

Three examples of Brownian flows on \mathbb{R}

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Abstract. We show that the only flow solving the stochastic differential equation (SDE) on \mathbb{R}

$$dX_t = 1_{\{X_t > 0\}} W^+(dt) + 1_{\{X_t < 0\}} dW^-(dt),$$

where W^+ and W^- are two independent white noises, is a coalescing flow we will denote by φ^\pm . The flow φ^\pm is a Wiener solution of the SDE. Moreover, $K^+ = \mathbb{E}[\delta_{\varphi^\pm} | W^+]$ is the unique solution (it is also a Wiener solution) of the SDE

$$K_{s,t}^+ f(x) = f(x) + \int_s^t K_{s,u} (1_{\mathbb{R}^+} f')(x) W^+(du) + \frac{1}{2} \int_s^t K_{s,u} f''(x) du$$

for $s < t$, $x \in \mathbb{R}$ and f a twice continuously differentiable function. A third flow φ^+ can be constructed out of the n -point motions of K^+ . This flow is coalescing and its n -point motion is given by the n -point motions of K^+ up to the first coalescing time, with the condition that when two points meet, they stay together. We note finally that $K^+ = \mathbb{E}[\delta_{\varphi^+} | W^+]$.

Résumé. Nous montrons que le seul flot solution de l'équation différentielle stochastique (EDS) sur \mathbb{R}

$$dX_t = 1_{\{X_t > 0\}} W^+(dt) + 1_{\{X_t < 0\}} dW^-(dt),$$

où W^+ et W^- sont deux bruits blancs indépendants, est un flot coalescent que nous noterons φ^\pm . Le flot φ^\pm est une solution Wiener de l'équation. De plus, $K^+ = \mathbb{E}[\delta_{\varphi^\pm} | W^+]$ est l'unique solution (c'est aussi une solution Wiener) de l'EDS

$$K_{s,t}^+ f(x) = f(x) + \int_s^t K_{s,u} (1_{\mathbb{R}^+} f')(x) W^+(du) + \frac{1}{2} \int_s^t K_{s,u} f''(x) du$$

pour tout $s < t$, $x \in \mathbb{R}$ et f une fonction deux fois continûment mesurable. Un troisième flot φ^+ peut être construit à partir des mouvements à n points de K^+ . Ce flot est coalescent et ses mouvements à n points sont donnés par les mouvements à n points de K^+ jusqu'au premier temps de coalescence, avec comme condition que lorsque deux points se rencontrent, ils restent confondus. On remarquera finalement que $K^+ = \mathbb{E}[\delta_{\varphi^+} | W^+]$.

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1. Introduction

Our purpose in this paper is to study two very simple one dimensional SDE's which can be completely solved, although they do not satisfy the usual criteria. This study is done in the framework of stochastic flows exposed in

[11–13] and [20] (see also [16] and [17] in an SPDE setting). There is still a lot to do to understand the nature of these flows, even if one consider only Brownian flows, i.e. flows whose one-point motion is a Brownian motion. As in our previous study of Tanaka's equation (see [14] and also [7,8]), it is focused on the case where the singularity is located at an interface between two half lines (see also [4] and references therein where a flow related to the skew Brownian motion is studied and also [18] where pathwise uniqueness is proved for a perturbed Tanaka's SDE). It should be generalizable to various situations. The first SDE represents the motion of particles driven by two independent white noises W^+ and W^- . All particles on the positive half-line are driven by W^+ and therefore move parallel until they hit 0. W^- drives in the same way the particles while they are on the negative side. What should happen at the origin is a priori not clear, but we should assume particles do not spend a positive measure of time there. The SDE can therefore be written

$$dX_t = 1_{\{X_t > 0\}} W^+(dt) + 1_{\{X_t < 0\}} W^-(dt)$$

and will be shown to have a strong, i.e. $\sigma(W^+, W^-)$ (Wiener) measurable, solution which is a coalescing flow of continuous maps. This is the only solution in any reasonable sense, even if one allows an extension of the probability space, i.e. other sources of randomness than the driving noises, to define the flow.

If we compare this result with the one obtained in [14] for Tanaka's equation, in which the construction of the coalescing flow requires an additional countable family of independent Bernoulli variable attached to local minima of the noise W , the Wiener measurability may seem somewhat surprising. A possible intuitive interpretation is the following: In the case of Tanaka's equation, if we consider the image of zero, a choice has to be made at the beginning of every excursion outside of zero of the driving reflected Brownian motion. In the case of our SDE, an analogous role is played by excursions of W^+ and W^- . These excursions have to be taken at various levels different of 0, but the essential point is that at given levels they a.s. never start at the same time. And if we could a priori neglect the effect of the excursions of height smaller than some positive ε , the motion of a particle starting at zero would be perfectly determined.

The second SDE is a transport equation, which cannot be induced by a flow of maps. The matter is dispersed according to the heat equation on the negative half line and is driven by W^+ on the positive half line. A solution is easily constructed by integrating out W^- in the solution of our first equation. We will prove that also in this case, there is no other solution, even on an extended probability space.

The third flow is not related to an SDE. It is constructed in a similar way as Arratia flow (see [1,5,6,12,20]) is constructed: the n -point motion is given by independent Brownian motions that coalesce when they meet (without this condition the matter is dispersed according to the heat equation on the line). Using the same procedure, each particle is driven on the negative half line by an independent white noises and on the positive half line by W^+ . This procedure allows to define a coalescing flow of maps, which is not Wiener measurable (i.e. not $\sigma(W^+)$ -measurable).

2. Notation, definitions and results

2.1. Notation

- For $n \geq 1$, $C(\mathbb{R}^+; \mathbb{R}^n)$ (resp. $C_b(\mathbb{R}^+; \mathbb{R}^n)$) denotes the space of continuous (resp. bounded continuous) functions $f: \mathbb{R}^+ \rightarrow \mathbb{R}^n$.
- For $n \geq 1$, $C_0(\mathbb{R}^n)$ is the space of continuous functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ converging to 0 at infinity. It is equipped with the norm $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|$.
- For $n \geq 1$, $C_0^2(\mathbb{R}^n)$ is the space of twice continuously differentiable functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ converging to 0 at infinity as well as their derivatives. It is equipped with the norm $\|f\|_{2,\infty} = \|f\|_\infty + \sum_i \|\partial_i f\|_\infty + \sum_{i,j} \|\partial_i \partial_j f\|_\infty$.
- For a metric space M , $\mathcal{B}(M)$ denotes the Borel σ -field on M .
- For $n \geq 1$, $\mathcal{M}(\mathbb{R}^n)$ (resp. $\mathcal{M}_b(\mathbb{R}^n)$) denotes the space of measurable (resp. bounded measurable) functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$.
- We denote by F the space $\mathcal{M}(\mathbb{R})$. It will be equipped with the σ -field generated by $f \mapsto f(x)$ for all $x \in \mathbb{R}$.
- $\mathcal{P}(\mathbb{R})$ denotes the space of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The space $\mathcal{P}(\mathbb{R})$ is equipped with the topology of narrow convergence. For $f \in \mathcal{M}_b(\mathbb{R})$ and $\mu \in \mathcal{P}(\mathbb{R})$, μf or $\mu(f)$ denotes $\int_{\mathbb{R}} f d\mu = \int_{\mathbb{R}} f(x) \mu(dx)$.

- A kernel is a measurable function K from \mathbb{R} into $\mathcal{P}(\mathbb{R})$. Denote by E the space of all kernels on \mathbb{R} . For $f \in \mathcal{M}_b(\mathbb{R})$, $Kf \in \mathcal{M}_b(\mathbb{R})$ is defined by $Kf(x) = \int_{\mathbb{R}} f(y)K(x, dy)$. For $\mu \in \mathcal{P}(\mathbb{R})$, $\mu K \in \mathcal{P}(\mathbb{R})$ is defined by $(\mu K)f = \mu(Kf)$. If K_1 and K_2 are two kernels then $K_1 K_2$ is the kernel defined by $(K_1 K_2)f(x) = K_1(K_2 f)(x) (= \int f(z)K_1(x, dz)K_2(y, dz))$. The space E will be equipped with \mathcal{E} the σ -field generated by the mappings $K \mapsto \mu K$, for every $\mu \in \mathcal{P}(\mathbb{R})$.
- We denote by Δ (resp. $\Delta^{(n)}$ for $n \geq 1$) the Laplacian on \mathbb{R} (resp. on \mathbb{R}^n), acting on twice differentiable functions f on \mathbb{R} (resp. on \mathbb{R}^n) and defined by $\Delta f = f''$ (resp. $\Delta^{(n)} f = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f$).

2.2. Definitions: Stochastic flows and n -point motions

Definition 2.1. A measurable stochastic flow of mappings (SFM) φ on \mathbb{R} , defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, is a family $(\varphi_{s,t})_{s < t}$ such that

- (1) For all $s < t$, $\varphi_{s,t}$ is a measurable mapping from $(\Omega \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$;
- (2) For all $h \in \mathbb{R}$, $s < t$, $\varphi_{s+h,t+h}$ is distributed like $\varphi_{s,t}$;
- (3) For all $s < t < u$ and all $x \in \mathbb{R}$, a.s. $\varphi_{s,u}(x) = \varphi_{t,u} \circ \varphi_{s,t}(x)$, and $\varphi_{s,s}$ equals the identity;
- (4) For all $f \in C_0(\mathbb{R})$, and $s \leq t$, we have

$$\lim_{(u,v) \rightarrow (s,t)} \sup_{x \in \mathbb{R}} \mathbb{E}[(f \circ \varphi_{u,v}(x) - f \circ \varphi_{s,t}(x))^2] = 0;$$

- (5) For all $f \in C_0(\mathbb{R})$, $x \in \mathbb{R}$, $s < t$, we have

$$\lim_{y \rightarrow x} \mathbb{E}[(f \circ \varphi_{s,t}(y) - f \circ \varphi_{s,t}(x))^2] = 0;$$

- (6) For all $s < t$, $f \in C_0(\mathbb{R})$, $\lim_{|x| \rightarrow \infty} \mathbb{E}[(f \circ \varphi_{s,t}(x))^2] = 0$.

Definition 2.2. A measurable stochastic flow of kernels (SFK) K on \mathbb{R} , defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, is a family $(K_{s,t})_{s < t}$ such that

- (1) For all $s < t$, $K_{s,t}$ is a measurable mapping from $(\Omega \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}))$ to $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R})))$;
- (2) For all $h \in \mathbb{R}$, $s < t$, $K_{s+h,t+h}$ is distributed like $K_{s,t}$;
- (3) For all $s < t < u$ and all $x \in \mathbb{R}$, a.s. $K_{s,u}(x) = K_{s,t} K_{t,u}(x)$, and $K_{s,s}$ equals the identity;
- (4) For all $f \in C_0(\mathbb{R})$, and $s \leq t$, we have

$$\lim_{(u,v) \rightarrow (s,t)} \sup_{x \in \mathbb{R}} \mathbb{E}[(K_{u,v} f(x) - K_{s,t} f(x))^2] = 0;$$

- (5) For all $f \in C_0(\mathbb{R})$, $x \in \mathbb{R}$, $s < t$, we have

$$\lim_{y \rightarrow x} \mathbb{E}[(K_{s,t} f(y) - K_{s,t} f(x))^2] = 0;$$

- (6) For all $s < t$, $f \in C_0(\mathbb{R})$, $\lim_{|x| \rightarrow \infty} \mathbb{E}[(K_{s,t} f(x))^2] = 0$.

The law of a SFK (resp. of a SFM) is a probability measure on $(\prod_{s < t} E, \otimes_{s \leq t} \mathcal{E})$ (resp. on $(\prod_{s < t} F, \otimes_{s \leq t} \mathcal{F})$). A SFK K will be called a SFM when $K_{s,t}(x) = \delta_{\varphi_{s,t}(x)}$ for some SFM φ .

Definition 2.3. Let $(P_t^{(n)}, n \geq 1)$ be a family of Feller semigroups, respectively defined on \mathbb{R}^n and acting on $C_0(\mathbb{R}^n)$ as well as on bounded continuous functions. We say that this family is consistent as soon as

- (1) for all $n \geq 1$, all permutation σ of $\{1, \dots, n\}$ and all $f \in C_0(\mathbb{R}^n)$,

$$P_t^{(n)}(f^\sigma) = (P_t^{(n)} f)^\sigma,$$

where $f^\sigma(x_1, \dots, x_n) = f(x_{\sigma_1}, \dots, x_{\sigma_n})$;

(2) for all $k \leq n$ and all $f \in C_0(\mathbb{R}^k)$, we have

$$P_t^{(n)}(f \circ \pi_{k,n}) = (P_t^{(k)} f) \circ \pi_{k,n},$$

where $\pi_{k,n} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is such that $\pi_{k,n}(x) = (x_1, \dots, x_k)$,

such that $x_i \leq y_i$ for $i \leq n$. We will denote by $\mathbb{P}_x^{(n)}$ the law of the Markov process associated with $P_t^{(n)}$ and starting from $x \in \mathbb{R}^n$. This Markov process will be called the n -point motion of this family of semigroups.

A general result (see Theorem 2.1 in [12]) states that there is a one to one correspondence between laws of SFK K and consistent family of Feller semigroups $P^{(n)}$, with the semigroup $P^{(n)}$ defined by $P_t^{(n)} = E[K_{0,t}^{\otimes n}]$. It can be viewed as a generalization of De Finetti's Theorem (see [20]).

2.3. Definition: White noises

Let $C : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a covariance function on \mathbb{R} , i.e. C is symmetric and $\sum_{i,j} \lambda_i \lambda_j C(x_i, x_j) \geq 0$ for all finite family $(\lambda_i, x_i) \in \mathbb{R}^2$. Assuming that the reproducing Hilbert space H_C associated to C is separable, there exists $(e_i)_{i \in I}$ (with I at most countable) an orthonormal basis of H_C such that $C(x, y) = \sum_{i \in I} e_i(x)e_i(y)$.

Definition 2.4. A white noise of covariance C is a centered Gaussian family of real random variables $(W_{s,t}(x), s < t, x \in \mathbb{R})$, such that

$$E[W_{s,t}(x)W_{u,v}(y)] = |[s, t] \cap [u, v]| \times C(x, y).$$

A standard white noise is a centered Gaussian family of real random variables $(W_{s,t}, s < t)$ such that

$$E[W_{s,t}W_{u,v}] = |[s, t] \cap [u, v]|.$$

Starting with $(W^i)_{i \in I}$ independent standard white noises, one can define $W = (W_{s,t})_{s < t}$ a white noise of covariance C by the formula

$$W_{s,t}(x) = \sum_{i \in I} W_{s,t}^i e_i(x),$$

which is well defined in L^2 .

Although $W_{s,t}$ doesn't belong to H_C , one can recover $W_{s,t}^i$ out of W by $W_{s,t}^i = \langle W_{s,t}, e_i \rangle$. Indeed, for any given i , e_i is the limit as $n \rightarrow \infty$ in H_C of $e_i^n = \sum_k \lambda_k^n C_{x_k^n}$ where for all n , (λ_k^n, x_k^n) is a finite family in \mathbb{R}^2 . Note that

$$\|e_i^n - e_i^m\|_{H_C}^2 = \sum_{k,\ell} \lambda_k^n \lambda_\ell^m C(x_k^n, x_\ell^m).$$

Denote $W_{s,t}^{n,i} = \sum_k \lambda_k^n W_{s,t}(x_k^n)$. Then for n and m ,

$$E[(W_{s,t}^{n,i} - W_{s,t}^{m,i})^2] = (t - s) \|e_i^n - e_i^m\|_{H_C}^2.$$

Thus one can define $W_{s,t}^i$ as the limit in L^2 of $W_{s,t}^{n,i}$. Then one easily checks that $W_{s,t}(x) = \sum_i W_{s,t}^i(x)e_i(x)$ a.s.

For K a SFK (resp. φ a SFM or W a white noise), we denote for all $s \leq t$ by $\mathcal{F}_{s,t}^K$ (resp. $\mathcal{F}_{s,t}^\varphi$ or $\mathcal{F}_{s,t}^W$) the σ -field generated by $\{K_{u,v}; s \leq u \leq v \leq t\}$ (resp. by $\{\varphi_{u,v}; s \leq u \leq v \leq t\}$ or $\{W_{u,v}; s \leq u \leq v \leq t\}$). A white noise W is said to be a \mathcal{F}^K (resp. \mathcal{F}^φ -white noise) if $\mathcal{F}_{s,t}^W \subset \mathcal{F}_{s,t}^K$ for all $s < t$ (resp. $\mathcal{F}_{s,t}^W \subset \mathcal{F}_{s,t}^\varphi$ for all $s < t$). In all the following, all σ -fields will be completed by negligible events.

2.4. Definition: The $(\frac{1}{2}\Delta, C)$ -SDE

Let φ be a SFM and W a \mathcal{F}^φ -white noise of covariance C . Then if for all $s < t$ and all $x \in \mathbb{R}$, we have

$$\varphi_{s,t}(x) = x + \sum_{i \in I} \int_s^t e_i \circ \varphi_{s,u}(x) W^i(du) \tag{1}$$

then (φ, W) is said to solve the SDE (1). Since this SDE is determined by the covariance C , we will more simply say that (φ, W) solves the C -SDE driven by W . Note that to find SFM's solutions of the C -SDE for which the one-point motion is a Brownian motion, we will need to assume that $C(x, x) = 1$ for all $x \in \mathbb{R}$.

In all the following, we will be interested in constructing SFM and SFK for which the one-point motion is a Brownian motion. Adding this condition, the C -SDE will be called the $(\frac{1}{2}\Delta, C)$ -SDE, since $\frac{1}{2}\Delta f = \frac{1}{2}f''$ for $f \in C^2(\mathbb{R})$ is the generator of the Brownian motion on \mathbb{R} .

The notion of solution of this SDE can be extended to stochastic flows of kernels: let K be a SFK and $W = \sum_i e_i W^i$, a \mathcal{F}^K -white noise of covariance C , then (K, W) is said to solve the $(\frac{1}{2}\Delta, C)$ -SDE driven by W if

$$K_{s,t}f(x) = f(x) + \sum_{i \in I} \int_s^t K_{s,u}(e_i f')(x) W^i(du) + \frac{1}{2} \int_s^t K_{s,u} f''(x) du \tag{2}$$

for all $f \in C_0^2(\mathbb{R})$, $s < t$ and $x \in \mathbb{R}$. Note that when C is continuous, then identity (2) implies that W is a \mathcal{F}^K -white noise (see Section 5 and Lemma 5.3 in [12]). To find kernel solutions of the $(\frac{1}{2}\Delta, C)$ -SDE, we will only need to assume that $C(x, x) \leq 1$ for all $x \in \mathbb{R}$. (This condition comes from the fact that for all $t \geq 0$,

$$P_t f^2 = E[K_{0,t} f^2(x)] \geq E[(K_{0,t} f)^2(x)] = P_t^{(2)} f^{\otimes 2} f(x, x)$$

and that

$$A f^2(x) - A^{(2)} f^{\otimes 2}(x, x) = (1 - C(x, x))(f'(x))^2,$$

where $A = \frac{1}{2}\Delta$ is the generator of P_t and $A^{(2)}$ is the generator of $P_t^{(2)}$.) Having C strictly less than one means the flow is a mixture of stochastic transport and deterministic heat flow. Note also that in this case there are no SFM's solution of the $(\frac{1}{2}\Delta, C)$ -SDE.

Taking the expectation in (2), we see that for a solution (K, W) of the $(\frac{1}{2}\Delta, C)$ -SDE, the one-point motion of K is a standard Brownian motion. Note that if K is a SFK of the form δ_φ , with φ a SFM, then (K, W) is a solution of the $(\frac{1}{2}\Delta, C)$ -SDE if and only if (φ, W) solves (1), and we must have $C(x, x) = 1$ for all $x \in \mathbb{R}$.

In the following we will make the assumption that $C(x, x) \leq 1$ for all $x \in \mathbb{R}$.

A solution (K, W) of the $(\frac{1}{2}\Delta, C)$ -SDE is called a Wiener solution if for all $s < t$, $\mathcal{F}_{s,t}^K = \mathcal{F}_{s,t}^W$. A SFM φ will be called coalescing when for all x, y , $T = \inf\{t > 0; \varphi_{0,t}(x) = \varphi_{0,t}(y)\}$ is finite a.s. A SFK K will be called diffusive when it is not a SFM.

We will say the $(\frac{1}{2}\Delta, C)$ -SDE has a unique solution when all its solutions (K, W) have the same distribution.

2.5. Martingale problems related to the $(\frac{1}{2}\Delta, C)$ -SDE

Let (K, W) be a solution of the $(\frac{1}{2}\Delta, C)$ -SDE. Denote by $P_t^{(n)}$, the semigroup associated with the n -point motion of K , and by $\mathbb{P}_x^{(n)}$ the law of the n -point motion started from $x \in \mathbb{R}^n$.

Proposition 2.5. *Let $X^{(n)}$ be distributed like $\mathbb{P}_x^{(n)}$ with $x \in \mathbb{R}^n$. Then it is a solution of the martingale problem:*

$$f(X_t^{(n)}) - \int_0^t A^{(n)} f(X_s^{(n)}) ds \tag{3}$$

is a martingale for all $f \in C_0^2(\mathbb{R}^n)$, where

$$A^{(n)} f(x) = \frac{1}{2} \Delta^{(n)} f(x) + \sum_{i < j} C(x_i, x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x).$$

Proof. When f is of the form $f_1 \otimes \dots \otimes f_n$, with f_1, \dots, f_n in $C_0^2(\mathbb{R})$, then it is easy to verify (3) when (K, W) solves (2). This extends to the linear space spanned by such functions, and by density, to $C_0^2(\mathbb{R}^n)$. (A proof of this fact can be derived from the observation that, on the torus, Sobolev theorem shows that trigonometric polynomials are dense in C^2 .) \square

Remark 2.6. If one can prove uniqueness for these martingale problems, then this implies there exists at most one solution to the $(\frac{1}{2}\Delta, C)$ -SDE. (Indeed, using Itô's formula, the expectation of any product of $K_{s,t} f(x)$ and $W_{s,t}(y)$'s can be expressed in terms of $\mathbb{P}_t^{(n)}$'s.)

Remark 2.7. When C is smooth, then the martingale problem (3) is well posed. This implies that the n -point motion of K is uniquely determined and that the $(\frac{1}{2}\Delta, C)$ -SDE has a unique solution, which is a Wiener solution. If one assumes in addition that $C(x, x) = 1$, then the usual theory applies, and the solution is a flow of diffeomorphisms.

2.6. Statement of the main theorems

Theorem 2.8. Let $C_{\pm}(x, y) = 1_{\{x>0\}} 1_{\{y>0\}} + 1_{\{x<0\}} 1_{\{y<0\}}$. Then there is a unique solution (K, W) of the $(\frac{1}{2}\Delta, C_{\pm})$ -SDE. Moreover, this solution is a Wiener solution and K is a coalescing SFM.

Denote by φ^{\pm} the coalescing flow defined in Theorem 2.8. The white noise W of covariance C_{\pm} can be written in the form $W = W^+ 1_{\mathbb{R}^+} + W^- 1_{\mathbb{R}^-}$, with W^+ and W^- two independent standard white noises. Then (2) is equivalent to

$$\varphi_{s,t}^{\pm}(x) = x + \int_s^t 1_{\{\varphi_{s,u}^{\pm}(x)>0\}} W^+(du) + \int_s^t 1_{\{\varphi_{s,u}^{\pm}(x)<0\}} W^-(du). \tag{4}$$

A consequence of this theorem, with Proposition 4.1 below, is Theorem 2.9 below. In this theorem, the solutions of the SDE (5) are in the usual sense, that is they are single paths. After completing this work, we were informed by H. Hajri that a result proved in [18] and in [2] implies that pathwise uniqueness holds for this SDE.

Theorem 2.9. The SDE driven by B^+ and B^- , two independent Brownian motions,

$$dX_t = 1_{\{X_t>0\}} dB_t^+ + 1_{\{X_t<0\}} dB_t^- \tag{5}$$

has a unique solution. Moreover, this solution is a strong solution.

Proof. We recall that saying (5) has a unique solution means that for all $x \in \mathbb{R}$, there exists one and only one probability measure \mathbb{Q}_x on $C(\mathbb{R}^+; \mathbb{R}^3)$ such that under $\mathbb{Q}_x(d\omega)$, the canonical process $(X, B^+, B^-)(\omega) = \omega$ satisfies (5), with B^+ and B^- two independent Brownian motions, and X a Brownian motion started at x . Proposition 4.1 states that (5) has a unique solution. Since (4) holds, one can take $(X_t, B_t^+, B_t^-) = (\varphi_{0,t}^{\pm}(x), W_{0,t}^+, W_{0,t}^-)$ for this solution.

A solution (X, B^+, B^-) is a strong solution if X is measurable with respect to the σ -field generated by B^+ and B^- , completed by the events of probability 0. Since φ^{\pm} is a Wiener solution, one can conclude. \square

Theorem 2.10. Let $C_+(x, y) = 1_{\{x>0\}} 1_{\{y>0\}}$.

- (i) There is a unique solution (K^+, W) solution of the $(\frac{1}{2}\Delta, C_+)$ -SDE.
- (ii) The flow K^+ is diffusive and is a Wiener solution.
- (iii) The flow K^+ can be obtained by filtering φ^{\pm} with respect to the noise generated by W^+ : $K_{s,t}^+ = \mathbb{E}[\delta_{\varphi_{s,t}^{\pm}} | W^+]$.

Theorem 2.11. *There exists a unique coalescing SFM φ^+ such that its n -point motions coincide with the n -point motions of K^+ before hitting $\Delta_n = \{x \in \mathbb{R}^n; \exists i \neq j, x_i = x_j\}$. Moreover*

(i) (φ^+, W^+) is not a solution of the $(\frac{1}{2}\Delta, C_\pm)$ -SDE but it satisfies

$$\int_s^t 1_{\{\varphi_{s,u}^+(x) > 0\}} d\varphi_{s,u}^+(x) = \int_s^t 1_{\{\varphi_{s,u}^+(x) > 0\}} W^+(du).$$

(ii) K^+ can be obtained by filtering φ^+ with respect to the noise generated by W^+ : $K_{s,t}^+ = E[\delta_{\varphi_{s,t}^+} | W^+]$.

Remark 2.12. *Following [12], it should be possible to prove that the linear part of the noise generated by φ^+ is the noise generated by W^+ .*

We refer to Section 3.2 for more precise definitions of noises, extension of noises, filtering by a subnoise, and linear part of a noise.

3. General results

In this section C is any covariance function on \mathbb{R} satisfying $C(x, x) \leq 1$ for all $x \in \mathbb{R}$.

3.1. Chaos decomposition of Wiener solutions

Proposition 3.1. *Let (K, W) be a Wiener solution of the $(\frac{1}{2}\Delta, C)$ -SDE. Then for all $s < t$, f a bounded measurable function on \mathbb{R} and $x \in \mathbb{R}$, a.s.,*

$$K_{s,t}f(x) = \sum_{n \geq 0} J_{s,t}^n f(x) \tag{6}$$

with J^n defined by $J_t^0 = P_t$, where P_t is the heat semigroup on \mathbb{R} , and for $n \geq 0$,

$$J_{s,t}^{n+1} f(x) = \sum_i \int_s^t J_{s,u}^n ((P_{t-u}f)' e_i)(x) W^i(du). \tag{7}$$

This implies that there exists at most one Wiener solution to the $(\frac{1}{2}\Delta, C)$ -SDE.

Proof. We essentially follow the proof of Theorem 3.2 in [11], with a minor correction at the end noticed by Bertrand Micaux during his Ph.D. [15].

Let us first remark that the stochastic integral $\sum_i \int_0^t K_s(e_i(P_{t-s}f)')(x) W^i(ds)$ do converge in L^2 for all bounded measurable function f since (using in the fourth inequality that $C(x, x) \leq 1$ for all $x \in \mathbb{R}$)

$$\begin{aligned} & \sum_i E \left[\left(\int_s^t K_{s,u} ((P_{t-u}f)' e_i)(x) W^i(du) \right)^2 \right] \\ & \leq \sum_i \int_s^t E [(K_{s,u} ((P_{t-u}f)' e_i)(x))^2] du \\ & \leq \sum_i \int_s^t E [K_{s,u} ((P_{t-u}f)' e_i)^2(x)] du \\ & \leq \int_s^t P_{u-s} \left(((P_{t-u}f)')^2 \sum_i e_i^2 \right)(x) du \\ & \leq \int_s^t P_{u-s} ((P_{t-u}f)')^2(x) du \end{aligned}$$

which is finite (since, using that $\frac{d}{du}(\mathbb{P}_{u-s}(\mathbb{P}_{t-u}f)^2) = \mathbb{P}_{u-s}((\mathbb{P}_{t-u}f)')^2$, it is equal to $\mathbb{P}_{t-s}f^2(x) - (\mathbb{P}_{t-s}f)^2(x)$).

Take f a C^3 function with compact support. For $t > 0$, denote $K_{0,t}$ simply K_t . Then for $t > 0$, $n \geq 1$ and $x \in \mathbb{R}$

$$\begin{aligned} K_t f(x) - \mathbb{P}_t f(x) &= \sum_{k=0}^{n-1} (K_{t(k+1)/n}(\mathbb{P}_{t(1-(k+1)/n)}f) - K_{tk/n}(\mathbb{P}_{t(1-k/n)}f)) \\ &= \sum_{k=0}^{n-1} (K_{t(k+1)/n} - K_{tk/n})(\mathbb{P}_{t(1-(k+1)/n)}f) \\ &\quad + \sum_{k=0}^{n-1} K_{tk/n}(\mathbb{P}_{t(1-(k+1)/n)}f - \mathbb{P}_{t(1-k/n)}f) \\ &= \sum_{k=0}^{n-1} \sum_i \int_{tk/n}^{t(k+1)/n} K_s(e_i(\mathbb{P}_{t(1-(k+1)/n)}f)')(x) W^i(ds) \\ &\quad + \sum_{k=0}^{n-1} \int_{tk/n}^{t(k+1)/n} K_s\left(\frac{1}{2}(\mathbb{P}_{t(1-(k+1)/n)}f)''\right)(x) ds \\ &\quad - \sum_{k=0}^{n-1} K_{tk/n}(\mathbb{P}_{t(1-k/n)}f - \mathbb{P}_{t(1-(k+1)/n)}f) \end{aligned}$$

since (K, W) solves the $(\frac{1}{2}\Delta, C)$ -SDE. This last expression implies that for $t > 0$, $n \geq 1$ and $x \in \mathbb{R}$,

$$K_t f(x) - \mathbb{P}_t f(x) - \sum_i \int_0^t K_s(e_i(\mathbb{P}_{t-s}f)')(x) W^i(ds) = \sum_{k=1}^3 B_k(n)$$

with

$$\begin{aligned} B_1(n) &= \sum_{k=0}^{n-1} \sum_i \int_{tk/n}^{t(k+1)/n} K_s(e_i(\mathbb{P}_{t(1-(k+1)/n)}f - \mathbb{P}_{t-s}f)')(x) W^i(ds), \\ B_2(n) &= - \sum_{k=0}^{n-1} K_{tk/n} \left(\mathbb{P}_{t(1-k/n)}f - \mathbb{P}_{t(1-(k+1)/n)}f - \frac{t}{2n}(\mathbb{P}_{t(1-(k+1)/n)}f)'' \right)(x), \\ B_3(n) &= \sum_{k=0}^{n-1} \int_{tk/n}^{t(k+1)/n} (K_s - K_{tk/n}) \left(\frac{1}{2}(\mathbb{P}_{t(1-(k+1)/n)}f)'' \right)(x) ds. \end{aligned}$$

The terms in the expression of $B_1(n)$ being orthogonal,

$$\begin{aligned} \mathbb{E}[(B_1(n))^2] &= \sum_{k=0}^{n-1} \sum_i \int_{tk/n}^{t(k+1)/n} \mathbb{E}[(K_s(e_i(\mathbb{P}_{t(1-(k+1)/n)}f - \mathbb{P}_{t-s}f)'))^2(x)] ds \\ &\leq \sum_{k=0}^{n-1} \sum_i \int_{tk/n}^{t(k+1)/n} \mathbb{P}_s(e_i(\mathbb{P}_{t(1-(k+1)/n)}f - \mathbb{P}_{t-s}f)')^2(x) ds \\ &\leq \sum_{k=0}^{n-1} \int_{tk/n}^{t(k+1)/n} \mathbb{P}_s((\mathbb{P}_{t(1-(k+1)/n)}f - \mathbb{P}_{t-s}f)')^2(x) ds, \end{aligned}$$

where we have used Jensen inequality in the second inequality and the fact that $\sum_i e_i^2(x) = C(x, x) \leq 1$ in the last inequality. This last term is less than $n \int_0^{t/n} \|\mathbb{P}_s f' - f'\|_\infty^2 ds = O(\|f''\|_\infty^2 t^2/n)$.

By using triangular inequality, $E[(B_2(n))^2]^{1/2}$ is less than

$$\begin{aligned} & \sum_{k=0}^{n-1} E \left[K_{tk/n} \left(\mathbb{P}_{t(1-k/n)} f - \mathbb{P}_{t(1-(k+1)/n)} f - \frac{t}{2n} (\mathbb{P}_{t(1-(k+1)/n)} f)'' \right)^2 (x) \right]^{1/2} \\ & \leq \sum_{k=0}^{n-1} \left[\mathbb{P}_{tk/n} \left(\mathbb{P}_{t(1-k/n)} f - \mathbb{P}_{t(1-(k+1)/n)} f - \frac{t}{2n} (\mathbb{P}_{t(1-(k+1)/n)} f)'' \right)^2 (x) \right]^{1/2}. \end{aligned}$$

Using moreover that $\|\mathbb{P}_{t/n} f - f - t/(2n) f''\|_\infty = O((t/n)^{3/2} \|f'''\|_\infty)$, we get that $E[(B_2(n))^2]^{1/2} = O(t^{3/2} n^{-1/2} \|f'''\|_\infty)$.

By using again triangular inequality, $E[(B_3(n))^2]^{1/2}$ is less than

$$\begin{aligned} & \sum_{k=0}^{n-1} E \left[\left(\int_{tk/n}^{t(k+1)/n} (K_s - K_{tk/n}) \left(\frac{1}{2} (\mathbb{P}_{t(1-(k+1)/n)} f)'' \right) (x) ds \right)^2 \right]^{1/2} \\ & \leq \sum_{k=0}^{n-1} \left(\frac{t}{n} \int_{tk/n}^{t(k+1)/n} E \left[\left((K_s - K_{tk/n}) \left(\frac{1}{2} (\mathbb{P}_{t(1-(k+1)/n)} f)'' \right) (x) \right)^2 ds \right] \right)^{1/2}. \end{aligned}$$

For f a Lipschitz function and $0 \leq s < t$, we have (with (X, Y) the two point motion of K of law $\mathbb{P}_{(x,x)}^{(2)}$ and $E_{(x,x)}^{(2)}$ the expectation with respect to $\mathbb{P}_{(x,x)}^{(2)}$)

$$\begin{aligned} E[(K_t f - K_s f)^2(x)] &= E_{(x,x)}^{(2)} [(f(X_t) - f(X_s))(f(Y_t) - f(Y_s))] \\ &\leq (t - s) (\text{Lip}(f))^2 \end{aligned}$$

with $\text{Lip}(f)$ the Lipschitz constant of f . From this estimate, and since $\text{Lip}(\mathbb{P}_{t(1-(k+1)/n)} f'') \leq \text{Lip}(f'')$, we deduce that $E[(B_3(n))^2]^{1/2} = O(t^{3/2} \text{Lip}(f'') n^{-1/2})$. The estimates we gave for $E[(B_i(n))^2]$, for $i \in \{1, 2, 3\}$, implies that for f a C^3 function with compact support,

$$K_t f(x) = \mathbb{P}_t f(x) + \sum_i \int_0^t K_s (e_i(\mathbb{P}_{t-s} f)')(x) W^i(ds).$$

This implies that

$$K_{s,t} f(x) = \mathbb{P}_{t-s} f(x) + \sum_i \int_s^t K_{s,u} ((\mathbb{P}_{t-u} f)')(x) W^i(du). \tag{8}$$

Iterating n times relation (8), we get that

$$K_{s,t} f(x) = \sum_{k=0}^n J_{s,t}^k f(x) + R_{s,t}^n f(x),$$

where $R_{s,t}^n f(x)$ is a L^2 random variable whose chaos expansion is such that its n first Wiener chaoses are 0. This implies that for all k , $J_{s,t}^k f(x)$ is the k th Wiener chaos of $K_{s,t} f(x)$. Thus, for f a C^3 function with compact support, the Wiener chaos expansion of $K_{s,t} f(x)$ is given by (6). This extends to all bounded measurable functions. \square

3.2. Filtering a SFK by a subnoise

We follow here Section 3 in [12].

Definition 3.2. A noise consists of a separable probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a one-parameter group $(T_h)_{h \in \mathbb{R}}$ of \mathbb{P} -preserving L^2 -continuous transformations of Ω and a family $\mathcal{F}_{s,t}, \leq s \leq t \leq \infty$ of sub- σ -fields of \mathcal{A} such that:

- (a) T_h sends $\mathcal{F}_{s,t}$ onto $\mathcal{F}_{s+h,t+h}$ for all $h \in \mathbb{R}$ and $s \leq t$,
- (b) $\mathcal{F}_{s,t}$ and $\mathcal{F}_{t,u}$ are independent for all $s \leq t \leq u$,
- (c) $\mathcal{F}_{s,t} \wedge \mathcal{F}_{t,u} = \mathcal{F}_{s,u}$ for all $s \leq t \leq u$.

Moreover, we will assume that, for all $s \leq t$, $\mathcal{F}_{s,t}$ contains all \mathbb{P} -negligible sets of $\mathcal{F}_{-\infty,\infty}$, denoted \mathcal{F} .

A subnoise \bar{N} of N is a noise $(\Omega, \mathcal{A}, \mathbb{P}, (T_h), \bar{\mathcal{F}}_{s,t})$, with $\bar{\mathcal{F}}_{s,t} \subset \mathcal{F}_{s,t}$. Note that a subnoise is characterized by a σ -field invariant by T_h for all $h \in \mathbb{R}$, $\bar{\mathcal{F}} = \bar{\mathcal{F}}_{-\infty,\infty}$. In this case, we will say that \bar{N} is the noise generated by $\bar{\mathcal{F}}$.

A linear representation of N is a family of real random variables $X = (X_{s,t}; s \leq t)$ such that:

- (a) $X_{s,t} \circ T_h = X_{s+h,t+h}$ for all $s \leq t$ and $h \in \mathbb{R}$,
- (b) $X_{s,t}$ is $\mathcal{F}_{s,t}$ -measurable for all $s \leq t$,
- (c) $X_{r,s} + X_{s,t} = X_{r,t}$ a.s., for all $r \leq s \leq t$.

Define \mathcal{F}^{lin} be the σ -field generated by the random variables $X_{s,t}$ where X is a linear representation of N and $s \leq t$. Define N^{lin} to be the subnoise of N generated by \mathcal{F}^{lin} . This noise is called the linear part of the noise N .

Let \mathbb{P}^0 be the law of a SFK, it is a law on $(\Omega^0, \mathcal{A}^0) = (\prod_{s < t} E, \otimes_{s < t} \mathcal{E})$, and let K be the canonical SFK of law \mathbb{P}^0 . For $h \in \mathbb{R}$, define $T_h^0: \Omega \rightarrow \Omega$ by $T_h^0(\omega^0)_{s,t} = \omega^0_{s+h,t+h}$. For $s < t$, $\bar{\mathcal{F}}_{s,t}^0$ is the σ -field $\mathcal{F}_{s,t}^0$ completed by \mathbb{P}^0 -negligible sets of \mathcal{A}^0 . This defines a noise N^0 called the noise of the SFK K .

Let \bar{N} be a subnoise of N^0 . In Section 3.2 in [12], a SFK \bar{K} is defined as the filtering of K with respect to \bar{N} : for $s < t$, $f \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$, \bar{K} is such that

$$\bar{K}_{s,t} f(x) = \mathbb{E}[K_{s,t} f(x) | \bar{\mathcal{F}}_{s,t}] = \mathbb{E}[K_{s,t} f(x) | \bar{\mathcal{F}}].$$

Suppose now \mathbb{P}_0 is the law of a SFK that solves the $(\frac{1}{2}\Delta, C)$ -SDE. Writing C in the form $C(x, y) = \sum_{i \in I} e_i(x)e_i(y)$, there is a \mathcal{F}^K -white noise $W = \sum_{i \in I} W^i e_i$ such that:

$$K_{s,t} f(x) = f(x) + \sum_{i \in I} \int_s^t K_{s,u}(e_i f')(x) W^i(du) + \frac{1}{2} \int_s^t K_{s,u} f''(x) du \tag{9}$$

(i.e. the $(\frac{1}{2}\Delta, C)$ -SDE is transported on the canonical space).

Proposition 3.3. Let $J \subset I$ and \bar{N} the noise generated by $\{W^j; j \in J\}$, and $\bar{W} = \sum_{j \in J} W^j e_j$. Let \bar{K} be the SFK obtained by filtering K with respect to \bar{N} . Then (9) is satisfied with K replaced by \bar{K} and W by \bar{W} (or I by J). If $\sigma(\bar{W}) \subset \sigma(\bar{K})$, then \bar{K} solves the $(\frac{1}{2}\Delta, \bar{C})$ -SDE with $\bar{C}(x, y) = \sum_{j \in J} e_j(x)e_j(y)$, and driven by \bar{W} .

Proof. This proposition reduces to prove that

$$\bar{K}_{s,t} f(x) = f(x) + \sum_{j \in J} \int_s^t \bar{K}_{s,u}(e_j f')(x) W^j(du) + \frac{1}{2} \int_s^t \bar{K}_{s,u} f''(x) du$$

which easily follows by taking the conditional expectation with respect to $\sigma(W^j; j \in J)$ in equation (9). □

4. $C_{\pm}(x, y) = \mathbf{1}_{\{x < 0\}}\mathbf{1}_{\{y < 0\}} + \mathbf{1}_{\{x > 0\}}\mathbf{1}_{\{y > 0\}}$

In this section, we will prove Theorem 2.8. In order to do this we will prove that the n -point motion of a solution is uniquely determined and has the coalescing property (when two points meet, they stay together).

In Section 4.2, we prove that such a n -point motion is uniquely determined up to the first time to of them meet.

In Section 4.3, we show that the process obtained from the two-point motion after deleting its excursions out of $D = \{(x, y) \in \mathbb{R}^2; xy \leq 0\}$, and stopped when it reaches $(0, 0)$ is a Brownian motion with oblique reflection at the boundary of D , stopped when it reaches $(0, 0)$.

In Section 4.4, after having shown that the two-point motion of a solution reaches $(0, 0)$ in finite time, we construct a consistent family of n -point motions verifying the coalescing property. This allows to construct a coalescing SFM and a solution to the $(\frac{1}{2}\Delta, C_{\pm})$ -SDE.

In Section 4.5, we show that any two point-motion of a solution cannot leave $\{(x, y) \in \mathbb{R}^2; x = y\}$. This is done by studying the process obtained from the two-point motion after deleting its excursions out of D .

4.1. The SDE $dX_t = \mathbf{1}_{\{X_t > 0\}} dB_t^+ + \mathbf{1}_{\{X_t < 0\}} dB_t^-$

Let B^1 and B^2 be two independent Brownian motions. For $x \in \mathbb{R}$, define $X^x, B^{x,+}$ and $B^{x,-}$ by

$$X_t^x = x + B_t^1, \tag{10}$$

$$B_t^{x,-} = \int_0^t \mathbf{1}_{\{X_s^x < 0\}} dX_s^x + \int_0^t \mathbf{1}_{\{X_s^x > 0\}} dB_s^2, \tag{11}$$

$$B_t^{x,+} = \int_0^t \mathbf{1}_{\{X_s^x < 0\}} dB_s^2 + \int_0^t \mathbf{1}_{\{X_s^x > 0\}} dX_s^x. \tag{12}$$

Then $X^x, B^{x,-}$ and $B^{x,+}$ are Brownian motions respectively started at $x, 0$ and 0 . Moreover $B^{x,-}$ and $B^{x,+}$ are independent, and we have

$$X_t^x = x + \int_0^t \mathbf{1}_{\{X_s^x > 0\}} dB_s^{x,+} + \int_0^t \mathbf{1}_{\{X_s^x < 0\}} dB_s^{x,-}. \tag{13}$$

Denote by \mathbb{Q}_x the law of $(X^x, B^{x,-}, B^{x,+})$.

Proposition 4.1. *Let $x \in \mathbb{R}$, X, B^+ and B^- be real random processes such that B^+ and B^- are independent Brownian motions. Then if*

$$X_t = x + \int_0^t \mathbf{1}_{\{X_s > 0\}} dB_s^+ + \int_0^t \mathbf{1}_{\{X_s < 0\}} dB_s^-, \tag{14}$$

the process (X, B^+, B^-) has for law \mathbb{Q}_x .

Proof. Assume (X, B^+, B^-) satisfy (14), with B^+ and B^- two independent Brownian motions. Let

$$B_t = \int_0^t \mathbf{1}_{\{X_s < 0\}} dB_s^+ + \mathbf{1}_{\{X_s > 0\}} dB_s^-.$$

Observe that $X - x$ and B are two independent Brownian motions. Moreover

$$B_t^- = \int_0^t \mathbf{1}_{\{X_s < 0\}} dX_s + \int_0^t \mathbf{1}_{\{X_s > 0\}} dB_s, \tag{15}$$

$$B_t^+ = \int_0^t \mathbf{1}_{\{X_s < 0\}} dB_s + \int_0^t \mathbf{1}_{\{X_s > 0\}} dX_s. \tag{16}$$

Note that (X, B^+, B^-) is defined out of $(X - x, B)$ exactly in the same way as $(X^x, B^{x,+}, B^{x,-})$ is defined out of (B^1, B^2) . This implies the proposition $((X, B^+, B^-)$ is distributed like $(X^x, B^{x,+}, B^{x,-})$). \square

4.2. Construction of the n -point motions up to $T^{(n)}$

Let $x \notin \Delta_n = \{x \in \mathbb{R}^n; \exists i \neq j, x_i = x_j\}$. For convenience, we will assume that $x_1 < x_2 < \dots < x_n$ and set i the integer such that $x_i \leq 0 < x_{i+1}$, with $i = 1$ when $x_1 > 0$ and $i = n$ when $x_n \leq 0$. Let B^1 and B^2 be two independent Brownian motions. In the following construction, X_i follows B^1 . Out of X_i and B^2 , we construct B^+ and B^- two independent Brownian motions, and for $j > i$ (resp. for $j < i$), X_j follows B^+ (resp. B^-), this until the first time τ when X_{i+1} or X_{i-1} hits 0. After time τ , we follow the same procedure by replacing i by $i - 1$ when $X_{i-1}(\tau) = 0$ or by $i + 1$ when $X_{i+1}(\tau) = 0$. More precisely, define for $t > 0$, the processes

$$\begin{aligned} X_i^0(t) &= x_i + B_t^1, \\ B_t^{0,-} &= \int_0^t 1_{\{X_i^0(s) < 0\}} dB_s^1 + \int_0^t 1_{\{X_i^0(s) > 0\}} dB_s^2, \\ B_t^{0,+} &= \int_0^t 1_{\{X_i^0(s) < 0\}} dB_s^2 + \int_0^t 1_{\{X_i^0(s) > 0\}} dB_s^1, \\ X_j^0(t) &= x_j + B_t^{0,-} \quad \text{for } j \leq i - 1, \\ X_j^0(t) &= x_j + B_t^{0,+} \quad \text{for } j \geq i + 1. \end{aligned}$$

Set

$$\tau_1 = \inf\{t > 0: X_{i-1}^0(t) = 0 \text{ or } X_{i+1}^0(t) = 0\}$$

and set for $t \leq \tau_1$, $(X, B^-, B^+)(t) = (X^0, B^{0,-}, B^{0,+})(t)$. Set $i_1 = i + 1$ if $X_{i+1}(\tau_1) = 0$ and $i_1 = i - 1$ if $X_{i-1}(\tau_1) = 0$.

Then $X_1(\tau_1) < X_2(\tau_1) < \dots < X_{i_1}(\tau_1) = 0 < \dots < X_n(\tau_1)$ (it can be shown this is almost surely true, although actually the same proof would work even if two points could coalesce at τ_1). Assume now that $(\tau_k)_{k \leq \ell}$ and $(X, B^-, B^+)(t)$ have been defined for $t \leq \tau_\ell$ such that a.s.

- $(\tau_k)_{1 \leq k \leq \ell}$ is an increasing sequence of stopping times with respect to the filtration associated to X ;
- $X_1(\tau_k) < \dots < X_n(\tau_k)$ for $1 \leq k \leq \ell$;
- for all k , there exists an integer i_k such that $X_{i_k}(\tau_k) = 0$.

We then define $\tau_{\ell+1}$ and $(X_t, B_t^-, B_t^+)_{t \in]\tau_\ell, \tau_{\ell+1}]}$ as are defined τ_1 and $(X_t, B_t^-, B_t^+)_{t \leq \tau_1}$ by replacing i by i_ℓ , x by X_{τ_ℓ} and (B^1, B^2) by $(B_{\tau_\ell+}^1, -B_{\tau_\ell}^1, B_{\tau_\ell+}^2, -B_{\tau_\ell}^2)$. Let $T = \lim_{\ell \rightarrow \infty} \tau_\ell$. Note that $T = \inf\{t \geq 0; X_t \in \Delta_n\}$ (with the convention $\inf \emptyset = +\infty$). Denote by $\mathbb{P}_x^{(n),0}$ the law of $(X_t)_{t \leq T}$.

Lemma 4.2. *Let $X^{(n)}$ be a solution to the martingale problem (3), with $X_0^{(n)} = x$. Let $T^{(n)} = \inf\{t; X_t^{(n)} \in \Delta_n\}$. Then $(X_t^{(n)})_{t \leq T^{(n)}}$ is distributed like $\mathbb{P}_x^{(n),0}$.*

Proof. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be such that $x_1 < \dots < x_n$. Again, let i be such that $x_i \leq 0 < x_{i+1}$, with $i = 1$ when $x_1 > 0$ and $i = n$ when $x_n \leq 0$. Let $X^{(n)}$ be a solution of the martingale problem. This implies that for all j , $X_j^{(n)}$ is a Brownian motion and for all j and k ,

$$\langle X_j^{(n)}, X_k^{(n)} \rangle_t = \int_0^t C_\pm(X_j^{(n)}(s), X_k^{(n)}(s)) ds. \tag{17}$$

Let B^0 be a Brownian motion, independent of $X^{(n)}$. Set

$$\tau_1 = \inf\{t > 0: \exists j \neq i, X_j^{(n)}(t) = 0\}.$$

Define for $t \leq \tau_1$,

$$B_t^+ = X_{i+1}^{(n)}(t) - x_{i+1}, \quad \text{when } i \leq n - 1,$$

$$B_t^+ = \int_0^t 1_{\{X_i^{(n)}(s) > 0\}} dX_i^{(n)}(s) + \int_0^t 1_{\{X_i^{(n)}(s) < 0\}} dB_s^0, \quad \text{when } i = n.$$

Define also for $t \leq \tau_1$

$$B_t^- = X_{i-1}^{(n)}(t) - x_{i-1}, \quad \text{when } i \geq 2,$$

$$B_t^- = \int_0^t 1_{\{X_i^{(n)}(s) < 0\}} dX_i^{(n)}(s) + \int_0^t 1_{\{X_i^{(n)}(s) > 0\}} dB_s^0(s), \quad \text{when } i = 1.$$

Define for $t \leq \tau_1$,

$$B_t^1 = X_i^{(n)}(t) - x_i$$

and

$$B_t^2 = \int_0^t 1_{\{X_i^{(n)}(s) > 0\}} dB_s^- + \int_0^t 1_{\{X_i^{(n)}(s) < 0\}} dB_s^+.$$

Note that for $t \leq \tau_1$, $\langle B^1, B^2 \rangle_t = 0$ and that

$$X_j^{(n)}(t) = x_j + B_t^- \quad \text{for } j < i,$$

$$X_j^{(n)}(t) = x_j + B_t^+ \quad \text{for } j > i.$$

Assume now that $(\tau_k)_{k \leq \ell}$ and $(B_t^1, B_t^2, B_t^+, B_t^-)_{t \leq \tau_\ell}$ have been defined such that a.s.

- $(\tau_k)_{1 \leq k \leq \ell}$ is an increasing sequence of stopping times with respect to the filtration associated to $X^{(n)}$;
- $X_{\tau_k}^{(n)} \notin \Delta_n$ for $1 \leq k \leq \ell$;
- for all $1 \leq k \leq \ell$, there exists an integer i_k such that $X_{i_k}^{(n)}(\tau_k) = 0$.

We then define $\tau_{\ell+1}$ and $(B_t^1, B_t^2, B_t^+, B_t^-)_{\tau_\ell < t \leq \tau_{\ell+1}}$ as are defined τ_1 and $(B_t^1, B_t^2, B_t^+, B_t^-)_{0 < t \leq \tau_1}$ by replacing i by i_ℓ , x by $X^{(n)}(\tau_\ell)$. Note that $T^{(n)} = \lim_{\ell \rightarrow \infty} \tau_\ell$. Define for $t \geq T^{(n)}$, $B_t^1 = B_{T^{(n)}}^1 + X_1^{(n)}(t) - X_1^{(n)}(T^{(n)})$ and $B_t^2 = B_{T^{(n)}}^2 + B_t^0 - B_{T^{(n)}}^0$. Then B^1 and B^2 are two independent Brownian motions. We finally remark that $(X_t^{(n)})_{t \leq T^{(n)}}$ can be defined out of B^1 and B^2 in the same way $\mathbb{P}_x^{(n),0}$ is defined. This proves the lemma. \square

4.3. Brownian motions with oblique reflection

Let X and B be two independent Brownian motions, with X started at $x \in \mathbb{R}$. Let Y be a Brownian motion started at y defined by

$$Y_t = y + \int_0^t 1_{\{X_s < 0\}} dB_s + \int_0^t 1_{\{X_s > 0\}} dX_s.$$

Let $A_t = \int_0^t 1_{\{X_s < 0\}} ds$ and $\kappa_t = \inf\{s > 0; A_s > t\}$. Then define $(X^r, Y^r)_t = (X, Y)_{\kappa_t}$, it is a continuous process taking its values in $\{x \leq 0\}$. Denote by L_t the local time at 0 of X_t . Then X^r is a Brownian motion in \mathbb{R}^- reflected at 0 and that if $L_t^r = \frac{1}{2}L_{\kappa_t}$, $B_t^1 = X_t^r - L_t^r - X_0^r$ is a Brownian motion.

Lemma 4.3. Set $B_t^2 = \int_0^{\kappa_t} 1_{\{X_s < 0\}} dB_s$. Then B^2 is a Brownian motion independent of B^1 and we have

$$Y_t^r = B_t^2 - L_t^r + Y_0^r.$$

Thus (X^r, Y^r) is a Brownian motion in $\{x \leq 0\}$ reflected at $\{x = 0\}$ with oblique reflection with angle of reflection equal to $\pi/4$, i.e.

$$(X_t^r, Y_t^r) = (X_0^r, Y_0^r) + (B_t^1, B_t^2) - (L_t^r, L_t^r)$$

(note that $(X_0^r, Y_0^r) = (x, y)$ if $x \leq 0$ and $(X_0^r, Y_0^r) = (0, y - x)$ if $x > 0$).

Proof. Note that $B_t^1 = \int_0^{\kappa_t} 1_{\{X_s < 0\}} dX_s$. Since $\langle B^1 \rangle_t = \langle B^2 \rangle_t = \int_0^{\kappa_t} 1_{\{X_s < 0\}} ds$ and since $\langle B^1, B^2 \rangle_t = 0$, B^1 and B^2 are two independent Brownian motions. Let ε be a small positive parameter and define the sequences of stopping times σ_k^ε and τ_k^ε such that $\sigma_0^\varepsilon = 0$ and for $k \geq 0$

$$\begin{aligned} \tau_k^\varepsilon &= \inf\{t \geq \sigma_k^\varepsilon; X_t = 0\}, \\ \sigma_{k+1}^\varepsilon &= \inf\{t \geq \tau_k^\varepsilon; X_t = \varepsilon\}. \end{aligned}$$

Note that $X_0^r = x$ if $x \leq 0$ and $X_0^r = 0$ if $x > 0$, and for $t > 0$

$$X_t^r = \sum_{k \geq 1} (X_{\tau_k^\varepsilon \wedge \kappa_t} - X_{\tau_{k-1}^\varepsilon \wedge \kappa_t}) + X_{\tau_0^\varepsilon \wedge \kappa_t},$$

with $X_{\tau_0^\varepsilon \wedge \kappa_t} = X_t$ if $x \leq 0$ and $\kappa_t < \tau_0^\varepsilon$ and $X_{\tau_0^\varepsilon \wedge \kappa_t} = 0$ if $\kappa_t > \tau_0^\varepsilon$ (which holds if $x > 0$). We have that $X_t^r = -L_t^{\varepsilon,r} + B_t^{\varepsilon,1} + X_0^r$ with

$$\begin{aligned} L_t^{\varepsilon,r} &= - \sum_{k \geq 1} (X_{\tau_k^\varepsilon \wedge \kappa_t} - X_{\sigma_k^\varepsilon \wedge \kappa_t}), \\ B_t^{\varepsilon,1} &= \sum_{k \geq 1} (X_{\sigma_k^\varepsilon \wedge \kappa_t} - X_{\tau_{k-1}^\varepsilon \wedge \kappa_t}) + (X_{\tau_0^\varepsilon \wedge \kappa_t} - X_0^r). \end{aligned}$$

Then $L_t^{\varepsilon,r}$ and $B_t^{\varepsilon,1}$ both converges in probability respectively towards L_t^r and B_t^1 (see Theorem 2.23 in Chapter 6 of [10]).

Note now that $Y_0^r = y$ if $x \leq 0$ and $Y_0^r = y - x$ if $x > 0$, and for $t > 0$

$$Y_t^r = \sum_{k \geq 1} (Y_{\tau_k^\varepsilon \wedge \kappa_t} - Y_{\tau_{k-1}^\varepsilon \wedge \kappa_t}) + Y_{\tau_0^\varepsilon \wedge \kappa_t},$$

with $Y_{\tau_0^\varepsilon \wedge \kappa_t} = Y_t$ if $x \leq 0$ and $\kappa_t < \tau_0^\varepsilon$ and $Y_{\tau_0^\varepsilon \wedge \kappa_t} = y - x$ if $\kappa_t > \tau_0^\varepsilon$ (which holds if $x > 0$). Since when X_t is positive, $X_t - Y_t$ remains constant, $Y_t^r = -L_t^{\varepsilon,r} + B_t^{\varepsilon,2} + Y_0^r$, with

$$B_t^{\varepsilon,2} = \sum_{k \geq 1} (Y_{\sigma_k^\varepsilon \wedge \kappa_t} - Y_{\tau_{k-1}^\varepsilon \wedge \kappa_t}) + (Y_{\tau_0^\varepsilon \wedge \kappa_t} - Y_0^r).$$

It remains to observe that $B_t^{\varepsilon,2}$ converges in probability towards B_t^2 . □

Denote now by (X, Y) a solution of the martingale problem (3), for $n = 2$. Set $T = \inf\{s \geq 0; X_s = Y_s\}$. Let $D^+ = \{(x, y) \in \mathbb{R}^2; x \leq 0 \text{ and } y \geq 0\}$, $D^- = \{(x, y) \in \mathbb{R}^2; x \geq 0 \text{ and } y \leq 0\}$ and $D = D^+ \cup D^-$. Let $A_t^\pm = \int_0^t 1_{\{(X_s, Y_s) \in D^\pm\}} ds$ and $A_t = A_t^+ + A_t^-$. Let $\kappa_t^\pm = \inf\{s; A_s^\pm > t\}$ and $\kappa_t = \inf\{s; A_s > t\}$. Set for $t \leq A_T^\pm$, $(X_t^{r,\pm}, Y_t^{r,\pm}) = (X, Y)(\kappa_t^\pm)$ and for $t \leq A_T$, $(X_t^r, Y_t^r) = (X, Y)(\kappa_t)$. Note that if $X_0 > Y_0$ (resp. if $X_0 < Y_0$) we have for $t \leq A_T$ (which then equals A_T^-), $(X^r, Y^r) = (X^{r,-}, Y^{r,-})$ (resp. for $t \leq A_T$ (which then equals A_T^+), $(X^r, Y^r) = (X^{r,+}, Y^{r,+})$). Denote by $L_t(X)$ and by $L_t(Y)$ the local times at 0 of X and of Y . And denote $\frac{1}{2}L_{\kappa_t}(X)$ and $\frac{1}{2}L_{\kappa_t}(Y)$ by L_t^1 and by L_t^2 .

The proof of Lemma 4.3 can be adapted to prove that:

Lemma 4.4. *The process (X^r, Y^r) is a Brownian motion in D , with oblique reflection at the boundary of angle of reflection equal to $\pi/4$, and stopped when it hits $(0, 0)$. More precisely, there exists B^1 and B^2 two independent Brownian motions such that for all $t \leq A_T$, when $X_0 < Y_0$,*

$$\begin{aligned} X_t^r &= X_0^r + B_t^1 - L_t^1 + L_t^2, \\ Y_t^r &= Y_0^r + B_t^2 - L_t^1 + L_t^2 \end{aligned}$$

and when $X_0 > Y_0$,

$$\begin{aligned} X_t^r &= X_0^r + B_t^1 + L_t^1 - L_t^2, \\ Y_t^r &= Y_0^r + B_t^2 + L_t^1 - L_t^2 \end{aligned}$$

with $(X_0^r, Y_0^r) = (X_0, Y_0)$ if $(X_0, Y_0) \in D$, $(X_0^r, Y_0^r) = (0, Y_0 - X_0)$ if $Y_0 < X_0 < 0$ or if $0 < X_0 < Y_0$, and $(X_0^r, Y_0^r) = (X_0 - Y_0, 0)$ if $0 < Y_0 < X_0$ or if $X_0 < Y_0 < 0$.

This process is a special case of the class of processes studied in [21,22]: the process (X^r, Y^r) is a Brownian motion in the wedge D , with angle $\xi = \pi/2$ and angles of reflection $\theta_1 = \theta_2 = \pi/4$. An important parameter in the study of these processes is $\alpha = (\theta_1 + \theta_2)/\xi = 1$.

The fact that the reflected Brownian motion is a semimartingale on $[0, \tau_0]$, where τ_0 is the first time to hit the origin, follows from Theorem 1 of [22]. An alternative proof, in a more general multi-dimensional and state-dependent coefficient setting, is given in Theorem 1.4 (property 3) of [19].

The non-semimartingale property of the Markovian extension spending no time at zero follows from Theorem 5 (with $\alpha = 1$) of [22]. Two alternative proofs, the first of which also generalizes to more general diffusions and higher dimensions, are given in Theorem 3.1 of [9] and Proposition 4.13 of [3]. These results are essentially related to the study of the local times which is done in the next section. We choose not to derive the results of this study from these references in order to keep their proofs self contained and reasonably short.

4.4. Coalescing n -point motions

Without loss of generality, assume that $(X_0^r, Y_0^r) \in D^+$. Let $U_t = \frac{Y_t^r + X_t^r}{\sqrt{2}}$ and $V_t = \frac{Y_t^r - X_t^r}{\sqrt{2}}$. Then V is a Brownian motion stopped when it hits 0. This implies in particular that $T_0 = \inf\{t \geq 0; X_t^r = Y_t^r = 0\}$ is finite a.s. Denote by $L^r := L^1 + L^2$ the local time at 0 of (X^r, Y^r) at the boundary $\{x = 0\} \cup \{y = 0\}$.

Lemma 4.5. $\mathbb{P}(L_{T_0}^r < \infty) = 1$.

Proof. Fix $M > 0$ and denote $T_M = \inf\{t \geq 0; V_t = M\}$. Note that the process $(U_t, V_t)_{t \leq T_0}$ is a Brownian motion with oblique reflection in $\{-v \leq u \leq v\}$, stopped when it hits $(0, 0)$. For $t < T_0$, one has that

$$\begin{aligned} U_t &= U_0 + W_t^1 + \sqrt{2} \int_0^t (1_{\{U_s = -V_s\}} - 1_{\{U_s = V_s\}}) dL_s^r, \\ V_t &= V_0 + W_t^2, \end{aligned}$$

where $W^1 = \frac{B^2 + B^1}{\sqrt{2}}$ and $W^2 = \frac{B^2 - B^1}{\sqrt{2}}$ are two independent Brownian motions. Let $h(u, v) = \frac{u^2 + v^2}{2v}$. Then, when $u = \pm v$, $\partial_v h(u, v) = 0$ and $\partial_u h(u, v) = \pm 1$. Note also that in $\{-v < u < v\}$,

$$\Delta h(u, v) = \frac{u^2 + v^2}{v^3} \leq \frac{2}{v}.$$

Applying Itô's formula, we get for $t < T_0 \wedge T_M$ that

$$h(U_t, V_t) \leq h(U_0, V_0) + \int_0^t \frac{ds}{V_s} - \sqrt{2}L_t^r + M_t,$$

where M_t is a local martingale with quadratic variation given by

$$\langle M \rangle_t = \frac{1}{4} \int_0^t \left(1 + \frac{U_s^2}{V_s^2} \right)^2 ds.$$

Since this quadratic variation is dominated by t , $(M_t)_{t \leq T_0}$ is a martingale. This implies that

$$\mathbb{E}[h((U, V)_{t \wedge T_0 \wedge T_M})] \leq \mathbb{E}[h(U_0, V_0)] + \mathbb{E} \left[\int_0^{t \wedge T_0 \wedge T_M} \frac{ds}{V_s} \right] - \sqrt{2} \mathbb{E}[L_{t \wedge T_0 \wedge T_M}].$$

It is well known that $\mathbb{E}[\int_0^{T_0 \wedge T_M} \frac{ds}{V_s}] < \infty$. For example, using Itô's formula we get for $t < T_0 \wedge T_M$,

$$d(V_t \log(V_t)) = (1 + \log(V_t)) dV_t + \frac{dt}{2V_t}.$$

Taking the limit as $t \rightarrow \infty$, using dominated and monotone convergence theorems (using that $h(u, v)$ is dominated by M on $\{-v \leq u \leq v \leq M\}$), we get

$$\mathbb{E}[h((U, V)_{T_0 \wedge T_M})] \leq \mathbb{E}[h(U_0, V_0)] + \mathbb{E} \left[\int_0^{T_0 \wedge T_M} \frac{ds}{V_s} \right] - \sqrt{2} \mathbb{E}[L_{T_0 \wedge T_M}^r].$$

We then must have that $\mathbb{E}[L_{T_0 \wedge T_M}^r] < \infty$. So a.s., $L_{T_0 \wedge T_M}^r < \infty$ for all $M > 0$. This proves the lemma. □

Lemma 4.6. *Consider again (X_t, Y_t) a solution of the martingale problem (3) for $n = 2$. Then $T = \inf\{t \geq 0; X_t = Y_t\}$ is finite a.s.*

Proof. Suppose $Y_0 > X_0$, so that $(X_0^r, Y_0^r) \in D^+$. To simplify a little, we will also assume that $(X_0, Y_0) \in D$. Note first that when $T < \infty$, we must have $X_T = Y_T = 0$. Since we are only interested to (X, Y) up to T , when $T < \infty$ we will replace $(X_{T+t}, Y_{T+t})_{t>0}$ by $(B_t, B_t)_{t>0}$, with B a Brownian motion independent of $(X_t, Y_t)_{t \leq T}$. Fix $\varepsilon > 0$ and define the sequences of stopping times σ_k^ε and τ_k^ε , by $\tau_0^\varepsilon = 0$ and for $k \geq 1$

$$\begin{aligned} \sigma_k^\varepsilon &= \inf\{t \geq \tau_{k-1}^\varepsilon, X_t = \varepsilon \text{ or } Y_t = -\varepsilon\}, \\ \tau_k^\varepsilon &= \inf\{t \geq \sigma_k^\varepsilon, X_t = 0 \text{ or } Y_t = 0\}. \end{aligned}$$

Then all these stopping times are finite a.s. Let us also remark that $T \notin \bigcup_k [\sigma_k^\varepsilon, \tau_k^\varepsilon]$ and that $\sum_k (\tau_k^\varepsilon \wedge T - \sigma_k^\varepsilon \wedge T)$ converges a.s. towards $T - T_0$ (i.e. the time spent by (X, Y) in D^c before T). Take $\alpha > 0$. Denoting $Z = (X, Y)$, $Z_{\tau_k^\varepsilon}$ is a Markov chain. Note that $N_\varepsilon = \inf\{k, Z_{\tau_k^\varepsilon} = (0, 0)\}$ is a stopping time for this Markov chain. Moreover it is clear that $(\tau_k^\varepsilon - \sigma_k^\varepsilon)_{k \geq 1}$ is a sequence of independent variables, and independent of the Markov chain $Z_{\tau_k^\varepsilon}$. And we have for all $k \geq 1$, $\mathbb{E}[e^{-\alpha(\tau_k^\varepsilon - \sigma_k^\varepsilon)}] = e^{-\varepsilon\sqrt{2\alpha}}$. Hence

$$\begin{aligned} \mathbb{E}[e^{-\sum_k \alpha(\tau_k^\varepsilon \wedge T - \sigma_k^\varepsilon \wedge T)}] &= \mathbb{E}[e^{-\sum_{k=1}^{N_\varepsilon-1} \alpha(\tau_k^\varepsilon - \sigma_k^\varepsilon)}] \\ &= \sum_n \mathbb{P}[N_\varepsilon = n] e^{-(n-1)\varepsilon\sqrt{2\alpha}} \end{aligned}$$

which implies that

$$\mathbb{E}[e^{-\alpha \sum_k (\tau_k^\varepsilon \wedge T - \sigma_k^\varepsilon \wedge T)}] = \mathbb{E}[e^{-(N_\varepsilon-1)\varepsilon\sqrt{2\alpha}}].$$

By taking the limit as $\varepsilon \rightarrow 0$ in this equality, since $\varepsilon N_\varepsilon$ converges in probability towards $L_T(X) + L_T(Y)$. Denote $L_T := \frac{L_T(X) + L_T(Y)}{2}$. Note that $L_T = L_{T_0}^r$, so that we have

$$\mathbb{E}[e^{-\alpha(T-T_0)}] = \mathbb{E}[e^{-2\sqrt{2\alpha}L_{T_0}^r}].$$

Since $L_{T_0}^r$ is finite a.s., $T - T_0$ is also finite a.s. \square

Proposition 4.7. *There exists a unique consistent family $(\mathbb{P}_t^{(n)}, n \geq 1)$ of Feller semigroups on \mathbb{R} such that if $X^{(n)}$ is its associated n -point motion started from $x \in \mathbb{R}^n$ and $T^{(n)} = \inf\{t \geq 0; X_t^{(n)} \in \Delta_n\}$, then:*

- (i) $(X_t^{(n)})_{t \leq T^{(n)}}$ is distributed like $\mathbb{P}_x^{(n),0}$.
- (ii) For $t \geq T^{(n)}$, $X_t^{(n)} \in \Delta_n$.

Moreover, this family is associated to a coalescing SFM.

Proof. To prove the Feller property, it suffices to check condition (C) of Theorem 4.1 in [12]. Denoting by (X, Y) the two-point motion, condition (C) is verified as soon as for all positive t and ε (denoting $d(x, y) = |y - x|$)

$$\lim_{d(x,y) \rightarrow 0} \mathbb{P}_{(x,y)}^{(2)}[d(X_t, Y_t) > \varepsilon] = 0. \quad (18)$$

Assume $0 < y - x = \alpha < \varepsilon$. Then

$$\begin{aligned} \mathbb{P}_{(x,y)}^{(2)}[d(X_t, Y_t) > \varepsilon] &= \mathbb{P}_{(x,y)}^{(2)}[Y_t - X_t > \varepsilon \text{ and } t < T] \\ &\leq \mathbb{P}_{(x,y)}^{(2)}\left[\sup_{t \leq T_0} (Y_t^r - X_t^r) > \varepsilon\right] \\ &\leq \mathbb{P}_{(x,y)}^{(2)}\left[\sup_{t \leq T_0} V_t > \frac{\varepsilon}{\sqrt{2}}\right]. \end{aligned}$$

This last probability is equal to the probability that a Brownian motion started at $\alpha/\sqrt{2}$ hits $\varepsilon/\sqrt{2}$ before hitting 0, which is equal to α/ε . This implies (18). \square

Proposition 4.8. *Let φ be the SFM associated to $\mathbb{P}^{(n)}$. Then there exist W^+ and W^- two independent white noises, $\sigma(\varphi)$ -measurable, such that (4) is satisfied.*

Proof. Define for $s < t$, $W_{s,t}^\pm = \lim_{x \rightarrow \pm\infty} (\varphi_{s,t}(x) - x)$. This limit exists a.s. since one can check (using the martingale problem) that for $y > x > 0$ (resp. $y < x < 0$) and $t \leq \tau_s^x = \inf\{u > s; \varphi_{s,u}(x) = 0\}$, $\varphi_{s,t}(x) - x = \varphi_{s,t}(y) - y$. Using that $\langle W_{s,\cdot}^\pm, \varphi_{s,\cdot}(y) \rangle_t = \lim_{x \rightarrow \pm\infty} \langle \varphi_{s,\cdot}(x), \varphi_{s,\cdot}(y) \rangle_t = \int_s^t \mathbf{1}_{\{\pm\varphi_{s,u}(y) > 0\}} du$, it is then easy to check (4). \square

4.5. Uniqueness

Let $\tilde{\mathbb{P}}^{(n)}$ be another consistent family solving the martingale problem (3). By uniqueness of the solution of this martingale problem up to the first coalescing time, if this consistent family is different to the family $\mathbb{P}_t^{(n)}$, then the n -point motion should leave Δ_n . After a time change, one may assume it spends no time in Δ_n . Denote by (\tilde{X}, \tilde{Y}) the associated two-point motion starting from $(0, 0)$. Denote $\tilde{A}_t = \int_0^t \mathbf{1}_{\{(\tilde{X}_s, \tilde{Y}_s) \in D\}} ds$, $\tilde{\kappa}_t = \inf\{s \geq 0; \tilde{A}_s \geq t\}$ and $(\tilde{X}_t^r, \tilde{Y}_t^r) = (\tilde{X}, \tilde{Y})_{\tilde{\kappa}_t}$. Denote $L_t(\tilde{X})$ and $L_t(\tilde{Y})$ the local times at 0 of \tilde{X} and of \tilde{Y} . Define also \tilde{L}_t by $\frac{1}{2}(L_t(\tilde{X}) + L_t(\tilde{Y}))$ and \tilde{L}_t^r by $\tilde{L}_{\tilde{\kappa}_t}^r$. Denote $\tilde{U}^r = \frac{\tilde{Y}^r + \tilde{X}^r}{\sqrt{2}}$ and $\tilde{V}^r = \left| \frac{\tilde{Y}^r - \tilde{X}^r}{\sqrt{2}} \right|$.

Lemma 4.9. *For all $a > 0$, $\tilde{T}_a = \inf\{t > 0; d(\tilde{X}_t, \tilde{Y}_t) = a\}$ is infinite a.s.*

Proof. Let $\tilde{T}_a^r = \inf\{t > 0; \tilde{V}_t^r = \frac{a}{\sqrt{2}}\}$. Then one has $\tilde{A}_{\tilde{T}_a^r} = \tilde{T}_a^r$. Let $\varepsilon < \frac{a}{\sqrt{2}}$. Define σ_n^ε and τ_n^ε by $\tau_0^\varepsilon = 0$ and for $n \geq 1$,

$$\begin{aligned} \sigma_n^\varepsilon &= \inf\{t \geq \tau_{n-1}^\varepsilon; \tilde{V}_t^r = \varepsilon\}, \\ \tau_n^\varepsilon &= \inf\{t \geq \sigma_n^\varepsilon; \tilde{V}_t^r = 0\}. \end{aligned}$$

Recall $h(u, v) = \frac{u^2+v^2}{2v}$. Applying Itô's formula on the time intervals $[\sigma_n^\varepsilon \wedge \tilde{T}_a^r, \tau_n^\varepsilon \wedge \tilde{T}_a^r]$, denoting $\tilde{Z}^r = (\tilde{U}^r, \tilde{V}^r)$, and using $\Delta h(u, v) \geq \frac{1}{v}$ for the second term,

$$h(\tilde{Z}_{\tilde{T}_a^r}^r) - h(0) \geq \sum_{n \geq 1} (h(\tilde{Z}_{\tau_n^\varepsilon \wedge \tilde{T}_a^r}^r) - h(\tilde{Z}_{\sigma_{n-1}^\varepsilon \wedge \tilde{T}_a^r}^r)) \tag{19}$$

$$+ \sum_{n \geq 1} \int_{\sigma_n^\varepsilon \wedge \tilde{T}_a^r}^{\tau_n^\varepsilon \wedge \tilde{T}_a^r} \frac{1}{2\tilde{V}_s^r} ds \tag{20}$$

$$- \sum_{n \geq 1} (\tilde{L}_{\tau_n^\varepsilon \wedge \tilde{T}_a^r}^r - \tilde{L}_{\sigma_n^\varepsilon \wedge \tilde{T}_a^r}^r) \tag{21}$$

$$+ \sum_{n \geq 1} (\tilde{M}_{\tau_n^\varepsilon \wedge \tilde{T}_a^r}^{n,\varepsilon} - \tilde{M}_{\sigma_n^\varepsilon \wedge \tilde{T}_a^r}^{n,\varepsilon}), \tag{22}$$

where for all $n \geq 1$ and all $\varepsilon > 0$, $\tilde{M}^{n,\varepsilon}$ is a martingale whose quadratic variation is such that $\frac{d}{dt} \langle \tilde{M}^{n,\varepsilon} \rangle_t \leq 1$. Since $\int_0^{T_r^B} \frac{ds}{|B_s|} = \infty$ a.s. (see Corollary 6.28 Chapter 3 in [10]), with T_r^B the first time a Brownian motion B hits $-a/\sqrt{2}$ or $a/\sqrt{2}$, we get that term (20) goes to ∞ as $\varepsilon \rightarrow 0$. Terms (19) is positive and is dominated by $\varepsilon \times \#\{n; \tau_n^\varepsilon \leq \tilde{T}_a^r\}$ which converges in probability towards $L_{\tilde{T}_a^r}^0$, with L^0 the local time at 0 of \tilde{V} , which is a reflected Brownian motion. Denote by M^ε the martingale defined by

$$M_t^\varepsilon = \sum_{n \geq 1} (\tilde{M}_{\tau_n^\varepsilon \wedge t}^{n,\varepsilon} - \tilde{M}_{\sigma_n^\varepsilon \wedge t}^{n,\varepsilon}).$$

Then if $\varepsilon' < \varepsilon$, $\langle M^\varepsilon - M^{\varepsilon'} \rangle_t = \langle M^{\varepsilon'} \rangle_t - \langle M^\varepsilon \rangle_t$. Using that $\frac{d}{dt} \langle \tilde{M}^{n,\varepsilon} \rangle_t \leq 1$, we get $\langle M^\varepsilon - M^{\varepsilon'} \rangle_{\tilde{T}_a^r} \leq \sum_{n \geq 1} (\sigma_n^\varepsilon \wedge \tilde{T}_a^r - \tau_{n-1}^\varepsilon \wedge \tilde{T}_a^r)$. Since $E[\sum_{n \geq 1} (\sigma_n^\varepsilon \wedge \tilde{T}_a^r - \tau_{n-1}^\varepsilon \wedge \tilde{T}_a^r)]$ converges towards 0 as $\varepsilon \rightarrow 0$, we get that $M_{\tilde{T}_a^r}^\varepsilon$ is a Cauchy sequence in $L^2(\tilde{\mathbb{P}}_{(0,0)}^{(2)})$ and thus (22), which is equal to $M_{\tilde{T}_a^r}^\varepsilon$, converges in $L^2(\tilde{\mathbb{P}}_{(0,0)}^{(2)})$. This implies that term (21) converges in probability towards ∞ . Since it also converges towards $\tilde{L}_{\tilde{T}_a^r}^r$. One concludes that $\tilde{L}_{\tilde{T}_a^r}^r = \infty$ a.s. Since $\tilde{T}_a = \tilde{\kappa}_{\tilde{T}_a^r}$, $\tilde{L}_{\tilde{T}_a^r}^r = \tilde{L}_{\tilde{T}_a}$ and therefore $\tilde{L}_{\tilde{T}_a} = \infty$ a.s. Since \tilde{X} and \tilde{Y} are two Brownian motions, one must have $\tilde{T}_a = \infty$ a.s. □

This lemma implies that after hitting $\Delta_2 = \{x = y\}$, the two-point motion stays in Δ_2 . Thus $\tilde{\mathbb{P}}^{(2)} = \mathbb{P}^{(2)}$. The same argument applies to prove that $\tilde{\mathbb{P}}^{(n)} = \mathbb{P}^{(n)}$. This proves that there exists only one flow which is a SFK solution of the $(\frac{1}{2}\Delta, C_\pm)$ -SDE. Moreover this solution is a SFM φ^\pm .

To prove φ^\pm is a Wiener solution, define \bar{K} the SFK obtained by filtering φ^\pm with respect to the noise N^\pm generated by W^+ and W^- . Then, like for Proposition 4.8 we have $\sigma(W^+, W^-) \subset \sigma(\bar{K})$, and applying Proposition 3.3, \bar{K} also solves the $(\frac{1}{2}\Delta, C_\pm)$ -SDE driven by $W^+1_{\mathbb{R}^+} + W^-1_{\mathbb{R}^-}$. Since there exists a unique solution, \bar{K} is distributed like δ_{φ^\pm} . The noise of \bar{K} being N^\pm , this implies that φ^\pm is measurable with respect to $\sigma(W^+, W^-)$, i.e. φ^\pm is a Wiener solution. This concludes the proof of Theorem 2.8.

5. $C_+(x, y) = \mathbf{1}_{\{x>0\}}\mathbf{1}_{\{y>0\}}$

In this section, we prove Theorem 2.10 and Theorem 2.11.

First (in Section 5.1), by filtering the solution of the $(\frac{1}{2}\Delta, C_\pm)$ -SDE, we construct the Wiener solution to the $(\frac{1}{2}\Delta, C_+)$ -SDE.

In Section 5.2, we determine the n -point motion of the $(\frac{1}{2}\Delta, C_\pm)$ -SDE up to the first time two of them meet.

In Section 5.3, associating to the family of n -point motions of the Wiener solution a consistent family of coalescing n -point motions, we construct a coalescing SFM. Using Section 5.2, this implies Theorem 2.11.

In Section 5.4, we prove the uniqueness of the solution of the $(\frac{1}{2}\Delta, C_+)$ -SDE. We first remark that this holds if and only if the two-point motion of a solution is the one of the Wiener solution. We then show there is at most one possible two-point motion.

5.1. *The Wiener solution obtained by filtering*

Let φ^\pm be the SFM solution of the $(\frac{1}{2}\Delta, C_\pm)$ -SDE, and N^+ the noise generated by W^+ . Define K^+ the SFK obtained by filtering φ^\pm with respect to N^+ . Note that for $x \in \mathbb{R}$, $s \in \mathbb{R}$ and $t \in [s, \tau_s(x)]$ with $\tau_s(x) = \inf\{u; W_{s,u}^+ = -x\}$, we have $\varphi_{s,t}^\pm(x) = x + W_{s,t}^+$ and thus that $K_{s,t}^+(x) = \delta_{x+W_{s,t}^+}$. This clearly implies that $W = W^+1_{\mathbb{R}^+}$ is a \mathcal{F}^{K^+} -white noise. Then (see Section 3.2) K^+ is a Wiener solution of the $(\frac{1}{2}\Delta, C_+)$ -SDE driven $W = W^+1_{\mathbb{R}^+}$. Proposition 3.1 implies it is the unique Wiener solution.

5.2. *Construction of the n-point motions*

Let $D_n = \{x \in \mathbb{R}^n; \exists i \neq j, x_i = x_j \geq 0\}$. Let $x \in \mathbb{R}^n \setminus D_n$.

The n -point motion has the property that at any time, the points located on the positive half line will move parallel to W^+ and the points located on the negative half line will follow independent Brownian motions. Its law is determined up to the first time it hits D_n . Indeed, denote $I_- = \{j; x_j \leq 0\}$ and $I_+ = \{j; x_j > 0\}$. Let i be such that $x_i = \max\{x_j; j \in I_-\}$ when $I_- \neq \emptyset$ and $x_i = \min\{x_j; 1 \leq j \leq n\}$ when $I_- = \emptyset$. Let (B^1, \dots, B^n, B) be $n + 1$ independent Brownian motions. Define for $t > 0$, the processes

$$\begin{aligned} X_j^0(t) &= x_j + B^j(t) \quad \text{for } j \in I_- \cup \{i\}, \\ B_t^{0,+} &= \int_0^t 1_{\{X_i^0(s) < 0\}} dB_s + \int_0^t 1_{\{X_i^0(s) > 0\}} dB_s^i, \\ X_j^0(t) &= x_j + B_t^{0,+} \quad \text{for } j \in I_+ \setminus \{i\}. \end{aligned}$$

Set

$$\tau_1 = \inf\{t > 0; \exists j \neq i, X_j^0(t) = 0\}$$

and set for $t \leq \tau_1$, $X_t = X_t^0$ and $B_t^+ = B_t^{0,+}$.

Assume now that $(\tau_k)_{k \leq \ell}$ and $(X(t), W^+(t))_{t \leq \tau_\ell}$ have been defined such that a.s.

- $(\tau_k)_{1 \leq k \leq \ell}$ is an increasing sequence of stopping times with respect to the filtration associated to X ;
- $X_{\tau_k} \notin D_n$ for $1 \leq k \leq \ell$;
- for all $k \leq \ell$, there exists an integer i_k such that $X_{i_k}(\tau_k) = 0$.

We then define $(X_t, B_t^+)_{\tau_\ell < t \leq \tau_{\ell+1}}$ as is defined $(X_t, B_t^+)_{0 < t \leq \tau_1}$ by replacing i by i_ℓ , x by X_{τ_ℓ} and (B^1, \dots, B^n, B) by $(B_{\tau_\ell+}^1, \dots, B_{\tau_\ell+}^n, B_{\tau_\ell+}) - (B_{\tau_\ell}^1, \dots, B_{\tau_\ell}^n, B_{\tau_\ell})$. Let $T = \lim_{\ell \rightarrow \infty} \tau_\ell$. Denote by $\mathbb{P}_x^{(n),0}$ the law of $(X_t)_{t \leq T}$.

Lemma 5.1. *Let $X^{(n)}$ be a solution to the martingale problem (3), with $X_0^{(n)} = x$. Let $T^{(n)} = \inf\{t; \exists i \neq j, X_t^{(n)} \in D_n\}$. Then $(X_t^{(n)})_{t \leq T^{(n)}}$ is distributed like $\mathbb{P}_x^{(n),0}$.*

Proof. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus D_n$. As before $I_- = \{j; x_j \leq 0\}$ and $I_+ = \{j; x_j > 0\}$. Let i be such that $x_i = \max\{x_j; j \in I_-\}$ when $I_- \neq \emptyset$ and $x_i = \min\{x_j; j \in I\}$ when $I_- = \emptyset$. Let $X^{(n)}$ be a solution of the martingale problem. This implies that for all j , $X_j^{(n)}$ is a Brownian motion and for all $j \neq k$,

$$\langle X_j^{(n)}, X_k^{(n)} \rangle_t = \int_0^t 1_{\{X_j^{(n)}(s) > 0\}} 1_{\{X_k^{(n)}(s) > 0\}} ds. \tag{23}$$

Let $(B^{0,1}, \dots, B^{0,n}, B^0)$ be $n + 1$ independent Brownian motions, and independent of $X^{(n)}$. Define (B^1, \dots, B^n) by

$$B_t^j = \int_0^t 1_{\{X_j^{(n)}(s) < 0\}} dX_j^{(n)}(s) + \int_0^t 1_{\{X_j^{(n)}(s) > 0\}} dB_s^{0,j}.$$

Note that (B^1, \dots, B^n) are n independent Brownian motions.

Set

$$\tau_1 = \inf\{t > 0; \exists j \neq i, X_j^{(n)}(t) = 0\}.$$

Define for $t \leq \tau_1$, $B_t = B_t^0$ when $I_+ = \emptyset$ and

$$B_t = \int_0^t 1_{\{X_i^{(n)}(s) < 0\}} dX_k^{(n)}(s) + \int_0^t 1_{\{X_i^{(n)}(s) > 0\}} dB_s^0$$

when $I_+ \neq \emptyset$ and where $k \in I_+$ (one can choose for example k such that $x_k = \max\{x_j; j \in I_+\}$). Define also for $t \leq \tau_1$

$$B_t^+ = \int_0^t 1_{\{X_i^{(n)}(s) < 0\}} dB_s + \int_0^t 1_{\{X_i^{(n)}(s) > 0\}} dB_s^i.$$

Note that for $t \leq \tau_1$, $\langle B, B^j \rangle_t = 0$ for all j and that

$$X_j^{(n)}(t) = x_j + B_t^j \quad \text{for } j \in I_- \cup \{i\},$$

$$X_j^{(n)}(t) = x_j + B_t^+ \quad \text{for } j \in I_+ \setminus \{i\}.$$

Assume now that $(\tau_k)_{k \leq \ell}$ and $(B_t, B_t^+)_{t \leq \tau_\ell}$ have been defined such that a.s.

- $(\tau_k)_{1 \leq k \leq \ell}$ is an increasing sequence of stopping times with respect to the filtration associated to $X^{(n)}$;
- $X_{\tau_k}^{(n)} \notin \Delta_n$ for $1 \leq k \leq \ell$;
- for all $1 \leq k \leq \ell$, there exists an integer i_k such that $X_{i_k}^{(n)}(\tau_k) = 0$.

We then define $(B_t, B_t^+)_{t \in]\tau_\ell, \tau_{\ell+1}[}$ as is defined $(B_t, B_t^+)_{0 < t \leq \tau_1}$ by replacing i by i_ℓ , x by $X_{\tau_\ell}^{(n)}$. Note that $T^{(n)} = \lim_{\ell \rightarrow \infty} \tau_\ell$. Define for $t \geq T^{(n)}$, $B_t = B_{T^{(n)}} + B_t^0 - B_{T^{(n)}}^0$. Then (B^1, \dots, B^n, B) are n independent Brownian motions.

We finally remark that $(X_t^{(n)})_{t \leq T^{(n)}}$ is defined $\mathbb{P}_x^{(n),0}$. This proves the lemma. □

5.3. Proof of Theorem 2.11

Denote by $\mathbb{P}^{(n)}$ the family of consistent Feller semigroups associated to K^+ . To this family of semigroups, we associate a unique consistent family of coalescing n -point motions, $\mathbb{P}^{(n),c}$ (see Theorem 4.1 in [12]), with the property that the law of this n -point motion before hitting Δ_n is described by Lemma 5.1. These semigroups are Fellerian and since the two-point motion hits $\Delta_2 = \{x = y\}$, the family of semigroups are associated to a coalescing SFM φ^+ .

Proof of (i): follow the proof of Proposition 4.8.

Proof of (ii): It is a consequence of Theorem 4.2 in [12].

5.4. Proof of Theorem 2.10

The existence (and uniqueness) of a Wiener solution K^+ was proved in Section 5.1. By construction, (iii) holds. The SFK K^+ is diffusive since the two-point motion clearly leaves Δ_2 (it behaves as a Brownian motion in $D = \{x < 0$ or $y < 0\}$). So (ii) is proved. To finish the proof of (i), it remains to show that K^+ is the unique solution. Let K be another solution. The flow obtained by filtering K with respect to W^+ is a Wiener solution. Since there exists at most

one Wiener solution, this flow is distributed like K^+ . For simplicity we also denote this flow K^+ . Denote by $P^{(2)}$ (resp. $P^{(2,+)}$) the semigroup of the two-point of K (resp. of K^+). Note that if $P^{(2)} = P^{(2,+)}$, then $K = K^+$. Indeed,

$$\begin{aligned} E[(K_{s,t}f(x) - K_{s,t}^+f(x))^2] &= P_{t-s}^{(2)}(f \otimes f)(x, x) - 2E[K_{s,t}f(x)K_{s,t}^+f(x)] \\ &\quad + P_{t-s}^{(2,+)}(f \otimes f)(x, x) \\ &= P_{t-s}^{(2)}(f \otimes f)(x, x) - 2E[K_{s,t}^+f(x)K_{s,t}^+f(x)] \\ &\quad + P_{t-s}^{(2,+)}(f \otimes f)(x, x) \\ &= P_{t-s}^{(2)}(f \otimes f)(x, x) - P_{t-s}^{(2,+)}(f \otimes f)(x, x) \\ &= 0. \end{aligned}$$

Denote

$$\begin{aligned} D^r &= \{(x, y) \in \mathbb{R}^2; x \leq 0 \text{ or } y \leq 0\}, \\ D^+ &= \{(x, y) \in \mathbb{R}^2; x \geq 0 \text{ and } y \geq 0\}, \\ B &= \{(x, y) \in \mathbb{R}^2; x = 0 \text{ and } y \geq 0\} \\ &\quad \cup \{(x, y) \in \mathbb{R}^2; y = 0 \text{ and } x \geq 0\} \end{aligned}$$

so that B is the boundary of D^r and D^+ , and $\mathbb{R}^2 = D^r \cup B \cup D^+$.

Let $Z = (X, Y)$ be a solution to the martingale problem:

$$f(Z_t) - \int_0^t \frac{1}{2} \Delta f(Z_s) ds - \int_0^t 1_{\{Z_s \in D^+\}} \frac{\partial^2}{\partial x \partial y} f(Z_s) ds \tag{24}$$

is a martingale for all $f \in C_0^2(\mathbb{R}^2)$. For $\varepsilon > 0$, let

$$B^\varepsilon = \{z \in D^+; d(z, B) = \varepsilon\}$$

with d the Euclidean distance on \mathbb{R}^2 (note that when $z = (x, y) \in D^+$, $d(z, B) = x \wedge y$).

Define $A_t^r = \int_0^t 1_{\{Z_s \in D^r\}} ds$ and $A_t^+ = \int_0^t 1_{\{Z_s \in D^+\}} ds$, the amount of time Z spends in D^r and D^+ up to time t . Denote by κ_t^r and by κ_t^+ the inverses of A_t^r and of A_t^+ :

$$\begin{aligned} \kappa_t^r &= \inf\{s \geq 0; A_s^r > t\}, \\ \kappa_t^+ &= \inf\{s \geq 0; A_s^+ > t\}. \end{aligned}$$

Denote by Z^r and by Z^+ the processes in D^r and D^+ defined by

$$Z_t^r = Z_{\kappa_t^r} \quad \text{and} \quad Z_t^+ = Z_{\kappa_t^+}.$$

Define the sequences of stopping times σ_k^ε and τ_k^ε by $\sigma_0^\varepsilon = 0$ and for $k \geq 0$

$$\begin{aligned} \tau_k^\varepsilon &= \inf\{t \geq \sigma_k^\varepsilon; Z_t \in B\}, \\ \sigma_{k+1}^\varepsilon &= \inf\{t \geq \tau_k^\varepsilon; Z_t \in B^\varepsilon\}. \end{aligned}$$

It is easy to see that for $k \geq 1$, in the time interval $[\sigma_k^\varepsilon, \tau_k^\varepsilon]$, $Z_t \in D^+$, $Y_t - X_t$ remains constant and that $d(Z, B)$ is a Brownian motion stopped when it hits 0 (note that $d(Z, B) = X \wedge Y$). Thus, if $R_t = d(Z_t^+, B)$, then $(R_t)_{t \leq A_\infty^+}$ is a Brownian motion instantaneously reflected at 0 (since $R_t - R_0 = \lim_{\varepsilon \rightarrow 0} \sum_{k \geq 0} (d(Z_{\sigma_k^\varepsilon \wedge \kappa_t^+}^\varepsilon, B) - d(Z_{\tau_k^\varepsilon \wedge \kappa_t^+}^\varepsilon, B))$).

Denote now by L_t^+ the local time at 0 of R_t . Then, for any random times T , $L_{A_T^+}^+$ is the limit in probability as $\varepsilon \rightarrow 0$ of $\varepsilon \times N_T^\varepsilon$ where

$$N_T^\varepsilon = \max\{k; \sigma_k^\varepsilon < T\}.$$

Denote also $L_t^r = L_{A_{\kappa_t^r}^+}^+$.

Lemma 5.2. *For all $f \in C_0^2(\mathbb{R}^2)$ (resp. $f \in C^2(\mathbb{R}^2)$), we have*

$$f(Z_t^r) - \frac{1}{2} \int_0^t \Delta f(Z_s^r) ds - \int_0^t (\partial_x + \partial_y) f(Z_s^r) dL_s^r \tag{25}$$

is a martingale (resp. a local martingale).

Proof. Take $f \in C_0^2(\mathbb{R}^2)$ and assume $Z_0 \in D^+$ (we leave to the reader the case $Z_0 \notin D^+$). Then

$$f(Z_t^r) - f(Z_0^r) = \sum_{k \geq 1} (f(Z_{\tau_k^\varepsilon \wedge \kappa_t^r}) - f(Z_{\sigma_k^\varepsilon \wedge \kappa_t^r})) \tag{26}$$

$$+ \sum_{k \geq 0} (f(Z_{\sigma_{k+1}^\varepsilon \wedge \kappa_t^r}) - f(Z_{\tau_k^\varepsilon \wedge \kappa_t^r})). \tag{27}$$

Note that for $k \geq 1$ and $\kappa_t^r > \sigma_k^\varepsilon$ (which implies $\kappa_t^r > \tau_k^\varepsilon$), $Z_{\tau_k^\varepsilon \wedge \kappa_t^r} - Z_{\sigma_k^\varepsilon \wedge \kappa_t^r} = -\varepsilon v$, with $v = (1, 1)$. Using Taylor expansion, we have that the first term (26) is equal to

$$-\varepsilon \sum_{k=1}^{N_{\kappa_t^r}^\varepsilon} (\partial_x + \partial_y) f(Z_{\sigma_k^\varepsilon}) + O(\varepsilon^2 \times N_{\kappa_t^r}^\varepsilon).$$

More precisely, $O(\varepsilon^2 \times N_{\kappa_t^r}^\varepsilon) \leq \frac{\varepsilon^2 N_{\kappa_t^r}^\varepsilon}{2} \|f\|_{2,\infty}$. This implies that $O(\varepsilon^2 \times N_{\kappa_t^r}^\varepsilon)$ converges in probability towards 0 as $\varepsilon \rightarrow 0$ (recall $\varepsilon \times N_{\kappa_t^r}^\varepsilon$ converges towards $L_t^r = L_{A_{\kappa_t^r}^+}^+$). Thus the first term converges towards

$$- \int_0^t (\partial_x + \partial_y) f(Z_s^r) dL_s^r$$

(it is obvious if $(\partial_x + \partial_y) f(z) = 1$ when $z \in B$). Indeed: we claim that for H a bounded continuous process, $\varepsilon \sum_{k=1}^{N_t^\varepsilon} H_{\sigma_k^\varepsilon}$ converges in probability towards $\int_0^t H_s dL_s$. This holds for $H = \sum_i H_i 1_{[t_{i-1}, t_i]}$. Every bounded continuous process can be approached by a sequence of processes $H^n = \sum_i H_i 2^{-n} 1_{[(i-1)2^{-n}, i2^{-n}]}$. Thus we prove the claim by density. Then we apply this result by taking $H_s = (\partial_x + \partial_y) f(Z_s)$ to prove the convergence of the first term (26).

Denote by M^f the martingale (24). Then the second term (27) is equal to

$$\sum_{k \geq 1} (M_{\sigma_k^\varepsilon \wedge \kappa_t^r}^f - M_{\tau_{k-1}^\varepsilon \wedge \kappa_t^r}^f) \tag{28}$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_{\tau_{k-1}^\varepsilon \wedge \kappa_t^r}^{\sigma_k^\varepsilon \wedge \kappa_t^r} \Delta f(Z_s) 1_{\{Z_s \in D\}} ds \tag{29}$$

$$+ \sum_{k \geq 1} \int_{\tau_{k-1}^\varepsilon \wedge \kappa_t^r}^{\sigma_k^\varepsilon \wedge \kappa_t^r} \frac{\partial^2}{\partial_x \partial_y} f(Z_s) 1_{\{Z_s \in D^+\}} ds. \tag{30}$$

Note that

$$(29) = \frac{1}{2} \int_0^t \Delta f(Z_s^r) ds.$$

Note that for a constant C depending only on f , and denoting $D_\varepsilon^+ = \{z \in D^+; d(z, B) \leq \varepsilon\}$,

$$|(30)| \leq C \int_0^{\kappa_t^r} 1_{\{Z_s \in D_\varepsilon^+\}} ds$$

which converges a.s. towards 0 as $\varepsilon \rightarrow 0$.

Denote by M_t^ε the term (28). It is a martingale for all $\varepsilon > 0$ and one can check that there exists a constant $C < \infty$ depending only on f such that

$$\begin{aligned} \frac{d}{dt} \langle M^f \rangle_t &= 1_{\{Z_t \in D^r\}} |\nabla f(Z_t)|^2 + 1_{\{Z_t \in D^+\}} |(\partial_x + \partial_y) f(Z_t)|^2 \\ &\leq C. \end{aligned}$$

Moreover, for $\varepsilon' < \varepsilon$,

$$\begin{aligned} \langle M^\varepsilon - M^{\varepsilon'} \rangle_t &= \langle M^\varepsilon \rangle_t - \langle M^{\varepsilon'} \rangle_t \\ &= \sum_{k \geq 0} \int_{\tau_k^\varepsilon \wedge \kappa_t^r}^{\sigma_{k+1}^\varepsilon \wedge \kappa_t^r} \frac{d}{dt} \langle M^f \rangle_t - \sum_{k' \geq 0} \int_{\tau_{k'}^{\varepsilon'} \wedge \kappa_t^r}^{\sigma_{k'+1}^{\varepsilon'} \wedge \kappa_t^r} \frac{d}{dt} \langle M^f \rangle_t. \end{aligned}$$

Since $\{s \geq 0; Z_s \in D^r \cup D_\varepsilon^+\} \supset \bigcup_{k \geq 0} [\tau_k^\varepsilon, \sigma_{k+1}^\varepsilon] \supset \bigcup_{k' \geq 0} [\tau_{k'}^{\varepsilon'}, \sigma_{k'+1}^{\varepsilon'}] \supset \{s \geq 0; Z_s \in D^r\}$, we get that

$$\langle M^\varepsilon - M^{\varepsilon'} \rangle_t \leq C \int_0^{\kappa_t^r} 1_{\{Z_s \in D_\varepsilon^+\}} ds.$$

Since this converges a.s. towards 0 as $\varepsilon \rightarrow 0$, uniformly in $\varepsilon' < \varepsilon$, we have that M^ε is a Cauchy sequence in the space of square integrable martingales. Thus M^ε converges towards a martingale M . This proves the lemma. \square

This lemma implies that Z^r is a Brownian motion with oblique reflection in the wedge D^r (see [21], it corresponds to the case $\xi = 3\pi/2$, $\theta_1 = \theta_2 = \pi/4$ and $\alpha = 1/3$): $\int_0^t 1_{\{Z_s^r=0\}} ds = 0$ and Z^r is a solution to the sub-martingale problem:

$$f(Z_t^r) - \frac{1}{2} \int_0^t \Delta f(Z_s^r) ds \tag{31}$$

is a sub-martingale for all f constant in the neighborhood of 0 and $f \in C_0^2(D)$ such that $(\partial_x + \partial_y) f(z) \geq 0$ for $z \in D$.

In [21], it is proved that for all initial value Z_0^r , there is a unique solution to this sub-martingale problem. Thus the law of Z^r is uniquely determined by Z_0^r .

Applying also the lemma to the function $f(x, y) = x$, we see that $X_t^r + L_t^r$ is a local martingale (it is actually a true martingale). This implies that X^r is a semimartingale and that in the Doob–Meyer decomposition of X^r , $-L^r$ is its compensator. Thus L^r can be recovered from X^r . Note that this gives a proof that Z^r is a semimartingale (see also [22]).

Note that $L_{A_t^r}^r = L_{A_t^+}^+$. Thus, if $A_\infty^+ = \infty$ then $L_{A_\infty^+}^+ = \infty$ (since $\{0\}$ is recurrent for R) and $L_{A_\infty^r}^r = \infty$. Therefore $A_\infty^+ = \infty$ implies that $A_\infty^r = \infty$. On the converse, if $A_\infty^r = \infty$, then $L_\infty^r = \infty$ (since B is recurrent for Z^r) and $L_{A_\infty^+}^+ = \infty$. Therefore $A_\infty^r = \infty$ implies that $A_\infty^+ = \infty$. Since $A_\infty^r + A_\infty^+ = \infty$, we must have $A_\infty^r = A_\infty^+ = \infty$. It is easy to see that the processes Z^r and R are independent. We have that

$$Z_t = Z_{A_t^r}^r + R_{A_t^+} v$$

with $v = (1, 1)$.

Set $L_t = L_{A_t^r}^r (= L_{A_t^+}^+)$. Define T_ℓ , T_ℓ^r and T_ℓ^+ by

$$T_\ell = \inf\{t \geq 0; L_t > \ell\},$$

$$T_\ell^r = \inf\{t \geq 0; L_t^r > \ell\},$$

$$T_\ell^+ = \inf\{t \geq 0; L_t^+ > \ell\}.$$

Then $T_\ell = A_{T_\ell}^r + A_{T_\ell}^+ = T_\ell^r + T_\ell^+$. Thus the process L_t is $\sigma(Z^r, R)$ -measurable since $L_t = \inf\{\ell; T_\ell > t\} = \inf\{\ell; T_\ell^r + T_\ell^+ > t\}$. Now, $A_t^r = T_{L_t}^r$ and $A_t^+ = T_{L_t}^+$. Since

$$Z_t = Z_{A_t^r}^r + R_{A_t^+} v,$$

we see that Z is $\sigma(Z^r, R)$ -measurable. Thus, the law of Z is uniquely determined. This proves that $\mathbf{P}^{(2)} = \mathbf{P}^{(2,+)}$.

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