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Weakly nonlinear stochastic CGL equations

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Dedicated to Claude Bardos on his 70th birthday

Abstract. We consider the linear Schrödinger equation under periodic boundary conditions, driven by a random force and damped by a quasilinear damping:

$$\frac{\mathrm{d}}{\mathrm{d}t}u + \mathrm{i}\big(-\Delta + V(x)\big)u = \nu\big(\Delta u - \gamma_R|u|^{2p}u - \mathrm{i}\gamma_I|u|^{2q}u\big) + \sqrt{\nu}\eta(t,x). \qquad (*)$$

The force η is white in time and smooth in x; the potential $V(x)$ is typical. We are concerned with the limiting, as $\nu \to 0$, behaviour of solutions on long time-intervals $0 \le t \le \nu^{-1}T$, and with behaviour of these solutions under the double limit $t \to \infty$ and $\nu \to 0$. We show that these two limiting behaviours may be described in terms of solutions for the *system of effective equations for* $(*)$ which is a well posed semilinear stochastic heat equation with a non-local nonlinearity and a smooth additive noise, written in Fourier coefficients. The effective equations do not depend on the Hamiltonian part of the perturbation $-\mathrm{i}\gamma_I|u|^{2q}u$ (but depend on the dissipative part $-\gamma_R|u|^{2p}u$). If p is an integer, they may be written explicitly.

Résumé. Nous considérons l'équation de Schrödinger linéaire avec les conditions aux limites périodiques, perturbée par une force aléatoire et amortie par un terme quasi linéaire:

$$\frac{\mathrm{d}}{\mathrm{d}t}u + \mathrm{i}\big(-\Delta + V(x)\big)u = \nu\big(\Delta u - \gamma_R|u|^{2p}u - \mathrm{i}\gamma_I|u|^{2q}u\big) + \sqrt{\nu}\eta(t,x). \qquad (*)$$

La force η est un processus aléatoire blanc en temps t et lisse en x; le potentiel $V(x)$ est typique. Nous étudions le comportement asymptotique des solutions sur de longs intervalles de temps $0 \le t \le \nu^{-1}T$, quand $\nu \to 0$, et le comportement des solutions quand $t \to \infty$ et $\nu \to 0$. Nous démontrons qu'on peut décrire ces deux comportements asymptotiques en termes des solutions du *système d'équations effectives pour* $(*)$. Ce dernier est une équation de la chaleur avec un terme quasi linéaire non local et une force aléatoire lisse additive, qui est écrite dans l'espace de Fourier. Les équations ne dépendent pas de la partie hamiltonienne de la perturbation $-\mathrm{i}\gamma_I|u|^{2q}u$ (mais elles dépendent de la partie dissipative $-\gamma_R|u|^{2p}u$). Si p est un entier, on peut écrire ces équations explicitement.

MSC: 35Q56; 60H15

Keywords: Complex Ginzburg–Landau equation; Small nonlinearity; Stationary measures; Averaging; Effective equations

0. Introduction

In [9,10] we considered the KdV equation on a circle, perturbed by a random force and a viscous damping. There we suggested auxiliary *effective equations* which are well posed and describe long-time behaviour of solutions for the perturbed KdV through a kind of averaging.

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In this work we apply the method of [9,10] to a weakly nonlinear situation when the unperturbed equation is not an integrable nonlinear PDE (e.g. KdV), but a linear Hamiltonian PDE with a generic spectrum. Since analytic properties of the latter are easier and better understood then those of the former, in the weakly nonlinear situation we understand better properties of the effective system and its relation with the original equation. Accordingly we can go further in analysis of long time behaviour of solutions.

More precisely, we are concerned with ν-small dissipative stochastic perturbations of the space-periodic linear Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u + \mathrm{i}\big(-\Delta u + V(x)\big)u = 0, \quad x \in \mathbb{T}^d, \tag{0.1}$$

i.e. with equations

$$\frac{\mathrm{d}}{\mathrm{d}t}u + \mathrm{i}Au = \nu\big(\Delta u - \gamma_R f_p\big(|u|^2\big)u - i\gamma_I f_q\big(|u|^2\big)u\big) + \sqrt{\nu}\eta(t,x), \quad x \in \mathbb{T}^d, \tag{0.2}$$

where $\eta(t,x) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{\infty} b_j \boldsymbol{\beta}_j(t) e_j(x)$. Here $Au = A_V u = -\Delta u + V(x)u$ and the potential $V(x) \geq 1$ is sufficiently smooth; the real numbers p, q are nonnegative, the functions $f_p(r)$ and $f_q(r)$ are the monomials $|r|^p$ and $|r|^q$, smoothed out near zero, and the constants γ_R, γ_I satisfy

$$\gamma_R, \gamma_I \geq 0, \qquad \gamma_R + \gamma_I = 1. \tag{0.3}$$

If $\gamma_R = 0$, then due to the usual difficulty with the zero-mode of a solution u, the term Δu in the r.h.s. should be modified to $\Delta - u$. The functions $\{e_j(x), j \geq 1\}$ in the definition of the random force form the real trigonometric base of $L_2(\mathbb{T}^d)$, the real numbers b_j decay sufficiently fast to zero when j grows, and $\{\boldsymbol{\beta}_j(t), j \geq 1\}$, are the standard complex Wiener processes. So the noise η is white in time and sufficiently smooth in x. It is convenient to pass to the slow time $\tau = \nu t$ and write the equation as

$$\dot{u} + \nu^{-1}\mathrm{i}Au = \Delta u - \gamma_R f_p\big(|u|^2\big)u - \mathrm{i}\gamma_I f_q\big(|u|^2\big)u + \eta(\tau, x), \tag{0.4}$$

where $\dot{u} = \mathrm{d}u/\mathrm{d}\tau$. The equation is supplemented with the initial condition

$$u(0,x) = u_0(x). \tag{0.5}$$

It is known that under certain restrictions on p, q and d the problem (0.4), (0.5) has a unique solution $u^\nu(\tau,x)$, $\tau \geq 0$, and Eq. (0.4) has a unique stationary measure μ^ν. We review these results in Section 1 (there attention is given to the 1d case, while higher-dimensional equations are only briefly discussed).

Let $\{\varphi_k, k \geq 1\}$, and $\{\lambda_k, k \geq 1\}$, be the eigenfunctions and eigenvalues of A_V, $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. We say that a potential V is *nonresonant* if $\sum_{j=1}^{\infty} \lambda_j s_j \neq 0$ for every finite nonzero integer vector $(s_1, s_2, \ldots)$. In Sections 1.4, 1.5 we show that nonresonant potentials are typical both in the sense of Baire and in the sense of measure. Assuming that V is nonresonant we are interested in two questions:

Q1. What is the limiting behaviour as $\nu \to 0$ of solutions $u^\nu(\tau,x)$ on long time-intervals $0 \leq \tau \leq T$?
Q2. What is the limiting behaviour of the stationary measure μ^ν as $\nu \to 0$?

For any complex function $u(x), x \in \mathbb{T}^d$, denote by $\Psi(u) = v = (v_1, v_2, \ldots)$ the complex vector of its Fourier coefficients with respect to the basis $\{\varphi_k\}$, i.e. $u(x) = \sum v_j \varphi_j$. Denote

$$I_j = \frac{1}{2}|v_j|^2, \qquad \varphi_j = \operatorname{Arg} v_j, \quad j \geq 1. \tag{0.6}$$

Then $(I, \varphi) \in \mathbb{R}_+^\infty \times \mathbb{T}^\infty$ are the action-angles for the linear Eq. (0.1). The v- and (I,φ)-variables are convenient to study the two questions above. Writing (0.4) in the (I,φ)-variables we arrive at the following system:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}I_j = \cdots, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_j = \nu^{-1}\lambda_j + \cdots, \tag{0.7}$$

where the dots stand for terms of order one (stochastic and deterministic). We have got slow/fast stochastic equations to which the principle of averaging is formally applicable (e.g., see [2,14] for the classical deterministic averaging and [4,8] for the stochastic averaging). Denoting $I_j^\nu(\tau) = I_j(u^\nu(\tau))$ and averaging in φ the I-equations in (0.7), using the rules of the stochastic calculus [4,8] and following the arguments in [10], we show in Section 2 that along sequences $\nu_j \to 0$ we have the convergences

$$\mathcal{D}\big(I^{\nu_j}(\cdot)\big) \rightharpoonup \mathcal{D}\big(I^0(\cdot)\big), \tag{0.8}$$

where the limiting process $I^0(\tau)$, $0 \le \tau \le T$, is a weak solution of the averaged I-equations. As in the KdV-case the averaged equations are singular and we do not know if their solution is unique. So we do not know if the convergence (0.8) holds as $\nu \to 0$. To continue the analysis we write Eq. (0.4) in the v-variables

$$\dot v_k + \mathrm{i}\nu^{-1}\lambda_k v_k = P_k(v) + \sum_{j\ge 1} B_{kj}\dot{\boldsymbol{\beta}}_j(\tau), \tag{0.9}$$

where the drift P_k and the dispersion B_{kj} are constructed in terms of the r.h.s. of Eq. (0.4) and the transformation Ψ. It turns out that the Hamiltonian term $-\mathrm{i}\gamma_I f_q(|u|^2)u$ contributes to $P(v)$ a term which disappears in the averaged I-equations. We remove it from $P(v)$ and denote the rest $\tilde P(v)$. For any vector $\theta = (\theta_1, \theta_2, \ldots) \in \mathbb{T}^\infty$ denote by Φ_θ the linear transformation of the space of complex vectors v which multiplies each component v_j by $e^{\mathrm{i}\theta_j}$. Following [9] we average the vector field $\tilde P$ by actions of the transformations Φ_θ and get the *effective drift* $R(v) = \int_{\mathbb{T}^\infty} \Phi_{-\theta}\tilde P(\Phi_\theta v)\,\mathrm{d}\theta$. In Section 3.1 we show that

$$R_k(v) = -\lambda_k v_k + R_k^0(v), \tag{0.10}$$

where $R^0(v)$ is a smooth locally Lipschitz nonlinearity.

Since the noise in (0.9) is additive (i.e., the matrix B is v-independent), then the construction of the effective dispersion, given in [9] for a non-additive noise, simplifies significantly and defines the effective noise for Eq. (0.9) whose kth component equals $(\sum_l b_l^2\Psi_{kl}^2)^{1/2}\,\mathrm{d}\boldsymbol{\beta}_k(\tau)$. Accordingly the effective equations for (0.4) become

$$\dot v_k = R_k(v)\,\mathrm{d}\tau + \left(\sum_l b_l^2\Psi_{kl}^2\right)^{1/2}\mathrm{d}\boldsymbol{\beta}_k(\tau), \quad k \ge 1. \tag{0.11}$$

By construction this system is invariant under rotations: if $v(\tau)$ is its weak solution, then $\Phi_\theta v(\tau)$ also is a weak solution. Due to (0.10) this is the heat equation $\dot u = -Au$ for a complex function $u(\tau, x)$, perturbed by a non-local smooth nonlinearity and a nondegenerate smooth noise, written in terms of the complex Fourier coefficients v_j. It turns out to be a monotone equation, so its solution is unique (see Section 3.2).

In particular, if in (0.4) $p = 1$, then the system of effective equations takes the form

$$\dot v_k = -v_k\Big((\lambda_k - M_k) + \gamma_R\sum |v_l|^2 L_{kl}\Big)\,\mathrm{d}\tau + \left(\sum_l b_l^2\Psi_{kl}^2\right)^{1/2}\mathrm{d}\boldsymbol{\beta}_k(\tau), \quad k \ge 1, \tag{0.12}$$

where $M_k = \int V(x)\varphi_k^2(x)\,\mathrm{d}x$ and $L_{kl} = (2 - \delta_{kl})\int \varphi_k^2(x)\varphi_l^2(x)\,\mathrm{d}x$. See Example 3.1 (the calculations, made there for $d = 1$, remain the same for $d \ge 2$).

It follows directly from the construction of effective equations that actions $\{I(v_k(\tau)) = \frac{1}{2}|v_k(\tau)|^2, k \ge 1\}$ of any solution $v(\tau)$ of (0.11) is a solution of the system of averaged I-equations. On the contrary, every solution $I^0(\tau)$ of the averaged I-equations, obtained as a limit (0.8), can be lifted to a weak solution of (0.11). Using the uniqueness we get

Theorem 0.1. *Let $I^\nu(\tau) = I(u^\nu(\tau))$, where $u^\nu(\tau)$, $0 \le \tau \le T$, is a solution of* (0.4), (0.5). *Then* $\lim_{\nu\to 0}\mathcal{D}(I^\nu(\cdot)) = \mathcal{D}(I^0(\cdot))$, *where $I^0(\tau)$, $0 \le \tau \le T$, is a weak solution of the averaged I-equations. Moreover, there exists a unique solution $v(\tau)$ of* (0.11) *such that $v(0) = v_0 = \Psi(u_0)$ and $\mathcal{D}(I(v(\cdot)) = \mathcal{D}(I^0(\cdot))$, where $I(v(\tau))_j = \frac{1}{2}|v_j(\tau)|^2$.*

The solutions $I^0(\tau)$ and $v(\tau)$ satisfy some apriori estimates, see Theorem 3.5. Concerning distribution of the angles $\varphi(u^\nu(\tau))$ and their joint distribution with the actions see Section 2.4.

Now let μ^ν be the unique stationary measure for Eq. (0.4) and u'^ν be a corresponding stationary solution, $\mathcal{D}(u'^\nu(\tau)) \equiv \mu^\nu$. As above, along sequences $\nu_j \to 0$ the actions $I'^{\nu_j}(\tau) = I(u'^{\nu_j}(\tau))$ converge in distribution to a stationary solution $I'(\tau)$ of the averaged I-equations. This solution can be lifted to a stationary weak solutions $v'(\tau)$ of effective Eqs (0.11). Since that system is monotone, then its stationary measure m is unique. So the limit above holds as $\nu \to 0$. As the effective system is rotation invariant, then in the (I, φ)-variables its unique stationary measure has the form $\mathrm{d}m = m_I(\mathrm{d}I) \times \mathrm{d}\varphi$, where $\mathrm{d}\varphi = \mathrm{d}\varphi_1\, \mathrm{d}\varphi_2 \cdots$ is the Haar measure on $\mathbb{T}^\infty$. It turns out that the measure $\lim_{\nu\to 0} \mu^\nu$ also has the rotation-invariant form and we arrive at the following result (see Theorem 4.3, 4.4 for a precise statement):

Theorem 0.2. *When $\nu \to 0$ we have the convergences $\mathcal{D}(I(u'^\nu(\cdot)) \rightharpoonup \mathcal{D}I(v'(\cdot))$ and $\Psi \circ \mu^\nu \rightharpoonup m$, where $\mathrm{d}m = m_I(\mathrm{d}I) \times \mathrm{d}\varphi$.*

Accordingly every solution $u^\nu(\tau)$ of (0.2) obeys the following double limit

$$\lim_{\nu\to 0} \lim_{t\to\infty} \mathcal{D}\big(u^\nu(t)\big) = \Psi^{-1} \circ m. \tag{0.13}$$

By Theorems 0.1 and 0.2, the actions $I(u^\nu(\tau))$ of a solution u^ν of (0.4), (0.5) converge in distribution to those of a solution $v(\tau)$ of the effective system (0.11) with $v(0) = \Psi(u_0)$, both for $0 \le \tau \le T$ and when $\tau \to \infty$. We conjecture that this convergence hold for each $\tau \ge 0$, uniformly in τ (the space of measures is equipped with the Wasserstein distance).

In Example 4.6 we discuss Theorem 0.2 for equations with $p = 1$, when the effective equations become (0.12). In particular, we show that Theorem 0.2 implies that in Eqs (0.2) with small ν there is no direct or inverse cascade of energy.

In Example 4.5 we discuss Theorem 0.2 for the case $\gamma_R = 0$ (when the nonlinear part of the perturbation is Hamiltonian) and its relation to the theory of weak turbulence.

We note that the effective Eqs (0.11) depend on the potential $V(x)$ in a regular way and are well defined without assuming that $V(x)$ is nonresonant (cf. Eqs (0.12)). In particular, if $V^M(x) \to 1$ as $M \to \infty$, where each $V^M(x) \ge 1$ is a nonresonant potential, then in (0.13) $m^M \rightharpoonup m(1)$, where $m(1)$ is a unique stationary measure for Eq. (0.12) with $V(x) \equiv 1$. In this equation $\Psi_{kl} = \delta_{k,l}$, $M_k \equiv 1$ and the constants L_{kl} can be written down explicitly.

In Section 5 we show that Theorems 0.1, 0.2 remain true for 1d equations with non-viscous damping (when Δu in the l.h.s. of (0.2) is removed, but $\gamma_R > 0$).

Inviscid limit

A stationary measure μ^ν for Eq. (0.4) also is stationary for the fast-time Eq. (0.2). Let $U^\nu(t)$ be a corresponding stationary solution, $\mathcal{D}U^\nu(t) \equiv \mu^\nu$. It is not hard to see that the system of solutions $U^\nu(t)$ is tight on any finite time-interval $[0, \tilde{T}]$. Let $\{U^{\nu_j}, \nu_j \to 0\}$, be a converging subsequence, i.e.

$$\mathcal{D}(U^{\nu_j}) \rightharpoonup Q^*, \qquad \mu^{\nu_j} \rightharpoonup \mu^*.$$

Then μ^* is an invariant measure for the linear Eq. (0.1) and $Q^* = \mathcal{D}(U^*(\cdot))$, where $U^*(t), 0 \le t \le \tilde{T}$, is a stationary process such that $\mathcal{D}(U^*(t)) \equiv \mu^*$ and every trajectory of U^* is a solution of (0.1). The limit $\mathcal{D}(U^{\nu_j}) \rightharpoonup \mathcal{D}(U^*)$ is the *inviscid limit for Eq.* (0.2). Equation (0.1) has plenty of invariant measures: if we write it in the action-angle variables (0.6), then every measure of the form $m(\mathrm{d}I) \times \mathrm{d}\varphi$ is invariant (see [12] for the more complicated inviscid limit for nonlinear Schrödinger equation). Theorem 0.2 explains which one is chosen by Eq. (0.2) for the limit $\lim_{\nu\to 0} \mu^\nu$.

The inviscid limit for the damped/driven KdV equation, studied in [9,10] is similar: the limit of the stationary measures for the perturbed equations is a stationary measure of the corresponding effective equations. Due to a complicated structure of the nonlinear Fourier transform which integrates KdV, uniqueness of their invariant measure is not proved yet. So the final results concerning the damped/driven KdV are less complete than those for the weakly perturbed CGL equation in Theorem 0.2.

Finally consider the damped/driven 2d Navier–Stokes equations with a small viscosity ν and a random force, similar to the forces above and proportional to $\sqrt{\nu}$:

$$v_t' - \nu\Delta v + (v\cdot\nabla)v + \nabla p = \sqrt{\nu}\eta(t,x); \quad \operatorname{div} v = 0,\, v \in \mathbb{R}^2,\, x \in \mathbb{T}^2. \tag{0.14}$$

It is known that (0.14) has a unique stationary measure μ^ν, the family of measures $\{\mu^\nu, 0<\nu\le 1\}$ is tight, and every limiting measure $\lim_{\nu_j\to 0}\mu^{\nu_j}$ is a non-trivial invariant measure for the 2d Euler Eq. $(0.14)_{\nu=0}$, see Section 5.2 of [13]. Hovewer it is nonclear if the limiting measure is unique and how to single it out among all invariant measures of the Euler equation. The research [9,10] was motivated by the belief that the damped/driven KdV is a model for (0.14). Unfortunately, we still do not know up to what extend the description of the inviscid limit for the damped/driven KdV and for weakly nonlinear CGL in terms of the effective equations is relevant for the inviscid limit of the 2d hydrodynamics.

Agreements

Analyticity of maps $B_1 \to B_2$ between Banach spaces B_1 and B_2, which are the real parts of complex spaces B_1^c and B_2^c, is understood in the sense of Fréchet. All analytic maps which we consider possess the following additional property: for any R a map analytically extends to a complex $(\delta_R > 0)$-neighbourhood of the ball $\{|u|_{B_1} < R\}$ in B_1^c.

Notations

χ_A stands for the indicator function of a set A (equal 1 in A and equal 0 outside A). By $\varkappa(t)$ we denote various functions of t such that $\varkappa(t)\to 0$ when $t\to\infty$, and by $\varkappa_\infty(t)$ denote functions $\varkappa(t)$ such that $\varkappa(t) = \mathrm{o}(t^{-N})$ for each N. We write $\varkappa(t) = \varkappa(t;R)$ to indicate that $\varkappa(t)$ depends on a parameter R.

1. Preliminaries

1.1. *Apriori estimates*

We consider the 1d CGL equation on a segment $[0,\pi]$ with a conservative linear part of order one and a small nonlinearity. The equation is supplemented with Dirichlet boundary conditions which we interpret as odd 2π-periodic boundary conditions. Introducing the slow time $\tau = \nu t$ (cf. the Introduction) we write the equation as follows:

$$\begin{aligned} &\dot u + \mathrm{i}\nu^{-1}\big(-u_{xx} + V(x)u\big) = \varkappa u_{xx} - \gamma_R|u|^{2p}u - \mathrm{i}\gamma_I|u|^{2q}u + \frac{\mathrm{d}}{\mathrm{d}\tau}\sum_{i=1}^\infty b_j\boldsymbol{\beta}_j(\tau)e_j(x),\\ &u(x)\equiv u(x+2\pi)\equiv -u(-x). \end{aligned} \tag{1.1}$$

Here $\dot u = \frac{\mathrm{d}}{\mathrm{d}\tau}u$, $p,q\in\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ (only for simplicity, see next section), $\varkappa > 0$, constants γ_R and γ_I satisfy (0.3) and $\mathbb{R}\ni V(x)\ge 0$ is a sufficiently smooth even 2π-periodic function, $\{e_j, j\ge 1\}$ is the sine-basis,

$$e_j(x) = \frac{1}{\sqrt{\pi}}\sin jx,$$

and $\boldsymbol{\beta}_j$, $j\ge 1$, are standard independent complex Wiener processes. That is, $\boldsymbol{\beta}_j(\tau) = \beta_j(\tau) + \mathrm{i}\beta_{-j}(\tau)$, where $\beta_{\pm j}(\tau)$ are standard independent real Wiener processes. Finally, the real numbers b_j all are nonzero and decay when j grows in such a way that $B_1 < \infty$, where

$$B_r := 2\sum_{j=1}^\infty j^{2r}b_j^2 \le \infty \quad \text{for } r\ge 0.$$

By $\mathcal{H}^r$, $r\in\mathbb{R}$ we denote the Sobolev space of order r of complex odd periodic functions and provide it with the homogeneous norm $\|\cdot\|_r$,

$$\|u\|_r^2 = \sum_{l=1}^\infty |u_l|^2 l^{2r} \quad \text{for } u(x) = \sum_{l=1}^\infty u_l e_l(x), \quad \|u\|_0 = \|u\|$$

(if $r \in \mathbb{N}$, then $\|u\|_r = |\frac{\partial^r u}{\partial x^r}|_{L_2}$).

Let $u(t,x)$ be a solution of (1.1) such that $u(0,x) = u_0$. Applying Ito's formula to $\frac{1}{2}\|u\|^2$ we get that

$$\mathrm{d}\left(\frac{1}{2}\|u\|^2\right) = \left(-\gamma_R |u|_{2p+2}^{2p+2} - \varkappa \|u\|_1^2 + \frac{1}{2}B_0\right)\mathrm{d}\tau + \mathrm{d}M(\tau), \tag{1.2}$$

where $M(\tau)$ is the martingale $\int_0^\tau \sum b_j u_j \cdot \mathrm{d}\boldsymbol{\beta}_j(\tau)$. Here $|u|_r$ stands for the L_r-norm, $1 \le r \le \infty$, and for complex numbers z_1, z_2 we denote by $z_1 \cdot z_2$ their real scalar product,

$$z_1 \cdot z_2 = \operatorname{Re} z_1 \overline{z_2}.$$

So $(u_j + \mathrm{i}u_{-j}) \cdot (\mathrm{d}\beta_j + \mathrm{i}\,\mathrm{d}\beta_{-j}) = u_j\,\mathrm{d}\beta_j + u_{-j}\,\mathrm{d}\beta_{-j}$. From (1.2) we get in the usual way (e.g., see Section 2.2.3 in [13]) that

$$\mathbf{E}\mathrm{e}^{\rho_\varkappa \|u(\tau)\|^2} \le C\big(\varkappa, B_0, \|u_0\|\big) \quad \forall \tau \ge 0 \tag{1.3}$$

for a suitable $\rho_\varkappa > 0$, uniformly in $\nu > 0$.

Denoting

$$\mathcal{E}(\tau) = \frac{1}{2}\big\|u(\tau)\big\|^2 + \gamma_R \int_0^\tau |u|_{2p+2}^{2p+2}\,\mathrm{d}s + \frac{\varkappa}{2}\int_0^\tau \|u\|_1^2\,\mathrm{d}s$$

and noting that the characteristic of the martingale M is $\langle M\rangle(\tau) = \sum b_j^2 |u_j|^2 \tau \le b_M^2 \|u\|^2 \tau$, where $b_M = \max|b_j|$, we get from (1.2) that

$$\begin{aligned}\mathcal{E}(\tau) &\le \frac{1}{2}\|u_0\|^2 + \frac{1}{2}B_0\tau + M(\tau) - \frac{\varkappa}{2}\int_0^\tau \|u\|_1^2\,\mathrm{d}s \\ &\le \frac{1}{2}\|u_0\|^2 + \frac{1}{2}B_0\tau + \varkappa^{-1}b_M^2\Big[\big(\varkappa b_M^{-2} M(\tau)\big) - \frac{1}{2}\big\langle \varkappa b_M^{-2} M\big\rangle(\tau)\Big].\end{aligned}$$

Applying in a standard way the exponential supermartingale estimate to the term in the square bracket in the r.h.s. (e.g., see [13], Section 2.2.3), we get that

$$\mathbf{P}\left\{\sup_{\tau \ge 0}\left(\mathcal{E}(\tau) - \frac{1}{2}B_0\tau\right) \ge \frac{1}{2}\|u_0\|^2 + \rho\right\} \le \mathrm{e}^{-2\varkappa\rho b_M^{-2}} \tag{1.4}$$

for any $\rho > 0$.

Now let us re-write Eq. (1.1) as follows:

$$\dot{u} + \mathrm{i}\nu^{-1}\big(-u_{xx} + V(x)u + \nu\gamma_I |u|^{2q}u\big) = \varkappa u_{xx} - \gamma_R |u|^{2p}u + \frac{\mathrm{d}}{\mathrm{d}\tau}\sum b_j \beta_j(\tau) e_j. \tag{1.5}$$

The l.h.s. is a Hamiltonian system with the hamiltonian $-\nu^{-1}H(u)$,

$$H(u) = \frac{1}{2}\langle Au, u\rangle + \gamma_I \frac{\nu}{2q+2}\int |u|^{2q+2}\,\mathrm{d}x, \qquad A = -\frac{\partial^2}{\partial x^2} + V(x).$$

For any $j \in \mathbb{N}$ we denote

$$\|u\|_r'^{\,2} = \big\langle A^r u, u\big\rangle.$$

Then $\mathrm{d}H(u)(v) = \langle Au, v\rangle + \gamma_I \nu \langle |u|^{2q}u, v\rangle$ and

$$\frac{1}{2} \cdot 2\sum_{j=1}^{\infty} b_j^2\,\mathrm{d}^2 H(u)(e_j, e_j) = \frac{1}{2}B_1' + \gamma_I \nu X(\tau),$$

where

$$B_r' = 2\sum b_j^2 \|e_j\|_r'^2 = 2\sum b_j^2 \lambda_j^r \quad \forall r,$$

and

$$X(\tau) = 2q \operatorname{Re} \int \left(|u|^{2q-2} u^2 \sum_j b_j^2 e_j(x)^2 \right) \mathrm{d}x + \int |u|^{2q} \sum_j b_j^2 e_j(x)^2 \, \mathrm{d}x$$
$$\le C B_0 \big|u(\tau)\big|_{2q}^{2q}.$$

Therefore applying Ito's formula we get that

$$\mathrm{d}H\big(u(\tau)\big) = \Big(-\gamma_R \big\langle Au, |u|^{2p} u \big\rangle + \varkappa \langle Au, u_{xx} \rangle - \gamma_I \nu \gamma_R \int |u|^{2p+2q+2} \, \mathrm{d}x$$
$$+ \varkappa \gamma_I \nu \big\langle |u|^{2q} u, u_{xx} \big\rangle + B_1' + \gamma_I \nu X(\tau) \Big) \mathrm{d}\tau + \mathrm{d}M(\tau), \tag{1.6}$$

where $\mathrm{d}M(\tau) = \sum b_j \langle Au + \gamma_I \nu |u|^{2q} u, e_j \rangle \cdot \mathrm{d}\boldsymbol{\beta}_j(\tau)$.

Denoting $U_q(x) = \frac{1}{q+1} u^{q+1}$ and $U_p(x) = \frac{1}{p+1} u^{p+1}$, we have

$$\big\langle |u|^{2q} u, u_{xx} \big\rangle \le - \int |u_x|^2 |u|^{2q} \, \mathrm{d}x = - \left\| \frac{\partial}{\partial x} U_q \right\|^2,$$

and a similar relation holds for q replaced by p. Accordingly,

$$\mathrm{d}H\big(u(\tau)\big) \le -\frac{1}{2} \bigg(\varkappa \|u\|_2^2 + \gamma_R \left\| \frac{\partial}{\partial x} U_p \right\|^2 + \varkappa \gamma_I \nu \left\| \frac{\partial}{\partial x} U_q \right\|^2$$
$$+ \nu \gamma_I \gamma_R \int |u|^{2p+2q+2} \, \mathrm{d}x - C_\varkappa \|u\|^2 - 2B_1' \bigg) \mathrm{d}\tau + \mathrm{d}M(\tau), \tag{1.7}$$

where $C_\varkappa$ may be chosen independent from $\varkappa$ if $\gamma_R > 0$. Considering relations on $H(u)^m$, $m \ge 1$, which follow from (1.7) and (1.6), using (1.4) and arguing by induction we get that

$$\mathbf{E} \bigg(\sup_{0 \le t \le T} H\big(u(t)\big)^m + \frac{\varkappa}{2} \int_0^T H^{m-1}(u) \|u\|_2^2 \, \mathrm{d}s \bigg)$$
$$\le H(u_0)^m + C_m(\varkappa, T, B_1) \big(1 + \|u_0\|^{c_m}\big), \tag{1.8}$$

$$\mathbf{E} H\big(u(t)\big)^m \le C_m(\varkappa, B_1) \big(1 + H(u_0)^m + \|u_0\|^{c_m}\big) \quad \forall t > 0, \tag{1.9}$$

for any m. Estimates (1.8) in a traditional way (cf. [5,12,16,18]) imply that Eq. (1.1) is *regular in space* $\mathcal{H}^1$ in the sense that for any $u_0 \in \mathcal{H}^1$ it has a unique strong solution, satisfying (1.4), (1.8).

1.2. *Stationary measures*

The a-priori estimates on solutions of (1.1) and the Bogolyubov–Krylov argument (e.g., see in [13]) imply that Eq. (1.1) has a stationary measure μ^ν, supported by space $\mathcal{H}^2$. Now assume that

$$b_j \ne 0 \quad \forall j. \tag{1.10}$$

Then the approaches, developed in the last decade to study the 2d stochastic Navier–Stokes equations, apply to (1.1) and allow to prove that under certain restrictions on the equation the stationary measure μ^ν is unique. In particular this is true if $\gamma_I = 0$ (the easiest case), or if $p \geq q$ and $\gamma_R \neq 0$ (see [16]), or if $\gamma_R = 0$ and $p = 1$ (see [18]). In this case any solution $u(t)$ of (1.1) with $u(0) = u_0 \in \mathcal{H}^1$ satisfies

$$\mathcal{D}u(t) \rightharpoonup \mu^\nu \quad \text{as } t \to \infty. \tag{1.11}$$

This convergence and (1.3), (1.9) imply that

$$\int e^{\rho_\varkappa \|u\|^2} \mu^\nu(\mathrm{d}u) \leq C(\varkappa, B), \tag{1.12}$$

$$\int \|u\|_1^{2m} \mu^\nu(\mathrm{d}u) \leq C_m(\varkappa, B_1) \quad \forall m. \tag{1.13}$$

1.3. *Multidimensional case*

In this section we briefly discuss a multidimensional analogy of Eq. (1.1):

$$\begin{aligned} \dot{u} + \mathrm{i}\nu^{-1} A u = \Delta u - \gamma_R f_p\big(|u|^2\big)u - \mathrm{i}\gamma_I f_q\big(|u|^2\big)u \\ + \frac{\mathrm{d}}{\mathrm{d}\tau} \sum_{j=1}^{\infty} b_j \boldsymbol{\beta}_j(\tau) e_j(x), \quad u = u(\tau, x), x \in \mathbb{T}^d. \end{aligned} \tag{1.14}$$

Here $Au = -\Delta u + V(x)u$, $V \in C^N(\mathbb{T}^d, \mathbb{R})$ and $V(x) \geq 1$. The numbers γ_I, γ_R satisfy (0.3). Functions $f_p \geq 0$ and $f_q \geq 0$ are real-valued smooth and

$$f_p(t) = t^p \quad \text{for } t \geq 1, \qquad f_q(t) = t^q \quad \text{for } t \geq 1,$$

where $p, q \geq 0$. If $\gamma_R = 0$, then the term Δu in the r.h.s. should be modified to $\Delta - u$. By $\{e_k, k \geq 1\}$, we denote the usual trigonometric basis of the space $L_2(\mathbb{T}^d)$ (formed by all functions $\pi^{-d/2} f_{s_1}(x_1) \cdots f_{s_d}(x_d)$, where each $f_s(x)$ is $\sin sx$ or $\cos sx$), parameterised by natural numbers. These are eigen-functions of the Laplacian, $-\Delta e_r = \lambda_r e_r$. We assume that

$$B'_{N_1} = 2 \sum_k \lambda_k^{N_1} b_k^2 < \infty, \tag{1.15}$$

where $N_1 = N_1(d)$ is sufficiently large. In this section we denote by $(\mathcal{H}^r, \|\cdot\|_r)$ the Sobolev space $\mathcal{H}^r = H^r(\mathbb{T}^d, \mathbb{C})$, regarded as a real Hilbert space, and $\langle\cdot,\cdot\rangle$ stands for the real L_2-scalar product.

Noting that $(f_p(|u|^2)u - |u|^{2p}u)$ and $(f_q(|u|^2)u - |u|^{2q}u)$ are bounded Lipschitz functions with compact support we immediately see that the a-priori estimates from Section 1.1 remain true for solutions of (1.14). Accordingly, for any $u_0 \in \mathcal{H}^1 \cap L_{2q+2}$ Eq. (1.1) has a solution $u(t, x)$ such that $u(0, x) = u_0$, satisfying (1.3), (1.8), (1.9).

Now assume that

$$p, q < \infty \quad \text{if } d = 1, 2, \qquad p, q < \frac{2}{d-2} \quad \text{if } d \geq 3. \tag{1.16}$$

Applying Ito's formula to the processes $\langle A^m u(\tau), u(\tau)\rangle^n$, $m, n \geq 1$, using (1.3), (1.8), (1.9) and arguing by induction (first in n and next in m) we get that

$$\begin{aligned} &\mathbf{E}\bigg(\sup_{0\leq\tau\leq T} \big\|u(\tau)\big\|'^{2n}_{2m} + \int_0^T \big\|u(s)\big\|'^{2}_{2m+1} \big\|u(s)\big\|'^{2n-2}_{2m} \,\mathrm{d}s\bigg) \\ &\quad \leq \|u_0\|'^{2n}_{2m} + C(m, n, T)\big(1 + \|u_0\|^{c_{m,n}}\big), \end{aligned} \tag{1.17}$$

$$\mathbf{E}\big\|u(\tau)\big\|'^{2n}_{2m} \leq C(m, n) \quad \forall \tau \geq 0, \tag{1.18}$$

for each m and n, where $C(m,n,T)$ and $C(m,n)$ also depends on $|V|_{C^N}$ and B_{N_1} (see (1.15)), and $N = N(m)$, $N_1 = N_1(m)$.

Relations (1.17) with $m = m_0 \geq 1$ in the usual way (cf. [5,12,16,18]) imply that Eq. (1.14) is *regular in the space* $\mathcal{H}^{m_0} \cap L_{2q+2}$ in the sense that for any $u_0 \in \mathcal{H}^{m_0} \cap L_{2q+2}$ it has a unique strong solution $u(t,x)$, equal u_0 at $t = 0$, and satisfying estimates (1.3), (1.17) with $m = m_0$ for any n. By the Bogolyubov–Krylov argument this equation has a stationary measure μ^ν, supported by the space $\mathcal{H}^{m_0} \cap L_{2q+2}$, and a corresponding stationary solution $u^\nu(\tau)$, $\mathcal{D}u^\nu(\tau) \equiv \mu^\nu$, also satisfies (1.3) and (1.18) with $m = m_0$.

If (1.10) holds and (1.16) is replaced by a stronger assumption, then a stationary measure is unique. If $\gamma_I = 0$, the uniqueness readily follows, for example, from the abstract theorem in [13]. In [18] this assertion is proved if

$$\gamma_R = 0 \quad \text{and} \quad q \leq 1 \quad \text{if } d = 1, \qquad q < 1 \quad \text{if } d = 2, \qquad q \leq 2/d \quad \text{if } d = 3. \tag{1.19}$$

In [16] it is established if

$$p = q, \qquad \gamma_R, \gamma_I > 0 \quad \text{if } d = 1, 2, \quad \text{and} \quad p = q < \frac{2}{d-2}, \qquad \gamma_R, \gamma_I > 0 \quad \text{if } d \geq 3; \tag{1.20}$$

the argument of that work also applies if $p > q$.

Note that when $\gamma_R = 0$ or when $p < q$ (i.e., when the nonlinear damping is weaker than the conservative term), the assumptions (1.19), (1.20), needed for the uniqueness of the stationary measure, are much stronger than the assumptions (1.16), needed for the regularity. This gap does not exist (at least it shrinks a lot) if the random force in Eq. (1.14) is not white in time, but is a kick-force. See in [11] the abstract theorem and its application to the CGL equations.

1.4. *Spectral properties of A_V: One-dimensional case*

As in Section 1.1 we denote $A_V = A = -\partial^2/\partial x^2 + V(x)$, where the potential $V(x) \geq 0$ belongs to the space C_e^N of C^N-smooth even and 2π-periodic functions, $N \geq 1$. Let $\phi_1, \phi_2, \ldots$ be the L_2-normalised complete system of real eigenfunctions of A_V with the eigenvalues $1 \leq \lambda_1 < \lambda_2 < \cdots$. Consider the linear mapping

$$\Psi : \mathcal{H} \ni u(x) \mapsto v = (v_1, v_2, \ldots) \in \mathbb{C}^\infty,$$

defined by the relation $u(x) = \sum v_k \phi_k(x)$. In the space of complex sequences v we introduce the norms

$$|v|_{h^m}^2 = \sum_{k \geq 1} |v_k|^2 \lambda_k^m, \quad m \in \mathbb{R},$$

and denote $h^m = \{v \mid |v|_{h^m} < \infty\}$. Due to the Parseval identity, $\Psi : \mathcal{H} \to h^0$ is a unitary isomorphism. By $\{\Psi_{km}, k, m \geq 1\}$ we denote the matrix of Ψ with respect to the basis $\{e_j\}$ in $\mathcal{H}$ and the standard basis in h^0. Since Ψ maps real vectors to real, its matrix has real entries.

For any $m \in \mathbb{N}$ we have $|v|_{h^m}^2 = \langle A^m u(x), u(x) \rangle$. So the norms $|v|_{h^m}$ and $\|u\|_m$ are equivalent for $m = 0, \ldots, N$. Since $\Psi^* = \Psi^{-1}$, then the norms are equivalent for integer $|m| \leq N$. By interpolation they are equivalent for all real $|m| \leq N$. So

$$\text{the maps} \quad \Psi : \mathcal{H}^m \to h^m, \quad |m| \leq N, \quad \text{are isomorphisms.} \tag{1.21}$$

Denote $G = \Psi^{-1} : h^m \to \mathcal{H}^m$. Then

$$\Psi \circ A \circ G = \operatorname{diag}\{\lambda_k, k \geq 1\} =: \widehat{A}.$$

Consider the operator

$$\mathcal{L} := \Psi \circ (-\Delta) \circ G = \Psi \circ (A - V) \circ G = \widehat{A} - \Psi \circ V \circ G =: \widehat{A} - \mathcal{L}^0. \tag{1.22}$$

By (1.21) $\mathcal{L}^0 = \Psi \circ V \circ G$ defines bounded linear maps

$$\mathcal{L}^0 : h^m \to h^m \quad \forall |m| \le N, \tag{1.23}$$

and in the space h^0 it is selfadjoint.

For any finite M consider the mapping

$$\Lambda^M : C_e^N \to \mathbb{R}^M, \qquad V(x) \mapsto (\lambda_1, \dots, \lambda_M).$$

Since the eigenvalues λ_j are different, this mapping is analytic. As $\nabla\lambda_j(V) = \phi_j(x)^2$ and the functions $\phi_1^2, \phi_2^2, \dots$ are linearly independent by the classical result of G. Borg (1946), then for any $V \in C_e^N$ the linear mapping

$$\mathrm{d}\Lambda^M(V) : C_e^N \to \mathbb{R}^M \quad \text{is surjective} \tag{1.24}$$

(all this result may be found in [17]; e.g. see there p. 46 for Borg's theorem). In the space C_e^N consider a Gaussian measure μ_K with a nondegenerate correlation operator K (so for the quadratic function $f(V) = \langle V, \xi\rangle_{L_2}\langle V, \eta\rangle_{L_2}$ we have $\int f(V)\mu_K(\mathrm{d}V) = \langle K\xi, \eta\rangle$). Relation (1.24) easily implies

Lemma 1.1. *For any $M \ge 1$ the measure $\Lambda^M \circ \mu_K$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^M$.*

We will call a vector $\Lambda \in \mathbb{R}^\infty$ *nonresonant* if for any nonzero integer vector s of finite length we have

$$\Lambda \cdot s \ne 0. \tag{1.25}$$

A potential $V(x)$ is called nonresonant if its spectrum $\Lambda(V) = (\lambda_1, \lambda_2, \dots)$ is nonresonant. The nonresonant potentials are defined in C_e^N by a countable family of open dense relations (1.25). So

$$\text{the nonresonant potentials form a subset of } C_e^N \text{ of the second Baire category.} \tag{1.26}$$

Applying Lemma 1.1 we also get

$$\text{the nonresonant potentials form a subset of } C_e^N \text{ of full } \mu_K \text{ measure} \tag{1.27}$$

for any Gaussian measure μ_K as above.

The nonresonant vectors Λ are important because of the following version of the Kronecker–Weyl theorem:

Lemma 1.2. *Let $f \in C^{n+1}(\mathbb{T}^n)$ for some $n \in \mathbb{N}$. Then for any nonresonant vector Λ we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f\big(q_0 + t\Lambda^n\big)\,\mathrm{d}t = (2\pi)^{-n}\int f\,\mathrm{d}x, \quad \Lambda^n = (\Lambda_1, \dots, \Lambda_n),$$

uniformly in $q_0 \in \mathbb{T}^n$. The rate of convergence depends on n, Λ and $|f|_{C^{n+1}}$.

Proof. Let us write $f(q)$ as the Fourier series $f(q) = \sum f_s \mathrm{e}^{\mathrm{i}s\cdot q}$. Then for each nonzero s we have $|f_s| \le C_{n+1}|f|_{C^{n+1}}|s|^{-n-1}$. So for any $\varepsilon > 0$ we may find $R = R_\varepsilon$ such that $|\sum_{|s|>R} f_s\mathrm{e}^{\mathrm{i}s\cdot q}| \le \frac{\varepsilon}{2}$ for each q. Now it suffices to show that

$$\left|\frac{1}{T}\int_0^T f_R\big(q_0 + t\Lambda^n\big)\,\mathrm{d}t - f_0\right| \le \frac{\varepsilon}{2} \quad \forall T \ge T_\varepsilon \tag{1.28}$$

for a suitable T_ε, where $f_R(q) = \sum_{|s|\le R} f_s \mathrm{e}^{\mathrm{i}s\cdot q}$. But

$$\left|\frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}s\cdot(q_0+t\Lambda^n)}\,\mathrm{d}t\right| \le \frac{2}{T|s\cdot\Lambda^n|}$$

for each nonzero s. Therefor the l.h.s. of (1.28) is

$$\leq \frac{2}{T}\Big(\inf_{|s|\leq R}\big|s\cdot\Lambda^n\big|\Big)^{-1}\sum|f_s|\leq T^{-1}|f|_{C^0}C(R,\Lambda).$$

Now the assertion follows. □

1.5. *Spectral properties of A_V: Multi-dimensional case*

Now let, as in Section 1.3, $A=A_V$ be the operator $A=-\Delta+V(x)$, $x\in\mathbb{T}^d$, where $1\leq V(x)\in C^N(\mathbb{T}^d)$. Let $\{\phi_k(x),k\geq 1\}$ be its L_2-normalised eigenfunctions and $\{\lambda_k,k\geq 1\}$, be the corresponding eigenvalues, $1\leq\lambda_1\leq\lambda_2\leq\cdots$. For any $M\geq 1$ denote by $F_M\subset C^N(\mathbb{T}^d)$ the open domain

$$F_M=\{V\mid \lambda_1<\lambda_2<\cdots<\lambda_M\}.$$

Its complement F_M^c is a real analytic variety in $C^N(\mathbb{T}^d)$ of codimension ≥ 2, so F_M is connected (see [6] and references therein). The functions $\lambda_1,\dots,\lambda_M$ are analytic in F_M. Let us fix any nonzero vector $s\in\mathbb{Z}^\infty$ such that $s_l=0$ for $l>M$. The set

$$Q_s=\big\{V\in F_M\mid \Lambda(V)\cdot s=0\big\}$$

clearly is closed in F_M. Since the function $\Lambda(V)\cdot s$ is analytic in F_M, then either $Q_s=F_M$, or Q_s is nowhere dense in F_M. Theorem 1 from [6] immediately implies that $Q_s\neq F_M$, so (1.26) also holds true in the case we consider now.

Let μ_K be a Gaussian measure with a nondegenerate correlation operator, supported by the space $C^N(\mathbb{T}^d)$. As $\Lambda(V)\cdot s$ is a non-trivial analytic function on F_M and F_M^c is an analytic variety of positive codimension, then $\mu_K(Q_s)=0$ (e.g., see Theorem 1.6 in [1]). Since this is true for any M and any s as above, then the assertion (1.27) also is true.

2. Averaging theorem

The approach and the results of this section apply both to Eqs (1.1) and (1.14). We present it for Eq. (1.1) and at Section 2.5 discuss small changes, needed to treat (1.14). Everywhere below T is an arbitrary fixed positive number.

2.1. *Preliminaries*

In Eq. (1.1) with $u\in\mathcal{H}^1$ we pass to the v-variables, $v=\Psi(u)\in h^1$:

$$\dot v_k+\mathrm{i}\nu^{-1}\lambda_k v_k=P_k(v)\,\mathrm{d}\tau+\sum_{j\geq 1}B_{kj}\,\mathrm{d}\boldsymbol{\beta}_j(\tau),\quad k\geq 1. \tag{2.1}$$

Here $B_{kj}=\Psi_{kj}b_j$ (a matrix with real entries, operating on complex vectors), and

$$P_k=P_k^1+P_k^2+P_k^3, \tag{2.2}$$

where P^1, P^2 and P^3 are, correspondingly, the linear, dissipative and Hamiltonian parts of the perturbation:

$$P^1(v)=\varkappa\Psi\circ\frac{\partial^2}{\partial x^2}u,\qquad P^2(v)=-\gamma_R\Psi\big(|u|^{2p}u\big),\qquad P^3(v)=-\mathrm{i}\gamma_I\Psi\big(|u|^{2q}u\big),$$

where $u=G(v)$. We will refer to Eqs (2.1) as to the *v-equations.*

For $k\geq 1$ let us denote $I_k=I(v_k)=\frac12|v_k|^2$ and $\varphi_k=\varphi(v_k)=\operatorname{Arg}v_k\in S^1$, where $\varphi(0)=0\in S^1$. Consider the mappings

$$\Pi_I: h^r\ni v\mapsto I=(I_1,I_2,\dots)\in h^r_{I+},\qquad \Pi_\varphi: h^r\ni v\mapsto\varphi=(\varphi_1,\varphi_2,\dots)\in\mathbb{T}^\infty.$$

Here h^r_{I+} is the positive octant in the space

$$h^r_I = \left\{ I \;\Big|\; |I|_{h^r_I} = 2\sum_j j^{2r}|I_j| < \infty \right\}.$$

We will write

$$\Pi_I\bigl(\Psi(u)\bigr) = I(u), \qquad \Pi_\varphi\bigl(\Psi(u)\bigr) = \varphi(u), \qquad (\Pi_I \times \Pi_\varphi)\bigl(\Psi(u)\bigr) = (I \times \varphi)(u).$$

The mapping $I : \mathcal{H}^r \to h^r_I$ is 2-homogeneous continuous, while the mappings $\varphi : \mathcal{H}^r \to \mathbb{T}^\infty$ and $(I \times \varphi) : \mathcal{H}^r \to h^r_I \times \mathbb{T}^\infty$ are Borel-measurable and discontinuous (the torus $\mathbb{T}^\infty$ is given the Tikhonov topology and a corresponding distance).

Now let us pass in Eq. (2.1) from the complex variables v_k to the real variables $I_k \geq 0, \varphi_k \in S^1$:

$$\mathrm{d}I_k(\tau) = (v_k \cdot P_k)(v)\,\mathrm{d}\tau + Y_k^2\,\mathrm{d}\tau + \sum_l \Psi_{kl} b_l (v_k \cdot \mathrm{d}\boldsymbol{\beta}_l), \quad Y_k = \sqrt{\sum b_l^2 \Psi_{kl}^2}, \tag{2.3}$$

and

$$\begin{aligned}
\mathrm{d}\varphi_k(\tau) \;&=\; \Bigl(\nu^{-1}\lambda_k + |v_k|^{-2}(\mathrm{i}v_k)\cdot P_k - |v_k|^{-2}\sum_l b_l(\Psi_{kl}\cdot v_k)(\Psi_{kl}\cdot \mathrm{i}v_k)\Bigr)\,\mathrm{d}\tau \\
&\quad + \sum_l |v_k|^{-2} b_l \Psi_{kl}(\mathrm{i}v_k \cdot \mathrm{d}\boldsymbol{\beta}_l) \\
&=: \bigl(\nu^{-1}\lambda_k + G_k(v)\bigr)\,\mathrm{d}\tau + \sum_l g_{kl}(v)\Bigl(\frac{\mathrm{i}v_k}{|v_k|}\cdot \mathrm{d}\boldsymbol{\beta}_l(\tau)\Bigr).
\end{aligned} \tag{2.4}$$

Due to (1.22), (1.23)

$$P(v) = \varkappa \widehat{A} v + P^0(v), \qquad P^0 : h^r \to h^r \quad \forall \frac{1}{2} < r \leq N,$$

where the map P^0 is real analytic. The mapping $P^0(v)$ and its differential $\mathrm{d}P^0(v)$ both have a polynomial growth in $|v|_{h^r}$. For any vector $v = (v_1, v_2, \ldots) \in h^0$ we denote $v^m = (v_1, \ldots, v_m) \in \mathbb{C}^m$ and identify v^m with the vector $(v_1, \ldots, v_m, 0, \ldots)$. Then

$$\bigl|v - v^m\bigr|_{h^{r-1/3}} \leq C m^{-1/3} |v|_{h^r}$$

since $\lambda_l \sim |l|^{2l}$. Therefore

$$\bigl|P(v) - P\bigl(v^m\bigr)\bigr|_{h^{r-2-1/3}} \leq m^{-1/3} Q\bigl(|v|_{h^r}\bigr),$$

where Q is a polynomial.

The functions G_k and g_{kl} are singular as $v_k = 0$ and satisfy the following estimates:

$$\bigl|G_k(v)\chi_{\{|v_k|>\delta\}}\bigr| \leq \delta^{-1} Q_k\bigl(|v|_{h^r}\bigr), \tag{2.5}$$

$$\bigl|g_{kl}(v)\chi_{\{|v_k|>\delta\}}\bigr| \leq C\delta^{-1} b_l, \tag{2.6}$$

where Q_k is a polynomial.

For any vector $\theta = (\theta_1, \theta_2, \ldots) \in \mathbb{T}^\infty$ we denote by Φ_θ the unitary rotation

$$\Phi_\theta : h^r \to h^r, \qquad v \mapsto v_\theta, \quad \text{where } v_{\theta j} = \mathrm{e}^{\mathrm{i}\theta_j} v_j \quad \forall j.$$

By $\langle F\rangle$ etc. we denote the averaged functions, $\langle F\rangle(v)=\int_{\mathbb{T}^\infty}F(\Phi_\theta v)\,\mathrm{d}\theta$. They are φ-independent, so $\langle F\rangle=\langle F\rangle(\Pi_I(v))$. The functions $\langle P\rangle,\langle F\rangle,\dots$ also satisfy the estimates above. So

$$\left|\left\langle(v_k\cdot P_k)\right\rangle\left(I^m\right)-\left\langle(v_k\cdot P_k)\right\rangle(I)\right|\le m^{-1/3}C_kQ\left(|I|_{h_I^1}\right),$$

where Q is a polynomial.

Since the dispersion matrix $\{B_{kj}\}$ is nondegenerate, then repeating for Eqs (2.1) and (2.3) the arguments from Section 7 in [10] (also see Section 6.2 in [9]), we get

Lemma 2.1. *Let $v^\nu(\tau)$ be a solution of (2.1) and $I^\nu(\tau)=I(v^\nu(\tau))$. Then for any $k\ge 1$ the following convergence hold uniformly in $\nu>0$:*

$$\int_0^T\mathbf{P}\left\{I_k^\nu(\tau)\le\delta\right\}\mathrm{d}\tau\to 0\quad\text{as }\delta\to 0.\tag{2.7}$$

(Certainly the rate of the convergence depends on k.)

2.2. *The theorem*

Let us abbreviate

$$h^1=h,\qquad h_I^1=h_I,\qquad C\left([0,T],h_{I+}\right)=\mathcal{H}_I,$$

where h_{I+} is the positive octant $\{I\in h_I\mid I_j\ge 0\ \forall j\}$. Fix any $u_0\in h$. Due to estimates (1.8), (1.9) and Eqs (2.3), the set of laws $\{\mathcal{D}(I^\nu(\cdot))\}$, $0<\nu\le 1$, is tight in $\mathcal{H}_I$. Denote by Q^0 any limiting measure as $\nu=\nu_j\to 0$, i.e.

$$\mathcal{D}\left(I^{\nu_j}(\cdot)\right)\rightharpoonup Q^0\quad\text{as }\nu_j\to 0.$$

Let us consider the averaged drift $(\langle(v_k\cdot P_k)\rangle(I)+Y_k^2)\,\mathrm{d}\tau$ for Eq. (2.3). We have

$$\left\langle(v_k\cdot P_k)\right\rangle(v)=\int_{\mathbb{T}^\infty}\left(\mathrm{e}^{\mathrm{i}\theta_k}v_k\right)\cdot P_k(\Phi_\theta v)\,\mathrm{d}\theta=v_k\cdot R_k'(v),\tag{2.8}$$

where $R_k'=\int_{\mathbb{T}^\infty}(\mathrm{e}^{-\mathrm{i}\theta_k}P_k(\Phi_\theta v))\,\mathrm{d}\theta$ (note that $\langle(v_k\cdot P_k)\rangle$ depends only on $I=\Pi_I(v)$, while $R_k'(v)$ depends on v). The diffusion matrix for (2.3) is $\{A_{kr},k,r\ge 1\}$, where

$$A_{kr}(v)=\sum_l(\Psi_{kr}b_lv_k)\cdot(\Psi_{rl}b_lv_r)=\sum_l b_l^2(v_k\cdot v_r)\Psi_{kl}\Psi_{rl}.$$

Its average is

$$\begin{aligned}\langle A_{kr}\rangle(v)&=\sum_l b_l^2\int_{\mathbb{T}^\infty}\mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}(\theta_k-\theta_r)}v_k\bar{v}_r\right)\Psi_{kl}\Psi_{rl}\,\mathrm{d}\theta\\&=\delta_{kr}|v_k|^2Y_k^2,\quad Y_k=\left(\sum_l b_l^2|\Psi_{kl}|^2\right)^{1/2}.\end{aligned}\tag{2.9}$$

Due to (1.21),

$$\sum_k Y_k^2k^{2m}\le C_mB_m\quad\forall m\le N.\tag{2.10}$$

Our first goal is to prove the following averaging theorem:

Theorem 2.2. *The measure Q^0 is a solution of the martingale problem in the space $h_I = h_I^1$ with the drift $(\langle v_k \cdot P_k\rangle(I) + Y_k^2)\,\mathrm{d}\tau$ and the diffusion matrix $\langle A_{kr}\rangle(I)$. That is, $Q^0 = \mathcal{D}(I^0(\cdot))$, where the process $I^0(\tau)$ is a weak (in the sense of stochastic analysis) solution of the system of averaged equations*

$$\mathrm{d}I_k = \big(\big\langle (v_k \cdot P_k)\big\rangle(I) + Y_k^2\big)\,\mathrm{d}\tau + \sum_r \big(\sqrt{\langle A\rangle}\big)_{kr}(I)\,\mathrm{d}\beta_r(\tau), \quad k \ge 1; \tag{2.11}$$

$I(0) = I_0 = \Pi_I(v_0)$. Moreover,

$$\mathbf{E} \sup_{0\le\tau\le T} \big|I^0(\tau)\big|_{h_I}^n \le C_n\big(\|u_0\|_1^{2n} + 1\big) \quad \forall n, \tag{2.12}$$

$$\mathbf{E} \int_0^T \big|I^0(\tau)\big|_{h_I^2}\,\mathrm{d}\tau \le C\big(\|u_0\|_1^2 + 1\big). \tag{2.13}$$

Proof. The crucial step of the proof is to establish the following lemma:

Lemma 2.3. *Let $\tilde F(v)$ be an analytic function on the space $h = h^1$ which extends to an analytic function on $h^{2/3}$ of a polynomial growth. Then*

$$\mathfrak{A}^\nu := \mathbf{E} \max_{0\le\tau\le T} \left| \int_0^\tau \big(\tilde F\big(I^\nu(s), \varphi^\nu(s)\big) - \langle \tilde F\rangle\big(I^\nu(s)\big)\big)\,\mathrm{d}s \right| \to 0 \quad \text{as } \nu \to 0. \tag{2.14}$$

The lemma is proved below in Section 2.3, following the argument in [10]. Now we derive from it the theorem. Let us equip the space $\mathcal{H}_I$ with the Borel sigma-algebra $\mathcal{F}$, the natural filtration of sigma-algebras $\{\mathcal{F}_\tau, 0 \le \tau \le T\}$ and the probability Q^0.

Let us denote $F_k(v) = (v_k \cdot P_k)(v) + Y_k^2$. The fact that the processes $I_k^\nu(\tau) - \int_0^\tau F_k(v^\nu(s))\,\mathrm{d}s$ are martingales (see (2.3)), the convergence $\mathcal{D}(I^{\nu_j}(\cdot)) \rightharpoonup Q^0$ and Lemma 2.3 with $\tilde F = F_k$ imply that the processes

$$Z_k(\tau) = I_k(\tau) - \int_0^\tau \big(\langle (v_k \cdot P_k)\rangle\big(I(s) + Y_k^2\big)\big)\,\mathrm{d}s, \quad k \ge 1,$$

are Q^0-martingales, cf. Section 6 of [10].

Similar to (2.14) we find that

$$\mathbf{E} \max_{0\le\tau\le T} \left| \int_0^\tau \big(\tilde F\big(I^\nu(s), \varphi^\nu(s)\big) - \langle \tilde F\rangle\big(I^\nu(s)\big)\big)\,\mathrm{d}s \right|^4 \to 0 \quad \text{as } \nu \to 0.$$

Then using the same arguments as before, we see that the processes $Z_k(\tau)Z_j(\tau) - \int_0^\tau \langle A_{kj}\rangle(I(s))\,\mathrm{d}s$ also are Q^0-martingales. That is, Q^0 is a solution of the martingale problem with the drift $\langle F_k\rangle + Y_k^2$ and the diffusion $\langle A\rangle$. Hence, Q^0 is a law of a weak solution of Eq. (2.11). See [19].

Estimates (2.12), (2.13) follow from (1.8) and the basic properties of the weak convergence since $\|u\|_m^2 \sim |v|_{h^m}^2 = |\Pi_I(v)|_{h_I^m}$. □

2.3. *Proof of Lemma 2.3*

Fix any $m \ge 1$ and denote by $I^{\nu,m}, \varphi^{\nu,m}$ etc. the vectors, formed by the first m components of the infinite vectors I^ν, φ^ν, etc. Below R stands for a suitable function of ν such that $R(\nu) \to \infty$ as $\nu \to 0$, but

$$\nu R^n \to 0 \quad \text{as } \nu \to 0, \forall n. \tag{2.15}$$

Denote by $\Omega_R = \Omega_R^\nu$ the event

$$\Omega_R = \Big\{ \sup_{0\le\tau\le T} \big|v^\nu(\tau)\big|_{h_1} \le R \Big\}.$$

Then, by (1.9), $\mathbf{P}(\Omega_R^c) \le \varkappa_\infty(R)$ uniformly in ν (see the Notations). We denote

$$\mathbf{P}_{\Omega_R}(Q) = \mathbf{P}(\Omega_R \cap Q), \qquad \mathbf{E}_{\Omega_R}(f) = \mathbf{E}(f\chi_{\Omega_R}).$$

Since for $|v|_{h^1} \le R$ we have $|v - v^m|_{h^{2/3}} \le C(R)m^{-1/3}$ and since $\tilde{F}$ is Lipschitz on $h^{2/3}$ uniformly on bounded sets, then

$$\mathfrak{A}^\nu \le \varkappa_\infty(R) + C_k(R)m^{-2/3} + \mathbf{E}_{\Omega_R} \max_{0\le\tau\le T}\left|\int_0^\tau \left(\tilde{F}\left(I^{\nu,m}, \varphi^{\nu,m}\right) - \langle\tilde{F}\rangle^m\left(I^{\nu,m}\right)\right) \mathrm{d}s\right|.$$

Here $\langle\tilde{F}\rangle^m$ stands for averaging of the function $\mathbb{T}^m \ni I^m \mapsto \tilde{F}(I^m, 0, \ldots)$. So it remains to estimate for any m and R an analogy $\mathfrak{A}^\nu_{m,R}$ of the quantity $\mathfrak{A}^\nu$ for the finite-dimensional process $I^{\nu,m}(\tau)$ on the event Ω_R (where its norm is $\le R$),

$$\mathfrak{A}^\nu_{m,R} = \mathbf{E}_{\Omega_R} \max_{0\le\tau\le T}\left|\int_0^\tau \left(\tilde{F}\left(I^{\nu,m}, \varphi^{\nu,m}\right) - \langle\tilde{F}\rangle^m\left(I^{\nu,m}\right)\right) \mathrm{d}s\right|.$$

Consider a partition of $[0, T]$ by the points

$$\tau_j = \tau_0 + jL, \quad 0 \le j \le K,$$

where τ_K is the last point τ_j in $[0, T)$. The diameter L of the partition is

$$L = \sqrt{\nu},$$

and the non-random phase $\tau_0 \in [0, L)$ will be chosen later. Denoting

$$\eta_l = \int_{\tau_l}^{\tau_{l+1}} \left(\tilde{F}\left(I^{\nu,m}, \varphi^{\nu,m}\right) - \langle\tilde{F}\rangle^m\left(I^{\nu,m}\right)\right) \mathrm{d}s, \quad 0 \le l \le K-1, \tag{2.16}$$

we see that

$$\mathfrak{A}^\nu_{m,R} \le LC(R) + \mathbf{E}_{\Omega_R} \sum_{l=0}^{K-1} |\eta_l|, \tag{2.17}$$

so it remains to estimate $\mathbf{E}_{\Omega_R} \sum |\eta_l|$. We have

$$\begin{aligned}
|\eta_l| \le{}& \left|\int_{\tau_l}^{\tau_{l+1}} \left(\tilde{F}\left(I^{\nu,m}(s), \varphi^{\nu,m}(s)\right) - \tilde{F}\left(I^{\nu,m}(\tau_l), \varphi^{\nu,m}(\tau_l) + \nu^{-1}\Lambda^m(s-\tau_l)\right)\right) \mathrm{d}s\right| \\
&+ \left|\int_{\tau_l}^{\tau_{l+1}} \left(\tilde{F}\left(I^{\nu,m}(\tau_l), \varphi^{\nu,m}\left(\tau_l + \nu^{-1}\Lambda^m(s-\tau_l)\right)\right) - \langle\tilde{F}\rangle^m\left(I^{\nu,m}(\tau_l)\right)\right) \mathrm{d}s\right| \\
&+ \left|\int_{\tau_l}^{\tau_{l+1}} \left(\langle\tilde{F}\rangle^m\left(I^{\nu,m}(\tau_l)\right) - \langle\tilde{F}\rangle^m\left(I^{\nu,m}(s)\right)\right) \mathrm{d}s\right| =: \Upsilon_l^1 + \Upsilon_l^2 + \Upsilon_l^3.
\end{aligned}$$

To estimate the quantities Υ_l^j we first optimise the choice of the phase τ_0. Consider the events $\mathcal{E}_l$, $1 \le l \le K$,

$$\mathcal{E}_l = \left\{I_k^\nu(\tau_l) \le \gamma\right\}, \quad \text{where } \gamma \ge \nu^a, a = 1/10. \tag{2.18}$$

By Lemma 2.1 and the Fubini theorem we can choose $\tau_0 \in [0, L)$ in such a way that

$$K^{-1} \sum_{l=0}^{K-1} \mathbf{P}(\mathcal{E}_l) = \varkappa\left(\gamma^{-1}; R, m\right).$$

For any l consider the event

$$Q_l = \Big\{ \sup_{\tau_l \le \tau \le \tau_{l+1}} \big| I^\nu(\tau) - I^\nu(\tau_l) \big|_{h_I} \ge P_1(R) L^{1/3} \Big\},$$

where $P_1(R)$ is a suitable polynomial. It is not hard to verify (cf. [10]) that $\mathbf{P}(Q_l) \le \varkappa_\infty(L^{-1})$. Setting

$$\mathcal{F}_l = \mathcal{E}_l \cup Q_l$$

we have that

$$\frac{1}{K} \sum_{l=0}^{K-1} \mathbf{P}(\mathcal{F}_l) \le \varkappa\big(\gamma^{-1}; R, m\big) + \varkappa\big(\nu^{-1/2}; m\big) =: \tilde{\varkappa}.$$

Accordingly,

$$\frac{1}{K} \sum_{l=0}^{K-1} \big| (\mathbf{E}_{\mathcal{F}_l}) \Upsilon_l^j \big| \le \frac{P(R)}{K} \sum_{l=0}^{K-1} \mathbf{P}(\mathcal{F}_l) \le P(R) \tilde{\varkappa} := \tilde{\varkappa}_1, \quad j = 1, 2, 3.$$

Similar, since for $\omega \in \Omega_R$ the integrand in (2.16) is $\le Q(R)$, then

$$\frac{1}{K} \sum_l |\mathbf{E}_{\Omega_R} \eta_l - \mathbf{E}_{\Omega_R \setminus \mathcal{F}_l} \eta_l| \le \tilde{\varkappa} Q(R). \tag{2.19}$$

If $\omega \in \Omega_R \setminus \mathcal{F}_l$, then for $\tau \in [\tau_l, \tau_{l+1}]$ we have that $I_k^\nu(\tau_l) \ge \gamma - P_1(R) L^{1/3} \ge \frac{1}{2}\gamma$, if ν is small. This relation and (2.4), (2.5), (2.6) imply that

$$\begin{aligned} &\mathbf{P}_{\Omega_R \setminus \mathcal{F}_l} \Big\{ \big| \varphi^{\nu,m}(s) - \big(\varphi^{\nu,m}(\tau_l) + \nu^{-1} \Lambda^m (s - \tau_l) \big) \big| \ge \nu^a \text{ for some } s \in [\tau_l, \tau_{l+1}] \Big\} \\ &\quad \le \varkappa_\infty\big(\nu^{-1}; R, m\big) \end{aligned}$$

(we recall that $\gamma \ge \nu^{1/10}$). Accordingly,

$$\Big(\sum_l \mathbf{E}_{\Omega_R \setminus \mathcal{F}_l} \Upsilon_l^1 \Big) \le C \nu^a + \varkappa_\infty\big(\nu^{-1}; R, m\big). \tag{2.20}$$

It is clear that

$$\Big(\sum_l \mathbf{E}_{\Omega_R \setminus \mathcal{F}_l} \Upsilon_l^3 \Big) \le P(R) L^{1/3} = P(R) \nu^{1/6}. \tag{2.21}$$

So it remains to estimate the expectation of $\sum \Upsilon_l^2$. For any $\omega \in \Omega_R \setminus \mathcal{F}_l$ abbreviate

$$F(\psi) = \tilde{F}\big(I^{\nu,m}(t_l), \varphi^{\nu,m}(t_l) + \psi \big), \quad \psi \in \mathbb{T}^m,$$

where in the r.h.s. ψ is identified with the vector $(\psi, 0, \ldots) \in \mathbb{T}^\infty$. We can write Υ_l^2 as

$$\Upsilon_l^2 = \bigg| \int_{\tau_l}^{\tau_{l+1}} F\big(\nu^{-1} \Lambda^m (s - \tau_l)\big)\, \mathrm{d}s - \langle F \rangle \bigg| = L \bigg| \frac{\nu}{L} \int_0^{\nu^{-1} L} F\big(\Lambda^m t\big)\, \mathrm{d}t - \langle F \rangle \bigg|.$$

Since the function $F(\psi)$ is analytic and the vector Λ is nonresonant, then by Lemma 1.1 $\Upsilon_l^2 \le L \varkappa(\nu^{-1} L; m, R, \gamma, \Lambda)$. Therefore

$$\Big(\sum_l \mathbf{E}_{\Omega_R \setminus \mathcal{F}_l} \Upsilon_l^2 \Big) \le \varkappa\big(\nu^{-1/2}; m, R, \gamma, \Lambda\big). \tag{2.22}$$

Now (2.17), (2.19) and (2.20)–(2.22) imply that

$$\mathfrak{A}^\nu \le \varkappa_\infty(R) + C(R)m^{-1/3} + \varkappa\big(\nu^{-a}; R, m\big) + \varkappa\big(\gamma^{-1}; R, m\big) + C\nu^a + P(R)\nu^{1/6} + \varkappa\big(\nu^{-1/2}; m, R, \gamma, \Lambda\big).$$

Choosing first R large, then m large and next γ small and ν small in such a way that (2.15) and (2.18) hold, we make the r.h.s. arbitrarily small. This proves the lemma.

2.4. *Joint distribution of actions and angles*

Denote $\tilde\mu_s^\nu = \mathcal{D}(I^\nu(s), \varphi^\nu(s)) = (I \times \varphi) \circ \mathcal{D}(u^\nu(s))$, where $u^\nu(s), 0 \le s \le T$, is a solution of (1.1) and (I^ν, φ^ν) is a solution of the system (2.3), (2.4). For any $f \in L_1(0, T)$, $f \ge 0$, such that $\int f = 1$, set $\tilde\mu^\nu(f) = \int_0^T f(s)\tilde\mu_s^\nu \, ds$. Also let us denote $m^0(f) = \int_0^T f(s)\mathcal{D}(I^0(s))\, ds$; this is a measure on h_{I+}.

Theorem 2.4. *For any f as above,*

$$\tilde\mu^{\nu_j}(f) \rightharpoonup m^0(f) \times d\varphi \quad \textit{as } \nu_j \to 0. \tag{2.23}$$

Proof. For a piecewise constant function f the convergence follows from Theorem 2.2 and Lemma 2.3 since by the lemma, for any $0 \le T_1 < T_2 \le T$, the integral $\int_{T_1}^{T_2} \tilde F(I^\nu(s), \varphi^\nu(s))\, ds$ is close to $\int_{T_1}^{T_2} \langle \tilde F\rangle(I^\nu(s))\, ds$, and by the theorem the integral $\int_{T_1}^{T_2} \langle \tilde F\rangle(I^\nu(s))\, ds$ is close to $\int_{T_1}^{T_2} \langle \tilde F\rangle(I^0(s))\, ds = \int_{\mathbb{T}^\infty}\int_{T_1}^{T_2} \tilde F(I^0(s), \psi)\, ds\, d\psi$ (we are applying the lemma and the theorem on segments $[0, T_1]$ and $[0, T_2]$).

To get the convergence for a general function f we approximate it by piecewise constant functions. See Section 2 of [9] for details. □

2.5. *Multidimensional case*

Let (2.1) be not Eq. (1.1), but Eq. (1.14), written in the v-variables. Assume that $V \in C^N$, where N is sufficiently big, and (1.15), (1.16) hold. Now we should consider (2.1) as an equation in a space $h^r, r > d/2$. The maps $P^1 : h^r \to h^r$ and $P^2 : h^r \to h^r$ are smooth and the differentials $d^m P^1(v) : h^r \times \cdots \times h^r \to h^r$ are poly-linear mappings such that their norms are bounded by polynomials of $|v|_{h^r}$. This allows to apply to Eq. (2.1) the method of [10][2] in the same way as in Sections 2.3–2.4 and establish validity of Theorems 2.2 and 2.4.

3. Effective equations and uniqueness of limit

Let (2.1) be Eq. (1.1) or Eq. (1.14), written in the v-variables, and (2.11) – the corresponding averaged equation. Accordingly, by h we denote either the space h^1 as in Section 1, or the space $h^r, r > d/2$, as in Section 1.3. For simplicity we assume that p and q in (1.14) are integers. If they are not, then in the calculations below the nonlinearities $|u|^{2p}u$ and $|u|^{2q}u$ should be modified by Lipschitz terms which cause no extra difficulties.

3.1. *Effective equations*

Let us write the averaged drift $\langle v_k \cdot P_k\rangle$ and the averaged diffusion $\langle A_{kr}\rangle$ in the form (2.8) and (2.9), respectively. Using (2.2) we write the term $R'(v)$ in (2.8) as

$$R_k'(v) = \sum_{m=1}^{3} \int e^{-i\theta_k} P_k^m(\Phi_\theta v)\, d\theta =: \sum_{m=1}^{3} R_k^m(v), \quad k \ge 1.$$

[2]In was assumed in [10] that the relevant maps and vector-fields are analytic. This analyticity was imposed only for simplicity. Sufficiently high smoothness and polynomial estimates on the corresponding high order differentials are sufficient for all construction of [10].

By (1.22) and (1.23),

$$
\begin{aligned}
R^1(v) &= \varkappa \int \Phi_{-\theta}\Psi\left(\frac{\partial^2}{\partial x^2}G(\Phi_\theta v)\right)\mathrm{d}\theta \\
&= -\varkappa \int \Phi_{-\theta}\widehat{A}\Phi_\theta v\,\mathrm{d}\theta + \varkappa\int \Phi_{-\theta}\mathcal{L}^0(\Phi_\theta v)\,\mathrm{d}\theta \\
&= -\varkappa\widehat{A} + \varkappa R^0(v),
\end{aligned}
\tag{3.1}
$$

$$
R^0(v) = \int \Phi_{-\theta}\mathcal{L}^0(\Phi_\theta v)\,\mathrm{d}\theta,
$$

since $\hat{A}$ commutes with the rotations Φ_θ. The operator R^0 is bounded and selfadjoint in h^0. For any v we have

$$
\left\langle R^1(v), v\right\rangle = \varkappa\int \langle \Delta G\Phi_\theta v, G\Phi_\theta v\rangle\,\mathrm{d}\theta \le -C\varkappa|v|_{h^1}^2,
\tag{3.2}
$$

since $\|G\Phi_\theta v\|_1 \sim |\Phi_\theta v|_{h^1} = |v|_{h^1}$. Writing in (3.1) $\mathcal{L}^0(v) = \Psi\circ V\circ G(v)$ as

$$
\mathcal{L}^0(v) = \varkappa\nabla\big(h^2\circ G\big)(v), \qquad h^2(u) = \frac{1}{2}\int V(x)\big|u(x)\big|^2\,\mathrm{d}x,
$$

we have $R^0(v) = \nabla\langle h^2\circ G\rangle(v)$. Since

$$
\begin{aligned}
\big\langle h^2\circ G\big\rangle(v) &= \frac{1}{2}\sum_{j,l}\int_{\mathbb{T}^\infty}\big\langle V(x)\mathrm{e}^{\mathrm{i}\theta_j}v_j\varphi_j(x), \mathrm{e}^{\mathrm{i}\theta_l}v_l\varphi_l(x)\big\rangle\,\mathrm{d}\theta \\
&= \frac{1}{2}\sum_l |v_l|^2 M_l, \quad M_l = \langle V\varphi_l, \varphi_l\rangle,
\end{aligned}
$$

then $R^0 = \mathrm{diag}\{M_l, l\ge 1\}$. Accordingly,

$$
R^1 = \varkappa\,\mathrm{diag}\{-\lambda_l + M_l, l\ge 1\} > 0, \quad M_l = \langle V\varphi_l, \varphi_l\rangle.
\tag{3.3}
$$

The term R^2 is defined as an integral with the integrand

$$
\Phi_{-\theta}P^2\Phi_\theta(v) = -\gamma_R\Phi_{-\theta}\Psi\big(|u|^{2p}u\big)\big|_{u=G\circ\Phi_\theta v} =: F_\theta(v).
$$

Writing $f^p(|u|^2)u$[3] as $\nabla h^p(u)$, where $h^p(u) = \int F^p(|u|^2)\,\mathrm{d}x$, $(F^p)' = \frac{1}{2}f^p$, and denoting $G\circ\Phi_\theta = L_\theta$, we have

$$
F_\theta(v) = -\gamma_R L_\theta^*\nabla h^p(u)|_{u=L_\theta(v)} = -\gamma_R\nabla\big(h^p\circ L_\theta(v)\big).
$$

So

$$
R^2(v) = -\gamma_R\nabla_v\left(\int_{\mathbb{T}^\infty}\big(h^p\circ G\big)(\Phi_\theta v)\,\mathrm{d}\theta\right) = \gamma_R\nabla_v\big\langle h^p\circ G\big\rangle.
\tag{3.4}
$$

Similar $R^3(v) = -\mathrm{i}\gamma_I\nabla_v\langle h^q\circ G\rangle$ (since the operator $G\circ\Phi_\theta$ is complex-linear). As $\langle h^q\circ G\rangle$ is a function solely of the actions $(I_1, I_2, \ldots)$, then $\nabla_{v_k}\langle h^q\circ G\rangle\in\mathbb{C}$ is a vector, real-proportional to v_k. Therefore $v_k\cdot R_k^3(v) = 0$ for each k. That is,

$$
\big\langle(v_k\cdot P_k)\big\rangle(v) = v_k\cdot R_k^1(v) + v_k\cdot R_k^2(v),
\tag{3.5}
$$

[3] If $d=1$ and p is an integer, then $f^p(|u|^2) = |u|^{2p}$.

where R^1 and R^2 are defined by (3.3) and (3.4). Now we set

$$R(v) = R^1(v) + R^2(v)$$

and consider the following system of stochastic equations:

$$\mathrm{d}v_k(\tau) = R_k(v)\,\mathrm{d}\tau + Y_k\,\mathrm{d}\boldsymbol{\beta}_k, \quad k \geq 1. \tag{3.6}$$

Equations (3.6) are called the *system of effective equations*.

***Example 3.1** ($p=1$). Now $h^1(u) = \frac{1}{4}\int |u|^4\,\mathrm{d}x$. So*

$$h^1 \circ G(v) = \frac{1}{4}\int \left|\sum_k v_k \varphi_k(x)\right|^4 \mathrm{d}x = \frac{1}{4}\sum_{k_1,k_2,k_3,k_4} v_{k_1} v_{k_2} \bar{v}_{k_3} \bar{v}_{k_4} \int \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\,\mathrm{d}x.$$

Since

$$\langle v_{k_1} v_{k_2} \bar{v}_{k_3} \bar{v}_{k_4}\rangle = \begin{cases} |v_{k_1}|^2 |v_{k_2}|^2 & \text{if } k_1 = k_3,\ k_2 = k_4 \text{ or } k_1 = k_4,\ k_2 = k_3, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\langle h^1 \circ G(v)\rangle = \frac{1}{2}\sum_{k_1 \neq k_2} |v_{k_1}|^2 |v_{k_2}|^2 L'_{k_1 k_2} + \frac{1}{4}\sum_k |v_k|^4 L'_{kk}, \tag{3.7}$$

where $L'_{k_1k_2} = \int \varphi_{k_1}^2 \varphi_{k_2}^2\,\mathrm{d}x$. So that

$$\begin{aligned} R_k^2(v) &= -\gamma_R \nabla_{v_k} \langle h^1 \circ G\rangle(v) = -\gamma_R v_k \left(|v_k|^2 L'_{kk} + 2\sum_{l \neq k} |v_l|^2 L'_{kl} \right) \\ &= -\gamma_R v_k \sum_l |v_l|^2 L_{kl}. \end{aligned}$$

Here $L_{kk} = L'_{kk}$ and $L_{kl} = 2L'_{kl}$ if $k \neq l$. So the system of effective equations becomes

$$\mathrm{d}v_k = -v_k\left(\varkappa(\lambda_k - M_k) + \gamma_R \sum_l |v_l|^2 L_{kl}\right)\mathrm{d}\tau + Y_k\,\mathrm{d}\boldsymbol{\beta}_k, \quad k \geq 1. \tag{3.8}$$

If $v(\tau) = \{v_k(\tau), k \geq 1\}$ satisfies (3.6), then for $I_k = I(v_k(\tau))$ we have

$$\mathrm{d}I_k(\tau) = v_k \cdot R_k(v)\,\mathrm{d}\tau + Y_k^2\,\mathrm{d}\tau + Y_k v_k \cdot \mathrm{d}\boldsymbol{\beta}_k, \quad k \geq 1. \tag{3.9}$$

By (3.5) the drift in this system equals $(\langle v_k \cdot P_k\rangle(I) + Y_k^2)\,\mathrm{d}\tau$, while the diffusion matrix is $\delta_{kr}|v_k|^2 Y_k^2 = \langle A_{kr}\rangle$. So system (3.9) has the same set of weak (= martingale) solutions as (2.11), see [19]. We have got

Proposition 3.2. *Let $v(\tau)$ be a weak solution of (3.6) such that $v(0) = v_0$ and*

$$\mathbf{E} \sup_{0 \leq \tau \leq T} \left|v(\tau)\right|_h^{2n} \leq C|v_0|_h^{2n} + C(n,T) \quad \forall n. \tag{3.10}$$

Then $\varPi_I(v(\tau))$ is a weak solution of the system (2.11), satisfying (2.12) and such that $I(0) = I_0$.

The 'right' inverse statement to Proposition 3.2 is given by the following

Proposition 3.3. *Let $I^0(\tau)$ be a weak solution of the averaged Eqs (2.11), constructed in Theorem 2.2. Then there exists a weak solution $v^0(\tau)$ of (3.6) such that $v(0) = v_0$, satisfying (3.10), and such that $\mathcal{D}(\varPi_I(v^0(\cdot))) = \mathcal{D}(I^0(\cdot))$.*

That is, the solutions of Eq. (2.11) which can be obtained as limits (when $\nu \to 0$) of actions $I^\nu(u(\tau))$ of solutions for (1.1) (or (1.14)) are those which can be covered by "regular" solutions of (3.6). For a proof we refer to Section 3 of [9], where the assertion is established in a similar but more complicated situation.

System (3.6) is invariant under rotations Φ_θ:

Proposition 3.4. *Let $v(\tau)$ be a weak solution of (3.6), satisfying (3.10). Then, for any $\theta \in \mathbb{T}^\infty$, $\Phi_\theta v(\tau)$ is a weak solution of (3.6), satisfying (3.10).*

Proof. Applying Φ_θ to (3.6) we get that

$$\mathrm{d}(\Phi_\theta v) = \Phi_\theta R(v)\,\mathrm{d}\tau + \Phi_\theta Y\,\mathrm{d}\boldsymbol{\beta}(\tau), \quad Y = \operatorname{diag}\{Y_k\}.$$

The vector fields $R^1(v)$ and $R^2(v)$ both are obtained by averaging and have the form $R^j(v) = \int \Phi_{-\theta} F^j(\Phi_\theta v)\,\mathrm{d}\theta$. So they commute with the rotations, as well as their sum $R(v)$, and we have

$$\mathrm{d}(\Phi_\theta v) = R(\Phi_\theta v)\,\mathrm{d}\theta + Y\,\mathrm{d}\big(\Phi_\theta \boldsymbol{\beta}(\tau)\big).$$

Since $\mathcal{D}\Phi_\theta \boldsymbol{\beta}(\tau) = \mathcal{D}\boldsymbol{\beta}(\tau)$, then the assertion follows. □

3.2. *The uniqueness*

Let $v^1(\tau)$ and $v^2(\tau)$ be solutions of the effective system (3.6). Denoting $v = v^1 - v^2$, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\big|v(\tau)\big|_{h^0}^2 \le -\varkappa |v|_{h^1}^2 + \big\langle R^2\big(v^1\big) - R^2\big(v^2\big), v^1 - v^2\big\rangle.$$

Consider the last term, denoting $v_\theta^j = \Phi_\theta v^j, u_\theta^j = G(v_\theta^j)$. Since $R^2(v)$ is an integral over $\mathbb{T}^\infty$ with the integrand $-\gamma_R \Phi_{-\theta}\Psi(|u_\theta|^{2p}u_\theta)$, where $u_\theta = G(\Phi_\theta(v))$, then

$$\begin{aligned}\big\langle R^2\big(v^1\big) - R^2\big(v^2\big), v^1 - v^2\big\rangle &= -\gamma_R \int \big\langle \Psi\big(\big|u_\theta^1\big|^{2p}u_\theta^1 - \big|u_\theta^2\big|^{2p}u_\theta^2\big), \Phi_\theta v^1 - \Phi_\theta v^2\big\rangle \mathrm{d}\theta \\ &= -\gamma_R \int \big\langle \big(\big|u_\theta^1\big|^{2p}u_\theta^1 - \big|u_\theta^2\big|^{2p}u_\theta^2\big), u_\theta^1 - u_\theta^2\big\rangle \mathrm{d}\theta.\end{aligned}$$

The integrand in the r.h.s. is nonnegative. So

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}|v|_{h^0}^2 \le -C\varkappa |v|_{h^1}^2 \tag{3.11}$$

(i.e., the effective system (3.6) is strongly monotone). Therefore a strong solution of the system (3.6) is unique. By the Yamada–Watanabe argument (see [7]) a weak solution also is unique. We have got

Theorem 3.5. *Let $I^\nu(\tau) = I(u^\nu(\tau))$, where $u^\nu(\tau)$ is a solution of Eq. (1.1) or of Eq. (1.14) and $u^\nu(0) = u_0$. Then*

$$\mathcal{D}\big(I^\nu(\cdot)\big) \rightharpoonup Q^0 \quad \text{as } \nu \to 0$$

in the space $\mathcal{H}_I$, where Q^0 is a weak solution of (2.11), satisfying (2.12), (2.13). There exists a unique weak solution $v(\tau)$ of the effective Eqs (3.6), satisfying (3.10), such that $v(0) = \Psi(u_0)$ and $\mathcal{D}(\Pi_I(v(\cdot)) = Q^0$.

We note that by (2.10), (3.11) (and since $R(0) = 0$), for any random initial data $v(0) = v_0$, independent from the random force and satisfyng $\mathbf{E}|v_0|_{h^0}^2 < \infty$, Eq. (3.6) has a unique strong solution.

4. Stationary solutions

4.1. *Averaging*

Again, let (2.1) be Eq. (1.1) or Eq. (1.14), written in the v-variables, and (2.11) be the corresponding averaged equation. Accordingly, by h we denote either the space h^1 as in Section 1, or the space h^r, $r > d/2$, as in Section 1.3. Assume that the corresponding u-equation is regular in the space $\mathcal{H}^r$ (e.g., $d = 1$ or the assumptions, given at the end of Section 1.3 are fulfilled), and that it has a unique stationary measure μ^ν (see Sections 1.2, 1.3).

Let $u'^\nu(\tau)$ be a stationary in time solution of Eq. (1.1), $\mathcal{D}(u'^\nu(\tau)) \equiv \mu^\nu$. By estimates in Section 1 the set of laws $\mathcal{D}(I'^\nu(\cdot))$, where $I'^\nu = I(u'^\nu(\tau))$, is compact in h_I. Let Q' be any limiting measure as $\nu_j \to 0$. Clearly it is stationary in τ. The same argument that was used to prove Theorem 2.2 (cf. [10]) imply that Q' is a stationary solution of the averaged equation:

Proposition 4.1. *The measure Q' is the law of a process $I'(\tau), 0 \le \tau \le T$, which is a stationary weak solution of the averaged Eq. (2.11). It meets estimates (2.12), (2.13), and the stationary measure $\pi = \mathcal{D}(I'(0))$ satisfies $\int |I|_{h_I^2} \pi(\mathrm{d}I) < \infty$.*

The measures $(I \times \varphi) \circ \mu^\nu = \mathcal{D}(I'^\nu(s), \varphi'^\nu(s))$ satisfies (2.23) for the same reason as in Section 2.4. Since the measure μ^ν is independent from s, then now

$$\mathcal{D}\big(I'^\nu(s), \varphi'^\nu(s)\big) \rightharpoonup \pi \times \mathrm{d}\varphi \quad \text{as } \nu_j \to 0. \tag{4.1}$$

In the stationary case relation (2.7) implies that

$$\mathbf{P}\big\{I'^\nu_k(\tau) < \delta\big\} \to 0 \quad \text{as } \delta \to 0, \tag{4.2}$$

uniformly in ν. In particular,

$$\pi\{I \mid I_k = 0\} = 0 \quad \forall k. \tag{4.3}$$

4.2. *Lifting to effective equations*

To study the limiting measure π further we lift it to a stationary measure of the effective system (3.6). We start with

Lemma 4.2. *System (3.6) has at most one stationary measure m such that $\int |v|^2_{h^0} m(\mathrm{d}v) < \infty$.*

Proof. Assume that there are two stationary measures and consider the corresponding two stationary solutions of Eq. (3.6). Their difference $v(\tau)$ satisfies (3.11). So a.s. it converges to zero as $\tau \to \infty$. Accordingly the two measures are equal. □

Let $v(\tau)$ be a stationary solution of (3.6), $\mathcal{D}(v(\tau)) \equiv m$. By Proposition 3.4, $\Phi_\theta(v(\tau))$ also is a (weak) stationary solution. So $\mathcal{D}(\Phi_\theta v(\tau)) = \Phi_\theta \circ m$ is a stationary measure for (3.6). Since it is unique, then

$$\Phi_\theta \circ m = m \quad \forall \theta \in \mathbb{T}^\infty.$$

Accordingly, $\Pi_\varphi \circ m$ is a rotation-invariant measure on $\mathbb{T}^\infty$, i.e. $\Pi_\varphi \circ m = \mathrm{d}\varphi$. This implies that in the (I, φ)-variables the measure m has the form

$$\mathrm{d}m = m_I(\mathrm{d}I) \times \mathrm{d}\varphi. \tag{4.4}$$

Proposition 4.1 applies to any time-interval $[0, T]$. So, replacing the sequence $\nu_j \to 0$ by a suitable subsequence $\nu_{j'} \to 0$ we construct a stationary process $I'(\tau)$, $\tau \ge 0$, such that $I'^{\nu_{j'}}(\tau)$ converges to $I'(\tau)$ in distribution on any finite time-interval. Using Proposition 3.3 we construct a solution $v'(\tau)$ of (3.6) such that $\mathcal{D}(\Pi_I(v'(\tau)) \equiv \pi$. Since $\mathcal{D}(v'(\tau)) \rightharpoonup m$ as $\tau \to \infty$, then

$$\pi = \Pi_I \circ m. \tag{4.5}$$

That is, the measure π is independent from the sequence ν_j. We have got

Theorem 4.3. *If* (1.10) *holds, then $I \circ \mu^\nu \rightharpoonup \pi = \Pi_I \circ m$, where m is the unique stationary measure of the effective system.*

In view of (4.1), (4.4) and (4.5),

$$(I \times \varphi) \circ \mu^\nu \rightharpoonup (\Pi_I \times \Pi_\varphi) \circ m \quad \text{as } \nu \to 0.$$

Denote $h_+ = \{v \in h \mid v_j \neq 0\ \forall j\}$. By (4.2) and (4.3) $(\Psi \circ \mu^\nu)(h_+) = 1$ and $m(h_+) = 1$. So the convergence above implies that

Theorem 4.4. *If* (1.10) *holds, then $\mu^\nu \rightharpoonup G \circ m$ as $\nu \to 0$.*

Example 4.5 (Hamiltonian perturbations). *If in* (1.14) *$\gamma_R = 0$, i.e. if the nonlinear term of the perturbation is Hamiltonian, then the effective system is the linear equation*

$$\mathrm{d}v(\tau) = R^1(v)\,\mathrm{d}\tau + Y\,\mathrm{d}\boldsymbol{\beta},$$

where R^1 is defined in (3.3) *and $Y = \operatorname{diag}\{Y_k, k \geq 1\}$. Let $v(0) = 0$. Then $v(\tau)$ is the diagonal complex Gaussian process*

$$v(\tau) = \int_0^\tau \mathrm{e}^{(\tau-s)R^1} Y\,\mathrm{d}\boldsymbol{\beta}(s), \qquad R^1 = \varkappa\big(\widehat{A} - R^0\big).$$

So the stationary measure for the effective system, $\mathcal{D}v(\infty)$, is a direct sum of independent complex Gaussian measures with zero mean and the dispersions $\varkappa^{-1}Y_k^2/(\lambda_k - M_k)$, $k \geq 1$.

The fact that a Hamiltonian nonlinearity produces no effect in the first order averaging (i.e. for the slow time $\tau \lesssim 1$) is well known in the theory of weak turbulence. To produce a non-trivial effect, the Hamiltonian term $-\mathrm{i}\gamma_I f_q(|u|^2)u$ should be scaled by the additional factor $\nu^{-1/2}$, and for the weak turbulence theory to apply to calculate this effect we should send the size of the x-torus to infinity when $\nu \to 0$, see [15].

Example 4.6 **(*$p = 1$*, *continuation*).** *If $p = 1$, then the effective equations become*

$$\mathrm{d}v_k = -v_k\left(\varkappa(\lambda_k - M_k) + \gamma_R \sum_l |v_l|^2 L_{kl}\right)\mathrm{d}\tau + Y_k\,\mathrm{d}\boldsymbol{\beta}_k. \tag{4.6}$$

Assume that the random force in (1.1) (*or in* (1.14)) *is small and is mostly concentrated at a frequency j_*. That is,*

$$b_{j_*} = \varepsilon < 1, \qquad 0 < b_l \ll \varepsilon \quad \text{if } l \neq j_*.$$

Then the numbers Y_k are of order ε and are concentrated close to j_*, i.e.,

$$Y_{j_*} \sim \varepsilon, \qquad Y_l \leq \varepsilon C_N |l - j_*|^{-N} \qquad \forall l, N.$$

So if $v(\tau)$ is a stationary solution of the effective equations and $E_k = \frac{1}{2}\mathbf{E}|v_k(\tau)|^2$, then

$$E_{j_*} \sim \varepsilon^2 \lambda_{j_*}^{-1}, \qquad E_l \leq \varepsilon^2 C_N \lambda_{j_*}^{-1} |l - j_*|^{-N} \quad \forall l, N.$$

That is, the systems (1.1) *and* (1.14) *exhibit no inverse or direct cascade of energy. For other polynomial systems* (1.1) *and* (1.14) *situation is the same. Certainly this is not surprising since by imposing the non-resonance condition we removed from the system resonances, responsible for the two energy cascades.*

5. Equations with non-viscous damping

Following Debussche–Odasso [3] we now discuss Eqs (1.1) with non-viscous damping, i.e. with $\varkappa = 0$ but with $\gamma_R > 0$ and $p = 0$ (Debussche–Odasso considered the case $p = 0, q = 1$):

$$\begin{aligned} &\dot{u} + \mathrm{i}\nu^{-1}\bigl(-u_{xx} + V(x)u\bigr) = -\gamma_R u - \mathrm{i}\gamma_I |u|^{2q} u + \frac{\mathrm{d}}{\mathrm{d}\tau}\sum b_j \boldsymbol{\beta}_j(\tau) e_j(x), \\ &u(x) \equiv u(x + 2\pi) \equiv -u(-x); \end{aligned} \tag{5.1}$$

$$u(0) = u_0. \tag{5.2}$$

Estimates (1.4), (1.8) and (1.9) are valid with $\varkappa = 0$. Jointly with an analogy of estimate (1.17) with $\varkappa = 0, m = 1$ they imply that for $u_0 \in \mathcal{H}^2$ the set of actions $I^\nu(\tau) = I(u^\nu(\tau))$ of solutions for (5.1), (5.2) is tight in $\mathcal{H}_I$. As in Section 2, any limiting measure $Q^0 = \lim \mathcal{D}(I^{\nu_j}(\cdot))$ is a law of a weak solution $I^0(\tau)$ of the averaged Eqs $(2.11)_{\varkappa=0}$ with $I(0) = I_0 = I(u_0)$. Constructions of Section 3 remain true, so $I^0(\tau)$ may be lifted to a weak solution $v^0(\tau)$ of the effective Eqs $(3.6)_{\varkappa=0,p=0}$. Now $R^1 = 0$ and, repeating constructions of Example 3.1 we see that $R_k^2(v) = -\gamma_R v_k$. So the effective equations become the linear system

$$\mathrm{d}v_k(\tau) = -\gamma_R v_k \,\mathrm{d}\tau + Y_k \,\mathrm{d}\boldsymbol{\beta}_k. \tag{5.3}$$

This system has a unique solution $v(\tau)$ such that $v(0) = v_0 = \Psi(u_0)$. So

$$\lim_{\nu\to 0} \mathcal{D}\bigl(I^\nu(\cdot)\bigr) = \mathcal{D}\Pi_I\bigl(v(\cdot)\bigr).$$

Due to the result of [3], Eq. (5.1) has a unique stationary measure μ^ν. Repeating arguments from Example 4.5, we see that when $\nu \to 0$, the measures $\Psi \circ \mu^\nu$ converge to the unique stationary measure of Eq. (5.3) which is

$$m = \mathcal{D}\int_{-\infty}^{0} \operatorname{diag}\bigl\{\mathrm{e}^{-s\gamma_R} Y_k\bigr\}\,\mathrm{d}\boldsymbol{\beta}_k(s).$$

This is a direct sum of independent complex Gaussian measures with zero mean and the dispersion $Y_k^2/\gamma_R, k \geq 1$. So every solution $u(\tau)$ of (5.1) satisfies the Gaussian limit

$$\lim_{\nu\to 0}\lim_{\tau\to\infty} \mathcal{D}u(\tau) = G \circ m.$$

If we replace in (5.1) the linear damping by the nonlinear term $-\gamma_R|u|^2 u$, then the effective system (5.3) should be replaced by the nonlinear system (4.6) with $\lambda_k = M_k = 0$. In this case the limiting measure is non-Gaussian.

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