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Superdiffusivity for Brownian Motion in a Poissonian potential with long range correlation I: Lower bound on the volume exponent

Hubert Lacoin

CEREMADE, Place du Maréchal De Lattre De Tassigny, 75775 Paris Cedex 16, France. E-mail: lacoin@ceremade.dauphine.fr

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Abstract. We study trajectories of d-dimensional Brownian Motion in Poissonian potential up to the hitting time of a distant hyperplane. Our Poissonian potential V is constructed from a field of traps whose centers location is given by a Poisson Point Process and whose radii are IID distributed with a common distribution that has unbounded support; it has the particularity of having long-range correlation. We focus on the case where the law of the trap radii ν has power-law decay and prove that superdiffusivity hold under certain condition, and get a lower bound on the volume exponent. Results differ quite much with the one that have been obtained for the model with traps of bounded radii by Wühtrich (*Ann. Probab.* **26** (1998) 1000–1015, *Ann. Inst. Henri Poincaré Probab. Stat.* **34** (1998) 279–308): the superdiffusivity phenomenon is enhanced by the presence of correlation.

Résumé. Dans cet article, nous étudions les trajectoires d'un mouvement brownien dans $\mathbb{R}^d$ évoluant dans un potentiel poissonien jusqu'au temps d'atteinte d'un hyper-plan situé loin de l'origine. Le potentiel poissonien V que nous considerons est construit à partir d'un champs de pièges dont les centres sont déterminés par un processus de Poisson et dont les rayons sont des variables aléatoires IID. Nous concentrons notre étude sur le cas particulier ou la loi des rayons des pièges à une queue polynomiale et nous prouvons que les trajectoires ont un caractère surdiffusif quand certaines conditions sont vérifées et nous donnons une borne inférieure pour l'exposant de volume. Les résultats sont sensiblement différents de ceux obtenus dans le cas ou les pièges sont à rayon bornés par Wühtrich (*Ann. Probab.* **26** (1998) 1000–1015, *Ann. Inst. Henri Poincaré Probab. Stat.* **34** (1998) 279–308) : le phénomène de surdiffusivité est renforcé par la présence de corrélations.

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1. Introduction

1.1. *Brownian Motion and Poissonian traps*

This paper studies a model of Brownian Motion in a random potential. Given a random function V defined on $\mathbb{R}^d$ and $\lambda, \beta > 0$ (β being the inverse temperature) we study trajectories of a Brownian Motion $(B_t)_{t\geq 0}$ killed at (space-dependent) rate $\beta(\lambda + V(B_t))$ conditioned to survive up to the hitting time of a distant hyperplane.

The potential V that we considered is buildt from a Poisson Point Process on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity $\mathcal{L} \times \nu$ where $\mathcal{L}$ is the Lebesgue measure, and ν is a probability measure. We call it $\omega := \{(\omega_i, r_i) \mid i \in \mathbb{N}\}$. The definition of V is $V(\cdot) := \sum_{i\in\mathbb{N}} (r_i)^{-\gamma} W((r_i)^{-1}(\cdot - \omega_i))$ where W is a non-negative function with compact support (for the sake of simplicity we restrict ourselves to $W = \mathbf{1}_{B(0,1)}$, where $B(0,1)$ denotes the Euclidian ball). The potential can be seen as a superposition of traps centered on the points ω_i, and with IID random radii r_i. We are specifically interested in the case where ν has unbounded support and a tail with power law decay.

This model is very similar to the ones studied in [15,16,18–20] (see also the monograph of Sznitman [17] for a full acquaintance with the subject), the only difference is that we allow the traps to have random radii. The crucial difference is that the potential we consider has long-range spacial correlation i.e. that the value of V at two distant point are not independent but have correlation that decays like a power of the distance. Another situation where one has a correlated potential V is when W is not compactly supported (see e.g. [4,12]) but this out of our scope.

1.2. *Superdiffusivity and volume exponent*

A typical trajectory of a Brownian Motion killed with homogeneous rate λ and conditioned to survive till it hits a distant hyperplane looks like the following: The motion along the direction that is orthogonal to the hyperplane (call it e_1) is ballistic (with speed $1/\sqrt{2\lambda}$) but the motion along the $d-1$ other coordinate is diffusive, and for that reason trajectories tend to stay in a tube centered on the axis $\mathbb{R}e_1$ of diameter $\sqrt{L}$ where L is the distance between the motions starting point and the hyperplane that has to be hit.

Adding a non-homogenous term to the killing rate makes the problem much harder to analyze and changes this behavior in some cases: physicists predicts that when β is large (at low temperature) or when $d<4$ for every β, transversal fluctuation of the trajectories are superdiffusive i.e. of an amplitude L^{ξ} for some $\xi\in(1/2,1)$ that is called *the volume exponent*. The aim of the paper is to show that spatial correlation in V enhances that phenomenon.

2. Model and results

2.1. *Model*

Let us make formal the definition we gave for the model. We consider

$$\omega:=\big\{(\omega_i,r_i)\mid i\in\mathbb{N}\big\} \tag{2.1}$$

a Poisson Point Process in $\mathbb{R}^d\times\mathbb{R}^+$ (we index the points in the Poisson Point Process in an arbitrary deterministic way, e.g. such that $|\omega_i|$ is an increasing sequence, $|\cdot|$ being the Euclidian norm on $\mathbb{R}^d$) whose intensity is given by $\mathcal{L}\times\nu$ where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^d$ and $\nu=\nu_\alpha$ is the probability measure on $\mathbb{R}^+$ defined by

$$\forall r\geq 1,\quad \nu\big([r,\infty]\big)=r^{-\alpha} \tag{2.2}$$

for some $\alpha>0$ (which is a parameter of the model). We denote by $\mathbb{P}$, and $\mathbb{E}$ its associated probability law and expectation.

Given $\gamma>0$, let V^ω, $\mathbb{R}^d\to\mathbb{R}_+$ be defined as

$$V^\omega(x):=\sum_{i=1}^{\infty}r_i^{-\gamma}\mathbf{1}_{\{|x-\omega_i|\leq r_i\}}. \tag{2.3}$$

Note that $V^\omega(x)<\infty$, for almost every realization of ω and for every x if and only if the condition

$$\alpha+\gamma-d>0 \tag{2.4}$$

is fulfilled, and we always consider it to be so in the sequel. This construction is natural way to get a potential with long range correlation that decays like a power of the distance constructed from a Poisson Point Process. Indeed with this setup,

$$\mathbb{E}\big[V^\omega(0)V^\omega(x)\big]\asymp x^{d-\alpha-\gamma}. \tag{2.5}$$

Given $x\in\mathbb{R}^d$, let $\mathbf{P}_x$ (and $\mathbf{E}_x$ the associated expectation) denote the law $B=(B_t)_{t\geq 0}$, standard Brownian Motion starting from x and set $\mathbf{P}:=\mathbf{P}_0$. Given $L>0$ set

$$\mathcal{H}_L:=\{L\}\times\mathbb{R}^{d-1}. \tag{2.6}$$

Given any closed set $A \subset \mathbb{R}^d$ let T_A denote the hitting time of A. Given $\lambda > 0$, $\beta > 0$, the probability for a Brownian Motion killed with rate $\beta(V+\lambda)$ to survive till it hits $\mathcal{H}_L$ is equal to

$$Z_L^\omega := \mathbf{E}\left[\exp\left(-\int_0^{T_{\mathcal{H}_L}} \beta\big(V^\omega(B_t)+\lambda\big)\,\mathrm{d}t\right)\right]. \tag{2.7}$$

The law of the trajectories conditioned to survival μ_L^ω is given by

$$\frac{\mathrm{d}\mu_L^\omega}{\mathrm{d}\mathbf{P}}(B) := \frac{1}{Z_L^\omega}\exp\left(-\int_0^{T_{\mathcal{H}_L}} \beta\big(V^\omega(B_t)+\lambda\big)\,\mathrm{d}t\right). \tag{2.8}$$

In what follows, we consider only the case $\beta = 1$ as temperature does not play any role in our results.

2.2. *Review of known results*

Let us turn to a rigorous definition of the volume exponent. For $\xi > 0$ one defines $\mathcal{C}_L^\xi$ the be a tube of cubic section, of width L^ξ and centered on $\mathbb{R}e_1$, where $e_1 = (1, 0, \dots, 0)$:

$$\mathcal{C}_L^\xi := \mathbb{R} \times \left[-L^\xi/2, L^\xi/2\right]^{d-1}, \tag{2.9}$$

and the event

$$\mathcal{A}_L^\xi := \left\{B \mid \forall t \in [0, T_{\mathcal{H}_L}], A_t \in \mathcal{C}_L^\xi\right\}. \tag{2.10}$$

In words, A_L^ξ is the event: "B has transversal fluctuation of amplitude less than L^ξ."

We define the volume exponent as

$$\xi_0 := \sup\left\{\xi > 0 \,\Big|\, \lim_{L\to\infty} \mathbb{E}\big[\mu_L\big(\mathcal{A}_L^\xi\big)\big] = 0\right\}. \tag{2.11}$$

It is expected to coincide with

$$\xi_1 := \inf\left\{\xi > 0 \,\Big|\, \lim_{L\to\infty} \mathbb{E}\big[\mu_L\big(\mathcal{A}_L^\xi\big)\big] = 1\right\}. \tag{2.12}$$

In particular if $V \equiv 0$ one has $\xi_1 = \xi_0 = 1/2$.

Let us recall what are the conjecture and known result for the volume exponent for the model of Brownian Motion in Poissonian Obstacles studied in [15–20] and for related model. In the remainder of this section, ξ_0 relates more to the general notion of volume exponent that to the strict definition given above and in the different results that are cited, definitions may differ: When $d \geq 4$ whether $\xi_0 > 1/2$ or not should depend on the temperature i.e. on the value of β: at high temperature (low β) trajectories should be diffusive and satisfy invariance principle whereas at low temperature (high β) trajectories are believed to be superdiffusive. In the low temperature phase, the value of ξ_0 should not depend on β. Diffusivity at high temperature has been proved for a discrete version of this model by Ioffe and Velenik [5] and it is reasonable to think that their technique can adapt to the Brownian case when correlation have bounded range. Prior to that, similar results had been proved for directed polymer in random environment that can be considered as a simplified version of the model (see e.g. [2,3]). Superdiffusivity at low-temperature is a much more challenging issue: physicists have no clear prediction for the value of ξ_0 and no mathematical progress towards proving $\xi_0 > 1/2$ has been made so far.

In any dimension, the value of ξ is conjectured be related to the fluctuation of $\log Z_L^\omega$ around its mean: If the variance asymptotically satisfies

$$\operatorname{Var}\log Z_L^\omega \approx L^{2\chi}, \tag{2.13}$$

then one should have the scaling relation

$$\chi = 2\xi_0 - 1. \tag{2.14}$$

The heuristic reason for this is that $L^{2\xi-1}$ is the entropic cost for moving L^{ξ} away from the axis $\mathbb{R}e_1$, whereas the energetic gain one might expect for such a move is L^{χ}. The volume exponent corresponds to the value of ξ for which cost and gain are balanced.

When $d \leq 3$, there is no phase transition and trajectories are expected to be superdiffusive for all β. It is not very clear what it means when $d = 3$ but for the two-dimensional case, physicists predicts on heuristic ground that $\xi_0 = 2/3$ and $\chi = 1/3$. This is conjectured to hold not only for Brownian Motion in Poissonian Obstacles but for a whole family of two-dimensional models called the KPZ universality class (directed last passage percolation, first passage percolation, directed polymers…). In fact the conjecture goes much further and includes a description of the the scaling limit (see e.g. the seminal paper of Kardar, Parisi and Zhang [7]).

A lot of efforts have been made to bring that conjecture on rigorous ground. In fact, it has even been proved that $\xi_0 = 2/3$ for some very specific models in the KPZ universality class:

– Directed last passage percolation in $1+1$ dimension with exponential environment by Johansson [6].
– Directed polymer in $1+1$ dimension with log-Gamma environment and specific boundary condition by Seppalainen [14].

These two results have in common that they have been proved by using exact computation that are specific to the model. Note that a similar result has been proved for the conjectured scaling limit of this model, in [1].

Another approach has been to look for more robust method using the idea of energy vs. entropy competition. In [19,20], Wühtrich proved that $\xi_0 \geq 3/5$ for $d = 2$ and that $\xi_0 \leq 3/4$ in all dimension (with a definition for ξ_0 that is slightly differs of the one we present here). In [18], he proved a rigorous version of the scaling identity $\chi = 2\xi_0 - 1$. Similar results had been proved before for first passage percolation by Licea, Newman and Piza [10] and after for directed polymers by Peterman [13] and Méjane [11].

In [8], we have investigated the effect of transversal correlation in the environment for directed polymers, and in particular their effect on the volume exponent. There it is shown that in any dimension, if environment correlations decay like a small power of the distance then, superdiffusivity holds. More precisely that if the correlation decays like the inverse-distance to the power θ, then $\xi_0 \geq 3/(4+\theta)$. In some cases it shows in particular that $\xi_0 > 2/3$ which indicates that KPZ conjecture does not holds in that case. The bound $\xi \leq 3/4$ of [11] remains valid.

Here we study the effect of isotropic correlation (and therefore it seemed natural to it in for an undirected model), and we have not found in the literature any prediction about what the value of ξ_0 should be.

2.3. *Main result*

We present a lower bound on ξ_0 for our model with correlation.

Set

$$\begin{aligned} \bar{\xi}(d,\alpha,\gamma) &:= \frac{1}{\alpha - d + 1} \quad \text{if } \gamma \leq \alpha - d, \\ \bar{\xi}(d,\alpha,\gamma) &:= \frac{3}{3+\alpha+2\gamma-d} \quad \text{if } \gamma \geq \alpha - d. \end{aligned} \tag{2.15}$$

Theorem 2.1 (Lower bound for the volume exponent). *For any choice of* α, d, γ, *one has*

$$\xi_0 \geq \bar{\xi}(d,\alpha,\gamma) \vee (1/2), \tag{2.16}$$

where ξ_0 *is the quantity defined in* (2.11).

Remark 2.2. *In some cases, the lower bound that we get for* ξ_0 *is larger than* $3/4$, *which contrasts with all the results that we have reviewed in the previous section and indicates that isotropic correlation enhance superdiffusivity in a more drastic way than transversal ones. The above result gives a necessary condition for having superdiffusivity*: $\gamma < \alpha - d$ *and* $\alpha - d < 1$ *or* $\gamma > \alpha - d$ *and* $\alpha + 2\gamma - d < 3$.

Remark 2.3. *The definition of the volume exponent that we use is different of the one used in* [19] *which is slightly weaker. Combining techniques used in* [8] *and here one could prove also that* $\xi_0 \geq 3/5$ *when* $d = 2$ *for any value of* α *and* γ *for this definition of* ξ_0.

2.4. *Further questions*

We prove in this paper that for a class of correlated environment, the trajectories have superdiffusive behavior and that the bound $\xi \le 3/4$ that is valid for the uncorrelated model [19] is not valid here and can be beaten. Therefore one would be interested to find an upper bound (< 1) for ξ. We have addressed this issue in companion paper [9]. In some special cases (when either $\gamma = \alpha - d > 1/3$) one can even prove that the lower bound that we prove here is optimal and give the exact value of $\xi_0 = \xi_1$.

The result that we present concerns the so-called point-to-plane model. A similar result should hold for the point-to-point model. The method that we use in Sections 3.3 and 3.4 are quite robust and could be easily adapted to the other setup but getting something similar to what is done in Section 3.2 seems more difficult and challenging and we are not able to do it yet. One can still get a non-optimal result by using another construction inspired by what is done in [20], we present it in the Appendix.

For the Brownian directed polymer in correlated environment, in [8], it is shown that either superdiffusivity holds at all temperature or that one has diffusivity at high temperature (except for some special limiting cases). For the model presented here one would like to show something similar e.g. that diffusivity holds if correlation have fast-decay at infinity (decay like a large power of the inverse-distance) and the amplitude of V is small. For the moment this is quite out of reach and the methods used in [5] do not seem to adapt to this case.

3. Proof of Theorem 2.1

3.1. *Sketch of proof*

In order to make the strategy of the proof clear we need to introduce some notation. One defines

$$\bar{C}_L^{\xi} := [L/2, L] \times \left[-L^{\xi}/2, L^{\xi}/2\right]^{d-1} \tag{3.1}$$

and

$$\widetilde{C}_L^{\xi} := \left\{x \in \mathbb{R}^d \mid d(x, \bar{C}_L) \le 2\sqrt{d} L^{\xi}\right\} = \bigcup_{y \in \bar{C}_L} B\left(y, 2\sqrt{d} L^{\xi}\right), \tag{3.2}$$

where for a closed set $A \subset \mathbb{R}^d$, and $x \in \mathbb{R}^d$, $d(x, A)$ denotes the Euclidean distance between x and A, i.e.

$$d(x, A) := \min_{y \in A} |y - x| \tag{3.3}$$

($|\cdot|$ is the Euclidean norm), and $B(x, r)$, $r \ge 0$ is the Euclidean ball of radius r. Let $\mathcal{B}_L^{\xi}$ be the set of trajectories that avoids the set $\widetilde{C}_L^{\xi}$:

$$\mathcal{B}_L^{\xi} := \left\{B \mid \forall t \in [0, T_{\mathcal{H}_L}], B_t \notin \widetilde{C}_L^{\xi}\right\}. \tag{3.4}$$

Note that, as Brownian trajectories are continuous

$$\mathcal{B}_L^{\xi} \cap \mathcal{A}_L^{\xi} = \varnothing. \tag{3.5}$$

The first step of our proof (Section 3.2) is inspired by [10]. We prove a result much weaker that Theorem 2.1 by using a simple geometric argument combined to rotational invariance: that with probability close to one,

$$\mu_L^{\omega}\left(\mathcal{A}_L^{\xi}\right) \le e^{L^{2\xi-1}(\log L)^3} \mu_L^{\omega}\left(\mathcal{B}_L^{\xi}\right), \tag{3.6}$$

or equivalently, that with probability close to one,

$$Z_L^{\omega}\left(\mathcal{A}_L^{\xi}\right) \le e^{L^{2\xi-1}(\log L)^3} Z_L^{\omega}\left(\mathcal{B}_L^{\xi}\right), \tag{3.7}$$

where for an event A, we use the notation

$$Z_L^\omega(A) := Z_L^\omega \times \mu_L^\omega(A) = \mathbf{E}\left[\exp\left(\int_0^{T_{\mathcal{H}_L}} \big(V^\omega(B_t)+\lambda\big)\,\mathrm{d}t\right)\mathbf{1}_A\right]. \tag{3.8}$$

Then we modify slightly the environment ($\omega \to \widetilde{\omega}$) by adding additional traps whose radii are in $(\sqrt{d}L^\xi, 2\sqrt{d}L^\xi)$, and whose centers are in the region $\bar{C}_L^\xi$. The second step of the proof (Section 3.3) is to show that typical realization of $\widetilde{\omega}$ are roughly the same as typical realization of ω.

Finally, we notice that adding these traps lowers the value of $Z_L^\omega(\mathcal{A}_L^\xi)$ but that $Z_L^\omega(\mathcal{B}_L^\xi) = Z_L^{\widetilde{\omega}}(\mathcal{B}_L^\xi)$ (adding these traps changes the values taken by V only in the region $\widetilde{C}_L^\xi$ that the trajectory in the event $\mathcal{B}_L^\xi$ do not visit). The third step of the proof (Section 3.4) is to show that with our choice of $\widetilde{\omega}$ and ξ, one has with large probability

$$Z_L^{\widetilde{\omega}}\big(\mathcal{A}_L^\xi\big) \le \mathrm{e}^{-L^{2\xi-1+\varepsilon}} Z_L^\omega\big(\mathcal{A}_L^\xi\big) \tag{3.9}$$

for some $\varepsilon > 0$, which combined with (3.7), gives the result with $\widetilde{\omega}$ instead of ω. The fact that ω and $\widetilde{\omega}$ look typically the same allows to conclude.

We explain in the course of the proof the reasons for our choices of $\widetilde{\omega}$ and how we obtain the condition on ξ.

3.2. *Using rotational invariance*

For $\theta \in (-\pi/2, \pi/2)$, let R_θ denote following the rotation of $\mathbb{R}^d$

$$(x_1, x_2, x_3, \dots, x_d) \mapsto (x_1\cos\theta - x_2\sin\theta, x_2\cos\theta + x_1\sin\theta, x_3, \dots, x_d). \tag{3.10}$$

Set

$$\mathcal{H}_L^\theta := R_\theta[\mathcal{H}_L] \quad \text{and} \quad \mathcal{C}_L^{\theta,\xi} = R_\theta \mathcal{C}_L^\xi \tag{3.11}$$

(the image of the sets $\mathcal{H}_L$ resp. $\mathcal{C}_L^{\theta,\xi}$ for R_θ). One defines in the same fashion the event $\mathcal{A}_L^{\theta,\xi}$ as

$$\mathcal{A}_L^{\theta,\xi} := \big\{B \mid \forall t \in [0, T_{\mathcal{H}_L^\theta}], B_t \in \mathcal{C}_L^{\theta,\xi}\big\}. \tag{3.12}$$

Note that if $B \in \mathcal{A}_L^{\theta,\xi}$ if and only if $R_{-\theta}B \in \mathcal{A}_L^\xi$.

One proves the following

Proposition 3.1. *For any $\xi \in (0,1)$ set $\theta = \theta(L,\xi) := 10\sqrt{d}L^{\xi-1}$. Then one has that for any $N \le \delta\theta^{-1}$ (for some fixed small enough $\delta > 0$),*

$$\mathbb{P}\Big[Z_L^\omega\big(\mathcal{A}_L^\xi\big) \ge \mathrm{e}^{2N^2\theta^2 L\sqrt{\log L}} \max_{i\in\{-N,\dots,N\}\setminus\{0\}} Z_L^\omega\big(\mathcal{A}_L^{i\theta,\xi} \cap \mathcal{B}_L^\xi\big)\Big] \le \frac{1}{L} + \frac{1}{N}. \tag{3.13}$$

In particular, setting $N := \log L$ one has

$$\mathbb{P}\big[Z_L^\omega\big(\mathcal{A}_L^{\theta,\xi}\big) \ge \mathrm{e}^{(\log L)^3 L^{2\xi-1}} Z_L^\omega\big(\mathcal{B}_L^\xi\big)\big] \le \frac{2}{\log L}. \tag{3.14}$$

We split the proof of the proposition into two lemmas: The first lemma allows to compare almost deterministically $Z_L^\omega(\mathcal{A}_L^{\theta,\xi})$ with $Z_L^{R_{-\theta}(\omega)}(\mathcal{A}_L^\xi)$ (which by rotation invariance of ω is distributed like $Z_L^\omega(\mathcal{A}_L^\xi)$). $R_\theta(\omega)$ denotes the image of the Poisson Point Process ω by R_θ, i.e. (recall (2.1))

$$R_\theta(\omega) := \big\{(R_\theta\omega_i, r_i) \mid i \in \mathbb{N}\big\}. \tag{3.15}$$

Lemma 3.2. *Set $\xi \in (0,1)$, θ such that $|\theta| \geq 10\sqrt{d}L^{\xi-1}$ and $|\theta| \leq \delta$ for some $\delta > 0$, and ω that satisfies $\max_{x\in[-L^2,L^2]^d} V^{\omega}(x) \leq \log L$. Then for all sufficiently large L one has*

$$Z_L^{\omega}\big(\mathcal{A}_L^{\theta,\xi} \cap \mathcal{B}_L^{\xi}\big) > Z_L^{R_{-\theta}(\omega)}\big(\mathcal{A}_L^{\xi}\big)\exp\big(-2\theta^2 L\sqrt{\log L}\big). \tag{3.16}$$

The second lemma estimates the probability that $Z_L^{\omega}(\mathcal{A}_L^{\xi})$ has the largest value among the different $(Z_L^{R_{i\theta}(\omega)}(\mathcal{A}_L^{\xi}))_{i\in\{-N,\dots,N\}}$. The argument comes from [13].

Lemma 3.3. *For any value of θ and any N*

$$\mathbb{P}\Big[Z_L^{\omega}\big(\mathcal{A}_L^{\xi}\big) > \max_{i\in\{-N,\dots,N\}\setminus\{0\}} Z_L^{R_{i\theta}(\omega)}\big(\mathcal{A}_L^{\xi}\big)\Big] \leq \frac{1}{N}. \tag{3.17}$$

The proof for of the proposition from the lemmas is straightforward with the use of Lemma A.1 that ensures that with probability $1/N$ the assumption on V in Lemma 3.2 is satisfied.

Proof of Lemma 3.2. By symmetry we can assume $\theta > 0$. The assumptions we have on θ guarantees that on the event $\mathcal{A}_L^{\theta,\xi}$, $T_{\mathcal{H}_L^{\theta}} < T_{\mathcal{H}_L}$, and that $\mathcal{C}_L^{\theta,\xi} \cap \widetilde{\mathcal{C}}_L^{\xi} = \varnothing$ (see Fig. 1). Therefore using the strong Markov property for Brownian Motion,

$$\begin{aligned} Z_L^{\omega}\big(\mathcal{A}_L^{\theta,\xi} \cap \mathcal{B}_L^{\xi}\big) = \mathbf{E}\bigg[&\exp\bigg(-\int_0^{T_{\mathcal{H}_L^{\theta}}}\big(V^{\omega}(B_t)+\lambda\big)\,\mathrm{d}t\bigg) \\ &\times \mathbf{1}_{\mathcal{A}_L^{\theta,\xi}}\mathbf{E}_{B_{T_{\mathcal{H}_L^{\theta}}}}\bigg[\exp\bigg(-\int_0^{T_{\mathcal{H}_L}}\big(V^{\omega}(B_t)+\lambda\big)\,\mathrm{d}t\bigg)\mathbf{1}_{\{\forall s\leq T_{\mathcal{H}_L}, B_s\notin\widetilde{\mathcal{C}}_L^{\xi}\}}\bigg]\bigg]. \end{aligned} \tag{3.18}$$

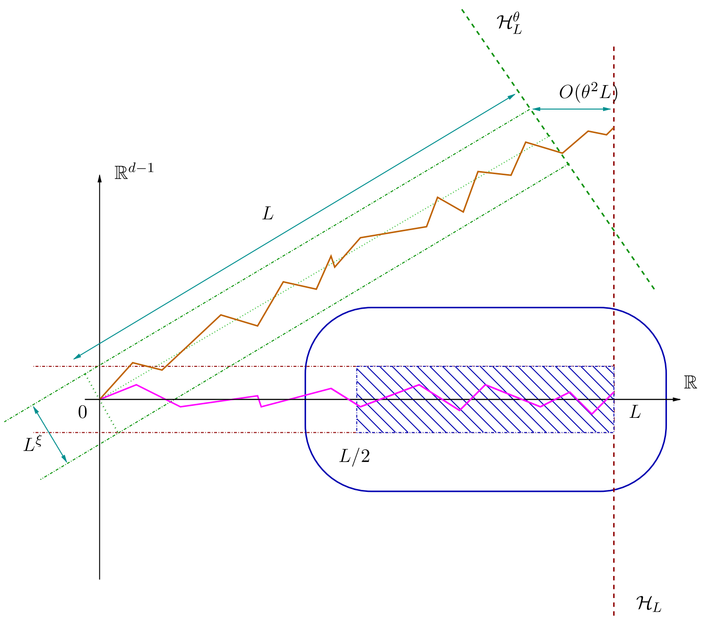

Fig. 1. Projection of the model along the two first coordinates. The two tubes represented are $\mathcal{C}_L^{\xi}$ and $\mathcal{C}_L^{\xi,\theta}$. The shadowed region is $\bar{\mathcal{C}}_L^{\xi}$, and this is where ω is modified. The full line that encircles $\bar{\mathcal{C}}_L^{\xi}$ denote the limit of $\widetilde{\mathcal{C}}_L^{\xi}$, the region where we may have $V^{\widetilde{\omega}} \neq V^{\omega}$. The two trajectories represent typical trajectories of $\mathcal{A}_L^{\xi}$ and $\mathcal{A}_L^{\xi,\theta} \cap \mathcal{B}_L^{\xi}$. One can see on the picture that if θ is chosen large enough ($\theta \geq CL^{\xi-1}$ where C is a constant depending on d), one the event $\mathcal{A}_L^{\theta,\xi}$, the hitting time of $\mathcal{H}_L$ is larger than the hitting time of $\mathcal{H}_L^{\theta}$. Moreover the tube $\mathcal{C}_L^{\xi,\theta}$ and the set $\widetilde{\mathcal{C}}_L^{\xi}$ are disjoint. Standard trigonometry allows to say that the maximal distance between a point in $\mathcal{C}_L^{\xi} \cap \mathcal{H}_L^{\theta}$ and $\mathcal{H}_L$ is $\mathrm{O}(\theta^2 L)$.

On the event $\mathcal{A}_L^{\theta,\xi}$, one has $B_{T_{\mathcal{H}_L^\theta}} \in (\mathcal{H}_L^\theta \cap \mathcal{C}_L^{\theta,\xi})$. Therefore, the right-hand side of (3.18) is smaller than

$$\begin{aligned}&\mathbf{E}\biggl[\exp\biggl(-\int_0^{T_{\mathcal{H}_L^\theta}}\bigl(V^\omega(B_t)+\lambda\bigr)\,\mathrm{d}t\biggr)\mathbf{1}_{\mathcal{A}_L^{\theta,\xi}}\biggr]\\ &\quad\times\max_{x\in\mathcal{H}_L^\theta\cap\mathcal{C}_L^{\theta,\xi}}\mathbf{E}_x\biggl[\exp\biggl(-\int_0^{T_{\mathcal{H}_L}}\bigl(V^\omega(B_t)+\lambda\bigr)\,\mathrm{d}t\biggr)\mathbf{1}_{\{\forall s\le T_{\mathcal{H}_L},B_s\notin\widetilde{\mathcal{C}}_L^\xi\}}\biggr].\end{aligned}\tag{3.19}$$

The first term on the above product is equal to $Z_L^{R_{-\theta}(\omega)}(\mathcal{A}_L^\xi)$. By the assumption one has on V ($V\le\log L$ in the ball of radius L^2), the second term is larger than

$$\max_{x\in\mathcal{H}_L^\theta\cap\mathcal{C}_L^{\theta,\xi}}\mathbf{E}_x\bigl[\mathrm{e}^{-T_{\mathcal{H}_L}(\log L+\lambda)}\mathbf{1}_{\{\forall s\le T_{\mathcal{H}_L},B_s\notin\widetilde{\mathcal{C}}_L^\xi,|B_s|\le L^2\}}\bigr].\tag{3.20}$$

Hence the lemma is proved if one can show that for all x in $\mathcal{H}_L^\theta\cap\mathcal{C}_L^{\theta,\xi}$

$$\mathbf{E}_x\bigl[\mathrm{e}^{-T_{\mathcal{H}_L}(\log L+\lambda)}\mathbf{1}_{\{\forall s\le T_{\mathcal{H}_L},B_s\notin\widetilde{\mathcal{C}}_L^\xi,|B_s|\le L^2\}}\bigr]\ge\exp\bigl(-2\sqrt{\log L}\,\theta^2L\bigr).\tag{3.21}$$

From our assumptions on θ, for L large enough one has

$$\max_{x\in\mathcal{H}_L^\theta\cap\mathcal{C}_L^{\theta,\xi}}d(x,\mathcal{H}_L)=\bigl(L^\xi/2\bigr)\sin\theta+L(1-\cos\theta)\le L\theta^2,\tag{3.22}$$

and

$$\min_{x\in\mathcal{H}_L^\theta\cap\mathcal{C}_L^{\theta,\xi}}d\bigl(x,\widetilde{\mathcal{C}}_L^\xi\bigr)\ge\sin\theta L-\bigl(1+\sqrt{d}\bigr)L^\xi\ge L\theta^2\tag{3.23}$$

(these inequality comes from the assumption one has taken for θ and trigonometry).

If one consider a d-dimensional cube whose edges are parallel to the coordinate axis centered at x and of side-length $d(x,\mathcal{H}_L)$, then with $\mathbf{P}$ probability $1/2d$, the exit time of the cube for a Brownian Motion started from x is equal to $T_{\mathcal{H}_L}$. Moreover, if L is large enough then this cube does not intersect $\widetilde{\mathcal{C}}_L^\xi$ (cf. (3.22) and (3.23)) and lies within the ball of radius L^2. Hence (using symmetries of the cube)

$$\mathbf{E}_x\bigl[\mathrm{e}^{-T_{\mathcal{H}_L}(\log L+\lambda)}\mathbf{1}_{\{\forall s\le T_{\mathcal{H}_L},B_s\notin\widetilde{\mathcal{C}}_L^\xi,|B_s|\le L^2\}}\bigr]\ge\frac{1}{2d}\mathbf{E}\bigl[\mathrm{e}^{-\mathcal{T}_{d(x,\mathcal{H}_L)}(\log L+\lambda)}\bigr],\tag{3.24}$$

where

$$\mathcal{T}_r:=\inf\bigl\{t\ge0,\|B_t\|_\infty=r\bigr\},\tag{3.25}$$

and $\|x\|_\infty=\max_{i=1,\dots,d}|x_i|$ is the l_∞ norm on $\mathbb{R}^d$. The hitting time $\mathcal{T}_r$ is stochastically dominated by τ_r the first hitting time of r by a one dimensional Brownian Motion. And one has

$$\mathbf{P}(\tau_r\le s)=2\int_{|x|>(r/\sqrt{s})}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x.\tag{3.26}$$

Hence one has for L large enough

$$\mathbf{E}_x\bigl[\mathrm{e}^{-T_{\mathcal{H}_L}(\log L+\lambda)}\mathbf{1}_{\{\forall s\le T_{\mathcal{H}_L},B_s\notin\widetilde{\mathcal{C}}_L^\xi\text{ and }|B_s|\le L^2\}}\bigr]\ge\frac{1}{2d}\mathbf{E}\bigl[\mathrm{e}^{-\tau_{L\theta^2}(\log L+\lambda)}\bigr]>\mathrm{e}^{-2\theta^2L\sqrt{\log L}}.\tag{3.27}$$

□

Proof of Lemma 3.3. Note that $Z_L^{R_{i\theta}(\omega)}(\mathcal{A}_L^\xi)$, $i \in \{-N,\dots,N\}$ are identically distributed variables. However they are not exchangeable, and therefore the statement is not that obvious. As

$$\sum_{k\in\{1,\dots,N\}} \mathbb{P}\Big[Z_L^{R_{k\theta}\omega}(\mathcal{A}_L^\xi) > \max_{i\in\{1,\dots,N\}\setminus\{k\}} Z_L^{R_{i\theta}\omega}(\mathcal{A}_L^\xi)\Big] \le 1 \tag{3.28}$$

there exists some $k_0 \in \{1, N\}$ such that

$$\mathbb{P}\Big[Z_L^{R_{k_0\theta}\omega}(\mathcal{A}_L^\xi) > \max_{i\in\{1,\dots,N\}\setminus\{k_0\}} Z_L^{R_{i\theta}(\omega)}(\mathcal{A}_L^\xi)\Big] \le \frac{1}{N}. \tag{3.29}$$

Hence by rotational invariance of ω and $V(\omega)$

$$\mathbb{P}\Big[Z_L^{\omega}(\mathcal{A}_L^\xi) > \max_{i\in\{1-k,\dots,N-k\}\setminus\{0\}} Z_L^{R_{i\theta}(\omega)}(\mathcal{A}_L^\xi)\Big] \le \frac{1}{N}. \tag{3.30}$$

$\square$

3.3. *Change of environment*: *Adding traps in* $\bar{C}_L^\xi$

With ω we construct a second environment $\widetilde{\omega}$ that has more traps with radius $\approx L^\xi$ in the region $\bar{C}_L^\xi$. The aim of this section is to show that typical event for ω are also typical for $\widetilde{\omega}$.

We construct ω and $\widetilde{\omega}$ on the same probability space and for convenience denote by $\mathbb{P}$ their joint probability. Recall that ω is Poisson Point Process in $\mathbb{R}^d \times \mathbb{R}^+$ with intensity $\mathcal{L} \times \nu$. Then define $\widehat{\omega}$ to be a Poisson Point Process on $\mathbb{R}^d \times \mathbb{R}^+$ independent of ω with intensity

$$L^{-((d+1+\alpha)\xi+1)/2}\widetilde{\mathcal{L}}, \tag{3.31}$$

where $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}(\xi, L)$ denotes the Lebesgue measure on the set

$$\bar{C}_L^\xi \times \big[\sqrt{d}L^\xi, 2\sqrt{d}L^\xi\big] \tag{3.32}$$

and define

$$\widetilde{\omega} = \omega + \widehat{\omega}, \tag{3.33}$$

which is a Poisson Point Process on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity

$$\mathcal{L} \times \nu + L^{-((d+1+\alpha)\xi+1)/2}\widetilde{\mathcal{L}}. \tag{3.34}$$

Lemma 3.4. *Assume that*

$$\xi(d-1-\alpha) + 1 > 0. \tag{3.35}$$

Then, there exists a constant C not depending on L such that for any event A one has

$$\mathbb{P}(\omega \in A) \le C\sqrt{\mathbb{P}(\widetilde{\omega} \in A)}. \tag{3.36}$$

Before going to the proof, we explain why the result holds: For ω the number of points in $\widetilde{C}_L^\xi \times [\sqrt{d}L^\xi, 2\sqrt{d}L^\xi]$ is a Poisson variable of mean

$$\frac{(1-2^{-\alpha})}{2} d^{-\alpha/2} L^{(d-1-\alpha)\xi+1}. \tag{3.37}$$

The fluctuation around the mean are therefore of order $L^{((d-1-\alpha)\xi+1)/2}$. The number of points in process $\widehat{\omega}$ is a Poisson variable of mean $L^{-((d+1+\alpha)\xi+1)/2} \times \frac{\sqrt{d}L^{d\xi+1}}{2}$ (intensity $\times$ volume). Therefore the number of points one add to ω to get $\widetilde{\omega}$ is of the same order as the fluctuation for the number of point of ω in $\mathcal{C}_L^\xi$, and for that reason the two process should typically look the same.

The result would not hold if $\widehat{\omega}$ had an intensity of a larger order.

Proof of Lemma 3.4. Let Q resp. $\widetilde{Q}$ denote the law of ω resp. $\widetilde{\omega}$ under $\mathbb{P}$. For a function f, we denote by $Q(f)$ resp. $\widetilde{Q}(f)$ expectation w.r.t. Q resp. $\widetilde{Q}$. Note that $\widetilde{Q}$ is absolutely continuous with respect to Q and one has

$$\frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}Q}(\omega) = \prod_{\{(\omega_i,r_i)\in\mathcal{C}'_L\times[\sqrt{d}L^\xi,2\sqrt{d}L^\xi]\}} \bigl(1+(\alpha)^{-1}r_i^{1+\alpha}L^{-((d+1+\alpha)\xi+1)/2}\bigr)\mathrm{e}^{-(\sqrt{d}/2)L^{((d-1-\alpha)\xi+1)/2}}. \tag{3.38}$$

For any event A by Cauchy–Schwartz inequality, on has

$$Q(A) = \widetilde{Q}\biggl(\frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\mathbf{1}_A\biggr) \le \sqrt{\widetilde{Q}\biggl[\biggl(\frac{Q}{\widetilde{Q}}\biggr)^2\biggr]}\sqrt{\widetilde{Q}(A)}. \tag{3.39}$$

What is left to show is that the first term in the right-hand side remains bounded with L. One has

$$\begin{aligned} Q\biggl(\frac{Q}{\widetilde{Q}}\biggr) &:= \exp\biggl(\frac{\sqrt{d}}{2}L^{((d-1-\alpha)\xi+1)/2}\biggr) \\ &\quad\times Q\biggl(\prod_{\{(\omega_i,r_i)\in\mathcal{C}'_L\times[\sqrt{d}L^\xi,2\sqrt{d}L^\xi]\}} \frac{1}{1+(\alpha)^{-1}r_i^{1+\alpha}L^{-((d+1+\alpha)\xi+1)/2}}\biggr). \end{aligned} \tag{3.40}$$

And

$$\begin{aligned} &Q\biggl(\prod_{\{(\omega_i,r_i)\in\mathcal{C}'_L\times[\sqrt{d}L^\xi,2\sqrt{d}L^\xi]\}} \frac{1}{1+(\alpha)^{-1}r_i^{1+\alpha}L^{-((d+1+\alpha)\xi+1)/2}}\biggr) \\ &\quad= \exp\biggl(-\frac{L^{(d-1)\xi+1}}{2}\int_{\sqrt{d}L^\xi}^{2\sqrt{d}L^\xi} \frac{L^{-((d+1+\alpha)\xi+1)/2}\,\mathrm{d}r}{1+(\alpha)^{-1}r^{1+\alpha}L^{-((d+1+\alpha)\xi+1)/2}}\biggr). \end{aligned} \tag{3.41}$$

Note that the quantity $r^{1+\alpha}L^{-((d+1+\alpha)\xi+1)/2}$ is small uniformly in the domain of integration (by the assumption $\xi(d-1-\alpha)+1>0$) so that

$$\begin{aligned} &\int_{\sqrt{d}L^\xi}^{2\sqrt{d}L^\xi} \frac{\mathrm{d}r}{1+(\alpha)^{-1}r^{1+\alpha}L^{-((d+1+\alpha)\xi+1)/2}} \\ &\quad= \sqrt{d}L^\xi - \bigl(1+\mathrm{o}(1)\bigr)L^{-((d+1+\alpha)\xi+1)/2}\int_{\sqrt{d}L^\xi}^{2\sqrt{d}L^\xi} \alpha^{-1}r^{1+\alpha}\,\mathrm{d}r = \sqrt{d}L^\xi + \mathrm{O}\bigl(L^{((\alpha+3-d)\xi-1)/2}\bigr). \end{aligned} \tag{3.42}$$

Putting everything together one gets

$$Q\biggl(\frac{Q}{\widetilde{Q}}\biggr) = \exp\bigl(\mathrm{O}(1)\bigr). \tag{3.43}$$

□

3.4. *The effect of the change of measure*

In this section we estimate the difference between $\log Z_L^{\widetilde{\omega}}(\mathcal{A}_L^\xi)$ and $\log Z_L^{\omega}(\mathcal{A}_L^\xi)$.

Proposition 3.5. *Suppose that*

$$(d-1-\alpha)\xi+1>0. \tag{3.44}$$

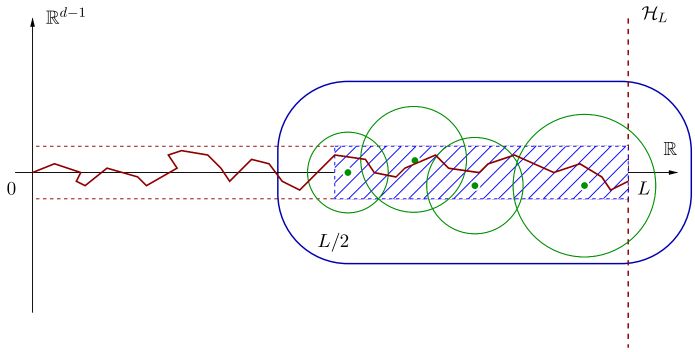

Fig. 2. The shadowed region is $\bar{C}_L^\xi$, and this is where ω is modified. The full line that encircles $\bar{C}_L^\xi$ denote the limit of $\widetilde{C}_L^\xi$, the region where the values taken by $V^{\widetilde{\omega}}$ and V^ω may differ. A typical trajectory of $\mathcal{A}_L^\xi$ is represented. The four dots denote $\widehat{\omega}_i$, $i \in 1,\dots,4$, the center of the traps that have been added. The zone of influence of these traps $B(\widehat{\omega}_i, \widehat{r}_i)$ are represented as circles. The definition of $\widehat{\omega}$ implies that the radius of the traps are large enough to cover all the width of the tube $\mathcal{C}_L^\xi$ on a segment of length L^ξ.

Then, for any $\varepsilon > 0$, *with probability tending to one when* N *goes to infinity*

$$\log Z_L^\omega\bigl(\mathcal{A}_L^\xi\bigr) - \log Z_L^{\widetilde{\omega}}\bigl(\mathcal{A}_L^\xi\bigr) \geq L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon}. \tag{3.45}$$

The idea of the proof is quite simple (see Fig. 2). Making the change of environment $\omega \to \widetilde{\omega}$, we add roughly $L^{((d-1-\alpha)\xi+1)/2}$ traps of radius larger than $\sqrt{d}L^\xi$. The traps we add are wide enough so that every trajectory in $\mathcal{A}_L^\xi$ has to go through every one of them (this explains our choice of adding only traps of large radius).

Under μ_L^ω, trajectories are roughly ballistic, so that they should typically spend a time of order L^ξ in each trap. As the traps are of radius $\approx L^\xi$, they modify the potential by $L^{-\xi\gamma}$. Therefore, for most trajectories in $B \in \mathcal{A}_L^\xi$, one should have

$$\int_0^{T_{\mathcal{H}_L}} \bigl(V^{\widetilde{\omega}} - V^\omega\bigr)(B_t)\,\mathrm{d}t = \int_0^{T_{\mathcal{H}_L}} \bigl(V^{\widehat{\omega}}\bigr)(B_t)\,\mathrm{d}t \approx L^\xi \times L^{-\xi\gamma} \times \#\{\text{traps in } \widehat{\omega}\} \approx L^{((d+1-\alpha-2\gamma)\xi+1)/2}, \tag{3.46}$$

which heuristically explains the result.

To make this sketch rigorous, the main point is to give a proof of the fact that each trajectory spend a time of order L^ξ in each trap. This is the aim of Proposition 3.6.

For a given function $V : \mathbb{R}^d \to \mathbb{R}^+$ define the probability measure $\bar{\mu}^V$ by

$$\frac{\mathrm{d}\bar{\mu}^V}{\mathrm{d}\mathbf{P}}(B) := \frac{1}{Z_L^V(\mathcal{A}_L^\xi)} \mathrm{e}^{-\int_0^{T_{\mathcal{H}_L}} (\lambda + V(B_t))\,\mathrm{d}t} \mathbf{1}_{\mathcal{A}_L^\xi}, \tag{3.47}$$

where $Z_L^V(\mathcal{A}_L^\xi)$ is defined in the same way as $Z_L^\omega(\mathcal{A}_L^\xi)$ with V^ω replaced by V.

Given a in $[0, L]$ one wants to check that most trajectories of $\mathcal{A}_L^\xi$ spend a reasonable amount of time in the slice of the tube

$$\bigl[a, a + L^\xi\bigr] \times \bigl[-L^{\xi/2}, L^\xi/2\bigr]^{d-1}. \tag{3.48}$$

Let $B^{(1)}$ denote the first coordinate of B.

Proposition 3.6. *For any non-negative function* V *such that* $V(x) \leq \log L$ *for all* x *such that* $|x| \leq L^2$, *for any* $\varepsilon > 0$ *for* L *large enough, and for any* $a \in [0, L - L^\xi]$,

$$\bar{\mu}^V\biggl(\int_0^{T_{\mathcal{H}_L}} \mathbf{1}_{\{B_t^{(1)} \in [a, a+L^\xi]\}}\,\mathrm{d}t \leq L^{\xi-\varepsilon}\biggr) \leq \mathrm{e}^{-L^\xi}. \tag{3.49}$$

Remark 3.7. *The result above simply states that under the polymer measure, the cost for the motion to be superbalistic is roughly the same as for Brownian Motion. Even though the statement is quite natural, the proof we present contains some technicalities due to the fact the Brownian Motion is confined in a tube and one has to control what happens close to the boundary. However the underlying idea is quite simple.*

We postpone the proof of this statement at the end of the section and use it to prove Proposition 3.5.

Let $\mathcal{N}$ be the number of point in $\widehat{\omega}$ (this a Poisson variable of mean $(\sqrt{d}/2)L^{((d-1-\alpha)\xi+1)/2}$). We choose to index them in an arbitrary way so that one can write

$$\widehat{\omega} := \big\{(\widehat{\omega}_k n, \widehat{r}_k), k \in \{1, \dots, \mathcal{N}\}\big\}. \tag{3.50}$$

For $i \in \{0, \dots, \mathcal{N}\}$, define

$$\omega^i := \omega \cup \big\{(\widehat{\omega}_k, \widehat{r}_k), k \in \{1, \dots, i\}\big\}. \tag{3.51}$$

Note that $\omega^0 = \omega$ and $\omega^{\mathcal{N}} := \widetilde{\omega}$, so that $(\omega^i)_{0\le i\le N}$ is an interpolating sequence between ω and $\widetilde{\omega}$.

Lemma 3.8. *Given $\varepsilon > 0$, there exists a constant c such that for all L large enough, for every environment $\widetilde{\omega}$ that satisfies*

$$\forall |x| \le L^2, \quad V^{\widetilde{\omega}}(x) \le \log L, \tag{3.52}$$

for all $i \in \{1, \dots, \mathcal{N}\}$

$$Z_L^{\omega^i}\big(\mathcal{A}_L^{\xi}\big) \le Z_L^{\omega^{i-1}}\big(\mathcal{A}_L^{\xi}\big)\mathrm{e}^{-cL^{\xi(1-\gamma)-\varepsilon}}. \tag{3.53}$$

As a consequence

$$Z_L^{\widetilde{\omega}}\big(\mathcal{A}_L^{\xi}\big) \le \exp\big(-c\mathcal{N}L^{\xi(1-\gamma)-\varepsilon}\big)Z_L^{\omega}\big(\mathcal{A}_L^{\xi}\big). \tag{3.54}$$

Proof. As ω_i lies in $\bar{C}_L^{\xi}$ and $\widehat{r}_i \ge \sqrt{d}L^{\xi}$, there exists $a_i \in (L/2, L - L^{\xi})$ such that

$$\big[a_i, a_i + L^{\xi}\big] \times \big[-L^{\xi}/2, L^{\xi}/2\big]^{d-1} \subset \mathcal{B}(\widehat{\omega}_i, \widehat{r}_i). \tag{3.55}$$

By assumption, for $|x| \le L^2$ and for all $i \le \mathcal{N}$, $V^{\omega^i}(x) \le V^{\widetilde{\omega}}(x) \le \log L$. Therefore one can apply Proposition 3.6. Using $\widehat{r}_i \le 2\sqrt{d}L^{\xi}$ one gets

$$\begin{aligned}
\frac{Z_{L,\beta}^{\omega^i}(\mathcal{A}_L^{\xi})}{Z_{L,\beta}^{\omega^{i-1}}(\mathcal{A}_L^{\xi})} &= \bar{\mu}^{V^{\omega^{i-1}}}\big(\mathrm{e}^{-\int_0^{T_{\mathcal{H}_L}} \widehat{r}_i^{-\gamma}\mathbf{1}_{\{B_t\in\mathcal{B}(\widehat{\omega}_i,\widehat{r}_i)\}}\,\mathrm{d}t}\big) \\
&\le \bar{\mu}^{V^{\omega^{i-1}}}\big(\mathrm{e}^{-\int_0^{T_{\mathcal{H}_L}} (2\sqrt{d}L^{\xi})^{-\gamma}\mathbf{1}_{\{B_t^{(1)}\in[a_i,a_i+L^{\xi}]\}}\,\mathrm{d}t}\big) \\
&\le \bar{\mu}^{V^{\omega^{i-1}}}\bigg(\int_0^{T_{\mathcal{H}_L}} \mathbf{1}_{\{B_t^{(1)}\in[a_i,a_i+L^{\xi}]\,\mathrm{d}t\le L^{\xi-\varepsilon}\}}\bigg) + \mathrm{e}^{-L^{\xi-\varepsilon}(2\sqrt{d}L^{\xi})^{-\gamma}} \\
&\le \mathrm{e}^{-L^{\xi}} + \mathrm{e}^{-L^{\xi-\varepsilon}(2\sqrt{d}L^{\xi})^{-\gamma}} \le \mathrm{e}^{-cL^{\xi(1-\gamma)-\varepsilon}}.
\end{aligned} \tag{3.56}$$

□

Proposition 3.5 is immediate consequence of (3.54) and the fact that $\mathcal{N}$ is a Poisson variable of mean $(\sqrt{d}/2) \times L^{((d-1-\alpha)\xi+1)/2}$.

Proof of Proposition 3.6. Define

$$T_1 := \inf\{t \geq 0\colon\ B_t^{(1)} = a + L^{\xi}/2\},$$
$$T_2 := \inf\{t \geq T_1\colon\ \big|B_t^{(1)} - \big(a + L^{\xi}/2\big)\big| = L^{\xi}/2\}. \tag{3.57}$$

For all $t \in [T_1, T_2]$, $B_t^{(1)} \in [a, a + L^{\xi}]$ and thus it is sufficient to prove that with large $\bar{\mu}^V$ probability $T_2 - T_1$ is large. Set

$$U_1 := B_{T_1} \quad \text{and} \quad U_2 := B_{T_2}. \tag{3.58}$$

Note that conditionally on $U_1 = x$, $U_2 = y$, the law of $(B_t)_{t\in[T_1,T_2]}$ under $\bar{\mu}^V$ is independent of the rest of the motion, and that (recall that $\mathbf{P}_x$ is the law of a standard motion started from x)

$$\bar{\mu}^V\big(T_2 - T_1 \leq L^{\xi-\varepsilon} \mid U_1 = x; U_2 = y\big)$$
$$= \frac{\mathbf{E}_x[\mathrm{e}^{-\int_0^{T_2}(\lambda+V(B_t))\,\mathrm{d}t}\mathbf{1}_{\{\forall t\in(0,T_2), B_t\in\mathcal{C}_L\}} \mid B_{T_2} = y]}{\mathbf{E}_x[\mathrm{e}^{-\int_0^{T_2}(\lambda+V(B_t))\,\mathrm{d}t}\mathbf{1}_{\{\forall t\in(0,T_2), B_t\in\mathcal{C}_L\}} \mid B_{T_2} = y]} =: \frac{R(x,y)}{Q(x,y)}. \tag{3.59}$$

We are to show that this is small uniformly in the choice of x and y. Our way to estimate both term in the fraction is to suppress inhomogeneity due to the potential V: using the assumption $0 \leq V \leq \log N$ we can replace $V(B_t)$ by 0 in R and by $\log L$ in Q.

Let $\bar{x}$ resp. $\bar{y}$ be the projection of the $d-1$ last coordinate of x resp. y on $\mathbb{R}^{d-1}$, and let $p_t^*(\cdot,\cdot)$ denote the heat kernel on $[-L^{\xi/2}, L^{\xi/2}]^{d-1}$ with Dirichlet boundary condition. One has

$$R(x,y) \leq \int_{(0,L^{\xi-\varepsilon})} p_t^*(\bar{x},\bar{y})\mathrm{e}^{-\lambda t}\mathbf{P}_x(T_2 \in \mathrm{d}t),$$
$$Q(x,y) \geq \int_{(0,\infty)} p_t^*(\bar{x},\bar{y})\mathrm{e}^{-(\lambda+\log L)t}\mathbf{P}_x(T_2 \in \mathrm{d}t). \tag{3.60}$$

Note that uniformly on $t \leq L^{2\xi-\varepsilon}$, when L gets large

$$\mathbf{P}_x(T_2 \leq t) = \mathbf{P}\Big(\max_{s\in[0,1]}\big|B_s^{(1)}\big| \geq L^{\xi}/\big(2\sqrt{t}\big)\Big) = 2\big(1 + \mathrm{o}(1)\big)\mathbf{P}\Big(\max_{s\in[0,1]} B_s^{(1)} \geq L^{\xi}/\big(2\sqrt{t}\big)\Big)$$
$$= \frac{4(1+\mathrm{o}(1))}{\sqrt{2\pi}}\int_{L^{\xi}/(2\sqrt{t})}^{\infty} \mathrm{e}^{-u^2/2}\,\mathrm{d}u = \frac{8\sqrt{t}(1+\mathrm{o}(1))}{L^{\xi}\sqrt{2\pi}}\mathrm{e}^{-L^{2\xi}/(8t)}. \tag{3.61}$$

Using the estimates on heat kernel from the Appendix (Lemma A.2) one obtains by using integration by part that for large L

$$R(x,y) \leq A(\bar{x},\bar{y})\int_{(0,L^{\xi-\varepsilon})} \frac{1}{t^2}\mathbf{P}_x(T_2 \in \mathrm{d}t)$$
$$= \big(1 + \mathrm{o}(1)\big)A(\bar{x},\bar{y})\int_0^{L^{\xi-\varepsilon}} \frac{16}{t^{5/2}\sqrt{2\pi}L^{\xi}}\mathrm{e}^{-L^{2\xi}/(8t)}\,\mathrm{d}t \leq A(\bar{x},\bar{y})\mathrm{e}^{-L^{\xi+\varepsilon}/16} \tag{3.62}$$

and that

$$Q(x,y) \geq A(\bar{x},\bar{y})\int_0^{\infty} \mathrm{e}^{-(\lambda+\log L)t}\mathrm{e}^{-L^{2\xi}t-\pi^2/(2t)}\mathbf{P}_x(T_2 \in \mathrm{d}t)$$
$$\geq A(\bar{x},\bar{y})\int_{L^{\xi}/2}^{L^{\xi}} \mathrm{e}^{-(\lambda+\log L)t}\mathrm{e}^{-L^{2\xi}/t-\pi^2 t/2}\mathbf{P}_x(T_2 \in \mathrm{d}t)\,\mathrm{d}t$$
$$\geq A(\bar{x},\bar{y})\mathrm{e}^{-2L^{\xi}(\log L)}, \tag{3.63}$$

where

$$A(u,v) := \prod_{i=1}^{d-1} \min\bigl((u_i + L^{\xi/2}), (L^{\xi/2} - u_i)\bigr) \min\bigl((v_i + L^{\xi/2}), (L^{\xi/2} - v_i)\bigr), \tag{3.64}$$

which (recall (3.59)) gives the result. □

3.5. *Proof of Theorem* 2.1

Now we can use the results of all the previous sections to get the main theorem. Consider $\xi < \bar{\xi}(d,\alpha,\gamma)$. One can check that it satisfies both

$$\begin{aligned} &\frac{(d+1-\alpha-2\gamma)\xi+1}{2} > 2\xi - 1, \\ &(d-1-\alpha)\xi + 1 > 0. \end{aligned} \tag{3.65}$$

Then with probability going to one (cf. Proposition 3.1), one has

$$\log Z_L^\omega\bigl(\mathcal{A}_L^\xi\bigr) \le \log Z_L^\omega\bigl(\mathcal{B}_L^\xi\bigr) + L^{2\xi-1}(\log L)^3. \tag{3.66}$$

Then combining this with Proposition 3.5 and $Z_L^\omega(\mathcal{B}_L^\xi) = Z_L^{\widetilde{\omega}}(\mathcal{B}_L^\xi)$ one get that with probability tending to one

$$\log Z_L^{\widetilde{\omega}}\bigl(\mathcal{A}_L^\xi\bigr) \le \log Z_L^{\widetilde{\omega}}\bigl(\mathcal{B}_L^\xi\bigr) + L^{2\xi-1}(\log L)^3 - L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon}. \tag{3.67}$$

Hence using (3.65) and choosing ε small enough, one gets that with probability tending to one when $L \to \infty$

$$\mu_L^{\widetilde{\omega}}\bigl(\mathcal{A}_L^\xi\bigr) \le \exp\bigl(-L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon}/2\bigr). \tag{3.68}$$

Using Lemma 3.4 for the event $A := \{\omega \mid \mu_L^\omega(\mathcal{A}_L^\xi) > \exp(-L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon}/2)\}$ one gets that

$$\lim_{L\to\infty} \mathbb{P}\bigl[\mu_L^\omega\bigl(\mathcal{A}_L^\xi\bigr) > \exp\bigl(-L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon}/2\bigr)\bigr] = 0. \tag{3.69}$$

Which ends the proof.

Appendix A: Estimates

Lemma A.1. *For all L large enough*

$$\mathbb{P}\Bigl[\max_{x\in[-L^2,L^2]^d} V^\omega(x) > \log L\Bigr] \le 1/L. \tag{A.1}$$

Proof. By translation invariance, it is sufficient to show that

$$\mathbb{P}\Bigl[\max_{x\in[-1,1]^d} V^\omega(x) \ge \log L\Bigr] \le \frac{1}{L^{2d+1}}. \tag{A.2}$$

Note that

$$\max_{x\in[-1,1]^d} V^\omega(x) \le \max_{x\in B(0,\sqrt{d})} V^\omega(x) \le \sum_{i=0}^{N} r_i^{-\gamma} \mathbf{1}_{|\omega_i| \le r_i + \sqrt{d}} := X_1. \tag{A.3}$$

Using standard properties of Poisson Point Processes one gets that

$$\mathbb{E}\big[\exp(aX_1)\big] = \exp\left(\int_1^\infty \alpha^{-1}\sigma_d\big(r+\sqrt{d}\big)^{d-\alpha-1}\big(\mathrm{e}^{ar^{-\gamma}}-1\big)\,\mathrm{d}r\right) < \infty, \tag{A.4}$$

where σ_d is the volume of the unit d-dimensional Euclidian ball. Therefore one has

$$\mathbb{P}\Big[\max_{x\in B(0,1)} V^\omega(x) \geq \log L\Big] \leq \mathbb{E}\big[\mathrm{e}^{(2d+2)X_1-(2d+2)\log L}\big] \tag{A.5}$$

and the right-hand side is less than L^{-2d+1} for L large enough. □

Let p_t^* be the heat kernel on $[-L^\xi/2, L^\xi/2]^{d-1}$ with Dirichlet boundary condition. For x in $[-L^\xi/2, L^\xi/2]$ set

$$A(x) := \prod_{i=1}^d \min\big((x_i + L^\xi/2), (L^\xi/2 - x_i)\big). \tag{A.6}$$

Lemma A.2. *One has that for every $t \geq 0$*

$$p_t^*(x,y) \leq A(x)A(y)/t^2 \tag{A.7}$$

and for all t large enough (*where large enough does not depend on L*)

$$p_t^*(x,y) \geq A(x)A(y)\mathrm{e}^{-L^{2\xi}/t - t\pi^2/2}. \tag{A.8}$$

Proof. First we remark that due to the product structure of the Kernel, it is sufficient to treat the one dimensional case $(d-1)=1$. One considers first $p_t^{0,*}$ the heat kernel on $[0,1]$ with Dirichlet boundary condition. Diffusive scaling gives

$$p_t^*(x,y) = L^{-\xi} p^*_{L^{-2\xi}t}\big(L^{-\xi}x + 1/2, L^{-\xi}y + 1/2\big). \tag{A.9}$$

A decomposition of the Dirac distribution δ_x onto the base of eigenfunction of $\Delta/2$ with Dirichlet boundary condition gives

$$\begin{aligned} p_t^{0,*}(x,y) &= \sum_{k=1}^\infty 2\sin(k\pi x)\sin(k\pi y)\mathrm{e}^{-(k\pi)^2 t} \\ &\leq \min(x,1-x)\min(y,1-y)\sum_{k=1}^\infty 2\pi^2k^2\mathrm{e}^{-(k\pi)^2t/2} \\ &\leq \frac{8}{\pi^2t^2}\min(x,1-x)\min(y,1-y), \end{aligned} \tag{A.10}$$

which once rescaled, gives the desired upper bound. Now we perform a lower bound on $p_t^{0,*}$ for large t, indeed using similar computation one gets that there exist a constant C such that of all $t \geq 1$

$$\left|\sum_{k=2}^\infty 2\sin(k\pi x)\sin(k\pi y)\mathrm{e}^{-(k\pi)^2t/2}\right| \leq C\min(x,1-x)\min(y,1-y)\mathrm{e}^{-2\pi^2 t}. \tag{A.11}$$

Hence for t large enough

$$\begin{aligned} p_t^{0,*}(x,y) &\geq 2\sin(\pi x)\sin(\pi y)\mathrm{e}^{-\pi^2t/2} - \left|\sum_{k=2}^\infty 2\sin(k\pi x)\sin(k\pi y)\mathrm{e}^{-(k\pi)^2t/2}\right| \\ &\geq \min(x,1-x)\min(y,1-y)\mathrm{e}^{-\pi^2t/2}. \end{aligned} \tag{A.12}$$

However this is not sufficient to get directly by rescaling the lower bound for $p_t^*(x,y)$ for $t \le L^{2\xi}$.

To do so define x^* (and y^* is defined similarly) as

$$\begin{aligned} &x^* := x \quad \text{if } x \in \big[-\big(L^\xi - 1\big)/2/2, \big(L^\xi - 1\big)/2\big],\\ &x^* := \big(L^\xi - 1\big)/2 \quad \text{if } x \ge \big(L^\xi - 1\big)/2,\\ &x^* := -\big(L^\xi - 1\big)/2 \quad \text{if } x \le -\big(L^\xi - 1\big)/2. \end{aligned} \tag{A.13}$$

One uses the following comparison argument

$$\begin{aligned} p_t^*(x,y)\,\mathrm{d}y &= \mathbf{P}_x\big[B_s \in \big[-L^\xi/2, L^{\xi/2}\big]; \forall s \in [0,t], B_t \in \mathrm{d}y\big]\\ &\ge \mathbf{P}_x\big[\big|B_s - sy^* + (1-s)x^*\big| \le 1/2; \forall s \in [0,1], B_t \in \mathrm{d}y\big]\\ &= \mathrm{e}^{1/t(-(y-x)(x^*-y^*)+(x^*-y^*)^2/2)} p_t^{0,*}\big(x - x^* + 1/2, y - y^* + 1/2\big)\,\mathrm{d}y, \end{aligned} \tag{A.14}$$

where last inequality is just Girsanov Path Transform. Then we use (A.12) to get the result. □

Appendix B: The point-to-point model

B.1. *Result*

In this section we show that a weakened version of our result holds for the so-called point-to-point model. Consider ω and V^ω defined as for the other model. Given $y \in \mathbb{R}^d$ we define $B(y) := B(y,1)$ to be the Euclidian ball of radius one centered on y and T_y to be the first hitting time of $B(y)$. For $L > 0$ set $y_L := (L, 0, \dots, 0)$ and

$$Z_L^\omega := \mathbf{E}\big[\mathrm{e}^{-\int_0^{T_{y_L}} (\lambda + V^\omega(B_t))\,\mathrm{d}t} \mathbf{1}_{\{T_{y_L} < \infty\}}\big]. \tag{B.1}$$

The associated path measure μ_L^ω is given by

$$\frac{\mathrm{d}\mu_L^\omega}{\mathrm{d}P} := \frac{1}{Z_L^\omega} \mathrm{e}^{-\int_0^{T_{y_L}} (\lambda + V(B_t))\,\mathrm{d}t} \mathbf{1}_{T_{y_L} < \infty}. \tag{B.2}$$

Consider $\mathcal{C}_L^\xi$ as in (2.9) and define

$$\mathcal{A}_L^\xi := \big\{T_{y_L} < \infty, \forall t \in [0, T_y], B_t \in \mathcal{C}_L^\xi\big\}. \tag{B.3}$$

One has the following weakened version of Theorem 2.1

Theorem B.1. *For any*

$$\xi < \frac{1}{1 + \alpha + 2\gamma - d} \tag{B.4}$$

one has

$$\lim_{L\to\infty} \mathbb{E}\big[\mu_L^\omega\big(\mathcal{A}_L^\xi\big)\big] = 0. \tag{B.5}$$

B.2. Sketch of proof

One defines (in analogy with (3.1), (3.2))

$$\bar{C}_L^{\xi} := [L/4, 3L/4] \times \left[-L^{\xi}/2, L^{\xi}/2\right]^{d-1} \tag{B.6}$$

and

$$\widetilde{C}_L^{\xi} := \left\{x \in \mathbb{R}^d \mid d(x, \bar{C}_L) > 2\sqrt{d}L^{\xi}\right\} = \bigcup_{y \in \bar{C}_L} B\left(y, 2\sqrt{d}L^{\xi}\right). \tag{B.7}$$

Let $\mathcal{B}_L^{\xi}$ be the set of trajectories that avoids the set $\widetilde{C}_L^{\xi}$:

$$\mathcal{B}_L^{\xi} := \left\{B \mid \forall t \in [0, T_{y_L}], B_t \notin \widetilde{C}_L^{\xi}\right\}. \tag{B.8}$$

The strategy we use here, is to get first a weak comparison between $Z_L^{\omega}(\mathcal{B}_L^{\xi})$ and $Z_L^{\omega}(\mathcal{A}_L^{\xi})$ similar to the one of Proposition 3.1, and then to upgrade it by modifying the environment, adding traps of radius $(\sqrt{d}L^{\xi}, 2\sqrt{d}L^{\xi})$ in $\bar{C}_L^{\xi}$.

Fix $\xi < \frac{1}{1+\alpha+2\gamma-d}$. We assume that is bounded by $\log L$ on $B(0, L^2)$ (that happens with probability larger than $1 - 1/L$ according to Lemma A.1).

Let $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$ be the two first coordinate vector in $\mathbb{R}^d$'s canonical base. For $i \in \mathbb{N}$ one defines

$$\mathcal{C}_{L,i}^{\xi} + \mathcal{C}_L^{\xi} + i\left(1 + 2\sqrt{d}\right)L^{\xi} e_2. \tag{B.9}$$

And set

$$\begin{aligned} &x_{i,L} := i\left(1 + 2\sqrt{d}\right)L^{\xi} e_2, \qquad y_{i,L} := Le_1 + i\left(1 + 2\sqrt{d}\right)L^{\xi} e_2, \\ &\mathcal{D}_{i,L}^{(1)} := \bigcup_{\alpha \in [0,1]} B(\alpha x_{i,L}, 2), \qquad \mathcal{D}_{i,L}^{(2)} := \mathcal{D}_{i,L}^{(1)} + Le_1. \end{aligned} \tag{B.10}$$

The set $\mathcal{D}_i^{(1)}$ resp. $\mathcal{D}_i^{(2)}$ is the set of points whose distance to the segment $[0, x_{i,L}]$ resp. $[y_L, y_{i,L}]$ less than two.

Finally ρ denote the translation of vector $(1 + 2\sqrt{d})L^{\xi} e_2$.

Consider the family of events

$$\begin{aligned} \mathcal{A}_{L,i}^{\xi} := \big\{ &T_{x_{i,L}} < T_{y_{i,L}} < T_{y_L} < \infty, \forall t \in (T_{x_{i,L}}, T_{y_{i,L}}), B_t \in \mathcal{C}_L^{\xi}, \\ &\forall t < T_{x_{i,L}}, B_t \in \mathcal{D}_{i,L}^{(1)}, \forall t \in (T_{y_{i,L}}, T_{y_L}), B_t \in \mathcal{D}_{i,L}^{(2)}\big\}. \end{aligned} \tag{B.11}$$

Note that these events are disjoint (see Fig. 3), and that for all $i \neq 0$, $\mathcal{A}_{L,i}^{\xi} \subset \mathcal{B}_L^{\xi}$, and hence

$$Z_L^{\omega}\left(\mathcal{B}_L^{\xi}\right) \geq \sum_{i \in \{-\log L, \dots, \log L\} \setminus \{0\}} Z_L^{\omega}\left(A_{L,i}^{\xi}\right). \tag{B.12}$$

Then by using the Markov property at time $T_{x_{i,L}}$ and $T_{y_{i,L}}$, one gets that

$$\begin{aligned} Z_L^{\omega}\left(A_{L,i}^{\xi}\right) \geq\; &\mathbf{E}\left[\mathrm{e}^{-\int_0^{T_{x_{L,i}}} (\lambda + V^{\omega}(B_t))\,\mathrm{d}t} \mathbf{1}_{\{T_{x_{i,L}} < \infty; \forall t < T_{x_{i,L}}, B_t \in \mathcal{D}_{i,L}^{(1)}\}}\right] \\ &\times \inf_{u \in B(x_{L_i})} \mathbf{E}_u\left[\mathrm{e}^{-\int_0^{T_{y_{L,i}}} (\lambda + V^{\omega}(B_t))\,\mathrm{d}t} \mathbf{1}_{\{T_{y_{i,L}} < \infty; \forall t \leq T_{y_{i,L}}, B_t \in \mathcal{C}_{L,i}^{\xi}\}}\right] \\ &\times \inf_{v \in B(y_{L_i})} \mathbf{E}_v\left[\mathrm{e}^{-\int_0^{T_{y_L}} (\lambda + V^{\omega}(B_t))\,\mathrm{d}t} \mathbf{1}_{\{T_{y_L} < \infty; \forall t \leq T_{y_L}, B_t \in \mathcal{D}_{i,L}^{(2)}\}}\right]. \end{aligned} \tag{B.13}$$

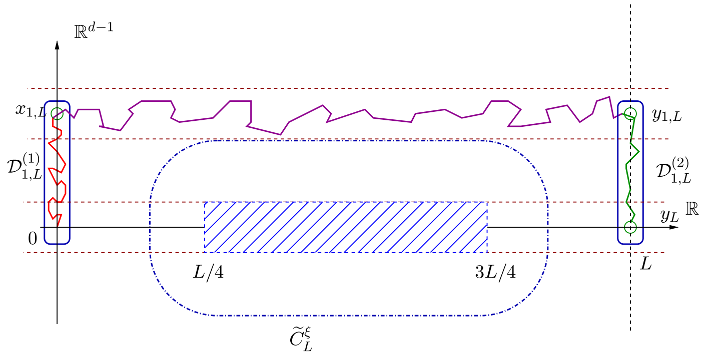

Fig. 3. Here is a two-dimensional projection of a trajectory in $\mathcal{A}^{\xi}_{L,1}$. First it reaches $B(x_{L,1})$ while staying in the narrow tube $\mathcal{D}^{(1)}$ then it goes from $B(x_{L,1})$ to $B(y_{L,1})$ and stays in the tube $\mathcal{C}^{\xi}_{L,1}$ (dashed line), and then stays in $\mathcal{D}^{(1)}$ till it hits $B(y)$. It appears clearly on the figure that such trajectories cannot hit $\widetilde{C}^{\xi}_L$ whose limits are represented by the thick dashed line (the shadowed region is C^{ξ}_L). The three parts of the path corresponding to three terms in the decomposition (B.13) are draw in different colors.

Then one remarks that the first and third term can be bounded by using standard tubular estimate for Brownian Motion (see for instance (1.11) of [15]). Indeed in $D^{(j)}_{i,L}$ under our assumption the potential is less than $\log L$, there exists C such that both terms are larger than $\exp(-CiL^{\xi}\log L)$ for and L large enough.

As for the second term, using Proposition 2.2 in [17], Chapter 5, one get that it is larger than

$$e^{-C\log L}\mathbf{E}_{x_{L_i}}\left[e^{-\int_0^{T_{y_{L,i}}}(\lambda+V^{\omega}(B_t))\,dt}\mathbf{1}_{\{T_{y_{i,L}}<\infty;\forall t\le T_{y_{i,L}},B_t\in\mathcal{C}^{\xi}_{L,i}\}}\right]=e^{-C\log L}Z^{\rho^{-i}\omega}_L\big(\mathcal{A}^{\xi}_L\big). \tag{B.14}$$

We write the conclusion of this as

Lemma B.2. *With probability larger than* $1-1/L$

$$\log Z^{\omega}_L\big(\mathcal{B}^{\xi}_L\big)\ge\max_{i\in\{-\log L,\dots,\log L\}\setminus\{0\}}\log Z^{\rho^{-i}\omega}_L\big(\mathcal{A}^{\xi}_L\big)-C'\big(\log L^2\big)L^{\xi}. \tag{B.15}$$

The above lemma plays the role of Lemma 3.2 for the point-to-plane model. The reason why the result we obtain at the end is not as good as for the point-to-plane model is that the $L^{2\xi-1}$ of Lemma 3.2 is replaced by L^{ξ} here.

One can also get an equivalent of Lemma 3.3 (the proof being exactly analogous).

Lemma B.3.

$$\mathbb{P}\left[\log Z^{\omega}_L\big(\mathcal{A}^{\xi}_L\big)>\max_{i\in\{-\log L,\dots,\log L\}}\log Z^{\rho^{-i}\omega}_L\big(\mathcal{A}^{\xi}_L\big)\right]\le\frac{1}{\log L} \tag{B.16}$$

and thus of Proposition 3.1.

Proposition B.4.

$$\mathbb{P}\left[\log Z^{\omega}_L\big(\mathcal{A}^{\xi}_L\big)>\log Z^{\omega}_L\big(\mathcal{B}^{\xi}_L\big)+C'\big(\log L^2\big)L^{\xi}\right]\le\frac{2}{\log L}. \tag{B.17}$$

The rest of the proof being exactly similar to the point-to-plane case we survey it very briefly. We consider $\widetilde{\omega}$ constructed just as in Section 3.3 but with the definiton of $\bar{C}^{\xi}$ replaced by (B.6) (here is important to notice that our choice for ξ implies $(d-1-\alpha)\xi+1>0$). Obviously Lemma 3.4 is still valid with this modifications. Then proof of Proposition 3.5 can be adapted without difficulties and one gets that, for every $\varepsilon>0$, with probability tending to one,

$$\log Z^{\widetilde{\omega}}_L\big(\mathcal{A}^{\xi}_L\big)\le\log Z^{\omega}_L\big(\mathcal{A}^{\xi}_L\big)-L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon} \tag{B.18}$$

while $\log Z_L^{\widetilde{\omega}}(\mathcal{B}_L^\xi) = \log Z_L^{\omega}(\mathcal{B}_L^\xi)$. This together with Proposition B.4 implies

$$\log Z_L^{\widetilde{\omega}}\big(\mathcal{B}_L^\xi\big) \geq \log Z_L^{\widetilde{\omega}}\big(\mathcal{A}_L^\xi\big) - C'\big(\log L^2\big)L^\xi + L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon}. \tag{B.19}$$

With our choice of ξ, for ε sufficiently small, $L^{((d+1-\alpha-2\gamma)\xi+1)/2-\varepsilon} - C'(\log L^2)L^\xi$ tends to infinity, and that ends the proof.

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