Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2013, Vol. 49, No. 1, 252–269
DOI: 10.1214/12-AIHP492

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Characterizations of processes with stationary and independent increments under G-expectation[1]

Yongsheng Song

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China. E-mail: yssong@amss.ac.cn

Received 26 October 2010; revised 8 April 2012; accepted 17 April 2012

Abstract. Our purpose is to investigate properties for processes with stationary and independent increments under G-expectation. As applications, we prove the martingale characterization of G-Brownian motion and present a pathwise decomposition theorem for generalized G-Brownian motion.

Résumé. Notre but est d'étudier des propriétés de processus à accroissements stationnaires et indépendants sous une G-espérance. Comme application, nous démontrons la caractérisation de la martingale de G-mouvement Brownien et fournissons un théorème de décomposition trajectorielle pour le G-mouvement Brownien généralisé.

MSC: 60G10; 60G17; 60G48; 60G51

Keywords: Stationary increments; Independent increments; Martingale characterization; Decomposition theorem; G-Brownian motion; G-expectation

1. Introduction

Recently, motivated by the modelling of dynamic risk measures, Shige Peng ([3–5]) introduced the notion of a G-expectation space. It is a generalization of probability spaces (with their associated linear expectation) to spaces endowed with a nonlinear expectation. As the counterpart of Wiener space in the linear case, the notion of G-Brownian motion was introduced under the nonlinear G-expectation.

Recall that if $\{A_t\}$ is a continuous process over a probability space $(\Omega, \mathcal{F}, P)$ with stationary, independent increments and finite variation, then there exists some constant c such that $A_t = ct$. However, it is not the case in the G-expectation space $(\Omega_T, L_G^1(\Omega_T), \hat{E})$. A counterexample is $\{\langle B\rangle_t\}$, the quadratic variation process for the coordinate process $\{B_t\}$, which is a G-Brownian motion. We know that $\{\langle B\rangle_t\}$ is a continuous, increasing process with stationary and independent increments, but it is not deterministic.

The process $\{\langle B\rangle_t\}$ is very important in the theory of G-expectation, which shows, in many aspects, the difference between probability spaces and G-expectation spaces. For example, we know that for a probability space continuous local martingales with finite variation are trivial processes. However, [4] proved that in a G-expectation space all processes in form of $\int_0^t \eta_s \,\mathrm{d}\langle B\rangle_s - \int_0^t 2G(\eta_s)\,\mathrm{d}s$, $\eta \in M_G^1(0,T)$ (see Section 2 for the definitions of the function $G(\cdot)$ and the space $M_G^1(0,T)$), are nontrivial G-martingales with finite variation (in fact, they are even nonincreasing) and continuous paths. [4] also conjectured that any G-martingale with finite variation should have such representation. Up to

[1] Supported by NCMIS; Youth Grant of National Science Foundation (No. 11101406/A011002); the National Basic Research Program of China (973 Program) (No. 2007CB814902); Key Lab of Random Complex Structures and Data Science, Chinese Academy of Sciences (Grant No. 2008DP173182).

now, some properties of the process $\{\langle B\rangle_t\}$ remain unknown. For example, we know that, if $G(x)=\frac{1}{2}\sup_{\underline{\sigma}\le\sigma\le\overline{\sigma}}\sigma^2 x$ generates the G-expectation, we have $\underline{\sigma}^2(t-s)\le\langle B\rangle_t-\langle B\rangle_s\le\overline{\sigma}(t-s)$ for all $s<t$, but we do not know whether $\{\frac{\mathrm{d}}{\mathrm{d}s}\langle B\rangle_s\}$ belongs to $M_G^1(0,T)$. This is a very important property since $\{\frac{\mathrm{d}}{\mathrm{d}s}\langle B\rangle_s\}\in M_G^1(0,T)$ would imply that the representation mentioned above of G-martingales with finite variation is not unique.

For the case of a probability space, a continuous local martingale $\{M_t\}$ is a standard Brownian motion if and only if the quadratic variation process $\langle M\rangle_t=t$. However, it's not the case for G-Brownian motion since its quadratic variation process is only an increasing process with stationary and independent increments. How can we give a characterization for G-Brownian motion?

In this article, we shall prove that if $A_t=\int_0^t h_s\,\mathrm{d}s$ (respectively $A_t=\int_0^t h_s\,\mathrm{d}\langle B\rangle_s$) is a process with stationary, independent increments and $h\in M_G^1(0,T)$ (respectively $h\in M_G^{\beta,+}(0,T)$, for some $\beta>1$), then there exists some constant c such that $h\equiv c$. As applications, we prove the following conclusions (Question 1 and 3 are put forward by Prof. Shige Peng in private communications):

1. $\{\frac{\mathrm{d}}{\mathrm{d}s}\langle B\rangle_s\}\notin M_G^1(0,T)$.
2. (Martingale characterization)

   A symmetric G-martingale $\{M_t\}$ is a G-Brownian motion if and only if its quadratic variation process $\{\langle M\rangle_t\}$ has stationary and independent increments;

   A symmetric G-martingale $\{M_t\}$ is a G-Brownian motion if and only if its quadratic variation process $\langle M\rangle_t=c\langle B\rangle_t$ for some $c\ge 0$.

   The sufficiency of the second assertion is trivial, but not the necessity.
3. *Let $\{X_t\}$ be a generalized G-Brownian motion with zero mean, then we have the following decomposition:*

$$X_t=M_t+L_t,$$

   where $\{M_t\}$ is a (symmetric) G-Brownian motion, and $\{L_t\}$ is a nonpositive, nonincreasing G-martingale with stationary and independent increments.

This article is organized as follows: In Section 2 we recall some basic notions and results of G-expectation and the related space of random variables. In Section 3 we characterize processes with stationary and independent increments. In Section 4, as application, we prove the martingale characterization of G-Brownian motion and present a decomposition theorem for generalized G-Brownian motion. In Section 5 we present some properties for G-martingales with finite variation.

2. Preliminary

We recall some basic notions and results of G-expectation and the related space of random variables. More details of this section can be found in [3–8].

Definition 2.1. *Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω with $c\in\mathcal{H}$ for all constants c. $\mathcal{H}$ is considered as the space of "random variables." A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E}:\mathcal{H}\to R$ satisfying the following properties: For all $X,Y\in\mathcal{H}$, we have*

(a) *Monotonicity: If $X\ge Y$ then $\hat{E}(X)\ge\hat{E}(Y)$.*
(b) *Constant preserving: $\hat{E}(c)=c$.*
(c) *Sub-additivity: $\hat{E}(X)-\hat{E}(Y)\le\hat{E}(X-Y)$.*
(d) *Positive homogeneity: $\hat{E}(\lambda X)=\lambda\hat{E}(X)$, $\lambda\ge 0$.*

$(\Omega,\mathcal{H},\hat{E})$ is called a sublinear expectation space.

Definition 2.2. *Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1,\mathcal{H}_1,\hat{E}_1)$ and $(\Omega_2,\mathcal{H}_2,\hat{E}_2)$. They are called identically distributed, denoted by $X_1\sim X_2$, if $\hat{E}_1[\varphi(X_1)]=\hat{E}_2[\varphi(X_2)]$, for all $\varphi\in C_{l,\mathrm{Lip}}(R^n)$, where $C_{l,\mathrm{Lip}}(R^n)$ is the space of real continuous functions defined on R^n such that*

$$\left|\varphi(x)-\varphi(y)\right|\le C\left(1+|x|^k+|y|^k\right)|x-y|,\quad \text{for all } x,y\in R^n,$$

where k and C depend only on φ.

Definition 2.3. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$, under $\hat{E}(\cdot)$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{b,\mathrm{Lip}}(R^m \times R^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}]$.*

Definition 2.4 (G-normal distribution). *A d-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normal distributed if for every $a, b \in R_+$ we have*

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(A) := \frac{1}{2}\hat{E}\big[(AX, X)\big] : S_d \to R,$$

where S_d denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot) : S_d \to R$ is a monotonic, sublinear mapping on S_d and $G(A) = \frac{1}{2}\hat{E}[(AX, X)] \leq \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} \mathrm{Tr}(\gamma A). \tag{2.1}$$

If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta \,\mathrm{Tr}(A - B)$ for any $A \geq B$, we call the G-normal distribution nondegenerate. This is the case we consider throughout this article.

Definition 2.5. (i) *Let $\Omega_T = C_0([0, T]; R^d)$ be endowed with the supremum norm and $\{B_t\}$ be the coordinate process. Set $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, \ldots, B_{t_n}) | n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{l,\mathrm{Lip}}(R^{d\times n})\}$. G-expectation is a sublinear expectation defined by*

$$\hat{E}[X] = \tilde{E}\big[\varphi(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_m - t_{m-1}}\xi_m)\big],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \ldots, \xi_n$ are identically distributed d-dimensional G-normally distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$ such that ξ_{i+1} is independent of $(\xi_1, \ldots, \xi_i)$ for every $i = 1, \ldots, m-1$. $(\Omega_T, \mathcal{H}_T^0, \hat{E})$ is called a G-expectation space.

(ii) *Let us define the conditional G-expectation $\hat{E}_t$ of $\xi \in \mathcal{H}_T^0$ knowing $\mathcal{H}_t^0$, for $t \in [0, T]$. Without loss of generality we can assume that ξ has the representation $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$ with $t = t_i$, for some $1 \leq i \leq m$, and we put*

$$\begin{aligned}&\hat{E}_{t_i}\big[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})\big]\\ &\quad = \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),\end{aligned}$$

where

$$\tilde{\varphi}(x_1, \ldots, x_i) = \hat{E}\big[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})\big].$$

Define $\|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$. Then for all $t \in [0, T]$, $\hat{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}_T^0$ with respect to the norm $\|\cdot\|_{1,G}$ and therefore can be extended continuously to the completion $L_G^1(\Omega_T)$ of $\mathcal{H}_T^0$ under the norm $\|\cdot\|_{1,G}$.

Let $L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) | n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,\mathrm{Lip}}(R^{d\times n})\}$, where $C_{b,\mathrm{Lip}}(R^{d\times n})$ denotes the set of bounded Lipschitz functions on $R^{d\times n}$. [1] proved that the completions of $C_b(\Omega_T)$, $\mathcal{H}_T^0$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same; we denote them by $L_G^p(\Omega_T)$.

Definition 2.6. (i) *We say that* $\{X_t\}$ *on* $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ *is a process with independent increments if for any* $0 < t < T$ *and* $s_0 \le \cdots \le s_m \le t \le t_0 \le \cdots \le t_n \le T$,

$$(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}) \perp (X_{s_1} - X_{s_0}, \dots, X_{s_m} - X_{s_{m-1}}).$$

(ii) *We say that* $\{X_t\}$ *on* $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ *with* $X_t \in L_G^1(\Omega_t)$ *for every* $t \in [0, T]$ *is a process with independent increments* w.r.t. the filtration *if for any* $0 < s < T$ *and* $s_0 \le \cdots \le s_m \le s \le t_0 \le \cdots \le t_n \le T$,

$$(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}) \perp (B_{s_1} - B_{s_0}, \dots, B_{s_m} - B_{s_{m-1}}).$$

Remark 2.7. (i) *Let* $\xi \in L_G^1(\Omega_T)$. *If there exists* $s \in [0, T]$ *such that for any* $s_0 \le \cdots \le s_m \le s$, $\xi \perp (B_{s_1} - B_{s_0}, \dots, B_{s_m} - B_{s_{m-1}})$, *then we have* $\hat{E}_s(\xi) = \hat{E}(\xi)$. *In fact, there is no loss of generality, we assume* $\hat{E}(\xi) = 1$ *and* $C \ge \xi \ge \varepsilon$ *for some* $C, \varepsilon > 0$. *Set* $\eta = \hat{E}_s(\xi)$. *For any* $n \in N$, *we have*

$$\hat{E}(\eta^{n+1}) = \hat{E}(\eta^n \xi).$$

Since $\xi \perp \eta^n$, *we have*

$$\hat{E}(\eta^{n+1}) = \hat{E}(\eta^n) = \cdots = \hat{E}(\eta) = 1.$$

By this, we have

$$\eta \le 1, \quad q.s.$$

On the other hand, we have

$$\hat{E}[(\eta - 1)^2] = \hat{E}[\eta(\eta - 2)] + 1 = \hat{E}[\eta(\xi - 2)] + 1.$$

Since $\xi - 2 \perp \eta$, *we have*

$$\hat{E}[(1 - \eta)^2] = \hat{E}(1 - \eta).$$

By Theorem 2.12 *below, there exists* $P \in \mathcal{P}$ *such that*

$$E_P[(1 - \eta)^2] = \hat{E}[(1 - \eta)^2].$$

Noting that

$$E_P(1 - \eta) \le \hat{E}(1 - \eta) = \hat{E}[(1 - \eta)^2] = E_P[(1 - \eta)^2] \le E_P(1 - \eta),$$

we have

$$E_P[(1 - \eta)^2] = E_P(1 - \eta).$$

By this, we have

$$\eta^2 = \eta, \quad P\text{-}a.s.$$

Since $\eta \ge \varepsilon$, *we have* $\eta = 1$, P-*a.s. So we have*

$$\hat{E}[(1 - \eta)^2] = E_P[(1 - \eta)^2] = 0.$$

(ii) *Let* $\{X_t\}$ *on* $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ *be a process with stationary and independent increments and let* $c = \hat{E}(X_T)/T$. *If* $\hat{E}(X_t) \to 0$ *as* $t \downarrow 0$, *then for any* $0 \le s < t \le T$, *we have* $\hat{E}(X_t - X_s) = c(t - s)$.

Definition 2.8. *Let $\{X_t\}$ be a d-dimensional process defined on $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ such that*:

(i) $X_0 = 0$;
(ii) *$\{X_t\}$ is a process with stationary and independent increments w.r.t. the filtration*;
(iii) $\lim_{t\to 0} \hat{E}[|X_t|^3]t^{-1} = 0$.

Then $\{X_t\}$ is called a generalized G-Brownian motion.
If in addition $\hat{E}(X_t) = \hat{E}(-X_t) = 0$ for all $t \in [0, T]$, $\{X_t\}$ is called a (symmetric) G-Brownian motion.

Remark 2.9. (i) *Clearly, the coordinate process $\{B_t\}$ is a (symmetric) G-Brownian motion and its quadratic variation process $\{\langle B\rangle_t\}$ is a process with stationary and independent increments (w.r.t. the filtration).*

(ii) [4] *gave a characterization for the generalized G-Brownian motion: Let $\{X_t\}$ be a generalized G-Brownian motion. Then*

$$X_{t+s} - X_t \sim \sqrt{s}\xi + s\eta \quad \text{for } t, s \geq 0, \tag{2.2}$$

where (ξ, η) is G-distributed (see, e.g., [6] *for the definition of G-distributed random vectors). In fact, this characterization presented a decomposition of generalized G-Brownian motion in the sense of distribution. In this article, we shall give a pathwise decomposition for the generalized G-Brownian motion.*

Let $H_G^0(0, T)$ be the collection of processes of the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0, T]$, $N \geq 1$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{]t_j, t_{j+1}]}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \ldots, N-1$. For every $\eta \in H_G^0(0, T)$, let $\|\eta\|_{H_G^p} = \{\hat{E}(\int_0^T |\eta_s|^2\, ds)^{p/2}\}^{1/p}$, $\|\eta\|_{M_G^p} = \{\hat{E}(\int_0^T |\eta_s|^p\, ds)\}^{1/p}$ and denote by $H_G^p(0, T)$, $M_G^p(0, T)$ the completions of $H_G^0(0, T)$ under the norms $\|\cdot\|_{H_G^p}$, $\|\cdot\|_{M_G^p}$ respectively.

Definition 2.10. *For every $\eta \in H_G^0(0, T)$ with the form*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{]t_j, t_{j+1}]}(t),$$

we define

$$I(\eta) = \int_0^T \eta(s)\, dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

By B–D–G inequality (see Proposition 4.3 *in* [10] *for this inequality under G-expectation), the mapping $I : H_G^0(0, T) \to L_G^p(\Omega_T)$ is continuous under $\|\cdot\|_{H_G^p}$ and thus can be continuously extended to $H_G^p(0, T)$.*

Definition 2.11. (i) *A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G-martingale if $\hat{E}_s(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both G-martingales, we call $\{M_t\}$ a symmetric G-martingale.*

(ii) *A random variable $\xi \in L_G^1(\Omega_T)$ is called symmetric if $\hat{E}(\xi) + \hat{E}(-\xi) = 0$.*

A G-martingale $\{M_t\}$ is symmetric if and only if M_T is symmetric.

Theorem 2.12 ([1,2]). *There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that*

$$\hat{E}(\xi) = \max_{P \in \mathcal{P}} E_P(\xi) \quad \text{for all } \xi \in \mathcal{H}_T^0.$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.

Remark 2.13. (i) *Let $(\Omega^0, \mathcal{F}^0, P^0)$ be a probability space and $\{W_t\}$ be a d-dimensional Brownian motion under P^0. Let $F^0 = \{\mathcal{F}_t^0\}$ be the augmented filtration generated by W.* [1] *proved that*

$$\mathcal{P}_M := \left\{ P_h | P_h = P^0 \circ X^{-1}, X_t = \int_0^t h_s \, \mathrm{d}W_s, h \in L^2_{F^0}\big([0,T]; \Gamma^{1/2}\big) \right\}$$

is a set that represents $\hat{E}$, where $\Gamma^{1/2} := \{\gamma^{1/2} | \gamma \in \Gamma\}$ and Γ is the set in the representation of $G(\cdot)$ in the formula (2.1).

(ii) *For the* 1*-dimensional case, i.e., $\Omega_T = C_0([0,T], R)$,*

$$L^2_{F^0} := L^2_{F^0}\big([0,T]; \Gamma^{1/2}\big) = \big\{ h | h \text{ is adapted w.r.t. } F^0 \text{ and } \underline{\sigma} \le h_s \le \overline{\sigma} \big\},$$

where $\overline{\sigma}^2 = \hat{E}(B_1^2)$ and $\underline{\sigma}^2 = -\hat{E}(-B_1^2)$.

$$G(a) = 1/2 \hat{E}\big[a B_1^2\big] = 1/2 \big[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-\big] \quad \text{for } a \in R.$$

(iii) *Set $c(A) = \sup_{P \in \mathcal{P}_M} P(A)$, for $A \in \mathcal{B}(\Omega_T)$. We say $A \in \mathcal{B}(\Omega_T)$ is a polar set if $c(A) = 0$. If an event happens except on a polar set, we say the event happens q.s.*

3. Characterization of processes with stationary and independent increments

In what follows, we only consider the G-expectation space $(\Omega_T, L_G^1(\Omega_T), \hat{E})$ with $\Omega_T = C_0([0,T], R)$ and $\overline{\sigma}^2 = \hat{E}(B_1^2) > -\hat{E}(-B_1^2) = \underline{\sigma}^2 > 0$.

Lemma 3.1. *For $\zeta \in M_G^1(0,T)$ and $\varepsilon > 0$, let*

$$\zeta_t^\varepsilon = \frac{1}{\varepsilon} \int_{(t-\varepsilon)^+}^t \zeta_s \, \mathrm{d}s$$

and

$$\zeta_t^{\varepsilon,0} = \sum_{k=1}^{k_\varepsilon - 1} \frac{1}{\varepsilon} \int_{(k-1)\varepsilon}^{k\varepsilon} \zeta_s \, \mathrm{d}s \, 1_{]k\varepsilon,(k+1)\varepsilon]}(t),$$

where $t \in [0,T]$, $k_\varepsilon \varepsilon \le T < (k_\varepsilon + 1)\varepsilon$. Then as $\varepsilon \to 0$

$$\big\| \zeta^\varepsilon - \zeta \big\|_{M_G^1(0,T)} \to 0 \quad \text{and} \quad \big\| \zeta^{\varepsilon,0} - \zeta \big\|_{M_G^1(0,T)} \to 0.$$

Proof. The proofs of the two cases are similar. Here we only prove the second case. Our proof starts with the observation that for any $\zeta, \zeta' \in M_G^1(0,T)$

$$\big\| \zeta^{\varepsilon,0} - \zeta'^{\varepsilon,0} \big\|_{M_G^1(0,T)} \le \big\| \zeta - \zeta' \big\|_{M_G^1(0,T)}. \tag{3.1}$$

By the definition of the space $M_G^1(0,T)$, we know that for every $\zeta \in M_G^1(0,T)$, there exists a sequence of processes $\{\zeta^n\}$ with

$$\zeta_t^n = \sum_{k=0}^{m_n - 1} \xi_{t_k^n}^n 1_{]t_k^n, t_{k+1}^n]}(t)$$

and $\xi^n_{t^n_k} \in L_{ip}(\Omega_{t^n_k})$ such that

$$\|\zeta - \zeta^n\|_{M^1_G(0,T)} \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

It is easily seen that for every n

$$\|\zeta^{n;\varepsilon,0} - \zeta^n\|_{M^1_G(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.3}$$

Thus we get

$$\begin{aligned}
&\|\zeta^{\varepsilon,0} - \zeta\|_{M^1_G(0,T)} \\
&\quad \le \|\zeta^{\varepsilon,0} - \zeta^{n;\varepsilon,0}\|_{M^1_G(0,T)} + \|\zeta^n - \zeta^{n;\varepsilon,0}\|_{M^1_G(0,T)} + \|\zeta^n - \zeta\|_{M^1_G(0,T)} \\
&\quad \le 2\|\zeta^n - \zeta\|_{M^1_G(0,T)} + \|\zeta^n - \zeta^{n;\varepsilon,0}\|_{M^1_G(0,T)}.
\end{aligned}$$

The second inequality follows from (3.1). Combining (3.2) and (3.3), first letting $\varepsilon \to 0$, then letting $n \to \infty$, we have

$$\|\zeta^{\varepsilon,0} - \zeta\|_{M^1_G(0,T)} \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

Theorem 3.2. *Let $A_t = \int_0^t h_s \, \mathrm{d}s$ with $h \in M^1_G(0,T)$ be a process with stationary and independent increments* (*w.r.t. the filtration*). *Then we have $h \equiv c$ for some constant c.*

Proof. Let $\overline{c} := \hat{E}(A_T)/T \ge -\hat{E}(-A_T)/T =: \underline{c}$. For $n \in N$, set $\varepsilon = T/(2n)$, and define $h^{T/(2n),0}$ as in Lemma 3.1. Then we have

$$\begin{aligned}
&\|h - h^{T/(2n),0}\|_{M^1_G(0,T)} \\
&\quad = \hat{E}\left[\sum_{k=0}^{2n-1} \int_{kT/(2n)}^{(k+1)T/(2n)} \left|h_s - h_s^{T/(2n),0}\right| \mathrm{d}s\right] \\
&\quad \ge \hat{E}\left[\sum_{k=1}^{n-1} \int_{2kT/(2n)}^{(2k+1)T/(2n)} \left(h_s - h_s^{T/(2n),0}\right) \mathrm{d}s\right] \\
&\quad = \hat{E}\left[\sum_{k=1}^{n-1}\left(\int_{2kT/(2n)}^{(2k+1)T/(2n)} h_s \, \mathrm{d}s - \int_{(2k-1)T/(2n)}^{2kT/(2n)} h_s \, \mathrm{d}s\right)\right] \\
&\quad = \hat{E}\sum_{k=1}^{n-1}\left[(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)T/2n})\right].
\end{aligned}$$

Consequently, from the condition of independence of the increments and their stationarity, we have

$$\begin{aligned}
&\|h - h^{T/(2n),0}\|_{M^1_G(0,T)} \\
&\quad \ge \sum_{k=1}^{n-1} \hat{E}\left[(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)T/2n})\right] \\
&\quad = \sum_{k=1}^{n-1} (\overline{c} - \underline{c})T/(2n) \\
&\quad = (\overline{c} - \underline{c})(n-1)T/(2n).
\end{aligned}$$

So by Lemma 3.1, letting $n \to \infty$, we have $\overline{c} = \underline{c}$. Furthermore, we note that $M_t := A_t - \overline{c}t$ is a G-martingale. In fact, for $t > s$, we see

$$\begin{aligned}
&\hat{E}_s(M_t) \\
&\quad = \hat{E}_s(M_t - M_s) + M_s \\
&\quad = \hat{E}(M_t - M_s) + M_s \\
&\quad = M_s.
\end{aligned}$$

The second equality is due to the independence of increments of M w.r.t. the filtration.

So $\{M_t\}$ is a symmetric G-martingale with finite variation, from which we conclude that $M_t \equiv 0$, hence that $A_t = \overline{c}t$. □

Corollary 3.3. *Assume $\overline{\sigma} > \underline{\sigma} > 0$. Then we have that $\{\frac{\mathrm{d}}{\mathrm{d}s}\langle B\rangle_s\} \notin M_G^1(0,T)$.*

Proof. The proof is straightforward from Theorem 3.2. □

Corollary 3.4. *There is no symmetric G-martingale $\{M_t\}$ which is a standard Brownian motion under G-expectation (i.e. $\langle M\rangle_t = t$).*

Proof. Let $\{M_t\}$ be a symmetric G-martingale. If $\{M_t\}$ is also a standard Brownian motion, by Theorem 4.8 in [10] or Corollary 5.2 in [11], there exists $\{h_s\} \in M_G^2(0,T)$ such that

$$M_t = \int_0^t h_s \,\mathrm{d}B_s$$

and

$$\int_0^t h_s^2 \,\mathrm{d}\langle B\rangle_s = t.$$

Thus we have $\frac{\mathrm{d}}{\mathrm{d}s}\langle B\rangle_s = h_s^{-2} \in M_G^1(0,T)$, which contradicts the conclusion of Corollary 3.3. □

Proposition 3.5. *Let $A_t = \int_0^t h_s \,\mathrm{d}s$ with $h \in M_G^1(0,T)$ be a process with independent increments. Then A_t is symmetric for every $t \in [0,T]$.*

Proof. By arguments similar to those in the proof of Theorem 3.2, we have

$$\begin{aligned}
&\big\|h - h^{T/(2n),0}\big\|_{M_G^1(0,T)} \\
&\quad \geq \hat{E}\sum_{k=0}^{n-1}\big[(A_{(2k+1)T/2n} - A_{2kT/2n}) - (A_{2kT/2n} - A_{(2k-1)^+T/2n})\big] \\
&\quad = \sum_{k=0}^{n-1}\big\{\hat{E}(A_{(2k+1)T/2n} - A_{2kT/2n}) + \hat{E}\big[-(A_{2kT/2n} - A_{(2k-1)^+T/2n})\big]\big\}.
\end{aligned}$$

The right side of the first inequality is only the sum of the odd terms. Summing up the even terms only, we have

$$\begin{aligned}
&\big\|h - h^{T/(2n),0}\big\|_{M_G^1(0,T)} \\
&\quad \geq \sum_{k=0}^{n-1}\big\{\hat{E}(A_{(2k+2)T/2n} - A_{(2k+1)T/2n}) + \hat{E}\big[-(A_{(2k+1)T/2n} - A_{2kT/2n})\big]\big\}.
\end{aligned}$$

Combining the above inequalities, we have

$$
\begin{aligned}
&2\|h-h^{T/(2n),0}\|_{M_G^1(0,T)}\\
&\quad\ge\sum_{k=0}^{2n-1}\{\hat E[A_{(k+1)T/2n}-A_{kT/2n}]+\hat E[-(A_{(k+1)T/2n}-A_{kT/2n})]\}\\
&\quad\ge\hat E\sum_{k=0}^{2n-1}[A_{(k+1)T/2n}-A_{kT/2n}]+\hat E\sum_{k=0}^{2n-1}[-(A_{(k+1)T/2n}-A_{kT/2n})]\\
&\quad=\hat E(A_T)+\hat E(-A_T).
\end{aligned}
$$

Thus by Lemma 3.1, letting $n\to\infty$, we have $\hat E(A_T)+\hat E(-A_T)=0$, which means that A_T is symmetric. □

For $n\in N$, define $\delta_n(s)$ in the following way:

$$
\delta_n(s)=\sum_{i=0}^{n-1}(-1)^i 1_{]\frac{iT}{n},\frac{(i+1)T}{n}]}(s)\quad\text{for all } s\in[0,T].
$$

In [12] we proved that $\lim_{n\to\infty}\hat E(\int_0^T\delta_n(s)h_s\,\mathrm{d}s)=0$ for $h\in M_G^1(0,T)$.

Let $\mathcal{F}_t=\sigma\{B_s|s\le t\}$ and $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$.

In the following, we shall use some notations introduced in Remark 2.13.

For every $P\in\mathcal{P}_M$ and $t\in[0,T]$, set $\mathcal{A}_{t,P}:=\{Q\in\mathcal{P}_M|Q_{|\mathcal{F}_t}=P_{|\mathcal{F}_t}\}$. Proposition 3.4 in [9] gave the following result: For $t\in[0,T]$, assume $\xi\in L_G^1(\Omega_T)$ and $\eta\in L_G^1(\Omega_t)$. Then $\eta=\hat E_t(\xi)$ if and only if for every $P\in\mathcal{P}_M$

$$
\eta=\operatorname*{ess\,sup}_{Q\in\mathcal{A}_{t,P}}{}^{P}E_Q(\xi|\mathcal{F}_t),\qquad P\text{-a.s.},
$$

where $\operatorname{ess\,sup}^P$ denotes the essential supremum under P.

Theorem 3.6. *Let $A_t=\int_0^t h_s\,\mathrm{d}\langle B\rangle_s$ be a process with stationary, independent increments (w.r.t. the filtration) and $h\in M_G^{1,+}(0,T)$. If $A_T\in L_G^\beta(\Omega_T)$ for some $\beta>1$, we have $A_t=c\langle B\rangle_t$ for some constant $c\ge0$.*

Proof. For the readability, we divide the proof into several steps:

Step 1. Set $K_t:=\int_0^t h_s\,\mathrm{d}s$. We claim that K_T is symmetric.

Step 1.1. Let $\overline{\mu}=\hat E(A_T)/T$ and $\underline{\mu}=-\hat E(-A_T)/T$. First, we shall prove that $\frac{\overline{\mu}}{\overline{\sigma}^2}=\frac{\underline{\mu}}{\underline{\sigma}^2}$.

Actually, for any $0\le s<t\le T$, we have

$$
\hat E_s\left(\int_s^t h_r\,\mathrm{d}r\right)=\hat E_s\left(\int_s^t\theta_r^{-1}\,\mathrm{d}A_r\right)\ge\frac{1}{\overline{\sigma}^2}\hat E_s\left(\int_s^t\mathrm{d}A_r\right)=\frac{\overline{\mu}}{\overline{\sigma}^2}(t-s)\quad\text{q.s.},
$$

where the inequality holds due to $\theta_s:=\frac{\mathrm{d}\langle B\rangle_s}{\mathrm{d}s}\le\overline{\sigma}^2$, q.s. Noting that $\underline{\mu}t-A_t$ is nonincreasing by Lemma 4.3 in Section 4 since it is a G-martingale with finite variation, we have, for every $\eta\in L^2_{F^0}$, P_η-a.s.,

$$
\begin{aligned}
&\hat E_s\left(\int_s^t h_r\,\mathrm{d}r\right)\\
&\quad=\operatorname*{ess\,sup}_{Q\in\mathcal{A}_{t,P_\eta}}{}^{P_\eta}E_Q\left(\int_s^t h_r\,\mathrm{d}r\,\middle|\,\mathcal{F}_s\right)
\end{aligned}
$$

$$= \operatorname*{ess\,sup}_{Q \in \mathcal{A}_{t,P_\eta}}{}^{P_\eta} E_Q\left(\int_s^t \theta_r^{-1}\, \mathrm{d}A_r \Big| \mathcal{F}_s\right)$$

$$\geq \underline{\mu} \operatorname*{ess\,sup}_{Q \in \mathcal{A}_{t,P_\eta}}{}^{P_\eta} E_Q\left(\int_s^t \theta_r^{-1}\, \mathrm{d}r \Big| \mathcal{F}_s\right)$$

$$= \frac{\underline{\mu}}{\underline{\sigma}^2}(t-s).$$

So $\hat{E}_s(\int_s^t h_r\, \mathrm{d}r) \geq \max\{\frac{\overline{\mu}}{\overline{\sigma}^2}, \frac{\underline{\mu}}{\underline{\sigma}^2}\}(t-s) =: \overline{\lambda}(t-s)$, q.s.

On the other hand,

$$\hat{E}_s\left(-\int_s^t h_r\, \mathrm{d}r\right) = \hat{E}_s\left(\int_s^t -\theta_r^{-1}\, \mathrm{d}A_r\right) \geq \frac{1}{\underline{\sigma}^2}\hat{E}_s\left(-\int_s^t \mathrm{d}A_r\right) = -\frac{\underline{\mu}}{\underline{\sigma}^2}(t-s), \quad \text{q.s.}$$

and for every $\eta \in L^2_{F^0}$, P_η-a.s.,

$$\hat{E}_s\left(-\int_s^t h_r\, \mathrm{d}r\right)$$

$$= \operatorname*{ess\,sup}_{Q \in \mathcal{A}_{t,P_\eta}}{}^{P_\eta} E_Q\left(-\int_s^t h_r\, \mathrm{d}r \Big| \mathcal{F}_s\right)$$

$$= \operatorname*{ess\,sup}_{Q \in \mathcal{A}_{t,P_\eta}}{}^{P_\eta} E_Q\left(-\int_s^t \theta_r^{-1}\, \mathrm{d}A_r \Big| \mathcal{F}_s\right)$$

$$\geq \overline{\mu} \operatorname*{ess\,sup}_{Q \in \mathcal{A}_{t,P_\eta}}{}^{P_\eta} E_Q\left(-\int_s^t \theta_r^{-1}\, \mathrm{d}r \Big| \mathcal{F}_s\right)$$

$$= -\frac{\overline{\mu}}{\overline{\sigma}^2}(t-s)$$

since $A_t - \overline{\mu}t$ is nonincreasing. So

$$\hat{E}_s\left(-\int_s^t h_r\, \mathrm{d}r\right) \geq -\min\left\{\frac{\overline{\mu}}{\overline{\sigma}^2}, \frac{\underline{\mu}}{\underline{\sigma}^2}\right\}(t-s) =: -\underline{\lambda}(t-s), \quad \text{q.s.}$$

Noting that

$$\hat{E}\left(\int_0^T \delta_{2n}(s) h_s\, \mathrm{d}s\right)$$

$$= \hat{E}\left[\int_0^{(2n-1)T/(2n)} \delta_{2n}(s) h_s\, \mathrm{d}s + \hat{E}_{(2n-1)T/(2n)}\left(-\int_{(2n-1)T/(2n)}^T h_s\, \mathrm{d}s\right)\right]$$

$$\geq (-\underline{\lambda})\frac{T}{2n} + \hat{E}\left[\int_0^{(2n-2)T/(2n)} \delta_{2n}(s) h_s\, \mathrm{d}s + \hat{E}_{(2n-2)T/(2n)}\left(\int_{(2n-2)T/(2n)}^{(2n-1)T/(2n)} h_s\, \mathrm{d}s\right)\right]$$

$$\geq \frac{\overline{\lambda} - \underline{\lambda}}{2n}T + \hat{E}\left[\int_0^{(2n-2)T/(2n)} \delta_{2n}(s) h_s\, \mathrm{d}s\right],$$

we have

$$\hat{E}\left(\int_0^T \delta_{2n}(s) h_s\, \mathrm{d}s\right) \geq \frac{\overline{\lambda} - \underline{\lambda}}{2}T.$$

So

$$0=\lim_{n\to\infty}\hat{E}\left(\int_0^T \delta_{2n}(s)h_s\,\mathrm{d}s\right)\geq\frac{\overline{\lambda}-\underline{\lambda}}{2}T$$

and $\frac{\overline{\mu}}{\overline{\sigma}^2}=\frac{\underline{\mu}}{\underline{\sigma}^2}=:\lambda$.

Step 1.2. For every $\eta\in L^2_{F^0}$, $E_{P_\eta}(K_T)=\lambda T$, which implies that K_T is symmetric.

Step 1.2.1. We now introduce some notations: For $0\leq s<t\leq T$ and $\eta\in L^2_{F^0}$, set $\overline{\eta}=\overline{\sigma}$, $\underline{\eta}=\underline{\sigma}$, $\eta^*=\sqrt{\frac{\overline{\sigma}^2+\underline{\sigma}^2}{2}}$ on $]s,t]$ and $\overline{\eta}=\underline{\eta}=\eta^*=\eta$ on $]s,t]^c$. For $n\in N$, set $\eta^n_r=\sum_{i=0}^{n-1}(\underline{\sigma}1_{]t_{2i},t_{2i+1}]}(r)+\overline{\sigma}1_{]t_{2i+1},t_{2i+2}]}(r))$ on $]s,t]$ and $\eta^n=\eta$ on $]s,t]^c$, where $t_j=s+\frac{j}{2n}(t-s),\ j=0,\ldots,2n$.

Step 1.2.2. $E_{P_{\eta^n}}(\int_s^t(h_r-\lambda)\,\mathrm{d}r|\mathcal{F}_s)\to 0$, P_η-a.s., as $n\to\infty$.

Actually, we have, P_η-a.s.,

$$\overline{\mu}(t-s)=\hat{E}_s\left(\int_s^t h_r\,\mathrm{d}\langle B\rangle_r\right)\geq E_{P_{\overline{\eta}}}\left(\int_s^t h_r\,\mathrm{d}\langle B\rangle_r\Big|\mathcal{F}_s\right)=\overline{\sigma}^2E_{P_{\overline{\eta}}}\left(\int_s^t h_r\,\mathrm{d}r\Big|\mathcal{F}_s\right).$$

So

$$E_{P_{\overline{\eta}}}\left(\int_s^t h_r\,\mathrm{d}r\Big|\mathcal{F}_s\right)\leq\lambda(t-s),\quad P_\eta\text{-a.s.}\tag{3.4}$$

By similar arguments we have that

$$E_{P_{\underline{\eta}}}\left(\int_s^t h_r\,\mathrm{d}r\Big|\mathcal{F}_s\right)\geq\lambda(t-s),\quad P_\eta\text{-a.s.}\tag{3.5}$$

Let's compute the following conditional expectations:

$$\begin{aligned}&E_{P_{\eta^n}}\left(\int_s^t(h_r-\lambda)\delta_{2n}(r)\,\mathrm{d}r\Big|\mathcal{F}_s\right)\\&=E^{\mathcal{F}_s}_{P_{\eta^n}}\left[\sum_{i=0}^{n-1}\left\{E^{\mathcal{F}_{t_{2i}}}_{P_{\eta^n}}\int_{t_{2i}}^{t_{2i+1}}(h_r-\lambda)\,\mathrm{d}r+E^{\mathcal{F}_{t_{2i+1}}}_{P_{\eta^n}}\int_{t_{2i+1}}^{t_{2i+2}}(\lambda-h_r)\,\mathrm{d}r\right\}\right]\\&=:E^{\mathcal{F}_s}_{P_{\eta^n}}\left[\sum_{i=0}^{n-1}(A_i+B_i)\right],\end{aligned}$$

where $\delta_{2n}(r)=\sum_{i=0}^{n-1}(1_{]t_{2i},t_{2i+1}]}(r)-1_{]t_{2i+1},t_{2i+2}]}(r))$, $t_j=s+\frac{j}{2n}(t-s),\ j=0,\ldots,2n$;

$$E_{P_{\eta^n}}\left(\int_s^t(h_r-\lambda)\,\mathrm{d}r\Big|\mathcal{F}_s\right)=E^{\mathcal{F}_s}_{P_{\eta^n}}\left[\sum_{i=0}^{n-1}(A_i-B_i)\right].$$

By (3.4) and (3.5) (noting that η and s,t are all arbitrary), we conclude that $A_i,B_i\geq 0$, P_{η_n}-a.s. So

$$\left|E_{P_{\eta^n}}\left(\int_s^t(h_r-\lambda)\,\mathrm{d}r\Big|\mathcal{F}_s\right)\right|\leq E_{P_{\eta^n}}\left(\int_s^t(h_r-\lambda)\delta_{2n}(r)\,\mathrm{d}r\Big|\mathcal{F}_s\right),\quad P_\eta\text{-a.s.}$$

Noting that

$$E_{P_{\eta^n}}\left(\int_s^t(h_r-\lambda)\delta_{2n}(r)\,\mathrm{d}r\Big|\mathcal{F}_s\right)\leq\hat{E}_s\left[\int_s^t(h_r-\lambda)\delta_{2n}(r)\,\mathrm{d}r\right],\quad P_\eta\text{-a.s.}$$

and

$$\hat{E}_s\left[\int_s^t (h_r-\lambda)\delta_{2n}(r)\,\mathrm{d}r\right]\to 0 \quad \text{q.s., as } n\to\infty,$$

we have $E_{P_{\eta^n}}(\int_s^t (h_r-\lambda)\,\mathrm{d}r|\mathcal{F}_s)\to 0$, P_η-a.s., as $n\to\infty$.

Step 1.2.3. For any $\xi\in L_G^1(\Omega_t)$, $E_{P_{\eta^n}}(\xi|\mathcal{F}_s)\to E_{P_{\eta^*}}(\xi|\mathcal{F}_s)$, P_η-a.s., as $n\to\infty$.

In fact, for $\xi=\varphi(B_{s_1}-B_{s_0},\dots,B_{s_m}-B_{s_{m-1}})\in L_{ip}(\Omega_t)$, the conclusion is obvious. For general $\xi\in L_G^1(\Omega_t)$, there exists a sequence $\{\xi^m\}\subset L_{ip}(\Omega_t)$ such that $\hat{E}[|\xi^m-\xi|]=\hat{E}[\hat{E}_s(|\xi^m-\xi|)]\to 0$. So we can assume $\hat{E}_s(|\xi^m-\xi|)\to 0$ q.s.

Then, P_η-a.s., we have

$$\begin{aligned}
&\left|E_{P_{\eta^n}}(\xi|\mathcal{F}_s)-E_{P_{\eta^*}}(\xi|\mathcal{F}_s)\right|\\
&\quad\le \left|E_{P_{\eta^n}}(\xi|\mathcal{F}_s)-E_{P_{\eta^n}}\left(\xi^m|\mathcal{F}_s\right)\right|+\left|E_{P_{\eta^n}}\left(\xi^m|\mathcal{F}_s\right)-E_{P_{\eta^*}}\left(\xi^m|\mathcal{F}_s\right)\right|\\
&\qquad+\left|E_{P_{\eta^*}}\left(\xi^m|\mathcal{F}_s\right)-E_{P_{\eta^*}}(\xi|\mathcal{F}_s)\right|\\
&\quad\le 2\hat{E}_s\left(\left|\xi^m-\xi\right|\right)+\left|E_{P_{\eta^n}}\left(\xi^m|\mathcal{F}_s\right)-E_{P_{\eta^*}}\left(\xi^m|\mathcal{F}_s\right)\right|.
\end{aligned}$$

First letting $n\to\infty$, then letting $m\to\infty$, we have $E_{P_{\eta^n}}(\xi|\mathcal{F}_s)\to E_{P_{\eta^*}}(\xi|\mathcal{F}_s)$, P_η-a.s. So combining Step 1.2.2 and Step 1.2.3, we have

$$E_{P_{\eta^*}}\left(\int_s^t h_r\,\mathrm{d}r\middle|\mathcal{F}_s\right)=\lambda(t-s), \qquad P_\eta\text{-a.s.} \tag{3.6}$$

Step 1.2.4. For $0\le s<t\le T$, $\eta\in L^2_{F^0}$, $\sigma\in[\underline{\sigma},\overline{\sigma}]$, set $\eta^\sigma=\sigma$ on $]s,t]$ and $\eta^\sigma=\eta$ on $]s,t]^c$. We have

$$E_{P_{\eta^\sigma}}\left(\int_s^t h_r\,\mathrm{d}r\middle|\mathcal{F}_s\right)=\lambda(t-s), \qquad P_\eta\text{-a.s.}$$

In fact, Step 1.2.2–Step 1.2.3 proved the following fact: If (3.4), (3.5) hold for some $\sigma,\sigma'\in[\underline{\sigma},\overline{\sigma}]$, then (3.6) holds for $\sqrt{\frac{\sigma^2+\sigma'^2}{2}}$. So by repeating the Step 1.2.2–Step 1.2.3, we get the desired result.

Step 1.2.5. For any simple process $\eta\in L^2_{F^0}$, $E_{P_\eta}(K_T)=\lambda T$.

Let $\eta_r=\sum_{i=0}^{m-1}\eta_{t_i}1_{]t_i,t_{i+1}]}(r)\in L^2_{F^0}$ with $\eta_{t_i}=\sum_{j=1}^{n_i}a_j^i 1_{A_j^i}$ an $\mathcal{F}^0_{t_i}$ measurable simple function, where $\{t_0,\dots,t_m\}$ is a given partition of $[0,T]$. Set $X_t=\int_0^t\eta_r\,\mathrm{d}W_r$. Let $F^X=\{\mathcal{F}_t^X\}$ be the filtration generated by X.

Fix $0\le i<m$. Set $\eta_s^{j,\varepsilon}=\eta_s 1_{[0,t_i+\varepsilon]}(s)+a_j^i 1_{]t_i+\varepsilon,T]}(s)$ and $X_t^{j,\varepsilon}=\int_0^t\eta_s^{j,\varepsilon}\,\mathrm{d}W_s$ for $\varepsilon>0$ small enough. Let $F^{X^{j,\varepsilon}}=\{\mathcal{F}_t^{X^{j,\varepsilon}}\}$ be the filtration generated by $X^{j,\varepsilon}$. Then

$$E_{P_\eta}\left(\int_{t_i+\varepsilon}^{t_{i+1}}h_r\,\mathrm{d}r\right)=E_{P^0}\left(\int_{t_i+\varepsilon}^{t_{i+1}}h_r\circ X\,\mathrm{d}r\right)=E_{P^0}\left[E_{P^0}\left(\int_{t_i+\varepsilon}^{t_{i+1}}h_r\circ X\,\mathrm{d}r\middle|\mathcal{F}^X_{t_i+\varepsilon}\right)\right].$$

Since $A_j^i\in\mathcal{F}^X_{t_i+\varepsilon}=\mathcal{F}^{X^{j,\varepsilon}}_{t_i+\varepsilon}$ and $X_t=\sum_{j=0}^{n_i}X_t^{j,\varepsilon}1_{A_j^i}$ on $[0,t_{i+1}]$, we have

$$\begin{aligned}
&E_{P^0}\left(\int_{t_i+\varepsilon}^{t_{i+1}}h_r\circ X\,\mathrm{d}r\middle|\mathcal{F}^X_{t_i+\varepsilon}\right)\\
&\quad=\sum_{j=1}^{n_i}E_{P^0}\left(1_{A_j^i}\int_{t_i}^{t_{i+1}}h_r\circ X^{j,\varepsilon}\,\mathrm{d}r\middle|\mathcal{F}^X_{t_i+\varepsilon}\right)\\
&\quad=\sum_{j=1}^{n_i}1_{A_j^i}E_{P^0}\left(\int_{t_i}^{t_{i+1}}h_r\circ X^{j,\varepsilon}\,\mathrm{d}r\middle|\mathcal{F}^{X^{j,\varepsilon}}_{t_i+\varepsilon}\right).
\end{aligned}$$

Noting that

$$E_{P^0}\left(\int_{t_i+\varepsilon}^{t_{i+1}} h_r \circ X^{j,\varepsilon}\,\mathrm{d}r \middle| \mathcal{F}_{t_i+\varepsilon}^{X^{j,\varepsilon}}\right) = E_{P_{\eta^{j,\varepsilon}}}\left(\int_{t_i+\varepsilon}^{t_{i+1}} h_r\,\mathrm{d}r \middle| \mathcal{F}_{t_i+\varepsilon}\right) \circ X^{j,\varepsilon} = \lambda(t_{i+1}-t_i-\varepsilon) \quad P^0\text{-a.s.},$$

by Step 1.2.4, we have $E_{P_\eta}(\int_{t_i}^{t_{i+1}} h_r\,\mathrm{d}r) = \lambda(t_{i+1}-t_i)$ and $E_{P_\eta}(K_T) = \lambda T$.

Step 2. $h \equiv \lambda$.

Let $M_t = \int_0^t h_r\,\mathrm{d}\langle B\rangle_s - \int_0^t 2G(h_s)\,\mathrm{d}s$ and $N_t = \int_0^t h_s\,\mathrm{d}\langle B\rangle_s - \overline{\mu}t$. As is mentioned in the Introduction, [4] proved that $\{M_t\}$ is a G-martingale. Since $\{\int_0^t h_s\,\mathrm{d}\langle B\rangle_s\}$ is a process with stationary and independent increments w.r.t. the filtration, we know that $\{N_t\}$ is also a G-martingale. Let $L_t = \hat{E}_t(\overline{\mu}T - \overline{\sigma}^2 K_T)$. Then $\{L_t\}$ is a symmetric G-martingale since K_T is symmetric. By the symmetry of $\{L_t\}$ we have

$$M_t = \hat{E}_t(M_T) = \hat{E}_t(L_T + N_T) = L_t + N_t.$$

By the uniqueness of the G-martingale decomposition, we get $L \equiv 0$ and $h \equiv \lambda$. □

Remark 3.7. *Clearly,* $h \in M_G^\beta(0,T)$ *for some* $\beta > 1$ *implies* $A_T = \int_0^T h_s\,\mathrm{d}\langle B\rangle \in L_G^\beta(\Omega_T)$.

4. Characterization of the *G*-Brownian motion

A version of the martingale characterization for the G-Brownian motion was given in [13], where only symmetric G-martingales with Markovian property were considered. Here we shall present a martingale characterization in a quite different form, which is a natural but nontrivial generalization of the classical case in a probability space.

Theorem 4.1 (Martingale characterization of the *G*-Brownian motion).

Let $\{M_t\}$ *be a symmetric* G*-martingale with* $M_T \in L_G^\alpha(\Omega_T)$ *for some* $\alpha > 2$ *and* $\{\langle M\rangle_t\}$ *a process with stationary and independent increments* (*w.r.t. the filtration*). *Then* $\{M_t\}$ *is a* G*-Brownian motion*:

Let $\{M_t\}$ *be a* G*-Brownian motion on* $(\Omega_T, L_G^1(\Omega_T), \hat{E})$. *Then there exists a positive constant* c *such that* $\langle M\rangle_t = c\langle B\rangle_t$.

Proof. By Corollary 5.2 in [11], there exists $h \in M_G^2(0,T)$ such that $M_t = \int_0^t h_s\,\mathrm{d}B_s$. So $\langle M\rangle_t = \int_0^t h_s^2\,\mathrm{d}\langle B\rangle_s$. By the assumption, we know that $\langle M\rangle_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$. By Theorem 3.6, there exists some constant $c \ge 0$ such that $h^2 \equiv c$. Thus by Theorem 2.12 and Remark 2.13, $\{M_t\}$ is a G-Brownian motion with M_t distributed as $N(0, [c\underline{\sigma}^2 t, c\overline{\sigma}^2 t])$.

On the other hand, if $\{M_t\}$ is a G-Brownian motion on $(\Omega_T, L_G^1(\Omega_T))$, then $\{M_t\}$ is a symmetric G-martingale. By the above arguments, we have $\langle M\rangle_t = c\langle B\rangle_t$ for some positive constant c. □

Let

$$\mathcal{H} = \left\{a \middle| a(t) = \sum_{k=0}^{n-1} a_{t_k} 1_{]t_k, t_{k+1}]}(t), n \in N, 0 = t_0 < t_1 < \cdots < t_n = T\right\}$$

and $H = \{a \in \mathcal{H} | \lambda[a = 0] = 0\}$, where λ is the Lebesgue measure.

Lemma 4.2. *Let* $\{L_t\}$ *be a process with absolutely continuous paths. Assume that there exist real numbers* $\underline{c} \le \overline{c}$ *such that* $\underline{c}(t-s) \le L_t - L_s \le \overline{c}(t-s)$ *for any* $s < t$. *Let* $C(a) = \overline{c}a^+ - \underline{c}a^-$ *for any* $a \in R$. *If*

$$\hat{E}\left(\int_0^T a(s)\,\mathrm{d}L_s\right) = \int_0^T C\big(a(s)\big)\,\mathrm{d}s \quad \textit{for all } a \in \mathcal{H},$$

we have that $\{L_t\}$ *is a process with stationary and independent increments such that* $\underline{c}t = -\hat{E}(-L_t) \le \hat{E}(L_t) = \overline{c}t$, *i.e., its distribution is determined by* $\underline{c}, \overline{c}$.

Proof. It suffices to prove the lemma for the case $\underline{c} < \overline{c}$. For any $a \in H$, let

$$\theta_s^a = \overline{c}1_{[a(s)\geq 0]} + \underline{c}1_{[a(s)<0]}.$$

By assumption,

$$\hat{E}\left(\int_0^T a(s)\,\mathrm{d}L_s\right) = \int_0^T a(s)\theta_s^a\,\mathrm{d}s.$$

On the other hand, by Theorem 2.12, there exists some weakly compact subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that

$$\hat{E}(\xi) = \max_{P\in\mathcal{P}} E_P(\xi) \quad \text{for all } \xi \in L_G^1(\Omega_T),$$

which means that there exists $P_a \in \mathcal{P}$ such that

$$E_{P_a}\left(\int_0^T a(s)\,\mathrm{d}L_s\right) = \int_0^T a(s)\theta_s^a\,\mathrm{d}s.$$

By the assumption for $\{L_t\}$, we have $P_a\{L_t = \int_0^t \theta_s^a\,\mathrm{d}s$, for all $t \in [0, T]\} = 1$. From this we have

$$\hat{E}\big[\varphi(L_{t_1} - L_{t_0}, \dots, L_{t_n} - L_{t_{n-1}})\big] \geq \varphi\left(\int_{t_0}^{t_1} \theta_s^a\,\mathrm{d}s, \dots, \int_{t_{n-1}}^{t_n} \theta_s^a\,\mathrm{d}s\right)$$

for any $\varphi \in C_b(R^n)$ and $n \in N$. Consequently,

$$\begin{aligned}
&\hat{E}\big[\varphi(L_{t_1} - L_{t_0}, \dots, L_{t_n} - L_{t_{n-1}})\big] \\
&\quad \geq \sup_{a\in H} \varphi\left(\int_{t_0}^{t_1} \theta_s^a\,\mathrm{d}s, \dots, \int_{t_{n-1}}^{t_n} \theta_s^a\,\mathrm{d}s\right) \\
&\quad = \sup_{c_1,\dots,c_n\in[\underline{c},\overline{c}]} \varphi\big(c_1(t_1 - t_0), \dots, c_n(t_n - t_{n-1})\big).
\end{aligned}$$

The converse inequality is obvious. Thus $\{L_t\}$ is a process with stationary and independent increments such that $\underline{c}t = -\hat{E}(-L_t) \leq \hat{E}(L_t) = \overline{c}t$. □

Lemma 4.3. *Let $\{L_t\}$ be a G-martingale with finite variation and $L_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$. Then $\{L_t\}$ is nonincreasing. Particularly, $L_t \leq L_0 = \hat{E}(L_T)$.*

Proof. By Theorem 4.5 in [10], we know $\{L_t\}$ has the following decomposition

$$L_t = \hat{E}(L_T) + M_t + K_t,$$

where $\{M_t\}$ is a symmetric G-martingale and $\{K_t\}$ is a nonpositive, nonincreasing G-martingale. Since both $\{L_t\}$ and $\{K_t\}$ are processes with finite variation, we get $M_t \equiv 0$. Therefore, we have $L_t = \hat{E}(L_T) + K_t \leq \hat{E}(L_T) = L_0$. □

Theorem 4.4. *Let $\{X_t\}$ be a generalized G-Brownian motion with zero mean. Then we have the following decomposition:*

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a symmetric G-Brownian motion, and $\{L_t\}$ is a nonpositive, nonincreasing G-martingale with stationary and independent increments.

Proof. Clearly $\{X_t\}$ is a G-martingale. By Theorem 4.5 in [10], we have the following decomposition

$$X_t = M_t + L_t,$$

where $\{M_t\}$ is a symmetric G-martingale, and $\{L_t\}$ is a nonpositive, nonincreasing G-martingale. Noting that $X_t \in L_G^3(\Omega_T)$ from the definition of generalized G-Brownian motion, we know that $M_t, L_t \in L_G^{\beta}(\Omega_T)$ for any $1 \le \beta < 3$ by Theorem 4.5 in [10].

In the sequel, we first prove that $\{L_t\}$ is a process with stationary and independent increments. Noting that $\hat{E}(-L_t) = \hat{E}(-X_t) = ct$ for some positive constant c since $\{X_t\}$ is a process with stationary and independent increments, we claim that $-L_t - ct$ is a G-martingale. To prove this, it suffices to show that for any $t > s$, $\hat{E}_s[-(L_t - L_s)] = c(t-s)$. In fact, since $\{M_t\}$ is a symmetric G-martingale, we have

$$\hat{E}_s\big[-(L_t - L_s)\big] = \hat{E}_s\big[-(X_t - M_t - X_s + M_s)\big] = \hat{E}_s\big[-(X_t - X_s)\big].$$

Noting that $\{X_t\}$ is a process with independent increments (w.r.t. the filtration),

$$\hat{E}_s\big[-(X_t - X_s)\big] = \hat{E}\big[-(X_t - X_s)\big] = c(t-s).$$

Combining this with Lemma 4.3, we have $-(L_t - L_s) - c(t-s) \le 0$ for any $s < t$. On the other hand, for any $a \in \mathcal{H}$, noting that $\{M_t\}$ is a symmetric G-martingale, we have

$$\hat{E}\left[\int_0^T a(s)\,\mathrm{d}L_s\right] = \hat{E}\left[\int_0^T a(s)\,\mathrm{d}X_s\right] = \hat{E}\left[\sum_{k=0}^{n-1} a_{t_k}(X_{t_{k+1}} - X_{t_k})\right].$$

Since $\{X_t\}$ is a process with stationary, independent increments, we have

$$\begin{aligned}
&\hat{E}\left[\int_0^T a(s)\,\mathrm{d}L_s\right] \\
&\quad = \sum_{k=0}^{n-1} \hat{E}\big[a_{t_k}(X_{t_{k+1}} - X_{t_k})\big] \\
&\quad = \sum_{k=0}^{n-1} c a_{t_k}^-(t_{k+1} - t_k) \\
&\quad = \int_0^T c a^-(s)\,\mathrm{d}s = \int_0^T C\big(a(s)\big)\,\mathrm{d}s,
\end{aligned}$$

where $C(a(s))$ is defined as in Lemma 4.2 with $\overline{c} = 0$, $\underline{c} = -c$. By Lemma 4.2, $\{L_t\}$ is a process with stationary and independent increments.

Now we are in a position to show that $\{M_t\}$ is a (symmetric) G-Brownian motion. To this end, by Theorem 4.1, it suffices to prove that $\{\langle M\rangle_t\}$ is a process with stationary and independent increments (w.r.t. the filtration). For $n \in N$, let

$$X_t^n = \sum_{k=0}^{2^n-1} X_{kT/2^n} 1_{]kT/2^n,(k+1)T/2^n]}(t)$$

and

$$\Omega_t^n(X) = \sum_{k=0}^{2^n-1} (X_{(k+1)t/2^n} - X_{kt/2^n})^2.$$

Observing that $\Omega_t^n(X) = X_t^2 - 2\int_0^t X_s^n \,\mathrm{d}X_s$, we have

$$\begin{aligned}
&\left|\Omega_t^n(X) - \Omega_t^{m+n}(X)\right| \\
&\quad \le 2\left(\left|\int_0^t \left(X_s^n - X_s^{m+n}\right)\mathrm{d}M_s\right| + \left|\int_0^t \left(X_s^n - X_s^{m+n}\right)\mathrm{d}L_s\right|\right) \\
&\quad = 2\left(|I| + |II|\right)
\end{aligned}$$

for any $n, m \in N$. It's easy to check that

$$\hat{E}\left(|II|\right) \le c\int_0^t \hat{E}\left(\left|X_s^n - X_s^{m+n}\right|\right)\mathrm{d}s \to 0 \quad \text{as } m, n \to \infty.$$

Noting that

$$\begin{aligned}
I &= \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^m-1}(X_{it/2^n+jt/2^{n+m}} - X_{it/2^n})(M_{it/2^n+(j+1)t/2^{n+m}} - M_{it/2^n+jt/2^{n+m}}) \\
&= \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^m-1} I_i^j,
\end{aligned}$$

we get

$$\hat{E}\left(I^2\right) \le \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^m-1}\hat{E}\left[\left(I_i^j\right)^2\right].$$

Let's estimate the expectation $\hat{E}[(I_i^j)^2]$:

$$\begin{aligned}
&\hat{E}\left[\left(I_i^j\right)^2\right] \\
&\quad = \hat{E}\left[(X_{it/2^n+jt/2^{n+m}} - X_{it/2^n})^2(M_{it/2^n+(j+1)t/2^{n+m}} - M_{it/2^n+jt/2^{n+m}})^2\right] \\
&\quad \le 2\hat{E}\left[(X_{it/2^n+jt/2^{n+m}} - X_{it/2^n})^2\left\{(X_{it/2^n+(j+1)t/2^{n+m}} - X_{it/2^n+jt/2^{n+m}})^2\right.\right. \\
&\qquad \left.\left. + (L_{it/2^n+(j+1)t/2^{n+m}} - L_{it/2^n+jt/2^{n+m}})^2\right\}\right].
\end{aligned}$$

Noting that $-c(t-s) \le L_t - L_s \le 0$, we have

$$\hat{E}\left[\left(I_i^j\right)^2\right] \le \hat{E}\left[(X_{it/2^n+jt/2^{n+m}} - X_{it/2^n})^2\left\{(X_{it/2^n+(j+1)t/2^{n+m}} - X_{it/2^n+jt/2^{n+m}})^2 + c^2\frac{t^2}{2^{2(n+m)}}\right\}\right].$$

By (2.2), $\hat{E}[(X_t - X_s)^2] \le C_1|t-s|$ for some constant C_1. From the condition of independent increments of X, we have $\hat{E}[(I_i^j)^2] \le C\frac{j}{2^{2(n+m)}}$ for some constant C, hence that $\hat{E}(I^2) \to 0$, and finally that $\hat{E}(|\Omega_t^n(X) - \Omega_t^{m+n}(X)|) \to 0$ as $m, n \to \infty$. Then

$$\langle X\rangle_t := \lim_{L_G^1(\Omega_T), n\to\infty} \Omega_t^n$$

is a process with stationary and independent increments (w.r.t. the filtration). Noting that $\langle M\rangle_t = \langle X\rangle_t$, $\langle M\rangle_t$ is also a process with stationary and independent increments (w.r.t. the filtration). □

5. *G*-martingales with finite variation

Proposition 5.1. *Let $\eta \in M_G^1(0,T)$ with $|\eta| \equiv c$ for some constant c. Then*

$$K_t := \int_0^t \eta_s \,\mathrm{d}\langle B\rangle_s - \int_0^t 2G(\eta_s)\,\mathrm{d}s \tag{5.1}$$

is a process with stationary and independent increments. Moreover, for fixed c, all processes in the above form have the same distribution.

Proof. Since $-c(\overline{\sigma}^2 - \underline{\sigma}^2)(t-s) \leq K_t - K_s \leq 0$ for any $s < t$, by Lemma 4.2, it suffices to prove that for any $a \in \mathcal{H}$

$$\hat{E}\left(\int_0^T a_s \,\mathrm{d}K_s\right) = \int_0^T C(a_s)\,\mathrm{d}s,$$

where $C(a_s)$ is defined as in Lemma 4.2 with $\overline{c} = 0$, $\underline{c} = -c(\overline{\sigma}^2 - \underline{\sigma}^2)$. In fact, noting that

$$\int_0^T a_s \,\mathrm{d}K_s \leq \int_0^T 2G(a_s\eta_s)\,\mathrm{d}s - \int_0^T 2a_s G(\eta_s)\,\mathrm{d}s = \int_0^T C(a_s)\,\mathrm{d}s,$$

we have

$$\hat{E}\left(\int_0^T a_s \,\mathrm{d}K_s\right) \leq \int_0^T C(a_s)\,\mathrm{d}s.$$

On the other hand, we have

$$\hat{E}\left(\int_0^T a_s \,\mathrm{d}K_s\right) \geq -\hat{E}\left\{-\left[\int_0^T 2G(a_s\eta_s)\,\mathrm{d}s - \int_0^T 2a_s G(\eta_s)\,\mathrm{d}s\right]\right\} = \int_0^T C(a_s)\,\mathrm{d}s.$$

So $\{K_t\}$ is a process with stationary and independent increments and its distribution is determined by c. □

Just like the conjecture by Shige Peng for the representation of G-martingales with finite variation, we guess that any G-martingale with stationary, independent increments and finite variation should have the form of (5.1). At the end we present a characterization for G-martingales with finite variation.

Proposition 5.2. *Let $\{M_t\}$ be a G-martingale with $M_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$. Then $\{M_t\}$ is a G-martingale with finite variation if and only if $\{f(M_t)\}$ is a G-martingale for any nondecreasing $f \in C_{b,\mathrm{Lip}}(R)$.*

Proof. Necessity. Assume $\{M_t\}$ is a G-martingale with finite variation. By Lemma 4.3, we know that $\{M_t\}$ is nonincreasing. By Theorem 5.4 in [11], there exists a sequence $\{\eta_t^n\} \subset H_G^0(0,T)$ such that

$$\hat{E}\Big[\sup_{t\in[0,T]} \big|M_t - L_t(\eta^n)\big|^\beta\Big] \to 0$$

as n goes to infinity, where $L_t(\eta^n) = \int_0^t \eta_s^n \,\mathrm{d}\langle B\rangle_s - \int_0^t 2G(\eta_s^n)\,\mathrm{d}s$. It suffices to prove that for any $\eta \in H_G^0(0,T)$ and nondecreasing $f \in C_b^2(R)$, $f(L_t(\eta))$ is a G-martingale. In fact,

$$\begin{aligned} f\big(L_t(\eta)\big) &= f(L_0) + \int_0^t f'\big(L_s(\eta)\big)\,\mathrm{d}L_s(\eta) \\ &= f(L_0) + \int_0^t f'\big(L_s(\eta)\big)\eta_s \,\mathrm{d}\langle B\rangle_s - \int_0^t 2f'\big(L_s(\eta)\big)G(\eta_s)\,\mathrm{d}s. \end{aligned}$$

Since $f'(L_s(\eta)) \geq 0$ and $f'(L_s(\eta))\eta_s \in M_G^1(0,T)$, we conclude that

$$f\big(L_t(\eta)\big) = f(L_0) + L_t\big(f'\big(L(\eta)\big)\eta\big)$$

is a G-martingale.

Sufficiency. Assume $\{f(M_t)\}$ is a G-martingale for any nondecreasing $f \in C_{b,\mathrm{Lip}}(R)$. Let $X_t := \arctan M_t$. Then $\{X_t\}$ is a bounded G-martingale and $\{f(X_t)\}$ is a G-martingale for any nondecreasing $f \in C_{b,\mathrm{Lip}}(R)$. By Theorem 4.5 in [10], we know $\{X_t\}$ has the following decomposition

$$X_t = \hat{E}(X_T) + N_t + K_t,$$

where $\{N_t\}$ is a symmetric G-martingale and $\{K_t\}$ is a nonpositive, nonincreasing G-martingale. Then by Itô's formula

$$\mathrm{e}^{\alpha X_t} = \mathrm{e}^{\alpha X_0} + \alpha \int_0^t \mathrm{e}^{\alpha X_s}\,\mathrm{d}X_s + \frac{\alpha^2}{2}\int_0^t \mathrm{e}^{\alpha X_s}\,\mathrm{d}\langle N\rangle_s.$$

For any $\alpha > 0$, by assumption, $\mathrm{e}^{\alpha X_t}$ is a G-martingale. So $L_t := \int_0^t \mathrm{e}^{\alpha X_s}\,\mathrm{d}K_s + \frac{\alpha}{2}\int_0^t \mathrm{e}^{\alpha X_s}\,\mathrm{d}\langle N\rangle_s$ is a G-martingale with finite variation. By Lemma 4.3, L_t is nonincreasing, by which we conclude that $K_t + \frac{\alpha}{2}\langle N\rangle_t$ is nonincreasing. So

$$\frac{\alpha}{2}\hat{E}\big(\langle N\rangle_T\big) \leq \hat{E}(-K_T) \quad \text{for all } \alpha > 0.$$

By this, we conclude that $\hat{E}(\langle N\rangle_T) = 0$ and $N_t \equiv 0$. Then $X_t = \hat{E}(X_T) + K_t$ is nonincreasing, and consequently, M_t is nonincreasing. □

Particularly, Proposition 5.2 provides a method to convert G-martingales with finite variation into bounded G-martingales with finite variation.

Acknowledgment

I am very grateful to the conscientious anonymous referee for his very serious and very valuable comments.

References

[1] L. Denis, M. Hu and S. Peng. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion pathes. *Potential Anal.* **34** (2011) 139–161. MR2754968

[2] M. Hu and S. Peng. On representation theorem of G-expectations and paths of G-Brownian motion. *Acta Math. Appl. Sin. Engl. Ser.* **25** (2009) 539–546. MR2506990

[3] S. Peng. G-expectation, G-Brownian motion and related stochastic calculus of Itô type. In *Stochastic Analysis and Applications* 541–567. *Abel Symp.* **2**. Springer, Berlin, 2007. MR2397805

[4] S. Peng. G-Brownian motion and dynamic risk measure under volatility uncertainty. Available at arXiv:0711.2834v1 [math.PR], 2007.

[5] S. Peng. Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. *Stochastic Process. Appl.* **118** (2008) 2223–2253. MR2474349

[6] S. Peng. A new central limit theorem under sublinear expectations. Available at arXiv:0803.2656v1 [math.PR], 2008.

[7] S. Peng. Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations. *Sci. China Ser. A* **52** (2009) 1391–1411. MR2520583

[8] S. Peng. Nonlinear expectations and stochastic calculus under uncertainty. Available at arXiv:1002.4546v1 [math.PR], 2010.

[9] M. Soner, N. Touzi and J. Zhang. Martingale representation theorem under G-expectation. *Stochastic Process. Appl.* **121** (2011) 265–287. MR2746175

[10] Y. Song. Some properties on G-evaluation and its applications to G-martingale decomposition. *Sci. China Math.* **54** (2011) 287–300. MR2771205

[11] Y. Song. Properties of hitting times for G-martingales and their applications. *Stochastic Process. Appl.* **121** (2011) 1770–1784. MR2811023

[12] Y. Song. Uniqueness of the representation for G-martingales with finite variation. *Electron. J. Probab.* **17** (2012) 1–15.

[13] J. Xu and B. Zhang. Martingale characterization of G-Brownian motion. *Stochastic Process. Appl.* **119** (2009) 232–248. MR2485026