Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2013, Vol. 49, No. 2, 307–339
DOI: 10.1214/11-AIHP461

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Random hysteresis loops

Gioia Carinci

Dipartimento di Matematica Pura e Applicata, Università dell'Aquila, via Vetoio 1, 67100 L'Aquila, Italy. E-mail: gioia.carinci@univaq.it

Received 3 September 2010; revised 27 October 2011; accepted 4 November 2011

Abstract. Dynamical hysteresis is a phenomenon which arises in ferromagnetic systems below the critical temperature as a response to adiabatic variations of the external magnetic field. We study the problem in the context of the mean-field Ising model with Glauber dynamics, proving that for frequencies of the magnetic field oscillations of order $N^{-2/3}$, N the size of the system, the "critical" hysteresis loop becomes random.

Résumé. L'hystérésis dynamique est un phénomène qu'on observe dans les systèmes ferromagnétiques au-dessous de la temperature critique, en réponse à des variations adiabatiques du champ magnétique extérieur. Nous étudions le problème dans le contexte du modéle d'Ising de champ moyen avec la dynamique de Glauber, en montrant que, pour des fréquences d'oscillations du champ magnétique d'ordre de $N^{-2/3}$, avec N la taille du système, la boucle d'hystérésis « critique » devient aléatoire.

MSC: 82C20

Keywords: Hysteresis; Ising; Mean field Glauber dynamics; Macroscopic fluctuations

1. Introduction

Hysteresis appears when a time dependent magnetic field $h = h(t)$ is applied to a ferromagnet whose temperature is kept fixed below the critical value. The origin of the phenomenon lies in the fact that, at the equilibrium, at each value of the external magnetic field h may not correspond a unique value of the magnetization m of the system. The value of $m(t)$ is, thus, not determined by $h(t)$ alone but also by the previous history of the input.

The phenomenon has been widely studied and modelled. Most classical theories (see for example [5,6,22]) consider hysteresis from a static point of view, by modelling it through integral operators not depending on the velocity of variation of the external input.

A dynamical approach to the study of the phenomenon has been proposed for the first time by Rao et al. [20] in the early nineties. The new theory aroused great interest and a number of experimental, numerical and theoretical works appeared on the argument in the last twenty years, investigating the response of the system to adiabatic oscillations of the magnetic field. They analyse, in particular, the dependence of shapes and areas of the hysteresis loops on amplitude and frequency of the input oscillations. Most of these results are essentially numerical. Monte Carlo simulations have widely been used to study the hysteretic response of a nearest-neighbor ferromagnetic Ising model (see for instance [1,10,11,15–17,20,23]). On the other hand, several theoretical and numerical results are concerned with those known as mean-field models (see [1,9,20,21]). In these models the dynamics is reduced to a single differential equation of the order parameter (the uniform magnetization $m(t)$). These equations govern the dynamics of the magnetization in stochastic spin models in the limit of infinite system volume. Therefore they neglect both thermal fluctuations and finite system size effects. A first rigorous analysis of the effects of the stochastic fluctuations on the properties of the hysteresis cycles has been carried out by B. Genz and N. Berglund in a series of papers of about ten years ago

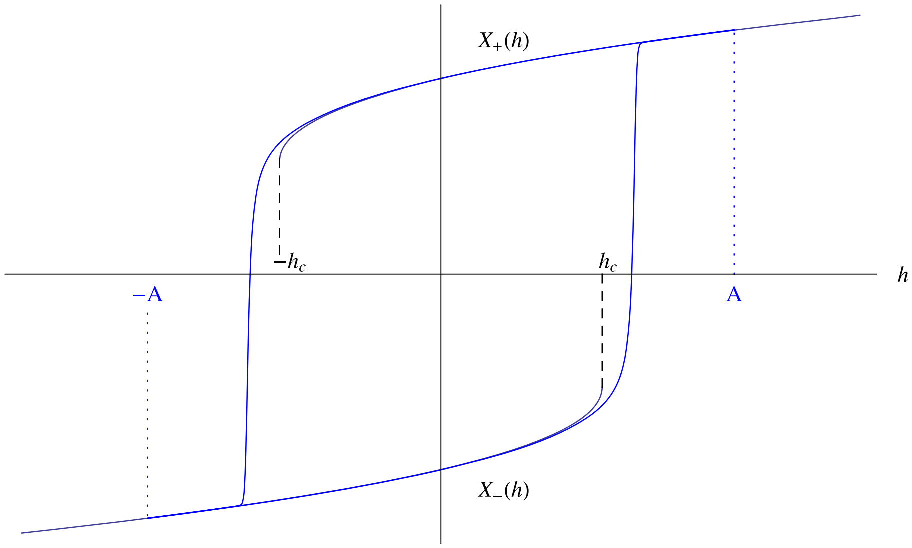

Fig. 1. The picture shows the dependence of the two equilibrium branches $X_\pm(h)$ (black lines) on the external magnetic field h. Adiabatic oscillations of the magnetic field of amplitude $A > h_c$ yield the typical hysteresis loop (blue line).

[2–4]. They model the thermal fluctuations by adding a stochastic noise to a mean-field type equation. They consider a Langevin equation with a Ginzburg–Landau potential:

$$\mathrm{d}x = \big(F(x) + h\big)\,\mathrm{d}t + N^{-1/2}\,\mathrm{d}w(t), \qquad F(x) = x - x^3, \tag{1.1}$$

where $w(t)$ is the standard Brownian motion. We give to $N > 0$ the physical interpretation of the total number of spin sites in a ferromagnetic system. Then, in the large N regime, equation (1.1) can be thought of as a continuous counterpart of our Ising spin dynamics (see Section 2).

In the present paper we shall study the problem for the Glauber process in the Curie–Weiss model, from which (1.1) is inspired.

Let $h_c > 0$ be the "coercive magnetic field" value, then for $|h| \le h_c$ the magnetization density of the ferromagnet may have two equilibrium values, $X_+(h)$ and $X_-(h)$ (see Fig. 1). The upper branch $X_+(h)$ continues past h_c while it disappears for $h < -h_c$; the opposite holds for the lower branch $X_-(h)$. Let us apply, now, a slowly oscillating magnetic field $h(t)$. We denote, respectively, by A and ω the amplitude and the frequency of the oscillations (we choose, for instance, $h(t) = -A\cos(\omega t)$). Let $m(t)$ be the magnetization observed at time t and choose initially $m(0) = X_+(h(0))$. In the adiabatic (quasi-static) regime, where ω is very small, the following is observed. If $A \le h_c$ then $m(t) \approx X_+(h(t))$ for any $t \ge 0$. If $A > h_c$, $m(t)$ traces out the so called hysteresis loop, in the sense that $m(t) \in \{X_+(h(t)), X_-(h(t))\}$ (approximately), jumping from the upper to the lower branch when $h(t)$ crosses $-h_c$ and the opposite when $h(t)$ crosses h_c. A sharp statement (which avoids the above approximated statements) can be obtained in "the adiabatic limit" where $\omega \to 0$.

The pediod of the magnetic field oscillations is of order ω^{-1}, thus, in the adiabatic regime the natural time-scale of the dynamics is very long. In long time intervals other phenomena may appear which in short time intervals are negligible and which may invalidate the picture. In the context of (1.1) $X_\pm(h)$ are identified with the locally stable solutions of the stationary equation $F(x) = -h$. If h is constant, say $h \in (0, h_c)$, then $X_-(h)$ is metastable and, on a time interval which diverges exponentially with N (as $N \to \infty$), there is tunneling from $X_-(h)$ to $X_+(h)$. Thus, if ω is exponentially small with N, the oscillations period is exponentially long with N, and then stochastic jumps between the two branches occur, essentially perturbing the hysteresis loop. We intend to consider a different regime for the frequency ω, i.e. we take $\omega = N^{-\kappa}$, $\kappa > 0$. We shall concentrate here on the critical amplitude case $A = h_c$. In such a case the deterministic equation (i.e. (1.1) without the Brownian term) predicts that the magnetization $m(t)$ tracks always the upper branch $X_+(h(t))$, where it was initially. [3] proves that, with the addition of the stochastic effects, there exists a critical value for κ, $\kappa = \frac{2}{3}$. If $\kappa < \frac{2}{3}$ the dynamics is still governed by the deterministic equation, i.e. the magnetization tracks the upper branch, in the adiabatic limit. Whereas, if $\kappa > \frac{2}{3}$ there is hysteresis, thus the magnetization jumps to the lower branch as soon as $h = -h_c$ and then back to the upper one when $h = h_c$ and so forth. In the present work we will prove $\kappa = \frac{2}{3}$ to be the critical value even in our Ising spin context. We shall concentrate here on the critical case $\kappa = \frac{2}{3}$ which is not covered by the analysis in [3,4]. We will see that for $\kappa = \frac{2}{3}$ the hysteresis loop becomes truly random. There is a positive and not one probability to leave the upper-lower branch at $\pm h_c$. Our future aim is to extend our analysis to the Kac potential case by taking into account spatial effects.

2. Definitions and results

The mean field Ising model

The configuration space is $\{-1, 1\}^N$, $N \in \mathbb{N}$; its elements are denoted by $\sigma = \{\sigma(i), i = 1, \ldots, N\}$, $\sigma(i)$ the spin at site i. By

$$m_N = m_N(\sigma) := \frac{1}{N} \sum_{i=1}^{N} \sigma(i) \tag{2.1}$$

we denote the magnetization density of the configuration σ, so that $m_N \in \mathcal{M}_N$,

$$\mathcal{M}_N := \frac{1}{N}\{-N, -N+2, \ldots, N-2, N\}.$$

The mean field Hamiltonian is

$$H_{h,N}(\sigma) := N\left(-\frac{m_N(\sigma)^2}{2} - h m_N(\sigma)\right)$$

and the mean field Gibbs measure at the inverse temperature $\beta > 0$ is the probability $G_{\beta,h,N}$ on $\{-1, 1\}^N$ given by

$$G_{\beta,h,N}(\sigma) := \frac{\mathrm{e}^{-\beta H_{h,N}(\sigma)}}{Z_{\beta,h,N}},$$

where the partition function $Z_{\beta,h,N}$ is the normalization factor.

For an introduction to the mean field Ising model see Section 4.1 of [19].

The Glauber dynamics

A Glauber dynamics for the Ising system is the Markov process on $\{-1, 1\}^N$ with generator

$$Lf(\sigma) := \sum_{i=1}^{N} c(i, \sigma; h)\big(f(\sigma_i) - f(\sigma)\big), \tag{2.2}$$

where $\sigma_i(j) = \sigma(j)$ for $i \neq j$ and $\sigma_i(i) = -\sigma(i)$; $c(i, \sigma; h) > 0$, the spin flip intensity at i, is given by the formula

$$c(i, \sigma; h) = \frac{\mathrm{e}^{-\beta[H_{h,N}(\sigma^{(i)}) - H_{h,N}(\sigma)]}}{\mathrm{e}^{-\beta H_{h,N}(\sigma^{(i)})} + \mathrm{e}^{-\beta H_{h,N}(\sigma)}}$$

with $\sigma^{(i)}$ the configuration obtained from σ by flipping the spin at i. For more details on the Glauber dynamics for mean field Ising systems see Section 5.1 of [19].

$h = h(t)$ is a smooth function of time, hence $\sigma(t)$ is a time non-homogeneous Markov process. Since the Hamiltonian depends on σ via $m_N(\sigma)$, the process $\{m_N(\sigma_t), t \geq 0\}$ is itself Markov with state space $\mathcal{M}_N$ and generator $\mathcal{L}$ given by

$$\mathcal{L}_h f(x) := c^+(x, h)\big[f(x + 2/N) - f(x)\big] + c^-(x, h)\big[f(x - 2/N) - f(x)\big] \tag{2.3}$$

with

$$c^{\pm}(x,h)=\frac{N}{2}(1\mp x)\hat{c}^{\pm}(x,h),\quad \hat{c}^{\pm}(x,h)=\frac{\mathrm{e}^{\pm\beta[h+(x\pm 1/N)]}}{\mathrm{e}^{-\beta[h+(x\pm 1/N)]}+\mathrm{e}^{\beta[h+(x\pm 1/N)]}},$$

for $x\in\mathcal{M}_N$. When h is time independent there is a unique invariant measure (see Section 5.1.2 of [19]) which is the marginal $\mu_{\beta,h,N}$ of $G_{\beta,h,N}$ on the magnetization density m_N defined in (2.1). $\mu_{\beta,h,N}$ is then the probability on $\mathcal{M}_N$ given by

$$\mu_{\beta,h,N}(x):=\frac{\mathrm{e}^{-\beta N\phi_{\beta,h,N}(x)}}{Z_{\beta,h,N}},\quad x\in\mathcal{M}_N,$$

where

$$\phi_{\beta,h,N}(x):=-\frac{x^2}{2}-hx-\frac{\mathcal{S}_N(x)}{\beta}$$

and

$$\mathrm{e}^{N\mathcal{S}_N(x)}:=\mathrm{card}\big(\sigma\in\{-1,1\}^N:\ m_N(\sigma)=x\big).$$

If $x_N\in\mathcal{M}_N$, $x_N\to x\in[-1,1]$ as $N\to\infty$ then $\phi_{\beta,h,N}(x_N)\to\phi_{\beta,h}(x)$ where

$$\phi_{\beta,h}(x)=-\frac{x^2}{2}-hx-\frac{\mathcal{S}(x)}{\beta}$$

and

$$\mathcal{S}(x)=-\frac{1-x}{2}\log\frac{1-x}{2}-\frac{1+x}{2}\log\frac{1+x}{2}.$$

The mean field phase transitions

For any $\beta\le 1$ and any $h\in\mathbb{R}$ the mean field free energy density (see Section 4.1.2 of [19]) $\phi_{\beta,h}(x)$ is a convex function of x (absence of phase transitions). If instead $\beta>1$ (see Fig. 2) there is $h_c>0$ such that, for any $|h|<h_c$, $\phi_{\beta,h}(x)$ is a double well function of x with local minima at $X_+(h)>X_-(h)$ and local maximum at $X_0(h)\in(X_-(h),X_+(h))$; $X_\pm(h)$ and $X_0(h)$ are solutions of the mean field equation:

$$x=\tanh\big\{\beta(x+h)\big\},$$

$X_+(h)$ is the absolute minimum for $h\ge 0$ and $X_-(h)$ for $h\le 0$, then only at $h=0$ there are two absolute minima and thus a phase transition; for $h\in(0,h_c)$, $X_+(h)$ is the only pure phase while $X_-(h)$ is a metastable state, the opposite holds for negative fields. When $h\to -h_c$, $X_+(h)-X_0(h)\to 0$ and the limit $x_c:=X_+(-h_c)$ of $X_+(h)$ is an inflection point for the function $\phi_{\beta,-h_c}(x)$. By symmetry the analogous picture describes $X_-(h)$ when $h\to h_c$.

The macroscopic mean field dynamics

The infinite volume dynamics is governed by the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t}=F(x,h),\quad F(x,h):=-x+\tanh\big\{\beta(x+h)\big\},\tag{2.4}$$

in the following sense. Let $m_N(t)$ be the process of generator $\mathcal{L}_{h(t)}$ (see (2.3)), $h(t)$ a smooth function of t, which starts from $m_N^0\in\mathcal{M}_N$. We suppose that $m_N^0\to x^0\in[-1,1]$ as $N\to\infty$ and denote by $\mathcal{P}_N$ the law of $m_N(t)$, $t\ge 0$. We have the following result.

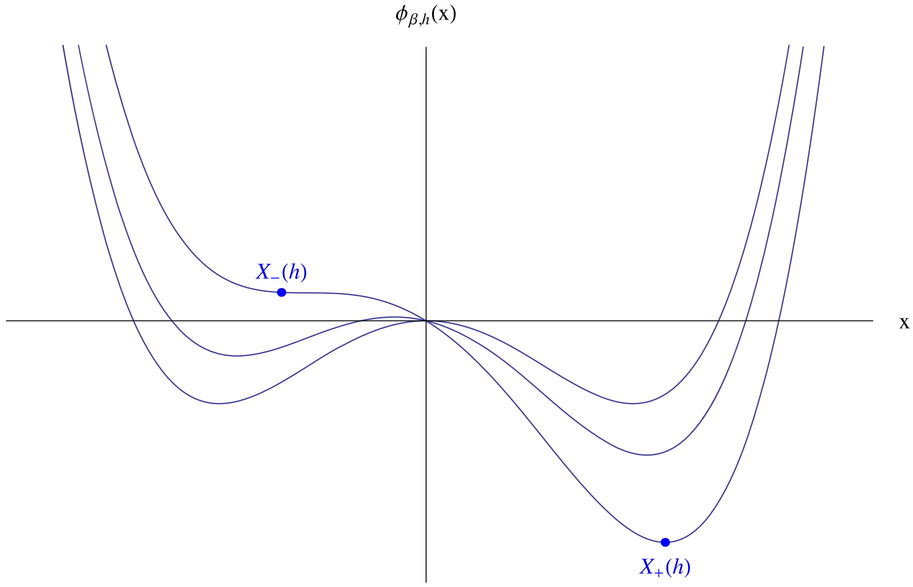

Fig. 2. The picture shows some profiles of $\phi_{\beta,h}(x)$ according to different values of h, for $\beta > 1$.

Theorem 2.1. *With the above notation, for any* $\delta > 0$ *and any* $T > 0$,

$$\lim_{N\to\infty} \mathcal{P}_N \left\{ \sup_{t\le T} \left| m_N(t) - x(t) \right| \ge \delta \right\} = 0, \tag{2.5}$$

where $x(t)$ *is the unique solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F\bigl(x, h(t)\bigr), \quad x(0) = x^0. \tag{2.6}$$

The proof of Theorem 2.1 is omitted. The proof in the case of constant h can be found, for instance, in Section 5.1.5 of [19], the proof easily extends to the present case.

The adiabatic limit

Let

$$h(t) := -h_c \cos t \tag{2.7}$$

we denote by $x_\omega(t)$ the solution of (2.6) with $h = h(\omega t)$ and initial condition $x_\omega(0) = X_+(-h_c)$. We omit the proof that

Theorem 2.2. *For any* $\tau > 0$

$$\lim_{\omega\to 0} \sup_{t\le\omega^{-1}\tau} \left| x_\omega(t) - X_+\bigl(h(\omega t)\bigr) \right| = 0. \tag{2.8}$$

Theorem 2.2 proves that, for oscillations of critical amplitude h_c, in the adiabatic limit $\omega \to 0$ there is not hysteresis (see Fig. 3). The relevant time scale is $t = \omega^{-1}\tau$ and the limit evolution is

$$\lim_{\omega\to 0} x_\omega\bigl(\omega^{-1}\tau\bigr) = X_+\bigl(h(\tau)\bigr). \tag{2.9}$$

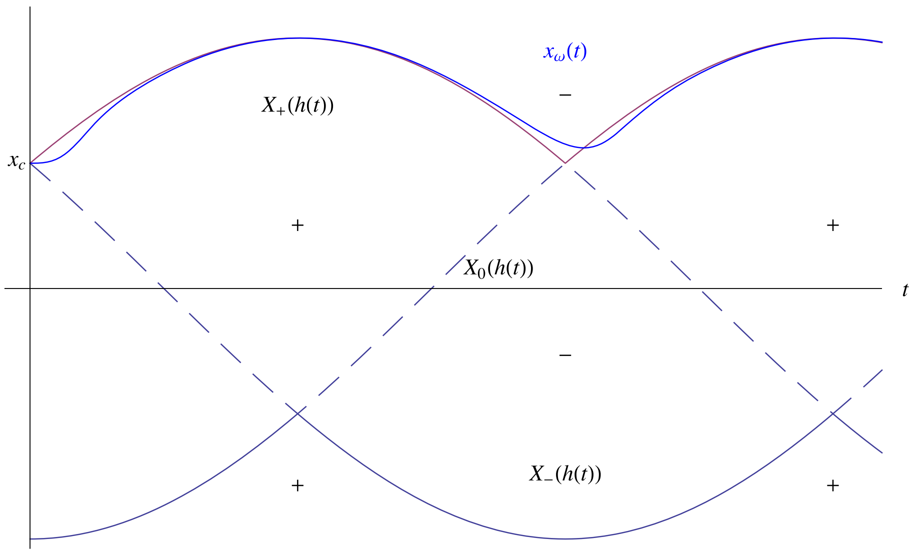

Fig. 3. The function $x_\omega(t)$ (blue line) for small values of ω tracks the positive branch $X_+(h(t))$ (black line).

The main theorem

Theorem 2.1 asserts that the dynamics in the macroscopic limit $N \to \infty$ on finite time intervals is described by the deterministic mean field evolution equation (2.6). When ω is small with N, the period of the magnetic field oscillations is large with N. Therefore the behavior exhibited by (2.9) in the adiabatic limit may not correspond to what the Glauber process does for large but finite N. As it will turn out, it all depends on the way $\omega \to 0$ as $N \to \infty$. As stated in the Introduction, the critical case is $\omega = N^{-2/3}$ to which we restrict hereafter (the origin of the factor $2/3$ will become clear from the proofs but it will also be explained in Section 3 in a heuristic way).

There are criticalities for values of the magnetic field in a neighborhood of $\pm h_c$. Since h is a periodic function of time and the process is invariant under change of sign we shall restrict ourselves to study the behavior in a semi-period. We consider $t \in N^{2/3}[-\frac{\pi}{2}, \frac{\pi}{2}]$ and suppose $h = h_N(t)$, with

$$h_N(t) := h\big(N^{-2/3}t\big) = -h_c \cos\big(N^{-2/3}t\big) \tag{2.10}$$

so that the critical time is set at $t = 0$. We shall denote by $\mathcal{P}_N$ the law of the process $m_N(t), t \in N^{2/3}[-\frac{\pi}{2}, \frac{\pi}{2}]$ of generator $\mathcal{L}_{h_N(t)}$, with $m_N(-N^{2/3}\frac{\pi}{2}) = m_N^0$. We choose such initial value in a neighborhood of size $N^{-1/2+\gamma}$, $\gamma > 0$, of the positive branch, i.e. $|m_N^0 - X_+(0)| \le N^{-1/2+\gamma}$ (since $h_N(-N^{2/3}\frac{\pi}{2}) = 0$). The main result is given by the following theorem. It provides the probability, for large N, to find the magnetization in a neighborhood of one of the two equilibrium branches $X_\pm(h_N(t))$, respectively, before and after the critical time $t = 0$.

Theorem 2.3 (Main theorem). *Consider the events*

$$\mathcal{H}_\gamma^\pm(I) := \Big\{\sup_{t\in I}\big|m_N(t) - X_\pm\big(h_N(t)\big)\big| \le N^{-1/2+\gamma}\Big\}, \quad I \subseteq \mathbb{R}, \gamma > 0. \tag{2.11}$$

There is $p_- \in (0,1)$ so that for any $\gamma, \eta > 0$ and $\gamma' > \gamma$, if $|m_N^0 - X_+(0)| \le N^{-1/2+\gamma}$ then

$$\lim_{N\to\infty} \mathcal{P}_N\Big\{\mathcal{H}_{\gamma'}^+\Big(N^{2/3}\Big[-\frac{\pi}{2}, -\eta\Big]\Big)\Big\} = 1, \tag{2.12}$$

$$\lim_{N\to\infty} \mathcal{P}_N\Big\{\mathcal{H}_\gamma^\pm\Big(N^{2/3}\Big[\eta, \frac{\pi}{2}\Big]\Big)\Big\} = p_\pm, \tag{2.13}$$

where $p_+ = 1 - p_-$.

The critical interval is $N^{2/3}(-\eta, \eta)$, $\eta > 0$ arbitrarily small. Equation (2.12) shows that, in the limit as $N \to \infty$, the magnetization remains, almost surely, in a neighborhood of size $N^{-1/2+\gamma'}$, $\gamma' > \gamma$, of the positive branch before the criticality (i.e. for $t < -\eta N^{2/3}$). Equation (2.13) provides the behavior after the criticality (for $t > \eta N^{2/3}$), it states that there exists a non-trivial probability to find the magnetization either in the positive or in the negative equilibrium branches.

The result can be iterated, as the same arguments can be repeated every time the process runs into a criticality. The macroscopic dynamics is no more deterministic since, at every step there is a positive probability for the magnetization to jump or not, and the hysteresis loops observed become, in this sense, random.

3. Outline of proof

The proof of (2.12) is simple. Indeed, if we fix $h > -h_c + \varepsilon$, for some $\varepsilon > 0$, and the magnetization is initially in a neighborhood of $X_+(h)$, then $m_N(t)$ has a drift towards $X_+(h)$. Therefore, with large probability, it stays in a neighborhood of size $N^{-1/2+\gamma'}$ (as $N^{-1/2}$ is the strength of the noise) of the positive branch. Only after a longer (exponential) time, tunneling to the negative branch will be observed. In our case h is not fixed but it is so slowly varying that the above argument remains valid as long as $h(t) > -h_c + \varepsilon$, for some $\varepsilon > 0$ (see Section 7). When h approaches $-h_c$ the above picture is wrong because at $-h_c$ the value x_c is stationary but not stable. Lack of stability and slow changes of the frequency make the noise competitive with the drift (for the special choice $\omega = N^{-2/3}$) as we are going to see.

Scalings

In order to understand the scalings let us go back to the stochastic ODE (1.1). Let the magnetic field oscillate as $h(\omega t) = -h_c \cos(\omega t)$, by expanding to leading orders $F(x) + h$ ($F(x)$ given in (2.4)) around x_c, $-h_c$ (i.e. for $x - x_c$ and ωt both small) we get approximately

$$\mathrm{d}x = \left\{ h_c \frac{(\omega t)^2}{2} + \frac{F''(x_c)}{2}(x - x_c)^2 \right\} \mathrm{d}t + N^{-1/2}\,\mathrm{d}w(t). \tag{3.1}$$

We scale $y = \omega^a(x - x_c)$ and $\tau = \omega^b t$, thus

$$\omega^{-a}\,\mathrm{d}y = \left\{ \omega^{-2b} \frac{h_c \omega^2 \tau^2}{2} + \omega^{-2a} \frac{F''(x_c) y^2}{2} \right\} \omega^{-b}\,\mathrm{d}\tau + N^{-1/2}\omega^{-b/2}\,\mathrm{d}w(\tau) \tag{3.2}$$

which becomes independent of ω and N if

$$\omega^{a-b/2} N^{-1/2} = 1, \quad 2 + a - 3b = 0, a + b = 0, \tag{3.3}$$

which yields $\omega = N^{-2/3}$.

The same scalings apply to our case as we shall prove using extensively martingales techniques. In order to get rid of constants in the final equation, it is convenient to introduce suitable coefficients in the scaling transformation (3.2), we define, thus, the process

$$Y_N(t) = \nu N^{1/3}\big(m_N\big(\mu N^{1/3} t\big) - x_c\big) \tag{3.4}$$

with

$$\mu = \left(\frac{2}{\beta h_c x_c} \right)^{1/4} \quad \text{and} \quad \nu = (\beta x_c)^{3/4} \left(\frac{2}{h_c} \right)^{1/4}. \tag{3.5}$$

We shall study the process $Y_N(t)$ in a time interval which starts from time $-T$, letting $T \to +\infty$ after $N \to \infty$. The proof of (2.12) can be extended (see Section 7) till time $-\mu T N^{1/3}$ (which is the microscopic time corresponding to time $-T$ for $Y_N(\cdot)$) in the following sense:

Theorem 3.1. *There is $c > 0$ so that, for any T large enough, $\varepsilon > 0$ small enough,*

$$\limsup_{N\to\infty} \mathcal{P}_N\big\{\big|Y_N(-T) - T\big| \le \varepsilon\big\} \ge 1 - \mathrm{e}^{-c\varepsilon^2 T}. \tag{3.6}$$

One of the main points in the proof of (2.13) will be to show (see Sections 4 and 8) that the law of $Y_N(t)$ converges, as $N \to \infty$, to the law of the stochastic ODE

$$\mathrm{d}Y(t) = \big[t^2 - Y^2(t)\big]\,\mathrm{d}t + \xi\,\mathrm{d}w_t, \quad \xi = \frac{2}{\beta}\mu\nu^2, \tag{3.7}$$

which is (modulo multiplicative coefficients) the same as (3.1) with parameters as in (3.3). Due to the quadratic dependence on Y the solution can blow up in a finite time, therefore the process is defined with values on $\mathbb{R} \cup \{-\infty\}$, with the convention that, if $Y(t) = -\infty$, then $Y(t') = -\infty$ for all $t' \ge t$. The drift in (3.7) vanishes on the two straight lines $Y = \pm t$. It is negative for $Y < -|t|$ and it points towards $|t|$ for $Y > -|t|$. A more careful analysis shows that there is a critical trajectory $y^*(t) < 0$ solution of the deterministic version (i.e. with $\xi = 0$) of (3.7) such that any deterministic solution which starts above the critical curve is exponentially asymptotic to (t, t) as $t \to \infty$.

We denote by $\mathcal{P}_{-T,y}$ the law on $\mathbb{R} \cup \{-\infty\}$ of the solution $Y(t)$, $t > -T$, of (3.7) starting from $Y(-T) = y$, $T > 0$. In Section 5 we prove the following theorem.

Theorem 3.2. *Let $\mathcal{P}$ be the probability law with support on solutions $Y(t)$ of (3.7) such that*

$$\lim_{t\to-\infty} \big|Y(t) + t\big| = 0 \quad \mathcal{P}\text{-a.s.} \tag{3.8}$$

then there exist $p_\pm \in (0, 1)$, $p_+ = 1 - p_-$, such that

$$\mathcal{P}\Big\{\text{there is } t\text{: } Y(t) = -\infty\Big\} = p_- \quad \text{and} \quad \mathcal{P}\Big\{\lim_{t\to\infty} \big|Y(t) - t\big| = 0\Big\} = p_+. \tag{3.9}$$

For any $\varepsilon > 0$ small enough, for any bounded continuous function $g(y)$ with compact support and any $t \in \mathbb{R}$,

$$\lim_{T\to\infty} \mathbf{1}_{|y-T|\le\varepsilon} \mathbb{E}_{\mathcal{P}_{-T,y}}\big[g\big(Y(t)\big)\big] = \mathbb{E}_{\mathcal{P}}\big[g\big(Y(t)\big)\big]. \tag{3.10}$$

Moreover there exists $c > 0$ such that, for any T large enough, ε small enough,

$$\mathcal{P}\big\{\big|Y(-T) - T\big| \le \varepsilon\big\} > 1 - \mathrm{e}^{-c\varepsilon^2 T}. \tag{3.11}$$

Thus with $\mathcal{P}$ probability one either $Y(t)$ blows up in a finite time or it is asymptotic to t as $t \to \infty$, both events having non-zero probability. The next goal is to extend the above result to the finite N process $Y_N(t)$. For $T > 0$ we define the rectangle:

$$\mathcal{R}_T = \big\{(t, y) \in \mathbb{R}^2\text{: } t \in [-T, T], |y| \le 2T\big\} \tag{3.12}$$

and, for $\varepsilon \in (0, 1)$,

$$\partial\mathcal{R}_T^+ := \{T\} \times [T - \varepsilon, T + \varepsilon], \qquad \partial\mathcal{R}_T^- := [-T, T] \times \{-2T\}, \tag{3.13}$$

$\partial\mathcal{R}_T^\pm \subseteq \partial\mathcal{R}_T$. For the processes $Y(t)$ such that $(-T, Y(-T)) \in \mathcal{R}_T$, we denote by τ_T the first exit time from $\mathcal{R}_T$

$$\tau_T := \inf\big\{t \ge -T\text{: } Y(t) \notin \mathcal{R}_T\big\} \tag{3.14}$$

and define the sets

$$\mathcal{E}_T^\pm = \big\{Y\text{: } \big(\tau_T, Y(\tau_T)\big) \in \partial\mathcal{R}_T^\pm\big\}. \tag{3.15}$$

We shall prove in Section 5

Proposition 3.3. *Let $Y(t), t \geq -T$, be a solution of* (3.7) *starting at $-T$ from y: $|y-T| \leq \varepsilon$, $\varepsilon > 0$ small enough, then*

$$\lim_{T\to\infty} \mathcal{P}_{-T,y}\{Y \in \mathcal{E}_T^+ \cup \mathcal{E}_T^-\} = 1 \tag{3.16}$$

moreover

$$\lim_{T\to\infty} \mathcal{P}_{-T,y}\Big\{\lim_{t\to\infty}\big|Y(t)-t\big| = 0 | Y \in \mathcal{E}_T^+\Big\} = 1 \tag{3.17}$$

and

$$\lim_{T\to\infty} \mathcal{P}_{-T,y}\{\text{there is } t\text{: } Y(t) = -\infty | Y \in \mathcal{E}_T^-\} = 1. \tag{3.18}$$

The following corollary is a direct consequence of Theorem 3.2 and Proposition 3.3:

Corollary 3.4. *For $Y(t)$ as in the previous proposition we have*

$$\lim_{T\to\infty} \big|\mathcal{P}_{-T,y}\{Y \in \mathcal{E}_T^\pm\} - p_\pm\big| = 0. \tag{3.19}$$

Let $\mathcal{P}_{N,-T,y}$ be the law of $Y_N(t)$ given $Y_N(-T) = y$. Using martingale convergence theorems, in Section 8 we prove the following result.

Proposition 3.5. *For $Y(t)$ solution of* (3.7) *starting at $-T$ from y: $|y-T| \leq \varepsilon$, ε small enough, we have*

$$\lim_{N\to\infty} \mathcal{P}_{N,-T,y}\{Y_N \in \mathcal{E}_T^\pm\} = \mathcal{P}_{-T,y}\{Y \in \mathcal{E}_T^\pm\} \tag{3.20}$$

and

$$\lim_{N\to\infty} \mathcal{P}_{N,-T,y}\{Y_N \in \mathcal{E}_T^+ \cup \mathcal{E}_T^-\} = \mathcal{P}_{-T,y}\{Y \in \mathcal{E}_T^+ \cup \mathcal{E}_T^-\}. \tag{3.21}$$

Proposition 3.5 allows us to extend the results obtained for $Y(t)$ to the finite N process $Y_N(t)$. Finally in Section 8 we prove the following proposition that is the last ingredient to conclude the proof of Theorem 2.3.

Proposition 3.6. *For any $\eta, \gamma > 0$,*

$$\lim_{T\to\infty}\lim_{N\to\infty} \mathcal{P}_N\left\{\mathcal{H}_\gamma^\pm\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\Big| Y_N \in \mathcal{E}_T^\pm\right\} = 1. \tag{3.22}$$

4. Limit dynamics in the critical region

The study of the limit behavior as $N \to \infty$ of the spin-flip evolution defined in Section 2 is based on some martingale theorems. In our dynamics we have two natural martingales:

$$M_{N,T}(t) = m_N(t) - m_N\big(-\mu T N^{1/3}\big) - \int_{-\mu T N^{1/3}}^t \mathcal{F}_N\big(m_N(s), h_N(s)\big)\,\mathrm{d}s, \tag{4.1}$$

where $\mathcal{F}_N(x,h) := \mathcal{L}_h x$, $T > 1$, and

$$M_{N,T}^2(t) - \int_{-\mu T N^{1/3}}^t \mathcal{G}_N\big(m_N(s), h_N(s)\big)\,\mathrm{d}s \tag{4.2}$$

with $\mathcal{G}_N(x,h) := \mathcal{L}_h x^2 - 2x\mathcal{L}_h x$.

In the following lemma we prove that, for large N, the function $\mathcal{F}_N(x,h)$ is well approximated by the infinite volume drift $F(x,h) = -x + \tanh\{\beta(x+h)\}$ (see the infinite volume equation (2.4)).

Lemma 4.1. *There exists $c > 0$ such that, for any $x \in [-1, 1]$, $|h| \le h_c$, N large enough,*

$$\left|\mathcal{F}_N(x,h) - F(x,h)\right| \le \frac{c}{N} \tag{4.3}$$

and, for $\Lambda(x,h) = 1 - x\tanh\{\beta(x+h)\}$,

$$\left|N\mathcal{G}_N(x,h) - 2\Lambda(x,h)\right| \le \frac{c}{N}. \tag{4.4}$$

Proof. We have

$$\mathcal{F}_N(x,h) = \frac{2}{N}\bigl(c^+(x,h) - c^-(x,h)\bigr) = \bigl(\hat{c}^+(x,h) - \hat{c}^-(x,h)\bigr) - x\bigl(\hat{c}^+(x,h) + \hat{c}^-(x,h)\bigr)$$

then there exists $c > 0$ such that

$$\left|\hat{c}^+(x,h) + \hat{c}^-(x,h)\right| \le \frac{c}{N} \quad \text{and} \quad \left|\bigl(\hat{c}^+(x,h) - \hat{c}^-(x,h)\bigr) - \tanh\bigl\{\beta(x+h)\bigr\}\right| \le \frac{c}{N} \tag{4.5}$$

for any N large enough, that yields (4.3). Now

$$\mathcal{G}_N(x,h) = \frac{4}{N^2}\bigl[c^+(x,h) + c^-(x,h)\bigr]$$

thus

$$2 - N\mathcal{G}_N(x,h) = 2\bigl[1 - \bigl(\hat{c}^+(x,h) + \hat{c}^-(x,h)\bigr)\bigr] + 2x\bigl[\hat{c}^+(x,h) - \hat{c}^-(x,h)\bigr]$$

then (4.4) follows from (4.5). □

Let $Y(t)$, $t \ge -T$, be the solution of (3.7) starting from $Y(-T) = y$, and τ_T be the first exit time from the rectangle $\mathcal{R}_T$ (see (3.12) and (3.14)). We denote by $\mathcal{P}^*_{-T,y}$ the law of the stopped process $Y(t \wedge \tau_T)$ on $\mathcal{D}[-T,T]$. We call $\tau_{N,T}$ the corresponding stopping time for the finite N-process $Y_N(t)$ (see (3.4)) and denote by $\mathcal{P}^*_{N,-T,y}$ the law of the corresponding stopped process. We are going to prove (see Proposition 4.3) the convergence of $\mathcal{P}^*_{N,-T,y}$ to $\mathcal{P}^*_{-T,y}$ for suitable T, y. Let $\mathcal{D}[-T,T]$ be the space of functions on $[-T,T]$ that are right-continuous and have left-hand limits. The convergence results in this section are meant in the sense of the Skorohod metric on $\mathcal{D}[-T,T]$. For more details on the space $\mathcal{D}[-T,T]$ and the weak convergence on $\mathcal{D}[-T,T]$ see Chapter 3 of [7].

For the martingale $\hat{M}_{N,T}(t) := \nu N^{1/3} M_{N,T}(\mu N^{1/3}(t \wedge \tau_{N,T}))$,

$$\hat{M}_{N,T}(t) = Y_N(t \wedge \tau_{N,T}) - Y_N(-T) - \nu\mu N^{2/3}\int_{-T}^{t\wedge\tau_{N,T}} \mathcal{F}_N\bigl(m\bigl(\mu s N^{1/3}\bigr), h_N\bigl(\mu s N^{1/3}\bigr)\bigr)\,\mathrm{d}s$$

we have the following result

Proposition 4.2. *Let $w(t)$ be the standard Brownian motion and $\xi := \frac{2}{\beta}\mu\nu^2$, then*

$$\hat{M}_{N,T}(t) \xrightarrow{\mathcal{D}} \xi w(T + t \wedge \tau_T) \quad \text{as } N \to \infty. \tag{4.6}$$

Proof. By (4.2), the quadratic variation of $\hat{M}_{N,T}(t)$ is given by

$$\hat{V}_{N,T}(t) := \nu^2\mu N \int_{-T}^{t\wedge\tau_{N,T}} \mathcal{G}_N\bigl(m_N\bigl(\mu s N^{1/3}\bigr), h_N\bigl(\mu s N^{1/3}\bigr)\bigr)\,\mathrm{d}s$$

thus, for $\Lambda(m,h)$ as in Lemma 4.1, by (4.4), there exists $c > 0$ such that

$$\sup_{t\ge -T}\left|\hat{V}_{N,T}(t) - 2\nu^2\mu\int_{-T}^{t\wedge\tau_{N,T}} \Lambda\bigl(m_N\bigl(\mu s N^{1/3}\bigr), h_N\bigl(\mu s N^{1/3}\bigr)\bigr)\,\mathrm{d}s\right| \le cN^{-1}$$

for any N large enough. In a neighborhood of $(x_c, -h_c)$,

$$\Lambda(x,h) = \frac{1}{\beta} + \mathcal{O}(h+h_c) + \mathcal{O}(x-x_c)$$

moreover, for $t < N^{2/3}$ there exists $c > 0$ such that $|h_N(t) + h_c| \le c(tN^{-2/3})^2$ for any N large enough. We have, thus

$$\sup_{-T\le s\le \tau_{N,T}} \left| \Lambda\big(m_N\big(\mu s N^{1/3}\big), h_N\big(\mu s N^{1/3}\big)\big) - \frac{1}{\beta}\right| \le cN^{-1/3}$$

for a suitable $c > 0$ then

$$\sup_{t\ge -S} \big|\hat{V}_{N,T}(t) - \xi(T + t\wedge \tau_{N,T})\big| \le cN^{-1/3}. \tag{4.7}$$

We have $\tau_{N,T} \xrightarrow{\mathcal{P}} \tau_T$, for $N\to\infty$, hence, by (4.7),

$$\hat{V}_{N,T}(t) \xrightarrow{\mathcal{P}} \xi(T + t\wedge\tau_T) \quad \text{as } N\to\infty$$

thus (4.6) follows since $\hat{M}_{N,T}(-T) = 0$ and $\hat{M}_{N,T}(t)$ has at most discontinuities of order $N^{-2/3}$ (see [7] and [18]). □

Proposition 4.3. *For any $T, y > 0$ such that $y < 2T$, $\mathcal{P}^*_{N,-T,y}$ converges to $\mathcal{P}^*_{-T,y}$ as $N\to\infty$.*

Proof. As usual with martingale problems, we first need to prove tightness and then to identify the limiting points by proving that they satisfy a martingale equation which has unique solution. By Proposition 4.2 follows the tightness of $\hat{M}_{N,T}(t)$. It remains to prove the tightness of

$$\Gamma_{N,T}(t) = \nu\mu N^{2/3} \int_{-T}^{t\wedge\tau_{N,T}} \mathcal{F}_N\big(m_N\big(\mu s N^{1/3}\big), h_N\big(\mu s N^{1/3}\big)\big)\,\mathrm{d}s.$$

We use the Chensov moment condition, indeed there exists c such that, for all $t > s \ge -T$,

$$\mathbb{E}_{\mathcal{P}_{N,-T,y}}\big[\big|\Gamma_{N,T}(t) - \Gamma_{N,T}(s)\big|^2\big] \le c|t-s|^2, \tag{4.8}$$

where (4.8) holds after using the Cauchy–Schwartz inequality, being the integrated function in L^2. It follows that the stopped process $Y_N(t\wedge\tau_{N,T})$ is tight and, consequently, its law $\mathcal{P}_{N,-T,y}$ converges by subsequences. Moreover, any limiting point has support on $\mathcal{C}([-T,T],\mathbb{R})$, this follows from the fact that the jumps of Y_N are $\pm N^{-2/3}$.

By (4.3), we can approximate the term $\mathcal{F}_N(x,h)$ in (4.1) with $F(x,h)$ unless errors of order N^{-1}. We perform the Taylor expansion of $F(x,h)$ in a neighborhood of $(x_c,-h_c)$. Being $F(x_c,-h_c) = \partial F/\partial x(x_c,-h_c) = 0$, the leading terms are the first order in $(h+h_c)$ and the second order in $(x-x_c)$, we have

$$F(x,h) = (h+h_c) - \beta x_c(x-x_c)^2 + \mathcal{O}\big((h+h_c)(x-x_c)\big) + \mathcal{O}\big((h+h_c)^2\big) + \mathcal{O}\big((x-x_c)^3\big).$$

On the other hand, for $tN^{-2/3}$ vanishingly small as $N\to\infty$, $h_N(t) = -h_c + h_c t^2 N^{-4/3}/2 + \mathcal{O}((tN^{-2/3})^4)$, thus there exists c such that

$$\sup_{t\in\mu N^{1/3}[-T,\tau_{N,T}]} \left|\mathcal{F}_N\big(m_N(t), h_N(t)\big) - \left\{\frac{h_c}{2}t^2N^{-4/3} - \beta x_c\big(m_N(t) - x_c\big)^2\right\}\right| \le cN^{-1} \tag{4.9}$$

for N large enough, then, by (4.9),

$$\sup_{t\ge -T}\left|\hat{M}_{N,T}(t) - Y_N(t\wedge\tau_{N,T}) + Y_N(-T) + \int_{-T}^{t\wedge\tau_{N,T}} \left\{\frac{h_c}{2}\nu\mu^3 s^2 - \beta x_c\mu\nu^{-1}Y_N^2(s)\right\}\mathrm{d}s\right| \le cN^{-1/3}. \tag{4.10}$$

For our choice of μ and ν (see (3.4)), the integrand in (4.10) becomes $s^2 - Y_N^2(s)$. From (4.10) and Proposition 4.2 we deduce that any limiting point satisfies a martingale relation that uniquely defines a process which is the law of the solution of (3.7). □

5. Behavior of the limit process

In this section we are going to investigate the behavior of a generic solution $Y(t)$ of the SDE

$$\mathrm{d}Y(t) = \left[t^2 - Y^2(t)\right]\mathrm{d}t + \xi\,\mathrm{d}w_t, \quad \xi > 0. \tag{5.1}$$

For any fixed $t_0 \in \mathbb{R} \cup \{-\infty\}$, $y_0 \in \mathbb{R}$, we denote by $\mathcal{P}_{t_0,y_0}$ the probability law of the process $Y(t), t \geq t_0$, solution of (5.1) starting from y_0 at time t_0. Moreover we denote by $\mathcal{P}$ the law of $Y(t), t \in \mathbb{R}$, solution of (5.1) conditioned to $|Y(t) + t| \to 0$ as $t \to -\infty$.

Deterministic analysis

One of the preliminary steps for the study of (3.7) is the analysis of the related deterministic equation

$$y'(t) = t^2 - y^2(t). \tag{5.2}$$

Proposition 5.1 is proved in Section 2.3 of [8], it concerns the asymptotic behavior for $t \to \infty$ of a generic solution $y(t)$ of (5.2).

Proposition 5.1. *There exists a decreasing solution $y^*(t)$ of (5.2) such that $-t > y^*(t) > -\sqrt{t^2+1}$, for any $t \geq 0$. Let $y(t)$ be the solution of (5.2) starting at time $t_0 \geq 0$ from $y_0 \in \mathbb{R}$,*

- *if $y_0 > y^*(t_0)$, then, for any $\delta \in (0,1)$ there exists $t_\delta \geq t_0$ such that $|y(t) - t| \leq \frac{1}{2(1-\delta)t}$ for any $t \geq t_\delta$;*
- *if $y_0 < y^*(t_0)$, then $y(t)$ is decreasing for $t \geq 0$ and it explodes to $-\infty$ in a finite time.*

Asymptotic behavior of $Y(t)$ for $t \to \infty$

In this first part of the section we prove the following theorem.

Theorem 5.2. *Consider the sets*

$$E^+ := \left\{Y : \lim_{t\to\infty}\left|Y(t) - t\right| = 0\right\} \quad \textit{and} \quad E^- := \left\{Y\colon \textit{there is } t\colon\ Y(t) = -\infty\right\} \tag{5.3}$$

then $\mathcal{P}_{t_0,y_0}\{Y \in E^+ \cup E^-\} = 1$ for any $t_0 \in \mathbb{R} \cup \{-\infty\}$, $y_0 \in \mathbb{R}$.

The proof of Theorem 5.2 consists of three parts. We define the stopping time

$$\Pi := \inf\left\{t\colon\ Y(t) = -\infty\right\}$$

then $\Pi \in \mathbb{R} \cup \{+\infty\}$. We fix $T > 0$ large enough, suppose $\Pi > T$ and study the behavior of $Y(t)$ for $t \geq T$. In Proposition 5.3 we prove that if $Y(t)$ is in a neighborhood of $y^*(t)$ at time T then $Y(t)$ escapes from it $\mathcal{P}$-a.s. In Propositions 5.5 and 5.7 we prove that the probability for the events $Y \in E^\mp$ to occur is close to the probability that $Y(t)$ leaves such a critical neighborhood, respectively, from below or from above. Unless further indications, in this section we mean, by c, a positive constant not depending on T.

We will denote by $y^*(t)$ the solution of the ODE (5.2) defined in Proposition 5.1, and define the process $z^*(t) := Y(t) - y^*(t)$. $z^*(t)$ verifies the equation

$$\mathrm{d}z^*(t) = -z^*(t)\left(z^*(t)t + 2y^*(t)\right)\mathrm{d}t + \xi\,\mathrm{d}w(t). \tag{5.4}$$

For any fixed $\delta > 0$ small enough, we define the stopping time $\tau^*_{T,\delta} := \inf\{t \geq T\colon\ |z^*(t)| \geq \delta\}$.

Proposition 5.3. *For any $T > 0$, $\delta > 0$ small enough,*

$$\mathbf{1}_{\Pi > T}\mathcal{P}_{T,Y(T)}\left\{\tau^*_{T,\delta} < \infty\right\} = 1. \tag{5.5}$$

Proof. Let us assume $\Pi > T$. We need to prove the assertion for the paths such that $|z^*(T)| < \delta$. Suitably applying the Ito's formula to (5.4), we get

$$\mathrm{d}z^{*2}(t) = \left[-2z^{*2}(t)\left(z^{*2}(t) + 2y^*(t)\right) + \xi^2\right]\mathrm{d}t + 2\xi z^*(t)\,\mathrm{d}w(t), \tag{5.6}$$

thus, for $T \le t \le \tau^*_{T,\delta}$

$$z^{*2}(t) \ge z^{*2}(T) + \xi^2 t + 2\xi \int_T^t z^*_s\,\mathrm{d}w_s, \tag{5.7}$$

the inequality descending since, for δ small enough, $-2z^{*2}_{t\wedge\tau^*_{T,\delta}}(z^*_{t\wedge\tau^*_{T,\delta}} + 2y^*_{t\wedge\tau^*_{T,\delta}}) \ge 0$.

The process $2\xi \int_T^{t\wedge\tau^*_{T,\delta}} z^*_s\,\mathrm{d}w_s$ is a continuous martingale, thus its expected value is constantly zero and

$$\mathbb{E}\left[\left(2\xi \int_T^{t\wedge\tau^*_{T,\delta}} z^*_s\,\mathrm{d}w_s\right)^2\right] = 4\xi^2 \int_T^t \mathbb{E}\left[z^{*2}_s \mathbf{1}_{s\le\tau^*_{T,\delta}}\right]\mathrm{d}s \le 4\xi^2\delta^2(t-T)$$

hence, by the Doob's inequality, for any $n \in \mathbb{N}$,

$$\mathcal{P}_{T,Y(T)}\left\{2\xi\left|\int_T^{(T+n^4)\wedge\tau^*_{T,\delta}} z^*_s\,\mathrm{d}w_s\right| \ge n^3\right\} \le \frac{4\xi^2\delta^2}{n^2}$$

thus, from the Borel–Cantelli Lemma and (5.7), $\mathcal{P}_{T,Y(T)}$-a.s., there exists $\tilde{n}$ such that, for $n \ge \tilde{n}$,

$$\delta^2 \ge z^{*2}\left(\left(T+n^4\right)\wedge\tau^*_{T,\delta}\right) > -n^3 + \xi^2\left(\left(T+n^4\right)\wedge\tau^*_{T,\delta}\right)$$

then $\tau^*_{T,\delta} \le (T+n^4) \vee (\delta^2+n^3)/\xi^2$, thus, for any $T > 0$

$$\mathcal{P}_{T,Y(T)}\left\{\tau^*_{T,\delta} < \infty\right\} \ge \mathcal{P}_{T,Y(T)}\left\{\liminf_{n\to+\infty}\left\{\tau^*_{T,\delta} \le \left(T+n^4\right)\right\}\right\} = 1$$

and (5.5) is proved. □

We omit the proof of the following lemma.

Lemma 5.4. *Let $t > s$, for any $\gamma > 0$, we have*

$$\frac{\mathrm{e}^{\gamma t^2}}{2\gamma t}\left[1 - \mathrm{e}^{-\gamma(t^2-s^2)/2}\right] \le \int_s^t \mathrm{e}^{\gamma u^2}\,\mathrm{d}u \le \frac{\mathrm{e}^{\gamma t^2}}{2\gamma t}\left(\frac{2\gamma s^2}{2\gamma s^2 - 1}\right) \tag{5.8}$$

for any $s > \frac{1}{\sqrt{2\gamma}}$, and

$$\frac{\mathrm{e}^{-\gamma s^2}}{2\gamma s}\left[1 - t^{-1}\mathrm{e}^{-\gamma(t^2-s^2)/2}\right]\left(\frac{2\gamma s^2}{2\gamma s^2 + 1}\right) \le \int_s^t \mathrm{e}^{-\gamma u^2}\,\mathrm{d}u \le \frac{\mathrm{e}^{-\gamma s^2}}{2\gamma s} \tag{5.9}$$

for any $s > 0$.

Proposition 5.5. *There exists $c > 0$ such that, for any T large enough, $\delta > 0$,*

$$\mathbf{1}_{\Pi>T}\mathbf{1}_{\tau^*_{T,\delta}<\infty,\, z^*(\tau^*_{T,\delta})<-\delta}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\left\{Y \notin E^-\right\} \le \mathrm{e}^{-cT}. \tag{5.10}$$

Proof. Suppose $\tau^*_{T,\delta} < \infty$ and $\Pi > T$, thus, consequently, $\Pi > \tau^*_{T,\delta}$. We denote by $\hat{y}(t)$ the solution of the ODE (5.2) starting at time $\tau^*_{T,\delta}$ from $y^*(\tau^*_{T,\delta}) - \delta/2$. From Proposition 5.1 we know that $\hat{y}(t)$ explodes to $-\infty$ in a finite time. Consider $\hat{z}(t) := Y(t) - \hat{y}(t)$, thus $\hat{z}(t)$ verifies the SDE

$$\mathrm{d}\hat{z}_t = -\hat{z}_t\big(\hat{z}_t + 2\hat{y}(t)\hat{z}_t\big)\,\mathrm{d}t + \xi\,\mathrm{d}w_t.$$

We can assume $\hat{z}(\tau^*_{T,\delta}) \le -\delta/2$ since $\mathbf{1}_{z^*(\tau^*_{T,\delta})<-\delta} \le \mathbf{1}_{\hat{z}(\tau^*_{T,\delta})\le-\delta/2}$. We have

$$\hat{z}(t) = \hat{z}\big(\tau^*_{T,\delta}\big)\mathrm{e}^{-2\int_{\tau^*_{T,\delta}}^t \hat{y}(s)\,\mathrm{d}s} - \int_{\tau^*_{T,\delta}}^t \hat{z}^2(u)\mathrm{e}^{-2\int_u^t \hat{y}(s)\,\mathrm{d}s}\,\mathrm{d}u + \xi\,\chi_{\tau^*_{T,\delta}}(t) \tag{5.11}$$

with

$$\hat{\chi}_{\tau^*_{T,\delta}}(t) := \int_{\tau^*_{T,\delta}}^t \mathrm{e}^{2\int_{\tau^*_{T,\delta}}^u \hat{y}(s)\,\mathrm{d}s}\,\mathrm{d}w_u \tag{5.12}$$

then

$$\mathbf{1}_{\hat{z}(\tau^*_{T,\delta})\le-\delta/2}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\big\{\hat{z}(t) \le \mathrm{e}^{-2\int_{\tau^*_{T,\delta}}^t \hat{y}(s)\,\mathrm{d}s}\big(\hat{\chi}_{\tau^*_{T,\delta}}(t) - \delta/2\big), \forall t \ge \tau^*_{T,\delta}\big\} = 1.$$

The probability law of $\hat{\chi}_{\tau^*_{T,\delta}}(t)|\tau^*_{T,\delta}$, $t \ge \tau^*_{T,\delta}$, is a centered Gaussian. Since $\hat{y}(t) \le -t$, $t \ge \tau^*_{T,\delta}$, we have

$$\mathbb{E}\big[\hat{\chi}^2_{\tau^*_{T,\delta}}(t)|\tau^*_{T,\delta}\big] = \int_{\tau^*_{T,\delta}}^t \mathrm{e}^{4\int_{\tau^*_{T,\delta}}^t \hat{y}(s)\,\mathrm{d}s}\,\mathrm{d}u \le \mathrm{e}^{2\tau^{*2}_{T,\delta}}\int_{\tau^*_{T,\delta}}^t \mathrm{e}^{-2u^2}\,\mathrm{d}u \le \frac{1}{4T}, \tag{5.13}$$

where the last inequality descends from (5.9), since $\tau^*_{T,\delta} \ge T$. Hence there exists $c > 0$, such that

$$\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\Big\{\sup_{t\ge\tau^*_{T,\delta}} \hat{z}(t) \ge 0\Big\} \le \mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\Big\{\sup_{t\ge\tau^*_{T,\delta}} \hat{\chi}_{\tau^*_{T,\delta}}(t) \ge \frac{\delta}{2}\Big\} \le \mathrm{e}^{-cT} \tag{5.14}$$

for any T large enough, where the second inequality follows from (5.13) and (A.2). Then we get (5.10). □

We denote by $y^+(t)$ the solution of (5.2) conditioned to $\lim_{t\to-\infty}|y^+(t)+t| = 0$ and define the process $z^+(t) := Y(t) - y^+(t)$. $z^+(t)$ satisfies the SDE

$$\mathrm{d}z^+(t) = -z^+(t)\big(z^+(t)t + 2y^+(t)\big)\,\mathrm{d}t + \xi\,\mathrm{d}w(t), \tag{5.15}$$

thus, for any $t_0 \in \mathbb{R}\cup\{-\infty\}$,

$$z^+(t) = z^+(t_0)\mathrm{e}^{-2\int_{t_0}^t y^+(s)\,\mathrm{d}s} - \int_{t_0}^t z^{+2}(u)\mathrm{e}^{-2\int_u^t y^+(s)\,\mathrm{d}s}\,\mathrm{d}u + \xi\,\chi^+_{t_0}(t) \tag{5.16}$$

with

$$\chi^+_{t_0}(t) := \int_{t_0}^t \mathrm{e}^{-2\int_u^t y^+(s)\,\mathrm{d}s}\,\mathrm{d}w_u. \tag{5.17}$$

We fix $\varepsilon > 0$ small enough and define the stopping time $\tau^+_{T,\varepsilon} := \inf\{t \ge T\colon |z^+(t)| \le \varepsilon\}$.

Lemma 5.6. *For $t_0 \in \mathbb{R}\cup\{-\infty\}$, $\chi^+_{t_0}(t)$ as in (5.17), $y_0 \in \mathbb{R}$, there exists $c > 0$ such that, for λ large enough,*

$$\mathcal{P}_{t_0,y_0}\Big\{\sup_{t\ge t_0} \chi^+_{t_0}(t)\big(\sqrt{|t|}\vee 1\big) > \lambda\Big\} \le \mathrm{e}^{-c\lambda^2}. \tag{5.18}$$

Proof. $\chi^+_{t_0}(t)$ is a centered Gaussian process of variance

$$\mathbb{E}\big[\chi^{+2}_{t_0}(t)\big] = \int_{t_0}^{t} \mathrm{e}^{-4\int_u^t y^+(s)\,\mathrm{d}s}\,\mathrm{d}u.$$

Let us suppose, at first, $t_0 < 0$. We know that $y^+(t) \geq -t$ for $t < 0$, thus, for $t_0 \leq t < 0$,

$$\mathbb{E}\big[\chi^{+2}_{t_0}(t)\big] \leq \mathrm{e}^{2t^2}\int_{|t|}^{|t_0|} \mathrm{e}^{-2u^2}\,\mathrm{d}u \leq \frac{1}{4|t|} \wedge 1,$$

where the second inequality follows from (5.9). A similar estimate can be obtained for $t \geq 0$ using (5.8), since $\inf_{t\in\mathbb{R}} y^+(t) > 0$ and $y^+(t) \geq t - 1/t$ for $t > 0$ large enough. Therefore, for any $t_0 \in \mathbb{R} \cup \{-\infty\}$, there exists $c > 0$ such that, for $t \geq t_0$,

$$\mathbb{E}\big[\chi^{+2}_{t_0}(t)\big] \leq c\left(\frac{1}{|t|} \wedge 1\right),$$

thus (5.18) follows from inequality (A.2). □

Proposition 5.7. *There exists $c > 0$ such that, for any T large enough, $\delta, \varepsilon > 0$,*

$$\mathbf{1}_{\Pi > T}\mathbf{1}_{\tau^*_{T,\delta}<\infty, z^*(\tau^*_{T,\delta})>\delta}\mathcal{P}_{\tau^*_{T,\delta}, Y(\tau^*_{T,\delta})}\big\{\tau^+_{T,\varepsilon} = +\infty\big\} \leq \mathrm{e}^{-cT}. \tag{5.19}$$

Proof. Suppose $\tau^*_{T,\delta} < \infty$ and $\Pi > T$, then $\Pi > \tau^*_{T,\delta}$. As in the proof of Proposition 5.3, we mainly make use of comparison arguments. We compare, by means of Lemma A.1, the process $z^+(t)$ with suitable Gaussian processes. Then use the inequality (A.2) to estimate the behavior of such Gaussian processes. We will avoid the details, let us see. Suppose $z^*(\tau^*_{T,\delta}) > \delta$ and $|z^+(\tau^*_{T,\delta})| > \varepsilon$, we need to distinguish two cases: $z^+(\tau^*_{T,\delta}) > \varepsilon$ and $z^+(\tau^*_{T,\delta}) < -\varepsilon$.

Consider the first case $z^+(\tau^*_{T,\delta}) > \varepsilon$, from (5.16), we have

$$\mathbf{1}_{z^+(\tau^*_{T,\delta})>\varepsilon}\mathcal{P}_{\tau^*_{T,\delta}, Y(\tau^*_{T,\delta})}\Big\{z^+(t) \leq z^+\big(\tau^*_{T,\delta}\big)\mathrm{e}^{-2\int_{\tau^*_{T,\delta}}^t y^+(s)\,\mathrm{d}s} + \xi\chi^+_{\tau^*_{T,\delta}}(t)\Big\} = 1,$$

thus

$$\begin{aligned}
&\mathbf{1}_{z^+(\tau^*_{T,\delta})>\varepsilon}\mathcal{P}_{\tau^*_{T,\delta}, Y(\tau^*_{T,\delta})}\Big\{\inf_{t\geq\tau^*_{T,\delta}} z^+(t) > \varepsilon\Big\} \\
&\quad\leq \mathbf{1}_{z^+(\tau^*_{T,\delta})>\varepsilon}\mathcal{P}_{\tau^*_{T,\delta}, Y(\tau^*_{T,\delta})}\Big\{\inf_{t\geq\tau^*_{T,\delta}}\Big(z^+\big(\tau^*_{T,\delta}\big)\mathrm{e}^{-2\int_{\tau^*_{T,\delta}}^t y^+(s)\,\mathrm{d}s} + \xi\chi^+_{\tau^*_{T,\delta}}(t)\Big) > \varepsilon\Big\} \\
&\quad\leq \mathrm{e}^{-cT},
\end{aligned} \tag{5.20}$$

where the last inequality is obtained by the use of Lemma 5.6.

We prove, now, the statement for the second case $z^+(\tau^*_{T,\delta}) < -\varepsilon$, $z^*(\tau^*_{T,\delta}) > \delta$. At first, we show that, with large probability, $z^*(t)$ reaches the line $\frac{3}{2}t$, i.e. that the stopping time $\tau'_T := \inf\{t \geq \tau^*_T\colon\ z^*(t) \geq \frac{3}{2}t\}$ is finite. We compare $z^*(t)$ with the process $v^+(t)$, solution of the linear problem

$$\mathrm{d}v^+(t) = \frac{t}{2}v^+(t)\,\mathrm{d}t + \xi\,\mathrm{d}w(t), \quad v^+\big(\tau^*_{T,\delta}\big) = z^*\big(\tau^*_{T,\delta}\big), \tag{5.21}$$

we have

$$v^+(t) = v^+\big(\tau^*_{T,\delta}\big)\mathrm{e}^{(1/4)(t^2-\tau^{*2}_{T,\delta})} + \xi\mathrm{e}^{t^2/4}\int_{\tau^*_{T,\delta}}^t \mathrm{e}^{-u^2/4}\,\mathrm{d}w_u. \tag{5.22}$$

Since $-z(z+2y_t^*) > tz/2$ for $0 \le z \le -3y_t^*/2$, by Lemma A.1, $z^*(t) \ge v^+(t)$, as long as $0 \le z^*(t) \le \frac{3}{2}t$. It is sufficient to apply the inequality (A.2) to $v^+(t)$ whose quadratic variation is easily estimable from (5.9) and (5.22) to show that

$$\mathbf{1}_{v^+(\tau^*_{T,\delta})>\delta}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\Big\{\inf_{t\ge\tau^*_{T,\delta}} v^+(t) \le 0\Big\} \le \mathrm{e}^{-cT}$$

and

$$\mathbf{1}_{v^+(\tau^*_{T,\delta})>\delta}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\Big\{\sup_{t\ge\tau^*_{T,\delta}} \frac{2v^+(t)}{3t} < 1\Big\} \le \mathrm{e}^{-cT}$$

hence

$$\mathbf{1}_{\tau^*_{T,\delta}<\tau'_T}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\{\tau'_T=\infty\} = \mathbf{1}_{\tau^*_{T,\delta}<\tau'_T}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\Big\{\sup_{t\ge\tau^*_{T,\delta}} \frac{2z^*(t)}{3t} < 1\Big\} \le \mathrm{e}^{-cT}. \tag{5.23}$$

For t large enough, $y^*(t) \ge t - 1/t$, thus, from (5.23), with $\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}$-probability greater than $1-\mathrm{e}^{-cT}$, there exists $\tau^*_T \le \tau'_T < \infty$ such that $z^+(\tau'_T) \ge -\tau'_T/2 - 1/\tau'_T$.

By an analogous comparison argument it is possible to prove that

$$\mathbf{1}_{z^+(\tau^*_T)<-\varepsilon,\tau'_T<\infty}\mathcal{P}_{\tau'_T,z^+(\tau'_T)}\Big\{\sup_{t\ge\tau^*_T} z^+(t) < -\varepsilon\Big\} \le \mathrm{e}^{-cT}. \tag{5.24}$$

Equation (5.19) follows, then, from (5.20), (5.23) and (5.24). □

Proposition 5.8. *There is $c>0$ such that, for any $\varepsilon>0$ small enough, T, λ large enough, $\lambda < \varepsilon\sqrt{T}$,*

$$\mathbf{1}_{\tau^+_{T,\varepsilon}<\infty}\mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\Big\{\inf_{s\ge\tau^+_{T,\varepsilon}}\sup_{t\ge s}\big|Y(t)-y^+(t)\big|\sqrt{t} > \lambda\Big\} \le \mathrm{e}^{-c\lambda^2}. \tag{5.25}$$

Proof. Let us suppose $\tau^+_{T,\varepsilon} < +\infty$, thus the relation (5.16) with $\tau^+_{T,\varepsilon}$ in place of t_0 holds, for $t \ge \tau^+_{T,\varepsilon}$. We apply Lemma 5.6 to the process $\chi^+_{\tau^+_{T,\varepsilon}}(t)$, thus, by symmetry, we get

$$\mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\Big\{\sup_{t\ge\tau^+_{T,\varepsilon}}\big|\chi^+_{\tau^+_{T,\varepsilon}}(t)\big|\sqrt{t} > \frac{\lambda}{2\xi}\Big\} \le \mathrm{e}^{-c\lambda^2} \tag{5.26}$$

for any λ large enough. Let us define the stopping time $\tau''_{T,\varepsilon} := \inf\{t \ge \tau^+_{T,\varepsilon}\colon |z^+(t)| > 2\varepsilon\}$. We have

$$\int_{\tau^+_{T,\varepsilon}}^t \mathrm{e}^{-2\int_u^t y^+(s)\,\mathrm{d}s}\,\mathrm{d}u \le t^2\mathrm{e}^{-t^2}\int_{\tau^+_{T,\varepsilon}}^t \frac{\mathrm{e}^{u^2}}{u^2}\,\mathrm{d}u \le \frac{c}{t}\wedge 1,$$

thus, with $\mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}$-probability greater than $1-2\mathrm{e}^{-c\lambda^2}$ we have

$$-\varepsilon\mathrm{e}^{-2\int_{\tau^+_{T,\varepsilon}}^t y(s)\,\mathrm{d}s} - \Big(c\frac{\varepsilon^2}{t}\wedge 1\Big) - \frac{\lambda}{2\sqrt{t}} \le z^+(t) \le \varepsilon\mathrm{e}^{-2\int_{\tau^+_{T,\varepsilon}}^t y^+(s)\,\mathrm{d}s} + \frac{\lambda}{2\sqrt{t}} \tag{5.27}$$

for $\tau^+_{T,\varepsilon} \le t \le \tau''_{T,\varepsilon}$. Assume $\lambda < \varepsilon\sqrt{T}$, thus, since $\tau^+_{T,\varepsilon} \ge T$, from (5.27) it follows that

$$\mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\{\tau''_{T,\varepsilon} < +\infty\} \le \mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\Big\{\sup_{\tau^+_{T,\varepsilon}\le t\le\tau''_{T,\varepsilon}}\big|z^+(t)\big| < 2\varepsilon\Big\} \le \mathrm{e}^{-c\varepsilon^2 T} \tag{5.28}$$

then, by (5.27) and (5.28), we have

$$\mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\Big\{\inf_{s\geq\tau^+_{T,\varepsilon}}\sup_{t\geq s}\big|z^+(t)\big|\sqrt{t}>\lambda\Big\}$$

$$\leq \mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\big\{\tau''_{T,\varepsilon}<+\infty\big\}+\mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\Big\{\inf_{s\geq\tau^+_{T,\varepsilon}}\sup_{t\geq s}\big|z^+(t)\big|\sqrt{t}>\lambda|\tau''_{T,\varepsilon}=\infty\Big\}\leq 2\mathrm{e}^{-c\lambda^2}$$

hence (5.25) is proved. □

Proposition 5.9. *There exists $c>0$ such that, for any T large enough, δ,ε small enough,*

$$\mathbf{1}_{\Pi>T}\mathbf{1}_{\tau^*_{T,\delta}<\infty,z^*(\tau^*_{T,\delta})>\delta}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\big\{Y\notin E^+\big\}\leq \mathrm{e}^{-c\varepsilon^2T}. \tag{5.29}$$

Proof. Assume $\Pi>T$, $\tau^*_{T,\delta}<\infty$ and $z^*(\tau^*_{T,\delta})>\delta$ then, for any $\varepsilon,\delta>0$ small enough, T large enough,

$$\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\big\{Y\notin E^+\big\}\leq \mathbb{E}\big[\mathbf{1}_{\tau^+_{T,\varepsilon}<\infty}\mathcal{P}_{\tau^+_{T,\varepsilon},Y(\tau^+_{T,\varepsilon})}\big\{Y\notin E^+\big\}\big]+\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\big\{\tau^+_{T,\varepsilon}=+\infty\big\}$$

thus (5.29) follows from Propositions 5.7 and 5.8. □

Conclusion of proof of Theorem 5.2. Let us suppose $T>t_0$, thus, from the definition of Π and Proposition 5.3, we have

$$\begin{aligned}\mathcal{P}_{t_0,y_0}\big\{Y\notin E^+\cup E^-\big\}&=\mathbb{E}\big[\mathbf{1}_{\Pi>T}\mathcal{P}_{T,Y(T)}\big\{Y\notin E^+\cup E^-\big\}\big]\\&=\mathbb{E}\big[\mathbf{1}_{\tau^*_{T,\delta}<\infty,\Pi>\tau^*_{T,\delta}}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\big\{Y\notin E^+\cup E^-\big\}\big].\end{aligned} \tag{5.30}$$

Equation (5.30) is bounded by

$$\begin{aligned}&\mathbb{E}\big[\mathbf{1}_{\Pi>T}\mathbf{1}_{\tau^*_{T,\delta}<\infty,z^*(\tau^*_{T,\delta})<-\delta}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\big\{Y\notin E^-\big\}\big]+\mathbb{E}\big[\mathbf{1}_{\Pi>T}\mathbf{1}_{\tau^*_{T,\delta}<\infty,z^*(\tau^*_{T,\delta})>\delta}\mathcal{P}_{\tau^*_{T,\delta},Y(\tau^*_{T,\delta})}\big\{Y\notin E^+\big\}\big]\\&\quad\leq \mathrm{e}^{-c\varepsilon^2T},\end{aligned} \tag{5.31}$$

where the inequality follows from Propositions 5.5 and 5.9, and holds for some $c>0$, for any T large enough, δ,ε small enough. The result follows from (5.31) by performing the limit for $T\to\infty$. □

Behavior of $Y(t)$ for $t\to-\infty$

In this part of the section we will provide some results for the behavior of $Y(t)$ for negative t, $|t|$ large enough.

Proposition 5.10. *Let $\mathcal{P}$ be the probability law defined at the beginning of this section. There is $c>0$ such that for T,λ large enough, $\lambda<\sqrt{T}$,*

$$\mathcal{P}\Big\{\sup_{t\leq -T}\big|Y(t)-y^+(t)\big|\sqrt{|t|}>\lambda\Big\}\leq \mathrm{e}^{-c\lambda^2}. \tag{5.32}$$

Proof. $z^+(t)$ satisfies the equation (5.16) even in the limit as $t_0\to-\infty$. $y^+(t)\to+\infty$ and $z^+(t)\to 0$ for $t\to-\infty$, $\mathcal{P}$-a.s., thus

$$z^+(t)=-\int_{-\infty}^t z^{+2}(u)\mathrm{e}^{-2\int_u^t y^+(s)\,\mathrm{d}s}\,\mathrm{d}u+\xi\chi^+_{-\infty}(t). \tag{5.33}$$

We use Lemma 5.6 with $t_0=-\infty$ to estimate the behavior of $\chi^+_{-\infty}(t)$, then the proof proceeds specularly to proof of Proposition 5.8. □

Proposition 5.11. *There is $c>0$ such that for T, S, λ large enough, $S<T$, $\lambda<\sqrt{S}$,*

$$\mathbf{1}_{|Y(T)-T|<\lambda/(2\sqrt{T})}\mathcal{P}_{-T,Y(T)}\Big\{\sup_{-T\le t\le -S}\big|Y(t)-y^+(t)\big|\sqrt{|t|}>\lambda\Big\}\le \mathrm{e}^{-c\lambda^2}. \tag{5.34}$$

Proof. The proof of (5.34) is almost the same of Proposition 5.10. □

Behavior of $Y(t)$ in bounded intervals

In this part of the section we study the behavior of solutions $Y(t)$ of (5.1) starting at time $-T$ from y: $|y-T|\le\varepsilon$, ε small enough. We recall that the stopping time $\tau_T\in[-T,T]$ is the first exit time of $Y(t)$ from the rectangle $\mathcal{R}_T$ (see (3.12) and (3.14)). Notice that the condition $|y-T|\le\varepsilon$ guaranties $(-T,Y(-T))\in\mathcal{R}_T$.

Lemma 5.12. *There exists $c>0$ such that, for any T large enough, ε small enough,*

$$\mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{-T,y}\big\{Y(\tau_T)=2T\big\}\le \mathrm{e}^{-cT^2}.$$

Proof. We have $y^+(-T)\ge T$, $y\le T+\varepsilon$, then $z^+(-T)=y-y^+(-T)\le\varepsilon$, hence, by (5.16),

$$\mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{-T,y}\big\{z^+(t)\le\varepsilon+\xi\chi^+_{-T}(t),\forall t\ge -T\big\}=1,$$

thus, since $\sup_{-T\le t\le T}y^+(t)<T$, we have

$$\begin{aligned}\mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{-T,y}\big\{Y(\tau_T)=2T\big\}&\le \mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{-T,y}\Big\{\sup_{-T\le t\le T}z^+(t)\ge T\Big\}\\ &\le \mathcal{P}_{-T,y}\Big\{\sup_{-T\le t\le T}\chi^+_{-T}(t)\ge\frac{T-\varepsilon}{\xi}\Big\}\le \mathrm{e}^{-cT^2},\end{aligned}$$

where the last inequality follows from (A.2) and Lemma 5.6. □

Lemma 5.13. *There exists $c>0$ such that, for any T large enough,*

$$\mathcal{P}_{\tau_T,-2T}\big\{\Pi\ge\tau_T+T^{-1}\big\}\le \mathrm{e}^{-cT^3}. \tag{5.35}$$

Proof. Consider the process $\tilde{y}(t)$, solution of the ODE (5.2) starting from $-\frac{3}{2}T$ at time τ_T. Let us consider, now, $\tilde{z}(t):=Y(t)-\tilde{y}(t)$, thus $\tilde{z}_{\tau_T}(\tau_T)=-\frac{T}{2}$. Using exactly the same arguments used in proof of Proposition 5.5 to show (5.14), it is possible to prove that

$$\mathcal{P}_{\tau_T,-2T}\Big\{\sup_{t\ge\tau_T}\tilde{z}(t)\ge 0\Big\}\le \mathrm{e}^{-cT^3}. \tag{5.36}$$

$\tilde{y}(t)$ lies below $y^*(t)$, then, from Proposition 5.1, we know that it explodes to $-\infty$. It is easy to show that $\tilde{y}(t)$ explodes within τ_T+T^{-1} (see Lemma 2.3.15 in [8]), then (5.35) easily follows from (5.36). □

Proof of Proposition 3.3. Let $Y(-T)=y$ with y: $|y-T|\le\varepsilon$, then, from Theorem 5.2 we have

$$\begin{aligned}\mathcal{P}_{-T,y}\big\{Y\notin\mathcal{E}^+_T\cup\mathcal{E}^-_T\big\}&=\mathcal{P}_{-T,y}\big\{Y\notin\mathcal{E}^+_T\cup\mathcal{E}^-_T, Y\in E^+\cup E^-\big\}\\ &\le \mathcal{P}_{-T,y}\big\{Y(T)\in\big[-2T,y^+(T)-\varepsilon\big)\cup\big(y^+(T)+\varepsilon,2T\big], Y\in E^+\big\} &(5.37)\\ &\quad+\mathcal{P}_{-T,y}\big\{Y(\tau_T)=2T\big\}. &(5.38)\end{aligned}$$

Lemma 5.12 provides a bound for the probability in (5.38) that assures its convergence to 0 as $T \to \infty$. The term (5.37) vanishes as $T \to \infty$ since, by Theorem 5.2, for any $\varepsilon > 0$ small enough,

$$\mathcal{P}_{-T,y}\left\{\inf_{T\geq 0}\sup_{t\geq T}\left|Y(t) - y^{+}(t)\right| > \varepsilon | Y \in E^{+}\right\} = 1$$

hence (3.16) follows. From Proposition 5.8 and Lemma 5.13 it follows that

$$\mathbf{1}_{(\tau_T, Y(\tau_T))\in\partial\mathcal{R}_T^{\pm}}\mathcal{P}_{\tau_T, Y(\tau_T)}\left\{Y \notin E^{\pm}\right\} \leq \mathrm{e}^{-c\varepsilon^2 T} \tag{5.39}$$

for some $c > 0$, thus (3.17) follows from (5.39) and Theorem 5.2. □

Proof of Theorem 3.2

We consider two processes $Y, \bar{Y}$ solutions of (5.1) starting from $Y(-T) = y$ and $\bar{Y}(-T) = \bar{y}$, with $y, \bar{y}$ such that $|y - T| \leq \varepsilon, |\bar{y} - T| \leq \varepsilon$ for some $\varepsilon > 0$ small enough. Without lost of generality, we can suppose $\bar{y} > y$. We denote by $\mathcal{Q}_{-T,y,\bar{y}}$ the probability law of the coupled process $(Y(t), \bar{Y}(t))$ by taking the same noise for Y and $\bar{Y}$.

Let us fix $S \in (1, T)$ and $\varepsilon > 0$ small enough and define the sets

$$\mathcal{A}_S = \mathcal{A}_{T,S,\varepsilon} := \left\{Y\colon \sup_{-T\leq t\leq -S}\left|Y(t) - y^{+}(t)\right| \leq \varepsilon\right\} \tag{5.40}$$

and

$$\mathcal{B}_S = \mathcal{B}_{S,\varepsilon} := \left\{Y\colon \sup_{t\geq S}\left|Y(t) - y^{+}(t)\right| \leq \varepsilon\right\}. \tag{5.41}$$

Let $\tau_S, \bar{\tau}_S$ be the first exit times respectively for the processes $Y(t)$ and $\bar{Y}(t)$ from the rectangle $\mathcal{R}_S$ defined in (3.12). We call Π and $\bar{\Pi}$ the times of explosion to $-\infty$ of Y and $\bar{Y}$. $Y(t)$ and $\bar{Y}(t)$ are well defined, thus, respectively for $t \leq \Pi$ and for $t \leq \bar{\Pi}$. We agree with the convention to define $Y(t) := -\infty$ for $t \geq \Pi$, $\bar{Y}(t) := -\infty$ for $t \geq \bar{Y}(t)$. We have the following results.

Lemma 5.14. *For any $S \in (1, T)$ and $\varepsilon > 0$ small enough*

$$\mathcal{Q}_{-T,y,\bar{y}}\left\{\lim_{T\to\infty}\sup_{t\geq -T}\bar{Y}(t) - Y(t) = 0 | Y, \bar{Y} \in \mathcal{A}_S \cap \mathcal{E}_S^{+} \cap \mathcal{B}_S\right\} = 1. \tag{5.42}$$

Proof. Let us assume $Y, \bar{Y} \in \mathcal{A}_S \cap \mathcal{E}_S^{+} \cap \mathcal{B}_S$. We denote by $v(t)$ the process $\bar{Y}(t) - Y(t)$, then $\mathrm{d}v = -v(Y + \bar{Y})\,\mathrm{d}t$, hence

$$v(t) = (\bar{y} - y)\mathrm{e}^{-\int_{-T}^{t}(Y(u)+\bar{Y}(u))\,\mathrm{d}u} \tag{5.43}$$

thus $v(t) > 0$ for any $t \geq -T$.

Since $Y, \bar{Y} \in \mathcal{A}_S$ and $y^{+}(t) \geq -t$ for $t < 0$, from (5.43) we have

$$0 \leq v(t) \leq \varepsilon\mathrm{e}^{2\int_{-T}^{t}(u+\varepsilon)\,\mathrm{d}u} \leq \varepsilon\mathrm{e}^{(S-\varepsilon)^2}\mathrm{e}^{-(T-\varepsilon)^2} \quad \text{for } -T \leq t \leq -S$$

thus

$$\lim_{T\to\infty}\sup_{-T\leq t\leq -S} v(t) = 0. \tag{5.44}$$

$Y, \bar{Y} \in \mathcal{E}_S^{+}$ implies $\bar{Y}(t), Y(t) \geq -2S$ for any $-S \leq t \leq S$, then, by (5.43),

$$0 \leq v(t) \leq v(-S)\mathrm{e}^{-\int_{-S}^{t}(Y(u)+\bar{Y}(u))\,\mathrm{d}u} \leq v(-S)\mathrm{e}^{8S^2} \quad \text{for } -S \leq t \leq S,$$

thus, by (5.44),

$$\lim_{T\to\infty}\sup_{-S\le t\le S} v(t)=0. \tag{5.45}$$

$Y,\bar Y\in\mathcal{B}_S$ thus $\bar Y(t), Y(t)\ge y^+(t)-\varepsilon>0$, for $t\ge S$, S large enough, then, from (5.43) we have

$$0\le v(t)\le v(S)\mathrm{e}^{-\int_S^t (Y(u)+\bar Y(u))\,\mathrm{d}u}\le v(S)\quad \text{for } t\ge S,$$

thus, from (5.45),

$$\lim_{T\to\infty}\sup_{t\ge S} v(t)=0 \tag{5.46}$$

then the lemma is proved. □

Lemma 5.15. *For any $S\in(1,T)$ and $\varepsilon>0$ small enough*

$$Q_{-T,y,\bar y}\Big\{\lim_{T\to\infty}\sup_{-T\le t\le \tau_S}\big|\bar Y(t)-Y(t)\big|=0\big|Y\in\mathcal{A}_S\cap\mathcal{E}_S^-,\bar Y\in\mathcal{A}_S\Big\}=1. \tag{5.47}$$

Proof. Let us assume $Y\in\mathcal{A}_S\cap\mathcal{E}_S^-$ and $\bar Y\in\mathcal{A}_S$. Consider the process $v(t)$ defined in the proof of the previous lemma, then, since $Y,\bar Y\in\mathcal{A}_S$, (5.44) holds also in the current case.

On the other hand $Y\in\mathcal{E}_S^-$ implies $\bar Y(t)\ge Y(t)\ge -2S$, then, since $|\tau_S|\le S$,

$$0\le v(t)\le v(-S)\mathrm{e}^{-\int_{-S}^t (Y(u)+\bar Y(u))\,\mathrm{d}u}\le v(-S)\mathrm{e}^{8S^2}\quad \text{for } -S\le t\le \tau_S,$$

thus, by (5.44), we have

$$\lim_{T\to\infty}\sup_{-S\le t\le \tau_S} v(t)=0\quad Q_{-T,y,\bar y}\text{-a.s.} \tag{5.48}$$

hence (5.47) follows. □

Proposition 5.16. *For any bounded continuous function $g(y)$ with compact support and for any fixed $\varepsilon>0$ small enough, $t\ge -T$, we have*

$$\lim_{T\to\infty}\mathbf{1}_{|y-T|\le\varepsilon}\mathbf{1}_{|\bar y-T|\le\varepsilon}\big|\mathbb{E}_{\mathcal{P}_{-T,y}}\big[g\big(Y(t)\big)\big]-\mathbb{E}_{\mathcal{P}_{-T,\bar y}}\big[g\big(\bar Y(t)\big)\big]\big|=0. \tag{5.49}$$

Proof. We define $G(t):=|g(Y(t))-g(\bar Y(t))|$ then we need to prove that

$$\lim_{T\to\infty}\mathbf{1}_{|y-T|\le\varepsilon}\mathbf{1}_{|\bar y-T|\le\varepsilon}\mathbb{E}_{\mathcal{Q}_{-T,y,\bar y}}\big[G(t)\big]=0. \tag{5.50}$$

Let us fix $S\in(1,T)$ large enough and $\varepsilon>0$ small enough, $y,\bar y$: $|y-T|\le\varepsilon$, $|\bar y-T|\le\varepsilon$. For $\mathcal{A}_S$ as in (5.40) we have

$$\begin{aligned}&\big|\mathbb{E}_{\mathcal{Q}_{-T,y,\bar y}}\big[G(t)\big]-\mathbb{E}_{\mathcal{Q}_{-T,y,\bar y}}\big[\mathbf{1}_{Y,\bar Y\in\mathcal{A}}G(t)\big]\big|\\&\quad\le 2\sup|g|\big(\mathcal{P}_{-T,y}\{Y\notin\mathcal{A}_S\}+\mathcal{P}_{-T,\bar y}\{\bar Y\notin\mathcal{A}_S\}\big)\\&\quad\le 4\sup|g|\mathrm{e}^{-c\varepsilon^2 S},\end{aligned} \tag{5.51}$$

where the last inequality follows from (5.34). We have

$$\begin{aligned}&\big|\mathbb{E}_{\mathcal{Q}_{-T,y,\bar y}}\big[\mathbf{1}_{Y,\bar Y\in\mathcal{A}_S}G(t)\big]-\mathbb{E}_{\mathcal{Q}_{-T,y,\bar y}}\big[\mathbf{1}_{Y,\bar Y\in\mathcal{A}_S\cap(\mathcal{E}_S^+\cup\mathcal{E}_S^-)}G(t)\big]\big|\\&\quad\le 2\sup|g|\big(\mathcal{P}_{-S,y}\big\{Y\notin\mathcal{E}_S^+\cup\mathcal{E}_S^-\big\}+\mathcal{P}_{-S,\bar y}\big\{\bar Y\notin\mathcal{E}_S^+\cup\mathcal{E}_S^-\big\}\big).\end{aligned} \tag{5.52}$$

For $\mathcal{B}_S$ as in (5.41) we have

$$\begin{aligned}
&\big|\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}\cap\mathcal{E}_S^+}G(t)\big]-\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}\cap\mathcal{B}_S\cap\mathcal{E}_S^+}G(t)\big]\big|\\
&\quad\le 2\sup|g|\big(\mathbf{1}_{|y-y^+(S)|\le\varepsilon}\mathcal{P}_{S,y}\{Y\notin\mathcal{B}_S\}+\mathbf{1}_{|\bar{y}-y^+(S)|\le\varepsilon}\mathcal{P}_{S,\bar{y}}\{\bar{Y}\notin\mathcal{B}_S\}\big)\\
&\quad\le 4\sup|g|\mathrm{e}^{-c\varepsilon^2S},
\end{aligned}\tag{5.53}$$

where the last inequality follows from (5.25).

Since g is bounded, it follows from (3.16), (5.51), (5.52) and (5.53) that for any $\zeta>0$ there exists S_0 such that, for any $T>S\ge S_0$, for any $t\ge -T$,

$$\big|\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[G(t)\big]-\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[(\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^+\cap\mathcal{B}_S}+\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^-})G(t)\big]\big|\le\zeta.\tag{5.54}$$

By the continuity of g and Lemma 5.14 it follows that

$$\mathcal{Q}_{-T,y,\bar{y}}\Big\{\lim_{T\to\infty}\sup_{t\ge -T}G(t)=0|Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^+\cap\mathcal{B}_S\Big\}=1,\tag{5.55}$$

thus

$$\lim_{T\to\infty}\mathbf{1}_{t\ge -T}\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{B}_S\cap\mathcal{E}_S^+}G(t)\big]=0.\tag{5.56}$$

On the other hand, by Lemma 5.15,

$$\mathcal{Q}_{-T,y,\bar{y}}\Big\{\lim_{T\to\infty}\sup_{-T\le t\le\tau_S}G(t)=0|Y\in\mathcal{A}_S\cap\mathcal{E}_S^-\bar{Y}\in\mathcal{A}_S\Big\}=1,\tag{5.57}$$

thus

$$\lim_{T\to\infty}\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{-T\le t\le\tau_S}\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^-}G(t)\big]=0.\tag{5.58}$$

We have

$$\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{\tau_S\le t\le\tau_S+S^{-1}}\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}\cap\mathcal{E}_S^-}G(t)\big]\le 2\sup|g|\mathcal{P}_{-T,y}\big\{t-S^{-1}\le\tau_S\le t\big\}\tag{5.59}$$

with the right-hand side term vanishing as $S\to\infty$.

Since $\bar{Y}(t)\ge Y(t)$, $\bar{\Pi}\ge\Pi$, thus, for $t\ge\bar{\Pi}$, $Y(t)=\bar{Y}(t)=-\infty$, then $G(t)=0$. It remains to estimate the term for $\tau_S+S^{-1}\le t\le\bar{\Pi}$. We have $Y(\tau_S)=-2S$ and $\bar{Y}(\tau_S)=-2S+v(\tau_S)$, with, by Lemma 5.15, $\lim_{T\to\infty}|v(\tau_S)|=0$ for $Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^-$. Hence for any fixed $\zeta>0$ arbitrarily small there is T_0 such that, for any $T>T_0$

$$\begin{aligned}
&\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{\tau_S+S^{-1}\le t\le\bar{\Pi}}\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^-}G(t)\big]\\
&\quad\le\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{\tau_S+S^{-1}\le\bar{\Pi}}\mathbf{1}_{\bar{Y}(\tau_S)\le -2S+\zeta}G(t)\big]\\
&\quad\le 2\sup|g|\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{\bar{Y}(\tau_S)\le -2S+\zeta}\mathcal{P}_{\tau_S,\bar{Y}(\tau_S)}\big\{\bar{\Pi}\ge\tau_S+S^{-1}\big\}\big]
\end{aligned}$$

then, by Lemma 5.13, for any $\zeta'>0$ there exists S_0 such that, for any $T>S>S_0$,

$$\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{\tau_S+S^{-1}\le t\le\bar{\Pi}}\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^-}G(t)\big]<\zeta'.\tag{5.60}$$

From (5.59) and (5.60) it follows that

$$\lim_{T\to\infty}\mathbb{E}_{\mathcal{Q}_{-T,y,\bar{y}}}\big[\mathbf{1}_{t\ge\tau_S}\mathbf{1}_{Y,\bar{Y}\in\mathcal{A}_S\cap\mathcal{E}_S^-}G(t)\big]=0,\tag{5.61}$$

then (5.50) follows from (5.54), (5.56), (5.58) and (5.61). $\square$

Corollary 5.17. *Let $Y(t), \hat{Y}(t)$ be solutions of (5.1) starting from $Y(-T)=y$, $\hat{Y}(-S)=\hat{y}$, $T>S$, then, for any function $g(y)$ as in the previous proposition*,

$$\lim_{S\to\infty}\lim_{T\to\infty}\left|\mathbb{E}_{\mathcal{P}_{-T,y}}\left[g\big(Y(t)\big)\right]\mathbf{1}_{|y-T|\le\varepsilon}-\mathbb{E}_{\mathcal{P}_{-S,\hat{y}}}\left[g\big(\hat{Y}(t)\big)\right]\mathbf{1}_{|\hat{y}-S|\le\varepsilon}\right|=0. \tag{5.62}$$

Proof. Suppose $T>S$, $|y-T|\le\varepsilon$, $|\hat{y}-S|\le\varepsilon$. We have

$$\mathbb{E}_{\mathcal{P}_{-T,y}}\left[g\big(Y(t)\big)\right]=\mathbb{E}_{\mathcal{P}_{-T,y}}\left[\mathbb{E}_{\mathcal{P}_{S,Y(-S)}}\left[g\big(Y(t)\big)\right]\right] \tag{5.63}$$

thus

$$\begin{aligned}&\left|\mathbb{E}_{\mathcal{P}_{-T,y}}\left[g\big(Y(t)\big)\right]-\mathbb{E}_{\mathcal{P}_{-T,y}}\left[\mathbb{E}_{\mathcal{P}_{-S,Y(-S)}}\left[g\big(Y(t)\big)\right]\mathbf{1}_{|Y(-S)-S|\le\varepsilon}\right]\right|\\&\quad\le\sup|g|\mathcal{P}_{-T,y}\left\{\left|Y(-S)-S\right|>\varepsilon\right\}\le\sup|g|\mathrm{e}^{-c\varepsilon^2S},\end{aligned} \tag{5.64}$$

where the last inequality in (5.64) follows from (5.34). On the other hand, from Proposition 5.16, for any $\zeta>0$ there exists S_0 such that, for any $T>S>S_0$,

$$\left|\mathbb{E}_{\mathcal{P}_{-T,y}}\left[\mathbb{E}_{\mathcal{P}_{S,Y(-S)}}\left[g\big(Y(t)\big)\right]\mathbf{1}_{|Y(-S)-S|\le\varepsilon}\right]-\mathbb{E}_{\mathcal{P}_{-S,\hat{y}}}\left[g\big(\hat{Y}(t)\big)\right]\mathbf{1}_{|\hat{y}-S|\le\varepsilon}\right|<\zeta \tag{5.65}$$

then (5.62) follows from (5.63), (5.64), (5.65) and the boundedness of g. □

Proposition 5.18. *Let $\mathcal{P}$ be the probability law defined at the beginning of this section, then the probabilities $p_\pm:=\mathcal{P}\{Y\in E^\pm\}$ are strictly positive.*

Proof. Let us prove, at first, the statement for E^-. By (5.33), $z^+(t)\le\xi\chi^+_{-\infty}(t)$ $\mathcal{P}$-a.s., thus, for $\gamma:=y^+(0)-y^*(0)>0$, $\Phi(x):=\frac{1}{\sqrt{2\pi}}\int_x^{+\infty}\mathrm{e}^{-z^2/2}\,\mathrm{d}z$, we have

$$\mathcal{P}\left\{z^*(0)<0\right\}=\mathcal{P}\left\{z^+(0)<-\gamma\right\}\ge\mathcal{P}\left\{\chi^+_{-\infty}(0)<-\gamma\xi^{-1}\right\}=\Phi\left(\gamma\xi^{-1}/\sqrt{\mathbb{E}\left[\chi^{+2}_{-\infty}(0)\right]}\right)\ge c>0 \tag{5.66}$$

since, by Lemma 5.6, $\mathbb{E}[\chi^{+2}_{-\infty}(0)]$ is bounded by a constant. From (5.4) it is easy to verify that

$$z^*(t)\le z^*(0)\mathrm{e}^{-2\int_0^t y^*(s)\,\mathrm{d}s}+\xi\chi^*_0(t),\qquad \chi^*_0(t):=\int_0^t\mathrm{e}^{-2\int_u^t y^*(s)\,\mathrm{d}s}\,\mathrm{d}w_u,$$

$\mathcal{P}$-a.s. Let us suppose $z^*(0)<0$, then $z^*(t)\le\xi\chi^*_0(t)$, thus

$$\mathcal{P}\left\{z^*(T)<-\delta\right\}\ge\mathcal{P}\left\{\chi^*_0(T)>-\delta\xi^{-1}\right\}=\Phi\left(\delta\xi^{-1}\sqrt{\mathbb{E}\left[\chi^{*2}_0(T)\right]}\right)\ge c>0 \tag{5.67}$$

for any $\delta>0$, hence, from Proposition 5.5, (5.66) and (5.67) it follows that $\mathcal{P}\{Y\in E^-\}>0$.

By the use of comparison arguments as in proof of Proposition 5.8 it is easily provable that there exist $\varepsilon_{\max}, c>0$ such that, for any $\varepsilon\le\varepsilon_{\max}$, if $|z^+(-T)|<\frac{\varepsilon}{2}$, then

$$\mathcal{P}_{-T,Y(-T)}\left\{\sup_{-T\le t\le T}\left|z^+(t)\right|<\varepsilon\right\}>\mathrm{e}^{-c/\varepsilon^2}. \tag{5.68}$$

To prove (5.68) it is sufficient to use the small balls inequality (A.5). Thus the claim for E^+ follows from Propositions 5.8 and 5.10 and (5.68). □

Conclusion of the proof of Theorem 3.2. The convergence result (3.10) is a direct consequence of Corollary 5.17. Equation (3.11) easily follows from Proposition 5.10.

The convergence of the probabilities $\mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{-T,y}\{Y\in E^\pm\}$ is a direct consequence of (3.10); finally, from (3.17) and Proposition 5.18 it follows that $p_-\in(0,1)$. □

6. Escape from criticality

In this section we study our N-finite dynamics assuming $Y_N \in \mathcal{E}_T^-$, i.e. $Y_N(\tau_{N,T}) = -2T$ or, equivalently,

$$m_N(\mathcal{T}_{N,T}) = x_c - \frac{2T}{\nu N^{1/3}}, \quad \mathcal{T}_{N,T} := \mu \tau_{N,T} N^{1/3} \in [-T, T]\mu N^{1/3}, \tag{6.1}$$

we recall the definition of $\mathcal{H}_\gamma^\pm(I)$ in (2.11) and prove the following result.

Proposition 6.1. *For any $\mu' > \mu$*

$$\lim_{T\to\infty} \lim_{N\to\infty} \mathcal{P}_N\{\mathcal{H}_0^-(\{\mu' T N^{1/3}\})|Y_N \in \mathcal{E}_T^-\} = 1. \tag{6.2}$$

We consider the stochastic process $x_N^*(t) := x_N^*(\mathcal{T}_{N,T})(t)$ defined as the solution of the ODE (2.6) with $h = h_N(t)$ and random initial condition $x_N^*(\mathcal{T}_{N,T}) = m_N(\mathcal{T}_{N,T})$. We prove that, for any $\mu' > \mu$ (μ as in (3.5)), $x_N^*(t)$ reaches $X_-(h_N(t))$ within the time $\mathcal{T}_{N,T} + \mu' T N^{1/3}$, then we show that, by tracking $x_N^*(t)$, our magnetization $m_N(t)$ approaches $X_-(h_N(t))$. We denote by $\mathcal{P}_{N,T}^-$ the probability law of $m_N(t)$ given $Y_N \in \mathcal{E}_T^-$. All the computations are done for $N > T$, N, T large enough. Unless further indications, we will denote by c a generic positive constant independent of N, T. In order to lighten notation, in this section we will omit the index N for the magnetization and simply write $m(t)$ and $x^*(t)$.

We define the stopping time

$$\hat{\mathcal{T}}_{N,T} := \inf\{t \ge \mathcal{T}_{N,T}: |m(t) - x^*(t)| > N^{-1/6}\}$$

and recall that $F(x) = -x + \tanh\{\beta(x+h)\}$, we have the following result

Lemma 6.2. *Let τ, τ' be two stopping times for $m(t)$ such that $\mathcal{T}_{N,T} < \tau < \tau' < \hat{\mathcal{T}}_{N,T}$ and $N > \tau' - \tau$ $\mathcal{P}_{N,T}^-$-a.s. There exists a function $\psi(t)$, such that*

$$\sup_{\mathcal{T}_{N,T} < t < \hat{\mathcal{T}}_{N,T}} \left|\psi(t) - \frac{\partial F}{\partial x}(x^*(t), h_N(t))\right| \le c N^{-1/6} \tag{6.3}$$

and, for $\gamma > 0$ small enough,

$$\mathcal{P}_{N,T}^-\left\{\sup_{\tau \le t \le \tau'} [|m(t) - x^*(t)| - \Theta_{\tau,\tau'}(t)] \le 0\right\} \ge 1 - cN^{-\gamma} \tag{6.4}$$

with

$$\Theta_{\tau,\tau'}(t) := |m(\tau) - x^*(\tau)| e^{\int_\tau^t \psi(u)\,du} + \frac{2(\tau'-\tau)^{1/2}}{N^{(1-\gamma)/2}}\left(1 + e^{\int_\tau^t \psi(u)\,du} \int_\tau^t |\psi(s)| e^{-\int_\tau^s \psi(u)\,du}\,ds\right). \tag{6.5}$$

Proof. Let us define the function $f(x,t) := x - x^*(t)$, then the process

$$\mathcal{M}(t) := f(m(t), t) - f(m(\mathcal{T}_{N,T}), \mathcal{T}_{N,T}) - \int_{\mathcal{T}_{N,T}}^t \left[\mathcal{L}_{h(s)} f + \frac{\partial f}{\partial s}\right](m(s), s)\,ds$$

is a martingale. For any τ as in the hypothesis, the process $\mathcal{M}_\tau(t) := \mathcal{M}(t) - \mathcal{M}(t \wedge \tau)$ is a martingale as well and

$$\mathcal{V}_\tau(t) := \int_{t\wedge\tau}^t [\mathcal{L}_{h(s)} f^2 - 2f\mathcal{L}_{h(s)} f](m(s), s)\,ds$$

is its quadratic variation. For any $t \ge \mathcal{T}_{N,T}$,

$$\mathbb{E}[\mathcal{M}_\tau^2(t\wedge\tau')|\tau,\tau'] = \mathbb{E}[\mathcal{V}_\tau(t\wedge\tau')|\tau,\tau'] \le cN^{-1}(t\wedge\tau' - t\wedge\tau) \le cN^{-1}(\tau' - \tau)$$

thus, by the Doob's inequality, for any $\gamma \in (0,1)$,

$$\mathcal{P}\left\{\sup_{\tau\le t\le\tau'}\left|\mathcal{M}_\tau(t)\right| \ge \frac{(\tau'-\tau)^{1/2}}{N^{(1-\gamma)/2}}\middle|\tau,\tau'\right\} \le cN^{-\gamma}$$

then

$$\begin{aligned}\mathcal{P}^-_{N,T}&\left\{\sup_{\tau\le t\le\tau'}\left|\mathcal{M}_\tau(t)\right| \ge \frac{(\tau'-\tau)^{1/2}}{N^{(1-\gamma)/2}}\right\}\\ &= \mathbb{E}_{\mathcal{P}^-_{N,T}}\left[\mathbf{1}_{\tau<\tau'}\mathcal{P}\left\{\sup_{\tau\le t\le\tau'}\left|\mathcal{M}_\tau(t)\right| \ge \frac{(\tau'-\tau)^{1/2}}{N^{(1-\gamma)/2}}\middle|\tau,\tau'\right\}\right] \le cN^{-\gamma}.\end{aligned} \tag{6.6}$$

Recall the initial condition (6.1), then there exists a function $\psi(t)$ satisfying (6.3) and such that

$$\left|\left[\mathcal{L}_{h(s)}f + \frac{\partial f}{\partial s}\right]\big(m(s),s\big) - \psi(s)\big(m(s)-x^*(s)\big)\right| \le cN^{-1} \tag{6.7}$$

for $t \ge \mathcal{T}_{N,T}$. For $\tau \le t \le \hat{\mathcal{T}}_{N,T}$ we define the process

$$R_\tau(t) := f\big(m(t),t\big) - f\big(m(\tau),\tau\big) - \int_\tau^t \psi(s) f\big(m(s),s\big)\,\mathrm{d}s - \mathcal{M}_\tau(t), \tag{6.8}$$

then, from (6.7),

$$\sup_{\tau\le t\le\hat{\mathcal{T}}_{N,T}} \frac{|R_\tau(t)|}{|t-\tau|} \le cN^{-1} \quad \mathcal{P}^-_{N,T}\text{-a.s.} \tag{6.9}$$

By treating (6.8) as an integral equation for $f(m(t),t)$ we find

$$\begin{aligned} f\big(m(t),t\big) &= f\big(m(\tau),\tau\big)e^{\int_\tau^t\psi(u)\,\mathrm{d}u} + \big[R_\tau(t)+\mathcal{M}_\tau(t)\big]\\ &\quad + e^{\int_\tau^t\psi(u)\,\mathrm{d}u}\int_\tau^t \big[R_\tau(s)+\mathcal{M}_\tau(s)\big]\psi(s)e^{-\int_\tau^s\psi(u)\,\mathrm{d}u}\,\mathrm{d}s, \quad \tau\le t\le\hat{\mathcal{T}}_{N,T}.\end{aligned}$$

From (6.6) and (6.9), assuming $N > \tau'-\tau$, we find

$$\mathcal{P}^-_{N,T}\left\{\sup_{\tau\le t\le\tau'}\left|R_\tau(t)+\mathcal{M}_\tau(t)\right| \le \frac{2(\tau'-\tau)^{1/2}}{N^{(1-\gamma)/2}}\right\} \ge 1 - cN^{-\gamma} \tag{6.10}$$

thus (6.4) follows. □

Lemma 6.3. *Let us fix $\delta > 0$ small enough and consider the stopping time*

$$\mathcal{T}'_{N,T} = \mathcal{T}'_{N,T,\delta} := \inf\big\{t \ge \mathcal{T}_{N,T}\colon\, x^*(t) \le x_c - \delta\big\}$$

then, $\mathcal{P}^-_{N,T}$-a.s, there exists $C_0 > 0$ such that $\mathcal{T}'_{N,T} - \mathcal{T}_{N,T} \le C_0T^{-1}N^{1/3}$ for any T, N large enough, and

$$x^*(t) \ge \hat{x}(t) := x_c - \frac{2T}{\nu N^{1/3} - 4\beta x_c T(t + T\mu N^{1/3})} \quad \textit{for } \mathcal{T}_{N,T} \le t \le \mathcal{T}'_{N,T}. \tag{6.11}$$

Proof. Recall the initial condition (6.1) for x^*, then, for $\mathcal{T}_{N,T} \le t \le 2\mu TN^{1/3}$ we have $-h_c \le h_N(t) \le -h_c + cT^2N^{-2/3}$, thus

$$0 \le F\big(x^*(t),h_N(t)\big) - F\big(x^*(t),h_c\big) \le cT^2N^{-2/3}$$

hence, for $\mathcal{T}_{N,T} \le t \le \mathcal{T}'_{N,T} \wedge 2\mu T N^{1/3}$,

$$-\beta x_c(1+c_0\delta)\big(x^*(t)-x_c\big)^2 \le F\big(x^*(t),h_N(t)\big) \le -\beta x_c(1-c_0\delta)\big(x^*(t)-x_c\big)^2 + cT^2N^{-2/3}$$

for a suitable $c_0 > 0$ independent of δ, then $x_1(t) \le x^*(t) \le x_2(t)$, $x_{1,2}(t)$ solutions of

$$x_1'(t) = -\beta x_c(1+c_0\delta)\big(x_1(t)-x_c\big)^2, \tag{6.12}$$

$$x_2'(t) = -\beta x_c(1-c_0\delta)\big(x_2(t)-x_c\big)^2 + cT^2N^{-2/3} \tag{6.13}$$

with $x_1(\mathcal{T}_{N,T}) = x_2(\mathcal{T}_{N,T}) = x^*(\mathcal{T}_{N,T}) = x_c - 2T/\nu N^{1/3}$. It is easy to check that

$$x_1(t) = x_c - \frac{1}{\nu N^{1/3}(2T)^{-1} - \beta x_c(1+c_0\delta)(t-\tau_0)}. \tag{6.14}$$

On the other hand $m_2(t)$ is a function blowing up at time

$$\mathcal{T}_{N,T} + C_0\frac{N^{1/3}}{T} \ll 2\mu T N^{1/3}$$

for a suitable C_0 possibly depending on δ. In particular we have $\mathcal{T}'_{N,T} < 2\mu T N^{1/3}$, thus the result follows. □

Lemma 6.4. *Let us fix $\delta > 0$ small enough and define the stopping time*

$$\mathcal{T}''_{N,T} = \mathcal{T}''_{N,T,\delta} := \inf\big\{t \ge \mathcal{T}'_{N,T}\colon\ x^*(t) \le X_-\big(h_N(t)\big) + \delta\big\}$$

then, $\mathcal{P}^-_{N,T}$-a.s., there exists $C_1 > 0$ such that $\mathcal{T}''_{N,T} - \mathcal{T}'_{N,T} \le C_1$ for any T, N large enough, and

$$x^*(t) \le X_-\big(h_N(t)\big) + \delta \quad \textit{for any } t \ge \mathcal{T}''_{N,T}. \tag{6.15}$$

Proof. Let $\bar{x}_1(t)$ and $\bar{x}_2(t)$ be the solutions of

$$\bar{x}_1'(t) = F\big(\bar{x}_1(t),h_c\big) \quad \text{and} \quad \bar{x}_2'(t) = F\big(\bar{x}_2(t),h_c\big) + cT^2N^{-2/3} \tag{6.16}$$

with $\bar{x}_1(\mathcal{T}'_{N,T}) = \bar{x}_2(\mathcal{T}'_{N,T}) = x^*(\mathcal{T}'_{N,T}) = x_c - \delta$. From Lemma 6.3 we know that $\mathcal{T}_{N,T} \le 2\mu T N^{1/3}$, $\mathcal{P}^-_{N,T}$-a.s., then $\bar{x}_1(t) \le X_+(h_N(t)) \le \bar{x}_2(t)$ for $\mathcal{T}'_{N,T} \le t \le \mathcal{T}'_{N,T} + \mu T N^{1/3}$.

Consider the stopping time $\tilde{\mathcal{T}}''_{N,T} := \inf\{t \ge \mathcal{T}'_{N,T}\colon\ \bar{x}_1(t) \le X_-(h_N(t)) + \delta/2\}$, then there exists $C_1 > 0$ such that $\tilde{\mathcal{T}}''_{N,T} - \mathcal{T}'_{N,T} \le C_1$. We denote by $\Delta\bar{x}(t)$ the non-negative function $\bar{x}_2(t) - \bar{x}_1(t)$, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\bar{x}(t) \le (\beta-1)\Delta\bar{x}(t) + c\frac{T^2}{N^{2/3}}, \qquad \Delta\bar{x}\big(\mathcal{T}'_{N,T}\big) = 0,$$

hence $\Delta\bar{x}(t) \le cT^2N^{-2/3}$ for any $t \le \mathcal{T}'_{N,T} + \mu T N^{1/3}$, then, in particular, $x^*(\tilde{\mathcal{T}}''_{N,T}) \le \bar{x}_2(\tilde{\mathcal{T}}''_{N,T}) \le \bar{x}_1(\tilde{\mathcal{T}}''_{N,T}) + cT^2N^{-2/3} \le X_-(h_N(\tilde{\mathcal{T}}''_{N,T})) + \delta$ for N large enough, then $\mathcal{T}''_{N,T} \le \tilde{\mathcal{T}}''_{N,T} \le \mathcal{T}'_{N,T} + C_1$. Equation (6.15) is thus proved. □

Lemma 6.5. *Consider the stopping time*

$$\mathcal{T}'''_{N,T} := \inf\big\{t \ge \mathcal{T}''_{N,T}\colon\ x^*(t) \le X_-\big(h_N(t)\big) + N^{-1/2}\big\}$$

then, $\mathcal{P}^-_{N,T}$-a.s., there exists $C_2 > 0$ such that $\mathcal{T}'''_{N,T} - \mathcal{T}''_{N,T} \le C_2 \ln N$ for any T, N large enough, and

$$\big|x^*(t) - X_-\big(h_N(t)\big)\big| \le N^{-1/2} \quad \textit{for any } \mathcal{T}'''_{N,T} \le t \le \frac{\pi}{2}N^{2/3}. \tag{6.17}$$

Proof. There exists $c>0$ such that

$$-\frac{\partial}{\partial x}F\big(X_-\big(h_N(t)\big),h_N(t)\big)=\beta\big[X_-\big(h_N(t)\big)^2-x_c^2\big]\geq c$$

for $\mathcal{T}''_{N,T}\leq t\leq\frac{\pi}{2}N^{2/3}$. We have $\delta\geq x^*(t)-X_-(h_N(t))\geq 0$ for $t\geq\mathcal{T}''_{N,T}$, then there exists $c_0>0$ not depending on δ such that

$$\mathcal{F}_N\big(x^*(t),h_N(t)\big)\leq -c(1-c_0\delta)\big(x^*(t)-X_-\big(h_N(t)\big)\big).$$

Let us call $\Delta x(t):=x^*(t)-X_-(h_N(t))\geq 0$, then, being $X_-(h_N(t))$ a not decreasing function for $0\leq t\leq\frac{\pi}{2}N^{2/3}$, there exists $c>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta x(t)\leq -c\Delta x(t)-\frac{\mathrm{d}}{\mathrm{d}t}X_-\big(h_N(t)\big)\leq -c\Delta x(t),\qquad \Delta x\big(\mathcal{T}''_{N,T}\big)=\delta,$$

hence $\Delta x(t)\leq\delta\mathrm{e}^{-c(t-\mathcal{T}''_{N,T})}$ for any $\mathcal{T}''_{N,T}\leq t\leq\frac{\pi}{2}N^{2/3}$, then follows the result. □

Proof of Proposition 6.1. The proof consists of three steps.

Step I. We prove, at first, that there exists $c>0$ such that, for any $\gamma>0$ small enough,

$$\mathcal{P}^-_{N,T}\big\{\big|m\big(\mathcal{T}'_{N,T}\big)-x^*\big(\mathcal{T}'_{N,T}\big)\big|\leq N^{-1/3+\gamma/2}\big\}\geq 1-cN^{-\gamma}. \tag{6.18}$$

We have $|x^*(t)-x_c|\leq\delta$ for $\mathcal{T}_{N,T}\leq t\leq\mathcal{T}'_{N,T}$, then there exists $c_0>0$ independent of δ such that

$$0\leq\frac{\partial}{\partial x}F\big(x^*(t),h_N(t)\big)\leq\beta x_c(1+c_0\delta)\big(x_c-x^*(t)\big)+cT^2N^{-2/3}$$

thus, in particular, there exists $c>0$ such that, $\mathcal{P}^-_{N,T}$-a.s., for any $\mathcal{T}_{N,T}\leq t\leq\mathcal{T}'_{N,T}\wedge\hat{\mathcal{T}}_{N,T}$,

$$\big|\psi(t)\big|=\psi(t)\leq c\big[\big(x_c-x^*(t)\big)+T^2N^{-2/3}\big]\leq c\big[\big(x_c-\hat{x}(t)\big)+T^2N^{-2/3}\big],$$

the last inequality descending from (6.11). For C_0 as in Lemma 6.3, referring to (6.5) for the definition of $\Theta_{\mathcal{T}_{N,T},\mathcal{T}'_{N,T}}(t)$, there exist $c,c'>0$ such that

$$\begin{aligned}\Theta_{\mathcal{T}_{N,T},\mathcal{T}'_{N,T}}(t)&\leq cN^{-1/3+\gamma/2}\exp\bigg\{\int_{\mathcal{T}_{N,T}}^t\psi(u)\,\mathrm{d}u\bigg\}\\&\leq c\frac{N^{-1/3+\gamma/2}}{\sqrt{T}}\exp\bigg\{c'\int_{\mathcal{T}_{N,T}}^t\bigg[\big(x_c-\hat{x}(s)\big)+\frac{T^2}{N^{2/3}}\bigg]\mathrm{d}s\bigg\}.\end{aligned}\tag{6.19}$$

Let us define $\Upsilon_T:=(\nu/4\beta x_cT-2\mu T)$, then, by the definition of $\hat{x}(t)$ in (6.11), the exponent in (6.19) is bounded by

$$c'\int_{\mathcal{T}_{N,T}}^t\bigg(\frac{1}{\Upsilon_TN^{1/3}-t}+\frac{T^2}{N^{2/3}}\bigg)\mathrm{d}s=c'\frac{T^2}{N^{2/3}}(t-\mathcal{T}_{N,T})\ln\frac{|\Upsilon_TN^{1/3}-t|}{|\Upsilon_TN^{1/3}-\mathcal{T}_{N,T}|}.$$

Since, by Lemma 5.31, $\mathcal{T}'_{N,T}<\mathcal{T}_{N,T}+C_0N^{1/3}/T$, $\mathcal{P}^-_{N,T}$-a.s., there exist $c,c',c''>0$ such that

$$\begin{aligned}\sup_{\mathcal{T}_{N,T}\leq t\leq\mathcal{T}'_{N,T}\wedge\hat{\mathcal{T}}_{N,T}}\Theta_{\mathcal{T}_{N,T},\mathcal{T}'_{N,T}}(t)&\leq cN^{-1/3+\gamma/2}\frac{t-\Upsilon_TN^{1/3}}{\mathcal{T}_{N,T}-\Upsilon_TN^{1/3}}\mathrm{e}^{c'TN^{-1/3}}\\&\leq c''N^{-1/3+\gamma/2}\end{aligned}$$

for T large enough, thus, by 6.4,

$$\mathcal{P}^-_{N,T}\Big\{\sup_{\mathcal{T}_{N,T}\le t\le \mathcal{T}'_{N,T}\wedge\hat{\mathcal{T}}_{N,T}}\big|m(t)-x^*(t)\big|\le cN^{-1/3+\gamma/2}\Big\}\ge 1-cN^{-\gamma}$$

in particular, with the same probability $\mathcal{T}'_{N,T}<\hat{\mathcal{T}}_{N,T}$, thus (6.18) follows.

Step II. We prove, now, that there exists $c>0$ such that, for any $\gamma>0$ small enough,

$$\mathcal{P}^-_{N,T}\big\{\big|m\big(\mathcal{T}''_{N,T}\big)-x^*\big(\mathcal{T}''_{N,T}\big)\big|\le cN^{-1/3+\gamma/2}\big\}\ge 1-cN^{-\gamma}. \tag{6.20}$$

We have $|\partial F(x^*(t),h_N(t))/\partial x|\le \max\{1,\beta-1\}:=c_\beta$, thus, by (6.15), there exists $c>0$ such that

$$\sup_{\mathcal{T}'_{N,T}\le t\le \mathcal{T}''_{N,T}\wedge\hat{\mathcal{T}}_{N,T}}\left|\int_{\mathcal{T}'_{N,T}}^t\psi(u)\,\mathrm{d}u\right|\le c.$$

We can use the same arguments of Step I, there exists $c>0$ such that

$$\sup_{\mathcal{T}'_{N,T}\le t\le \mathcal{T}''_{N,T}\wedge\hat{\mathcal{T}}_{N,T}}\Theta_{\mathcal{T}'_{N,T},\mathcal{T}''_{N,T}}(t)\le c\big(\big|m\big(\mathcal{T}'_{N,T}\big)-x^*\big(\mathcal{T}'_{N,T}\big)\big|+N^{-(1-\gamma)/2}\big)$$

$\mathcal{P}^-_{N,T}$-a.s., thus, by (6.4) and (6.18), we have

$$\mathcal{P}^-_{N,T}\Big\{\sup_{\mathcal{T}'_{N,T}\le t\le \mathcal{T}''_{N,T}\wedge\hat{\mathcal{T}}_{N,T}}\big|m(t)-x^*(t)\big|\le cN^{-1/3+\gamma/2}\Big\}\ge 1-cN^{-\gamma}$$

thus (6.20) follows since $\mathcal{T}''_{N,T}<\hat{\mathcal{T}}_{N,T}$ with the same probability.

Step III. We conclude the proof of the proposition. We have $|x^*(t)-X_-(h_N(t))|\le\delta$ for $t\ge\mathcal{T}''_{N,T}$, thus, for small δ,

$$\frac{\partial}{\partial x}F\big(x^*(t),h_N(t)\big)=\big(1+\mathcal{O}(\delta)\big)\frac{\partial}{\partial x}F\big(X_-\big(h_N(t)\big),h_N(t)\big).$$

On the other hand, there exists $c>0$ such that $\frac{\partial}{\partial x}F(X_-(h_N(t)),h_N(t))\le -c$, for any $\mathcal{T}''_{N,T}\le t\le\frac{\pi}{2}N^{2/3}$, hence there exists $c'>0$ such that

$$\sup_{\tau_2\le t\le\frac{\pi}{2}N^{2/3}\wedge\hat{\tau}}\big|\psi(t)\big|\le -c'.$$

Let us fix $\mathcal{T}''_{N,T}\le t_*\le\frac{\pi}{2}N^{2/3}\wedge\hat{\mathcal{T}}_{N,T}$, thus

$$\Theta_{\mathcal{T}''_{N,T},t_*}(t)=\big|m\big(\mathcal{T}''_{N,T}\big)-x^*\big(\mathcal{T}''_{N,T}\big)\big|\mathrm{e}^{\int_{\mathcal{T}''_{N,T}}^t\psi(u)\,\mathrm{d}u}+4\sqrt{t}N^{-(1-\gamma)/2}$$

then, by (6.20),

$$\mathcal{P}^-_{N,T}\big\{\Theta_{\mathcal{T}''_{N,T},t_*}(t)\le N^{-1/3+\gamma/2}\mathrm{e}^{-c'(t_*-\mathcal{T}''_{N,T})}+4\sqrt{t_*}N^{-(1-\gamma)/2}\big\}\ge 1-cN^{-\gamma}. \tag{6.21}$$

Let us fix, now, $\mu''>\mu'>\mu$ and choose $t_*=\mu''TN^{1/3}$, thus, by Lemmas 6.3 and 6.4, $t_*>\mathcal{T}''_{N,T}$, $\mathcal{P}^-_{N,T}$-a.s., hence, by (6.4) and (6.21) we get

$$\mathcal{P}^-_{N,T}\Big\{\sup_{\mathcal{T}''_{N,T}\le t\le t_*\wedge\hat{\mathcal{T}}_{N,T}}\big|m(t)-x^*(t)\big|\le TN^{-1/3+\gamma/2}\Big\}\ge 1-cN^{-\gamma} \tag{6.22}$$

then, in particular, with the same probability $\hat{\mathcal{T}}_{N,T} > t_*$. We have $\mathcal{T}''_{N,T} < \mu' T N^{1/3} < t_* \ \mathcal{P}^-_{N,T}$-a.s., then

$$\mathcal{P}^-_{N,T}\left\{\left|m\left(\mu' T N^{1/3}\right) - x^*\left(\mu' T N^{1/3}\right)\right| \le T N^{-1/3+\gamma/2}\right\} \ge 1 - cN^{-\gamma}. \tag{6.23}$$

On the other hand, by Lemma 6.5, $\mathcal{T}''_{N,T} < \mu' T N^{1/3} \ \mathcal{P}^-_{N,T}$-a.s., hence

$$\mathcal{P}^-_{N,T}\left\{\left|m\left(\mu' T N^{1/3}\right) - x^*\left(\mu' T N^{1/3}\right)\right| \le N^{-1/2}\right\} = 1 \tag{6.24}$$

thus (6.2) follows from (6.23) and (6.24). □

7. Behavior far from criticalities

In this section we give some results concerning the dynamics in the stable region. Theorem 7.1 provides a law for the behavior of $m_N(t)$ in $N^{2/3}[-\frac{\pi}{2}, -\eta]$ and $N^{2/3}[\eta, \frac{\pi}{2}]$, $\eta > 0$. Recall that $\mathcal{P}_N$ is the probability law of $m_N(t)$ in $N^{2/3}[-\frac{\pi}{2}, \frac{\pi}{2}]$ given $m_N(-\frac{\pi}{2}N^{2/3}) = m^0_N$. For any fixed $\eta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we denote by $\mathcal{P}^\eta_N$ the law of $m_N(t)$ in $N^{2/3}[\eta, \frac{\pi}{2}]$ given $m_N(\eta N^{2/3}) = m^0_N$. For $\mathcal{H}^\pm_\gamma(I)$, $I \subseteq \mathbb{R}$, as in (2.11), we prove the following result.

Theorem 7.1. *For any $\eta, \gamma > 0$ small enough and $\gamma' > \gamma > 0$, if $|m^0_N - X_+(0)| \le N^{-1/2+\gamma}$ then*

$$\lim_{N\to\infty} \mathcal{P}_N\left\{\mathcal{H}^+_{\gamma'}\left(N^{2/3}\left[-\frac{\pi}{2}, -\eta\right]\right)\right\} = 1. \tag{7.1}$$

For any $\eta, \gamma > 0$ small enough and $\gamma' > \gamma > 0$, if $|m^0_N - X_+(h_N(\eta N^{2/3}))| \le N^{-1/2+\gamma}$ then

$$\lim_{N\to\infty} \mathcal{P}^\eta_N\left\{\mathcal{H}^+_{\gamma'}\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right\} = 1. \tag{7.2}$$

For any $\eta \in [-\frac{\pi}{2}, \frac{\pi}{2})$, $\gamma' > \gamma$, if $|m^0_N - X_-(h_N(\eta N^{2/3}))| \le N^{-1/2+\gamma}$ then

$$\lim_{N\to\infty} \mathcal{P}^\eta_N\left\{\mathcal{H}^-_{\gamma'}\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right\} = 1. \tag{7.3}$$

Theorem 7.2 provides a connection between the critical and the stable regions.

Theorem 7.2. *There is $c > 0$ so that for any T large enough, $\gamma, \eta, \varepsilon > 0$*

$$\limsup_{N\to\infty} \mathcal{P}_N\left\{\left|Y_N(-T) - T\right| \ge \varepsilon \middle| \mathcal{H}^+_\gamma\left(\left\{-\eta N^{2/3}\right\}\right)\right\} \le \mathrm{e}^{-c\varepsilon^2 T} \tag{7.4}$$

and

$$\limsup_{N\to\infty} \mathcal{P}_N\left\{\left(\mathcal{H}^+_\gamma\left(\left\{\eta N^{2/3}\right\}\right)\right)^c \middle| \left|Y_N(T) - T\right| \le \varepsilon\right\} \le \mathrm{e}^{-c\varepsilon^2 T}. \tag{7.5}$$

For the proof of Theorems 7.1 and 7.2 see Section 2.5 in [8].

8. Conclusion of the proof of the main result

At this stage Theorem 2.3 is an almost direct consequence of Theorem 3.1, Propositions 3.5 and 3.6, that we are going to prove.

Proof of Theorem 3.1. Let us fix $\gamma, \varepsilon > 0$ small enough. Recalling that $\mathcal{P}_N$ is the law of $m(t)$ with $m(-\pi N^{2/3}/2) = m_N^0$, suppose $|m_N^0 - X_+(h_N(0))| \le N^{-1/2+\gamma}$, then, for any fixed $\eta > 0$,

$$\mathcal{P}_N\{|Y(-T) - T| > \varepsilon\} \le \mathcal{P}_N\{(\mathcal{H}_{2\gamma}^+(\{-\eta N^{2/3}\}))^c\} + \mathcal{P}_N\{|Y(-T) - T| > \varepsilon | \mathcal{H}_{2\gamma}^+(\{-\eta N^{2/3}\})\}$$

thus the result follows from (2.12) and (7.4). $\square$

Proof of Proposition 3.5. For $\mathcal{P}_{N,-T,y}^*$ and $\mathcal{P}_{-T,y}^*$ as defined in Section 4, $\mathcal{P}_{N,-T,y}\{Y_N \in \mathcal{E}_T^\pm\} = \mathcal{P}_{N,-T,y}^*\{Y_N \in \mathcal{E}_T^\pm\}$ and $\mathcal{P}_{-T,y}\{Y \in \mathcal{E}_T^\pm\} = \mathcal{P}_{-T,y}^*\{Y \in \mathcal{E}_T^\pm\}$, thus Proposition 3.5 follows directly from Proposition 4.3. $\square$

Proof of Proposition 3.6. For any $\eta, \gamma > 0$, $\mathcal{H}_\gamma^\pm(I)$, $I \subseteq \mathbb{R}$, as in (2.11), we have

$$\begin{aligned}
&\mathcal{P}_N\left\{\left(\mathcal{H}_\gamma^+\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right)^c \middle| Y_N \in \mathcal{E}_T^+\right\} \\
&\quad\le \mathcal{P}_N\left\{\left(\mathcal{H}_\gamma^+\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right)^c \middle| \mathcal{H}_{\gamma/2}^+(\{\eta N^{2/3}\})\right\} \\
&\qquad + \mathcal{P}_N\{(\mathcal{H}_{\gamma/2}^+(\{\eta N^{2/3}\}))^c || Y_N(T) - T| \le \varepsilon\}
\end{aligned}$$

then the plus case of (3.22) follows from (7.2) and (7.5). Analogously, for any $\eta, \gamma, \mu' > \mu$ independent of N, we have

$$\begin{aligned}
&\mathcal{P}_N\left\{\left(\mathcal{H}_\gamma^-\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right)^c \middle| Y_N \in \mathcal{E}_T^-\right\} \\
&\quad\le \mathcal{P}_N\left\{\left(\mathcal{H}_\gamma^-\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right)^c \middle| \mathcal{H}_{\gamma/2}^-(\{\mu' T N^{1/3}\})\right\} \\
&\qquad + \mathcal{P}_N\{(\mathcal{H}_{\gamma/2}^-(\{\mu' T N^{1/3}\}))^c | Y_N \in \mathcal{E}_T^-\}
\end{aligned}$$

thus the minus case of (3.22) follows from (6.2) and (7.3), since $\mu' T N^{-1/3} \ll \eta$ for large N. $\square$

Lemma 8.1. *We have*

$$\lim_{T\to\infty}\lim_{N\to\infty}\left|\mathcal{P}_N\{Y_N \in \mathcal{E}_T^\pm\} - \mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{N,-T,y}\{Y_N \in \mathcal{E}_T^\pm\}\right| = 0 \tag{8.1}$$

and

$$\lim_{T\to\infty}\lim_{N\to\infty}\left|\mathcal{P}_N\{Y_N \in \mathcal{E}_T^+ \cup \mathcal{E}_T^-\} - \mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{N,-T,y}\{Y_N \in \mathcal{E}_T^+ \cup \mathcal{E}_T^-\}\right| = 0. \tag{8.2}$$

Proof. We prove only (8.1). We show at first that, for any fixed y: $|y - T| \le \varepsilon$, $\varepsilon > 0$ small enough,

$$\lim_{T\to\infty}\lim_{N\to\infty}\left|\mathcal{P}_N\{Y_N \in \mathcal{E}_T^\pm | |Y_N(-T) - T| \le \varepsilon\} - \mathcal{P}_{N,-T,y}\{Y_N \in \mathcal{E}_T^\pm\}\right| = 0. \tag{8.3}$$

We have

$$\inf_{|y-T|\le\varepsilon}\mathcal{P}_{N,-T,y}\{\mathcal{E}_T^\pm\} \le \mathcal{P}_N\{\mathcal{E}_T^\pm | |Y(-T) - T| \le \varepsilon\} \le \sup_{|y-T|\le\varepsilon}\mathcal{P}_{N,-T,y}\{\mathcal{E}_T^\pm\}$$

thus, in order to prove (8.3), it is sufficient to show that, for any couple $y, \bar{y}$: $|y - T|, |\bar{y} - T| \le \varepsilon$,

$$\lim_{T\to\infty}\lim_{N\to\infty}\left|\mathcal{P}_{N,-T,y}\{\mathcal{E}_T^\pm\} - \mathcal{P}_{N,-T,\bar{y}}\{\mathcal{E}_T^\pm\}\right| = 0. \tag{8.4}$$

Equation (8.4) follows since, for $Y(t)$, $\bar{Y}(t)$ solutions of (5.1) starting at $-T$ respectively from y, $\bar{y}$, by Proposition 3.6 we have

$$\lim_{N\to\infty}\left|\mathcal{P}_{N,-T,y}\left\{Y_N \in \mathcal{E}_T^{\pm}\right\} - \mathcal{P}_{-T,y}\left\{Y \in \mathcal{E}_T^{\pm}\right\}\right| = 0 \tag{8.5}$$

and, by Proposition 5.16,

$$\lim_{T\to\infty}\left|\mathcal{P}_{-T,y}\left\{Y \in \mathcal{E}_T^{\pm}\right\} - \mathcal{P}_{-T,\bar{y}}\left\{\bar{Y} \in \mathcal{E}_T^{\pm}\right\}\right| = 0 \tag{8.6}$$

thus (8.4) follows from (8.5) and (8.6). We have, now

$$\left|\mathcal{P}_N\left\{Y_N \in \mathcal{E}_T^{\pm}\right\} - \mathcal{P}_N\left\{Y_N \in \mathcal{E}_T^{\pm} \middle| \left|Y_N(-T) - T\right| \le \varepsilon\right\}\right| \le 2\mathcal{P}_N\left\{\left|Y_N(-T) - T\right| > \varepsilon\right\}. \tag{8.7}$$

From Theorem 3.1 we know that the term in (8.7) is vanishingly small for large T, then (8.1) directly follows from (8.3). □

Conclusion of the proof of Theorem 2.3. We just need to prove (2.13) since the proof of (2.12) has been proved in Section 7 as a part of Theorem 7.1 (see (7.1)).

Let us suppose $|m_N^0 - X_+(0)| \le N^{1/2+\gamma}$. We have

$$\begin{aligned}
&\left|\mathcal{P}_N\left\{\mathcal{H}_\gamma^{\pm}\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right\} - \mathcal{P}_N\left\{Y_N \in \mathcal{E}_T^{\pm}\right\}\right| \\
&\quad \le \mathcal{P}_N\left\{\left(\mathcal{H}_\gamma^{\pm}\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right)^c \middle| Y_N \in \mathcal{E}_T^{\pm}\right\} \\
&\qquad + \mathcal{P}_N\left\{\left(\mathcal{H}_\gamma^{\mp}\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right)^c \middle| Y_N \in \mathcal{E}_T^{\mp}\right\} && (8.8)\\
&\qquad + \mathcal{P}_N\left\{Y_N \notin \mathcal{E}_T^{+} \cup \mathcal{E}_T^{-}\right\}. && (8.9)
\end{aligned}$$

From (3.16), (3.21) and (8.2) we have

$$\lim_{T\to\infty}\lim_{N\to\infty}\mathcal{P}_N\left\{Y_N \notin \mathcal{E}_T^{+} \cup \mathcal{E}_T^{-}\right\} = 0.$$

From Proposition 3.6 we know that the terms in (8.8) and (8.9) are vanishingly small for large T and N, thus, from (8.1) we have

$$\lim_{T\to\infty}\lim_{N\to\infty}\left|\mathcal{P}_N\left\{\mathcal{H}_\gamma^{\pm}\left(N^{2/3}\left[\eta, \frac{\pi}{2}\right]\right)\right\} - \mathbf{1}_{|y-T|\le\varepsilon}\mathcal{P}_{N,-T,y}\left\{Y_N \in \mathcal{E}_T^{\pm}\right\}\right| = 0. \tag{8.10}$$

Suppose $|y - T| \le \varepsilon$, $Y(t)$ as in Proposition 3.6, then

$$\left|\mathcal{P}_{N,-T,y}\left\{Y_N \in \mathcal{E}_T^{\pm}\right\} - p_\pm\right| \le \left|\mathcal{P}_{N,-T,y}\left\{Y_N \in \mathcal{E}_T^{\pm}\right\} - \mathcal{P}_{-T,y}\left\{Y \in \mathcal{E}_T^{\pm}\right\}\right| + \left|\mathcal{P}_{-T,y}\left\{Y \in \mathcal{E}_T^{\pm}\right\} - p_\pm\right| \tag{8.11}$$

then (2.13) follows from (3.19), (3.20), (8.10) and (8.11). □

Appendix

In this paper we mainly make use of techniques of comparison with Gaussian Processes. In this appendix we provide some Gaussian Inequalities and a comparison lemma.

Marcus–Shepp inequality for Gaussian processes

There is a classical result of Landau and Shepp [12] and Marcus and Shepp [14] that gives an estimate on the probability for a general centered Gaussian process of escaping from a large ball. If $G(t)$ is an a.s. bounded, centered Gaussian process of variance $\sigma^2(t)$, then

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^2}\ln\mathcal{P}\Big\{\sup_{t\in I}G(t)\geq\lambda\Big\}=-\frac{1}{2\sigma_I^2}\quad\text{with }\sigma_I^2:=\sup_{t\in I}\sigma^2(t).\tag{A.1}$$

An almost immediate consequence of (A.1) is that for any λ large enough, δ small enough,

$$\mathcal{P}\bigg\{\sup_t\frac{|G(t)|}{\sigma(t)}\geq\lambda\bigg\}\leq 2\mathrm{e}^{-\lambda^2/2(1-\delta)}.\tag{A.2}$$

Small deviations for Gaussian Markov processes

We give a result of Li (see [13]) dealing with the probability, for a Gaussian Markov process, of escaping from a small ball. Let $G(t)$ be a continuous centered Gaussian Markov process of covariance $\sigma(s,t)\neq 0$ for $t_0<s<t<t_1$. We can write $\sigma(s,t)=G(s)H(t)$ with $G,H>0$ and G/H non-decreasing on (t_0,t_1), then

$$\lim_{\varepsilon\to 0}\varepsilon^2\ln\mathcal{P}\Big\{\sup_{t_0<t\leq t_1}\big|G(t)\big|<\varepsilon\Big\}=-\frac{\pi^2}{8}\int_{t_0}^{t_1}\big(G'H-H'G\big)\,\mathrm{d}t.\tag{A.3}$$

We apply (A.3) to processes of the kind

$$G(t)=\int_{t_0}^{t}\mathrm{e}^{-\int_u^t a(s)\,\mathrm{d}s}\,\mathrm{d}w_u,\quad t_0\leq t\leq t_1,\tag{A.4}$$

we get

$$\lim_{\varepsilon\to 0}\varepsilon^2\log\mathcal{P}\Big\{\sup_{t_0<t\leq t_1}\big|G(t)\big|<\varepsilon\Big\}=-\frac{\pi^2}{8}\big(1-\mathrm{e}^{-\int_{t_0}^{t_1}a(s)\,\mathrm{d}s}\big).\tag{A.5}$$

Comparison with Gaussian processes

In the thesis we repeatedly make use of a comparison argument comparing the solution of a linear SDE with the solution of a more general SDE, let us see.

Let G_t be a solution of the problem

$$\mathrm{d}G_t=\big(a(t)G_t+b(t)\big)\,\mathrm{d}t+\xi\,\mathrm{d}w_t\tag{A.6}$$

with $a,b:\mathbb{R}^+\to\mathbb{R}$ bounded on bounded intervals and $\xi\in\mathbb{R}$, then $G(t)$ is a Gaussian process of the form

$$G(t)=G(t_0)\mathrm{e}^{\int_{t_0}^t a(s)\,\mathrm{d}s}+\int_{t_0}^t b(s)\mathrm{e}^{\int_s^t a(u)\,\mathrm{d}u}\,\mathrm{d}s+\xi\int_{t_0}^t\mathrm{e}^{\int_s^t a(u)\,\mathrm{d}u}\,\mathrm{d}w_s.$$

Consider, now, the processes $v(t)$ solution of

$$\mathrm{d}v_t=c(v_t,t)\,\mathrm{d}t+\xi\,\mathrm{d}w_t$$

with the same noise of (A.6), $c:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ globally Lipschitz.

Lemma A.1. *For $G(t), v(t)$ as above we define $\delta_t := c(G_t, t) - [a(t)G_t + b(t)]$, $\Delta_t := G_t - v_t$, and let $\tau \in \mathbb{R}^+$ be a generic random variable. Suppose*

$$\operatorname{sign}(\Delta_\tau) = \operatorname{sign}(\delta_\tau) \quad \textit{or} \quad \Delta_\tau = 0,$$

then

$$\operatorname{sign}(\Delta_t) = \operatorname{sign}(\delta_t) \quad \textit{for any } \tau \le t \le \inf\{s \ge \tau \colon \delta_s = 0\} \textit{ a.s.}$$

Proof. We have

$$\mathrm{d}\Delta_t = \bigl(a(t)\Delta_t + \delta_t\bigr)\,\mathrm{d}t$$

thus, for any $\tau \ge 0$

$$\Delta(t) = \Delta(\tau)\mathrm{e}^{\int_\tau^t a(s)\,\mathrm{d}s} + \int_\tau^t \delta(s)\mathrm{e}^{\int_s^t a(u)\,\mathrm{d}u}\,\mathrm{d}s$$

then follows the result. □

Acknowledgments

I am grateful to my advisor Errico Presutti for having suggested the problem and for relevant discussions. I acknowledge kind hospitality at the Department of Mathematics, University of Roma Tor Vergata. I finally thank the Department of Mathematics of the University of Modena and Reggio Emilia for partial financial support.

References

[1] M. Acharyya and B. K. Chakrabarti. Response of Ising systems to oscillating and pulsed fields: Hysteresis, ac, and pulse susceptibility. *Phys. Rev. B* **52** (1995) 6550–6568.
[2] N. Berglund and B. Gentz. Pathwise description of dynamic pitchfork bifurcations with additive noise. *Probab. Theory Related Fields* **122** (2002) 341–388. MR1892851
[3] N. Berglund and B. Gentz. A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential. *Ann. Appl. Probab.* **12** (2002) 1419–1470. MR1936599
[4] N. Berglund and B. Gentz. The effect of additive noise on dynamical hysteresis. *Nonlinearity* **15** (2002) 605–632. MR1901095
[5] G. Bertotti. *Hysteresis in Magnetism*. Academic Press, Boston, 1998.
[6] G. Bertotti and I. D. Mayergoyz. *The Science of Hysteresis, Mathematical Modeling and Applications*, Vol. I. Elsevier, Amsterdam, 2006. MR2307929
[7] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1999. MR1700749
[8] G. Carinci. Stochastic effects in critical regimes. Ph.D. thesis, Università degli Studi dell'Aquila, 2010.
[9] P. Jung, G. Gray, R. Roy and P. Mandel. Scaling law for dynamical hysteresis. *Phys. Rev. Lett.* **65** (1990) 1873–1876.
[10] G. Korniss, M. A. Novotny, P. A. Rikvold and C. J. White. Dynamic phase transition, universality, and finite-size scaling in the two-dimensional kinetic Ising model in an oscillating field. *Phys. Rev. E* **63** (2000) 016120.
[11] G. Korniss, M. A. Novotny and P. A. Rikvold. Absence of first-order transition and tricritical point in the dynamic phase diagram of a spatially extended bistable system in an oscillating field. *Phys. Rev. E* **66** (2002) 056127. MR1948573
[12] H. Landau and L. A. Shepp. On the supremum of a Gaussian process. *Sankhya A* **32** (1970) 369–378. MR0286167
[13] W. V. Li. Small deviations for Gaussian Markov processes under the sup-norm. *J. Theoret. Probab.* **12** (1999) 971–984. MR1729464
[14] M. B. Marcus and L. A. Shepp. Sample behaviour of Gaussian processes. In *Proceedings of the 6th Berkeley Symposium on Mathematics, Statistic and Probability*, Vol. 2 423–441. Univ. California Press, Berkeley, CA, 1972. MR0402896
[15] M. A. Novotny, P. A. Rikvold and S. W. Sides. Stochastic hysteresis and resonance in a kinetic Ising system. *Phys. Rev. E* **57** (1998) 6512–6533.
[16] M. A. Novotny, P. A. Rikvold and S. W. Sides. Kinetic Ising model in an oscillating field: Finite-size scaling at the dynamic phase transition. *Phys. Rev. Lett.* **81** (1998) 834–837.
[17] M. A. Novotny, P. A. Rikvold and S. W. Sides. Kinetic Ising model in an oscillating field: Avrami theory for the hysteretic response and finite-size scaling for the dynamic phase transition. *Phys. Rev. E* **59** (1999) 2710–2729.
[18] D. Pollard. *Convergence of Stochastic Processes*. Springer, New York, 1984. MR0762984
[19] E. Presutti. *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*. Springer, Berlin, 2009. MR2460018

[20] M. Rao, H. R. Krishnamurthy and R. Pandit. Magnetic hysteresis in two model spin systems. *Phys. Rev. B* **42-1** (1990) 856–884.
[21] T. Tomé and M. J. de Oliveira. Dynamic phase transition in the kinetic Ising model under a time-dependent oscillating field. *Phys. Rev. A* **41** (1990) 4251–4254.
[22] A. Visintin. *Differential Models of Hysteresis*. Springer, Berlin, 1994. MR1329094
[23] H. Zhu, S. Dong and J. M. Liu. Hysteresis loop area of the Ising model. *Phys. Rev. B* **70** (2004) 132403.