i.e., $\mathbf{P}(X > x) = \exp\{-\exp\{x/\rho\}\}$, $x > 0$, for $\rho \in (0, \infty)$ and the Weibull distribution, i.e., $\mathbf{P}(X > x) = \exp\{-x^\gamma\}$, $x > 0$ with $\gamma > 1$ for $\rho = \infty$.

For $\alpha \in (0, 2)$ let $\mathcal{F}_\alpha$ be the α-stable distribution with characteristic function

$$\phi_\alpha(u) = \begin{cases} \exp\big\{-\Gamma(1-\alpha)|u|^\alpha \exp\big\{\frac{-\mathrm{i}\pi\alpha}{2}\operatorname{sign} u\big\}\big\}, & \alpha \ne 1, \\ \exp\big\{\mathrm{i}u(1-\gamma) - \frac{\pi}{2}|u|(1 + 2\pi\mathrm{i}\log|u|\operatorname{sign} u)\big\}, & \alpha = 1. \end{cases}$$

Moreover, let

$$L_\alpha(t) := \exp\{\varphi(h_{\alpha t})\} \quad \text{and} \quad B_\alpha(t) := \exp\{t \cdot (h_{\alpha t} - \chi + \mathrm{o}(1))\},$$

where the error term of $B_\alpha(t)$ is chosen in a suitable way. Then we find our main result:

Theorem 1 (Stable limit laws). *Let $\varphi \in C^2$ be ultimately convex and Assumption F be satisfied. Then for $\alpha \in (0, 2)$,*

$$\sum_{x \in Q_{L_\alpha(t)}} \frac{u(t,x) - A(t)}{B_\alpha(t)} \overset{t\to\infty}{\Longrightarrow} \mathcal{F}_\alpha,$$

with

$$A(t) = \begin{cases} 0, & \text{if } \alpha \in (0,1), \\ \langle u(t,0) \rangle, & \text{if } \alpha \in (1,2), \\ \langle u(t,0) \mathbb{1}_{u(t,0) \le B_\alpha(t)} \rangle, & \text{if } \alpha = 1. \end{cases}$$

Furthermore, we find:

Theorem 2 (Strong law of large numbers). *Let Assumption H be satisfied, and $r(t)$ be so large that $\lim_{t\to\infty} \frac{1}{t} \times (\log|Q_{r(t)}| - H(2t) + 2H(t)) > 0$ then for every sequence $(t_n)_{n\in\mathbb{N}}$ satisfying $\sum_n \exp\{-t_n\} < \infty$,*

$$\frac{(1/|Q_{r(t_n)}|) \sum_{x \in Q_{r(t_n)}} u(t_n, x)}{\langle u(t_n, 0) \rangle} \overset{t_n\to\infty}{\longrightarrow} 1 \quad \textit{a.s.}$$

Notice that the necessary growth rate of Q for a WLLN to hold is the same as in Theorem 1 for $\alpha = 1$ and that the necessary growth rate of Q for a CLT to hold corresponds to $\alpha = 2$, see [3]. The growth rate in Theorem 2 is of the same order as in the CLT case. Notice, that Theorem 1 is closer to the i.i.d. case than to the case of a random walk among random obstacles as considered in [2], Theorem 3, where the limiting distributions are not stable laws, but infinite divisible distributions with Levy spectral functions that are not continuous. It seems as if the discrete character of the random walk mentioned in the comment on [2], Theorem 3, is more decisive for that model than for ours which can be reduced to the i.i.d. case by virtue of an appropriate coarse-graining.

To get a feeling for the numbers involved we give them for the two examples mentioned above in Table 1 below.

Notice that in the Weibull case we have

$$\log B_\alpha(t) = \frac{1}{\alpha} \left(\log \langle u(t,0)^\alpha \rangle + \log |Q_{L_\alpha(t)}| \right) + \mathrm{o}(t),$$

see [11], i.e., the asymptotics of $B_\alpha(t)$, $Q_{L_\alpha}(t)$ and the αth moment of u are closely linked. Because of (3) and our considerations in Section 2 this relationship seems to be true in the double exponential case, as well.

2. Stable limit laws

Let us explain our strategy of the proof of Theorem 1. We decompose the large box $Q_{L_\alpha(t)}$ into boxes $Q_{l(t)}^{(i)}$, $i = 1, \ldots, \lfloor |Q_{L_\alpha(t)}| / |Q_{l(t)}| \rfloor$ of much smaller size. In each subbox we approximate u by $u^{(i)}$, the solution with Dirichlet boundary conditions in $Q_{l(t)}^{(i)}$. In this way, we reduce the problem to the case of i.i.d. random variables. A spectral

Table 1

Distribution	$\varphi(x)$	$\log L_\alpha(t)$	$\log B_\alpha(t)$
Weibull	$x^\gamma, \gamma > 1$	$(\frac{\alpha t}{\gamma})^{\gamma/(\gamma-1)}$	$t(\frac{\alpha t}{\gamma})^{1/(\gamma-1)} - 2d\kappa\alpha t + \mathrm{o}(t)$
Double-exponential	$\exp\{x/\rho\}$	$\rho\alpha t$	$t\rho \log \rho\alpha t - \chi(\rho)\alpha t + \mathrm{o}(t)$

representation shows that $\sum_{x\in Q^{(i)}_{l(t)}} u^{(i)}(t,x)$ can be approximated by $\mathrm{e}^{t\lambda_1^{(i)}}$, where $\lambda_1^{(i)}$ is the principal Dirichlet eigenvalue of $\Delta+\xi$ in $Q^{(i)}_{l(t)}$. Then a classical result on stable limits for sums of t-dependent i.i.d. random variables yields the result.

Note that we cannot apply the results of [1] since they require the function φ to be normalized regularly varying, which is for instance not true in the important case of double-exponential tails. An alternative approach could be to adopt the techniques from [5].

Let us turn to the details. We first work on the $u^{(i)}$ and show in the end how to approximate u by $u^{(i)}$. We assume that $Q^{(i)}_{l(t)}$ are translated copies of $Q_{l(t)}$. We consider the solution $u^{(i)}$ to the PAM in $Q^{(i)}_{l(t)}$ with Dirichlet boundary conditions, i.e., $\xi(x)=-\infty$ for all $x\notin Q^{(i)}_{l(t)}$, where $l(t)=\max\{t^2\log^2 t, H(4t)\}$. The corresponding Laplacian will be denoted $\Delta^0_{Q^{(i)}_{l(t)}}$. By $\tau_U:=\inf\{t>0\colon X_t\in U\}$ we denote the first hitting time of a set U by a random walk X. The Feynman–Kac representation of $u^{(i)}$ reads

$$u^{(i)}(t,x)=\mathbb{E}_x\exp\left\{\int_0^\infty \xi(X_s)\,\mathrm{d}s\right\}\mathbb{1}_{\tau_{(Q^{(i)}_{l(t)})^c}>t},\qquad (t,x)\in[0,\infty)\times Q^{(i)}_{l(t)}.$$

Let $\lambda_1^{(i)},\dots,\lambda^{(i)}_{|Q_{l(t)}|}$ be the order statistics of the eigenvalues of the Anderson Hamiltonian $\Delta^0_{Q^{(i)}_{l(t)}}+\xi$ and $e_1^{(i)},\dots,e^{(i)}_{|Q_{l(t)}|}$ be the corresponding orthonormal basis. Then we have the following spectral representation

$$\sum_{x\in Q^{(i)}_{l(t)}} u^{(i)}(t,x)=\sum_{x,y\in Q^{(i)}_{l(t)}}\sum_{k=1}^{|Q_{l(t)}|}\mathrm{e}^{\lambda_k^{(i)}t}e_k^{(i)}(x)e_k^{(i)}(y),\qquad t\in[0,\infty).\tag{4}$$

For simplicity we have suppressed the time dependence of the eigenvalues and eigenvectors that arises because the boxes are time dependent. From Parseval's inequality, the fact that $l(t)$ is of subexponential order and the proof of Theorem 2.2 in [10] it follows that there exists $\widetilde{\varepsilon}^{(i)}(t)=\widetilde{\varepsilon}^{(i)}(\xi,t)=\mathrm{o}(1)$ such that

$$\sum_{x\in Q^{(i)}_{l(t)}} u^{(i)}(t,x)=\mathrm{e}^{t\mu_t^{(i)}},\quad\text{where } \mu_t^{(i)}=\mu_t^{(i)}(\xi)=\lambda_1^{(i)}+\widetilde{\varepsilon}^{(i)}(t).$$

Sometimes we will write μ_t instead of $\mu_t^{(i)}$, λ_1 for $\lambda_1^{(i)}$ and $\widetilde{\varepsilon}(t)$ for $\widetilde{\varepsilon}^{(i)}(t)$.

Remark. *The above already implies that for* $\log r(t)=\mathrm{o}(H(t))$ *the quenched setting is prominent in the following sense,*

$$\lim_{t\to\infty}\frac{\log u(t,0)}{\log\sum_{x\in Q_{r(t)}}u(t,x)}=1,\quad a.s.$$

In the next lemma we show how the distributions of μ_t and $\xi(0)$ are linked.

Lemma 3. *Let Assumption F be satisfied. Then for all functions h with* $\lim_{t\to\infty}|Q_{l(t)}|\mathbf{P}(\xi(0)>h(t))=0$ *there exists* $\varepsilon(t)=\varepsilon(\xi,t)=\mathrm{o}(1)$ *such that,*

$$\mathbf{P}\big(\mu_t>h(t)\big)\sim|Q_{l(t)}|\mathbf{P}\big(\xi(0)>h(t)+\chi-\varepsilon(t)\big),\quad t\to\infty.$$

Proof. In [10], Proof of Theorem 2.16, the authors show that the first eigenvalue of $\Delta^0_{Q_{l(t)}}+\xi$ satisfies

$$\lambda_1=\xi^{(1)}_{Q_{l(t)}}-\chi+\bar{\varepsilon}(t),$$

with $\bar{\varepsilon}(t) = \bar{\varepsilon}(\xi, t) = \mathrm{o}(1)$. Let

$$\varepsilon(t) := \widetilde{\varepsilon}(t) + \bar{\varepsilon}(t).$$

Then

$$\begin{aligned}\mathbf{P}\big(\mu_t > h(t)\big) &= \mathbf{P}\big(\xi^{(1)}_{Q_{l(t)}} > h(t) + \chi - \varepsilon(t)\big)\\ &= 1 - \big(1 - \mathbf{P}\big(\xi(0) > h(t) + \chi - \varepsilon(t)\big)\big)^{|Q_{l(t)}|}\\ &\sim 1 - \exp\big\{-|Q_{l(t)}|\mathbf{P}\big(\xi(0) > h(t) + \chi - \varepsilon(t)\big)\big\}\\ &\sim |Q_{l(t)}|\mathbf{P}\big(\xi(0) > h(t) + \chi - \varepsilon(t)\big), \quad t \to \infty.\end{aligned}$$

In the third line we use L'Hopital's rule. □

Let

$$\widetilde{\varphi}_t(x) = -\log \mathbf{P}(\mu_t > x),$$

and $\widetilde{h}_t$ a solution to

$$\sup_{h\in(0,\infty)} \big(th - \widetilde{\varphi}_t(h)\big) = t\widetilde{h}_t - \widetilde{\varphi}_t(\widetilde{h}_t). \tag{5}$$

It follows that $\widetilde{h}_t = h_t + \chi + \mathrm{o}(1)$ and if $\widetilde{\varphi}_t$ is ultimately convex then $\widetilde{h}_t$ is the unique solution to (5). Then, an application of the Laplace method yields

$$\big\langle u(t,0)\big\rangle \sim \big\langle u^{(i)}(t,0)\big\rangle \sim t\int_0^\infty \exp\big\{th - \widetilde{\varphi}(h)\big\}\,\mathrm{d}h = \exp\big\{\big[t\widetilde{h}_t - \widetilde{\varphi}(\widetilde{h}_t)\big]\big(1+\mathrm{o}(1)\big)\big\}.$$

The first asymptotics follow from [11], Proposition 7. Hence, we obtain together with Lemma 3 that

$$\log B_\alpha(t) = t\widetilde{h}_{\alpha t} = \frac{1}{\alpha}\big(\log\big\langle u(t,0)^\alpha\big\rangle + \log|Q_{L_\alpha(t)}|\big)\big(1+\mathrm{o}(1)\big).$$

To prove convergence of $\sum_{i:Q^{(i)}_{l(t)}\subset Q_{L_\alpha(t)}}(\mathrm{e}^{t\mu_t^{(i)}} - \widetilde{A}(t))/B_\alpha(t)$, as $t\to\infty$, to an infinitely divisible distribution with characteristic function equal to

$$\phi(u) = \exp\bigg\{\mathrm{i}au - \frac{\sigma^2u^2}{2} + \int_{|x|>0}\bigg(\mathrm{e}^{\mathrm{i}ux} - 1 - \frac{\mathrm{i}ux}{1+x^2}\bigg)\,\mathrm{d}\widetilde{L}(x)\bigg\}, \tag{6}$$

we have to verify the following condition (see [16], Chapter IV).

Condition P.

(i) *Condition of infinite smallness*:

$$\lim_{t\to\infty}\max_{i:Q^{(i)}_{l(t)}\subset Q_{L_\alpha(t)}}\mathbf{P}\bigg(\frac{\mathrm{e}^{t\mu_t^{(i)}}}{B_\alpha(t)} \ge \varepsilon\bigg) = 0, \quad \varepsilon > 0.$$

(ii) *In all points x of continuity, the function $\widetilde{L}$ satisfies*:

$$\widetilde{L}(x) = -\lim_{t\to\infty}\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\mathbf{P}\bigg(\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)} > x\bigg).$$

(iii) *The constant* σ^2 *satisfies*:

$$\sigma^2 = \lim_{\tau\to 0}\limsup_{t\to\infty}\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\operatorname{Var}\bigg(\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))\le\tau}\bigg)$$
$$= \lim_{\tau\to 0}\liminf_{t\to\infty}\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\operatorname{Var}\bigg(\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))\le\tau}\bigg).$$

(iv) *For every* $\tau > 0$ *the constant* a *satisfies*:

$$\lim_{t\to\infty}\bigg\{\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\bigg\langle\frac{\mathrm{e}^{t\mu_t}}{B_\alpha(t)}\mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))\le\tau}\bigg\rangle - \frac{\widetilde{A}(t)}{B_\alpha(t)}\bigg\} = a + \int_0^\tau\frac{x^3}{1+x^2}\,\mathrm{d}L(x) - \int_\tau^\infty\frac{x}{1+x^2}\,\mathrm{d}L(x).$$

Items (i) and (ii) will follow from the next lemma, and (iii) and (iv) from the next proposition.

Lemma 4. *Let Assumption F be satisfied and* $\varphi\in C^2$ *be ultimately convex. Then*

$$\lim_{t\to\infty}\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\mathbf{P}\bigg(\mu_t > \frac{\log B_\alpha(t)}{t}+\frac{\log x}{t}\bigg) = x^{-\alpha}.$$

Proof. Lemma 3 and a first order Taylor expansion yield

$$\begin{aligned}
&\mathbf{P}\bigg(\mu_t > \frac{\log B_\alpha(t)}{t}+\frac{\log x}{t}\bigg)\\
&\quad\sim |Q_{l(t)}|\mathbf{P}\bigg(\xi(0) > \frac{\log B_\alpha(t)}{t}+\frac{\log x}{t}+\chi+\varepsilon(t)\bigg)\\
&\quad= \exp\bigg\{\log|Q_{l(t)}| - \varphi\bigg(\frac{\log B_\alpha(t)}{t}+\chi+\mathrm{o}(1)\bigg) - \varphi'\bigg(\frac{\log B_\alpha(t)}{t}+\chi+\mathrm{o}(1)\bigg)\frac{\log x}{t}+\mathrm{o}(1)\bigg\}.
\end{aligned}$$

Since φ is ultimately convex and ξ is unbounded from above we find that $\varphi''(h_{\alpha t}) = 1/h'_{\alpha t} = \mathrm{o}(t^2)$. From this we can conclude that the error term in the Taylor expansion above vanishes asymptotically. Moreover, by our choice of $l(t)$ and $B_\alpha(t)$ it follows that

$$\log|Q_{L_\alpha(t)}| = \varphi\bigg(\frac{\log B_\alpha(t)}{t}+\chi+\mathrm{o}(1)\bigg)\quad\text{and}\quad\lim_{t\to\infty}\frac{\varphi'((\log B_\alpha(t)/t)+\chi+\mathrm{o}(1))}{t} = \alpha.$$

□

Proposition 5. *Let Assumption F be satisfied and* $\varphi\in C^2$ *be ultimately convex. Then, for any* $\tau > 0$,

(i) *if* $p > \alpha$ *then*

$$\lim_{t\to\infty}\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\bigg\langle\frac{\mathrm{e}^{pt\mu_t}}{B_\alpha(t)^p}\mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))\le\tau}\bigg\rangle = \frac{\alpha}{p-\alpha}\tau^{p-\alpha};$$

(ii) *if* $p < \alpha$ *then*

$$\lim_{t\to\infty}\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\bigg\langle\frac{\mathrm{e}^{pt\mu_t}}{B_\alpha(t)^p}\mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))>\tau}\bigg\rangle = \frac{\alpha}{\alpha-p}\tau^{p-\alpha};$$

(iii) *if* $p = \alpha$ *then*

$$\lim_{t\to\infty}\frac{|Q_{L_\alpha(t)}|}{|Q_{l(t)}|}\bigg\langle\frac{\mathrm{e}^{pt\mu_t}}{B_\alpha(t)^p}(\mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))\le\tau} - \mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))\le 1})\bigg\rangle = \alpha\log\tau.$$

Proof. (i) Integration by parts yields

$$\big\langle \mathrm{e}^{pt\mu_t}\mathbb{1}_{(\mathrm{e}^{t\mu_t}/B_\alpha(t))\le\tau}\big\rangle = \int_0^{\widetilde{h}_{\alpha t}+\log\tau/t} \mathrm{e}^{tpx}\,\mathrm{d}\big(1-\bar{F}_{\mu_t}(x)\big)$$
$$= -\big[\mathrm{e}^{tpx-\widetilde{\varphi}(x)}\big]_{x=0}^{x=\widetilde{h}_{\alpha t}+\log\tau/t} + pt\int_0^{\widetilde{h}_{\alpha t}+\log\tau/t}\mathrm{e}^{tpx-\widetilde{\varphi}(x)}\,\mathrm{d}x.$$

Here $\bar{F}_{\mu_t}$ denotes the tail distribution function of μ_t. Similarly as in the proof of Lemma 4 we find with the help of a first order Taylor expansion of φ that uniformly in τ,

$$\widetilde{\varphi}\left(\widetilde{h}_{\alpha t}+\frac{\log\tau}{t}\right)\sim\widetilde{\varphi}(\widetilde{h}_{\alpha t})-\alpha\log\tau,\quad t\to\infty.$$

Substituting $x=\widetilde{h}_{\alpha t}+\frac{\log\tau}{t}u$ for $\tau\neq1$ we find that

$$pt\int_0^{\widetilde{h}_{\alpha t}+\log\tau/t}\mathrm{e}^{tpx-\widetilde{\varphi}(x)}\,\mathrm{d}x\sim\mathrm{e}^{pt\widetilde{h}_{\alpha t}-\widetilde{\varphi}(\widetilde{h}_{\alpha t})}p\log\tau\int_{-\infty\cdot\operatorname{sign}\log\tau}^{1}\mathrm{e}^{u(p-\alpha)\log\tau}\,\mathrm{d}u$$
$$\sim\frac{p}{p-\alpha}\mathrm{e}^{pt\widetilde{h}_{\alpha t}-\widetilde{\varphi}(\widetilde{h}_{\alpha t})+(p-\alpha)\log\tau},\quad t\to\infty.$$

Altogether this proves the claim.

(ii) and (iii) follow similarly. □

Overall we find:

Theorem 6. *Let Assumption F be satisfied and $\varphi\in C^2$ be ultimately convex. Then for $\alpha\in(0,2)$,*

$$\sum_{i:Q_{l(t)}^{(i)}\subset Q_{L_\alpha(t)}}\frac{\mathrm{e}^{t\mu_t^{(i)}}-\widetilde{A}(t)}{B_\alpha(t)}\overset{t\to\infty}{\Longrightarrow}\mathcal{F}_\alpha,$$

with

$$\widetilde{A}(t)=\begin{cases}0, & \text{if } \alpha\in(0,1),\\ \langle\mathrm{e}^{t\mu_t}\rangle, & \text{if } \alpha\in(1,2),\\ \langle\mathrm{e}^{t\mu_t}\mathbb{1}_{\mu_t\le1}\rangle, & \text{if } \alpha=1.\end{cases}$$

Proof. Since the $u^{(i)}$ are i.i.d., the $\mu_t^{(i)}$ are as well. Hence, we have to check the four points of Condition P. Items (i) and (ii) follow from Lemma 4. We find that $\widetilde{L}(x)=x^{-\alpha}$. It follows from Proposition 5 that $\sigma^2=0$. Furthermore, Proposition 5 together with [1], Proposition 6.4, yields the constant a from which we can deduce ϕ. The stability of the limit law follows from [16], Theorem IV.12, since $\sigma^2=0$ and $\widetilde{L}(x)=x^{-\alpha}$. □

Remark. *An infinitely divisible law with characteristic function as in (6) is stable if and only if either $\widetilde{L}\equiv0$ or $\sigma^2=0$ and $\widetilde{L}(x)=cx^{-\alpha}$, $c>0$, $\alpha\in(0,2)$, see [16], Theorem* IV.12.

We extend the functions $u^{(i)}$ to a function $\widetilde{u}:Q_{L_\alpha(t)}\to[0,\infty)$ by putting $\widetilde{u}(t,x)=u^{(i)}(t,x)$ for $x\in Q_{l(t)}^{(i)}$. Now it remains to show that

$$\sum_{x\in Q_{L_\alpha(t)}}\frac{u(t,x)-A(t)}{B_\alpha(t)}\quad\text{and}\quad\sum_{x\in Q_{L_\alpha(t)}}\frac{\widetilde{u}(t,x)-\widetilde{A}(t)/|Q_{l(t)}|}{B_\alpha(t)}=\sum_{i:Q_{l(t)}^{(i)}\subset Q_{L_\alpha(t)}}\frac{\exp\{t\mu_t^{(i)}\}-\widetilde{A}(t)}{B_\alpha(t)}$$

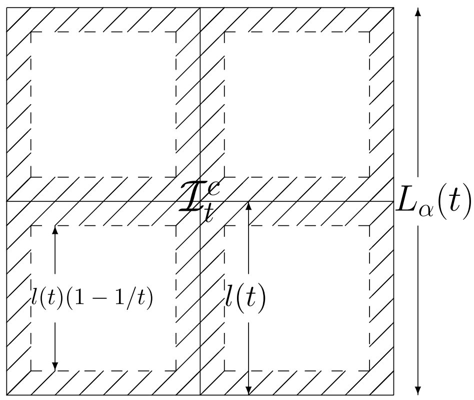

Fig. 1. Coarse-graining.

have the same α-stable limit distribution. To this end let $\mathcal{I}_t^c = \bigcup_{i: Q_{l(t)}^{(i)} \subset Q_{L_\alpha(t)}} Q_{l(t)}^{(i)} \setminus Q_{l(t)(1-1/t)}^{(i)}$ and $\mathcal{I}_t = Q_{L_\alpha(t)} \setminus \mathcal{I}_t^c$ (see Fig. 1).

Notice that

$$u(t,x) - \widetilde{u}(t,x) = \mathbb{E}_x \exp\left\{\int_0^\infty \xi(X_s)\,\mathrm{d}s\right\} \mathbb{1}_{\tau_{(Q_{l(t)}^{(i)})^c} \le t}, \quad (t,x) \in [0,\infty) \times Q_{l(t)}^{(i)}.$$

In the next lemma we show that those paths of the random walk in the Feynman–Kac formula that start in $\mathcal{I}_t$ and leave $Q_{l(t)}$ before time t are asymptotically negligible.

Lemma 7. *Almost surely*,

$$\lim_{t\to\infty} \sup_{x \in Q_{l(t)(1-1/t)}} \mathbb{E}_x \exp\left\{\int_0^\infty \xi(X_s)\,\mathrm{d}s\right\} \mathbb{1}_{\tau_{(Q_{l(t)})^c} \le t} = 0.$$

Proof. We find that

$$\begin{aligned}
&\sup_{x \in Q_{l(t)(1-1/t)}} \mathbb{E}_x \exp\left\{\int_0^\infty \xi(X_s)\,\mathrm{d}s\right\} \mathbb{1}_{\tau_{(Q_{l(t)})^c} \le t} \\
&\quad \le \exp\left\{t \sup_{x \in Q_{l(t)}} \xi(x)\right\} \mathbb{P}_0(\tau_{(Q_{l(t)/t})^c} \le t) \\
&\quad \le 2^{d+1} \exp\left\{t \cdot \mathrm{o}\big(\log\big(|Q_{l(t)}|\big)\big) - |Q_{l(t)/t}| \log\left(\frac{|Q_{l(t)/t}|}{d\kappa t}\right)\right\} \xrightarrow{t\to\infty} 0.
\end{aligned}$$

In the last inequality we use [10], Lemma 2.5 and Corollary 2.7. □

Lemma 8. *For all* $\varepsilon > 0$,

(i) *if* $\alpha \in (0,1]$ *then*

$$\lim_{t\to\infty} \mathbf{P}\left(\frac{1}{B_\alpha(t)} \sum_{x \in Q_{L_\alpha(t)}} \big[u(t,x) - \widetilde{u}(t,x)\big] > \varepsilon\right) = 0;$$

(ii) *if* $\alpha \in [1,2)$ *then*

$$\lim_{t\to\infty} \mathbf{P}\left(\frac{1}{B_\alpha(t)} \sum_{x \in Q_{L_\alpha(t)}} \big[u(t,x) - \langle u(t,x)\rangle - \widetilde{u}(t,x) + \langle\widetilde{u}(t,x)\rangle\big] > \varepsilon\right) = 0;$$

(iii) *if* $\alpha = 1$ *then*

$$\lim_{t\to\infty} \mathbf{P}\biggl(\sum_{x\in Q_{L_\alpha(t)}} \frac{u(t,x) - \langle u(t,x)\mathbb{1}_{u(t,x)\le B_\alpha(t)}\rangle - \widetilde{u}(t,x) + \langle \widetilde{u}(t,x)\mathbb{1}_{\widetilde{u}(t,x)\le B_\alpha(t)}\rangle}{B_\alpha(t)} > \varepsilon\biggr) = 0.$$

Proof. (i) From Lemma 7 and the fact that $|\mathcal{I}_t| < B_\alpha(t)$ for all t it follows that for $t\to\infty$,

$$\mathbf{P}\biggl(\frac{1}{B_\alpha(t)} \sum_{x\in Q_{L_\alpha(t)}} u(t,x) - \widetilde{u}(t,x) > \varepsilon\biggr) \sim \mathbf{P}\biggl(\frac{1}{B_\alpha(t)} \sum_{x\in \mathcal{I}_t^c} u(t,x) - \widetilde{u}(t,x) > \varepsilon\biggr).$$

By the definition of $B_\alpha(t)$ and by Markov's inequality it follows that

$$\begin{aligned}
\mathbf{P}\biggl(\frac{1}{B_\alpha(t)} \sum_{x\in \mathcal{I}_t^c} u(t,x) - \widetilde{u}(t,x) > \varepsilon\biggr) &\le \mathbf{P}\biggl(\sum_{x\in \mathcal{I}_t^c} \frac{u(t,x)}{|Q_{L_\alpha(t)}|^{1/\alpha}\langle u(t,0)^\alpha\rangle^{1/\alpha}} > \varepsilon\biggr)\\
&\le \frac{1}{\varepsilon^\alpha}\frac{\langle(\sum_{x\in \mathcal{I}_t^c} u(t,x))^\alpha\rangle}{|Q_{L_\alpha(t)}|\langle u(t,0)^\alpha\rangle}\\
&\le \frac{1}{\varepsilon^\alpha}\frac{|Q_{L_\alpha(t)}|}{|\mathcal{I}_t|}\frac{\langle u(t,0)^\alpha\rangle}{|Q_{L_\alpha(t)}|\langle u(t,0)^\alpha\rangle} \xrightarrow{t\to\infty} 0.
\end{aligned}$$

(ii) Similarly as in case (i) we find that asymptotically

$$\begin{aligned}
&\mathbf{P}\biggl(\frac{1}{B_\alpha(t)} \sum_{x\in Q_{L_\alpha(t)}} \bigl[u(t,x) - \bigl\langle u(t,x)\bigr\rangle - \widetilde{u}(t,x) + \bigl\langle \widetilde{u}(t,x)\bigr\rangle\bigr] > \varepsilon\biggr)\\
&\quad\le \frac{1}{\varepsilon^\alpha}\frac{\langle(\sum_{x\in \mathcal{I}_t^c} u(t,x))^\alpha\rangle}{|Q_{L_\alpha(t)}|\langle u(t,0)^\alpha\rangle} + \mathrm{o}(1).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\biggl\langle\biggl(\sum_{x\in \mathcal{I}_t^c} u(t,x)\biggr)^\alpha\biggr\rangle &\le \biggl\langle\biggl(\sum_{x\in \mathcal{I}_t^c}\biggl(u(t,x)^2 + \sum_{\substack{y\in \mathcal{I}_t^c:\\ |x-y|\le l(t)/t}} u(t,x)u(t,y) + \sum_{\substack{y\in \mathcal{I}_t^c:\\ |x-y|> l(t)/t}} u(t,x)u(t,y)\biggr)\biggr)^{\alpha/2}\biggr\rangle\\
&\le \sum_{x\in \mathcal{I}_t^c} \bigl|l(t)(1-1/t)\bigr|\bigl\langle u(t,x)^\alpha\bigr\rangle + \sum_{\substack{x,y\in \mathcal{I}_t^c:\\ |x-y|> l(t)/t}} \bigl\langle\bigl(u(t,x)u(t,y)\bigr)^{\alpha/2}\bigr\rangle.
\end{aligned}$$

The first summand can be treated as in case (i) whereas the second summand can be treated similarly as in the proof of Lemma 9.

(iii) follows analogously. □

Now we are able to prove Theorem 1.

Proof of Theorem 1. We only consider the case $\alpha \in (0,1)$. The other cases follow similarly. It follows from Lemma 8 that for every $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbf{P}\biggl(\sum_{i: Q_{l(t)}^{(i)}\subset Q_{L_\alpha(t)}} \biggl|\frac{\sum_{x\in Q_{l(t)}^{(i)}} u(t,x) - \exp\{t\mu_t^{(i)}\}}{B_\alpha(t)}\biggr| > \varepsilon\biggr) = 0,$$

while Theorem 6 states that under the same conditions as in Theorem 1,

$$\sum_{i: Q^{(i)}_{l(t)} \subset Q_{L_\alpha(t)}} \frac{\mathrm{e}^{t\mu^{(i)}_t}}{B_\alpha(t)} \overset{t\to\infty}{\Longrightarrow} \mathcal{F}_\alpha.$$

Therefore, the claim follows from Slutzky's theorem. □

Remark. *We expect that a similar result as Theorem 1 with the same stable limit distribution also holds for potential tails that are bounded from above as considered in* [4] *and* [12]. *However, since in that case we do not have such a close link between μ_t and $\xi^{(1)}_{Q_{l(t)}}$ we cannot determine the distribution of μ_t and therefore $L_\alpha(t) = -\log \mathbf{P}(\mu_t > \widetilde{h}_t)$ cannot be made as explicit as under Assumption* F.

For more heavy tailed potentials than considered in this paper the cumulant generating function H is not finite any more and annealed asymptotics do not exist. Therefore, our approach is not feasible it that situation. However, there are some recent papers on the PAM with localised initial condition δ_0 that derive scaling limit theorems for exponential tails, see [14] or Weibull tails with parameter $\gamma < 1$, see [17].

3. Strong law of large numbers

Recall that $l(t) = \max\{t^2 \log^2 t, H(4t)\}$ and that $x + Q_{l(t)}$ is the lattice box with centre x and sidelength $l(t)$.

Lemma 9. *Let Assumption* H *be satisfied and $r(t)$ be chosen as in Theorem* 2, *then*

$$\lim_{t\to\infty} \frac{1}{|Q_{r(t)}|^2} \sum_{\substack{x,y\in Q_{r(t)}: \\ |x-y|>2l(t)}} \left(\frac{\langle u(t,x)u(t,y)\rangle}{\langle u(t,0)\rangle^2} - 1 \right) = 0.$$

Proof. For $t > 0$ and $x \in Q_{r(t)}$ let

$$u^{(1)}(t,x) = \mathbb{E}_x \exp\left\{ \int_0^\infty \xi(X_s)\,\mathrm{d}s \right\} \mathbb{1}_{\tau_{(x+Q_{l(t)})^c} \ge t}$$

and

$$u(t,x,y) = \mathbb{E}_{x,y} \exp\left\{ \int_0^\infty \xi(X_s)\,\mathrm{d}s \right\} \exp\left\{ \int_0^\infty \xi(Y_s)\,\mathrm{d}s \right\} \mathbb{1}_{\tau^X_{(x+Q_{l(t)})^c} < t \text{ or } \tau^Y_{(y+Q_{l(t)})^c} < t},$$

where X and Y are two independent random walks starting in x, y, respectively, $\mathbb{E}_{x,y}$ is their joint expectation, and τ^X_A, τ^Y_A are their exit times from a set $A \subset \mathbb{Z}^d$, respectively. If $|x - y| > 2l(t)$ then $u^{(1)}(t,x)$ and $u^{(1)}(t,y)$ are independent, and hence

$$\sum_{\substack{x,y\in Q_{r(t)}: \\ |x-y|>2l(t)}} \left(\frac{\langle u(t,x)u(t,y)\rangle}{\langle u(t,0)\rangle^2} - 1 \right) = \sum_{\substack{x,y\in Q_{r(t)}: \\ |x-y|>2l(t)}} \frac{\langle u(t,x,y)\rangle}{\langle u(t,0)\rangle^2}.$$

Hölder's inequality and [9], Lemma 2.4 and Theorem 3.1, yield for all $x, y \in Q_{r(t)} \setminus Q_{l(t)}$,

$$\begin{aligned} \langle u(t,x,y)\rangle &\le \sqrt{\langle u(t,0)^4\rangle 2\mathbb{P}_x(\tau_{(x+Q_{l(t)})^c} < t)} \\ &\le \exp\left\{ \frac{1}{2}\big(l(t) - l(t)\log l(t)\big) + \mathrm{o}(t) \right\} \overset{t\to\infty}{\longrightarrow} 0. \end{aligned}$$

□

Proof of Theorem 2. By Chebyshev's inequality we find that for every $s > 0$,

$$\mathbf{P}\left(\sup_{t_n > s} \frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \left(\frac{u(t_n, x)}{\langle u(t_n, 0)\rangle} - 1\right) > \varepsilon\right)$$
$$\le \sum_{t_n > s} \frac{1}{\varepsilon^2} \operatorname{Var}\left(\frac{1}{|Q_{r(t_n)}|} \sum_{x \in Q_{r(t_n)}} \left(\frac{u(t_n, x)}{\langle u(t_n, 0)\rangle} - 1\right)\right).$$

As t tends to infinity it follows with Lemma 9 that

$$\begin{aligned}
\operatorname{Var}\left(\frac{1}{|Q_{r(t)}|} \sum_{x \in Q_{r(t)}} \left(\frac{u(t, x)}{\langle u(t, 0)\rangle} - 1\right)\right) &\sim \frac{1}{|Q_{r(t)}|^2} \sum_{\substack{x, y \in Q_{r(t)}: \\ |x-y| < 2l(t)}} \left(\frac{\langle u(t, x) u(t, y)\rangle}{\langle u(t, 0)\rangle^2} - 1\right) \\
&\sim \frac{1}{|Q_{r(t)}|} \sum_{x \in Q_{l(t)}} \left(\frac{\langle u(t, 0) u(t, x)\rangle}{\langle u(t, 0)\rangle^2} - 1\right) \\
&\le \frac{|Q_{l(t)}|}{|Q_{r(t)}|} \frac{\langle u(t, 0)^2\rangle}{\langle u(t, 0)\rangle^2} \\
&= \exp\bigl\{-\log|Q_{r(t)}| + H(2t) - 2H(t) + \mathrm{o}(t)\bigr\}.
\end{aligned}$$

The last asymptotics are due to (3). Now the claim follows because for our choice of $r(t)$,

$$\lim_{s \to \infty} \sum_{t_n > s} \exp\bigl\{-\log|Q_{r(t_n)}| + H(2t_n) - 2H(t_n) + \mathrm{o}(t_n)\bigr\} = 0.$$

□

Acknowledgement

We would like to thank Wolfgang König for various helpful comments on the first draft of this paper.

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