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Product of exponentials and spectral radius of random k-circulants

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Abstract. We consider $n \times n$ random k-circulant matrices with $n \to \infty$ and $k = k(n)$ whose input sequence $\{a_l\}_{l \geq 0}$ is independent and identically distributed (i.i.d.) random variables with finite $(2 + \delta)$ moment. We study the asymptotic distribution of the spectral radius, when $n = k^g + 1$. For this, we first derive the tail behaviour of the g fold product of i.i.d. exponential random variables. Then using this tail behaviour result and appropriate normal approximation techniques, we show that with appropriate scaling and centering, the asymptotic distribution of the spectral radius is Gumbel. We also identify the centering and scaling constants explicitly.

Résumé. Nous considérons des matrices aléatoires k-circulantes de taille $n \times n$ avec $n \to \infty$ et $k = k(n)$, dont les entrées $\{a_l\}_{l \geq 0}$ sont des variables aléatoires, de moment $(2 + \delta)$ fini, indépendantes et identiquement distribuées. Nous étudions la distribution asymptotique du rayon spectral, lorsque $n = k^g + 1$. Pour établir cette distribution asymptotique, nous calculons d'abord le comportement de la queue du produit de g variables aléatoires exponentielles i.i.d. Ensuite, en utilisant un résultat sur le comportement des queues et les techniques appropriées d'approximation normale, nous montrons que, après renormalisation et recentrage, la distribution limite est une distribution de Gumbel. Nous identifions explicitement les constantes de recentrage et de remise à l'échelle.

1. Introduction

Random matrices are matrices whose elements are random variables. In random matrix theory, one of the most important areas of research is the behaviour of the eigenvalues. In particular the spectral radius and spectral norm have been important objects of study.

The *spectral radius* sp(A) of any matrix A is defined as

$$\mathrm{sp}(A) := \max\{|\mu|\colon \mu \text{ is an eigenvalue of } A\},$$

where $|z|$ denotes the modulus of $z \in \mathbb{C}$.

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The *spectral norm* $\|A\|$ of any matrix A with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix A^*A:

$$\|A\| = \sqrt{\lambda_{\max}(A^*A)},$$

where A^* denotes the conjugate transpose of A. Therefore if A is an $n \times n$ normal matrix (e.g. if A is symmetric), with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$\|A\| = \mathrm{sp}(A) = \max_{1 \leq i \leq n} |\lambda_i|.$$

Here is a brief description of the existing results on these and related quantities for random matrices. Geman [20] considered an $n \times n$ random matrix $M_n = ((m_{ij}))$ where $\{m_{ij}\}$ are i.i.d. random variables with mean zero and showed that $\limsup_n \frac{1}{\sqrt{n}} \mathrm{sp}(M_n)$ is bounded almost surely under suitable assumptions. Later the same conclusion was obtained by Bai and Yin [4] under weaker assumptions. Silverstein [31] considered the case where the mean is nonzero and the fourth moment is finite, and showed that the spectral radius converges almost surely and also converges weakly to a normal distribution after proper scaling and centering.

Kostlan [22] gave a simple characterization of the moduli of the eigenvalues of M_n with complex Gaussian entries: the squared moduli of its eigenvalues are distributed as independent χ^2_{2i}, $1 \leq i \leq n$. Rider [27], using joint probability distribution of eigenvalues, showed that the spectral radius of M_n when the entries are i.i.d. complex Gausian with mean zero and variance $1/n$, converges to a Gumbel distribution with appropriate scaling and centering.

For work on spectral norm of (symmetric) Toeplitz matrices, see Bose and Sen [11], Meckes [25] and Adamczak [1]. Bryc, Dembo and Jiang [12] studied the almost sure convergence rate of spectral norm of Markov matrices.

Another related aspect to study is the behaviour of maximum and minimum eigenvalues. The literature in this area is extensive. Geman [19] proved that the largest eigenvalue of a sample covariance matrix converges almost surely under growth conditions on the moments of the underlying distribution. Yin, Bai and Krishnaiah [41] proved the same result under the finiteness of the fourth moment, and Bai, Silverstein and Yin [3] proved that the finitness of the fourth moment is also necessary for the existence of the limit. Bai and Yin [5] found necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of the Wigner matrix. Yin, Bai and Krishnaiah [40], Silverstein [29] and Bai and Yin [6] considered the almost sure limit of the smallest eigenvalue of a sample covariance matrix. Silverstein [30] found a necessary and sufficient condition for weak convergence of the largest eigenvalue of a sample covariance matrix to a nonrandom limit. The distributional convergence of the largest eigenvalue of Gaussian orthogonal, unitary and symplectic ensembles were studied by Tracy and Widom in a series of articles. See Tracy and Widom [37] for a brief survey of such results.

There are a few results in the literature for limiting behaviour of maximum eigenvalues of matrices with heavy tailed entries. See Soshnikov [32,33]; Auffinger, Ben Arous and Peche [2]. For results on spectral norm of circulant matrices and Toeplitz matrices with heavy tailed entries, see Bose, Hazra and Saha [8].

In general the eigenvalues of a random matrix are dependent and hence the behaviour of the spectral norm or the maximum eigenvalue is not apriori comparable to the behaviour of the maximum of i.i.d. random variables. However, for one class of patterned matrices the classical theory of extreme values of i.i.d. random variables can be applied after approximating the eigenvalues with i.i.d. random variables or their suitable functions. This is the class of k-circulant matrices for suitable values of k. In this article we study the limiting behaviour of spectral radius of a specific type of k-circulant matrices.

Suppose $\underline{a} = \{a_l\}_{l \geq 0}$ is a sequence of real numbers or random variables. For positive integers k and n, the k-circulant matrix with *input* sequence $\{a_l\}$ is defined as

$$A_{k,n}(\underline{a}) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-k} & a_{n-k+1} & \cdots & a_{n-k-1} \\ a_{n-2k} & a_{n-2k+1} & \cdots & a_{n-2k-1} \\ & & \vdots & \end{bmatrix}_{n \times n}.$$

We write $A_{k,n}(\underline{a}) = A_{k,n}$. The subscripts appearing in the matrix entries above are calculated modulo n. The $(j+1)$th row of $A_{k,n}$ is a right-circular shift of the jth row by k positions (equivalently, $k \bmod n$ positions). The value of k

may change with the increasing dimension of the matrix. Note that $A_{1,n}$ is the well-known circulant matrix (C_n) and $A_{n-1,n}$ is the *reverse circulant matrix* (RC_n). Without loss of generality, k may always be reduced modulo n.

These matrices and their block versions arise in many different areas of Mathematics and Statistics such as, multi-level supersaturated design of experiment (Georgiou and Koukouvinos [21], Chen and Liu [14]), spectra of De Bruijn graphs (Strok [35]) and, (0, 1)-matrix solutions to $A^m = J_n$ (Wu, Jia and Li [39]).

The initial advantage in dealing with this matrix is that a formula solution is known for its eigenvalues (see Zhou [42]). However, this formula depends heavily on the number theoretic relation between k and n. This makes it hard to study any type of spectral properties for arbitrary choice of these values.

The spectral norm and spectral radius of reverse circulant random matrices were considered in Bose, Hazra and Saha [7] and results on symmetric circulant (the usual circulant with symmetry imposed) matrices are available in Bose, Hazra and Saha [7] and Bryc and Sethuraman [13]. Bose, Mitra and Sen [9] considered the k-circulants with $n = k^2 + 1$ with i.i.d. entries having mean zero variance one and $E[|a_1|^{2+\delta}] < \infty$ for some $\delta > 0$ and showed that the spectral radius, appropriately scaled and centered, converges to the (standard) Gumbel distribution

$$\Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathbb{R}.$$

When the entries are i.i.d. Gaussian, this result is relatively easy to establish using some properties of the eigenvalues and the tail behaviour of $H_2(x) = P(E_1 E_2 > x)$ where E_1 and E_2 are i.i.d. standard exponential variables. When the entries are not necessarily Gaussian, appropriate normal approximation arguments were used.

We provide a significant extension of this result to the case $n = k^g + 1$ where g is any positive integer. In particular we show that as $n \to \infty$,

$$\frac{\text{sp}(n^{-1/2} A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda, \tag{1.1}$$

where $\{q = q(n), c_n, d_n\}$ are constants defined later in Theorem 4.

To prove (1.1), first assume that the entries are Gaussian. Once the result is established for Gaussian entries, the case of general entries is tackled by using appropriate normal approximation results. When the entries are Gaussian, we show that the result is easy to establish once we can derive the tail behaviour of

$$H_g(x) = P[E_1 E_2 \cdots E_g > x], \tag{1.2}$$

where $\{E_i\}$ are i.i.d. standard exponentials. The tail behaviour of $H_2(\cdot)$ is available in the literature. This has been obtained through different approaches. Springer and Thompson [34] and Lomnicki [24] used Mellin transform, Tang [36] used a clever substitution and Bose, Mitra and Sen [9] set up a second order differential equation for $H_2(\cdot)$ and observed that the solution is a modified Bessel function.

For $n > 2$, it is not hard to set up a higher order differential equation for $H_n(\cdot)$. However, this does not seem to provide the precise information that we need on the tail. We show in Theorem 1 that for any $n \geq 1$,

$$H_n(x) = C_n x^{\alpha_n} e^{-nx^{1/n}} g_n(x), \quad n \geq 1, \tag{1.3}$$

for some suitable sequence $\{\alpha_n, C_n\}$ and a sequence of functions $\{g_n(x) \to 1\}$ for every x. The proof is based on the so called Laplace asymptotics (see Erdélyi [17]) after an appropriate substitution similar to that used in Tang [36], in conjunction with induction on n. This result may be of independent interest. The tail behaviour of H_n when $\{E_i\}$ are not necessarily exponential is an open question. For information on the properties of product of general random variables, see Galambos and Simonelli [18].

The outline of the rest of the paper is as follows. In Theorem 1 of Section 2 we establish (1.3). This immediately implies that $(E_1 \cdots E_g)^{1/2g}$ belongs to the max domain of attraction of the Gumbel distribution. In Section 3 we give a brief description of the known results on spectral radius of circulant matrices and state our main theorem (Theorem 4) on the k-circulant.

In Sections 4.1–4.5 we develop the proof of the main result. In particular, we present the known formula for the eigenvalues of the k-circulant in Section 4.1 and more detailed properties for $n = k^g + 1$ in Section 4.2. Section 4.3 contains the properties of the eigenvalues when the input sequence is i.i.d. Gaussian. Section 4.4 has two preparatory lemmas on truncation and normal approximation. Section 4.5 has the final steps of the proof.

In Section 5.1, we remark about the case $sn = k^g + 1$. In Section 5.2, we deal with the case where the input sequence is an infinite order linear process and derive the limit for the maxima of the eigenvalues when scaled by the spectral density.

2. Tail of product and extreme values

It does not seem to be known when the product of i.i.d. random variables belongs to the max domain of attraction of the Gumbel distribution. However, for our purposes, we need only the exponential case that is given in the following theorem.

Theorem 1. *There exists constants $\{C_n, \alpha_n\}$ such that*

$$H_n(x) = C_n x^{\alpha_n} e^{-nx^{1/n}} g_n(x), \quad n \geq 1, \tag{2.1}$$

where for $n \geq 1$,

$$C_n = \frac{1}{\sqrt{n}} (2\pi)^{(n-1)/2}, \qquad \alpha_n = \frac{n-1}{2n} \quad and \quad g_n(x) \to 1 \quad as \ x \to \infty.$$

Proof. Since $H_1(x) = P[E_1 > x] = e^{-x}$, we have $C_1 = 1$, $\alpha_1 = 0$ and $g_1(x) = 1$ for all x. Now

$$
\begin{aligned}
H_2(x) &= \int_0^\infty e^{-y} e^{-x/y} \, dy \\
&= x^{1/2} \int_0^\infty e^{-x^{1/2}(t+1/t)} \, dt \quad \text{(substituting } y = tx^{1/2}) \\
&= x^{1/2} \int_0^\infty f(t) e^{-x^{1/2} g(t)} \, dt,
\end{aligned}
$$

where $f(t) = 1$ and $g(t) = t + \frac{1}{t}$. Note that g assumes a strict minimum at $t = 1$ and $f(1) = 1 \neq 0$. So applying Laplace asymptotics (see Section 2.4 of Erdélyi [17]) we have

$$H_2(x) = x^{1/2} e^{-x^{1/2} g(1)} f(1) \sqrt{\frac{2\pi}{x^{1/2} g''(1)}} g_2(x) = \sqrt{\pi} x^{1/4} e^{-2x^{1/2}} g_2(x),$$

where $g_2(x) \to 1$ as $x \to \infty$. Hence $C_2 = \sqrt{\pi} = \frac{1}{\sqrt{2}} (2\pi)^{1/2}$, $\alpha_k = 1$ and the result is true for $n = 2$. Now suppose (2.1) is true for $n = k$. We shall prove it for $n = k + 1$.

$$
\begin{aligned}
H_{k+1}(x) &= \int_0^\infty e^{-y} H_k\left(\frac{x}{y}\right) dy \\
&= C_k \int_0^\infty e^{-y} \left(\frac{x}{y}\right)^{\alpha_k} e^{-k(x/y)^{1/k}} g_k\left(\frac{x}{y}\right) dy \\
&= xkC_k \int_0^\infty e^{-(ks+x/s^k)} s^{k\alpha_k - k - 1} g_k(s^k) \, ds \quad \text{(substituting } x/y = s^k) \\
&= x^{(k\alpha_k+1)/(k+1)} kC_k \int_0^\infty e^{-(kt+1/t^k)x^{1/(k+1)}} t^{k\alpha_k - k - 1} g_k\left(t^k x^{k/(k+1)}\right) dt \quad \text{(substituting } s = x^{1/(k+1)} t) \\
&= x^{(k\alpha_k+1)/(k+1)} kC_k \int_0^\infty f(t) e^{-x^{1/(k+1)} g(t)} \, dt,
\end{aligned}
$$

where

$$f(t) = t^{k\alpha_k - k - 1} g_k\big(t^k x^{k/(k+1)}\big) \quad \text{and} \quad g(t) = kt + \frac{1}{t^k}.$$

Note that g assumes a strict minimum at $t = 1$ and $f(1) = g_k(x^{k/(k+1)}) \neq 0$, $g''(1) = k(k+1)$. Again applying Laplace asymptotics we have

$$H_{k+1}(x) = x^{(k\alpha_k+1)/(k+1)} kC_k e^{-x^{1/(k+1)} g(1)} f(1) \sqrt{\frac{2\pi}{x^{1/(k+1)} g''(1)}} h(x)$$

$$= x^{(k\alpha_k+1)(k+1)} kC_k e^{-(k+1)x^{1/(k+1)}} g_k\big(x^{k/(k+1)}\big) \sqrt{\frac{2\pi}{x^{1/(k+1)} k(k+1)}} h(x),$$

where $h(x) \to 1$ as $x \to \infty$. Substituting the values of α_k and C_k we get

$$H_{k+1}(x) = x^{k/(2(k+1))} \frac{1}{\sqrt{k+1}} (2\pi)^{k/2} e^{-(k+1)x^{1/(k+1)}} g_k\big(x^{k/(k+1)}\big) h(x)$$

$$= C_{k+1} x^{\alpha_{k+1}} e^{-(k+1)x^{1/(k+1)}} g_{k+1}(x),$$

where

$$\alpha_{k+1} = \frac{k}{2(k+1)}, \qquad C_{k+1} = \frac{1}{\sqrt{k+1}} (2\pi)^{k/2}, \qquad g_{k+1} = g_k\big(x^{k/(k+1)}\big) h(x)$$

and $g_{k+1}(x) \to 1$ as $x \to \infty$. Hence the result is true for $n = k + 1$ and this completes the proof. $\qquad\square$

The next two results shall be needed in the study of the spectral radius. The first is an easy consequence of Theorem 1 and standard calculations in extreme value theory as found in Rootzèn [28], Embrechts et al. [16] and Exercise 1.1.4 of Resnick [26]. We omit the details.

Lemma 1. *Let $\{X_n\}$ be a sequence of i.i.d. random variables where $X_i \stackrel{\mathcal{D}}{=} (E_1 E_2 \cdots E_k)^{1/2k}$ and $\{E_i\}_{1 \leq i \leq k}$ are i.i.d.* Exp(1) *random variables. Then*

$$\frac{\max_{1 \leq i \leq n} X_i - d_n}{c_n} \stackrel{\mathcal{D}}{\to} \Lambda,$$

where

$$c_n = \frac{1}{2k^{1/2}(\log n)^{1/2}}, \qquad d_n = \frac{\log C_k - ((k-1)/2)\log k}{2k^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{k}\right)^{1/2}\left[1 + \frac{(k-1)\log\log n}{4\log n}\right],$$

$$C_k = \frac{1}{\sqrt{k}} (2\pi)^{(k-1)/2}.$$

The next result is immediate from Lemma 1 and will be useful in the proof of Lemma 7.

Lemma 2. *Let $\{E_i\}$, c_n and d_n be as in Lemma 1. Let $\sigma_n^2 = n^{-c}$, $c > 0$. Then there exists some positive constant $K = K(x)$, such that for all large n we have*

$$P\big((E_1 E_2 \cdots E_k)^{1/2k} > \big(1 + \sigma_n^2\big)^{-1/2}(c_n x + d_n)\big) \leq \frac{K}{n}, \quad x \in \mathbb{R}.$$

3. Spectral radius of random k-circulant matrices

To put our main result (Theorem 4) in perspective, we now briefly describe the known results on various circulant type matrices when the entries do not have heavy tails. The following result can be derived following the argument given for symmetric Toeplitz matrix in Bose and Sen [11].

Theorem 2. *Consider the reverse circulant matrix RC_n with the input sequence $\{a_i\}$ which is i.i.d. with $\mathrm{E}(a_0) = \mu$ and $\mathrm{Var}(a_0) = 1$. Let $RC_n^0 = RC_n - \mu n u_n u_n^T$ with $u_n = (1, \dots, 1)^T$. If $\mu > 0$, then*

$$\frac{\|RC_n\|}{n} \to \mu \quad \text{almost surely} \quad \text{and} \quad \left\| \frac{RC_n^0}{\|RC_n\|} \right\| \to 0 \quad \text{almost surely}.$$

Similar results hold for circulant matrix (C_n) also.

Theorem 3 (Bose, Hazra and Saha [7]). *Consider the reverse circulant matrix RC_n and the circulant matrix C_n both with the input sequence $\{a_i\}$ which is i.i.d. with mean μ and $\mathrm{E}[|a_i|^{2+\delta}] < \infty$ for some $\delta > 0$.*

(a) *If $\mu \neq 0$ then,*

$$\frac{\|RC_n\| - |\mu|n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, 1).$$

(b) *If $\mu = 0$ then,*

$$\frac{\|(1/\sqrt{n})RC_n\| - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where

$$q = q(n) = \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad d_q = \sqrt{\ln q} \quad \text{and} \quad c_q = \frac{1}{2\sqrt{\ln q}}.$$

The above conclusions continue to hold for C_n also.

We now state the following significant generalisation of a result of Bose, Mitra and Sen [9] who proved the result for the case $g = 2$. See Sections 5.1 and 5.2 for extensions to the case $sn = k^g + 1$ and to the case of dependent entries.

Theorem 4. *Suppose $\{a_l\}_{l \geq 0}$ is an i.i.d. sequence of random variables with mean zero and variance 1 and $\mathrm{E}[|a_l|^\gamma] < \infty$ for some $\gamma > 2$. If $n = k^g + 1$ for some fixed positive integer g, then as $n \to \infty$,*

$$\frac{\mathrm{sp}(n^{-1/2}A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where $q = q(n) = \frac{n}{2g}$ and the normalizing constants c_n and d_n can be taken as follows

$$c_n = \frac{1}{2g^{1/2}(\log n)^{1/2}}, \qquad d_n = \frac{\log C_g - ((g-1)/2)\log g}{2g^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{g}\right)^{1/2}\left[1 + \frac{(g-1)\log\log n}{4\log n}\right],$$

$$C_g = \frac{1}{\sqrt{g}}(2\pi)^{(g-1)/2}.$$

Remark. *Bose and Sen [11] showed that the* limiting spectral distribution *(LSD) of the symmetric circulant and the reverse circulant exist if $\mathrm{E}(a_1^2) < \infty$. Bose, Mitra and Sen [10] showed that the LSD of the k-circulant exists for suitable values of k under the slightly stronger assumption $\mathrm{E}[|a_1|^{2+\delta}] < \infty$. This assumption helps to use normal approximation for non-Gaussian entries.*

For i.i.d. variables, it is well known that the finiteness of any moment is not necessary for its maximum to have a limit distribution. However, in our case, due to the approximation of the non-Gaussian case by the Gaussian case, we are compelled to use a moment condition. It is an open question if Theorem 4 remains true under weaker conditions and in particular if convergence to some extreme value distribution holds solely under appropriate tail behaviour of a_1.

4. Proof of Theorem 4

The proof is long and is developed in the following sections. In Section 4.1 we describe the eigenvalues of a general k-circulant matrix and in Section 4.2 we provide more detailed description of the eigenvalues of the k-circulant matrix for $n = k^g + 1$. In Section 4.3 we discuss some distributional properties of the eigenvalues of k-circulant matrix when the input sequence is i.i.d. Gaussian. Section 4.4 has two preparatory lemmas on truncation and normal approximation. Drawing on the developments of Sections 2, 4.2 and 4.4, we finish the proof in Section 4.5.

4.1. *Description of eigenvalues of a k-circulant*

The formula solution by Zhou [42], given below in Theorem 5, for the eigenvalues of a k-circulant is our starting point. A proof is also provided in Bose, Mitra and Sen [10]. Let

$$\omega = \omega_n := \cos(2\pi/n) + \mathrm{i}\sin(2\pi/n), \qquad \mathrm{i}^2 = -1 \quad \text{and} \quad \lambda_t = \sum_{l=0}^{n-1} a_l \omega^{tl}, \quad 0 \leq t < n. \tag{4.1}$$

Note that $\{\lambda_t, 0 \leq t < n\}$ are eigenvalues of the usual circulant matrix $A_{1,n}$. Let $p_1 < p_2 < \cdots < p_c$ be all the common prime factors of n and k. Then we may write,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}. \tag{4.2}$$

Here $\alpha_q, \beta_q \geq 1$ and n', k', p_q are pairwise relatively prime. For any positive integer m, let

$$\mathbb{Z}_m = \{0, 1, 2, \ldots, m - 1\}.$$

We introduce the following family of sets

$$S(x) := \left\{ xk^b \bmod n' \colon b \geq 0 \right\}, \quad x \in \mathbb{Z}_{n'}. \tag{4.3}$$

Note that $x \in S(x)$ for every x. Suppose $S(x) \cap S(y) \neq \varnothing$. Then, $xk^{b_1} = yk^{b_2} \bmod n'$ for some integers $b_1, b_2 \geq 1$. Multiplying both sides by $k^{g_x - b_1}$ we see that, $x \in S(y)$ so that, $S(x) \subseteq S(y)$. Hence, reversing the roles, $S(x) = S(y)$. Thus, the distinct sets in $\{S(x)\}_{x \in \mathbb{Z}_{n'}}$ forms a partition, called the *eigenvalue partition*, of $\mathbb{Z}_{n'}$. Denote the partitioning sets and their sizes by

$$\mathcal{P}_0 = \{0\}, \qquad \mathcal{P}_1, \ldots, \mathcal{P}_{\ell-1} \quad \text{and} \quad k_j = \#\mathcal{P}_j, \quad 0 \leq j < \ell. \tag{4.4}$$

Define

$$\Pi_j := \prod_{t \in \mathcal{P}_j} \lambda_{tn/n'}, \quad j = 0, 1, \ldots, \ell - 1. \tag{4.5}$$

Theorem 5 (Zhou [42]). *The characteristic polynomial of $A_{k,n}$ is given by*

$$\chi(A_{k,n})(\lambda) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} \left(\lambda^{k_j} - \Pi_j \right). \tag{4.6}$$

4.2. *Additional description of eigenvalues when $n = k^g + 1$*

Let $g_x = \#S(x)$. We call g_x the *order* of x. Note that $g_0 = 1$. It is easy to see that

$$g_x = \min\{b > 0 \colon b \text{ is an integer and } xk^b = x \bmod n'\} \tag{4.7}$$

and

$$S(x) = \{xk^b \bmod n' \colon 0 \le b < g_x\}.$$

Define

$$J_k := \{\mathcal{P}_i \colon \#\mathcal{P}_i = k\}, \qquad n_k := \#J_k, \qquad X(k) := \{x \colon x \in \mathbb{Z}_n \text{ and } x \text{ has order } k\},$$

$$v_{k,n} := \frac{1}{n}\#\{x \colon x \in \mathbb{Z}_n \text{ and } g_x < g_1\}. \tag{4.8}$$

Lemma 3. *The eigenvalues $\{\eta_i\}$ of the k-circulant with $n = k^g + 1$, $g \ge 2$, satisfy the following*:

(a) $\eta_0 = \sum_{t=0}^{n-1} a_t$, *is always an eigenvalue and if n is even, then $\eta_{n/2} = \sum_{t=0}^{n-1}(-1)^t a_t$, is also an eigenvalue and both have multiplicity one.*

(b) *For $x \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$, $g_x = g_1$ or $\frac{g_1}{b}$ for some $b \ge 2$ and $\frac{g_1}{2b}$ is an integer.*

(c) *For all large n, $g_1 = 2g$. Hence from (b), for $x \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$, $g_x = 2g$ or $\frac{2g}{b}$. The total number of eigenvalues corresponding to J_{2g} is*

$$2g \times \#J_{2g} = \#X(2g) \sim n.$$

(d) *$X(\frac{2g}{b}) = \varnothing$ for $2 \le b < g$, b even. If g is even then $X(\frac{2g}{g}) = X(2)$ is either empty or contains exactly two elements with eigenvalues*

$$\eta_l = |\lambda_l|, \qquad \eta_{n-l} = -|\lambda_l| \quad \text{for some } 1 \le l \le \frac{n}{2}.$$

(e) *Suppose b is odd, $3 \le b \le g$ and $\frac{g}{b}$ is an integer. For each $\mathcal{P}_j \in J_{2g/b}$ there are $\frac{2g}{b}$ eigenvalues given by the $\frac{2g}{b}$th roots of Π_j. Total number of eigenvalues corresponding to the set $J_{2g/b}$ is*

$$\frac{2g}{b} \times \#J_{2g/b} = \#X\left(\frac{2g}{b}\right) \sim \left(k^{g/b} + 1\right)\left(1 + n^{-a}\right) \quad \text{for some } a > 0.$$

There are no other eigenvalues.

Proof. Since $n = k^g + 1$, n and k are relatively prime and hence $n' = n$.

(a) $\mathcal{P}_0 = S(0) = \{0\}$ and the corresponding eigenvalue is $\eta_0 = \sum_{t=0}^{n-1} a_t$ with multiplicity one. Similarly if n is even then k is odd and hence $S(n/2) = \{\frac{n}{2}\}$, and the corresponding eigenvalue is $\eta_{n/2} = \sum_{t=0}^{n-1}(-1)^t a_t$ of multiplicity one.

(b) From (4.7) it is easy to see that g_x divides g_1 and hence $g_x = g_1$ or $g_x = \frac{g_1}{b}$ for some $b \ge 2$. Also for every integer $t \ge 0$, $tk^g = (-1 + n)t = -t \bmod n$. Hence λ_t and λ_{n-t} belong to same partition block $S(t) = S(n - t)$. Thus each $S(t)$ contains even number of elements, except for $t = 0, \frac{n}{2}$. Hence $\frac{g_1}{b}$ must be even, that is, $\frac{g_1}{2b}$ must be an integer.

(c) From Lemma 5(i) of Bose, Mitra and Sen [10], $g_1 = 2g$ for all but finitely many n and $v_{k,n} \to 0$ as $n \to \infty$. For each $\mathcal{P}_j \in J_{2g}$ we have $2g$ many eigenvalues and which are $2g$th roots of Π_j. Now the result follows from the fact that

$$n = 2g\#J_{2g} + nv_{k,n}.$$

(d) Suppose $b = 2$ and $x \in X(\frac{g_1}{2}) = X(\frac{2g}{2})$. Then $xk^{g_1/2} = xk^g = x \bmod n$. But $k^g = -1 \bmod n$ and so, $xk^g = -x \bmod n$. Therefore $2x = 0 \bmod n$ and x can be either 0 or $n/2$. But we have already seen in part (a) that $g_0 = g_{n/2} = 1$. Hence $X(\frac{2g}{2}) = \varnothing$.

Now suppose $b > 2$, even. From Lemma 3(ii) of Bose, Mitra and Sen [10], $\#X(\frac{2g}{b}) \le \gcd(k^{2g/b} - 1, k^g + 1)$ for $b \ge 3$. Now observe that for b even,

$$\gcd(k^{2g/b} - 1, k^g + 1) = \begin{cases} 1 & \text{if } k \text{ even,} \\ 2 & \text{if } k \text{ odd.} \end{cases}$$

So we have $\#X(\frac{2g}{b}) \le 2$ for $b > 2$ and b even.

Suppose if possible, there exist $x \in \mathbb{Z}_n$ such that $g_x = \frac{2g}{b}$. Then $\#S(x) = \frac{2g}{b}$ and for all $y \in S(x)$, $g_y = \frac{2g}{b}$. Hence

$$\#\left\{ y: g_y = \frac{2g}{b} \right\} \ge \frac{2g}{b} > 2 \quad \text{for } g > b > 2, b \text{ even.}$$

This contradicts the fact that $\#X(\frac{2g}{b}) \le 2$ for $g > b > 2$, b even. Hence $X(\frac{2g}{b}) = \varnothing$ for b even and $g > b > 2$.

If $b = g$ and it is even, then from previous discussion $\#X(\frac{2g}{g}) = 0$ or 2. In the later case there are exactly two elements in $\mathbb{Z}_n$ whose order is 2 and there will be only one partitioning set containing them. So corresponding eigenvalues will be

$$\eta_l = |\lambda_l|, \qquad \eta_{n-l} = -|\lambda_l| \quad \text{for some } 1 \le l \le \frac{n}{2}.$$

(e) We first show that for b odd,

$$\left(k^{g/b} + 1\right) - \sum_{\substack{b_i > b, b_i \text{ odd,} \\ g/b_i \text{ integer}}} \left(k^{g/b_i} + 1\right) \le \#X\left(\frac{2g}{b}\right) \le k^{g/b} + 1.$$

Note that (e) is a simple consequence of this. Let

$$Z_{n,b} = \left\{ x: x \in \mathbb{Z}_n \text{ and } xk^{2g/b} = x \bmod \left(k^g + 1\right) \right\}.$$

Then it is easy to see that

$$X\left(\frac{2g}{b}\right) \subseteq Z_{n,b}. \tag{4.9}$$

Let $x \in Z_{n,b}$ and $\frac{g}{b} = m$. Then

$$k^g + 1 \mid x\left(k^{2g/b} - 1\right)$$
$$\Rightarrow \quad k^{bm} + 1 \mid x\left(k^{2m} - 1\right)$$
$$\Rightarrow \quad k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1 \mid x\left(k^m - 1\right).$$

But $\gcd(k^m - 1, k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1) = 1$, and therefore x is a multiple of $(k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1)$. Hence

$$\#Z_{n,b} = \left\lfloor \frac{k^{bm} + 1}{(k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1)} \right\rfloor = k^m + 1 = k^{g/b} + 1$$

and combining with (4.9),

$$\#X\left(\frac{2g}{b}\right) \le \#Z_{n,b} = k^{g/b} + 1.$$

On the other hand, if $x \in Z_{n,b}$ then either $g_x = \frac{2g}{b}$ or $g_x < \frac{2g}{b}$. For the second case $g_x = \frac{2g}{b_i}$ for some $b_i > b$, b_i odd and therefore $x \in Z_{n,b_i}$. Hence

$$\#X\left(\frac{2g}{b}\right) \geq \#Z_{n,b} - \sum_{\substack{b_i > b, b_i \text{ odd}, \\ g/b_i \text{ integer}}} \#Z_{n,b_i} \geq \left(k^{g/b} + 1\right) - \sum_{\substack{b_i > b, b_i \text{ odd}, \\ g/b_i \text{ integer}}} \left(k^{g/b_i} + 1\right).$$

$\square$

4.3. *Properties of eigenvalues of Gaussian circulant matrices*

Suppose $\{a_l\}_{l \geq 0}$ are independent, mean zero and variance one random variables. Fix n. For $1 \leq t < n$, let us split λ_t into real and complex parts as $\lambda_t = a_{t,n} + ib_{t,n}$, that is,

$$a_{t,n} = \sum_{l=0}^{n-1} a_l \cos\left(\frac{2\pi tl}{n}\right), \qquad b_{t,n} = \sum_{l=0}^{n-1} a_l \sin\left(\frac{2\pi tl}{n}\right). \tag{4.10}$$

For $z \in \mathbb{C}$, $\bar{z}$ denotes its complex conjugate. For all $0 < t, t' < n$, the following identities can easily be verified using the orthogonality relations of sine and cosine functions.

$$\mathrm{E}(a_{t,n} b_{t,n}) = 0 \quad \text{and} \quad \mathrm{E}\left(a_{t,n}^2\right) = \mathrm{E}\left(b_{t,n}^2\right) = n/2,$$

$$\bar{\lambda}_t = \lambda_{n-t}, \qquad \mathrm{E}(\lambda_t \lambda_{t'}) = n\mathbb{I}(t + t' = n), \qquad \mathrm{E}\left(|\lambda_t|^2\right) = n.$$

The following lemma is due to Bose, Mitra and Sen [10].

Lemma 4. *Fix k and n. Suppose that $\{a_l\}_{0 \leq l < n}$ are i.i.d. standard normal random variables.*

(a) *For every n, $n^{-1/2} a_{t,n}, n^{-1/2} b_{t,n}$, $0 \leq t \leq n/2$, are i.i.d. normal with mean zero and variance $1/2$. Consequently, any subcollection $\{\Pi_{j_1}, \Pi_{j_2}, \ldots\}$ of $\{\Pi_j\}_{0 \leq j < \ell}$, so that no member of the corresponding partition blocks $\{\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots\}$ is a conjugate of any other, are mutually independent.*

(b) *Suppose $1 \leq j < \ell$ and $\mathcal{P}_j = n - \mathcal{P}_j$ and $n/2 \notin \mathcal{P}_j$. Then $n^{-n_j/2} \Pi_j$ are distributed as $(n_j/2)$-fold product of i.i.d. exponential random variables with mean one.*

4.4. *Truncation and normal approximation lemmata*

4.4.1. *Truncation*
From Section 4.2, $n = n'$ and $S(t) = S(n - t)$ except for $t = 0, n/2$. So for $\mathcal{P}_j \neq S(0), S(n/2)$, we can define $\mathcal{A}_j$ such that

$$\mathcal{P}_j = \{x: x \in \mathcal{A}_j \text{ or } n - x \in \mathcal{A}_j\} \quad \text{and} \quad \#\mathcal{A}_j = \frac{1}{2}\#\mathcal{P}_j. \tag{4.11}$$

For any sequence of random variables $b = \{b_l\}_{l \geq 0}$, define for $\mathcal{P}_k \in J_{2j}$

$$\beta_{b,j}(k) = \prod_{t \in \mathcal{A}_j} \left|\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} b_l \omega^{tl}\right|^2, \quad \text{where } \omega = \exp\left(\frac{2\pi i}{n}\right). \tag{4.12}$$

For each $n \geq 1$, define a triangular array of centered random variables $\{\bar{a}_l^{(n)}\}_{0 \leq l < n}$ by

$$\bar{a}_l = \bar{a}_l^{(n)} = a_l I_{|a_l| \leq n^{1/\gamma}} - \mathrm{E} a_l I_{|a_l| \leq n^{1/\gamma}}.$$

Lemma 5. *Assume $\mathrm{E}[|a_l|^\gamma] < \infty$ for some $\gamma > 2$. Then, almost surely,*

$$\max_{1 \leq j \leq q} \left(\beta_{a,g}(j)\right)^{1/2g} - \max_{1 \leq j \leq q} \left(\beta_{\bar{a},g}(j)\right)^{1/2g} = o(1).$$

Proof. Since $\sum_{l=0}^{n-1} \omega^{tl} = 0$ for $0 < t < n$, it follows that $\beta_{\bar{a},n}(j) = \beta_{\tilde{a},n}(j)$ where

$$\tilde{a}_l = \tilde{a}_l^{(n)} = \bar{a}_l + \mathrm{E}a_l I_{|a_l| \le n^{1/\gamma}} = a_l I_{|a_l| \le n^{1/\gamma}}.$$

By Borel–Cantelli lemma, $\sum_{t=0}^{\infty} |a_l| I_{|a_l|>l^{1/\gamma}}$ is finite a.s. and has only finitely many nonzero terms. Thus there exists a positive integer $N(\omega)$ such that

$$\sum_{t=0}^{n} |a_t - \tilde{a}_t| = \sum_{t=0}^{n} |a_t| I_{|a_t|>n^{1/\gamma}} \le \sum_{t=0}^{\infty} |a_t| I_{|a_t|>t^{1/\gamma}} = \sum_{t=0}^{N(\omega)} |a_t| I_{|a_t|>t^{1/\gamma}}. \tag{4.13}$$

It follows that for $n \ge \{N(\omega), |a_1|^{\gamma}, \ldots, |a_{N(\omega)}|^{\gamma}\}$ the left-hand side of (4.13) is zero. Consequently, for all n sufficiently large,

$$\beta_{a,g}(j) = \beta_{\bar{a},g}(j) \quad \text{a.s. for all } j \tag{4.14}$$

and the assertion follows immediately.   □

4.4.2. *Normal approximation*

For $d \ge 1$, and any distinct integers $i_1, i_2, \ldots, i_d$, from $\{1, 2, \ldots, \lceil \frac{n-1}{2} \rceil\}$, define

$$v_{2d}(l) = \left(\cos\left(\frac{2\pi i_j l}{n}\right), \sin\left(\frac{2\pi i_j l}{n}\right) : 1 \le j \le d \right)^T, \quad l \in \mathbb{Z}_n.$$

Let $\varphi_{\Sigma}(\cdot)$ denote the density of the $2d$-dimensional Gaussian vector having mean zero and covariance matrix Σ and let I_{2d} be the identity matrix of order $2d$.

Lemma 6 (Davis and Mikosch [15]). *Fix $d \ge 1$, $\gamma > 2$ and let $\tilde{p}_n$ be the density function of*

$$2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_{2d}(l),$$

where $\{N_l\}_{l \ge 0}$ is a sequence of i.i.d. $N(0,1)$ random variables, independent of $\{a_l\}_{l \ge 0}$ and $\sigma_n^2 = \mathrm{Var}(\bar{a}_0)s_n^2$. If $n^{-2c} \ln n \le s_n^2 \le 1$ with $c = 1/2 - (1-\delta)/\gamma$ for arbitrarily small $\delta > 0$, then

$$\tilde{p}_n(x) = \varphi_{(1+\sigma_n^2) I_{2d}}(x)(1 + \varepsilon_n) \quad \text{with } \varepsilon_n \to 0$$

holds uniformly for $\|x\|^3 = \mathrm{o}_d(n^{1/2-1/\gamma})$, $x \in \mathbb{R}^{2d}$.

Corollary 1. *Let $\gamma > 2$ and $\sigma_n^2 = n^{-c}$ where c is as in Lemma 6. Then for an measurable $B \subseteq \mathbb{R}^{2d}$,*

$$\left| \int_B \tilde{p}_n(x)\, dx - \int_B \varphi_{(1+\sigma_n^2) I_{2d}}(x)\, dx \right| \le \varepsilon_n \int_B \varphi_{(1+\sigma_n^2) I_{2d}}(x)\, dx + \mathrm{O}_d\big(\exp(-n^{\eta})\big)$$

for some $\eta > 0$ and uniformly over all the d-tuples of distinct integers $1 \le i_1 < i_2 < \cdots < i_d \le \lceil \frac{n-1}{2} \rceil$.

4.5. *Proof of Theorem 4: Concluding arguments*

To complete the proof we use the following lemmata whose proofs are given in the next section. Recall that $\{\beta_{x,g}(t)^{1/2g}\}$ are the eigenvalues corresponding to the set of partitions having cardinality $2g$. We derive the behaviour of their maximum in Lemma 7. Lemma 8 is technical and helps to conclude that the maximum of the remaining eigenvalues is negligible compared to the above.

Lemma 7.

$$\frac{\max_{1 \leq t \leq q} \beta_{a,g}(t)^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$ (4.15)

where d_q, c_q *are as in Lemma* 1, $q = q_n = \frac{n}{2g} - k_n$ *and* $\frac{k_n}{n} \to 0$ *as* $n \to \infty$. *As a consequence,*

$$\frac{\max_{1 \leq t \leq q} \beta_{a,g}(t)^{1/2g} - d_{n/2g}}{c_{n/2g}} \xrightarrow{\mathcal{D}} \Lambda.$$ (4.16)

Let

$$c_n(l) = \frac{1}{2l^{1/2}(\log n)^{1/2}}, \qquad d_n(l) = \frac{\log C_l - (l-1)/2 \log l}{2l^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{l}\right)^{1/2}\left[1 + \frac{(l-1)\log\log n}{4 \log n}\right],$$

$$C_l = \frac{1}{\sqrt{l}}(2\pi)^{(l-1)/2},$$

$$c_{n_{2j}} = c_{n_{2j}}(j), \qquad d_{n_{2j}} = d_{n_{2j}}(j), \qquad c_{n/2g} = c_{n/2g}(g) \quad \text{and} \quad d_{n/2g} = d_{n/2g}(g).$$

Lemma 8. *Let* $n = k^g + 1$ *If* $j < g$ *and for some* $a > 0$, $2jn_{2j} = (k^j + 1)(1 + n^{-a}) \sim n^{j/g}$ *or is finite, then there exists a constant* $K = K(j, g) \geq 0$ *such that,*

$$\frac{c_{n/2g}}{c_{n_{2j}}} \to K \quad \text{and} \quad \frac{d_{n/2g} - d_{n_{2j}}}{c_{n_{2j}}} \to \infty \quad \text{as } n \to \infty.$$

Proof of Theorem 4. If $\#\mathcal{P}_i = j$, then the eigenvalues corresponding to $\mathcal{P}_i$'s are the jth roots of Π_i and hence these eigenvalues have the same modulus. From Lemma 3, the possible values of $\#\mathcal{P}_i$ are $\{1, 2, 2g$ and $2g/b, 3 \leq b < g, b$ odd, $\frac{g}{b} \in \mathbb{Z}\}$. Recall from (4.12) that $\beta_{a,j}(k)$ is the modulus of the eigenvalue associated with the partition set $\mathcal{P}_k$, where $\#\mathcal{P}_k = 2j$.

In case of Gaussian entries it easily follows that $\beta_{a,j}(k)$ is the product of j exponential random variables and they are independent as k takes n_{2j} many distinct values. So from Lemma 1, if $n_{2j} \to \infty$ then the maximum of $\beta_{a,j}(k)^{1/2j}$ has a Gumbel limit. For more general entries the method as in the proof of Lemma 7 can be adopted to get the following limit:

$$\max_{1 \leq k \leq n_{2j}} \frac{\beta_{a,j}(k)^{1/2j} - d_{n_{2j}}}{c_{n_{2j}}} \xrightarrow{\mathcal{D}} \Lambda \quad \text{as } n_{2j} \to \infty,$$ (4.17)

where $c_{n_{2j}}$ and $d_{n_{2j}}$ are as above.

Let $x_n = c_n x + d_n$, $q = q_n = \frac{n}{2g}$ and $\mathcal{B} = \{b: b$ odd, $3 \leq b < g, \frac{g}{b} \in \mathbb{Z}\}$. Then

$$P\big(\mathrm{sp}\big(n^{-1/2}A_{k,n}\big) > x_q\big) \geq P\Big(\max_{j:\mathcal{P}_j \in J_{2g}} \beta_{a,g}(j)^{1/2g} > x_q\Big)$$

and

$$P\big(\mathrm{sp}\big(n^{-1/2}A_{k,n}\big) > x_q\big) \leq P\Big(\max_{j:\mathcal{P}_j \in J_{2g}} \beta_{a,g}(j)^{1/2g} > x_q\Big) + \sum_{b \in \mathcal{B}} P\Big(\max_{j:\mathcal{P}_j \in J_{2g/b}} \beta_{a,g/b}(j)^{b/2g} > x_q\Big)$$

$$+ P\left(\left|n^{-1/2}\sum_{l=0}^{n-1} a_l\right| > x_q\right) + P\left(\left|n^{-1/2}\sum_{l=0}^{n-1}(-1)^l a_l\right| > x_q\right)$$

$$+ P\Big(\max_{j:\mathcal{P}_j \in J_2} \beta_{a,2}(j)^{1/2} > x_q\Big)$$

$$=: A + B + C + D + E.$$

From Lemma 3, the term D appears only when $\frac{n}{2} \in \mathbb{Z}$ and the term E appears only if g is even and in that case J_2 contains only one element. It is easy to see that C, D and E tend to zero since we are taking maximum of single element.

Note that B is the sum of finitely many terms. Now suppose for $b \in \mathcal{B}$, we have some finite K_b such that

$$\frac{c_{n/2g}}{c_{n_{2g/b}}} \to K_b \quad \text{and} \quad \frac{d_{n/2g} - d_{n_{2g/b}}}{c_{n_{2g/b}}} \to \infty \quad \text{as } n \to \infty. \tag{4.18}$$

Then from observations (4.17) and (4.18) we get that the probability in B goes to zero. So it remains to check that whether (4.18) holds for $b \in \mathcal{B}$. But (4.18) holds from Lemmas 3(e) and 8.

Now the limit in A follows from Lemma 7, proving the result. $\qquad\square$

4.6. *Proofs of Lemmas 7 and 8*

Proof of Lemma 7. First assume that $\{a_l\}_{l \geq 0}$ are i.i.d. standard normal. Let $\{E_j\}_{j \geq 1}$ be i.i.d. standard exponentials. By Lemma 4, it easily follows that

$$\mathrm{P}\Big(\max_{1 \leq t \leq q} \big(\beta_{a,g}(t)\big)^{1/2g} > c_q x + d_q \Big) = \mathrm{P}\big((E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj})^{1/2g} > c_q x + d_q \text{ for some } 1 \leq j \leq q \big).$$

The lemma then follows in this special case from Corollary 1.

For the general case we break the proof into the following three steps and make use of the two results from Section 4.4. We shall prove the three steps later. Fix $x \in \mathbb{R}$.

Step 1. $\lim_{n \to \infty}[Q_1^{(n)} - Q_2^{(n)}] = 0$, *where*

$$Q_1^{(n)} := \mathrm{P}\Big(\max_{1 \leq j \leq q} \big(\beta_{\bar{a}+\sigma_n N, g}(j)\big)^{1/2g} > c_q x + d_q \Big),$$

$$Q_2^{(n)} := \mathrm{P}\Big(\max_{1 \leq j \leq q} \big(1 + \sigma_n^2\big)(E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj})^{1/2g} > c_q x + d_q \Big)$$

and $\{N_l\}_{l \geq 0}$ *is a sequence of i.i.d. standard normal random variables.*

Step 2.

$$\frac{\max_{1 \leq j \leq q} (\beta_{\bar{a}+\sigma_n N, g}(j))^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

Step 3.

$$\frac{\max_{1 \leq t \leq q} \beta_{\bar{a},g}(t)^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

Now combining Lemma 5 and Step 3, we can conclude

$$\frac{\max_{1 \leq t \leq q} \beta_{\bar{a},g}(t)^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

This completes the proof of first part, (4.15) of the lemma. By convergence of type theorem, the second part, (4.16) follows since the following hold. We omit the tedious algebraic details.

$$\frac{c_q}{c_{n/2g}} \to 1 \quad \text{and} \quad \frac{d_q - d_{n/2g}}{c_q} \to 0 \quad \text{as } n \to \infty. \tag{4.19}$$

Proof of Step 1. We approximate $Q_1^{(n)}$ by the simpler quantity $Q_2^{(n)}$ using Bonferroni's inequality. By Bonferroni's inequality, for all $m \geq 1$,

$$\sum_{j=1}^{2m}(-1)^{j-1}S_{j,n} \leq Q_1^{(n)} \leq \sum_{j=1}^{2m-1}(-1)^{j-1}S_{j,n}, \tag{4.20}$$

where

$$S_{j,n} = \sum_{1 \leq t_1 < t_2 < \cdots < t_j \leq q} P\big((\beta_{\bar{a}+\sigma_n N,g}(t_i))^{1/2g} > c_q x + d_q, i = 1, \ldots, j\big).$$

Similarly, we have

$$\sum_{j=1}^{2m}(-1)^{j-1}T_{j,n} \leq Q_2^{(n)} \leq \sum_{j=1}^{2m-1}(-1)^{j-1}T_{j,n}, \tag{4.21}$$

where

$$T_{j,n} = \sum_{1 \leq t_1 < t_2 < \cdots < t_j \leq q} P\big((1+\sigma_n^2)(E_{g(t_i-1)+1}E_{g(t_i-1)+2}\cdots E_{gt_i})^{1/2g} > c_q x + d_q, i = 1, \ldots, j\big).$$

Therefore, the difference between $Q_1^{(n)}$ and $Q_2^{(n)}$ can be bounded as follows:

$$\sum_{j=1}^{2m}(-1)^{j-1}(S_{j,n}-T_{j,n}) - T_{2m+1,n} \leq Q_1^{(n)} - Q_2^{(n)} \leq \sum_{j=1}^{2m-1}(-1)^{j-1}(S_{j,n}-T_{j,n}) + T_{2m,n} \tag{4.22}$$

for each $m \geq 1$. By independence and Lemma 2, there exists $K = K(x)$ such that

$$T_{j,n} \leq \binom{n}{j}\frac{K^j}{n^j} \leq \frac{K^j}{j!} \quad \text{for all } n, j \geq 1. \tag{4.23}$$

Consequently, $\lim_{j \to \infty} \limsup_n T_{j,n} = 0$.

Now fix $j \geq 1$. Let us bound the difference between $S_{j,n}$ and $T_{j,n}$. Let $\mathcal{A}_t$ defined in (4.11) be represented as $\mathcal{A}_t = \{e_t^1, e_t^2, \ldots, e_t^g\}$. Also note $e_t^1, e_t^2, \ldots, e_t^g \in \{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor\}$. For $1 \leq t_1 < t_2 < \cdots < t_j \leq q$, define

$$v_{2gj}(l) = \left(\cos\left(\frac{2\pi l e_{t_k}^1}{n}\right), \sin\left(\frac{2\pi l e_{t_k}^1}{n}\right), \cos\left(\frac{2\pi l e_{t_k}^2}{n}\right), \ldots, \sin\left(\frac{2\pi l e_{t_k}^g}{n}\right); 1 \leq k \leq j\right).$$

Note that $\{e_{t_k}^1, \ldots, e_{t_k}^g : 1 \leq k \leq j\}$ is a set of distinct integers in $\{1, 2, \ldots, \lfloor\frac{n}{2}\rfloor\}$. Then,

$$P\big((\beta_{\bar{a}+\sigma_n N,g}(t_i))^{1/2g} > c_q x + d_q, i = 1, \ldots, j\big) = P\left(2^{1/2}n^{-1/2}\sum_{l=0}^{n-1}(\bar{a}_l + \sigma_n N_l)v_{2gj}(l) \in B_n^{(j)}\right),$$

where

$$B_n^{(j)} := \left\{y \in \mathbb{R}^{2gj}: \prod_{l=1}^{g}(y_{2gt+2l-1}^2 + y_{2gt+2l}^2)^{1/2g} > 2^{1/2}(c_q x + d_q); 0 \leq t < j\right\}.$$

By Corollary 1 and the fact $N_1^2 + N_2^2 \overset{\mathcal{D}}{=} 2E_1$, we deduce that uniformly over all the d-tuples $1 \leq t_1 < t_2 < \cdots < t_j \leq q$,

$$\left|P\left(2^{1/2}n^{-1/2}\sum_{l=0}^{n-1}(\bar{a}_l + \sigma_n N_l)v_{2gj}(l) \in B_n^{(j)}\right) - P\left((1+\sigma_n^2)^{1/2}\left(\prod_{i=1}^{g}E_{g(t_m-1)+i}\right)^{1/2g} > c_q x + d_q, 1 \leq m \leq j\right)\right|$$

$$\leq \varepsilon_n P\big((1+\sigma_n^2)^{1/2}(E_{g(t_m-1)+1}E_{g(t_m-1)+2}\cdots E_{gt_m})^{1/2g} > c_q x + d_q, 1 \leq m \leq j\big) + O\big(\exp(-n^\eta)\big).$$

Therefore, as $n \to \infty$,

$$|S_{j,n} - T_{j,n}| \leq \varepsilon_n T_{j,n} + \binom{n}{j} \mathrm{O}\big(\exp(-n^\eta)\big) \leq \varepsilon_n \frac{K^j}{j!} + \mathrm{o}(1) \to 0, \tag{4.24}$$

where $\mathrm{O}(\cdot)$ and $\mathrm{o}(\cdot)$ are uniform over j. Hence using (4.20), (4.21), (4.23) and (4.24), we have

$$\limsup_n \big| Q_1^{(n)} - Q_2^{(n)} \big| \leq \limsup_n T_{2m+1,n} + \limsup_n T_{2m,n} \quad \text{for each } m \geq 1.$$

Letting $m \to \infty$, we conclude $\lim_{n\to\infty}[Q_1^{(n)} - Q_2^{(n)}] = 0$. This completes the proof of Step 1. □

Proof of Step 2. Since by Corollary 1,

$$\max_{1 \leq j \leq q} (E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj})^{1/2g} = \mathrm{O}_p\big((\ln n)^{1/2}\big) \quad \text{and} \quad \sigma_n^2 = n^{-c},$$

it follows that

$$\frac{(1 + \sigma_n^2)^{1/2} \max_{1 \leq j \leq q} (E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj})^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

and consequently,

$$\frac{\max_{1 \leq j \leq q} (\beta_{\bar{a}+\sigma_n N, g}(j))^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

This completes the proof of Step 2. □

Proof of Step 3. In view of Step 2, it suffices to show that

$$\max_{1 \leq j \leq q} \big(\beta_{\bar{a}+\sigma_n N, g}(j)\big)^{1/2g} - \max_{1 \leq j \leq q} \big(\beta_{\bar{a}, g}(j)\big)^{1/2g} = \mathrm{o}_p(c_q).$$

Note that

$$\beta_{\bar{a}+\sigma_n N, g}(j) = \prod_{k=1}^{g} \left| \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) \omega^{le_j^k} \right|^2 = \prod_{k=1}^{g} |\alpha_{j,k}|^2, \quad \text{say,}$$

and

$$\beta_{\bar{a}, g}(j) = \prod_{k=1}^{g} \left| \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \bar{a}_l \omega^{le_j^k} \right|^2 = \prod_{k=1}^{g} |\gamma_{j,k}|^2, \quad \text{say.}$$

Now by the inequality

$$\left| \prod_{i=1}^{g} a_i - \prod_{i=1}^{g} b_i \right| \leq \sum_{j=1}^{g} \left(\prod_{i=1}^{j-1} b_i \right) |a_j - b_j| \left(\prod_{i=j+1}^{g} a_i \right) \tag{4.25}$$

for nonnegative numbers $\{a_i\}$ and $\{b_i\}$, we have

$$\big| \beta_{\bar{a}+\sigma_n N, g}(j) - \beta_{\bar{a}, g}(j) \big| \leq \sum_{k=1}^{g} |\gamma_{j,1}|^2 \cdots |\gamma_{j,k-1}|^2 \big| |\alpha_{j,k}|^2 - |\gamma_{j,k}|^2 \big| |\alpha_{j,k+1}|^2 \cdots |\alpha_{j,g}|^2.$$

For any sequence of random variables $\{X_n\}_{n \geq 0}$, define

$$M_n(X) := \max_{1 \leq t \leq n} \left| n^{-1/2} \sum_{l=0}^{n-1} X_l \omega^{tl} \right|.$$

As a trivial consequence of Theorem 2.1 of Davis and Mikosch [15], we have

$$M_n^2(\sigma_n N) = O_p(\sigma_n \ln n) \quad \text{and} \quad M_n^2(\bar{a} + \sigma_n N) = O_p(\ln n).$$

Therefore $|\alpha_{j,k}| = O_p(\sqrt{\ln n})$. Now,

$$|\gamma_{j,k}| \leq |\alpha_{j,k}| + \sigma_n \left| \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} N_l \omega^{le_j^k} \right|$$

and therefore $|\gamma_{j,k}| = (1 + \sigma_n) O_p(\sqrt{\ln n}) = O_p(\sqrt{\ln n})$. So we have

$$
\begin{aligned}
\left| \max_{1 \leq j \leq q} \beta_{\bar{a} + \sigma_n N, g}(j) - \max_{1 \leq j \leq q} \beta_{\bar{a}, g}(j) \right| &\leq \max_{1 \leq j \leq q} \left| \beta_{\bar{a} + \sigma_n N, g}(j) - \beta_{\bar{a}, g}(j) \right| \\
&\leq \max_{1 \leq j \leq q} \sum_{k=1}^{g} \left(O_p(\ln n) \right)^{g-1} |\alpha_{j,k} - \gamma_{j,k}| \left(|\alpha_{j,k}| + |\gamma_{j,k}| \right) \\
&\leq O_p(\ln n)^{g-1} O_p(\sqrt{\ln n}) \max_{1 \leq j \leq q} \sum_{k=1}^{g} |\alpha_{j,k} - \gamma_{j,k}| \\
&\leq O_p(\ln n)^{g-1/2} g \sigma_n M_n(N) = o_p\left(n^{-c/4} (\ln n)^g \right).
\end{aligned}
$$

Hence

$$\left| \max_{1 \leq j \leq q} \left(\beta_{\bar{a} + \sigma_n N, g}(j) \right)^{1/2g} - \max_{1 \leq j \leq q} \left(\beta_{\bar{a}, g}(j) \right)^{1/2g} \right| \leq \left| \max_{1 \leq j \leq q} \beta_{\bar{a} + \sigma_n N, g}(j) - \max_{1 \leq j \leq q} \beta_{\bar{a}, g}(j) \right| \frac{1}{\xi^{1/2g}},$$

where ξ lies between $\max_{1 \leq j \leq q} \beta_{\bar{a} + \sigma_n N, g}(j)$ and $\max_{1 \leq j \leq q} \beta_{\bar{a}, g}(j)$. We know that

$$\frac{\max_{1 \leq j \leq q} \beta_{\bar{a} + \sigma_n N, g}(j)}{(\ln n)^g} \xrightarrow{\mathcal{P}} 1$$

and

$$\frac{|\max_{1 \leq j \leq q} \beta_{\bar{a} + \sigma_n N, g}(j) - \max_{1 \leq j \leq q} \beta_{\bar{a}, g}(j)|}{(\ln n)^g} \xrightarrow{\mathcal{P}} 0.$$

Therefore

$$\frac{\max_{1 \leq j \leq q} \beta_{\bar{a}, g}}{(\ln n)^g} \xrightarrow{\mathcal{P}} 1.$$

Hence

$$\frac{\xi}{(\ln n)^g} \xrightarrow{\mathcal{P}} 1 \quad \Rightarrow \quad \frac{\xi^{1-1/2g}}{(\ln n)^{g(1-1/2g)}} \xrightarrow{\mathcal{P}} 1 \quad \Rightarrow \quad \frac{1}{\xi^{1-1/2g}} = O_p\left((\ln n)^{1/2-g} \right).$$

Combining all these we have

$$\left| \max_{1 \leq j \leq q} \beta_{\bar{a} + \sigma_n N, g}(j)^{1/2g} - \max_{1 \leq j \leq q} \beta_{\bar{a}, g}(j)^{1/2g} \right| \leq o_p\left(n^{-c/4} (\ln n)^g \right) + O_p\left((\ln n)^{1/2-g} \right) \leq o_p(c_q).$$

This completes the proof of Step 3 and hence completes the proof of Lemma 7.   □

Proof of Lemma 8. First observe that if n_j is finite then the result holds trivially. If $n_{2j} = \frac{(k^j+1)(1+n^{-a})}{2j}$ then

$$\log n_{2j} = j \log k + \left(\frac{1}{n^a} + \frac{1}{n^{j/g}} \right)(1 + o(1)) - \log 2j$$

for some $a > 0$ and since $k = (n-1)^{1/g}$ we have

$$\frac{c_{n/2g}}{c_{n_{2j}}} \to \frac{j}{g} \quad \text{as } n \to \infty.$$

Similarly we get for some $a_0 > 0$,

$$\log \log n_{2j} = \log \log n^{j/g} + \left(\frac{1}{n^{a_0} \log n} \right)(1 + o(1)) - \log 2j.$$

Now observe that $\frac{d_{n/2g} - d_{n_{2j}}}{c_{n_{2j}}}$ can be broken into the following three parts say $J_i, i = 1, 2$ or 3.

$$J_1 = 2j^{1/2}(\log n_{2j})^{1/2} \left[\frac{\log C_g - (g-1)/2 \log g}{2g^{1/2}(\log n/2g)^{1/2}} - \frac{\log C_j - (j-1)/2 \log j}{2j^{1/2}(\log n_{2j})^{1/2}} \right] \to m_1 \quad \text{(some finite constant),}$$

$$J_2 = 2j^{1/2}(\log n_{2j})^{1/2} \left[\left(\frac{\log n/2g}{g} \right)^{1/2} - \left(\frac{\log n_{2j}}{j} \right)^{1/2} \right] \to m_2 \quad \text{(some finite constant),}$$

$$\begin{aligned}
J_3 &= 2j^{1/2}(\log n_{2j})^{1/2} \left[\frac{(g-1) \log \log n/2g}{4(g \log n/2g)^{1/2}} - \frac{(j-1) \log \log n_{2j}}{4(j \log n_{2j})^{1/2}} \right] \\
&= 2j^{1/2}(\log n_{2j})^{1/2} \left[\frac{(g-1) \log \log n/2g}{4(g \log n/2g)^{1/2}} - \frac{(j-1)\sqrt{g} \log \log n_{2j}}{4j(\log n/2g)^{1/2}} + o(1) \right] \\
&= \frac{j^{1/2}(\log n_{2j})^{1/2}}{2(g \log n/2g)^{1/2}} \left[(g-1) \log \log n/2g - \frac{(j-1)g}{j} \log \log n_{2j} + o(1) \right] \\
&= \frac{j^{1/2}(\log n_{2j})^{1/2}}{2(g \log n/2g)^{1/2}} \left[\left((g-1) - \frac{g(j-1)}{j} \right) \log \log n/2g + o(1) \right] \to \infty \quad \text{(since } g > j).
\end{aligned}$$

This completes the proof of Lemma 8.   □

5. Extensions of Theorem 4

5.1. k-circulants with $sn = k^g + 1$

Bose, Mitra and Sen [10] showed the existence of the limiting spectral distribution of the k-circulant matrix with $k^g = sn - 1$ assuming that $s = o(n^{p_1-1})$ where p_1 is the smallest prime factor of g. To derive the limit of the spectral radius, we need a slightly stronger assumption that $s = o(n^{p_1-1-\epsilon})$ for some $0 < \epsilon < p_1$ and $s > 1$. This is essential since $s = o(n^{p_1-1})$ implies $v_{k,n} \to 0$ which is not enough to deal with the maximum. We need the stronger result $v_{k,n} = o(n^{-a_1})$ for some $a_1 > 0$, so that these terms are negligible in the log scale that we have. Note that with the above conditions $s = o(n^{p_1-1})$ and $v_{k,n} = O(n^{-\epsilon/p_1})$.

Since $s > 1$ it easy to see from Lemma 3 in Bose, Mitra and Sen [10] that

$$\#X \left(\frac{2g}{b} \right) \leq \gcd \left(k^{2g/b} - 1, \frac{k^g + 1}{s} \right) \leq \gcd \left(k^{2g/b} - 1, k^g + 1 \right). \tag{5.1}$$

Also observe that,

$$\#\left\{x \colon x \in \mathbb{Z}_n \text{ and } x k^{2g/b} = x \bmod \left(\frac{k^g + 1}{s}\right)\right\} \geq \#Z_{n,b}. \tag{5.2}$$

From observations (5.1) and (5.2) it easily follows that Lemma 3(d) remains valid in this case. Further, for some $\alpha > 0$ we get that

$$1 \geq \frac{\#X(2g/b)}{k^{g/b} + 1} \geq 1 - k^{-g\alpha}\big(1 + \mathrm{o}(1)\big) = 1 - (sn)^{-\alpha}\big(1 + \mathrm{o}(1)\big) \geq 1 - n^{-\alpha}\big(1 + \mathrm{o}(1)\big).$$

Hence from the above discussions we have the following theorem.

Theorem 6. *Suppose $\{a_l\}_{l \geq 0}$ is an i.i.d. sequence of random variables with mean zero and variance 1 and $\mathrm{E}|a_l|^\gamma < \infty$ for some $\gamma > 2$. If $s \geq 1$ and $sn = k^g + 1$ where $s = \mathrm{o}(n^{p_1 - 1 - \epsilon})$, $0 < \epsilon < p_1$, and p_1 is the smallest prime factor of g, then as $n \to \infty$,*

$$\frac{\mathrm{sp}(n^{-1/2} A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda,$$

where $q = q(n) = \frac{n}{2g}$ and c_n and d_n are as defined in Theorem 4.

5.2. *k-circulant with dependent input*

Now let $\{a_n; n \geq 0\}$ be a two sided moving average process,

$$a_n = \sum_{i=-\infty}^{\infty} x_i \epsilon_{n-i}, \tag{5.3}$$

where $\{x_n, n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |x_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. with $\mathrm{E}(\epsilon_i) = 0$ and $\mathrm{E}(\epsilon_i^2) = 1$. Let $f(\omega)$, $\omega \in [0, 2\pi]$ be the spectral density of $\{a_n\}$. Note that if $\{a_n\}$ is i.i.d. with mean 0 and variance σ^2, then $f \equiv \frac{\sigma^2}{2\pi}$.

In this case, the variance of each eigenvalue is actually of the order of the spectral density at the corresponding ordinate. Thus it is meaningful to rescale by the spectral density. This is, for example, the approach taken by Walker [38], Davis and Mikosch [15], Lin and Liu [23] while studying the periodogram. This rescaling by the spectral density makes them approximately same variance and that makes it relatively easy to handle their maxima. Define,

$$\widetilde{\beta}_{a,j}(t) := \frac{\beta_{a,j}(t)}{\prod_{l \in \mathcal{A}_t} 2\pi f(\omega_l)} \quad \text{and} \quad \mathrm{M}\big(n^{-1/2} A_{k,n}, f\big) = \max_l \max_{j : \mathcal{P}_j \in J_l} \big(\widetilde{\beta}_{a,l}(j)\big)^{1/2l}.$$

Theorem 7. *Let $\{a_n\}$ be the two sided moving average process (5.3) where $\mathrm{E}(\epsilon_i) = 0$, $\mathrm{E}(\epsilon_i^2) = 1$, $\mathrm{E}|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and*

$$\sum_{j=-\infty}^{\infty} |x_j| |j|^{1/2} < \infty \quad \text{and} \quad f(\omega) > \alpha > 0 \quad \text{for all } \omega \in [0, 2\pi]. \tag{5.4}$$

Then as $n \to \infty$, with $q = q(n) = \frac{n}{2g}$ and c_q, d_q as defined in Theorem 4,

$$\frac{\mathrm{M}(n^{-1/2} A_{k,n}, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

Proof. Since we shall be using the bounds given in Walker [38] we define a few relevant notation for convenience. Define

$$I_{a,n}(\omega_j) = \frac{1}{n}\left|\sum_{l=1}^{n} a_l e^{i\omega_j l}\right|^2, \qquad I_{\epsilon,n}(\omega_j) = \frac{1}{n}\left|\sum_{l=1}^{n} \epsilon_l e^{i\omega_j l}\right|^2,$$

$$X(\omega_j) = \sum_{t=-\infty}^{\infty} x_t e^{i\omega_j t}, \qquad T_n(\omega_j) = I_{a,n}(\omega_j) - \left|X(\omega_j)\right|^2 I_{\epsilon,n}(\omega_j).$$

To prove the result we use following facts:

(i) From Walker [38], p. 112,

$$\max_{1 \le t \le n} \left|T_n(\omega_t)\right| = O_p\big(n^{-\delta}(\log n)^{1/2}\big).$$

(ii) From Davis and Mikosch [15],

$$\max_{1 \le t \le n} \left|I_{\epsilon,n}(\omega_t)\right| = O_p(\log n) \quad \text{and} \quad \max_{1 \le t \le n} \left|I_{a,n}(\omega_t)\right| = O_p(\log n).$$

Using these and inequality (4.25), it is easy to see that, for some $\delta_0 > 0$

$$\max_l \max_{j:\mathcal{P}_j \in J_l} \left|\widetilde{\beta}_{a,l}(t) - \beta_{\epsilon,l}(t)\right| = o_p\big(n^{-\delta_0}\big). \tag{5.5}$$

Now the results follows from Theorem 4 and (5.5). □

Acknowledgements

References

[1] R. Adamczak. A few remarks on the operator norm of random Toeplitz matrices. *J. Theoret. Probab.* **23** (2008) 85–108. MR2591905

[2] A. Auffinger, G. Ben Arous and S. Peche. Poisson convergence for the largest eigenvalues of heavy tailed random matrices. *Ann. Inst. H. Poincaré Probab. Statist.* **45** (2009) 859–610. MR2548495

[3] Z. D. Bai, J. Silverstein and Y. Q. Yin. A note on the largest eigenvalue of a large dimensional sample covariance matrix. *J. Multivariate Anal.* **26** (1988) 166–168. MR0963829

[4] Z. D. Bai and Y. Q. Yin. Limiting behavior of the norm of products of random matrices and two problems of Geman–Hwang. *Probab. Theory Related Fields* **73** (1986) 555–569. MR0863545

[5] Z. D. Bai and Y. Q. Yin. Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. *Ann. Probab.* **16** (1988) 1729–1741. MR0958213

[6] Z. D. Bai and Y. Q. Yin. Limit of the smallest eigenvalue of a large dimensional sample covariance matrix. *Ann. Probab.* **21** (1993) 1275–1294. MR1235416

[7] A. Bose, R. S. Hazra and K. Saha. Spectral norm of circulant type matrices. *J. Theoret. Probab.* **24** (2011) 479–516.

[8] A. Bose, R. S. Hazra and K. Saha. Spectral norm of circulant type matrices with heavy tailed entries. *Electron. Comm. Probab.* **15** (2010) 299–313. MR2670197

[9] A. Bose, J. Mitra and A. Sen. Large dimensional random *k* circulants. Technical Report R10/2008, Stat-Math Unit, Indian Statistical Institute, Kolkata, 2008. Available at www.isical.ac.in/~statmath.

[10] A. Bose, J. Mitra and A. Sen. Large dimensional random *k* circulants. *J. Theoret. Probab.* (2012). To appear. DOI:10.1007/s10959-010-0312-9.

[11] A. Bose and A. Sen. Spectral norm of random large dimensional noncentral Toeplitz and Hankel matrices. *Electron. Comm. Probab.* **12** (2007) 29–35. MR2284045

[12] W. Bryc, A. Dembo and T. Jiang. Spectral measure of large random Hankel, Markov and Toeplitz matrices. *Ann. Probab.* **34** (2006) 1–38. MR2206341

[13] W. Bryc and S. Sethuraman. A remark on the maximum eigenvalue for circulant matrices. In *High Dimensional Probabilities V: The Luminy Volume* 179–184. *IMS Collections* **5**. Institute of Mathematical Statistics, Beachwood, OH, 2009.

[14] J. Chen and M.-Q. Liu. Optimal mixed-level *k*-circulant supersaturated designs. *J. Statist. Plann. Inference* **138** (2008) 4151–4157. MR2455995

[15] R. A. Davis and T. Mikosch. The maximum of the periodogram of a non-Gaussian sequence. *Ann. Probab.* **27** (1999) 522–536. MR1681157

[16] P. Embrechts, C. Kluppelberg and T. Mikosch. *Modelling Extremal Events in Insurance and Finance*. Springer, Berlin, 1997. MR1458613

[17] A. Erdélyi. *Asymptotics Expansions*. Dover, New York, 1956. MR0078494

[18] J. Galambos and I. Simonelli. *Products of Random Variables: Applications to Problems of Physics and to Arithmetical Functions*. Marcel Dekker, New York, 2004. MR2077249

[19] S. Geman. A limit theorem for the norm of random matrices. *Ann. Probab.* **8** (1980) 252–261. MR0566592

[20] S. Geman. The spectral radius of large random matrices. *Ann. Probab.* **14** (1986) 1318–1328. MR0866352

[21] S. Georgiou and C. Koukouvinos. Multi-level *k*-circulant supersaturated designs. *Metrika* **64** (2006) 209–220. MR2259223

[22] E. Kostlan. On the spectra of Gaussian matrices. *Linear Algebra Appl.* **162/164** (1992) 385–388. MR1148410

[23] Z. Lin and W. Liu. On maxima of periodograms of stationary processes. *Ann. Statist.* **37** (2009) 2676–2695. MR2541443

[24] Z. A. Lomnicki. On the distribution of products of random variables. *J. Roy. Statist. Soc. Ser. B* **29** (1967) 513–524. MR0222928

[25] M. W. Meckes. On the spectral norm of a random Toeplitz matrix. *Electron. Comm. Probab.* **12** (2007) 315–325. MR2342710

[26] S. Resnick. *Extreme Values, Regular Variation and Point Processes. Applied Probability: A Series of the Applied Probability Trust* **4**. Springer, New York, 1987. MR0900810

[27] B. Rider. A limit theorem at the edge of a non-Hermitian random matrix ensemble. *J. Phys. A* **36** (2003) 3401–3409. MR1986426

[28] H. Rootzèn. Extreme value theory for moving average processes. *Ann. Probab.* **14** (1986) 612–652. MR0832027

[29] J. W. Silverstein. The smallest eigenvalue of a large dimensional Wishart matrix. *Ann. Probab.* **13** (1985) 1364–1368. MR0806232

[30] J. W. Silverstein. On the weak limit of the largest eigenvalue of a large dimensional sample covariance matrix. *J. Multivariate Anal.* **30** (1989) 307–311. MR1015375

[31] J. W. Silverstein. The spectral radii and norms of large-dimensional non-central random matrices. *Comm. Statist. Stochastic Models* **10** (1994) 525–532. MR1284550

[32] A. Soshnikov. Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails. *Electron. Comm. Probab.* **9** (2004) 82–91. MR2081462

[33] A. Soshnikov. Poisson statistics for the largest eigenvalues in random matrix ensembles. In *Mathematical Physics of Quantum Mechanics* 351–364. *Lecture Notes in Phys.* **690**. Springer, Berlin, 2006. MR2234922

[34] M. D. Springer and W. E. Thompson. The distribution of products of Beta, Gamma and Gaussian random variables. *SIAM J. Appl. Math.* **18** (1970) 721–737. MR0264733

[35] V. V. Strok. Circulant matrices and the spectra of de Bruijn graphs. *Ukrainian Math. J.* **44** (1992) 1446–1454. MR1213901

[36] Q. Tang. From light tails to heavy tails through multiplier. *Extremes* **11** (2008) 379–391. MR2453346

[37] C. A. Tracy and H. Widom. The distribution of the largest eigenvalue in the Gaussian ensembles: $\beta = 1, 2, 4$. In *Calogero–Moser–Sutherland Models* 461–472. Springer, New York, 2000. MR1844228

[38] A. M. Walker. Some asymtotic results for the periodogram of a stationary time series. *J. Austral. Math. Soc.* **5** (1965) 107–128. MR0177457

[39] Y. K. Wu, R. Z. Jia and Q. Li. *g*-circulant solutions to the $(0, 1)$ matrix equation $A^m = J_n$. *Linear Algebra Appl.* **345** (2002) 195–224. MR1883273

[40] Y. Q. Yin, Z. D. Bai and P. R. Krishnaiah. Limiting behavior of the eigenvalues of a multivariate *F* matrix. *J. Multivariate Anal.* **13** (1983) 508–516. MR0727036

[41] Y. Q. Yin, Z. D. Bai and P. R. Krishnaiah. On the limit of the largest eigenvalue of the large dimensional sample covariance matrix. *Probab. Theory Related Fields* **78** (1988) 509–521. MR0950344

[42] J. T. Zhou. A formula solution for the eigenvalues of *g* circulant matrices. *Math. Appl. (Wuhan)* **9** (1996) 53–57. MR1384201