Transience, recurrence and speed of diffusions with a non-Markovian two-phase "use it or lose it" drift

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Abstract. We investigate the transience/recurrence of a non-Markovian, one-dimensional diffusion process which consists of a Brownian motion with a non-anticipating drift that has two phases – a transient to $+\infty$ mode which is activated when the diffusion is sufficiently near its running maximum, and a recurrent mode which is activated otherwise. We also consider the speed of a diffusion with a two-phase drift, where the drift is equal to a certain non-negative constant when the diffusion is sufficiently near its running maximum, and is equal to a certain positive constant otherwise.

Résumé. Nous étudions la transience/récurrence d'un processus de diffusion non-Markovien à une dimension, consistant en un mouvement brownien avec une dérive non anticipative qui a deux phases – un mode transitoire à $+\infty$ qui est activé quand la diffusion est suffisamment proche du processus de son maximum, et un mode récurrent qui est activé dans le cas contraire. On considère également la vitesse d'une diffusion avec une dérive à deux phases, où la dérive est égale à une certaine constante positive lorsque la diffusion est suffisamment proche du processus de son maximum, et est égale à une certaine constante strictement positive dans le cas contraire.

1. Introduction and statement of results

Over the past fifteen years or so, there has been much interest in the study of the long term behavior of various random walks with non-Markovian transition mechanisms, such as random walks in random environment, self-avoiding random walk, edge or vertex reinforced random walks, and excited (so-called "cookie") random walks. See, for example, the monograph [13], and the survey articles [9] and [7], which include many references. Non-Markovian diffusion processes analogous to excited random walks have also been studied (see [2,3,12]), as well as so-called Brownian polymers, which are non-Markovian self-repelling diffusions, analogous to certain negatively reinforced random walks (see [1,4,5,8]). In this paper we investigate the transience/recurrence of a non-Markovian, one-dimensional diffusion process which consists of a Brownian motion with a non-anticipating drift that has two phases – a transient to $+\infty$ mode which is activated when the diffusion is sufficiently near its running maximum, and a recurrent mode which is activated otherwise. We also consider the speed of a diffusion with a two-phase drift, where the drift is equal to a certain non-negative constant when the diffusion is sufficiently near its running maximum, and is equal to a certain positive constant otherwise.

Let $b^T(x)$ and $b^R(x)$ be continuous functions on R which satisfy

$$\int_{-\infty}^{0} \exp\left(-\int_0^x 2b^T(y)\,dy\right) dx = \infty, \qquad \int_0^{\infty} \exp\left(-\int_0^x 2b^T(y)\,dy\right) dx < \infty; \qquad (1.1)$$

$$\int_{-\infty} \exp\left(-\int_0^x 2b^R(y)\,\mathrm{d}y\right)\mathrm{d}x = \infty, \qquad \int^{\infty} \exp\left(-\int_0^x 2b^R(y)\,\mathrm{d}y\right)\mathrm{d}x = \infty.$$

As is well known [10], the one-dimensional diffusion processes corresponding to the operators $\mathcal{L}^T \equiv \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b^T(x)\frac{\mathrm{d}}{\mathrm{d}x}$ and $\mathcal{L}^R \equiv \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b^R(x)\frac{\mathrm{d}}{\mathrm{d}x}$ are respectively transient to $+\infty$ and recurrent. Let $\gamma : [0, \infty) \to (0, \infty)$ be a continuous function satisfying

$$\gamma > 0, \qquad \gamma' < 1 \quad \text{and} \quad \lim_{x \to \infty}(x - \gamma(x)) = \infty.$$

For a continuous trajectory $x(\cdot) : [0, \infty) \to R$, let $x^*(t) = \max_{0 \le s \le t} x(s)$ denote its running maximum. We define a non-anticipating, non-Markovian drift $b(t, x(\cdot))$ by

$$b(t, x(\cdot)) = \begin{cases} b^T(x(t)), & \text{if } x(t) > x^*(t) - \gamma(x^*(t)); \\ b^R(x(t)), & \text{if } x(t) \le x^*(t) - \gamma(x^*(t)). \end{cases} \tag{1.2}$$

We consider the diffusion process $X(t)$ that satisfies the stochastic differential equation

$$X(t) = x_0 + W(t) + \int_0^t b(s, X(\cdot))\,\mathrm{d}s, \tag{1.3}$$

where $W(\cdot)$ is a Brownian motion. Existence and uniqueness for this stochastic differential equation follow from the standard theory for classical diffusion processes (see Section 3). We call the solution to (1.3) a diffusion with a two-phase "use it or lose it" drift.

For example, the process $X(t)$ might represent the price of a stock. As prices rise, people are encouraged to buy, creating a certain trend represented by the transient drift. In addition to this underlying trend, there is a random fluctuation represented by the Brownian motion. These random fluctuations might cause prices to slump. If the slump becomes sufficiently large, it discourages buying, which creates a new weaker trend, represented by the recurrent drift. When random fluctuations eventually result in prices rising to levels close to the previous high, the original stronger trend reasserts itself.

We call γ the "down-crossing" function. For the majority of the paper, we will consider the case that the down-crossing function γ is a constant. In this case, at any time t, the diffusion $X(t)$ will run in the transient mode if and only if $X(t) > X^*(t) - \gamma$, or equivalently, if and only if by time t the path $X(\cdot)$ has not down-crossed an interval of length γ whose left hand endpoint is larger than or equal to $X(t)$.

The various equivalent definitions of transience and recurrence for classical non-degenerate diffusion processes hold for the diffusion with the two-phase drift. (This will be clear from the construction in Section 3.) We state here the standard definitions, although we will use other equivalent definitions in the proofs. The diffusion with the two-phase drift is *recurrent* if for any pair of points x_0 and x_1, the process starting at x_0 almost surely returns to x_1 at arbitrarily large times. The diffusion with the two-phase drift is *transient to* $+\infty$ if starting at any x_0, the process almost surely satisfies $\lim_{t \to \infty} X(t) = \infty$.

Our first result concerns the case in which the transient drift is constant: $b^T(x) \equiv b > 0$, and the recurrent drift b^R satisfies a regularity condition; namely, that the drift $b^R \vee 0$ is also a recurrent drift. In this case, the diffusion with the two-phase drift is always recurrent.

Theorem 1. *Assume that the down-crossing function γ is constant. Let the transient drift be constant: $b^T(x) \equiv b > 0$, and assume that the recurrent drift b^R is such that the drift $b^R \vee 0$ is also recurrent. That is, assume that the condition satisfied by b^R in (1.1) is also satisfied by $b^R \vee 0$. Then the diffusion with the two-phase drift is recurrent.*

Remark. *Note that the diffusion with the two-phase drift is recurrent even if the recurrent drift b^R is just border line recurrent – for example, if for sufficiently large x, $b^R(x) = \frac{1}{2x}$ is the drift of the radial part of a two-dimensional Brownian motion.*

Maintaining the constant transient drift, but choosing the recurrent drift b^R to take on very large positive values at most locations, and compensating to insure recurrence by having it take on even much larger negative values at other locations, we can construct a diffusion with such a two-phase drift that is transient.

Theorem 2. *Assume that the down-crossing function γ is constant. Let the transient drift be constant: $b^T(x) = b > 0$. There exists a recurrent drift b^R such that the corresponding diffusion with the two-phase drift is transient.*

We continue to assume that the down-crossing function γ is constant. As noted above, $b(t, X(\cdot)) = b^T(X(t))$ if and only if by time t, the path $X(\cdot)$ has not down-crossed an interval of length γ whose left hand endpoint is larger than or equal to $X(t)$. Now if the transient diffusion corresponding to $\mathcal{L}^T$ is such that it almost surely eventually stops making down-crossings of length γ, then the diffusion in the two-phase environment will eventually stop making down-crossings of length γ and will eventually be driven just by the transient drift; consequently, it will be transient. In [11] we proved the following result.

Theorem P2. *Consider the diffusion process corresponding to the operator $\mathcal{L}^T$.*

(i) *If $b^T(x) \leq \frac{1}{2\gamma} \log x + \frac{1}{\gamma} \log^{(2)} x$ for sufficiently large x, then the diffusion almost surely makes γ-down-crossings for arbitrarily large times;*

(ii) *If $b^T(x) \geq \frac{1}{2\gamma} \log x + \frac{k}{\gamma} \log^{(2)} x$, for some $k > 1$ and for sufficiently large x, then the diffusion almost surely eventually stops making down-crossings of size γ.*

In light of Theorem P2 and the paragraph preceding it, in the case of a constant down-crossing function γ, if b^T satisfies the condition in part (ii) of the theorem, then the diffusion with the two-phase drift is transient, regardless of what the recurrent drift b^R is.

Continuing with a constant down-crossing function γ, we now let the recurrent diffusion be Brownian motion, that is, $b^R \equiv 0$, and determine what the threshold growth rate is on b^T that distinguishes between transience and recurrence for the diffusion with the two-phase drift.

Theorem 3. *Assume that the down-crossing function γ is constant. Let the recurrent diffusion be Brownian motion: $b^R \equiv 0$.*

(i) *If $b^T(x) \leq \frac{1}{2\gamma} \log^{(2)} x + \frac{1}{2\gamma} \log^{(3)} x$, for large x, then the diffusion with the two-phase drift is recurrent;*

(ii) *If $b^T(x) \geq \frac{1}{2\gamma} \log^{(2)} x + \frac{k}{2\gamma} \log^{(3)} x$, for large x, where $k > 1$, then the diffusion with the two-phase drift is transient.*

Remark. *In light of Theorem P2 and Theorem 3, it follows that if the down-crossing function γ is constant, the recurrent diffusion is Brownian motion, and the transient diffusion has drift b^T satisfying*

$$\frac{1}{2\gamma} \log^{(2)} x + \frac{k}{2\gamma} \log^{(3)} x \leq b^T(x) \leq \frac{1}{2\gamma} \log x + \frac{1}{\gamma} \log^{(2)} x$$

for large x and some $k > 1$, then the diffusion with the two-phase drift will be transient, and will alternate infinity often between the recurrent and the transient regimes.

We now consider the case that the recurrent diffusion is Brownian motion: $b^R \equiv 0$, that the transient drift is constant: $b^T(x) \equiv b$, but we allow the down-crossing function γ to grow with x. We determine the threshold growth rate on the down-crossing function that distinguishes between transience and recurrence for the diffusion with the two-phase drift.

Theorem 4. *Let the recurrent diffusion be Brownian motion: $b^R \equiv 0$, and let the transient drift be constant: $b^T(x) \equiv b$.*

(i) *If the down-crossing function γ satisfies $\gamma(x) \leq \frac{1}{2b} \log^{(2)} x + \frac{1}{2b} \log^{(3)} x$, for large x, then the diffusion with the two-phase drift is recurrent;*

(ii) *If the down-crossing function γ satisfies $\gamma(x) \geq \frac{1}{2b} \log^{(2)} x + \frac{k}{2b} \log^{(3)} x$, for large x, where $k > 1$, then the diffusion with the two-phase drift is transient.*

We now make one minor and one more significant change in the setup we have used until now. The minor change is that the diffusion coefficient will be a constant $a > 0$ instead of 1. The more significant change is that the two-phase drift $b(t, x(\cdot))$ will be given by (1.2), with $\gamma > 0$ constant, with the transient drift b^T replaced by the constant $b \geq 0$, and with the recurrent drift b^R replaced by the constant $c > 0$. (The case $b > 0$ and $c \in (0, b)$ is more in keeping with the main theme of this paper; namely introducing a slowdown when the process has strayed too far from its running maximum.) The limiting case $c = \infty$, described below, is interesting. Thus, we will consider the operator

$$L = \frac{1}{2}a\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b(t, X(\cdot))\frac{\mathrm{d}}{\mathrm{d}x},$$

$$\text{where } b(t, X(\cdot)) = \begin{cases} b \geq 0, & \text{if } X(t) > X^*(t) - \gamma; \\ c > 0, & \text{if } X(t) \leq X^*(t) - \gamma. \end{cases} \tag{1.4}$$

The next theorem gives the speed of the diffusion in this two-phase drift.

Theorem 5. *Consider the diffusion in the two-phase drift corresponding to the operator L in* (1.4). *The speed of the diffusion $X(t)$ with the two-phase drift is given by*

$$\lim_{t \to \infty} \frac{X(t)}{t} = \begin{cases} \left(\frac{c(\mathrm{e}^{2b\gamma/a}-1)}{c(\mathrm{e}^{2b\gamma/a}-1)+(b-c)(1-\mathrm{e}^{-2b\gamma/a})}\right)b & a.s., \quad \text{if } b > 0; \\ \frac{ca}{2\gamma c+a} & a.s., \quad \text{if } b = 0. \end{cases}$$

Remark. *Let $d(b, c, \gamma, a) \equiv \frac{c(\mathrm{e}^{2b\gamma/a}-1)}{c(\mathrm{e}^{2b\gamma/a}-1)+(b-c)(1-\mathrm{e}^{-2b\gamma/a})}$. In the case $c \in (0, b)$, we call $d(b, c, \gamma, a)$ the damping coefficient. It gives the fractional reduction in speed between a classical diffusion with drift b and the slowed down diffusion with two-phase drift – b when the process is less than distance γ from its running maximum, and c when the process is at least distance γ from its running maximum. Of course, it is clear that $d(b, c, \gamma, a)$ must always fall between $\frac{c}{b}$ and 1 when $c \in (0, b)$. We make the following observations:*

1. *When $b \to \infty$, the damping coefficient $d(b, c, \gamma, a)$ converges to 1 exponentially in b;*
2. *When $\gamma \to \infty$, $d(b, c, \gamma, a)$ converges to 1 exponentially in γ;*
3. *When $c \to 0$, the damping coefficient $d(b, c, \gamma, a)$ converges to 0 linearly in c;*
4. *When $c \to 0$ and $\gamma \to \infty$, the damping coefficient $d(b, c, \gamma, a)$ converges to 1 (respectively, converges to 0, remains bounded away from 0 and 1) if $c \exp(\frac{2b\gamma}{a})$ converges to ∞ (respectively, converges to 0, remains bounded away from 0 and ∞);*
5. *When $c \to 0$ and $b \to \infty$, the damping coefficient $d(b, c, \gamma, a)$ converges to 0 if $c \exp(\frac{2b\gamma}{a})$ remains bounded. Otherwise, the damping coefficient converges to 0 (respectively, converges to 1, remains bounded away from 0 and 1) if $\frac{c \exp(2b\gamma/a)}{b}$ converges to 0 (respectively, converges to ∞, remains bounded away from 0 and ∞).*
6. *The limiting case $c = \infty$ corresponds to the situation in which the value $X^*(t) - \gamma$ serves as a reflecting barrier for $X(t)$; thus, the process can never stray farther than a distance γ from its running maximum. The drift is always equal to b, but the reflecting barrier at $X^*(t) - \gamma$ speeds the process up. In the case $b > 0$, we have*

$$d(b, \infty, \gamma, a) = \frac{\mathrm{e}^{2b\gamma/a} - 1}{\mathrm{e}^{2b\gamma/a} + \mathrm{e}^{-2b\gamma/a} - 2} = 1 + \frac{1 - \mathrm{e}^{-2b\gamma/a}}{\mathrm{e}^{2b\gamma/a} + \mathrm{e}^{-2b\gamma/a} - 2}.$$

In the case $b = 0$, we have a Brownian motion that is never allowed to stray farther than a distance γ from its running maximum; its speed is $\frac{a}{2\gamma}$.

In Section 2, we present some preliminary information on down-crossings. In Section 3, we give an explicit representation of the diffusion with a two-phase drift in terms of classical diffusions. In Section 4 we give a workable analytic criterion for transience/recurrence which depends on an auxiliary discrete time, increasing Markov process. Sections 5–9 give the proofs of Theorems 1–5.

2. Preliminaries concerning down-crossings

From the definition of the down-crossing function γ, it follows that $x - \gamma(x)$ is increasing. Define the stopping time σ_γ on continuous paths $x(\cdot) : [0, \infty) \to R$ with the standard filtration $\mathcal{F}_t = \sigma(x(s) \colon 0 \le s \le t)$ by

$$\sigma_\gamma = \inf\{t \ge 0 \colon \exists s < t \text{ with } x(t) \le x(s) - \gamma(x(s))\} = \inf\{t \ge 0 \colon x(t) = x^*(t) - \gamma(x^*(t))\}.$$

The equality above follows from the fact that $x - \gamma(x)$ is increasing. Let

$$L^\gamma = x^*(\sigma_\gamma);$$
$$K^\gamma = x(\sigma_\gamma) = x^*(\sigma_\gamma) - \gamma(x^*(\sigma_\gamma)).$$

In the case that the down-crossing function γ is constant, σ_γ is the first time that the path $x(\cdot)$ completes a down-crossing of an interval of length γ. The interval that was down-crossed is $[K^\gamma, L^\gamma]$. In [11], for γ constant, L^γ was called the γ-*down-crossed onset location*. In this paper, we will use this terminology also for variable γ. We will call σ_γ the *first γ-down-crossed time*. For use a bit later, let $\tau_a = \inf\{t \ge 0 \colon x(t) = a\}$ denote the first hitting time of the point a.

Consider now the one-dimensional diffusion process $Y(t)$ which corresponds to the operator $\mathcal{L}^T$ and which is transient to $+\infty$. Denote probabilities for the process starting from x by P_x. Fixing a point z_0, let

$$u_T(x) = \int_{z_0}^x \exp\left(- \int_{z_0}^y 2b^T(r)\, \mathrm{d}r\right) \mathrm{d}y. \tag{2.1}$$

(The formula in the theorem below is independent of z_0, but this specification of z_0 will be useful later on.) The following result was proved in [11].

Theorem P1. *For the diffusion process corresponding to $\mathcal{L}^T$, and for constant γ, the distribution of the γ-down-crossed onset location L^γ is given by*

$$P_x(L^\gamma > x + y) = \exp\left(- \int_x^{x+y} \frac{u_T'(z)}{u_T(z) - u_T(z - \gamma)}\, \mathrm{d}z\right), \quad y > 0.$$

Remark 1. *In [11], where the notation is a bit different from here, the mathematical definition of σ_γ, and consequently also of L^γ, were written incorrectly. From the verbal description in [11], it is clear that the intended definition of L^γ is the one given here. All the proofs in [11] are based on the correct definitions given here.*

Remark 2. *Theorem P2 in Section 1 was proved in [11] as an application of Theorem P1.*

The same method of proof used to prove Theorem P1 can be used in the case of variable γ to obtain a corresponding formula for the distribution of L^γ.

Proposition 1. *For the diffusion process corresponding to $\mathcal{L}^T$, and for variable γ, the distribution of the γ-down-crossed onset location L^γ is given by*

$$P_x(L^\gamma > x + y) = \exp\left(- \int_x^{x+y} \frac{u_T'(z)}{u_T(z) - u_T(z - \gamma(z))}\, \mathrm{d}z\right), \quad y > 0.$$

3. A representation for diffusion with two-phase drift

Consider the diffusion $X(t)$ with the two-phase drift starting from x_0. Up until the first γ-down-crossed time σ_γ, the process is exactly the $Y(\cdot)$-process corresponding to the operator $\mathcal{L}^T$ and starting from x_0. We have $X(\sigma_\gamma) = Y(\sigma_\gamma) = K^\gamma$ and $X^*(\sigma_\gamma) = Y^*(\sigma_\gamma) = L^\gamma$, with L^γ distributed as in Proposition 1. Let

$$\hat{\tau}_{L^\gamma} = \inf\{t \ge 0 \colon X(\sigma_\gamma + t) = L^\gamma\}.$$

Then $\sigma_\gamma + \hat{\tau}_{L^\gamma}$ is the first time after σ_γ that the process $X(\cdot)$ returns to its running maximum L^γ. Let $Z^{R,z,T}(t)$ be the diffusion starting from z and corresponding to the operator $\mathcal{L}^{R,z,T} = \frac{1}{2}\frac{d^2}{dx^2} + b^{R,z,T}(x)\frac{d}{dx}$, where

$$b^{R,z,T}(x) = \begin{cases} b^R(x), & x \le z; \\ b^T(x), & x > z. \end{cases}$$

Then the distribution of $\{X(\sigma_\gamma + t), 0 \le t \le \hat{\tau}_{L^\gamma}\}$, conditioned on $K^\gamma = X(\sigma_\gamma) = z$ and $L^\gamma = X^*(\sigma_\gamma) = a$, is that of $\{Z^{R,z,T}(t), 0 \le t \le \tau_a\}$. Of course, $X(\sigma_\gamma + \hat{\tau}_{L^\gamma}) = X^*(\sigma_\gamma + \hat{\tau}_{L^\gamma}) = L^\gamma$. Starting from time $\sigma_\gamma + \hat{\tau}_{L^\gamma}$, when the process has returned to its running maximum, X again looks like the Y process, until it again performs a γ-down-crossing, at which point it becomes a $Z^{R,z,T}$ process for appropriate z until it returns to its running maximum, and everything is repeated again.

In light of the above description, $X(t)$ can be represented as follows. For each $x \in R$ and each $n \ge 1$, let $Y^{n,x}(\cdot)$ be a diffusion process corresponding to the operator $\mathcal{L}^T$ and starting from x. Make the processes independent for different pairs (n, x). Similarly, for each $z \in R$ and each $n \ge 1$, let $Z^{R,z,T,n}(\cdot)$ be a diffusion process corresponding to the operator $\mathcal{L}^{R,z,T}$ and starting from z. Make the processes independent for different pairs (n, z) and independent of the $Y^{n,x}$ processes. Let $\sigma_\gamma^{n,x}$ denote the first γ-down-crossing time for the process $Y^{n,x}$, and let $\tau_a^{n,z}$ denote the first hitting time of a for the process $Z^{R,z,T,n}$. Let $L_0^\gamma = x_0$ and then by induction, for $n \ge 1$, define L_n^γ to be the γ-down-crossed onset location for Y^{n,L_{n-1}^γ}. For $n \ge 1$, let K_n^γ correspond to L_n^γ as K^γ corresponds to L^γ. Then $X(\cdot)$ can be represented as

$$X(t) = Y^{1,x_0}(t), \quad 0 \le t \le \sigma_\gamma^{1,x_0};$$
$$X(t) = Z^{R,K_1^\gamma,T,1}(t), \quad \sigma_\gamma^{1,x_0} \le t \le \sigma_\gamma^{1,x_0} + \tau_{L_1^\gamma}^{1,K_1^\gamma}, \tag{3.1}$$

and for $n \ge 2$,

$$X\left(\sum_{j=1}^{n-1}\sigma_\gamma^{j,L_{j-1}^\gamma} + \sum_{j=1}^{n-1}\tau_{L_j^\gamma}^{j,K_j^\gamma} + t\right) = Y^{n,L_{n-1}^\gamma}(t), \quad 0 \le t \le \sigma_\gamma^{n,L_{n-1}^\gamma};$$

$$X\left(\sum_{j=1}^{n}\sigma_\gamma^{j,L_{j-1}^\gamma} + \sum_{j=1}^{n-1}\tau_{L_j^\gamma}^{j,K_j^\gamma} + t\right) = Z^{R,K_n^\gamma,T,n}(t), \quad 0 \le t \le \tau_{L_n^\gamma}^{n,K_n^\gamma}. \tag{3.2}$$

From the above representation, it is clear that existence and uniqueness for (1.3) follows from the standard theory for classical diffusion processes.

At those times s when $X(s)$ is running as a $Y^{n,L_{n-1}^\gamma}(\cdot)$-process, for some $n \ge 1$, we will say that $X(\cdot)$ is in the Y-mode, and at those times s when $X(s)$ is running as a $Z^{R,K_n^\gamma,T,n}(\cdot)$-process, for some $n \ge 1$, we will say that $X(\cdot)$ is in the Z-mode. We denote by $\mathcal{P}_{x_0}$ probabilities corresponding to the diffusion $X(t)$ with the two-phase drift starting from x_0, and by $\mathcal{E}_{x_0}$ the corresponding expectation.

Note that $\{L_n^\gamma\}_{n=0}^\infty$ is a monotone increasing Markov process whose transition distribution from a state x is the distribution of L^γ in Proposition 1. That is, the transition probability measure $p(x, \cdot)$ is given by

$$p(x, [x + y, \infty)) = \exp\left(-\int_x^{x+y} \frac{u_T'(z)}{u_T(z) - u_T(z - \gamma(z))} dz\right), \quad \text{for } y \ge 0. \tag{3.3}$$

We will use the notation $P_{x_0}^L$ to denote probabilities for this discrete time Markov process and will denote the corresponding expectation by $E_{x_0}^L$. The times $\{L_n^\gamma\}_{n=0}^\infty$ will be called "regeneration" points for $X(\cdot)$ because the $\mathcal{P}_{x_0}$-distribution of $\{X(\sum_{j=1}^{n-1}\sigma_\gamma^{j,L_{j-1}^\gamma} + \sum_{j=1}^{n-1}\tau_{L_j^\gamma}^{j,K_j^\gamma} + t), 0 \le t < \infty\}$ given that $L_{n-1}^\gamma = a$ is the same as the $\mathcal{P}_a$-distribution of $\{X(t), 0 \le t < \infty\}$.

4. A transience/recurrence criterion

From the construction in Section 3, the well-known equivalent alternative conditions for transience/recurrence, which hold for standard non-degenerate diffusion processes [10], are easily seen to hold for diffusions with a two-phase drift. Since it is clear that the process is either transient to $+\infty$ or recurrent, we can use the following criterion:

Transience: for some pair of points $z_0 < x_0$, one has $\mathcal{P}_{x_0}(\tau_{z_0} = \infty) > 0$;

Recurrence: for some pair of points $z_0 < x_0$, one has $\mathcal{P}_{x_0}(\tau_{z_0} = \infty) = 0$.

$$(4.1)$$

We choose the point z_0 so that $z_0 < x_0 - \gamma(x_0)$. Then it follows from the representation of the process $X(\cdot)$ that at time τ_{z_0} the process is in the Z-mode. Using this with the regeneration structure noted at the end of the previous section, it follows that

$$\mathcal{P}_{x_0}(\tau_{z_0} = \infty) = E_{x_0}^L G(\{L_n^\gamma\}_{n=1}^\infty), \tag{4.2}$$

where for any non-decreasing sequence $\{a_n\}_{n=1}^\infty$ satisfying $a_1 \geq x_0$, we define

$$G(\{a_n\}_{n=1}^\infty) = \prod_{n=1}^\infty P_{a_n - \gamma(a_n)}^{R, a_n - \gamma(a_n), T}(\tau_{a_n} < \tau_{z_0}), \tag{4.3}$$

and where $P_z^{R,z,T}$ denotes probabilities for a $Z^{R,z,T}$-processes starting at z. From (4.2) we conclude that $\mathcal{P}_{x_0}(\tau_{z_0} = \infty) = 0$ if and only if $G(\{L_n^\gamma\}_{n=1}^\infty) = 0$ a.s. $P_{x_0}^L$; thus, from (4.3) we have

$$\mathcal{P}_{x_0}(\tau_{z_0} = \infty) = 0 \quad \text{if and only if} \quad \sum_{n=1}^\infty \big(P_{L_n^\gamma - \gamma(L_n^\gamma)}^{R, L_n^\gamma - \gamma(L_n^\gamma), T}(\tau_{z_0} < \tau_{L_n^\gamma})\big) = \infty \quad \text{a.s. } P_{x_0}^L. \tag{4.4}$$

As is well known [10], the function $v(x) \equiv P_x^{R,z,T}(\tau_{z_0} < \tau_{z+c})$, for $z_0 \leq x \leq z+c$, satisfies $\mathcal{L}^{R,z,T} v = 0$ in $(z_0, z) \cup (z, z+c)$, $v(z_0) = 1$, $v(z+c) = 0$, $v(z^-) = v(z^+)$, and $v'(z^-) = v'(z^+)$. Solving this, one finds that $P_z^{R,z,T}(\tau_{z_0} < \tau_{z+c}) = v(z)$ is given by

$$
\begin{aligned}
&P_z^{R,z,T}(\tau_{z_0} < \tau_{z+c}) \\
&= \frac{\exp(-\int_{z_0}^z (2b^R)(y)\,\mathrm{d}y) \int_z^{z+c} \mathrm{d}y \exp(-\int_z^y 2b^T(r)\,\mathrm{d}r)}{\int_{z_0}^z \mathrm{d}y \exp(-\int_{z_0}^y 2b^R(r)\,\mathrm{d}r) + \exp(-\int_{z_0}^z (2b^R)(y)\,\mathrm{d}y) \int_z^{z+c} \mathrm{d}y \exp(-\int_z^y 2b^T(r)\,\mathrm{d}r)}.
\end{aligned}
\tag{4.5}
$$

Analogous to u_T in (2.1), define the function

$$u_R(x) = \int_{z_0}^x \exp\left(-\int_{z_0}^y 2b^R(r)\,\mathrm{d}r\right) \mathrm{d}y. \tag{4.6}$$

We then rewrite the somewhat unwieldy equation (4.5), which would become a lot more unwieldy below, in the form

$$P_z^{R,z,T}(\tau_{z_0} < \tau_{z+c}) = \frac{u_R'(z)(u_T(z+c) - u_T(z)) \exp(\int_{z_0}^z 2b^T(y)\,\mathrm{d}y)}{u_R(z) + u_R'(z)(u_T(z+c) - u_T(z)) \exp(\int_{z_0}^z 2b^T(y)\,\mathrm{d}y)}. \tag{4.7}$$

From (4.1), (4.4) and (4.7), we obtain the following transience/recurrence criterion for the diffusion with the two-phase drift.

Proposition 2. *Define*

$$H(s) = \frac{u_R'(s - \gamma(s))(u_T(s) - u_T(s - \gamma(s))) \exp(\int_{z_0}^{s-\gamma(s)} 2b^T(y)\,\mathrm{d}y)}{u_R(s - \gamma(s)) + u_R'(s - \gamma(s))(u_T(s) - u_T(s - \gamma(s))) \exp(\int_{z_0}^{s-\gamma(s)} 2b^T(y)\,\mathrm{d}y)}. \tag{4.8}$$

Let $\{L_n^\gamma\}_{n=0}^\infty$ be the monotone increasing Markov process with $L_0^\gamma = x_0$ and transition probability measure $p(x, \cdot)$ given by (3.3). *If*

$$\sum_{n=0}^\infty H(L_n^\gamma) = \infty \quad a.s., \tag{4.9}$$

then the diffusion with the two-phase drift is recurrent. Otherwise, it is transient.

5. Proof of Theorem 1

By the Ikeda–Watanabe comparison theorem [6], if we prove recurrence for the diffusion with the two-phase drift whose recurrent drift b^R is replaced by the drift $b^R \vee 0$, then original diffusion with the two-phase drift is also recurrent. By assumption, the drift $b^R \vee 0$ is also a recurrent drift. Thus, we may assume without loss of generality that the recurrent drift b^R is non-negative.

Since $b^T \equiv b$, we have from (2.1) that

$$u_T(x) = \frac{1}{2b}\left(1 - \exp(-2b(x - z_0))\right).$$

Using this along with the fact that γ is constant, we have

$$\left(u_T(s) - u_T(s - \gamma(s))\right) \exp\left(\int_{z_0}^{s - \gamma(s)} 2b^T(y)\,dy\right) = \frac{1}{2b}\left(1 - \exp(-2b\gamma)\right) \equiv c_{b,\gamma}, \tag{5.1}$$

and thus the formula for $H(s)$ in (4.8) simplifies to

$$H(s) = \frac{c_{b,\gamma} u_R'(s - \gamma)}{u_R(s - \gamma) + c_{b,\gamma} u_R'(s - \gamma)}. \tag{5.2}$$

From (4.6), it follows that u_R is increasing, and by the assumption that b^R is non-negative, it follows that u_R' is non-increasing. Thus, H is non-increasing.

We also have

$$\frac{u_T'(z)}{u_T(z) - u_T(z - \gamma(z))} = \frac{1}{c_{b,\gamma} \exp(2b\gamma)} \equiv \frac{1}{d_{b,\gamma}}, \tag{5.3}$$

and thus the increment measure $p(x, x + A)$, $A \subset [0, \infty)$, corresponding to the transition probability measure $p(x, \cdot)$ in (3.3) for the Markov process $\{L_n^\gamma\}_{n=0}^\infty$ is independent of x and is equal to the exponential density with parameter $\frac{1}{d_{b,\gamma}}$. Consequently, $L_n^\gamma - L_0^\gamma$ is the sum of n IID exponential random variables with the above parameter, and thus

$$\lim_{n \to \infty} \frac{L_n^\gamma}{n} = d_{b,\gamma} \quad a.s. \tag{5.4}$$

Using (5.4) along with the fact that H is non-increasing, if we show that $\sum_{n=1}^\infty H((d_{b,\gamma} + 1)n) = \infty$, then it follows that $\sum_{n=1}^\infty H(L_n^\gamma) = \infty$ a.s., and consequently, from Proposition 2 we conclude that the diffusion with the two-phase drift is recurrent. Since u_R' is non-increasing and u_R is increasing, it is easy to see from (5.2) that $\sum_{n=1}^\infty H((d_{b,\gamma} + 1)n) = \infty$ if and only if $\sum_{n=1}^\infty \hat{H}((d_{b,\gamma} + 1)n) = \infty$, where $\hat{H} = \frac{u_R'(s-\gamma)}{u_R(s-\gamma)}$. Since $\hat{H}$ is monotone, it follows that $\sum_{n=1}^\infty \hat{H}((d_{b,\gamma} + 1)n) = \infty$ if and only if $\int^\infty \frac{u_R'(s)}{u_R(s)}\,ds = \infty$, that is, if and only if $\lim_{s \to \infty} u_R(s) = \infty$. But this last inequality holds from (4.6) and (1.1) since b^R is a recurrent drift.

6. Proof of Theorem 2

As in the proof of Theorem 1, the random variables $\{L_n^\gamma - L_{n-1}^\gamma\}_{n=1}^\infty$ are IID and distributed according to the exponential distribution with parameter $\frac{1}{d_{b,\gamma}}$, and the function H is given by (5.2). In order to show transience, by Proposition 2, we need to show that $\sum_{n=0}^\infty H(L_n^\gamma) < \infty$ with positive probability. Recalling (4.6) and (1.1), to complete the proof, we will construct a function u_R which satisfies $u_R > 0$, $u_R' > 0$ and $\lim_{x\to\infty} u_R(x) = \infty$, and for which the above sum is almost surely finite. (The corresponding drift b^R will then be given by $-\frac{u_R''}{2u_R}$.)

For $j \geq 2$, define the interval $I_j = [x_0 - \gamma + j, x_0 - \gamma + j + \frac{1}{j^2}]$. Without loss of generality, assume that $x_0 - \gamma + 2 \geq 0$. We now show that

$$P_{x_0}^L\left(L_n^\gamma - \gamma \in \bigcup_{j=2}^\infty I_j\right) \leq \frac{c}{n^2} \tag{6.1}$$

for some $c > 0$. The distribution of $L_n^\gamma - x_0$ is that of the sum of n IID exponential random variables with parameter $\lambda \equiv \frac{1}{d_{b,\gamma}}$. Thus, its density function is $\frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x)$, $x \geq 0$. For an appropriate constant $C > 1$, we then have for $n \geq 3$,

$$\begin{aligned} P_{x_0}^L\left(L_n^\gamma - \gamma \in \bigcup_{j=2}^\infty I_j\right) &= \sum_{j=2}^\infty \int_{I_j} \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x)\,\mathrm{d}x \\ &\leq C \int_0^\infty \frac{\lambda^n x^{n-3}}{(n-1)!} \exp(-\lambda x)\,\mathrm{d}x = \frac{C\lambda^3}{(n-1)(n-2)}. \end{aligned} \tag{6.2}$$

Now (6.1) follows from (6.2).

We now construct a positive, strictly increasing C^2-function u_R whose derivative on $R - \bigcup_{j=2}^\infty I_j$ is uniformly bounded, and which satisfies $u_R(x) \geq x^2$, for $x \geq 2$. (Of course, to have this quadratic growth, u_R' must get very large at certain places on $\bigcup_{j=2}^\infty I_j$.) By the law of large numbers, $\lim_{n\to\infty} \frac{L_n^\gamma}{n} = d_{b,\gamma}$ a.s. By (6.1) and the lemma of Borel–Cantelli, $P_{x_0}^L(L_n^\gamma - \gamma \in \bigcup_{j=2}^\infty I_j \text{ i.o.}) = 0$. Using the facts noted in this paragraph, we conclude that

$$\sum_{n=0}^\infty H(L_n^\gamma) = \sum_{n=0}^\infty \frac{c_{b,\gamma} u_R'(L_n^\gamma - \gamma)}{u_R(L_n^\gamma - \gamma) + c_{b,\gamma} u_R'(L_n^\gamma - \gamma)} < \infty \quad \text{a.s.}$$

7. Proof of Theorem 3

Since $b^R = 0$, we have from (4.6) that $u^R(x) = x - z_0$. Also, γ is constant. Thus, from (4.8), we have

$$H(s) = \frac{\int_{s-\gamma}^s \mathrm{d}y \exp(-\int_{s-\gamma}^y 2b^T(r)\,\mathrm{d}r)}{s - \gamma - z_0 + \int_{s-\gamma}^s \mathrm{d}y \exp(-\int_{s-\gamma}^y 2b^T(r)\,\mathrm{d}r)}. \tag{7.1}$$

By comparison, it suffices to consider the case that

$$b^T(x) = \frac{1}{2\gamma} \log^{(2)} x + \frac{k}{2\gamma} \log^{(3)} x$$

for $x \geq z_0$, with z_0 large enough so that $\log^{(3)} z_0$ is defined. We need to show transience for $k > 1$ and recurrence for $k = 1$. In what follows, we will always assume that $k \geq 1$.

We have

$$(y - s + \gamma)\log^{(j)}(s - \gamma) \leq \int_{s-\gamma}^y \log^{(j)} r\,\mathrm{d}r \leq (y - s + \gamma)\log^{(j)} s, \quad s - \gamma \leq y \leq s. \tag{7.2}$$

Thus,

$$
\frac{\gamma - o(1/\log s)}{\log^{(2)} s + k \log^{(3)} s} \leq \int_{s-\gamma}^{s} dy \exp\left(-\int_{s-\gamma}^{y} 2b^T(r)\, dr\right)
$$
$$
\leq \frac{\gamma}{\log^{(2)}(s - \gamma) + k \log^{(3)}(s - \gamma)}, \quad \text{as } s \to \infty. \tag{7.3}
$$

From (7.1) and (7.3) we conclude that there exist constants $C_1, C_2 > 0$ such that

$$
\frac{C_1}{s \log^{(2)} s} \leq H(s) \leq \frac{C_2}{s \log^{(2)} s} \quad \text{for large } s. \tag{7.4}
$$

We now investigate the growth rate of the Markov process $\{L_n^\gamma\}_{n=0}^\infty$. Recall that given $L_j^\gamma = x$, the distribution of $L_{j+1}^\gamma - L_j^\gamma$ is the distribution given in (3.3). From (2.1) we have

$$
\frac{u_T'(x)}{u_T(x) - u_T(x - \gamma)} = \frac{\exp(-\int_{z_0}^{x} 2b^T(r)\, dr)}{\int_{x-\gamma}^{x} dy \exp(-\int_{z_0}^{y} 2b^T(r)\, dr)}
$$
$$
= \frac{\exp(-\int_{x-\gamma}^{x} 2b^T(r)\, dr)}{\int_{x-\gamma}^{x} dy \exp(-\int_{x-\gamma}^{y} 2b^T(r)\, dr)}. \tag{7.5}
$$

From (7.2) and the definition of b^T, we have

$$
\frac{c_1}{(\log x)(\log^{(2)} x)^k} \leq \exp\left(-\int_{x-\gamma}^{x} 2b^T(r)\, dr\right) \leq \frac{c_2}{(\log x)(\log^{(2)} x)^k} \quad \text{for large } x
$$

for constants $c_1, c_2 > 0$. Using this with (7.3) and (7.5), we conclude that there exist constants $C_3, C_4 > 0$ such that

$$
\frac{C_3}{(\log x)(\log^{(2)} x)^{k-1}} \leq \frac{u_T'(x)}{u_T(x) - u_T(x - \gamma)} \leq \frac{C_4}{(\log x)(\log^{(2)} x)^{k-1}} \quad \text{for large } x. \tag{7.6}
$$

We now prove recurrence in the case that $k = 1$. Let $\{x_n\}_{n=0}^\infty$ be a sequence of positive numbers with x_0 sufficiently large so that the bound in (7.6) holds for $x \geq x_0$, and let $s_n = \sum_{j=0}^{n} x_j$. We have $\int_{s_{n-1}}^{s_n} \frac{1}{\log z}\, dz \geq \frac{x_n}{\log s_n}$. Using this with (7.6) and (3.3), we have

$$
P_{x_0}^L\left(L_n^\gamma - L_{n-1}^\gamma \leq x_n | L_{n-1}^\gamma \leq s_{n-1}\right) \geq 1 - \exp\left(-\frac{C_3 x_n}{\log s_n}\right), \quad n \geq 1. \tag{7.7}
$$

Fix a large number M. With x_0 as above, we wish to select the sequence $\{x_n\}_{n=0}^\infty$ so that

$$
\frac{C_3 x_n}{\log s_n} = 2 \log(n + M) \quad \text{for } n \geq 1. \tag{7.8}
$$

We suppress the dependence of this sequence on M in the sequel. For the sequence $\{x_n\}_{n=0}^\infty$ satisfying (7.8), it follows from (7.7) that

$$
P_{x_0}^L\left(L_n^\gamma \leq s_n \text{ for all } n\right) \geq 1 - \sum_{n=1}^{\infty} \frac{1}{(n + M)^2}. \tag{7.9}
$$

Now (7.8) is a difference equation corresponding to the differential equation

$$
\frac{C_3 S'(t)}{\log S(t)} = 2 \log(t + M), \quad t \geq 1. \tag{7.10}
$$

Integrating, it follows that

$$\frac{C_3 S(t)}{\log S(t)} \leq 2(t + M)\big(\log(t + M) - 1\big) + c \tag{7.11}$$

for some constant c. If one substitutes $\frac{3}{C_3}(t + M)(\log(t + M))^2$ for $S(t)$ in the left hand side of (7.11), one finds that the resulting expression is larger than the right hand side of (7.11) for large t. Since the left hand side of (7.11) is increasing as a function of $S(t)$ (for $S(t) \geq e$), it follows that $S(t) \leq \frac{3}{C_3}(t + M)(\log(t + M))^2$, for sufficiently large t. It then follows that the solution $\{s_n\}_{n=0}^{\infty}$ to (7.8) satisfies

$$s_n \leq \hat{s}_n \equiv C(n + M)\big(\log(n + M)\big)^2, \quad n \geq 1 \tag{7.12}$$

for some $C > 0$.

Now from (7.9) and (7.12) we can conclude that with probability at least $1 - \sum_{n=1}^{\infty} \frac{1}{(n+M)^2}$, we have $L_n^{\gamma} \leq C(n + M)(\log(n + M))^2$, for all n. However, this is not good enough to prove recurrence when $k = 1$. In fact though, from (7.7), (7.9) and (7.12), we conclude that with probability at least $1 - \sum_{n=1}^{\infty} \frac{1}{(n+M)^2}$, $\{L_n^{\gamma}\}_{n=1}^{\infty}$ is no larger than $\{\hat{L}_n^{\gamma}\}_{n=1}^{\infty}$, where $\hat{L}_n^{\gamma} = x_0 + \sum_{i=1}^{n} \hat{Z}_i$, and $\{\hat{Z}_n\}_{n=1}^{\infty}$ is a sequence of independent random variables with $\hat{Z}_n \sim \text{Exp}(\frac{C_3}{\log \hat{s}_n})$. By Kolmogorov's strong law and (7.12), it follows that $\frac{\hat{L}_n^{\gamma}}{\sum_{j=1}^{n} \log \hat{s}_j}$ almost surely converges to $\frac{1}{C_3}$. Using this with (7.12), it follows that $\{\hat{L}_n^{\gamma}\}_{n=1}^{\infty}$ almost surely grows on the order $n \log n$. Consequently, with probability at least $1 - \sum_{n=1}^{\infty} \frac{1}{(n+M)^2}$, $\{L_n^{\gamma}\}_{n=0}^{\infty}$ grows on an order no larger than $n \log n$. Using this with (7.4), it follows that $\sum_{n=0}^{\infty} H(L_n^{\gamma}) = \infty$, with probability at least $1 - \sum_{n=1}^{\infty} \frac{1}{(n+M)^2}$. But as M is arbitrary, we conclude that this occurs with probability one, and thus by Proposition 2, we conclude that the diffusion with the two-phase drift is recurrent.

We now assume that $k > 1$ and prove transience. Chose x_0 sufficiently large so that the bound in (7.6) holds for $x \geq x_0$. From (7.6) and (3.3), we have

$$P_{x_0}^L\big(L_{n+1}^{\gamma} - L_n^{\gamma} \geq x | L_n^{\gamma}\big) \geq \exp\left(-\frac{C_4 x}{(\log L_n^{\gamma})(\log^{(2)} L_n^{\gamma})^{k-1}}\right). \tag{7.13}$$

Thus, it follows from the law of large numbers that

$$\lim_{n \to \infty} \frac{1}{n}\left|\left\{j \leq n\colon L_{j+1}^{\gamma} - L_j^{\gamma} \geq \frac{(\log L_j^{\gamma})(\log^{(2)} L_j^{\gamma})^{k-1}}{C_4}\right\}\right| \geq e^{-1}, \quad \text{a.s.} \tag{7.14}$$

The above inequality states that, asymptotically, at least the fraction $\frac{1}{e}$ of the increments $L_{j+1}^{\gamma} - L_j^{\gamma}$ will be of size at least $\frac{(\log L_j^{\gamma})(\log^{(2)} L_j^{\gamma})^{k-1}}{C_4}$. As such, it provides a lower bound on the growth rate of $\{L_n^{\gamma}\}_{n=0}^{\infty}$. Since the function $\frac{(\log y)(\log^{(2)} y)^{k-1}}{C_4}$ is increasing in y, the "worst case" scenario resulting in the least growth would occur if out of the first n increments, the first $[\frac{n}{e}]$ increments satisfied the above condition, and the rest did not. We can thus get a lower bound on the growth rate as follows. Making sure that $x_0 > e$, so that $\log^{(2)} x_0 > 0$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by

$$x_{n+1} = \frac{(\log s_n)(\log^{(2)} s_n)^{k-1}}{C_4}, \quad n \geq 0, \tag{7.15}$$

where $s_n = \sum_{j=0}^{n} x_j$. Then it follows from (7.14) that

$$P_{x_0}^{\gamma}\big(L_n^{\gamma} \geq s_{[n/(2e)]} \text{ for all large } n\big) = 1. \tag{7.16}$$

As we did in the recurrent case, we analyze the growth rate of $\{s_n\}_{n=0}^{\infty}$ by looking at the growth rate of the differential equation associated with the above difference equation for $\{s_n\}_{n=0}^{\infty}$. The differential equation is

$$\frac{S'(t)}{(\log S(t))(\log^{(2)} S(t))^{k-1}} = \frac{1}{C_4}.$$

Integrating this, the leading order term on the left is $\frac{S(t)}{(\log S(t))(\log^{(2)} S(t))^{k-1}}$, and thus, for large $S(t)$, we have

$$\frac{2S(t)}{(\log S(t))(\log^{(2)} S(t))^{k-1}} \geq \frac{1}{C_4}t + c \tag{7.17}$$

for some constant c. If one substitutes $\frac{1}{2C_4}t (\log t)(\log^{(2)} t)^{k-1}$ for $S(t)$ in (7.17), one finds that the resulting expression is smaller than the right hand side of (7.17) for large t. Since the left hand side of (7.17) is increasing as a function of $S(t)$ (for $S(t)$ sufficiently large), it follows that $S(t) \geq \frac{1}{2C_4}t (\log t)(\log^{(2)} t)^{k-1}$, for sufficiently large t. It then follows that the solution $\{s_n\}_{n=0}^{\infty}$ to (7.15) satisfies

$$s_n \geq Cn(\log n)\big(\log^{(2)} n\big)^{k-1}, \quad n \geq 3 \tag{7.18}$$

for some $C > 0$. We conclude from (7.16) and (7.18) that

$$P_{x_0}^{\gamma}\big(L_n^{\gamma} \geq C_0 n(\log n)\big(\log^{(2)} n\big)^{k-1} \text{ for all large } n\big) = 1 \tag{7.19}$$

for some $C_0 > 0$. From (7.19) and (7.4), it follows that $\sum_{n=0}^{\infty} H(L_n^{\gamma}) < \infty$ a.s., and thus by Proposition 2, we conclude that the diffusion with the two-phase drift is transient.

8. Proof of Theorem 4

By comparison, it suffices to consider the case that $\gamma(x) = \frac{1}{2b}\log^{(2)} x + \frac{k}{2b}\log^{(3)} x$, for $x \geq x_0$, with x_0 large enough so that $\log^{(3)} x_0$ is defined. We need to show recurrence in the case that $k = 1$, and transience in the case that $k > 1$.

Since $b^T \equiv b$, we have similar to (5.1),

$$\big(u_T(s) - u_T\big(s - \gamma(s)\big)\big) \exp\left(\int_{z_0}^{s-\gamma(s)} 2b^T(y)\,dy\right) = \frac{1}{2b}\big(1 - \exp\big(-2b\gamma(s)\big)\big). \tag{8.1}$$

Since $b^R = 0$, we have from (4.6) that $u^R(x) = x - z_0$. Thus, from (4.8) we have

$$H(s) = \frac{1 - \exp(-2b\gamma(s))}{2b(s - z_0 - \gamma(s)) + 1 - \exp(-2b\gamma(s))}. \tag{8.2}$$

Since $\gamma(s) = o(s)$, we conclude from (8.2) that there exist constants $C_1, C_2 > 0$ such that

$$\frac{C_1}{s} \leq H(s) \leq \frac{C_2}{s} \quad \text{for large } s. \tag{8.3}$$

We now investigate the growth rate of the Markov process $\{L_n^{\gamma}\}_{n=0}^{\infty}$. Recall that given $L_j^{\gamma} = x$, the distribution of $L_{j+1}^{\gamma} - L_j^{\gamma}$ is the distribution given in (3.3). Since $b^T \equiv b$, we have from (2.1) that $u_T(x) = \frac{1}{2b}(1 - \exp(-2b(x - z_0)))$. Thus,

$$\frac{u_T'(z)}{u_T(z) - u_T(z - \gamma(z))} = \frac{2b}{\exp(2b\gamma(z)) - 1}.$$

Plugging into this equation the formula for $\gamma(z)$ given above, we have

$$\frac{u_T'(z)}{u_T(z) - u_T(z - \gamma(z))} = \frac{2b}{(\log z)(\log^{(2)} z)^k - 1}. \tag{8.4}$$

It was shown in the proof of Theorem 3 that if (7.6) holds, then $\{L_n^{\gamma}\}_{n=0}^{\infty}$ grows at least on the order $n \log n(\log^{(2)} n)^{k-1}$. Thus, comparing (8.4) with (7.6), it follows that in the case at hand $\{L_n^{\gamma}\}_{n=0}^{\infty}$ grows at least on the order

$n \log n (\log^{(2)} n)^k$. The same method of proof used to prove that if (7.6) holds with $k = 1$, then $\{L_n^\gamma\}_{n=0}^\infty$ grows on an order no larger than $n \log n$, also shows that if (7.6) holds with $k > 1$, then $\{L_n^\gamma\}_{n=0}^\infty$ grows on an order no larger than $n \log n (\log^{(2)} n)^{k-1}$. Thus, again comparing (8.4) with (7.6), it follows that in the case at hand $\{L_n^\gamma\}_{n=0}^\infty$ grows on an order no larger than $n \log n (\log^{(2)} n)^k$. We conclude that $\{L_n^\gamma\}_{n=0}^\infty$ grows exactly on the order $n \log n (\log^{(2)} n)^k$. Using this with (8.3), it follow from Proposition 2 that the diffusion with the two-phase drift is recurrent if $k = 1$ and transient if $k > 1$.

9. Proof of Theorem 5

Recall that the first γ-down-crossed time for the process $X(t)$ is given by

$$\sigma_\gamma = \inf\{t \geq 0 : \exists s < t \text{ with } X(t) \leq X(s) - \gamma(X(s))\} = \inf\{t \geq 0 : X(t) = X^*(t) - \gamma(X^*(t))\},$$

and is a stopping time. Recall that $X^*(\sigma_\gamma)$ has been denoted by L^γ and that

$$\hat{\tau}_{L^\gamma} = \inf\{t \geq 0 : X(\sigma_\gamma + t) = L^\gamma\}$$

is the first time after σ_γ that the process $X(\cdot)$ returns to its running maximum L^γ. Thus, the process $X(\cdot)$ increases from x_0 to L^γ from time 0 to time $\sigma_\gamma + \hat{\tau}_{L^\gamma}$. At the regeneration point L^γ at time $\sigma_\gamma + \hat{\tau}_{L^\gamma}$, everything begins anew according to the same rules, and also, according to the same distribution, since the two phases of the drift are constants and thus independent of location. If follows from this and the law of large numbers, and the standard technique to go from stopping times to deterministic times, that

$$\lim_{t \to \infty} \frac{X(t)}{t} = \frac{\mathcal{E}_{x_0} L^\gamma - x_0}{\mathcal{E}_{x_0}(\sigma_\gamma + \hat{\tau}_{L^\gamma})} \quad \text{a.s.} \tag{9.1}$$

We will prove the theorem for the case $b > 0$. After the proof, we briefly comment on the change needed to treat the case $b = 0$. Recall from the construction in Section 3 that $X(t)$ is in the Y-mode up until time σ_γ. Then from time σ_γ until time $\sigma_\gamma + \hat{\tau}_{L^\gamma}$ it is in the Z-mode. Under the assumption of the theorem, the Y-mode corresponds to Brownian motion with a constant drift b. It follows from Doob's optional stopping theorem that $X(t \wedge \sigma_\gamma) - b(t \wedge \sigma_\gamma)$ is a martingale. Taking expectations, we obtain

$$\mathcal{E}_{x_0} X(\sigma_\gamma \wedge t) = x_0 + b \mathcal{E}_{x_0} \sigma_\gamma \wedge t. \tag{9.2}$$

There exists a constant $p_0 > 0$ such that a Brownian motion with constant drift b and starting from any x has probability p_0 of downcrossing the interval $[x - \gamma, x]$ within one unit of time. Thus, it follows that there exists a constant $c_0 \in (0, 1)$ such that

$$\mathcal{P}_{x_0}(\sigma_\gamma > t) \leq c_0^t \quad \text{for } t \geq 1. \tag{9.3}$$

We have

$$\mathcal{E}_{x_0}\big(X(\sigma_\gamma \wedge t); \sigma_\gamma > t\big) = \mathcal{E}_{x_0}\big(X(t); \sigma_\gamma > t\big) \leq \big(\mathcal{E}_{x_0} X^2(t)\big)^{1/2}\big(\mathcal{P}_{x_0}(\sigma_\gamma > t)\big)^{1/2}. \tag{9.4}$$

By comparison, $X(t)$ under $\mathcal{P}_{x_0}$ is stochastically dominated by $x_0 + \sqrt{a} W(t) + bt$, where W is a standard Brownian motion; thus,

$$\mathcal{E}_{x_0} X^2(t) \leq x_0^2 + at + b^2 t + 2x_0 bt. \tag{9.5}$$

Letting $t \to \infty$ in (9.2), and using (9.3)–(9.5), we obtain

$$\mathcal{E}_{x_0} X(\sigma_\gamma) = x_0 + b \mathcal{E}_{x_0} \sigma_\gamma. \tag{9.6}$$

In the case at hand, where the diffusion coefficient is a instead of 1, the function u_T is given by $\int_{z_0}^x \exp(-\int_{z_0}^y \frac{2b}{a}\, \mathrm{d}t) = \frac{a}{2b}(1 - \exp(-\frac{2b}{a}(x - z_0)))$. Similar to (5.1) and (5.3), it follows that under $\mathcal{E}_{x_0}$, $L^\gamma - x_0$ is distributed according to an exponential distribution with parameter $\frac{2b}{a\exp(2b\gamma/a)-1}$; thus

$$\mathcal{E}_{x_0} L^\gamma = x_0 + \frac{a(\exp(2b\gamma/a) - 1)}{2b}. \tag{9.7}$$

Using (9.6) and (9.7), along with the fact that $L^\gamma = X(\sigma_\gamma) + \gamma$, we have

$$\mathcal{E}_{x_0}\sigma_\gamma = \frac{1}{b}\left(\frac{a(\exp(2b\gamma/a) - 1)}{2b} - \gamma\right). \tag{9.8}$$

We now evaluate $\mathcal{E}_{x_0}\hat{\tau}_{L^\gamma}$. From the definition of the process and the fact that the two drift phases are constants and thus independent of location, it follows that under $\mathcal{E}_{x_0}$, the distribution of $\hat{\tau}_{L^\gamma}$ is the distribution of the first hitting time of γ by the diffusion process starting at 0 and corresponding to the operator $L^{a;c,0,b} \equiv \frac{1}{2}a\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b^{c,0,b}(x)\frac{\mathrm{d}}{\mathrm{d}x}$, where $b^{c,0,b}(x)$ is equal to b when $x > 0$ and is equal to c when $x \leq 0$. Let $E_0^{a;c,0,b}$ denote the expectation for this diffusion starting from 0. So

$$\mathcal{E}_{x_0}\hat{\tau}_{L^\gamma} = E_0^{a;c,0,b}\tau_\gamma. \tag{9.9}$$

We have $E_0^{a;c,0,b}\tau_\gamma = \lim_{N\to\infty} E_0^{a;c,0,b}\tau_\gamma \wedge \tau_{-N}$. As is well-known [10], $E_0^{a;c,0,b}\tau_\gamma \wedge \tau_{-N} = v_N(0)$, where v_N solves the equation

$$L^{a;c,0,b}v_N = -1 \quad \text{in } (-N, 0) \cup (0, \gamma);$$
$$v_N(-N) = v_N(\gamma) = 0;$$
$$v_N(0^-) = v_N(0^+), \qquad v_N'(0^-) = v_N'(0^+).$$

Solving this, we obtain

$$v_N(y) = \begin{cases} \frac{aA_N}{2c}\left(\exp\left(\frac{2cN}{a}\right) - \exp\left(-\frac{2cy}{a}\right)\right) - \frac{y+N}{c}, & -N \leq y \leq 0; \\ \frac{aD_N}{2b}\left(\exp\left(-\frac{2b\gamma}{a}\right) - \exp\left(-\frac{2by}{a}\right)\right) + \frac{\gamma-y}{b}, & 0 \leq y \leq \gamma, \end{cases}$$

where

$$A_N = \left(\frac{\gamma}{b} + \frac{N}{c} + \frac{a(b-c)}{2b^2c}\left(1 - \exp\left(-\frac{2b\gamma}{a}\right)\right)\right) \Big/ \left(\frac{a}{2c}\left(\exp\left(\frac{2cN}{a}\right) - 1\right) + \frac{a}{2b}\left(1 - \exp\left(-\frac{2b\gamma}{a}\right)\right)\right);$$
$$D_N = A_N + \frac{1}{b} - \frac{1}{c}.$$

From this we obtain

$$E_0^{a;c,0,b}\tau_\gamma = \lim_{N\to\infty} v_N(0) = \frac{\gamma}{b} + \frac{a(b-c)}{2b^2c}\left(1 - \exp\left(-\frac{2b\gamma}{a}\right)\right),$$

and thus from (9.9),

$$\mathcal{E}_{x_0}\hat{\tau}_{L^\gamma} = \frac{\gamma}{b} + \frac{a(b-c)}{2b^2c}\left(1 - \exp\left(-\frac{2b\gamma}{a}\right)\right). \tag{9.10}$$

The theorem for the case $b > 0$ now follows from (9.1), (9.7), (9.8) and (9.10).

In the case $b = 0$, to calculate the right hand side of (9.1), one computes $\mathcal{E}_{x_0} L^\gamma$ and $\mathcal{E}_{x_0}\hat{\tau}_{L^\gamma}$ by the same method used above in the case $b > 0$. To compute $\mathcal{E}_{x_0}\sigma_\gamma$, one uses the martingale $X^2(\sigma_\gamma \wedge t) - \sigma_\gamma \wedge t$. instead of the martingale $X(t \wedge \sigma_\gamma) - b(t \wedge \sigma_\gamma)$.

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