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Limit theorems for geometric functionals of Gibbs point processes

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Abstract. Observations are made on a point process Ξ in $\mathbb{R}^d$ in a window Q_λ of volume λ. The observation, or 'score' at a point x, here denoted $\xi(x, \Xi)$, is a function of the points within a random distance of x. When the input Ξ is a Poisson or binomial point process, the large λ limit theory for the total score $\sum_{x \in \Xi \cap Q_\lambda} \xi(x, \Xi \cap Q_\lambda)$, when properly scaled and centered, is well understood. In this paper we establish general laws of large numbers, variance asymptotics, and central limit theorems for the total score for Gibbsian input Ξ. The proofs use perfect simulation of Gibbs point processes to establish their mixing properties. The general limit results are applied to random sequential packing and spatial birth growth models, Voronoi and other Euclidean graphs, percolation models, and quantization problems involving Gibbsian input.

Résumé. On observe un processus ponctuel Ξ dans $\mathbb{R}^d$ dans une fenêtre Q_λ de volume λ. L'observation en un point x que l'on note $\xi(x, \Xi)$ est une fonction des points situés à une distance aléatoire de x. Quand Ξ est un processus de Poisson ponctuel ou Binomial, la limite pour λ grand de la somme totale $\sum_{x \in \Xi \cap Q_\lambda} \xi(x, \Xi \cap Q_\lambda)$ (convenablement recentrée et normalisée) est bien comprise. Dans ce papier, nous étudions cette somme totale quand Ξ est Gibbsien et prouvons la loi des grands nombres, la variance asymptotique et un théorème de la limite centrale. Les preuves reposent sur la simulation parfaite de processus ponctuels Gibbsiens pour établir leurs propriétés de mélange. Ces résultats généraux sont appliqués dans différents contextes comme des modèles de croissance et de percolation, des graphes de Voronoi et des problèmes de quantification pour des entrées Gibbsiennes.

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1. Introduction

Functionals of geometric structures often admit the representation

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}), \tag{1.1}$$

where $\mathcal{X} \subset \mathbb{R}^d$ is finite and where the function ξ, defined on pairs $(x, \mathcal{X})$, represents the 'score' or 'interaction' of x with respect to $\mathcal{X}$. When $\mathcal{X}$ consists of either λ i.i.d. random variables, $\lambda \in \mathbb{N}$, or Poisson points of intensity $\lambda > 0$, and when ξ satisfies the spatial dependency condition known as stabilization, the papers [2,20–25] develop the large

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λ limit theory for the properly normalized sums (1.1). The main goals of this paper are to (i) establish general weak laws of large numbers, variance asymptotics, and central limit theorems for (1.1) for Gibbsian input $\mathcal{X}$ on dilating volume λ windows as $\lambda \to \infty$ and (ii) apply the general results to deduce the limit theory of functionals of Gibbsian input arising in computational and discrete stochastic geometry.

Stabilization of scores ξ with respect to a reference Poisson point process $\mathcal{P}_\tau$ on $\mathbb{R}^d$ of constant intensity $\tau \in (0, \infty)$ is defined as follows. Say that ξ is translation invariant if $\xi(x, \mathcal{X}) = \xi(x + z, \mathcal{X} + z)$ for all $z \in \mathbb{R}^d$. Let $B_r(x)$ denote the Euclidean ball centered at x with radius $r \in \mathbb{R}^+ := [0, \infty)$. Letting $\mathbf{0}$ denote the origin of $\mathbb{R}^d$, a translation invariant ξ is *stabilizing* on $\mathcal{P}_\tau$ if there exists an a.s. finite random variable $R := R^\xi(\tau)$ (a 'radius of stabilization') such that

$$\xi\big(\mathbf{0}, \mathcal{P}_\tau \cap B_R(\mathbf{0})\big) = \xi\big(\mathbf{0}, \big(\mathcal{P}_\tau \cap B_R(\mathbf{0})\big) \cup \mathcal{A}\big)$$

for all locally finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_R(\mathbf{0})$. Here and elsewhere when $x \notin \mathcal{X}$, we write $\xi(x, \mathcal{X})$ for $\xi(x, \mathcal{X} \cup \{x\})$, unless notated otherwise.

For all $\lambda \geq 1$, consider the point measures

$$\mu_\lambda := \sum_{u \in \mathcal{P}_\tau \cap Q_\lambda} \xi(u, \mathcal{P}_\tau \cap Q_\lambda)\delta_{\lambda^{-1/d}u},$$

where δ_x denotes the unit point mass at x whereas Q_λ is the volume λ window $[-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$. Let $\mathcal{B}(Q_1)$ denote the class of all bounded $f : Q_1 \to \mathbb{R}$ and for all random measures μ on $\mathbb{R}^d$ we put $\langle f, \mu\rangle := \int f \, \mathrm{d}\mu$ and $\bar{\mu} := \mu - \mathbb{E}[\mu]$.

Stabilization of Borel measurable translation invariant ξ on $\mathcal{P}_\tau \cap Q_\lambda$, $\lambda \in [1, \infty]$, when combined with moment conditions on ξ, yields for all $f \in \mathcal{B}(Q_1)$ the law of large numbers [21,24]

$$\lim_{\lambda \to \infty} \lambda^{-1}\langle f, \mu_\lambda\rangle = \tau \mathbb{E}\big[\xi(\mathbf{0}, \mathcal{P}_\tau)\big] \int_{Q_1} f(x) \, \mathrm{d}x \quad \text{in } L^2 \tag{1.2}$$

and, under further conditions on the tail behavior of $R^\xi(\tau)$, variance asymptotics [2,20]

$$\lim_{\lambda \to \infty} \lambda^{-1} \operatorname{Var}\big[\langle f, \mu_\lambda\rangle\big] = \tau V^\xi(\tau) \int_{Q_1} f(x) \, \mathrm{d}x. \tag{1.3}$$

Here, for all $\tau > 0$

$$V^\xi(\tau) := \mathbb{E}\big[\xi(\mathbf{0}, \mathcal{P}_\tau)^2\big] + \tau \int_{\mathbb{R}^d} \mathbb{E}\big[\xi\big(\mathbf{0}, \mathcal{P}_\tau \cup \{z\}\big)\xi\big(z, \mathcal{P}_\tau \cup \{\mathbf{0}\}\big)\big] - \big(\mathbb{E}\xi(\mathbf{0}, \mathcal{P}_\tau)\big)^2 \, \mathrm{d}z.$$

Additionally [2,20], the finite-dimensional distributions $(\lambda^{-1/2}\langle f_1, \overline{\mu}_\lambda\rangle, \ldots, \lambda^{-1/2}\langle f_k, \overline{\mu}_\lambda\rangle)$, $f_1, \ldots, f_k \in \mathcal{B}(Q_1)$, converge to a mean zero Gaussian field with covariance kernel

$$(f, g) \mapsto \tau V^\xi(\tau) \int_{Q_1} f(x)g(x) \, \mathrm{d}x. \tag{1.4}$$

It is natural to ask whether analogs of (1.2)–(1.4) hold when $\mathcal{P}_\tau$ is replaced by weakly dependent input, including Gibbsian input $\mathcal{P}^{\beta\Psi}$, where β is the inverse temperature and where the potential Ψ is in some general class of potentials, e.g. the set $\boldsymbol{\Psi}^*$ of potentials consisting of (i) a pair potential function, (ii) a continuum Widom–Rowlinson potential, (iii) an area interaction potential, (iv) a hard-core potential, and (v) a potential generating a truncated Poisson point process (see below for further details of such potentials). Given $D \subset \mathbb{R}^d$ open and bounded, $\Psi \in \boldsymbol{\Psi}^*$, the distribution of $\mathcal{P}_D^{\beta\Psi} := \mathcal{P}^{\beta\Psi} \cap D$ has a Radon–Nikodym derivative with respect to the reference process $\mathcal{P}_\tau$ given by

$$\frac{\mathrm{d}\mathcal{L}(\mathcal{P}_D^{\beta\Psi})}{\mathrm{d}\mathcal{L}(\mathcal{P}_\tau \cap D)}(\mathcal{X}) := \frac{\exp(-\beta\Psi(\mathcal{X} \cap D))}{Z(\beta\Psi_D)}, \tag{1.5}$$

with $\mathcal{X}$ finite and $Z(\beta\Psi_D) := \mathbb{E}[\exp(-\beta\Psi(\mathcal{P}_\tau \cap D))]$ the normalizing constant.

We answer this question affirmatively and use a graphical construction of $\mathcal{P}_D^{\beta\Psi}$ to show that if $\Psi \in \boldsymbol{\Psi}^*$, then there is a range of τ and β such that the processes defined by (1.5) extend to Gibbs processes $\mathcal{P}^{\beta\Psi}$ on $\mathbb{R}^d$. The graphical construction of the extended process $\mathcal{P}^{\beta\Psi}$, while of separate interest, also shows for this range of τ and β that $\mathcal{P}^{\beta\Psi}$ is exponentially mixing, as are the weighted empirical measures $\sum_{x\in\mathcal{P}^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi})\delta_x$, provided ξ satisfies an exponential stabilization condition. This leads to the analogs of (1.2)–(1.4) when $\mathcal{P}_\tau$ is replaced by the infinite volume Gibbsian input $\mathcal{P}^{\beta\Psi}$, $\Psi \in \boldsymbol{\Psi}^*$. See Theorems 2.1–2.3.

Stabilizing functionals of geometric graphs over Gibbsian input on large cubes, as well as functionals of random sequential packing models defined by Gibbsian input on large cubes, consequently satisfy weak laws of large numbers, variance asymptotics, and central limit theorems as the cube size tends to infinity. Our general results also yield the limit theory for the total edge length of Voronoi tessellations with Gibbsian input. Precise theorems appear in Sections 5 and 6, which includes asymptotics for functionals of communication networks and continuum percolation models over Gibbsian point sets, as well as asymptotics for the distortion error arising in Gibbsian quantization of probability measures.

Terminology

Throughout Ψ denotes a translation and rotation invariant energy functional defined on finite point sets $\mathcal{X} \subset \mathbb{R}^d$, with values in $[0,\infty]$. By translation invariant we mean $\Psi(\mathcal{X}) = \Psi(y + \mathcal{X})$ for all $y \in \mathbb{R}^d$ and by rotation invariant we mean $\Psi(\mathcal{X}) = \Psi(\mathcal{X}')$ for all rotations $\mathcal{X}'$ of $\mathcal{X}$. Given a finite point set $\mathcal{X}$ in $\mathbb{R}^d$, and an open bounded set $D \subseteq \mathbb{R}^d$, we define $\Psi_D(\mathcal{X}) := \Psi(\mathcal{X} \cap D)$. We always assume that Ψ is *non-degenerate*, that is $\Psi(\{x\}) < \infty$ for all $x \in \mathbb{R}^d \cup \{\varnothing\}$. We also assume that

$$\Psi_D(\mathcal{X}) \le \Psi_D(\mathcal{X}') \quad \text{for } \mathcal{X} \subseteq \mathcal{X}' \tag{1.6}$$

and thus Ψ is *hereditary*, that is if $\Psi(\mathcal{X}) = \infty$ for some $\mathcal{X}$ then $\Psi(\mathcal{X}') = \infty$ for all $\mathcal{X}' \supseteq \mathcal{X}$. Given a potential Ψ define for finite $\mathcal{X} \subset \mathbb{R}^d$ the local energy function

$$\Delta(\mathbf{0}, \mathcal{X}) := \Delta^{\Psi}(\mathbf{0}, \mathcal{X}) := \Psi\big(\mathcal{X} \cup \{\mathbf{0}\}\big) - \Psi(\mathcal{X}), \quad \mathbf{0} \notin \mathcal{X}. \tag{1.7}$$

When both $\Psi(\mathcal{X} \cup \{\mathbf{0}\})$ and $\Psi(\mathcal{X})$ are ∞ we set $\Delta(\mathbf{0}, \mathcal{X}) := 0$. Note that $\Delta^{\Psi}(x, \mathcal{X}) = \Delta^{\Psi}(\mathbf{0}, \mathcal{X} - x)$ is the 'energy' required to insert x into the configuration $\mathcal{X}$ and the so-called conditional intensity $\exp(-\beta\Delta^{\Psi}(x, \mathcal{X}))$ determines the law of $\mathcal{P}^{\beta\Psi}$. Ψ has *finite interaction range* if there is $r^{\Psi} \in (0,\infty)$ such that for all finite $\mathcal{X} \subset \mathbb{R}^d$ we have

$$\Delta^{\Psi}(\mathbf{0}, \mathcal{X}) = \Delta^{\Psi}\big(\mathbf{0}, \mathcal{X} \cap B_{r^{\Psi}}(\mathbf{0})\big). \tag{1.8}$$

Gibbs point processes with potentials Ψ in the class $\boldsymbol{\Psi}^*$

(i) *Point processes with a pair potential function.* A large class of Gibbs point processes, known as pairwise interaction point processes [29], has Hamiltonian

$$\Psi(\mathcal{X}) := \sum_{i<j} \phi\big(|x_i - x_j|\big), \quad \mathcal{X} := \{x_i\}_{i=1}^n, \tag{1.9}$$

with $\phi : [0,\infty) \to [0,\infty)$ and where $|\cdot|$ denotes the Euclidean norm. We assume that either ϕ has compact support or that it satisfies the superstability condition

$$\phi(r) \le K_1 \exp(-K_2 r), \quad r \in [r_0, \infty) \tag{1.10}$$

and $\phi(r) = \infty$ for $r \in (0, r_0)$. Thus there is a hard-core exclusion forbidding the presence of two points within distance less than r_0, see [27]. The case of compact support includes the Strauss process, where $\phi(u) = \alpha\mathbf{1}(u \le r_0)$ for some $\alpha > 0$; see [1,29], and Section 10.4 of [5] for details.

(ii) *Point processes defined by the continuum Widom–Rowlinson model*. Consider the point process $\mathcal{P}^{\beta\Psi}$ defined in terms of the continuum Widom–Rowlinson model from statistical physics, also called the penetrable spheres mixture model (Section 10.4 of [5]). Here we have spheres of type A and B, with common radii equal to a, with interpenetrating spheres of similar types but hard-core exclusion between the two types. Let $\mathcal{X} := \{x_i\}_{i=1}^n$ be the centers of type A spheres and let $\mathcal{Y} := \{y_i\}_{i=1}^m$ be the centers of type B spheres. As in Section 10.4 of [5] we have

$$\Psi(\mathcal{X} \cup \mathcal{Y}) = \alpha_1 n + \alpha_2 m + \alpha_3, \quad d(\mathcal{X}, \mathcal{Y}) > 2a \tag{1.11}$$

and otherwise $\Psi(\mathcal{X} \cup \mathcal{Y}) = \infty$. Here and below α_i, $i = 1, 2, 3$ are positive constants.

(iii) *Area interaction point processes.* These are Gibbs-modified germ grain processes, where the grain shape is a fixed compact convex set K; see Section 2 of [11], [1], and Section 10.4 of [5] for details. As in [5], these processes have Hamiltonian

$$\Psi(\mathcal{X}) = \mathrm{Vol}\left(\bigcup_{i=1}^n (x_i \oplus K)\right) + \alpha_1 n + \alpha_2, \quad \mathcal{X} := \{x_i\}_{i=1}^n. \tag{1.12}$$

(iv) *Point processes given by the hard-core model.* A natural model falling into the framework of our theory is the hard-core model, extensively studied in statistical mechanics. In its basic version, the model conditions a Poisson point process to contain no two points at distance less than $2r_0$, with $r_0 > 0$ denoting a parameter of the model. This model has Hamiltonian

$$\Psi(\mathcal{X}) = \alpha_1 n + \alpha_2, \quad \mathcal{X} := \{x_i\}_{i=1}^n, \tag{1.13}$$

if no two points of $\mathcal{X}$ are within distance $2r_0$ and otherwise $\Psi(\mathcal{X}) = \infty$.

(v) *Truncated Poisson processes.* The hard-core gas is a particular example of a truncated Poisson process. A truncated Poisson process arises by conditioning a Poisson point process on a constraint event. For example, we may fix $k \in \mathbb{N}$ and $r_0 \in (0, \infty)$ and require that no ball of radius r_0 contain more than k points from the process. In this case,

$$\Psi(\mathcal{X}) = \infty \quad \text{if there is } x \in \mathbb{R}^d \text{ such that } \mathrm{card}\big(\mathcal{X} \cap B_{r_0}(x)\big) > k \tag{1.14}$$

and otherwise $\Psi(\mathcal{X}) = 0$.

2. Limit theory for stabilizing functionals on Gibbsian input

Poisson-like processes

A point process Ξ on $\mathbb{R}^d$ is stochastically dominated by the reference process $\mathcal{P}_\tau$ if for all Borel sets $B \subset \mathbb{R}^d$ and $n \in \mathbb{N}$ we have $\mathbb{P}[\mathrm{card}(\Xi \cap B) \ge n] \le \mathbb{P}[\mathrm{card}(\mathcal{P}_\tau \cap B) \ge n]$. We say that Ξ is *Poisson-like* if (i) Ξ is stochastically dominated by $\mathcal{P}_\tau$ and (ii) there exists $C_1 := C_1(\tau) \in (0, \infty)$ and $r_1 := r_1(\tau) \in (0, \infty)$ such that for all $r \in [r_1, \infty)$, $x \in \mathbb{R}^d$, and any point set $\mathcal{E}_r(x)$ in $B_r^c(x)$, the conditional probability of $B_r(x)$ not being hit by Ξ, given that $\Xi \cap B_r(x)^c$ coincides with $\mathcal{E}_r(x)$, satisfies

$$\mathbb{P}\big[\Xi \cap B_r(x) = \varnothing | \big\{\big(\Xi \cap B_r(x)^c\big) = \mathcal{E}_r(x)\big\}\big] \le \exp\big(-C_1 r^d\big). \tag{2.1}$$

Stochastic domination and (2.1) provide stochastic bounds on the number of points in large balls analogous to those satisfied by homogeneous Poisson point processes and thus the terminology Poisson-like. Lemma 3.3 below shows that Gibbs processes $\mathcal{P}^{\beta\Psi}$, $\Psi \in \boldsymbol{\Psi}^*$, are Poisson-like.

Stabilization

We next consider stabilization with respect to Poisson-like processes. Given a locally finite point set $\mathcal{X}$ and $z \in \mathbb{R}^d$, write $\mathcal{X}^z$ for $\mathcal{X} \cup \{z\}$.

Definition 2.1. *ξ is a stabilizing functional in the wide sense if for every Poisson-like process Ξ, all $x \in \mathbb{R}^d$, all $z \in \mathbb{R}^d \cup \{\varnothing\}$, and almost all realizations $\mathcal{X}$ of Ξ there exists $R := R^\xi(x, \mathcal{X}^z) \in (0, \infty)$ (a 'radius of stabilization') such that*

$$\xi\big(x, \mathcal{X}^z \cap B_R(x)\big) = \xi\big(x, \big(\mathcal{X}^z \cap B_R(x)\big) \cup \mathcal{A}\big) \tag{2.2}$$

for all locally finite point sets $\mathcal{A} \subseteq \mathbb{R}^d \setminus B_R(x)$.

Wide sense stabilization of ξ on Ξ implies that $\xi(x, \mathcal{X}^z)$ is wholly determined by the point configuration $\mathcal{X}^z \cap B_{R^\xi}(x)$. It also yields $\xi(x, \mathcal{X}^z \cap B_r(x)) = \xi(x, \mathcal{X}^z \cap B_{R^\xi}(x))$ for $r \in (R^\xi, \infty)$. Stabilizing functionals in the wide sense can thus be a.s. extended to the entire process Ξ^z, that is to say for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d \cup \{\varnothing\}$ we have

$$\xi\big(x, \Xi^z\big) = \lim_{r\to\infty} \xi\big(x, \Xi^z \cap B_r(x)\big) \quad \text{a.s.} \tag{2.3}$$

Given $s > 0$, $\varepsilon > 0$, and a Poisson-like process Ξ, define the tail probability

$$t(s;\varepsilon) := \sup_{y\in\mathbb{R}^d} \sup_{z\in\mathbb{R}^d\cup\{\varnothing\}} \mathbb{P}\Big[\sup_{x\in B_\varepsilon(y)\cap\Xi^z} R^\xi\big(x, \Xi^z\big) > s\Big].$$

Further, ξ is *exponentially stabilizing in the wide sense* if for every *Poisson-like* process Ξ we have

$$\limsup_{\varepsilon\to 0} \limsup_{s\to\infty} s^{-1} \log t(s;\varepsilon) < 0. \tag{2.4}$$

In the often studied setting of Poisson input $\mathcal{P}_\tau$ [2,22–24], where Poisson points in disjoint balls are independent, the scores $\xi(x, \mathcal{P}_\tau \cap B_{R^\xi(x,\mathcal{P}_\tau)}(x))$ and $\xi(y, \mathcal{P}_\tau \cap B_{R^\xi(y,\mathcal{P}_\tau)}(y))$ are independent, conditional on $B_{R^\xi(x,\mathcal{P}_\tau)}(x) \cap B_{R^\xi(y,\mathcal{P}_\tau)}(y) = \varnothing$. In the setting of Gibbsian input $\mathcal{P}^{\beta\Psi}$, this conditional independence fails, as Gibbs configurations on disjoint sets are in general dependent. We shall use perfect simulation of Gibbs point processes $\mathcal{P}^{\beta\Psi}$, $\Psi \in \boldsymbol{\Psi}^*$, to show that if ξ is exponentially stabilizing in the wide sense, then conditional independence holds provided the stabilization balls are enlarged to contain the so-called 'ancestor clan' of the stabilization ball. It follows that if ξ is exponentially stabilizing in the wide sense then the covariance of $\xi(\mathbf{0}, \mathcal{P}^{\beta\Psi})$ and $\xi(x, \mathcal{P}^{\beta\Psi})$ decays exponentially fast with $|x|$. This is a consequence of an exponential mixing property, called here 'exponential clustering,' as given in Lemma 3.4. Exponential decay of spatial correlations is central to extending the limit results (1.2)–(1.4) to Gibbsian input $\mathcal{P}^{\beta\Psi}$.

The wide sense exponential stabilization involves probabilistic tail bounds on stabilization radii uniformly over small neighborhoods of y rather than just at y itself; this assumption is of technical importance in the proof of exponential clustering. We are unaware of examples of natural functionals ξ exhibiting exponential decay of the stabilization radius just at y but not over its small neighborhoods. We are neither aware of interesting functionals which stabilize over Poisson samples but not over Poisson-like samples. For these reasons, *we shall abuse terminology and use the term 'stabilization' to mean 'stabilization in the wide sense*,' with a similar meaning for 'exponentially stabilizing.'

The next proposition, proved in Section 3, extends the definition of the local energy function Δ^Ψ, $\Psi \in \boldsymbol{\Psi}^*$, and the processes (1.5) to the infinite volume setting. Let $v_d := \pi^{d/2}[\Gamma(1 + d/2)]^{-1}$ be the volume of the unit ball in $\mathbb{R}^d$.

Proposition 2.1. (i) *For all $\Psi \in \boldsymbol{\Psi}^*$ and locally finite $\mathcal{X} \subset \mathbb{R}^d$, the local energy $\Delta^\Psi(\mathbf{0}, \mathcal{X}) := \lim_{r\to\infty} \Delta^\Psi(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0}))$ is well-defined.*

(*ii*) *For $\Psi \in \boldsymbol{\Psi}^*$ there is a regime $\mathcal{R}^\Psi \subset \mathbb{R}^+ \times \mathbb{R}^+$ such that if $(\tau, \beta) \in \mathcal{R}^\Psi$, then the processes defined by* (1.5) *extend to an infinite volume exponentially mixing Gibbs process $\mathcal{P}^{\beta\Psi} := \mathcal{P}^{\beta\Psi}_{\mathbb{R}^d}$. If Ψ has finite range r^Ψ then $(\tau, \beta) \in \mathcal{R}^\Psi$ whenever $\tau v_d \exp(-\beta m_0^\Psi)(r^\Psi + 1)^d < 1$, where*

$$m_0^\Psi := \inf_{\mathcal{X} \text{ locally finite}} \Delta^\Psi(\mathbf{0}, \mathcal{X}). \tag{2.5}$$

While the hereditary property (1.6) implies $m_0^\Psi \in [0, \infty)$, Section 3.1 shows that m_0^Ψ is strictly positive for some $\Psi \in \boldsymbol{\Psi}^*$. Recall that $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$, $\lambda \ge 1$. Given $\mathcal{P}^{\beta\Psi}$ and ξ, let μ_λ^ξ be the re-scaled empirical measure on Q_1, that is

$$\mu_\lambda^\xi := \mu_\lambda^{\xi,\beta\Psi} := \sum_{u\in\mathcal{P}^{\beta\Psi}\cap Q_\lambda} \xi\big(u, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \setminus u\big)\delta_{\lambda^{-1/d}u}. \tag{2.6}$$

Given $\Psi \in \boldsymbol{\Psi}^*$ and $p \in [0, \infty)$, we say that ξ satisfies the p-moment condition if for all $(\tau, \beta) \in \mathcal{R}^{\Psi}$,

$$\sup_{\lambda \in [1,\infty]} \sup_{u \in Q_\lambda} \sup_{\mathcal{Y} \in \mathcal{C}} \mathbb{E}\big|\xi\big(u, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \cup \mathcal{Y}\big)\big|^p < \infty, \tag{2.7}$$

where $\mathcal{C}$ denotes the collection of all finite point sets in $\mathbb{R}^d$. We now give general results extending (1.2)–(1.4) to Gibbsian input. Recall that $\bar{\mu}_\lambda^\xi := \mu_\lambda^\xi - \mathbb{E}[\mu_\lambda^\xi]$ and for all $x \in \mathbb{R}^d$ put

$$c^\xi(x) := c^{\xi,\beta\Psi}(x) := \mathbb{E}\xi\big(x, \mathcal{P}^{\beta\Psi}\big) \exp\big(-\beta\Delta\big(x, \mathcal{P}^{\beta\Psi}\big)\big). \tag{2.8}$$

Theorem 2.1 (WLLN). *Let $\Psi \in \boldsymbol{\Psi}^*$. Assume that ξ is stabilizing as at (2.2) and satisfies the p-moment condition (2.7) for some $p > 1$. For $(\tau, \beta) \in \mathcal{R}^{\Psi}$ and $f \in \mathcal{B}(Q_1)$ we have*

$$\lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E}\big[\langle f, \mu_\lambda^\xi \rangle\big] = \tau c^\xi(\mathbf{0}) \int_{Q_1} f(x)\,\mathrm{d}x. \tag{2.9}$$

If (2.7) is satisfied for some $p > 2$ then $\lim_{\lambda\to\infty} \lambda^{-1}\langle f, \mu_\lambda^\xi\rangle = \tau c^\xi(\mathbf{0}) \int_{Q_1} f(x)\,\mathrm{d}x$ in L^2.

In (2.9), both μ_λ^ξ and $c^\xi(\mathbf{0})$ depend on the reference intensity τ via $\mathcal{P}^{\beta\Psi}$, suppressed for notational brevity. Before stating variance asymptotics, for $\Psi \in \boldsymbol{\Psi}^*$ and $(\tau, \beta) \in \mathcal{R}^{\Psi}$ we put

$$c^\xi(x, y) := c^{\xi,\beta\Psi}(x, y) := \mathbb{E}\xi\big(x, \mathcal{P}^{\beta\Psi} \cup \{y\}\big)\xi\big(y, \mathcal{P}^{\beta\Psi} \cup \{x\}\big) \exp\big(-\beta\Delta\big(\{x, y\}, \mathcal{P}^{\beta\Psi}\big)\big), \tag{2.10}$$

where for all $x, y \in \mathbb{R}^d$ we write $\Delta(\{x, y\}, \mathcal{P}^{\beta\Psi}) := \Delta(x, \mathcal{P}^{\beta\Psi} \cup \{y\}) + \Delta(y, \mathcal{P}^{\beta\Psi})$. By stationarity and isotropy of $\mathcal{P}^{\beta\Psi}$, we may show $\Delta(\{x, y\}, \mathcal{P}^{\beta\Psi}) \stackrel{\mathcal{D}}{=} \Delta(y, \mathcal{P}^{\beta\Psi} \cup \{x\}) + \Delta(x, \mathcal{P}^{\beta\Psi})$ and so the distribution of $\Delta(\{x, y\}, \mathcal{P}^{\beta\Psi})$ does not depend on the order in which x and y are inserted into $\mathcal{P}^{\beta\Psi}$.

Theorem 2.2 (Variance asymptotics). *Let $\Psi \in \boldsymbol{\Psi}^*$. Assume that ξ is exponentially stabilizing as at (2.4) and satisfies the p-moment condition (2.7) for some $p > 2$. For $(\tau, \beta) \in \mathcal{R}^{\Psi}$ and $f \in \mathcal{B}(Q_1)$ we have*

$$\lim_{\lambda \to \infty} \lambda^{-1} \operatorname{Var}\big[\langle f, \mu_\lambda^\xi \rangle\big] = \tau v^\xi(\tau) \int_{Q_1} f(x)^2\,\mathrm{d}x, \tag{2.11}$$

where

$$v^\xi(\tau) := c^{\xi^2}(\mathbf{0}) + \tau \int_{\mathbb{R}^d} \big[c^\xi(\mathbf{0}, z) - c^\xi(\mathbf{0})c^\xi(z)\big]\,\mathrm{d}z < \infty. \tag{2.12}$$

The measures $\bar{\mu}_\lambda^\xi$, $\lambda \geq 1$, are in the domain of attraction of Gaussian white noise with scaling parameter $\lambda^{1/2}$. Here $N(0, \sigma^2)$ denotes a mean zero normal random variable with variance σ^2.

Theorem 2.3 (CLT). *Let $\Psi \in \boldsymbol{\Psi}^*$. Assume that ξ is exponentially stabilizing as at (2.4) and satisfies the p-moment condition (2.7) for some $p > 2$. For $(\tau, \beta) \in \mathcal{R}^{\Psi}$ and $f \in \mathcal{B}(Q_1)$ we have as $\lambda \to \infty$,*

$$\lambda^{-1/2}\langle f, \bar{\mu}_\lambda^\xi \rangle \xrightarrow{\mathcal{D}} N\left(0, \tau v^\xi(\tau) \int_{Q_1} f(x)^2\,\mathrm{d}x\right), \tag{2.13}$$

and the finite-dimensional distributions $(\lambda^{-1/2}\langle f_1, \bar{\mu}_\lambda^\xi\rangle, \ldots, \lambda^{-1/2}\langle f_m, \bar{\mu}_\lambda^\xi\rangle)$, $f_1, \ldots, f_m \in \mathcal{B}(Q_1)$, converge to those of a mean zero Gaussian field with covariance kernel

$$(f_1, f_2) \mapsto \tau v^\xi(\tau) \int_{Q_1} f_1(x) f_2(x)\,\mathrm{d}x.$$

If the limiting variance in (2.11) *is strictly positive*, *if* (2.7) *is satisfied for some* $p > 2$ *and if* $q \in (2, 3]$ *with* $q < p$, *then for all* $\lambda \geq 2$ *and all* $f \in \mathcal{B}(Q_1)$

$$\sup_{M \in \mathbb{R}} \left| \mathbb{P}\left[\frac{\langle f, \bar{\mu}_\lambda^\xi \rangle}{\sqrt{\mathrm{Var}[\langle f, \bar{\mu}_\lambda^\xi \rangle]}} \leq M \right] - \mathbb{P}\left[N(0,1) \leq M \right] \right| \leq C (\log \lambda)^{qd} \lambda^{1-q/2}. \tag{2.14}$$

Functionals with bounded perturbations

Theorems 2.1–2.3 are confined to translation invariant functionals ξ, but they extend to asymptotically negligible bounded perturbations of translation-invariant functionals, described as follows. Assume that ξ is a translation invariant functional, exponentially stabilizing in the wide sense, and let $\hat{\xi}(\cdot, \cdot; \lambda)$, $\lambda \geq 1$, be the family of functionals

$$\hat{\xi}(x, \mathcal{X}; \lambda) = \xi(x, \mathcal{X}) + \delta(x, \mathcal{X}; \lambda), \quad \lambda \geq 1. \tag{2.15}$$

The correction (perturbation) $\delta(\cdot, \cdot; \lambda)$ is not necessarily translation invariant but, for $\Psi \in \boldsymbol{\Psi}^{\boldsymbol{*}}$, $(\tau, \beta) \in \mathcal{R}^\Psi$, and $p > 0$ it satisfies

$$\lim_{\lambda \to \infty} \sup_{u \in Q_\lambda} \mathbb{E} \left| \delta\left(u, \mathcal{P}^{\beta\Psi} \cap Q_\lambda; \lambda\right) \right|^p = 0 \tag{2.16}$$

and it also satisfies the wide sense exponential stabilization with the same stabilization radius R^ξ as ξ. If these conditions hold, we say that $\hat{\xi}$ is an asymptotically negligible bounded perturbation of ξ or simply a *bounded perturbation of* ξ. The asymptotic behavior of a bounded perturbation of a translation invariant functional coincides with that of the functional itself, as seen by the next theorem.

Theorem 2.4. *Let* $\Psi \in \boldsymbol{\Psi}^*$ *and let* $(\tau, \beta) \in \mathcal{R}^\Psi$. *Assume that* $\hat{\xi}$ *is a bounded perturbation of* ξ. *If* ξ *and* $\hat{\xi}$ *satisfy the same moment and stabilization conditions*, *as in Theorems* 2.1–2.3, *then as* $\lambda \to \infty$, *the respective asymptotic means*, *variances and limiting distributions of* $\langle f, \mu_\lambda^{\hat{\xi}} \rangle$ *coincide with those of* $\langle f, \mu_\lambda^\xi \rangle$, $f \in \mathcal{B}(Q_1)$.

Remarks.

(i) *Point processes with marks*. *Let* $(\mathcal{A}, \mathcal{F}_\mathcal{A}, \mu_\mathcal{A})$ *be a probability space* (*the mark space*) *and consider the marked Gibbs point process* $\tilde{\mathcal{P}}^{\beta\Psi} := \{(x, a) \colon x \in \mathcal{P}^{\beta\Psi}, a \in \mathcal{A}\}$ *in the space* $\mathbb{R}^d \times \mathcal{A}$ *with law given by the product measure of the law of* $\mathcal{P}^{\beta\Psi}$ *and* $\mu_\mathcal{A}$. *The proofs of Theorems* 2.1–2.3 *go through with* $\mathcal{P}^{\beta\Psi}$ *replaced by* $\tilde{\mathcal{P}}^{\beta\Psi}$.

(ii) *Comparison with* [9–11]. *The results of* [9] *establish limit theory for functionals* ξ *of weakly dependent Gibbsian input*, *but essentially these results require* ξ *to have a non-random radius of stabilization*. *Theorems* 2.1–2.4 *extend* [9] *to functionals* ξ *having random radius of stabilization and give closed form expressions for limiting means and variances*. *The assertions of Proposition* 2.1(*ii*) *could be deduced from* [9–11] *for finite range* $\Psi \in \boldsymbol{\Psi}^*$ *but not for infinite range* Ψ *as at* (1.9).

(iii) *Comparison with functionals on Poisson input*. *Theorems* 2.1–2.4 *show that the established limit theory for stabilizing functionals on Poisson input* [2,20–25] *is insensitive to weakly interacting Gibbsian modifications of the input*. *Thus weak laws of large numbers and central limit theorems for functionals on homogeneous Poisson input given previously in* [2,20–25] *extend to analogous results for functionals* ξ *on processes* $\mathcal{P}^{\beta\Psi}$ *whenever* $(\tau, \beta) \in \mathcal{R}^\Psi$. *To make this extension more transparent*, *notice that if the input* $\mathcal{P}^{\beta\Psi}$ *is Poisson*, *then by Proposition* 2.1(*i*) *with* $\Psi \equiv 0$, *the local energies* $\Delta^\Psi(x, \mathcal{P}^{\beta\Psi})$ *and* $\Delta^\Psi(\{x, y\}, \mathcal{P}^{\beta\Psi})$ *vanish*. *Hence Theorem* 2.1 *extends the Poisson weak law of large numbers given in Theorem* 2.1 *of* [24], *Theorem* 2.2 *extends the variance asympotics of* [2] *and* [20], *and Theorem* 2.3 *extends the central limit theory of* [2,20,25].

(iv) *Numerical evaluation of limits*. *When* $\Psi \in \boldsymbol{\Psi}^*$, *Section* 3 *shows that the point process* $\mathcal{P}^{\beta\Psi}$ *is intrinsically algorithmic*; *this algorithmic scheme provides an exact* (*perfect*) *sampler* [11]. *It is computationally efficient and yields a numerical evaluation of the limits* (2.9) *and* (2.11).

(v) *Extensions and generalizations*. *The low reference intensity and/or high inverse temperature requirements imposed in our results are restrictive but cannot be avoided because for general* $\Psi \in \boldsymbol{\Psi}^*$ *the processes* $\mathcal{P}^{\beta\Psi}$ *exhibit a phase transition outside these regimes and the central limit theorem does not hold there*. *On the other hand*, *variance asymptotics* (2.11) *and asymptotic normality* (2.13) *hold under weaker stabilization assumptions such as power-law* *stabilization* (*see Penrose* [20]), *but the additional technical details obscure the main ideas of our approach*, *and thus we have not tried for the weakest possible stabilization conditions*.

3. Exponential clustering of perfectly simulated Gibbs processes

We develop a variant of perfect simulation techniques originating in [9–11] to establish a spatial mixing property of the simulated process $\mathcal{P}_D^{\beta\Psi}$. Spatial mixing does not readily follow from standard simulation methods; see e.g. Chapter 11 of [19]. The mixing property only holds for a range of τ and β, but this restriction appears unavoidable. On the other hand, that the perfect simulation applies to potentials having infinite range, including pair potentials, is possibly of independent interest.

More precisely, our goals here are to use perfect simulation to (i) show that if $\Psi \in \boldsymbol{\Psi}^*$, then there is a regime of τ and β for which the point processes $\mathcal{P}_D^{\beta\Psi}$ at (1.5) extend to infinite volume point processes realized as spatial interacting birth and death processes, (ii) prove Proposition 2.1, and (iii) deduce that the measures $\sum_{x\in\mathcal{P}^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi})\delta_x$ are exponentially mixing whenever ξ is exponentially stabilizing.

3.1. *Potentials with nearly finite range*

When Ψ does not satisfy the finite range condition (1.8), the determination of the conditional intensity $\exp(-\Delta^{\Psi}(x, \mathcal{X}))$ in general requires knowledge of infinite configurations $\mathcal{X}$, rendering it difficult to use it to algorithmically construct $\mathcal{P}_D^{\beta\Psi}$, $D \subseteq \mathbb{R}^d$. However if Δ^{Ψ} is well approximated by a finite range local energy function on an exponential scale, as in Definition 3.1 below, then we may algorithmically construct $\mathcal{P}_D^{\beta\Psi}$ as well as its infinite volume version $\mathcal{P}^{\beta\Psi}$; see Sections 3.2 and 3.3, respectively. Algorithmic constructions facilitate showing exponential clustering, as seen in Section 3.4.

Fix Ψ and write $\Delta(\cdot,\cdot) := \Delta^{\Psi}(\cdot,\cdot)$ as at (1.7). Assume for all $r \in (0,\infty)$ that there are non-negative, translation invariant functions $\Delta^{[r]}(\cdot,\cdot)$ and $\Delta_{[r]}(\cdot,\cdot)$ such that for all finite $\mathcal{X} \subset \mathbb{R}^d$

$$\Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr) \le \Delta(\mathbf{0}, \mathcal{X}) \le \Delta^{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr). \tag{3.1}$$

Assume that $\Delta^{[r]}$ and $\Delta_{[r]}$ are respectively decreasing and increasing in r, that is for all locally finite $\mathcal{X} \subset \mathbb{R}^d$, and all $r' > r$ we have

$$\Delta_{[r']}\bigl(\mathbf{0}, \mathcal{X} \cap B_{r'}(\mathbf{0})\bigr) \ge \Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr), \qquad \Delta^{[r']}\bigl(\mathbf{0}, \mathcal{X} \cap B_{r'}(\mathbf{0})\bigr) \le \Delta^{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr). \tag{3.2}$$

We set by convention $\Delta^{[0]}(\cdot,\cdot) := \infty$ and $\Delta_{[0]}(\cdot,\cdot) := 0$.

Definition 3.1. *Let Ψ be a translation and rotation invariant potential satisfying* (1.6). *Given $\beta > 0$, we say that $\beta\Psi$ has nearly finite range* (*equivalently $\mathcal{P}_D^{\beta\Psi}$ has nearly finite range for any bounded open D*) *if there is a decreasing continuous function $\psi^{(\beta)} : \mathbb{R}^+ \to [0,1]$ such that $\psi^{(\beta)}(0) = 1$, $\psi^{(\beta)}(r)$ decays exponentially in r, and for all $r \in (0,\infty)$ and locally finite $\mathcal{X} \subset \mathbb{R}^d$ we have*

$$\exp\bigl(-\beta\Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr)\bigr) - \exp\bigl(-\beta\Delta^{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr)\bigr) \le \psi^{(\beta)}(r). \tag{3.3}$$

Conditions (3.1)–(3.3) show that the sequence $\exp(-\beta\Delta(\mathbf{0}, \mathcal{X} \cap B_r(x)))$, $r = 1, 2, \dots$ is Cauchy. Thus for locally finite $\mathcal{X}$, $\operatorname{card}(\mathcal{X}) = \infty$, we define $\exp(-\beta\Delta(\mathbf{0}, \mathcal{X})) := \lim_{r\to\infty} \exp(-\beta\Delta(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})))$. The local energy function on infinite sets $\mathcal{X}$ is thus given by

$$\Delta^{[\infty]}(\mathbf{0}, \mathcal{X}) := \Delta(\mathbf{0}, \mathcal{X}) := \lim_{r\to\infty} \Delta\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr), \tag{3.4}$$

justifying the terminology 'nearly finite range' and proving Proposition 2.1(i).

Poisson point processes have nearly finite range, since in this case $\Psi \equiv 0$ and $\Delta \equiv 0$. Also, if Ψ has finite range $r^{\Psi} \in (0,\infty)$, then $\beta\Psi$, $\beta > 0$, has nearly finite range. Indeed, for $r \in (0, r^{\Psi}]$, we put $\Delta^{[r]}(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})) := \sup_{r\le\rho\le r^{\Psi}} \Delta(\mathbf{0}, \mathcal{X} \cap B_\rho(\mathbf{0}))$ and $\Delta_{[r]}(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})) := \inf_{r\le\rho\le r^{\Psi}} \Delta(\mathbf{0}, \mathcal{X} \cap B_\rho(\mathbf{0}))$, whereas for $r \in (r^{\Psi}, \infty)$ we put

$\Delta^{[r]} = \Delta_{[r]} = \Delta$. With these choices for $\Delta^{[r]}$ and $\Delta_{[r]}$, we have that $\beta\Psi$, $\beta > 0$, has nearly finite range by putting $\psi^{(\beta)} : \mathbb{R}^+ \to [0, 1]$ to equal one on $[0, r^\Psi]$ and to be the linear function decreasing down to zero on $[r^\Psi, r^\Psi + 1]$ and zero thereafter.

Lemma 3.1. *The potentials $\beta\Psi$, $\Psi \in \boldsymbol{\Psi}^*$ and $\beta > 0$, have nearly finite range.*

Proof. (i) *Point processes with a pair potential function.* If the pair potential ϕ in (1.9) has support in $[0, r_0]$, then Δ^Ψ has finite range with r^Ψ set to r_0. On the other hand, suppose ϕ has infinite range, but satisfies (1.10). In this set-up $\phi(r) = \infty$ for $r \in (0, r_0)$ and so we only consider configurations $\mathcal{X}$ where the hard-core exclusion condition is satisfied. We assert that $\beta\Psi$ has nearly finite range for any $\beta > 0$. Indeed, letting $A(r, r_0)$, $r > 0$, be the collection of finite point configurations $\mathcal{A}$ in $\mathbb{R}^d \setminus B_r(\mathbf{0})$ such that any two points in $\mathcal{A}$ are at distance at least r_0, put

$$p(r) := \sup_{\mathcal{A} \in A(r, r_0)} \sum_{y \in \mathcal{A}} \phi\bigl(|y|\bigr), \quad r \in [r_0, \infty)$$

and $p(r) = 0$ for $r \in (0, r_0)$. By condition (1.10), we have that $p(r)$ decays exponentially fast to 0 as $r \to \infty$. Since the minimum interaction coming from points outside $B_r(\mathbf{0})$ is 0, we put for all $r \in (0, \infty)$

$$\Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr) := \sum_{y \in \mathcal{X} \cap B_r(\mathbf{0})} \phi\bigl(|y|\bigr)$$

and

$$\Delta^{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr) := \sup_{\mathcal{A} \in A(r, r_0)} \Delta\bigl(\mathbf{0}, \bigl(\mathcal{X} \cap B_r(\mathbf{0})\bigr) \cup \mathcal{A}\bigr).$$

Then $\Delta_{[r]}$ and $\Delta^{[r]}$ are translation invariant and are respectively increasing and decreasing in r. Since the maximum interaction coming from configurations in $\mathbb{R}^d \setminus B_r(\mathbf{0})$ is bounded by $p(r)$, we have

$$\Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr) \le \Delta(\mathbf{0}, \mathcal{X}) \le \Delta^{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr) \le p(r) + \Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr).$$

For all $\beta > 0$ we have

$$\begin{aligned}
&\exp\bigl(-\beta\Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr)\bigr) - \exp\bigl(-\beta\Delta^{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr)\bigr) \\
&\quad \le \exp\bigl(-\beta\Delta_{[r]}\bigl(\mathbf{0}, \mathcal{X} \cap B_r(\mathbf{0})\bigr)\bigr)\bigl(1 - \exp\bigl(-\beta p(r)\bigr)\bigr) \le \min\bigl(1, \beta p(r)\bigr),
\end{aligned}$$

since $1 - \exp(-u) \le u$ for $u \in [0, 1]$. Thus $\beta\Psi$ has nearly finite range with $\psi^{(\beta)}(r) := \min(1, \beta p(r))$.

(ii) *Point processes defined by the continuum Widom–Rowlinson model.* With Ψ as in (1.11), notice that Ψ is nearly of finite range on configurations $\mathcal{X} \cup \mathcal{Y}$ where $d(\mathcal{X}, \mathcal{Y}) \le 2a$, since for all $r \in (0, \infty)$ we have $\Delta_{[r]}(\mathbf{0}, (\mathcal{X} \cup \mathcal{Y}) \cap B_r(\mathbf{0})) = \Delta^{[r]}(\mathbf{0}, (\mathcal{X} \cup \mathcal{Y}) \cap B_r(\mathbf{0})) = 0$. On the remaining configurations we may put $\Delta_{[r]}(\mathbf{0}, (\mathcal{X} \cup \mathcal{Y}) \cap B_r(\mathbf{0})) = \min(\alpha_1, \alpha_2)$ and $\Delta^{[r]}(\mathbf{0}, (\mathcal{X} \cup \mathcal{Y}) \cap B_r(\mathbf{0})) = \max(\alpha_1, \alpha_2)$ provided x is distant at least $2a$ from both $\mathcal{X}$ and $\mathcal{Y}$, and otherwise $\Delta_{[r]}(\mathbf{0}, (\mathcal{X} \cup \mathcal{Y}) \cap B_r(\mathbf{0})) = \Delta^{[r]}(\mathbf{0}, (\mathcal{X} \cup \mathcal{Y}) \cap B_r(\mathbf{0})) = \infty$. Here $m_0^\Psi = \min(\alpha_1, \alpha_2)$ where m_0^Ψ is at (2.5).

(iii) *Area interaction point processes.* The set difference $\bigcup_{i=1}^{n+1}(x_i \oplus K) \Delta \bigcup_{i=1}^{n}(x_i \oplus K)$ is a function of x_{n+1}, $\mathrm{diam}(K)$, and those x_i, $i \le n$, for which $|x_i - x_{n+1}| < \mathrm{diam}(K)$. For all $\beta > 0$, it follows that $\beta\Psi$, with Ψ as in (1.12), has finite range $r^\Psi := \mathrm{diam}(K)$ and so has nearly finite range. Here $m_0^\Psi = \alpha_1$.

(iv) *Point processes given by the hard-core model.* With Ψ as in (1.13), it follows for all $\beta > 0$ that $\beta\Psi$ has finite range with r^Ψ set to $2r_0$ and thus has nearly finite range. Here $m_0^\Psi = \alpha_1$.

(v) *Truncated Poisson processes.* Let Ψ be as in (1.14). Then $\beta\Psi$ has finite range with r^Ψ set to r_0 and thus has nearly finite range. □

3.2. *Graphical construction of nearly finite range Gibbs processes*

For $\mathcal{P}_D^{\beta\Psi}$ given in law by (1.5), $\Psi \in \boldsymbol{\Psi}^*$, we algorithmically construct Gibbs point processes $\mathcal{P}^{\beta\Psi}$ on $\mathbb{R}^d$. The perfect simulation, in the spirit of Fernández, Ferrari and Garcia [9–11], is valid for (τ, β) belonging to a regime in $\mathbb{R}^+ \times \mathbb{R}^+$ depending on Ψ and it makes use of the nearly finite range property of Ψ. The construction goes as follows. Let $D \subset \mathbb{R}^d$ be an open bounded set and let $(\rho_D(t))_{t\in\mathbb{R}}$ be a stationary homogeneous *free birth and death process* in D with the following dynamics:

- A new point $x \in D$ is born in $\rho_D(t)$ during the time interval $[t - \mathrm{d}t, t]$ with probability $\tau\,\mathrm{d}x\,\mathrm{d}t$,
- An existing point $x \in \rho_D(t)$ dies during the time interval $[t - \mathrm{d}t, t]$ with probability $\mathrm{d}t$, that is the lifetimes of points of the process are independent standard exponential.

The unique stationary and reversible measure for this process is the law of the point process $\mathcal{P}_\tau \cap D$.

Next consider the following *trimming* procedure performed on $(\rho_D(t))_{t\in\mathbb{R}}$, paralleling the ideas developed in [9–11]. Trimming requires that an attempted birth in the free birth and death process *pass an additional stochastic test to determine if it is an actual birth*. This goes as follows.

Given a potential Ψ with nearly finite range and $\beta > 0$, we put $\psi := \psi^{(\beta)}$, as in Definition 3.1. For a birth site of a point $x \in D$ at some time $t \in \mathbb{R}$, draw a random natural number η from the geometric distribution with parameter $1/2$, that is to say $\mathbb{P}[\eta = k] = 2^{-k}$, $k = 1, 2, \ldots$. Let $r_0 := 0$ and put $r_k := \psi^{-1}(2^{-k})$, $k = 1, 2, \ldots$ where for all $v \in (0, 1]$ we have $\psi^{-1}(v) := \inf_{u\in\mathbb{R}^+}\{\psi(u) = v\}$. Letting $\gamma_D^{\beta\Psi}(t-) \cap B_r(x)$ denote the set of accepted points in $\rho_D(t-) \cap B_r(x)$, we accept x with probability

$$2^\eta\big[\exp\big(-\beta\Delta^{[r_\eta]}\big(x, \gamma_D^{\beta\Psi}(t-) \cap B_{r_\eta}(x)\big)\big) - \exp\big(-\beta\Delta^{[r_{\eta-1}]}\big(x, \gamma_D^{\beta\Psi}(t-) \cap B_{r_{\eta-1}}(x)\big)\big)\big] \tag{3.5}$$

and we reject x with the complementary probability, provided the acceptance/rejection statuses of all points in $\rho_D(t-) \cap B_{r_\eta}(x)$ are determined, otherwise proceed recursively to determine the statuses of points in $\rho_D(t-) \cap B_{r_\eta}(x)$. The acceptance probability at (3.5) does in fact belong to $[0, 1]$, as needed for the above procedure to be well defined. Indeed, whereas non-negativity is trivial by monotonicity, to see that (3.5) does not exceed 1 we add to it the non-negative number $2^\eta[\exp(-\beta\Delta_{[r_{\eta-1}]}(x, \gamma_D^{\beta\Psi}(t-) \cap B_{r_{\eta-1}}(x))) - \exp(-\beta\Delta_{[r_\eta]}(x, \gamma_D^{\beta\Psi}(t-) \cap B_{r_\eta}(x)))]$ and using (3.3) we upper bound the resulting sum by $2^\eta(\psi(r_{\eta-1}) - \psi(r_\eta)) = 1$, as required.

Before discussing properties of this recursive construction, we must first ensure that it actually terminates. The acceptance status of a point x at its birth time t only depends on the status of points in $\rho_D(t-) \cap B_{r_\eta}(x)$, that is to say depends on *accepted* births before time t, still alive at time t, and belonging to $B_{r_\eta}(x) \cap D$. We call these points *ancestors* of x. In general, given a subset $B \subseteq D$, a time $t_0 \in \mathbb{R}$, and $\beta > 0$, we let $\mathbf{A}_B^{\beta\Psi}(t_0) \subset \mathbb{R}^d$ denote the set of accepted births in $\rho_D(t_0) \cap B$ (where the acceptance probability is given by (3.5)), their ancestors, the ancestors of their ancestors and so forth throughout all past generations. The set $\mathbf{A}_B^{\beta\Psi}(t_0)$ is the *ancestor clan* of B with respect to the birth and death process $(\rho_D(t))_{t\in\mathbb{R}}$ and is a 'backwards in time oriented percolation cluster,' where two nodes in space-time are linked with a directed edge if one is the ancestor of another. In order that our recursive status determination procedure terminates for all points of $\rho_D(t)$ in B, it suffices that the ancestor clan $\mathbf{A}_B^{\beta\Psi}(t_0)$ is a.s. finite for all $t_0 \in \mathbb{R}$. This is easily checked to be a.s. the case for each $B \subseteq D$ – indeed, since D is bounded, a.s. there exists some $s \in (-\infty, t_0)$ such that $\rho_D(s) = \varnothing$ and thus no ancestor clan of a point alive at time t_0 can go past s backwards in time.

Having defined the trimming procedure above, we recursively remove from $\rho_D(t)$ the points rejected at their birth, and *we write* $(\gamma_D^{\beta\Psi}(t))_{t\in\mathbb{R}}$ *for the resulting process*. Clearly, $(\gamma_D^{\beta\Psi}(t))_{t\in\mathbb{R}}$ is stationary since $\rho_D(t)$ is stationary and the acceptance/rejection procedure is time-invariant as well. The trimmed process $(\gamma_D^{\beta\Psi}(t))_{t\in\mathbb{R}}$ evolves according to the following dynamics:

(D1) Add a new point x in the volume element $\mathrm{d}x$ with intensity $\tau\exp(-\beta\Delta(x, \gamma_D^{\beta\Psi}(t)))\,\mathrm{d}x\,\mathrm{d}t$,
(D2) Remove an existing point with intensity $\mathrm{d}t$.

Indeed, by (3.5) the acceptance probability of a birth attempt $x \in D$ is

$$\begin{aligned}
&\sum_{k=1}^\infty \frac{1}{2^k}\cdot 2^k\big[\exp\big(-\beta\Delta^{[r_k]}\big(x, \gamma_D^{\beta\Psi}(t-) \cap B_{r_k}(x)\big)\big) - \exp\big(-\beta\Delta^{[r_{k-1}]}\big(x, \gamma_D^{\beta\Psi}(t-) \cap B_{r_{k-1}}(x)\big)\big)\big] \\
&\quad = \exp\big(-\beta\Delta\big(x, \gamma_D^{\beta\Psi}(t-)\big)\big),
\end{aligned}$$

where equality follows by (3.4) as required. These are the standard Monte-Carlo dynamics for $\mathcal{P}_D^{\beta\Psi}$ as given at (1.5) and the law of $\mathcal{P}_D^{\beta\Psi}$ is its unique invariant distribution. Consequently, similarly to [9–11], the point process $\gamma_D^{\beta\Psi}(t)$ coincides in law with $\mathcal{P}_D^{\beta\Psi}$ for all $t \in \mathbb{R}$. In the next section we shall see that $\gamma_D^{\beta\Psi}(t)$ represents the perfect simulation of an *infinite volume* measure in the finite window D.

To this end we use the perfect simulation of $\mathcal{P}_D^{\beta\Psi}$ to deduce that its ancestor clans (backwards oriented percolation clusters) have an exponentially decaying spatial diameter. More precisely, if Ψ has nearly finite range, we establish regimes involving τ and β for which there exists a constant $C_2 := C_2(\tau, \beta)$ such that for all $t > 0$, $M > 0$, $D \subset \mathbb{R}^d$, and all $B \subset D$,

$$\mathbb{P}\bigl[\operatorname{diam}\bigl(\mathbf{A}_B^{\beta\Psi}(t)\bigr) \geq M + \operatorname{diam}(B)\bigr] \leq C_2 \operatorname{Vol}(B) \exp\left(-\frac{M}{C_2}\right). \tag{3.6}$$

Let $D \subset \mathbb{R}^d$ be open and bounded. Looking backwards in time t, by the dynamics $(D1)$, an individual in the *trimmed* process $(\gamma_D^{\beta\Psi}(t))_{t\in\mathbb{R}}$ observes ancestors with intensity at most $\tau \exp(-\beta m_0^\Psi)\, \mathrm{d}t(\mathbb{E}r_\eta^d)v_d$ where we recall that v_d is the volume of the unit ball in $\mathbb{R}^d$ and m_0^Ψ is at (2.5). Note that $\mathbb{E}r_\eta^d < \infty$ by the exponential decay of ψ. Since the same individual vanishes with intensity $\mathrm{d}t$, the number of ancestors in the trimmed process is dominated by a subcritical continuous time branching process as soon as the above intensity is bounded by $\mathrm{d}t$. Thus, if

$$\mathcal{R}^\Psi := \bigl\{(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ \colon u v_d \exp\bigl(-v m_0^\Psi\bigr)\bigl(\mathbb{E}r_\eta^d\bigr) < 1\bigr\}, \tag{3.7}$$

then for $(\tau, \beta) \in \mathcal{R}^\Psi$, branching process arguments originating in Hall [15] and used later in [9], show the exponential decay of the diameter of the ancestor clan arising from a single point, and thus (3.6) holds. Thus $(\tau, \beta) \in \mathcal{R}^\Psi$ whenever $\tau v_d \exp(-\beta m_0^\Psi)(\mathbb{E}r_\eta^d) < 1$. When Ψ has finite range r^Ψ, then as seen already, we may choose $\psi^{(\beta)}$ such that it vanishes on $[r^\Psi + 1, \infty)$. In this case, $r_\eta \leq r^\Psi + 1$ always holds, and so $(\tau, \beta) \in \mathcal{R}^\Psi$ whenever $\tau v_d \exp(-\beta m_0^\Psi)(r^\Psi + 1)^d < 1$.

3.3. *The infinite volume Gibbs process $\mathcal{P}^{\beta\Psi}$*

Put $D_n := [-n, n]^d$. When $(\tau, \beta) \in \mathcal{R}^\Psi$ and when Ψ has nearly finite range, we construct the infinite volume limit (thermodynamic limit) for $\mathcal{P}_{D_n}^{\beta\Psi}$ as $n \to \infty$, which goes as follows. Consider the infinite volume version $\rho(t) := \rho_{\mathbb{R}^d}(t)$ of the free birth and death process, with dynamics those of $\rho_D(t)$ with D replaced by $\mathbb{R}^d$. Then $\rho(t)$ coincides in law with $\mathcal{P}_\tau$ for each $t \in \mathbb{R}$. Since the constant C_2 in (3.6) does not depend on D, (3.6) shows that the ancestor clans $\mathbf{A}_B^{\beta\Psi}(t)$ are a.s. finite, uniformly in subsets $B \subset \mathbb{R}^d$. Thus the above trimming procedure is also valid for the infinite volume process $(\rho(t))_{t\in\mathbb{R}}$, yielding the stationary *trimmed* process $\gamma^{\beta\Psi}(t) := \gamma_{\mathbb{R}^d}^{\beta\Psi}(t)$, $t \in \mathbb{R}$.

For $(\tau, \beta) \in \mathcal{R}^\Psi$, perfect simulation thus implies that $\gamma_{\mathbb{R}^d}^{\beta\Psi}(t)$ satisfies (3.6) for all $B \subset \mathbb{R}^d$ and has the dynamics (D1) and (D2) with D set to $\mathbb{R}^d$. The following lemma summarizes the key properties of this limit process, proving part of Proposition 2.1(ii).

Lemma 3.2. *Let Ψ have nearly finite range and let $(\tau, \beta) \in \mathcal{R}^\Psi$. The thermodynamic limit of $\mathcal{P}_{D_n}^{\beta\Psi}$ as $n \to \infty$ is the Gibbs point process $\mathcal{P}^{\beta\Psi}$ which coincides in law with $\gamma^{\beta\Psi}(0)$ and hence with $\gamma^{\beta\Psi}(t)$ for all t. Also, $(\tau, \beta) \in \mathcal{R}^\Psi$ whenever $\tau v_d \exp(-\beta m_0^\Psi)(\mathbb{E}r_\eta^d) < 1$. When Ψ has finite range r^Ψ it suffices that (τ, β) satisfy $\tau v_d \exp(-\beta m_0^\Psi)(r^\Psi + 1)^d < 1$.*

The next result implies that if ξ is stabilizing in the wide sense, then ξ is stabilizing on $\mathcal{P}^{\beta\Psi}$, $\Psi \in \boldsymbol{\Psi}^*$.

Lemma 3.3. *Let $\Psi \in \boldsymbol{\Psi}^*$. For $(\tau, \beta) \in \mathcal{R}^\Psi$ the Gibbs process $\mathcal{P}^{\beta\Psi}$ is Poisson-like.*

Proof. Fix $(\tau, \beta) \in \mathcal{R}^\Psi$. The stochastic domination by $\mathcal{P}_\tau$ comes from the relation $\gamma^{\beta\Psi}(0) \subseteq \rho(0)$ in the graphical construction of $\mathcal{P}^{\beta\Psi}$, where $\rho(0)$ coincides in law with $\mathcal{P}_\tau$. We now show that $\mathcal{P}^{\beta\Psi}$ satisfies (2.1). By the graphical construction, it suffices to show there is $C_1 := C_1(\tau, \beta)$ and $r_1 := r_1(\tau, \beta)$ such that for all $r \in [r_1, \infty)$, $x \in \mathbb{R}^d$, and point sets $\mathcal{E}_r(x) \subset B_r^c(x)$, we have

$$\mathbb{P}\big[\gamma^{\beta\Psi}(0) \cap B_r(x) = \varnothing | \big\{\big(\gamma^{\beta\Psi}(0) \cap B_r(x)^c\big) = \mathcal{E}_r(x)\big\}\big] \le \exp\big(-C_1 r^d\big). \tag{3.8}$$

Non-degeneracy (defined in Section 1) and translation invariance of Ψ imply that $\Delta(x, \varnothing) < \infty$ *uniformly* in $x \in \mathbb{R}^d$. Since Ψ has nearly finite range, conditions (3.1)–(3.3) show there are functions $\Delta^{[\,]}(\cdot, \cdot)$ and $r_0 \in (0, \infty)$ such that $\sigma := \sup_{x \in \mathbb{R}^d} \Delta^{[r_0]}(x, \varnothing) < \infty$. Let $r \in [2r_0, \infty)$ and let $B_{r_0}(y_1), \dots, B_{r_0}(y_k)$ be disjoint balls of radius r_0 in $B_r(x)$, $k := k(r, r_0) = \Omega((r/r_0)^d)$.

Let $F := \{(\gamma^{\beta\Psi}(0) \cap B_r(x)^c) = \mathcal{E}_r(x)\}$. Define the events $E_i(r_0) := \{\gamma^{\beta\Psi}(t) \cap B_{r_0}(y_i) = \varnothing\}$, $i = 1, \dots, k$. For $i = 1, 2, \dots$, let $F_i := \bigcap_{j=0}^{i} E_j(r_0)$, and let F_0 be the common probability space on which all random variables are defined. For all $i = 1, \dots, k$, let p_{10}^i be the probability of $B_{r_0}(y_i)$ becoming empty, conditional on $F \cap F_{i-1}$. Then $p_{10}^i \le \mathrm{d}t$ since $B_{r_0}(y_i)$ becomes empty only when the last point of $\gamma^{\beta\Psi}(0)$ in $B_{r_0}(y_i)$ dies, which happens with intensity at most $\mathrm{d}t$. Let p_{01}^i be the probability that ball $B_{r_0}(y_i)$ gets filled, conditional on $F \cap F_{i-1}$. Then by definition of σ we have $p_{01}^i \ge \tau \exp(-\beta\sigma) \operatorname{Vol}(B_{r_0/2}(y_i))\,\mathrm{d}t$ regardless of the status of $\gamma^{\beta\Psi}(0)$ in the balls $B_{r_0}(y_j)$, $j \ne i$. Let π_0^i be the probability of ball $B_{r_0}(y_i)$ being empty, conditional on $F \cap F_{i-1}$, that is $\pi_0^i := \mathbb{P}[E_i(r_0) | F \cap F_{i-1}]$. Similarly, let π_1^i be the probability that $B_{r_0}(y_i)$ contains a point from $\gamma^{\beta\Psi}(0)$, conditional on $F \cap F_{i-1}$, so that $\pi_1^i \le \mathbb{P}[E_i^c(r_0) | F \cap F_{i-1}]$. Detailed balance for the reversible process $\gamma^{\beta\Psi}(t)$, conditional on $F \cap F_{i-1}$, gives $\pi_0^i p_{01}^i = \pi_1^i p_{10}^i$ for all $i = 1, \dots, k$, implying

$$\mathbb{P}\big[E_i(r_0) | F \cap F_{i-1}\big] \cdot \tau \exp(-\beta\sigma) \operatorname{Vol}\big(B_{r_0/2}(y_i)\big)\,\mathrm{d}t \le \mathbb{P}\big[E_i^c(r_0) | F \cap F_{i-1}\big]\,\mathrm{d}t. \tag{3.9}$$

Let $\alpha := \tau \exp(-\beta\sigma) \operatorname{Vol}(B_{r_0/2}(y_1))/(1 + \tau \exp(-\beta\sigma) \operatorname{Vol}(B_{r_0/2}(y_1)))$ and note that since σ decreases with increasing r_0, we have $\alpha = \Omega(r_0^d)$. The inequality (3.9) yields $\mathbb{P}[E_i^c(r_0) | F \cap F_{i-1}] \ge \alpha$. Thus we have

$$\mathbb{P}\big[E_i(r_0) | F \cap F_{i-1}\big] \le 1 - \alpha, \quad i = 1, \dots, k.$$

It follows that

$$\mathbb{P}[F_k \mid F] = \prod_{i=1}^{k} \frac{\mathbb{P}[F_i \cap F]}{\mathbb{P}[F_{i-1} \cap F]} = \prod_{i=1}^{k} \mathbb{P}\big[E_i(r_0) \mid F \cap F_{i-1}\big] \le (1 - \alpha)^k.$$

Since $k = \Omega((r/r_0)^d)$ and $\alpha = \Omega(r_0^d)$ we obtain $\mathbb{P}[F_k \mid F] = \exp(-\Omega(r^d))$. This proves (3.8). □

Perfect simulation reveals localization properties for $\mathcal{P}^{\beta\Psi}$ which might not otherwise be apparent. The next section further exploits perfect simulation to show mixing properties of $\mathcal{P}^{\beta\Psi}$.

3.4. *Exponential clustering of weighted Gibbs measures*

When (3.6) holds, the measures $\sum_{x \in \mathcal{P}^{\beta\Psi}} \delta_x$ are exponentially spatially mixing in the sense that the total variation distance between the restriction of these measures to disjoint convex sets decays exponentially with the distance between these sets. This is a consequence of a more general mixing property for the weighted measures $\sum_{x \in \mathcal{P}^{\beta\Psi}} \xi(x, \mathcal{P}^{\beta\Psi}) \delta_x$, which goes as follows. Let Ξ be a point process on $\mathbb{R}^d$ with law given with respect to $\mathcal{P}_\tau$ and let ξ be a functional. Put $\mu^{\xi,\Xi} := \sum_{x \in \Xi} \xi(x, \Xi) \delta_x$ and for $D \subset \mathbb{R}^d$, let $\mu^{\xi,\Xi}|_D$ be the restriction of the measure $\mu^{\xi,\Xi}$ to D. Let $\mu_1 \otimes \mu_2$ denote the product measure of μ_1 and μ_2.

Definition 3.2. *We say that $\mu^{\xi,\Xi}$ exponentially clusters if there exists $C_3 > 0$ such that for all $k \ge 2$ and all $x_1, \dots, x_k \in \mathbb{R}^d$*

$$P\Big[\mu^{\xi,\Xi}|_{\bigcup_{i=1}^k B_1(x_i)} \ne \bigotimes_{i=1}^{k} \big(\mu^{\xi,\Xi}|_{B_1(x_i)}\big)\Big] \le k C_3 \exp\bigg(-\frac{1}{C_3} \min_{1 \le i \ne j \le k} \operatorname{dist}(x_i, x_j)\bigg).$$

Thus, under exponential clustering, the restriction of the measure $\mu^{\xi,\Xi}$ to a union of k balls behaves like a k-fold product measure with a small error. Write $\mu^{\xi,\beta\Psi}$ for $\mu^{\xi,\mathcal{P}^{\beta\Psi}}$.

Lemma 3.4. *Let $\Psi \in \boldsymbol{\Psi}^*$. If $(\tau,\beta) \in \mathcal{R}^\Psi$ and if ξ is exponentially stabilizing then $\mu^{\xi,\beta\Psi}$ exponentially clusters.*

Proof. For $y \in \mathbb{R}^d$, let $R_y := R^\xi(y, \mathcal{P}^{\beta\Psi})$ and let $E_y := \bigcup_{x\in\mathcal{P}^{\beta\Psi}\cap B_1(y)} B_{R_x}(x)$ be the 'stabilization region' for y. Let $(\tau,\beta) \in \mathcal{R}^\Psi$ and $x_1,\dots,x_k \in \mathbb{R}^d$. Put $E_i := E_{x_i}$, $1 \le i \le k$. Put $r_0 := \frac{1}{2}\min_{i\ne j} \operatorname{dist}(x_i,x_j)$ and assume without loss of generality that $r_0 > 1$. Recalling the notation of Section 3.2, let $\mathbf{A}_i := \mathbf{A}^{\beta\Psi}_{E_i}(0)$ denote the ancestor clan of the stabilization region for x_i at time $t=0$. Recalling that $\mathcal{P}^{\beta\Psi}$ coincides in law with $\gamma^{\beta\Psi}(0)$, the event $\mu^{\xi,\beta\Psi}|_{\bigcup_{i=1}^k B_1(x_i)} \ne \bigotimes_{i=1}^k (\mu^{\xi,\beta\Psi}|_{B_1(x_i)})$ is a subset of the event that at least one of the ancestor clans $\mathbf{A}_i$, $1 \le i \le k$, is not contained in the respective ball $B_{r_0}(x_i)$. Indeed, if all clans $\mathbf{A}_i$, $1 \le i \le k$, were contained in their respective balls $B_{r_0}(x_i)$, then the scores over points in $\mathcal{P}^{\beta\Psi} \cap B_1(x_i)$ would depend on disjoint and hence independent portions of the free birth-and-death process in the graphical construction. To complete the proof it suffices to show that $\mathbb{P}[\mathbf{A}_i \not\subseteq B_{r_0}(x_i)]$, $1 \le i \le k$, decays exponentially with r_0. This goes as follows.

By the wide sense exponential stabilization of ξ with respect to the Poisson-like process $\mathcal{P}^{\beta\Psi}$, there exist positive constants ε_0, s_0 and C_4 such that for all $\varepsilon \in (0,\varepsilon_0)$, $s \in (s_0,\infty)$ we have

$$\sup_{y\in\mathbb{R}^d} P\Big[\sup_{x\in B_\varepsilon(y)\cap\mathcal{P}^{\beta\Psi}} R^\xi\big(x,\mathcal{P}^{\beta\Psi}\big) > s\Big] \le \exp(-C_4 s).$$

Consequently, for each $1 \le i \le k$, the diameter of the union $\bigcup_{x\in\mathcal{P}^{\beta\Psi}\cap B_1(x_i)} B_{R_x}(x)$ of such balls also has exponentially decaying tails. Indeed, choose ε_0, s_0, and C_4 as above and cover $B_1(x_i)$ by $m = \mathrm{O}(\varepsilon_0^{-d})$ balls of radius ε_0 to conclude from the union bound that for all $M \in (s_0,\infty)$

$$\mathbb{P}\Big[\sup_{x\in\mathcal{P}^{\beta\Psi}\cap B_1(x_i)} R_x > M\Big] \le m\exp(-C_4 M). \tag{3.10}$$

Finally, note that

$$\begin{aligned}\mathbb{P}\big[\mathbf{A}_i \not\subseteq B_{r_0}(x_i)\big] &\le \mathbb{P}\big[\operatorname{diam}(\mathbf{A}_i) \ge 2r_0\big]\\ &\le \mathbb{P}\big[\operatorname{diam}(\mathbf{A}_i) \ge r_0 + \operatorname{diam}(E_i), \operatorname{diam}(E_i) \le r_0\big] + \mathbb{P}\big[\operatorname{diam}(E_i) \ge r_0\big].\end{aligned}$$

The exponential decay (3.6) implies that the first probability on the right-hand side decays exponentially fast with r_0. Since $\{\operatorname{diam}(E_i) \ge r_0\} \subset \{\sup_{x\in\mathcal{P}^{\beta\Psi}\cap B_1(x_i)} R_x > r_0\}$, the bound (3.10) implies that the second probability decays exponentially fast with r_0. Thus $\mathbb{P}[\mathbf{A}_i \not\subseteq B_{r_0}(x_i)]$ decays exponentially with r_0 uniformly for $i = 1,2,\dots,k$. This completes the proof of Lemma 3.4. □

Lemma 3.5. *Let $\Psi \in \boldsymbol{\Psi}^*$. If $(\tau,\beta) \in \mathbb{R}^\Psi$ and if ξ is exponentially stabilizing then there is a constant C_5 such that for all x and $z \in \mathbb{R}^d$, $\lambda \ge 1$, the absolute value of the difference of*

$$\mathbb{E}\xi\big(x,\big(\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\cup\{x+z\}\big)\xi\big(x+z,\big(\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\cup\{x\}\big)\exp\big(-\Delta\big(\{x,x+z\},\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\big)$$

and

$$\mathbb{E}\xi\big(x,\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\exp\big(-\Delta\big(x,\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\big)\mathbb{E}\xi\big(x+z,\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\exp\big(-\Delta\big(\{x+z\},\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\big)$$

is bounded by $C_5\exp(-|z|/C_5)$.

Proof. For $x \in \mathbb{R}^d$, let $D_x := \operatorname{diam}(\mathbf{A}^{\beta\Psi}_{E_x}(0))$, where E_x is the stabilization region for x, as given in the proof of Lemma 3.4. That lemma and its proof show that on the event $E := \{\max(D_x, D_{x+z}) \le |z|/2\}$, the two scores

$\xi(x, ((\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \cup \{x+z\}) \cap B_{D_x}(x))$ and $\xi(x+z, ((\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \cup \{x\}) \cap B_{D_{x+z}}(x+z))$ are independent. Likewise, on E, the point $x+z$ is not in the ancestor clan for x and so on E we have $\Delta(x, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \cup \{x+z\}) = \Delta(x, \mathcal{P}^{\beta\Psi} \cap Q_\lambda)$. On E we thus have $\exp(-\beta\Delta(\{x, x+z\}, \mathcal{P}^{\beta\Psi} \cap Q_\lambda)) = \exp(-\beta\Delta(x, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \cup \{x+z\})) \cdot \exp(-\beta\Delta(\{x+z\}, \mathcal{P}^{\beta\Psi} \cap Q_\lambda))$ is the product of independent random variables $\exp(-\beta\Delta(x, \mathcal{P}^{\beta\Psi} \cap Q_\lambda))$ and $\exp(-\beta\Delta(\{x+z\}, \mathcal{P}^{\beta\Psi} \cap Q_\lambda))$. The moment condition (2.7) and Hölder's inequality combine to give the result; see Lemma 4.1 in [2] and Lemma 4.2 in [20] for details. $\square$

4. Proof of main results

Throughout $\Psi \in \boldsymbol{\Psi}^*$ is a fixed potential, $(\tau, \beta) \in \mathcal{R}^\Psi$ is fixed, and recalling (2.6), we write μ_λ^ξ for $\mu_\lambda^{\xi,\beta\Psi}$.

Proof of Theorem 2.1. We first show (2.9). We have for $f \in \mathcal{B}(Q_1)$

$$\lambda^{-1}\mathbb{E}\big[\langle f, \mu_\lambda^\xi\rangle\big] = \lambda^{-1}\mathbb{E}\sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} f\big(\lambda^{-1/d}u\big)\xi\big(u, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \setminus u\big).$$

Given $\mathcal{P}^{\beta\Psi} \cap D$ in $\mathbb{R}^d \setminus \mathrm{d}u$, the conditional probability of observing an extra point of $\mathcal{P}^{\beta\Psi} \cap D$ in the volume element $\mathrm{d}u$, given that configuration without that point, equals $\tau\exp(-\beta\Delta(u, \mathcal{P}^{\beta\Psi} \cap D))\,\mathrm{d}u$ as determined by the dynamics (D1) of the construction of $\mathcal{P}^{\beta\Psi}$. Here $\tau\,\mathrm{d}u$ corresponds to the birth attempt intensity at u whereas $\exp(-\beta\Delta(u, \mathcal{P}^{\beta\Psi} \cap D))$ comes from the acceptance probability. By the integral characterization of Gibbs point processes, as in Chapter 6.4 of [19], it follows from the Georgii–Nguyen–Zessin formula that

$$\lambda^{-1}\mathbb{E}\big[\langle f, \mu_\lambda^\xi\rangle\big] = \lambda^{-1}\tau\int_{Q_\lambda} f\big(\lambda^{-1/d}u\big)\mathbb{E}\xi\big(u, \mathcal{P}^{\beta\Psi} \cap Q_\lambda\big)\exp\big(-\beta\Delta\big(u, \mathcal{P}^{\beta\Psi} \cap Q_\lambda\big)\big)\,\mathrm{d}u = \tau\int_{Q_1} f(x)c_\lambda^\xi(x)\,\mathrm{d}x,$$

where

$$c_\lambda^\xi(x) := \mathbb{E}\xi\big(\lambda^{1/d}x, \mathcal{P}^{\beta\Psi} \cap Q_\lambda\big)\exp\big(-\beta\Delta\big(\lambda^{1/d}x, \mathcal{P}^{\beta\Psi} \cap Q_\lambda\big)\big), \quad x \in Q_1. \tag{4.1}$$

By translation invariance of ξ and stationarity of $\mathcal{P}^{\beta\Psi}$ we have

$$c_\lambda^\xi(x) = \mathbb{E}\xi\big(\mathbf{0}, \mathcal{P}^{\beta\Psi} \cap \big(-\lambda^{1/d}x + Q_\lambda\big)\big)\exp\big(-\beta\Delta^\Psi\big(\mathbf{0}, \mathcal{P}^{\beta\Psi} \cap \big(-\lambda^{1/d}x + Q_\lambda\big)\big)\big).$$

For $x \in Q_1 \setminus \partial Q_1$, we have by (2.3) that $\lim_{\lambda\to\infty}\xi(\mathbf{0}, \mathcal{P}^{\beta\Psi} \cap (-\lambda^{1/d}x + Q_\lambda)) = \xi(\mathbf{0}, \mathcal{P}^{\beta\Psi})$ a.s. whereas Proposition 2.1(i) gives that $\lim_{\lambda\to\infty}\exp(-\beta\Delta^\Psi(\mathbf{0}, \mathcal{P}^{\beta\Psi} \cap (-\lambda^{1/d}x + Q_\lambda))) = \exp(-\beta\Delta^\Psi(\mathbf{0}, \mathcal{P}^{\beta\Psi}))$ always holds. By the moment assumption we get that $c_\lambda^\xi(x) \to c^\xi(\mathbf{0})$, and dominated convergence yields $\lambda^{-1}\mathbb{E}[\langle f, \mu_\lambda^\xi\rangle] \to \tau c^\xi(\mathbf{0})\int_{Q_1} f(x)\,\mathrm{d}x$, which gives (2.9). To get L^2 convergence when ξ satisfies the moment condition (2.7) for some $p > 2$, one can follow the approach of Penrose [20]. $\square$

Proof of Theorem 2.2. Using again the integral characterization of Gibbs point processes we have

$$\lambda^{-1}\operatorname{Var}\big[\langle f, \mu_\lambda^\xi\rangle\big] = \tau\int_{Q_1} f(x)^2 c_\lambda^{\xi^2}(x)\,\mathrm{d}x + \tau^2\lambda\int_{Q_1}\int_{Q_1}\big[c_\lambda^\xi(x,y) - c_\lambda^\xi(x)c_\lambda^\xi(y)\big]f(y)f(x)\,\mathrm{d}y\,\mathrm{d}x,$$

where $c_\lambda^{\xi^2}$ is as in (4.1) and where for $x, y \in Q_1$

$$\begin{aligned} c_\lambda^\xi(x,y) := {}& \mathbb{E}\xi\big(\lambda^{1/d}x, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \cup \big\{\lambda^{1/d}y\big\}\big)\xi\big(\lambda^{1/d}y, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \cup \big\{\lambda^{1/d}x\big\}\big) \\ & \times \exp\big(-\beta\Delta\big(\big\{\lambda^{1/d}x, \lambda^{1/d}y\big\}, \mathcal{P}^{\beta\Psi} \cap Q_\lambda\big)\big). \end{aligned}$$

Next, put $y = x + \lambda^{-1/d} z$, where z ranges over $-\lambda^{1/d} x + Q_\lambda$ and $\mathrm{d}y = \lambda^{-1}\,\mathrm{d}z$. This gives

$$\begin{aligned}\lambda^{-1}\operatorname{Var}\big[\langle f, \mu_\lambda^\xi\rangle\big] = \tau \int_{Q_1} f(x)^2 c_\lambda^{\xi^2}(x)\,\mathrm{d}x \\ + \tau^2 \int_{Q_1}\int_{-\lambda^{1/d}x+Q_\lambda} \big[c_\lambda^\xi\big(x, x+\lambda^{-1/d}z\big) \\ - c_\lambda^\xi(x) c_\lambda^\xi\big(x+\lambda^{-1/d}z\big)\big] f\big(x+\lambda^{-1/d}z\big) f(x)\,\mathrm{d}z\,\mathrm{d}x.\end{aligned} \tag{4.2}$$

We have for $x \in Q_1$ and $z \in -\lambda^{1/d}x + Q_\lambda$

$$\begin{aligned}c_\lambda^\xi\big(x, x+\lambda^{-1/d}z\big) = \mathbb{E}\xi\big(\lambda^{1/d}x, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \cup \big\{\lambda^{1/d}x + z\big\}\big)\xi\big(\lambda^{1/d}x + z, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \cup \big\{\lambda^{1/d}x\big\}\big) \\ \times \exp\big(-\beta\Delta\big(\big\{\lambda^{1/d}x, \lambda^{1/d}x + z\big\}, \mathcal{P}^{\beta\Psi} \cap Q_\lambda\big)\big).\end{aligned}$$

By translation invariance of ξ we obtain

$$\begin{aligned}c_\lambda^\xi\big(x, x+\lambda^{-1/d}z\big) = \mathbb{E}\xi\big(\mathbf{0}, \big(\mathcal{P}^{\beta\Psi} \cap \big(Q_\lambda - \lambda^{1/d}x\big)\big) \cup \{z\}\big)\xi\big(z, \big(\mathcal{P}^{\beta\Psi} \cap \big(Q_\lambda - \lambda^{1/d}x\big)\big) \cup \{\mathbf{0}\}\big) \\ \times \exp\big(-\beta\Delta\big(\{\mathbf{0}, z\}, \mathcal{P}^{\beta\Psi} \cap \big(Q_\lambda - \lambda^{1/d}x\big)\big)\big).\end{aligned}$$

By the convergence (2.3) and Proposition 2.1(i), when $x \in Q_1 \setminus \partial Q_1$ we get $\lim_{\lambda\to\infty} c_\lambda^\xi(x, x+\lambda^{-1/d}z) = c^\xi(\mathbf{0}, z)$ where $c^\xi(\cdot,\cdot)$ is at (2.10). Likewise, as in the proof of Theorem 2.1, for all $z \in \mathbb{R}^d$, we have $\lim_{\lambda\to\infty} c_\lambda^\xi(x+\lambda^{-1/d}z) = c^\xi(z)$ and in particular we have $\lim_{\lambda\to\infty} c_\lambda^\xi(x) = c^\xi(\mathbf{0})$. By Lemma 3.5, we have that $[c_\lambda^\xi(x, x+\lambda^{-1/d}z) - c_\lambda^\xi(x)c_\lambda^\xi(x+\lambda^{-1/d}z)]$ is dominated by an integrable function of z uniformly in x and λ. When f is continuous, it follows by dominated convergence that the double integral in (4.2) converges to

$$\begin{aligned}\lim_{\lambda\to\infty} \tau^2 \int_{Q_1}\int_{-\lambda^{1/d}x+Q_\lambda} \big[c_\lambda^\xi\big(x, x+\lambda^{-1/d}z\big) - c_\lambda^\xi(x)c_\lambda^\xi\big(x+\lambda^{-1/d}z\big)\big] f\big(x+\lambda^{-1/d}z\big) f(x)\,\mathrm{d}z\,\mathrm{d}x \\ = \tau^2 \int_{Q_1}\int_{\mathbb{R}^d} \big[c^\xi(\mathbf{0}, z) - c^\xi(\mathbf{0})c^\xi(z)\big] f(x)^2\,\mathrm{d}z\,\mathrm{d}x.\end{aligned} \tag{4.3}$$

As in the proof of Theorem 2.1, the first integral in (4.2) converges to $\tau c^{\xi^2}(\mathbf{0}) \int_{Q_1} f(x)^2\,\mathrm{d}x$, completing the proof of Theorem 2.2 when f is continuous.

More generally, for $f \in \mathcal{B}(Q_1)$, we may follow verbatim the arguments in the proof of Theorem 2.1 of Penrose [20], which we include for completeness. For $x, z \in \mathbb{R}^d$, put

$$g_\lambda^\xi(x, z) := c_\lambda^\xi\big(x, x+\lambda^{-1/d}z\big) - c_\lambda^\xi(x)c_\lambda^\xi\big(x+\lambda^{-1/d}z\big), \qquad g_\infty^\xi(\mathbf{0}, z) := c^\xi(\mathbf{0}, z) - c^\xi(\mathbf{0})c^\xi(z).$$

If $x \in Q_1$ is a Lebesgue point of f then for any $M > 0$ we have

$$\lim_{\lambda\to\infty} \int_{B_M(x)} g_\lambda^\xi(x, z)\big(f\big(x+\lambda^{-1/d}z\big) - f(x)\big)\,\mathrm{d}z = 0,$$

since by Lemma 3.5, $g_\lambda^\xi(x, z)$ is bounded uniformly in λ, x, z. Combining this limit with $\lim_{\lambda\to\infty} g_\lambda^\xi(x, z) = g_\infty^\xi(\mathbf{0}, z)$, the dominated convergence theorem gives

$$\lim_{\lambda\to\infty} \int_{B_M(x)} g_\lambda^\xi(x, z) f\big(x+\lambda^{-1/d}z\big)\,\mathrm{d}z = \int_{B_M(x)} g_\infty^\xi(\mathbf{0}, z) f(x)\,\mathrm{d}z. \tag{4.4}$$

By Lemma 3.5, $g_\lambda^\xi(x, z)$ decays to zero exponentially fast in $|z|$, from which it follows by boundedness of f that

$$\lim_{M\to\infty}\lim_{\lambda\to\infty} \int_{\mathbb{R}^d \setminus B_M(x)} \big|g_\lambda^\xi(x, z) f\big(x+\lambda^{-1/d}z\big) - g_\infty^\xi(x, z) f(x)\big|\,\mathrm{d}z = 0. \tag{4.5}$$

By the Lebesgue density theorem, almost every $x \in Q_1$ is a Lebesgue point of f. Thus (4.4) and (4.5) give for almost every $x \in Q_1$ that

$$\lim_{\lambda\to\infty} \int_{-\lambda^{1/d}x+Q_\lambda} g_\lambda^\xi(x,z) f\big(x+\lambda^{-1/d}z\big)\,\mathrm{d}z = \int_{\mathbb{R}^d} g_\infty^\xi(x,z) f(x)\,\mathrm{d}z.$$

By dominated convergence again, (4.3) holds for $f \in \mathcal{B}(Q_1)$, completing the proof of Theorem 2.2. □

Proof of Theorem 2.3. When f is continuous on Q_1 and when ξ satisfies the moment condition (2.7) for all $p \in (0,\infty)$, we may nearly verbatim follow the cumulant methods of Section 5 of [2] (these methods are clarified and further developed in Section 3 of [31], which provides the correct centering of the associated moment measures). The exponential clustering Lemma 3.4 replaces the clustering Lemma 5.2 of [2] and shows that all cumulants of $\langle f, \bar{\mu}_\lambda^\xi\rangle$ are of order λ. Hence, upon the $\lambda^{-k/2}$-re-scaling with k being the order of the cumulant, the cumulants of order higher than two vanish asymptotically and thus yield the required Gaussian limit; see [2,31] for details.

More generally, when $f \in \mathcal{B}(Q_1)$ and when ξ satisfies the moment condition (2.7) for some $p > 2$, the rate (2.14) holds by following *verbatim* the approach of [25], which is based on Stein's method. Indeed, Lemma 3.4 establishes the independence of μ_λ^ξ over distant discretized sub-cubes on a high probability event, as required by the dependency graph arguments of [25]. Combining (2.11) and (2.14) yields (2.13) for $f \in \mathcal{B}(Q_1)$. This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. For all $f \in \mathcal{B}(Q_1)$ we claim that

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\langle f, \mu_\lambda^{\hat{\xi}}\rangle &= \lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\langle f, \mu_\lambda^{\xi}\rangle + \lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\sum_{u\in\mathcal{P}^{\beta\Psi}\cap Q_\lambda} f_\lambda(u)\delta\big(u,\big(\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\setminus u;\lambda\big)\\ &= \lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\langle f, \mu_\lambda^{\xi}\rangle. \end{aligned} \tag{4.6}$$

Indeed, by (2.16) with $p=1$ the penultimate limit in (4.6) is $\mathrm{O}(\sup_{u\in Q_\lambda}\mathbb{E}|\delta(u,\mathcal{P}^{\beta\Psi}\cap Q_\lambda;\lambda)|) = \mathrm{o}(1)$. This shows that the mean asymptotics of Theorem 2.1 are unchanged if ξ is replaced by $\hat{\xi}$.

Next, when δ satisfies the moment condition (2.16) with $p=2$, we assert that for all $f \in \mathcal{B}(Q_1)$

$$\lim_{\lambda\to\infty} \lambda^{-1}\operatorname{Var}\Bigg[\sum_{u\in\mathcal{P}^{\beta\Psi}\cap Q_\lambda} f\big(\lambda^{-1/d}u\big)\delta\big(u,\big(\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\setminus u;\lambda\big)\Bigg] = 0. \tag{4.7}$$

This assertion is enough to conclude the proof. Indeed, with $f \in \mathcal{B}(Q_1)$ fixed, let

$$H_\lambda^\xi := \lambda^{-1/2}\sum_{u\in\mathcal{P}^{\beta\Psi}\cap Q_\lambda} f\big(\lambda^{-1/d}u\big)\xi\big(u,\big(\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\setminus u\big),$$

$$H_\lambda^\delta := \lambda^{-1/2}\sum_{u\in\mathcal{P}^{\beta\Psi}\cap Q_\lambda} f\big(\lambda^{-1/d}u\big)\delta\big(u,\big(\mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\setminus u;\lambda\big).$$

Now $\operatorname{Cov}(H_\lambda^\xi, H_\lambda^\delta) \le (\operatorname{Var}[H_\lambda^\xi]\operatorname{Var}[H_\lambda^\delta])^{1/2}$ and it follows by (2.11) and (4.7) that $\operatorname{Cov}(H_\lambda^\xi, H_\lambda^\delta) \to 0$ as $\lambda\to\infty$. Thus since $\operatorname{Var}[H_\lambda^\xi + H_\lambda^\delta] = \operatorname{Var}[H_\lambda^\xi] + \operatorname{Var}[H_\lambda^\delta] + 2\operatorname{Cov}(H_\lambda^\xi, H_\lambda^\delta)$ we get $\lim_{\lambda\to\infty}\operatorname{Var}[H_\lambda^\xi + H_\lambda^\delta] = \tau V^\xi(\tau)\int_{Q_1} f(x)^2\,\mathrm{d}x$, that is Theorem 2.2 is unchanged if ξ is replaced by $\hat{\xi}$. Also, (4.6) and (4.7) show that $H_\lambda^\delta \xrightarrow{\mathcal{D}} 0$ and thus Theorem 2.3 is likewise unchanged if ξ is replaced by $\hat{\xi}$.

To show the asserted limit (4.7) we argue as follows. Put for all $x, y \in Q_1$

$$c_\lambda^\delta(x) := \mathbb{E}\big[\delta\big(\lambda^{1/d}x, \mathcal{P}^{\beta\Psi}\cap Q_\lambda;\lambda\big)\exp\big(-\beta\Delta\big(\big\{\lambda^{1/d}x\big\}, \mathcal{P}^{\beta\Psi}\cap Q_\lambda\big)\big)\big]$$

and

$$c_\lambda^\delta(x,y) := \mathbb{E}\big[\delta\big(\lambda^{1/d}x, \big(\mathcal{P}^{\beta\Psi} \cup \big\{\lambda^{1/d}y\big\}\big) \cap Q_\lambda; \lambda\big)\delta\big(\lambda^{1/d}y, \big(\mathcal{P}^{\beta\Psi} \cup \big\{\lambda^{1/d}x\big\}\big) \cap Q_\lambda; \lambda\big)$$
$$\times \exp\big(-\beta\Delta\big(\big\{\lambda^{1/d}x, \lambda^{1/d}y\big\}, \mathcal{P}^{\beta\Psi} \cap Q_\lambda\big)\big)\big].$$

As in the proof of Theorem 2.2 we have

$$\operatorname{Var}\big[H_\lambda^\delta\big] = \tau \int_{Q_1} f(x)^2 c_\lambda^{\delta^2}(x)\,\mathrm{d}x + \tau^2\lambda \int_{Q_1}\int_{Q_1} \big[c_\lambda^\delta(x,y) - c_\lambda^\delta(x)c_\lambda^\delta(y)\big] f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Putting $y = x + \lambda^{-1/d}z$ gives

$$\begin{aligned}\operatorname{Var}\big[H_\lambda^\delta\big] &= \tau \int_{Q_1} f(x)^2 c_\lambda^{\delta^2}(x)\,\mathrm{d}x \\ &\quad + \tau^2 \int_{Q_1}\int_{-\lambda^{1/d}x+Q_\lambda} \big[c_\lambda^\delta\big(x, x+\lambda^{-1/d}z\big) - c_\lambda^\delta(x)c_\lambda^\delta\big(x+\lambda^{-1/d}z\big)\big] f\big(x+\lambda^{-1/d}z\big) f(x)\,\mathrm{d}z\,\mathrm{d}x. \end{aligned} \tag{4.8}$$

As $\lambda \to \infty$, the first integral in (4.8) goes to zero by (2.16) with $p = 2$ there. The assumed exponential stabilization of δ and Lemma 3.5 give that $[c_\lambda^\delta(x, x+\lambda^{-1/d}z) - c_\lambda^\delta(x)c_\lambda^\delta(x+\lambda^{-1/d}z)]$ is dominated by an integrable function of z. By (2.16) again with $p = 2$ and Cauchy–Schwarz, we get for all $x \in Q_1$ that $\lim_{\lambda\to\infty}[c_\lambda^\delta(x, x+\lambda^{-1/d}z) - c_\lambda^\delta(x)c_\lambda^\delta(x+\lambda^{-1/d}z)] = 0$ uniformly in $z \in -\lambda^{1/d}x + Q_\lambda$. Thus for all $x \in Q_1$ the dominated convergence theorem yields

$$\lim_{\lambda\to\infty}\int_{-\lambda^{1/d}x+Q_\lambda} \big[c_\lambda^\delta\big(x, x+\lambda^{-1/d}z\big) - c_\lambda^\delta(x)c_\lambda^\delta\big(x+\lambda^{-1/d}z\big)\big] f\big(x+\lambda^{-1/d}z\big) f(x)\,\mathrm{d}z = 0$$

and the bounded convergence theorem gives that the second integral in (4.8) goes to zero as $\lambda \to \infty$. This concludes the proof of Theorem 2.4. □

5. Applications

5.1. *RSA packing and spatial birth growth models with Gibbsian input*

Let $\mathcal{X} \subset \mathbb{R}^d$ be locally finite. Elements $x \in \mathcal{X}$ are assigned i.i.d. *time marks* τ_x, independent of $\mathcal{X}$ and distributed uniformly in $[0,1]$. Consider a sequence of unit volume d-dimensional Euclidean balls $B_1, B_2, \ldots$ with centers arriving sequentially at points $x \in \mathcal{X}$ and at arrival times τ_x. The first ball B_1 to arrive is *packed* and recursively, for $i = 2, 3, \ldots$ let the ith ball be packed if it does not overlap any ball in $B_1, B_2, \ldots, B_{i-1}$ which has already been packed. Define the *packing functional* $\xi(x, \mathcal{X})$ to be either 0 or 1, depending on whether the ball arriving at x is either packed or discarded.

When $\mathcal{X}$ is the realization of a Poisson point process on Q_λ, this packing process is known as random sequential adsorption (RSA) with Poisson input on Q_λ [8]. It is also possible to let the number of points falling into Q_λ tend to ∞, which gives rise to the RSA process with infinite input; in such cases, RSA packing terminates when it is no longer possible to pack additional balls. In dimension $d = 1$, this process is known as the Rényi car parking problem [26]. In the infinite input setting and when $d = 1$ Rényi [26] (respectively Dvoretzky and Robbins [7]) proved that the total number of parked balls satisfies a weak law of large numbers (respectively central limit theorem) as $\lambda \to \infty$; these results were shown to hold for all dimensions in [20] and [28].

Limit results for RSA packing generally assume that the input is either a Poisson or binomial point process. To the best of our knowledge, RSA packing problems with Gibbsian input $\mathcal{P}^{\beta\Psi}$ have not been considered, though it is natural to consider packing models with input satisfying some intrinsic repulsivity, as in the Widom–Rowlinson or hard core model. The following theorem widens the scope of the existing limit results for RSA packing. Assign to the points of $\mathcal{P}^{\beta\Psi}$ i.i.d. marks in $[0,1]$, thus putting us in the set-up of Remark (i) following Theorem 2.4. Given $\mathcal{P}^{\beta\Psi}$, define the packing measure $\mu_\lambda^{\xi,\beta\Psi}$ as in (2.6) and note that its total mass is the total number of balls packed on Q_λ from the collection of balls with centers in $\mathcal{P}^{\beta\Psi} \cap Q_\lambda$ and with distinct arrival times in $[0,1]$. Note that ξ is bounded and thus satisfies all moment conditions of Theorems 2.1–2.3.

Theorem 5.1. *The packing functional ξ is exponentially stabilizing as at* (2.4). *Consequently*, *if $\Psi \in \boldsymbol{\Psi}^*$ and $(\tau, \beta) \in \mathcal{R}^{\Psi}$*, *then $\bar{\mu}_{\lambda}^{\xi,\beta\Psi}$ satisfies the conclusions of Theorems* 2.1–2.3.

Remark. *As spelled out in* [23], *Theorem* 5.1 *also applies to related packing models*, *including* (i) *spatial birth growth models with Gibbsian input*, (ii) *RSA models with balls replaced by particles of random size whose spatial locations are described by Gibbsian input*, *and* (iii) *ballistic deposition models with Gibbsian input*.

Proof of Theorem 5.1. The approach in [23] shows that ξ is exponentially stabilizing on *Poisson-like* sets. Indeed, we may couple any Poisson-like set Ξ with the dominating Poisson point process $\mathcal{P}_\tau$ such that $\mathcal{P}_\tau$ contains Ξ a.s. The arguments in [23] show that the packing status of a point x in a configuration $\mathcal{X}$ depends on $\mathcal{X}$ only through its algorithmically determined sub-configuration $Cl[x, \mathcal{X}]$ referred to as the *causal cluster* of x in the presence of $\mathcal{X}$. The causal cluster $Cl[x, \mathcal{X}]$ is non-decreasing in $\mathcal{X}$. In particular, using $\Xi \subseteq \mathcal{P}_\tau$ we get $Cl[x, \Xi] \subseteq Cl[x, \mathcal{P}_\tau]$ a.s. for $x \in \Xi$. However, by the arguments in Section 4 of [23], the causal clusters generated by points of $\mathcal{P}_\tau$ exhibit exponential decay, and hence so do causal clusters of points in Ξ showing that the packing functional ξ is exponentially stabilizing in the wide sense on Poisson-like sets, in particular on $\mathcal{P}^{\beta\Psi}$. This completes the proof of Theorem 5.1. □

5.2. *Functionals of Euclidean graphs on Gibbsian input*

In many cases, showing exponential stabilization of functionals of geometric graphs over Poisson input [2,22], can be reduced to upper bounding the probability that regions in $\mathbb{R}^d$ are devoid of points by a term which decays exponentially with the volume of the region. When the underlying point set is Poisson, as in [2,22], then we obtain the desired exponential decay. When the input is Poisson-like, the desired exponential decay follows from condition (2.1). In this way the existing stabilization proofs for functionals over Poisson point sets carry over to functionals on Poisson-like input. This extends central limit theorems for functionals of Euclidean graphs on Poisson input to the corresponding central limit theorems for functionals defined over Gibbsian input. The following applications illustrate this.

(i) *k-nearest neighbors graph*. The k-nearest neighbors (undirected) graph on the vertex set $\mathcal{X}$, denoted $NG(\mathcal{X})$, is the graph obtained by including $\{x, y\}$ as an edge whenever y is one of the k points nearest to x and/or x is one of the k points nearest to y. The k-nearest neighbors (directed) graph on $\mathcal{X}$, denoted $NG'(\mathcal{X})$, is obtained by placing a directed edge between each point and its k-nearest neighbors. In case $\mathcal{X} = \{x\}$ is a singleton, x has no nearest neighbor and the *nearest neighbor* distance for x is set by convention to 0.

Total edge length of k-nearest neighbors graph. Given $x \in \mathbb{R}^d$ and a locally finite point set $\mathcal{X} \subset \mathbb{R}^d$, the *nearest neighbors length functional* $\xi(x, \mathcal{X})$ is one half the sum of the edge lengths of edges in $NG(\mathcal{X} \cup \{x\})$ which are incident to x. Define the point measure $\mu_{\lambda}^{\xi,\beta\Psi}$ as in (2.6) and note that its total mass is the total edge length of $NG(\mathcal{P}^{\beta\Psi} \cap Q_\lambda)$. The next result generalizes Theorem 6.1 of [22], which is restricted to nearest neighbor graphs defined on Poisson input.

Theorem 5.2. *The nearest neighbors length functional ξ is exponentially stabilizing as at* (2.4) *and satisfies the p-moment condition* (2.7) *for some $p > 2$. Consequently*, *if $\Psi \in \boldsymbol{\Psi}^*$ and $(\tau, \beta) \in \mathcal{R}^{\Psi}$*, *then $\bar{\mu}_{\lambda}^{\xi,\beta\Psi}$ satisfies the conclusions of Theorems* 2.1–2.3.

Proof. An easy modification of the proof of Lemma 6.1 of [22] shows that ξ is exponentially stabilizing on Poisson-like point sets. Moreover, Lemma 6.2 of [22] shows that ξ satisfies the p-moments condition (2.7) for all p, completing the proof of Theorem 5.2. □

Number of components in nearest neighbors graph. Let $k = 1$. Given a locally finite point set $\mathcal{X}$, define the *component count functional* $\xi^{[c]}(x, \mathcal{X})$ to be the reciprocal of the cardinality of the component in $NG(\mathcal{X} \cup \{x\})$ which contains x. Thus $\sum_{x \in \mathcal{X}} \xi^{[c]}(x, \mathcal{X} \setminus \{x\})$ denotes the total number of finite components of $NG(\mathcal{X})$. Put

$$\mu_{\lambda}^{\xi,\beta\Psi} := \sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} \xi^{[c]}\big(u, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \setminus u\big) \delta_{\lambda^{-1/d} u}.$$

Theorem 5.3. *The component count functional* $\xi^{[c]}$ *is exponentially stabilizing as at* (2.4) *and satisfies the* p*-moment condition* (2.7) *for some* $p > 2$. *Consequently, if* $\Psi \in \boldsymbol{\Psi}^*$ *and* $(\tau, \beta) \in \mathcal{R}^{\Psi}$, *then* $\bar{\mu}_{\lambda}^{\xi,\beta\Psi}$ *satisfies the conclusions of Theorems* 2.1–2.3.

Proof. We establish that $\xi^{[c]}$ is exponentially stabilizing on Poisson-like sets Ξ and appeal to Theorems 2.1–2.3. When $k = 1$, the Poisson-like properties of the input process and the methods of Häggström and Meester [14] and Kozakova, Meester, and Nanda [16] show there are no infinite clusters in $NG(\Xi)$. Moreover, the proofs of Theorems 1.1, 1.2 and Propositions 2.2 and 2.3 of [16] and property (2.1) of Poisson-like processes show that the finite clusters in $NG(\Xi)$ have (super)exponentially decaying cardinalities and diameters.

This may be seen as follows. Let $\mathcal{G}$ be the directed graph whose vertices are the points in Ξ and such that there is a directed edge from $x \in \Xi$ to $y \in \Xi$ if y is the nearest neighbor of x. Let $E(n, L, j)$ be the event that there are directed paths in $\mathcal{G}$, one from $\mathbf{0}$ to some s', containing exactly j points in Ξ besides $\mathbf{0}$, and one from some s to the same s', containing exactly $n - j$ points in Ξ besides s', and such that the Euclidean norm of s exceeds L. As in [16], the empty ball probabilities (2.1) give

$$\mathbb{P}\big[E(n, L, j)\big] \le \int_{S_j} \exp\left(-C \operatorname{Vol}\left(\bigcup_{i=1}^{j} B_i \cup \bigcup_{i=j+1}^{n} B_i'\right)\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where C is a constant depending on C_3, $B_i := B_{|x_i|}(s_i)$ is the open ball centered at s_i of radius $|x_i|$, with a similar definition for $B_i' := B_{|x_i|}(s_i)$. Here S_j are the points in $(\mathbb{R}^d)^n$ satisfying conditions (i)–(iii) on p. 533 of [16]. This inequality is the analog of (7) of [16], where there $C = 1$. This yields the crucial Proposition 2.2 of [16], provided W_i are now i.i.d. random variables on $\mathbb{R}^d$ with density proportional to $\exp(-C \operatorname{Vol}(B_{|w|}(\mathbf{0}))$.

We show exponential stabilization of $\xi^{[c]}$ as follows. Let $R_{(y)}, y \in \mathbb{R}^d$, be the maximal radius of the cluster in $NG(\Xi)$ intersecting $B_1(y)$. Then by the above results, there is a $c \in (0, \infty)$ such that uniformly in y we have $\mathbb{P}[R_{(y)} > t] \le c^{-1} \exp(-ct)$. Then

$$S(x) := \sup_{y \in B_{R_{(x)}}(x) \cap \Xi} R_{(y)}$$

has an exponentially decaying tail as well. Indeed, covering a ball of radius M with $\mathrm{O}(M^d)$ unit balls, we use the exponential decay of nearest neighbor cluster diameters and the union bound to obtain

$$\mathbb{P}\big[S(x) > M\big] \le \mathbb{P}\big[S(x) > M, R_{(x)} \le M\big] + \mathbb{P}[R_{(x)} > M] \le c^{-1} M^d \exp(-cM) + c^{-1} \exp(-cM)$$

for some $c > 0$, as required. To proceed, we can also show that $4S(x)$ is a radius of stabilization for $\xi^{[c]}$ at x; see the proof of Lemma 6.1 of [22]. Since $\xi^{[c]}$ trivially satisfies the p moments condition (2.7) for all p, the proof of Theorem 5.3 is complete. □

(ii) *Gibbs–Voronoi tessellations*. Given $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, the set of points in $\mathbb{R}^d$ closer to x than to any other point of $\mathcal{X}$ is the interior of a possibly unbounded convex polyhedral cell $C(x, \mathcal{X})$. The Voronoi tessellation induced by $\mathcal{X}$ is the collection of cells $C(x, \mathcal{X}), x \in \mathcal{X}$. When $\mathcal{X}$ is the realization of the Poisson point set $\mathcal{P}_\tau$, this generates the Poisson–Voronoi tessellation of $\mathbb{R}^d$. As in [6,17], it is useful to study the case when there are geometric hard-core interactions between cells, where the Hamiltonian is defined in terms of edges and faces of the Poisson–Voronoi tessellation. We adopt a different approach, first defining the Gibbs point process $\mathcal{P}^{\beta\Psi}$ and then the Voronoi tessellation of this process, sometimes called the Ord process [19]. Our general results establish the limit theory for the total edge length of the Gibbs–Delaunay tessellation, yielding a closed form expression for the mean Voronoi cell perimeter on a Gibbs point process. This goes as follows.

Given $\mathcal{X} \subset \mathbb{R}^d$, let $L(x, \mathcal{X})$ denote one half the total edge length of the *finite* length edges in the cell $C(x, \mathcal{X} \cup \{x\})$ (thus we do not take infinite edges into account). It is easy to see that L is exponentially stabilizing on Poisson-like sets Ξ. Indeed, when $d = 2$, it suffices to follow the arguments in the proof of Theorem 8.1 of [22], where the stabilization arguments involve finding a minimum edge length such that the 12 isosceles triangles with this edge length and with common vertex each have at least one point from Ξ in them. Since Ξ is Poisson-like we may follow the arguments in [22] verbatim to see that L stabilizes. See Section 6.3 of [20] for the case $d > 2$, where it is also shown that L satisfies the moment condition (2.7) for $p = 3$. Putting

$$\mu_\lambda^{L,\beta\Psi} := \sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} L\big(u, \big(\mathcal{P}^{\beta\Psi} \cap Q_\lambda\big) \setminus u\big)\delta_{\lambda^{-1/d}u}$$

we have thus proved the following theorem.

Theorem 5.4. *The Voronoi length functional L is exponentially stabilizing as at (2.4) and satisfies the p-moment condition (2.7) for some $p > 2$. Consequently, if $\Psi \in \boldsymbol{\Psi}^*$ and $(\tau, \beta) \in \mathcal{R}^\Psi$, then $\bar{\mu}_\lambda^{L,\beta\Psi}$ satisfies the conclusions of Theorems 2.1–2.3. In particular for all $f \in \mathcal{B}(Q_1)$ we have*

$$\lim_{\lambda\to\infty} \lambda^{-1}\mathbb{E}\big[\big\langle f, \mu_\lambda^{L,\beta\Psi}\big\rangle\big] = \tau\mathbb{E}\big[L\big(\mathbf{0}, \mathcal{P}^{\beta\Psi}\big)\exp\big(-\beta\Delta\big(\mathbf{0}, \mathcal{P}^{\beta\Psi}\big)\big)\big]\int_{Q_1} f(x)\,\mathrm{d}x.$$

Putting $\tau = 1$ and $f \equiv 1$ gives an asymptotic Gibbs–Voronoi cell length of $\mathbb{E}L(\mathbf{0}, \mathcal{P}^{\beta\Psi})\exp(-\beta\Delta(\mathbf{0}, \mathcal{P}^{\beta\Psi}))$, which should be compared with the asymptotic Poisson–Voronoi cell length $\mathbb{E}L(\mathbf{0}, \mathcal{P}_1)$. Since the Gibbs–Voronoi cells are more regular, we suspect that their mean cell length is larger, but we are unable to prove this for the $\mathcal{P}^{\beta\Psi}$ in this paper.

(iii) *Other proximity graphs*. There are further examples where showing exponential stabilization of functionals of geometric graphs (in the wide sense) involves upper bounding the probability that regions in $\mathbb{R}^d$ are devoid of Poisson-like points. Such estimates are available in the Poisson setting and it is not difficult to extend them to Poisson-like point sets. In this way, by modifying the methods of [22] (Sections 7 and 9) and [2] (Section 3.1), we obtain weak laws of large numbers and central limit theorems for the total edge length of the sphere of influence graph, the Delaunay graph, the Gabriel graph, and the relative neighborhood graph over Gibbsian input $\mathcal{P}^{\beta\Psi}$.

5.3. *Gibbsian continuum percolation*

Let $\mathcal{X}$ be a locally finite point set and connect all pairs of points which are at most a unit distance apart. The resulting graph is equivalent to the basic model of continuum percolation, in which one considers the union of the radius 1 balls centered at points of $\mathcal{X}$, see Section 12.10 in [13]. Let $\xi^{[c]}(x, \mathcal{X})$ be the reciprocal of the cardinality of the component in the percolation graph on $\mathcal{X} \cup \{x\}$ containing x, so that $N(\mathcal{X}) = \sum_{x\in\mathcal{X}} \xi^{[c]}(x, \mathcal{X})$ counts the number of finite components in G.

Section 9 of [22] discusses central limit theorems for $N(\mathcal{P}_\tau \cap Q_\lambda)$. Using Theorem 2.3 we generalize these results to obtain a central limit theorem for the number of components $N(\mathcal{P}^{\beta\Psi} \cap Q_\lambda)$ in the continuum percolation model on Gibbsian input in the subcritical regime, possibly of interest in the context of sensor networks on Gibbsian point sets. We assume $\Psi \in \boldsymbol{\Psi}^*$ and $(\tau, \beta) \in \mathcal{R}^\Psi$, with τ in the subcritical regime for continuum percolation (see Section 12.10 in [13]). We argue that $\xi^{[c]}$ is exponentially stabilizing on Poisson-like sets Ξ as follows. If τ is subcritical, then Ξ is also subcritical by stochastic domination. Consequently, the diameter of the connected cluster emanating from a given point has exponentially decaying tails; see [13]. This yields the required exponential stabilization upon noting that $\xi^{[c]}(x, \cdot)$ does not depend on point configurations outside the connected cluster at x. Moreover, $\xi^{[c]}$ is bounded above by one and thus satisfies the moments condition (2.7). Hence by Theorems 2.1–2.3, $N(\mathcal{P}^{\beta\Psi} \cap Q_\lambda)$ satisfies the weak law of large numbers and central limit theorem, exactly as in the statement of Theorem 5.3.

5.4. *Functionals on Gibbsian loss networks*

Fix an integer $m \in \mathbb{N}$. Attach to each point of the reference point process $\mathcal{P}_\tau$ a bounded, convex, deterministic grain K. Similar to the truncated Poisson process, let the potential Ψ be infinite whenever the grain K at one point has non-empty intersection with more than m other grains. This condition prohibits overcrowding, and, for more general repulsive models, one can put Ψ large and finite whenever the grain K at one point has non-empty intersection with a large number (some number less than m) of other grains. The resulting process $\mathcal{P}^{\beta\Psi}$, $(\tau, \beta) \in \mathcal{R}^\Psi$, whose existence follows by Lemma 3.2, represents a version of spatial loss networks appearing in mobile and wireless communications.

Let $\mathcal{K}$ be an open convex cone in $\mathbb{R}^d$ (a cone is a set that is invariant under dilations) with apex at the origin. In the context of communication networks, $(\mathcal{K} + x)$ represents the broadcast range of a transmitter at x. Given

$x, y \in \mathcal{P}^{\beta\Psi}$, we say that y is *connected to* x, written $x \to y$, if there is a sequence of points $\{x_i\}_{i=1}^n \in (\mathcal{K}+x) \cap \mathcal{P}^{\beta\Psi}$, $|x_i - x_{i+1}| \le 1$, $|x_1 - x| \le 1$ and $|y - x_n| \le 1$. If the length of this sequence does not exceed a given m, we write $x \to_m y$. We thus have $x \to y$ iff there is a path joining x to y, whose edges link points of $\mathcal{P}^{\beta\Psi}$ lying inside $\mathcal{K}$ and which are of at most unit length. For all $r > 0$ let $B_r^{\mathcal{K}}(x) := x + (\mathcal{K} \cap B_r(\mathbf{0}))$.

Coverage functionals. The functional

$$\xi\big(x, \mathcal{P}^{\beta\Psi} \setminus x\big) := \sup\big\{r \in \mathbb{R} : x \to y \text{ for all } y \in B_r^{\mathcal{K}}(x) \cap \mathcal{P}^{\beta\Psi} \text{ and } B_r^{\mathcal{K}}(x) \cap \mathcal{P}^{\beta\Psi} \neq \varnothing\big\}$$

determines the maximal coverage range of the network at x in the direction of the cone $\mathcal{K}$. The coverage measure is $\mu_\lambda^\xi := \sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} \xi(u, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \setminus u)\delta_{\lambda^{-1/d}u}$. When τ belongs to the subcritical regime for continuum percolation, $\mathcal{P}^{\beta\Psi}$ is in turn subcritical because of Poisson domination. Since the continuum percolation clusters generated by any Poisson-like set Ξ have exponentially decaying diameter, it follows that ξ stabilizes in the wide sense (recall the above proof for the number of components in the continuum percolation model) and that ξ has an exponential moment. Theorems 2.1 and 2.3 yield a weak law of large numbers and central limit theorem for the coverage measure μ_λ^ξ and the total coverage $\sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} \xi(u, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \setminus u)$.

Network reach functional. Say that the network has *reach at least* r at x if $x \to y$ for all $y \in B_r^{\mathcal{K}}(x) \cap \mathcal{P}^{\beta\Psi}$ and $B_r^{\mathcal{K}}(x) \cap \mathcal{P}^{\beta\Psi} \neq \varnothing$. Put $\xi_r(x, \mathcal{P}^{\beta\Psi} \setminus x) := 1$ if the network has reach at least r at x and otherwise put $\xi_r(x, \mathcal{P}^{\beta\Psi} \setminus x) := 0$. Theorems 2.1 and 2.3 yield a weak law of large numbers and central limit theorem for the total network reach $\sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} \xi_r(u, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \setminus u)$.

Number of customers obtaining coverage. Independently mark each point x of $\mathcal{P}^{\beta\Psi}$ with mark T (transmitter) with probability $p > 0$ and with mark R (receiver) with the complement probability. Define the reception functional $\xi(x, \mathcal{P}^{\beta\Psi} \setminus x)$ to be 1 if x is marked with T or (when x is marked with R) if $z \to x$ for some z in the transmitter set $\{z \in \mathcal{P}^{\beta\Psi}\colon z \text{ marked with } T\}$. Put $\xi(x, \mathcal{P}^{\beta\Psi} \setminus x)$ to be zero otherwise. Thus $\xi(x, \cdot)$ counts when a customer at x gets coverage. We are in the setting of Remark (i) following Theorem 2.4 and the limit theory for the sum $\sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} \xi(u, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \setminus u)$, which counts the total number of customers obtaining network coverage, is given by Theorems 2.1–2.3.

Connectivity functional. Given a broadcast range $r > 0$ and the transmitter set $\{z \in \mathcal{P}^{\beta\Psi}\colon z \text{ marked with } T\}$, let $c_r(x, \mathcal{P}^{\beta\Psi})$ be the minimum number, say m, such that every point in $y \in B_r^{\mathcal{K}}(x) \cap \mathcal{P}^{\beta\Psi}$ can be reached from some transmitter $z \in \mathcal{P}^{\beta\Psi}$ with m or fewer edges or hops, that is to say there exists a transmitter z such that $z \to_m y$ for all $y \in B_r^{\mathcal{K}}(x) \cap \mathcal{P}^{\beta\Psi}$. Thus all receivers in the broadcast range $r > 0$ can be linked to a transmitter in m or fewer hops. Small values of $c_r(x, \mathcal{P}^{\beta\Psi})$ represent high network connectivity; of course $c_r(x, \mathcal{P}^{\beta\Psi})$ can admit infinite values. Next, for $n > 0$ set $\xi_{r,n}(x, \mathcal{P}^{\beta\Psi}) := 1$ if $c_r(x, \mathcal{P}^{\beta\Psi}) \le n$ and 0 otherwise. For each $r, n > 0$, Theorems 2.1 and 2.3 provide a weak law of large numbers and central limit theorem for the connectivity functional $\sum_{u \in \mathcal{P}^{\beta\Psi} \cap Q_\lambda} \xi_{r,n}(u, (\mathcal{P}^{\beta\Psi} \cap Q_\lambda) \setminus u)$ as $\lambda \to \infty$.

6. Gibbsian quantization for non-singular probability measures

Quantization for probability measures concerns the best approximation of a d-dimensional probability measure P by a discrete measure supported by a set $\mathcal{X}_n$ having n atoms. It involves a partitioning problem of the underlying space and it arises in information theory, cluster analysis, stochastic processes, and mathematical models in economics [12]. The goal is to optimally represent P, here assumed non-singular with density h, with a point set $\mathcal{X}_n$, where optimality involves minimizing the L^r *stochastic quantization error* (or 'random distortion error'), $r \in (0, \infty)$, given by

$$I(\mathcal{X}_n) := \int_{\mathbb{R}^d} \Big(\min_{x \in \mathcal{X}_n} |y - x|\Big)^r P(\mathrm{d}y) = \sum_{x \in \mathcal{X}_n} \int_{C(x, \mathcal{X}_n)} |y - x|^r P(\mathrm{d}y).$$

As in Section 5.2(ii), $C(x, \mathcal{X})$ is the Voronoi cell around x with respect to $\mathcal{X}$.

The minimal quantization error is $\min_{\mathcal{X}_n} I(\mathcal{X}_n)$. When $\mathbb{E}|X|^p < \infty$ for some $p > r$, with X having distribution P, then the seminal work of Bucklew and Wise [3,12] shows that

$$\lim_{n \to \infty} n^{r/d} \min_{\mathcal{X}_n} I(\mathcal{X}_n) = Q_{r,d} \|h\|_{d/(d+r)}, \tag{6.1}$$

where $\|h\|_{d/(d+r)}$ denotes the $d/(d+r)$ norm of the density h and where the so-called *rth quantization coefficient* $Q_{r,d}$ is some positive constant not known to have a closed form expression.

When $\mathcal{X}_n$ consists of n i.i.d. random variables, the first order asymptotics for $I(\mathcal{X}_n)$ were first investigated by Zador [32] and later by Graf and Luschgy [12], Cohort [4], and Yukich [30], who also obtained central limit theorems. Letting $\mathcal{X}_n$ consist of n i.i.d. random variables with common density $h^{d/(d+r)}/\int h^{d/(d+r)}$, where h is the density of P, Zador's theorem shows $\lim_{n\to\infty} n^{r/d} I(\mathcal{X}_n) = v_d^{-r/d}\Gamma(1+r/d)\|h\|_{d/(d+r)}$ whence (see Proposition 9.3 in [12]) the upper bound

$$Q_{r,d} \le v_d^{-r/d}\Gamma(1+r/d). \tag{6.2}$$

Molchanov and Tontchev [18] discuss the possibility of quantization of P via Poisson point sets and our purpose here is to establish asymptotics of the quantization error on Gibbsian input. This is done as follows. For $\lambda > 0$ and a finite point configuration $\mathcal{X}$ we define $\mathcal{X}_{(\lambda)} := \lambda^{-1/d}\mathcal{X}$. We write $\tilde{\mathcal{X}} := \mathcal{X}\cap Q_1$ so that in particular $\tilde{\mathcal{X}}_{(\lambda)} := \lambda^{-1/d}\mathcal{X}\cap Q_1$. Consider the random point measures induced by the distortion arising from $\tilde{\mathcal{P}}^{\beta\Psi}_{(\lambda)}$, namely

$$\mu_\lambda^{\beta\Psi} := \sum_{x\in\tilde{\mathcal{P}}^{\beta\Psi}_{(\lambda)}} \int_{C(x,\tilde{\mathcal{P}}^{\beta\Psi}_{(\lambda)})} |y-x|^r P(\mathrm{d}y)\delta_x. \tag{6.3}$$

Clearly, when $f\equiv 1$ then $\langle f,\mu_\lambda^{\beta\Psi}\rangle$ gives another expression for the distortion $I(\tilde{\mathcal{P}}^{\beta\Psi}_{(\lambda)})$. On the other hand, if $f=\mathbf{1}(B)$, then $\langle f,\cdot\rangle$ measures the local distortion. This section establishes mean and variance asymptotics for $\langle f,\mu_\lambda^{\beta\Psi}\rangle$ as well as central limit theorems. Since the functional $(x,\mathcal{X})\mapsto\int_{C(x,\mathcal{X})}|y-x|^r h(y)\,\mathrm{d}y$ is not translation invariant for h not constant, we will appeal to Theorem 2.4. Recalling that τ is the reference intensity, we put

$$M^{\beta\Psi}(\tau) := \int_{C(\mathbf{0},\mathcal{P}^{\beta\Psi})} |w|^r\,\mathrm{d}w\exp\bigl(-\beta\Delta\bigl(\mathbf{0},\mathcal{P}^{\beta\Psi}\bigr)\bigr).$$

Note that $M^{\beta\Psi}(\tau)$ depends on τ through $\mathcal{P}^{\beta\Psi}$. Changing the order of integration we have

$$\begin{aligned}\mathbb{E}\bigl[M^{\beta\Psi}(\tau)\bigr] &= \mathbb{E}\biggl[\int_{\mathbb{R}^d}|w|^r\mathbf{1}\bigl(\mathcal{P}^{\beta\Psi}\cap B_{|w|}(w)=\varnothing\bigr)\exp\bigl(-\beta\Delta\bigl(\mathbf{0},\mathcal{P}^{\beta\Psi}\bigr)\bigr)\,\mathrm{d}w\biggr]\\ &= \int_{\mathbb{R}^d}|w|^r\mathbb{E}\bigl[\exp\bigl(-\beta\Delta\bigl(\mathbf{0},\mathcal{P}^{\beta\Psi}\bigr)\bigr)\mathbf{1}\bigl(\mathcal{P}^{\beta\Psi}\cap B_{|w|}(w)\bigr)=\varnothing)\bigr]\,\mathrm{d}w.\end{aligned} \tag{6.4}$$

In the special case $\tau=1$ and $\Psi\equiv 0$ (i.e. $\mathcal{P}^{\beta\Psi}\overset{\mathcal{D}}{=}\mathcal{P}_1$), we readily get $\mathbb{E}M^0(1)=\Gamma(1+r/d)v_d^{-r/d}$. More generally we have $\mathbb{E}M^0(\tau)=\tau^{-(1+r/d)}\Gamma(1+r/d)v_d^{-r/d}$. Put

$$\begin{aligned}V^{\beta\Psi}(\tau) &:= \mathbb{E}\bigl[M^{\beta\Psi}(\tau)^2\bigr]\\ &\quad+\tau\int_{\mathbb{R}^d}\biggl(\mathbb{E}\biggl[\int_{C(\mathbf{0},\mathcal{P}^{\beta\Psi}\cup\{y\})}|w|^r\,\mathrm{d}w\int_{C(y,\mathcal{P}^{\beta\Psi}\cup\{\mathbf{0}\})}|w-y|^r\,\mathrm{d}w\exp\bigl(-\beta\Delta\bigl(\{\mathbf{0},y\},\mathcal{P}^{\beta\Psi}\bigr)\bigr)\biggr]-\bigl(\mathbb{E}M^{\beta\Psi}(\tau)\bigr)^2\biggr)\,\mathrm{d}y.\end{aligned}$$

We now give the limit theory for $\langle f,\mu_\lambda^{\beta\Psi}\rangle$.

Theorem 6.1. *Assume that the density h of P is continuous on Q_1 and zero outside Q_1. Assume $\Psi\in\boldsymbol{\Psi}^*$ and $(\tau,\beta)\in\mathcal{R}^\Psi$. We have for each $f\in\mathcal{B}(Q_1)$*

$$\lim_{\lambda\to\infty}\lambda^{r/d}\bigl\langle f,\mu_\lambda^{\beta\Psi}\bigr\rangle=\tau\mathbb{E}\bigl[M^{\beta\Psi}(\tau)\bigr]\int_{Q_1}f(x)h(x)\,\mathrm{d}x\quad \textit{in } L^2 \tag{6.5}$$

and

$$\lim_{\lambda\to\infty}\lambda^{1+2r/d}\operatorname{Var}\big[\langle f,\mu_\lambda^{\beta\Psi}\rangle\big]=\tau V^{\beta\Psi}(\tau)\int_{Q_1}f(x)^2h(x)\,\mathrm{d}x. \tag{6.6}$$

The finite-dimensional distributions $\lambda^{-1/2+r/d}(\langle f_1,\overline{\mu}_\lambda^{\beta\Psi}\rangle,\dots,\langle f_k,\overline{\mu}_\lambda^{\beta\Psi}\rangle)$, $f_1,\dots,f_k\in\mathcal{B}(Q_1)$, *converge as* $\lambda\to\infty$ *to those of a mean zero Gaussian field with covariance kernel*

$$(f_1,f_2)\mapsto\tau V^{\beta\Psi}(\tau)\int_{Q_1}f_1(x)f_2(x)h(x)\,\mathrm{d}x,\quad f_1,f_2\in\mathcal{B}(Q_1).$$

When $f\equiv 1$ the right-hand side of (6.5) gives

$$\lim_{\lambda\to\infty}\lambda^{r/d}\langle 1,\mu_\lambda^{\beta\Psi}\rangle=\lim_{\lambda\to\infty}\lambda^{r/d}I\big(\tilde{\mathcal{P}}_{(\lambda)}^{\beta\Psi}\big)=\tau\mathbb{E}\big[M^{\beta\Psi}(\tau)\big]\quad\text{in }L^2.$$

The limit (6.1) is necessarily less than or equal to the right-hand side of the above, showing that in addition to the bound (6.2), the rth quantization coefficient $Q_{r,d}$ satisfies

$$Q_{r,d}\le\big(\|h\|_{d/(d+r)}\big)^{-1}\tau\mathbb{E}\big[M^{\beta\Psi}(\tau)\big].$$

Recall from our discussion above that when $\Psi\equiv 0$ (i.e. $\mathcal{P}^{\beta\Psi}$ is Poisson) and when $f\equiv 1$, then the right-hand side of (6.5) equals $\tau^{-r/d}v_d^{-r/d}\Gamma(1+r/d)$ and thus

$$Q_{r,d}\le\big(\|h\|_{d/(d+r)}\big)^{-1}\tau^{-r/d}v_d^{-r/d}\Gamma(1+r/d).$$

Whereas the distortion error (6.5) is relatively large for Poisson input, we expect that it can be made smaller by restricting to point sets which enjoy built-in repulsivity while keeping the same mean point density. Indeed, given a fixed mean number of test points it seems more economical to spread them equidistantly over the domain of target distribution than to allow for local surpluses of test points in some regions, which only result in wasting test resources with the quantization quality improvement considerably inferior to that which would be achieved should we shift the extraneous points to regions of lower test point concentration. In other words, the right-hand side of (6.5) for repulsive Gibbs point processes should be smaller than the corresponding distortion for the Poisson point process with the same point density. These seem to be natural and interesting questions, yet we cannot handle them with our techniques.

Proof of Theorem 6.1. We deduce Theorem 6.1 from Theorem 2.4. We first assume that the density h of P is bounded away from 0. Consider the following parametric family of functionals:

$$\hat{\xi}(x,\mathcal{X};\lambda):=\int_{C(x,\mathcal{X})}|y-x|^r\frac{h(\lambda^{-1/d}y)}{h(\lambda^{-1/d}x)}\,\mathrm{d}y, \tag{6.7}$$

where *without loss of generality we assume that* $C(x,\mathcal{X})$ *denotes the intersection of* $\lambda^{1/d}Q_1$ *and the Voronoi cell around* x *with respect to* $\mathcal{X}$, since $h(\lambda^{-1/d}y)$ vanishes off $\lambda^{1/d}Q_1$. Putting

$$\xi(x,\mathcal{X}):=\int_{C(x,\mathcal{X})}|y-x|^r\,\mathrm{d}y \tag{6.8}$$

and

$$\delta(x,\mathcal{X};\lambda):=\int_{C(x,\mathcal{X})}|y-x|^r\frac{h(\lambda^{-1/d}y)-h(\lambda^{-1/d}x)}{h(\lambda^{-1/d}x)}\,\mathrm{d}y \tag{6.9}$$

we obtain the bounded perturbed representation (2.15) for $\hat{\xi}(\cdot,\cdot;\lambda)$, namely we have $\hat{\xi}(x,\mathcal{X};\lambda)=\xi(x,\mathcal{X})+\delta(x,\mathcal{X};\lambda)$, with ξ translation invariant.

As in both Theorem 8.1 of [22] and Section 6.3 of [20], it is seen that both ξ and δ, as given in (6.8) and (6.9), stabilize exponentially on Poisson-like input with common stabilization radius determined by the diameter of the Voronoi cell around the input point. To see that $\delta(\cdot,\cdot;\lambda)$ also satisfies the bounded moments condition (2.16), we note that $\|1/h\|_\infty \le C$ gives for $x \in Q_\lambda$

$$\left|\delta(x,\mathcal{X};\lambda)\right| \le C\int_{C(x,\mathcal{X})} |y-x|^r \left|h\left(\lambda^{-1/d}y\right) - h\left(\lambda^{-1/d}x\right)\right| \mathrm{d}y. \tag{6.10}$$

Given a point set $\mathcal{X}$ recall that the Voronoi flower is $F(x,\mathcal{X}) := \bigcup_y B_{|y|}(y)$, where the union ranges over the vertices y belonging to the Voronoi cell $C(x,\mathcal{X})$. Since $C(x,\mathcal{X}) \subset F(x,\mathcal{X})$, (6.10) gives for all $p > 0$ and $\lambda \ge 1$

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left|\delta\left(x,\mathcal{P}^{\beta\Psi}\cap Q_\lambda;\lambda\right)\right|^p \le \sup_{x\in Q_\lambda} C\mathbb{E}\left[\int_{F(x,\mathcal{P}^{\beta\Psi})} |y-x|^r \left|h\left(\lambda^{-1/d}y\right) - h\left(\lambda^{-1/d}x\right)\right| \mathrm{d}y\right]^p.$$

Stationarity of $\mathcal{P}^{\beta\Psi}$ and uniform continuity of h give

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left|\delta\left(x,\mathcal{P}^{\beta\Psi}\cap Q_\lambda;\lambda\right)\right|^p \le C\mathbb{E}\left[\int_{F(\mathbf{0},\mathcal{P}^{\beta\Psi})} |y|^r \omega_h\left(\lambda^{-1/d}|y|\right) \mathrm{d}y\right]^p =: L(p,r,\lambda),$$

where $\omega_h(\cdot)$ is the modulus of continuity of h. Now $[\int_{F(\mathbf{0},\mathcal{P}^{\beta\Psi})} |y|^r \omega_h(\lambda^{-1/d}|y|)\,\mathrm{d}y]^p$ is dominated by an integrable random variable uniformly over λ, namely by a constant multiple of the $p(r+d)^{th}$ power of the diameter of the Voronoi flower on Poisson-like input, which decays exponentially fast. Since $[\int_{C(\mathbf{0},\mathcal{P}^{\beta\Psi})} |y|^r \omega_h(\lambda^{-1/d}|y|)\,\mathrm{d}y]^p$ a.s. converges to zero as $\lambda \to \infty$, the dominated convergence theorem implies that $\lim_{\lambda\to\infty} L(p,r,\lambda) = 0$. This gives (2.16) and hence $\hat{\xi}(\cdot,\cdot;\lambda)$ is a bounded perturbation of $\xi(\cdot,\cdot)$.

To proceed, we note that $\mu_\lambda^{\beta\Psi}$ defined at (6.3) satisfies for each $f \in \mathcal{B}(Q_1)$

$$\left\langle f,\mu_\lambda^{\beta\Psi}\right\rangle = \lambda^{-1-r/d}\left\langle fh,\mu_\lambda^{\hat{\xi},\beta\Psi}\right\rangle, \tag{6.11}$$

where $\mu_\lambda^{\hat{\xi},\beta\Psi}$ is the empirical measure for $\hat{\xi}$ defined at (2.6), that is to say

$$\mu_\lambda^{\hat{\xi},\beta\Psi} := \sum_{u\in\mathcal{P}^{\beta\Psi}\cap Q_\lambda} \hat{\xi}\left(u,\left(\mathcal{P}^{\beta\Psi}\cap Q_\lambda\right)\setminus u;\lambda\right)\delta_{\lambda^{-1/d}u}.$$

It is easily verified that ξ satisfies all assumptions of Theorems 2.1–2.3. Consequently, Theorem 2.4 can be applied for $\hat{\xi}$, which yields Theorem 6.1 via the formula (6.11) allowing us to translate results for $\mu_\lambda^{\hat{\xi},\beta\Psi}$ to the corresponding results for $\mu_\lambda^{\beta\Psi}$. This completes the proof of Theorem 6.1 for h bounded away from 0.

Assume now that h fails to be bounded away from 0 and, for $\varepsilon > 0$ put $h_\varepsilon := \max(h,\varepsilon)$ and let $\mu_{\lambda;\varepsilon}^{\beta\Psi}$ be the version of $\mu_\lambda^{\beta\Psi}$ with h replaced by h_ε. Using the definition of $\mu_\lambda^{\beta\Psi}$, and the exponential decay of the diameter of Voronoi cells on Poisson-like input we easily conclude that

$$\left|\mathbb{E}\left[\left\langle f,\mu_\lambda^{\beta\Psi}\right\rangle - \left\langle f,\mu_{\lambda;\varepsilon}^{\beta\Psi}\right\rangle\right]\right| = \mathrm{O}\left(\lambda^{-r/d}\varepsilon\right), \qquad \mathrm{Var}\left[\left\langle f,\mu_\lambda^{\beta\Psi}\right\rangle - \left\langle f,\mu_{\lambda;\varepsilon}^{\beta\Psi}\right\rangle\right] = \mathrm{O}\left(\lambda^{-1-2r/d}\varepsilon\right). \tag{6.12}$$

We may apply the first half of the current proof to h_ε, since h_ε is integrable and bounded away from 0 (though h_ε does not integrate to one, h_ε can be easily renormalized to do so). Using (6.12) we obtain the required expectation and variance asymptotics for $\langle f,\mu_\lambda^{\beta\Psi}\rangle$ as well as the L^2 weak law of large numbers. The remaining central limit theorem statement for $\langle f,\bar{\mu}_\lambda^{\beta\Psi}\rangle$ follows directly by Stein's method as in Theorem 2.3, which is not affected by h being not bounded away from 0. This completes the proof of Theorem 6.1. $\square$

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References

[1] A. J. Baddeley and M. N. M. van Lieshout. Area interaction point processes. *Ann. Inst. Statist. Math.* **47** (1995) 601–619. MR1370279
[2] Y. Baryshnikov and J. E. Yukich. Gaussian limits for random measures in geometric probability. *Ann. Appl. Probab.* **15** (2005) 213–253. MR2115042
[3] J. A. Bucklew and G. Wise. Multidimensional asymptotic quantization theory with rth power distortion measures. *IEEE Trans. Inf. Th.* **28** (1982) 239–247. MR0651819
[4] P. Cohort. Limit theorems for random normalized distortion. *Ann. Appl. Probab.* **14** (2004) 118–143. MR2023018
[5] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*, vol. II, 2nd edition. Springer, New York, 2008. MR2371524
[6] D. Dereudre and F. Lavancier. Practical simulation and estimation for Gibbs Delaunay–Voronoi tessellations with geometric hardcore interaction. *Comput. Statist. Data Anal.* **55** (2011) 498–519. MR2736572
[7] A. Dvoretzky and H. Robbins. On the "parking" problem. *MTA Mat Kut. Int. Kžl.* (Publications of the Math. Res. Inst. of the Hungarian Academy of Sciences) **9** (1964) 209–225. MR0173275
[8] J. W. Evans. Random and cooperative adsorption. *Rev. Modern Physics* **65** (1993) 1281–1329.
[9] R. Fernández, P. Ferrari and N. Garcia. Measures on contour, polymer or animal models. A probabilistic approach. *Markov Process. Related Fields* **4** (1998) 479–497. MR1677054
[10] R. Fernández, P. Ferrari and N. Garcia. Loss network representation of Peierls contours. *Ann. Probab.* **29** (2001) 902–937. MR1849182
[11] R. Fernández, P. Ferrari and N. Garcia. Perfect simulation for interacting point processes, loss networks and Ising models. *Stochastic Process. Appl.* **102** (2002) 63–88. MR1934155
[12] S. Graf and H. Luschgy. *Foundations of Quantization for Probability Distributions*. *Lecture Notes in Mathemmatics* **1730**. Springer, Berlin, 2000. MR1764176
[13] G. Grimmett. *Percolation*, 2nd edition. *Grundlehren der Mathematischen Wissenschaften* **321**. Springer, Berlin, 1999. MR1707339
[14] O. Häggström and R. Meester. Nearest neighbor and hard sphere models in continuum percolation. *Random Structures Algorithms* **9** (1996) 295–315. MR1606845
[15] P. Hall. On continuum percolation. *Ann. Probab.* **13** (1985) 1250–1266. MR0806222
[16] I. Kozakova, R. Meester and S. Nanda. The size of components in continuum nearest-neighbor graphs. *Ann. Probab.* **34** (2006) 528–538. MR2223950
[17] C. Lautensack and T. Sych. 3D image analysis of open foams using random tessellations. *Image Anal. Stereo.* **25** (2006) 87–93.
[18] I. Molchanov and N. Tontchev. Optimal Poisson quantisation. *Stat. Probab. Letters* **77** (2007) 1123–1132. MR2395070
[19] J. Møller and R. Waagepetersen. *Statistical Inference and Simulation for Spatial Point Processes*. Chapman and Hall, Boca Raton, 2004. MR2004226
[20] M. D. Penrose. Gaussian limits for random geometric measures. *Electron. J. Probab.* **12** (2007) 989–1035. MR2336596
[21] M. D. Penrose. Laws of large numbers in stochastic geometry with statistical applications. *Bernoulli* **13** (2007) 1124–1150. MR2364229
[22] M. D. Penrose and J. E. Yukich. Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.* **11** (2001) 1005–1041. MR1878288
[23] M. D. Penrose and J. E. Yukich. Limit theory for random sequential packing and deposition. *Ann. Appl. Probab.* **12** (2002) 272–301. MR1890065
[24] M. D. Penrose and J. E. Yukich. Weak laws of large numbers in geometric probability. *Ann. Appl. Probab.* **13** (2003) 277–303. MR1952000
[25] M. D. Penrose and J. E. Yukich. Normal approximation in geometric probability. In *Stein's Method and Applications* 37–58. A. D. Barbour and Louis H. Y. Chen (Eds.). *Lecture Note Series*. *Institute for Mathematical Sciences, National University of Singapore* **5**. Singapore Univ. Press, Singapore, 2005. Available at http://www.lehigh.edu/~jey0/publications.html. MR2201885
[26] A. Rényi. On a one-dimensional random space-filling problem. *MTA Mat Kut. Int. Kžl.* (Publications of the Math. Res. Inst. of the Hungarian Academy of Sciences) **3** (1958) 109–127. MR0104284
[27] D. Ruelle. Superstable interactions in classical statistical mechanics. *Comm. Math. Phys.* **18** (1970) 127–159. MR0266565
[28] T. Schreiber, M. D. Penrose and J. E. Yukich. Gaussian limits for multidimensional random sequential packing at saturation. *Comm. Math. Phys.* **272** (2007) 167–183. MR2291806
[29] D. Stoyan, W. Kendall and J. Mecke. *Stochastic Geometry and Its Applications*, 2nd edition. Wiley, New York, 1995. MR0895588
[30] J. E. Yukich. Limit theorems for multi-dimensional random quantizers. *Electron. Commun. Probab.* **13** (2008) 507–517. MR2447837
[31] J. E. Yukich. Limit theorems in discrete stochastic geometry. In *Stochastic Geometry, Spatial Statistics and Random Fields* 239–275. E. Spodarev (Ed.). *Lecture Notes in Mathematics* **2068**. Springer, Berlin, 2013.
[32] P. L. Zador. Asymptotic quantization error of continuous signals and the quantization dimension. *IEEE Trans. Inform. Theory* **28** (1982) 139–149. MR0651809