# Strong solutions for stochastic differential equations with jumps 

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Received 6 October 2009; revised 3 August 2010; accepted 25 August 2010


#### Abstract

General stochastic equations with jumps are studied. We provide criteria for the uniqueness and existence of strong solutions under non-Lipschitz conditions of Yamada-Watanabe type. The results are applied to stochastic equations driven by spectrally positive Lévy processes.


Résumé. Nous étudions des équations stochastiques générales avec sauts et proposons un critère qui garantit l'existence et l'unicité de solutions fortes sous des conditions de régularité de type Yamada-Watanabe. Les résultats sont appliqués à des équations stochastiques conduites par des processus de Lévy de sauts positifs.

MSC: Primary 60H10; 60H20; secondary 60J80
Keywords: Stochastic equation; Strong solution; Pathwise uniqueness; Non-Lipschitz condition

## 1. Introduction

The question of pathwise uniqueness for one-dimensional stochastic differential equations driven by one-dimensional Brownian motions has been resolved a long time ago by Yamada and Watanabe [8]; see also Barlow [1]. The same question can also be asked for stochastic differential equations driven by discontinuous Lévy noises. Let us consider the equation

$$
\begin{equation*}
\mathrm{d} x(t)=F(x(t-)) \mathrm{d} L_{t}, \quad t \geq 0 . \tag{1.1}
\end{equation*}
$$

Bass [2] and Komatsu [6] showed that if $\left\{L_{t}\right\}$ is a symmetric stable process with exponent $\alpha \in(1,2)$ and if $x \mapsto F(x)$ is a bounded function with modulus of continuity $z \mapsto \rho(z)$ satisfying

$$
\begin{equation*}
\int_{0+} \frac{1}{\rho(z)^{\alpha}} \mathrm{d} z=\infty \tag{1.2}
\end{equation*}
$$

then (1.1) admits a strong solution and the solution is pathwise unique. This condition is the analogue of the YamadaWatanabe criterion for the diffusion coefficient. In particular, if $F$ is Hölder continuous with exponent $1 / \alpha$, then the pathwise uniqueness holds for (1.1). The required Hölder exponent tends to $1 / 2$ as $\alpha \rightarrow 2$ and it tends to 1 (Lipschitz condition) as $\alpha \rightarrow 1$. When the integral in (1.2) is finite, Bass [2] constructed a continuous function $x \mapsto \phi(x)$ having continuity modulus $x \mapsto \rho(x)$ for which the pathwise uniqueness for (1.1) fails; see also [3].

[^0]The pathwise uniqueness and strong solutions for stochastic differential equations driven by spectrally positive Lévy noises were studied in [4]. Those equations arise naturally in the study of branching processes. A typical special continuous state branching process is the non-negative solution to the stochastic differential equation

$$
\begin{equation*}
\mathrm{d} x(t)=\sqrt[\alpha]{x(t-)} \mathrm{d} L_{t}, \quad t \geq 0 \tag{1.3}
\end{equation*}
$$

where $\left\{L_{t}\right\}$ is a Brownian motion (for $\alpha=2$ ) or a spectrally positive $\alpha$-table process (for $1<\alpha<2$ ). Note that the coefficient $x \mapsto \sqrt[\alpha]{x}$ in (1.3) is non-decreasing, non-Lipschitz and degenerate at the origin. More general stochastic equations with similar structures arise naturally in limit theorems of branching processes with interactions or/and immigration.

In this paper we consider a class of stochastic differential equations with jumps, which generalizes Eq. (1.3). This exploration can be regarded as a continuation of [4]. We extend the results of [4] in two directions. First of all, we notice that the pathwise uniqueness results proved in [4] for non-negative càdlàg solutions can easily be extended to any càdlàg solutions. This extended result is given in Proposition 3.1. Its proof, which is in fact the most involved stochastic part behind the results in this paper, goes through along the same lines as in [4].

The second direction is to apply the above result to formulate some criteria for the pathwise uniqueness and existence of strong solutions to general stochastic differential equations with jumps. We consider this to be the main part of this paper. The proofs in this part involve some analytical arguments that allow us to apply the general pathwise uniqueness criterion of Proposition 3.1. From those results we derive sufficient conditions for the existence and uniqueness of non-negative strong solutions under suitable additional assumptions.

We also give applications of our main results to stochastic equations driven by spectrally positive Lévy processes. These extend and improve substantially the results of [4]. As a consequence of one of those results we get the following counterpart of the theorem of Bass [2].

Theorem 1.1. Let $\left\{L_{t}\right\}$ be a spectrally positive stable process with exponent $\alpha \in(1,2)$ that is, there exists $c_{\alpha}$ such that

$$
\mathbf{E}\left[\mathrm{e}^{-u L(t)}\right]=\mathrm{e}^{-c_{\alpha} u^{\alpha} t}, \quad t \geq 0, u \geq 0 .
$$

Let $F$ be a non-decreasing function on $\mathbb{R}$ with modulus of continuity $z \mapsto \rho(z)$ satisfying

$$
\begin{equation*}
\int_{0+} \frac{1}{\rho(z)^{\alpha /(\alpha-1)}} \mathrm{d} z=\infty \tag{1.4}
\end{equation*}
$$

Also assume that there is a constant $K \geq 0$ such that

$$
|F(x)| \leq K(1+|x|), \quad x \in \mathbb{R} .
$$

Then there is a pathwise unique strong solution to (1.1).

By the above theorem, if $F$ is a non-decreasing function Hölder continuous with exponent $1-1 / \alpha$, then the pathwise uniqueness holds for (1.1). The required Hölder exponent tends to 0 as $\alpha \rightarrow 1$, which differs sharply from the criterion of Bass [2] for a symmetric stable noise. Note that this result is also consistent with the Yamada-Watanabe result in the sense that as $\alpha \rightarrow 2$ the critical Hölder exponent converges to $1 / 2$.

The organization of the paper is as follows. The main theorem is stated in Section 2. Its proof is provided in Section 3. In Section 4 a number of particular cases is considered, for example, SDEs with stable Lévy noises. Theorem 1.1 is a consequence of one of the results obtained in that section. Throughout this paper, we make the conventions

$$
\int_{a}^{b}=\int_{(a, b]} \quad \text { and } \quad \int_{a}^{\infty}=\int_{(a, \infty)} \quad \text { for } b \geq a \in \mathbb{R}
$$

## 2. Main strong uniqueness and existence results

Suppose that $\mu_{0}(\mathrm{~d} u)$ and $\mu_{1}(\mathrm{~d} u)$ are $\sigma$-finite measures on the complete separable metric spaces $U_{0}$ and $U_{1}$, respectively. Let $\left(\Omega, \mathcal{g}, \mathcal{g}_{t}, \mathbf{P}\right)$ be a filtered probability space satisfying the usual hypotheses. Let $\{B(t)\}$ be a standard $\left(\mathcal{q}_{t}\right)$-Brownian motion and let $\left\{p_{0}(t)\right\}$ and $\left\{p_{1}(t)\right\}$ be $\left(\mathcal{q}_{t}\right)$-Poisson point processes on $U_{0}$ and $U_{1}$ with characteristic measures $\mu_{0}(\mathrm{~d} u)$ and $\mu_{1}(\mathrm{~d} u)$, respectively. Suppose that $\{B(t)\},\left\{p_{0}(t)\right\}$ and $\left\{p_{1}(t)\right\}$ are independent of each other. Let $N_{0}(\mathrm{~d} s, \mathrm{~d} u)$ and $N_{1}(\mathrm{~d} s, \mathrm{~d} u)$ be the Poisson random measures associated with $\left\{p_{0}(t)\right\}$ and $\left\{p_{1}(t)\right\}$, respectively. Suppose in addition that:

- $x \mapsto \sigma(x)$ is a continuous function on $\mathbb{R}$;
- $x \mapsto b(x)$ is a continuous function on $\mathbb{R}$ having the decomposition $b=b_{1}-b_{2}$ with $b_{2}$ being continuous and non-decreasing;
- $(x, u) \mapsto g_{0}(x, u)$ is a Borel function on $\mathbb{R} \times U_{0}$ such that $x \mapsto g_{0}(x, u)$ is non-decreasing for every $u \in U_{0}$;
- $(x, u) \mapsto g_{1}(x, u)$ is a Borel function on $\mathbb{R} \times U_{1}$.

Let $\tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u)$ be the compensated measure of $N_{0}(\mathrm{~d} s, \mathrm{~d} u)$. By a solution of the stochastic equation

$$
\begin{align*}
x(t)= & x(0)+\int_{0}^{t} \sigma(x(s-)) \mathrm{d} B(s)+\int_{0}^{t} \int_{U_{0}} g_{0}(x(s-), u) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u) \\
& +\int_{0}^{t} b(x(s-)) \mathrm{d} s+\int_{0}^{t} \int_{U_{1}} g_{1}(x(s-), u) N_{1}(\mathrm{~d} s, \mathrm{~d} u) \tag{2.1}
\end{align*}
$$

we mean a càdlàg and $\left(g_{t}\right)$-adapted real process $\{x(t)\}$ that satisfies the equation almost surely for every $t \geq 0$. Since $x(s-) \neq x(s)$ for at most countably many $s \geq 0$, we can also use $x(s)$ instead of $x(s-)$ for the integrals with respect to $\mathrm{d} B(s)$ and $\mathrm{d} s$ on the right-hand side of (2.1). We say pathwise uniqueness holds for (2.1) if for any two solutions $\left\{x_{1}(t)\right\}$ and $\left\{x_{2}(t)\right\}$ of the equation satisfying $x_{1}(0)=x_{2}(0)$ we have $x_{1}(t)=x_{2}(t)$ almost surely for every $t \geq 0$. Let $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ be the augmented natural filtration generated by $\{B(t)\},\left\{p_{0}(t)\right\}$ and $\left\{p_{1}(t)\right\}$. A solution $\{x(t)\}$ of (2.1) is called a strong solution if $x(t)$ is measurable with respect to $\mathcal{F}_{t}$ for every $t \geq 0$; see [5], p. 163, or [7], p. 76.

Lemma 2.1. Suppose that $\left(z \wedge z^{2}\right) v(\mathrm{~d} z)$ is a finite measure on $(0, \infty)$ and define

$$
\begin{equation*}
\alpha_{\nu}=\inf \left\{\beta>1: \lim _{x \rightarrow 0+} x^{\beta-1} \int_{x}^{\infty} z v(\mathrm{~d} z)=0\right\} \tag{2.2}
\end{equation*}
$$

Then $1 \leq \alpha_{\nu} \leq 2$ and, for any $\alpha>\alpha_{\nu}$,

$$
\begin{equation*}
\lim _{x \rightarrow 0+} x^{\alpha-2} \int_{0}^{x} z^{2} v(\mathrm{~d} z)=0 \tag{2.3}
\end{equation*}
$$

Proof. By (2.2) it is clear that $\alpha_{v} \geq 1$. For $x>0$ let

$$
G(x)=\int_{x}^{\infty} z v(\mathrm{~d} z) \quad \text { and } \quad H(x)=\int_{0}^{x} z^{2} v(\mathrm{~d} z)
$$

Given $\varepsilon>0$, choose $a>0$ so that $H(a)<\varepsilon$. Then for $a \geq x>0$ we have

$$
x G(x)=x \int_{x}^{a} z v(\mathrm{~d} z)+x G(a) \leq \int_{x}^{a} z^{2} v(\mathrm{~d} z)+x G(a) \leq \varepsilon+x G(a)
$$

It follows that $\limsup _{x \rightarrow 0+} x G(x) \leq \varepsilon$. That proves $\lim _{x \rightarrow 0+} x G(x)=0$, and so $\alpha_{v} \leq 2$. Clearly, (2.3) holds for any $\alpha \geq 2$. By integration by parts,

$$
\begin{equation*}
H(x)=-\int_{0}^{x} z \mathrm{~d} G(z)=-x G(x)+\int_{0}^{x} G(z) \mathrm{d} z \tag{2.4}
\end{equation*}
$$

Thus we have

$$
\lim _{x \rightarrow 0+} \int_{0}^{x} G(z) \mathrm{d} z=\lim _{x \rightarrow 0+} H(x)+\lim _{x \rightarrow 0+} x G(x)=0
$$

Now suppose that $\alpha_{\nu}<\alpha<2$. In view of (2.2), for any $\varepsilon>0$ there exists $b>0$ so that $x^{\alpha-1} G(x)<\varepsilon$ for all $0<x \leq b$. Then (2.4) implies

$$
x^{\alpha-2} H(x) \leq x^{\alpha-2} \int_{0}^{x} G(z) \mathrm{d} z \leq x^{\alpha-2} \int_{0}^{x} \varepsilon z^{1-\alpha} \mathrm{d} z=\varepsilon(2-\alpha)^{-1},
$$

and hence $\lim _{x \rightarrow 0+} x^{\alpha-2} H(x)=0$.
Let us consider a set $U_{2} \subset U_{1}$ satisfying $\mu_{1}\left(U_{1} \backslash U_{2}\right)<\infty$. As in the proof of Proposition 2.2 in [4] one can show that the uniqueness/existence of strong solutions for (2.1) can be reduced to the same question for the equation with $U_{1}$ replaced by $U_{2}$. Then in what follows all conditions for the ingredients of (2.1) only involve $U_{2}$ instead of $U_{1}$. As usual, let us consider some growth conditions on the coefficients:
(2a) There is a constant $K \geq 0$ such that

$$
\begin{aligned}
& \sigma(x)^{2}+\int_{U_{0}} g_{0}(x, u)^{2} \mu_{0}(\mathrm{~d} u)+\int_{U_{2}} g_{1}(x, u)^{2} \mu_{1}(\mathrm{~d} u)+b(x)^{2}+\left(\int_{U_{2}}\left|g_{1}(x, u)\right| \mu_{1}(\mathrm{~d} u)\right)^{2} \\
& \quad \leq K\left(1+x^{2}\right), \quad x \in \mathbb{R} .
\end{aligned}
$$

We next introduce our main conditions on the modulus of continuity that are particularly useful in applications to stochastic equations driven by Lévy processes. The conditions are given as follows:
(2b) For each $m \geq 1$ there is a non-decreasing and concave function $z \mapsto r_{m}(z)$ on $\mathbb{R}_{+}$such that $\int_{0+} r_{m}(z)^{-1} \mathrm{~d} z=\infty$ and

$$
\left|b_{1}(x)-b_{1}(y)\right|+\int_{U_{2}}\left|l_{1}(x, y, u)\right| \mu_{1}(\mathrm{~d} u) \leq r_{m}(|x-y|)
$$

for $|x|,|y| \leq m$, where $l_{1}(x, y, u)=g_{1}(x, u)-g_{1}(y, u)$.
(2c) For each $m \geq 1$ there is a constant $p_{m}>0$, a non-decreasing function $z \mapsto \rho_{m}(z)$ on $\mathbb{R}_{+}$and a function $u \mapsto f_{m}(u)$ on $U_{0}$ such that

$$
\int_{0+} \rho_{m}(z)^{-2} \mathrm{~d} z=\infty, \quad \int_{U_{0}}\left[f_{m}(u) \wedge f_{m}(u)^{2}\right] \mu_{0}(\mathrm{~d} u)<\infty
$$

and

$$
|\sigma(x)-\sigma(y)| \leq \rho_{m}(|x-y|), \quad\left|g_{0}(x, u)-g_{0}(y, u)\right| \leq \rho_{m}(|x-y|)^{2 p_{m}} f_{m}(u)
$$

for all $|x|,|y| \leq m$ and $u \in U_{0}$.
For each $m \geq 1$ and the function $f_{m}$ defined in (2c) we define the constant

$$
\alpha_{m}:=\inf \left\{\beta>1: \lim _{x \rightarrow 0+} x^{\beta-1} \int_{U_{0}} f_{m}(u) 1_{\left\{f_{m}(u) \geq x\right\}} \mu_{0}(\mathrm{~d} u)=0\right\} .
$$

By Lemma 2.1 we have $1 \leq \alpha_{m} \leq 2$. Our first main theorem of this paper is the following theorem.
Theorem 2.2. Suppose that conditions ( $2 \mathrm{a}-\mathrm{c}$ ) hold with

$$
\begin{equation*}
p_{m}>1-1 / \alpha_{m} \quad \text { for } \alpha_{m}<2 \quad \text { or } \quad p_{m}=1 / 2 \quad \text { for } \alpha_{m}=2 \text {. } \tag{2.5}
\end{equation*}
$$

Then for any given $x(0) \in \mathbb{R}$, there exists a pathwise unique strong solution $\{x(t)\}$ to (2.1).

From the above theorem we may derive some results on non-negative solutions of (2.1). For that purpose let us consider the following conditions:
(2d) $\sigma(0)=0, b(0) \geq 0$ and $g_{0}(0, u)=0$ for $u \in U_{0}$, and $g_{1}(x, u)+x \geq 0$ for $x \in \mathbb{R}_{+}$and $u \in U_{1}$.
(2e) There is a constant $K \geq 0$ such that

$$
b(x)+\int_{U_{2}}\left|g_{1}(x, u)\right| \mu_{1}(\mathrm{~d} u) \leq K(1+x), \quad x \geq 0 .
$$

(2f) There is a non-decreasing function $x \mapsto L(x)$ on $\mathbb{R}_{+}$so that

$$
\sigma(x)^{2}+\int_{U_{0}}\left[\left|g_{0}(x, u)\right| \wedge g_{0}(x, u)^{2}\right] \mu_{0}(\mathrm{~d} u) \leq L(x), \quad x \geq 0 .
$$

By Proposition 2.1 of [4], under condition (2d) any solution of (2.1) with non-negative initial value remains nonnegative forever.

Theorem 2.3. Suppose that conditions (2b-f) hold with (2.5). Then for any given $x(0) \in \mathbb{R}_{+}$, there exists a pathwise unique non-negative strong solution $\{x(t)\}$ to (2.1).

Remark 2.4. Under the conditions of Theorem 2.3 we can actually conclude that for any given $x(0) \in \mathbb{R}_{+}$there is a pathwise unique strong solution to (2.1) and the solution is non-negative. That follows from Proposition 2.1 of [4].

Remark 2.5. Note that when $\alpha_{m}<2$ the assumptions of Theorems 2.2 and 2.3 are strictly weaker than Theorems 2.5 and 5.3 of [4]. In some particular cases the condition (2.5) can be weakened to $p_{m} \geq 1-1 / \alpha_{m}$, as in the case of stable driving noise. This is done in Theorem 4.2.

## 3. Proofs of Theorems 2.2 and 2.3

The crucial part of the proof of Theorem 2.2 is verifying the pathwise uniqueness for (2.1). As we have mentioned already it is enough to consider the equation

$$
\begin{align*}
x(t)= & x(0)+\int_{0}^{t} \sigma(x(s-)) \mathrm{d} B(s)+\int_{0}^{t} \int_{U_{0}} g_{0}(x(s-), u) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u) \\
& +\int_{0}^{t} b(x(s-)) \mathrm{d} s+\int_{0}^{t} \int_{U_{2}} g_{1}(x(s-), u) N_{1}(\mathrm{~d} s, \mathrm{~d} u) . \tag{3.1}
\end{align*}
$$

For a function $f$ defined on the real line $\mathbb{R}$, note

$$
\Delta_{z} f(x)=f(x+z)-f(x) \quad \text { and } \quad D_{z} f(x)=\Delta_{z} f(x)-f^{\prime}(x) z .
$$

We shall need the next result, which provides a criterion for the pathwise uniqueness. It extends the criterion of Theorem 3.1 in [4], where it was formulated just for non-negative solutions.

Proposition 3.1. Suppose that conditions (2b, c) hold. Then the pathwise uniqueness of solution to (3.1) holds if for each $m \geq 1$ there exists a sequence of non-negative and twice continuously differentiable functions $\left\{\phi_{k}\right\}$ with the following properties:
(i) $\phi_{k}(z) \mapsto|z|$ non-decreasingly as $k \rightarrow \infty$;
(ii) $0 \leq \phi_{k}^{\prime}(z) \leq 1$ for $z \geq 0$ and $-1 \leq \phi_{k}^{\prime}(z) \leq 0$ for $z \leq 0$;
(iii) $\phi_{k}^{\prime \prime}(z) \geq 0$ for $z \in \mathbb{R}$ and as $k \rightarrow \infty$,

$$
\phi_{k}^{\prime \prime}(x-y)[\sigma(x)-\sigma(y)]^{2} \rightarrow 0
$$

uniformly on $|x|,|y| \leq m$;
(iv) as $k \rightarrow \infty$,

$$
\int_{U_{0}} D_{l_{0}(x, y, u)} \phi_{k}(x-y) \mu_{0}(\mathrm{~d} u) \rightarrow 0
$$

uniformly on $|x|,|y| \leq m$, where $l_{0}(x, y, u)=g_{0}(x, u)-g_{0}(y, u)$.
Proof. For non-negative solutions the result was given in Theorem 3.1 of [4]. In what follows we will show that the proof in [4] goes through for any càdlàg solutions. Let $\left\{x_{1}(t)\right\}$ and $\left\{x_{2}(t)\right\}$ be any two solutions of (3.1) starting at $x_{1}(0)=x_{2}(0)=x_{0}$. For each $m \geq 1$ define $\tau_{m}=\inf \left\{t \geq 0:\left|x_{1}(t)\right| \geq m\right.$ or $\left.\left|x_{2}(t)\right| \geq m\right\}$, and $\zeta(t)=x_{1}(t)-x_{2}(t)$. Recall that $l_{i}(x, y, u)=g_{i}(x, u)-g_{i}(y, u), i=0,1$. By (3.1) and the Itô formula one can show

$$
\begin{aligned}
\phi_{k}\left(\zeta\left(t \wedge \tau_{m}\right)\right)= & \int_{0}^{t \wedge \tau_{m}} \phi_{k}^{\prime}(\zeta(s-))\left[b\left(x_{1}(s-)\right)-b\left(x_{2}(s-)\right)\right] \mathrm{d} s \\
& +\frac{1}{2} \int_{0}^{t \wedge \tau_{m}} \phi_{k}^{\prime \prime}(\zeta(s-))\left[\sigma\left(x_{1}(s-)\right)-\sigma\left(x_{2}(s-)\right)\right] \mathrm{d} s \\
& +\int_{0}^{t \wedge \tau_{m}} \mathrm{~d} s \int_{U_{2}} \Delta_{l_{1}\left(x_{1}(s-), x_{2}(s-), u\right)} \phi_{k}(\zeta(s-)) \mu_{1}(\mathrm{~d} u) \\
& +\int_{0}^{t \wedge \tau_{m}} \mathrm{~d} s \int_{U_{0}} D_{l_{0}\left(x_{1}(s-), x_{2}(s-), u\right)} \phi_{k}(\zeta(s-)) \mu_{0}(\mathrm{~d} u) \\
& +M_{m}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
M_{m}(t)= & \int_{0}^{t \wedge \tau_{m}} \phi_{k}^{\prime}(\zeta(s-))\left[\sigma\left(x_{1}(s-)\right)-\sigma\left(x_{2}(s-)\right)\right] \mathrm{d} B(s) \\
& +\int_{0}^{t \wedge \tau_{m}} \int_{U_{2}} \Delta_{l_{1}\left(x_{1}(s-), x_{2}(s-), u\right)} \phi_{k}(\zeta(s-)) \tilde{N}_{1}(\mathrm{~d} s, \mathrm{~d} u) \\
& +\int_{0}^{t \wedge \tau_{m}} \int_{U_{0}} \Delta_{l_{0}\left(x_{1}(s-), x_{2}(s-), u\right)} \phi_{k}(\zeta(s-)) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u) .
\end{aligned}
$$

Under conditions ( $2 \mathrm{~b}, \mathrm{c}$ ) it is easy to show that $\left\{M_{m}(t)\right\}$ is a martingale. Therefore, we can follow the same argument as in the proof of Theorem 3.1 of [4] to get that, as $k \rightarrow \infty$,

$$
\mathbf{E}\left[\left|\zeta\left(t \wedge \tau_{m}\right)\right|\right] \leq \int_{0}^{t} r_{m}\left(\mathbf{E}\left[\left|\zeta\left(s \wedge \tau_{m}\right)\right|\right]\right) \mathrm{d} s
$$

From this by standard argument we have $\mathbf{E}\left[\left|\zeta\left(t \wedge \tau_{m}\right)\right|\right]=0$ for every $t \geq 0$. Since $\left\{x_{1}(t)\right\}$ and $\left\{x_{2}(t)\right\}$ are càdlàg, we have that $\tau_{m} \rightarrow \infty$ as $m \rightarrow \infty$. Hence letting $m \rightarrow \infty$ and using the right continuity of $\{\zeta(t)\}$ we get the result.

To prove the pathwise uniqueness for (3.1) we need to introduce more notation and prove a lemma which will play a crucial role in the proofs. For each integer $m \geq 1$ we shall construct a sequence of functions $\left\{\phi_{k}\right\}$ that satisfies the properties required in Proposition 3.1. Although main ideas are similar to those in the proof of Theorem 3.2 of [4], we will go through the details for the sake of completeness. Let $1=a_{0}>a_{1}>a_{2}>\cdots>0$ be defined by

$$
\int_{a_{k}}^{a_{k-1}} \rho_{m}(z) \mathrm{d} z=k
$$

Let $x \mapsto \psi_{k}(x)$ be a non-negative continuous function on $\mathbb{R}$ satisfying $\int_{a_{k}}^{a_{k-1}} \psi_{k}(x) \mathrm{d} x=1$ and

$$
\begin{equation*}
0 \leq \psi_{k}(x) \leq 2 k^{-1} \rho_{m}(x)^{-2} 1_{\left(a_{k}, a_{k-1}\right)}(x) . \tag{3.2}
\end{equation*}
$$

For each $k \geq 1$ we define the non-negative and twice continuously differentiable function

$$
\phi_{k}(z)=\int_{0}^{|z|} \mathrm{d} y \int_{0}^{y} \psi_{k}(x) \mathrm{d} x, \quad z \in \mathbb{R}
$$

Note that although the sequences $\left\{a_{k}\right\},\left\{\phi_{k}\right\}$ and $\left\{\psi_{k}\right\}$ also depend on $m \geq 1$, we do not put this additional index to simplify the notation.

Lemma 3.2. Suppose that condition (2c) holds. Fix $m \geq 1$ and let $a_{k}, \phi_{k}$ and $\psi_{k}$ be defined as above. Then the sequence $\left\{\phi_{k}\right\}$ satisfies properties (i)-(iii) in Proposition 3.1 and for any $h>0$,

$$
\begin{align*}
\int_{U_{0}} & D_{l_{0}(x, y, u)} \phi_{k}(x-y) \mu_{0}(\mathrm{~d} u) \\
\leq & k^{-1} \rho_{m}(|x-y|)^{4 p_{m}-2} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \int_{U_{0}} f_{m}(u)^{2} 1_{\left\{f_{m}(u) \leq h\right\}} \mu_{0}(\mathrm{~d} u) \\
& +\rho_{m}(|x-y|)^{2 p_{m}} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \int_{U_{0}} f_{m}(u) 1_{\left\{f_{m}(u)>h\right\}} \mu_{0}(\mathrm{~d} u) . \tag{3.3}
\end{align*}
$$

Proof. By definition, the sequence $\left\{\phi_{k}\right\}$ satisfies properties (i) and (ii) in Proposition 3.1. Moreover, by (3.2) we get

$$
\begin{equation*}
\phi_{k}^{\prime \prime}(x)=\psi_{k}(|x|) \leq 2 k^{-1} \rho_{m}(|x|)^{-2} 1_{\left(a_{k}, a_{k-1}\right)}(|x|) \tag{3.4}
\end{equation*}
$$

for all $x \in \mathbb{R}$. This together with condition (2c) implies

$$
\phi_{k}^{\prime \prime}(x-y)[\sigma(x)-\sigma(y)]^{2} \leq \psi_{k}(|x-y|) \rho_{m}(|x-y|)^{2} \leq 2 / k
$$

for $|x|,|y| \leq m$. Thus $\left\{\phi_{k}\right\}$ also satisfies property (iii) in Proposition 3.1. Observe that

$$
\begin{equation*}
D_{z} \phi_{k}(x-y)=\Delta_{z} \phi_{k}(x-y)-\phi_{k}^{\prime}(x-y) z \leq|z| 1_{\left\{|x-y| \leq a_{k-1}\right\}} \tag{3.5}
\end{equation*}
$$

when $(x-y) z \geq 0$. By Taylor's expansion,

$$
D_{z} \phi_{k}(x-y)=z^{2} \int_{0}^{1} \phi_{k}^{\prime \prime}(x-y+t z)(1-t) \mathrm{d} t=z^{2} \int_{0}^{1} \psi_{k}(|x-y+t z|)(1-t) \mathrm{d} t .
$$

Then (3.4) and the monotonicity of $\zeta \mapsto \rho_{m}(\zeta)$ imply

$$
\begin{align*}
D_{z} \phi_{k}(x-y) & \leq 2 k^{-1} z^{2} \int_{0}^{1} \frac{(1-t) 1_{\left(a_{k}, a_{k-1}\right)}(|(x-y)+t z|)}{\rho_{m}(|(x-y)+t z|)^{2}} \mathrm{~d} t \\
& \leq k^{-1} z^{2} \rho_{m}(|x-y|)^{-2} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \tag{3.6}
\end{align*}
$$

when $(x-y) z \geq 0$ and $|x|,|y| \leq m$. Recall that $l_{0}(x, y, u)=g_{0}(x, u)-g_{0}(y, u)$. Since $x \mapsto g_{0}(x, u)$ is nondecreasing, for $|x|,|y| \leq m$ we get by (3.5) and (2c) that

$$
D_{l_{0}(x, y, u)} \phi_{k}(x-y) \leq\left|l_{0}(x, y, u)\right| 1_{\left\{|x-y| \leq a_{k-1}\right\}} \leq \rho_{m}(|x-y|)^{2 p_{m}} f_{m}(u) 1_{\left\{|x-y| \leq a_{k-1}\right\}} .
$$

Similarly, by (3.6) and (2c) we have

$$
\begin{aligned}
D_{l_{0}(x, y, u)} \phi_{k}(x-y) & \leq k^{-1} \rho_{m}(|x-y|)^{-2} l_{0}(x, y, u)^{2} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \\
& \leq k^{-1} \rho_{m}(|x-y|)^{4 p_{m}-2} f_{m}(u)^{2} 1_{\left\{|x-y| \leq a_{k-1}\right\}} .
\end{aligned}
$$

Then (3.3) follows immediately.

Proposition 3.3. Under the conditions (2b, c) and (2.5), the pathwise uniqueness holds for Eq. (3.1).
Proof. For $\alpha_{m}=2$ and $p_{m}=2$, the result was essentially proved in Theorem 3.3 of [4] for non-negative solutions. It follows along the same lines for all solutions. So we here only consider the case of $\alpha_{m}<2$ and $p_{m}>1-1 / \alpha_{m}$. By Lemma 3.2 we get that the sequence $\left\{\phi_{k}\right\}$ satisfies properties (i)-(iii) in Proposition 3.1. Moreover, for any $\beta>0$ we can take $h=\rho_{m}(|x-y|)^{2 \beta}$ in (3.3) to get

$$
\begin{aligned}
& \int_{U_{0}} D_{l_{0}(x, y, u)} \phi_{k}(x-y) \mu_{0}(\mathrm{~d} u) \\
& \leq k^{-1} \rho_{m}(|x-y|)^{2\left(2 p_{m}-1\right)} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \int_{U_{0}} f_{m}(u)^{2} 1_{\left\{f_{m}(u) \leq \rho_{m}(|x-y|)^{2 \beta}\right\}} \mu_{0}(\mathrm{~d} u) \\
& \quad+\rho_{m}(|x-y|)^{2 p_{m}} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \int_{U_{0}} f_{m}(u) 1_{\left\{f_{m}(u)>\rho_{m}(|x-y|)^{2 \beta}\right\}} \mu_{0}(\mathrm{~d} u) .
\end{aligned}
$$

Since $\lim _{k \rightarrow \infty} a_{k}=0$ and $\lim _{z \rightarrow 0+} \rho(z)=0$, for $\alpha_{m}<\alpha<2$ we use Lemma 2.1 to see

$$
\begin{align*}
\int_{U_{0}} D_{l_{0}(x, y, u)} \phi_{k}(x-y) \mu_{0}(\mathrm{~d} u) \leq & k^{-1} \rho_{m}(|x-y|)^{2\left(2 p_{m}-1\right)} \rho_{m}(|x-y|)^{2 \beta(2-\alpha)} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \\
& +\rho_{m}(|x-y|)^{2 p_{m}} \rho_{m}(|x-y|)^{2 \beta(1-\alpha)} 1_{\left\{|x-y| \leq a_{k-1}\right\}} \tag{3.7}
\end{align*}
$$

when $k \geq 1$ is sufficiently large. If we can choose $\beta$ and $\alpha$ in the way that

$$
2\left(2 p_{m}-1\right)+2 \beta(2-\alpha)>0 \quad \text { and } \quad 2 p_{m}+2 \beta(1-\alpha)>0
$$

the value on the right-hand side of (3.7) will tend to zero as $k \rightarrow \infty$. The requirement is equivalent to

$$
\frac{1-2 p_{m}}{2-\alpha}<\beta<\frac{p_{m}}{\alpha-1}
$$

which can be done as long as

$$
\frac{1-2 p_{m}}{2-\alpha}<\frac{p_{m}}{\alpha-1}
$$

or, equivalently, $p_{m}>1-1 / \alpha$. For that purpose it sufficient to have $p_{m}>1-1 / \alpha_{m}$. This gives property (iv) in Proposition 3.1 and hence the pathwise uniqueness for (3.1).

Proposition 3.4. Suppose that condition (2a) holds. Let $\{x(t)\}$ be a solution of (3.1) with $\mathbf{E}\left[x(0)^{2}\right]<\infty$. Then we have

$$
\begin{equation*}
\mathbf{E}\left[1+\sup _{0 \leq s \leq t} x(s)^{2}\right] \leq\left(1+6 \mathbf{E}\left[x(0)^{2}\right]\right) \exp \{6 K(4+t) t\} \tag{3.8}
\end{equation*}
$$

Proof. Let $\tau_{m}=\inf \{t \geq 0:|x(t)| \geq m\}$ for $m \geq 1$. Since $\{x(t)\}$ has càdlàg sample paths, we have $\tau_{m} \rightarrow \infty$ as $m \rightarrow \infty$. Let us rewrite (3.1) into

$$
\begin{aligned}
x(t)= & x(0)+\int_{0}^{t} \sigma(x(s-)) \mathrm{d} B(s)+\int_{0}^{t} \int_{U_{0}} g_{0}(x(s-), u) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u) \\
& +\int_{0}^{t} b(x(s-)) \mathrm{d} s+\int_{0}^{t} \int_{U_{2}} g_{1}(x(s-), u) \tilde{N}_{1}(\mathrm{~d} s, \mathrm{~d} u) \\
& +\int_{0}^{t} \mathrm{~d} s \int_{U_{2}} g_{1}(x(s-), u) \mu_{1}(\mathrm{~d} u) .
\end{aligned}
$$

By Doob's martingale inequalities we have

$$
\begin{aligned}
& \mathbf{E}\left[\sup _{0 \leq s \leq t} x\left(s \wedge \tau_{m}\right)^{2}\right] \\
& \leq \\
& \quad 6 \mathbf{E}\left[x(0)^{2}\right]+24 \mathbf{E}\left[\int_{0}^{t \wedge \tau_{m}} \sigma(x(s-))^{2} \mathrm{~d} s\right]+6 \mathbf{E}\left[\left(\int_{0}^{t \wedge \tau_{m}}|b(x(s-))| \mathrm{d} s\right)^{2}\right] \\
& \quad+24 \mathbf{E}\left[\int_{0}^{t \wedge \tau_{m}} \mathrm{~d} s \int_{U_{0}} g_{0}(x(s-), u)^{2} \mu_{0}(\mathrm{~d} u)\right]+24 \mathbf{E}\left[\int_{0}^{t \wedge \tau_{m}} \mathrm{~d} s \int_{U_{2}} g_{1}(x(s-), u)^{2} \mu_{1}(\mathrm{~d} u)\right] \\
& \quad+6 \mathbf{E}\left[\left(\int_{0}^{t \wedge \tau_{m}} \mathrm{~d} s \int_{U_{2}}\left|g_{1}(x(s-), u)\right| \mu_{1}(\mathrm{~d} u)\right)^{2}\right] \\
& \leq \\
& \quad 6 \mathbf{E}\left[x(0)^{2}\right]+6 K(4+t) \mathbf{E}\left[\int_{0}^{t \wedge \tau_{m}}\left(1+x(s-)^{2}\right) \mathrm{d} s\right]
\end{aligned}
$$

Then it is easy to see that

$$
t \mapsto F_{m}(t):=\mathbf{E}\left[\sup _{0 \leq s \leq t} x\left(s \wedge \tau_{m}\right)^{2}\right]
$$

is locally bounded on $[0, \infty)$. Since $s \mapsto x(s)$ has at most a countable number of jumps, from the above inequality we obtain

$$
\begin{aligned}
1+F_{m}(t) & \leq 1+6 \mathbf{E}\left[x(0)^{2}\right]+6 K(4+t) \mathbf{E}\left[\int_{0}^{t \wedge \tau_{m}}\left(1+x(s)^{2}\right) \mathrm{d} s\right] \\
& \leq 1+6 \mathbf{E}\left[x(0)^{2}\right]+6 K(4+t) \int_{0}^{t}\left[1+F_{m}(s)\right] \mathrm{d} s
\end{aligned}
$$

By Gronwall's inequality,

$$
\mathbf{E}\left[1+\sup _{0 \leq s \leq t} x\left(s \wedge \tau_{m}\right)^{2}\right] \leq\left(1+6 \mathbf{E}\left[x(0)^{2}\right]\right) \exp \{6 K(4+t) t\}
$$

Then (3.8) follows by Fatou's lemma.

Proof of Theorem 2.2. Step 1: Suppose that conditions (2b, c) and (2.5) hold. Instead of condition (2a), we here assume there is a constant $K \geq 0$ such that

$$
\begin{align*}
& \sigma(x)^{2}+b(x)^{2}+\sup _{u \in U_{0}}\left|g_{0}(x, u)\right|+\int_{U_{0}} g_{0}(x, u)^{2} \mu_{0}(\mathrm{~d} u) \\
& \quad+\int_{U_{2}} g_{1}(x, u)^{2} \mu_{1}(\mathrm{~d} u)+\left(\int_{U_{2}}\left|g_{1}(x, u)\right| \mu_{1}(\mathrm{~d} u)\right)^{2} \leq K, \quad x \in \mathbb{R} \tag{3.9}
\end{align*}
$$

Let $\left\{V_{n}\right\}$ be a non-decreasing sequence of Borel subsets of $U_{0}$ so that $\bigcup_{n=1}^{\infty} V_{n}=U_{0}$ and $\mu_{0}\left(V_{n}\right)<\infty$ for every $n \geq 1$. By the result on continuous-type stochastic equations, there is a weak solution to

$$
\begin{equation*}
x(t)=x(0)+\int_{0}^{t} \sigma(x(s)) \mathrm{d} B(s)+\int_{0}^{t} b(x(s)) \mathrm{d} s-\int_{0}^{t} \mathrm{~d} s \int_{V_{n}} g_{0}(x(s), u) \mu_{0}(\mathrm{~d} u) \tag{3.10}
\end{equation*}
$$

see, e.g., Ikeda and Watanabe [5], p. 169. By Proposition 3.3, the pathwise uniqueness holds for (3.10), thus the equation has a pathwise unique strong solution. Let $\left\{W_{n}\right\}$ be a non-decreasing sequence of Borel subsets of $U_{1}$ so
that $\bigcup_{n=1}^{\infty} W_{n}=U_{2}$ and $\mu_{1}\left(W_{n}\right)<\infty$ for every $n \geq 1$. Then for every integer $n \geq 1$ there is a unique strong solution $\left\{x_{n}(t): t \geq 0\right\}$ to

$$
\begin{aligned}
x(t)= & x(0)+\int_{0}^{t} \sigma(x(s-)) \mathrm{d} B(s)+\int_{0}^{t} \int_{V_{n}} g_{0}(x(s-), u) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u) \\
& +\int_{0}^{t} b(x(s-)) \mathrm{d} s+\int_{0}^{t} \int_{W_{n}} g_{1}(x(s-), u) N_{1}(\mathrm{~d} s, \mathrm{~d} u)
\end{aligned}
$$

As in the proof of Lemma 4.3 of [4] one can see the sequence $\left\{x_{n}(t)\right\}$ is tight in $D([0, \infty), \mathbb{R})$ - the space of càdlàg functions with the Skorohod topology. Following the proof of Theorem 4.4 of [4] it is easy to show that any limit point of the sequence is a weak solution to (3.1). This and Proposition 3.3 imply the existence and uniqueness of the strong solution to (3.1); see, e.g., [7], p. 104.

Step 2: Suppose that the original conditions (2a-c) and (2.5) hold. For each $m \geq 1$ let

$$
\chi_{m}(x)= \begin{cases}x & \text { if }|x| \leq m, \\ m & \text { if } x>m, \\ -m & \text { if } x<-m\end{cases}
$$

We consider the equation

$$
\begin{align*}
x(t)= & x(0)+\int_{0}^{t} \sigma\left(\chi_{m}(x(s-))\right) \mathrm{d} B(s)+\int_{0}^{t} b_{m}\left(\chi_{m}(x(s-))\right) \mathrm{d} s \\
& +\int_{0}^{t} \int_{U_{0}} \chi_{m} \circ g_{0}\left(\chi_{m}(x(s-)), u\right) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u)+\int_{0}^{t} \int_{U_{2}} g_{1}\left(\chi_{m}(x(s-)), u\right) N_{1}(\mathrm{~d} s, \mathrm{~d} u), \tag{3.11}
\end{align*}
$$

where

$$
b_{m}(x)=b(x)-\int_{U_{0}}\left[g_{0}(x, u)-\chi_{m} \circ g_{0}(x, u)\right] \mu_{0}(\mathrm{~d} u) .
$$

By the first step, there is a unique strong solution to (3.11). Then using Proposition 3.4 one can show as in the proof of Proposition 2.4 of [4] that there is a pathwise unique strong solution to (3.1). Hence as we have mentioned above, there is a pathwise unique strong solution to (2.1) (see Proposition 2.2 of [4] and its proof for analogous result).

Proof of Theorem 2.3. By Proposition 2.1 of [4] and Theorem 2.2 there is a pathwise unique non-negative strong solution $\left\{x_{m}(t)\right\}$ to the equation

$$
\begin{align*}
x(t)= & x(0)+\int_{0}^{t} \sigma\left(\chi_{m}(x(s-) \vee 0)\right) \mathrm{d} B(s)+\int_{0}^{t} b\left(\chi_{m}(x(s-) \vee 0)\right) \mathrm{d} s \\
& +\int_{0}^{t} \int_{U_{0}} \chi_{m} \circ g_{0}\left(\chi_{m}(x(s-) \vee 0), u\right) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u)+\int_{0}^{t} \int_{U_{2}} \chi_{m} \circ g_{1}(x(s-) \vee 0, u) N_{1}(\mathrm{~d} s, \mathrm{~d} u) . \tag{3.12}
\end{align*}
$$

By Proposition 2.3 of [4] the first moment of $\left\{x_{m}(t)\right\}$ is dominated by a locally bounded function on $[0, \infty)$ independent of $m \geq 1$. Then one can follow the proof of Proposition 2.4 of [4] to show there is a pathwise unique non-negative strong solution to

$$
\begin{align*}
x(t)= & x(0)+\int_{0}^{t} \sigma(x(s-) \vee 0) \mathrm{d} B(s)+\int_{0}^{t} b(x(s-) \vee 0) \mathrm{d} s \\
& +\int_{0}^{t} \int_{U_{0}} g_{0}((x(s-) \vee 0), u) \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} u) \\
& +\int_{0}^{t} \int_{U_{2}} g_{1}(x(s-) \vee 0, u) N_{1}(\mathrm{~d} s, \mathrm{~d} u) . \tag{3.13}
\end{align*}
$$

Now note that the non-negative solution to (3.13) is also the non-negative solution to (3.1). This and Proposition 3.3 imply that there is a pathwise unique non-negative strong solution to (3.1). This again as in the proof of Theorem 2.2 implies that there is a pathwise unique non-negative strong solution to (2.1).

Remark 3.5. The above proofs show it is unnecessary to assume the existence of the sequence $\left\{V_{n}\right\}$ in (4.b) of [4]. As a consequence, condition (5.b) of [4] is also unnecessary for the results in Section 5 of that paper.

## 4. Stochastic equations with Lévy noises

In this section, we give some applications of our main results to stochastic equations driven by Lévy processes. Let $(\sigma, b)$ be given as in Section 2 and let $\nu_{0}(\mathrm{~d} z)$ and $\nu_{1}(\mathrm{~d} z)$ be $\sigma$-finite Borel measures on $(0, \infty)$ satisfying

$$
\int_{0}^{\infty}\left(z \wedge z^{2}\right) \nu_{0}(\mathrm{~d} z)+\int_{0}^{\infty}(1 \wedge z) \nu_{1}(\mathrm{~d} z)<\infty
$$

Let $\alpha_{0}$ be the constant defined by (2.2) for the measure $\nu_{0}(\mathrm{~d} z)$. In addition, we suppose that:

- $x \mapsto h_{0}(x)$ is a continuous and non-decreasing function on $\mathbb{R}$;
- $x \mapsto h_{1}(x)$ is a continuous function on $\mathbb{R}$.

Suppose we have a filtered probability space $\left(\Omega, \mathcal{G}_{,} \mathcal{G}_{t}, \mathbf{P}\right)$ satisfying the usual hypotheses. Let $\{B(t)\}$ be an $\left(\mathcal{q}_{t}\right)$ Brownian motion and let $\left\{L_{0}(t)\right\}$ and $\left\{L_{1}(t)\right\}$ be ( $\left.\mathcal{g}_{t}\right)$-Lévy processes with exponents

$$
u \mapsto \int_{0}^{\infty}\left(\mathrm{e}^{\mathrm{i} u z}-1-\mathrm{i} u z\right) \nu_{0}(\mathrm{~d} z) \quad \text { and } \quad u \mapsto \int_{0}^{\infty}\left(\mathrm{e}^{\mathrm{i} u z}-1\right) \nu_{1}(\mathrm{~d} z),
$$

respectively. Suppose that $\{B(t)\},\left\{L_{0}(t)\right\}$ and $\left\{L_{1}(t)\right\}$ are independent of each other. Note that $\left\{L_{0}(t)\right\}$ is centered and $\left\{L_{1}(t)\right\}$ is non-decreasing. We introduce the conditions:
(4a) There is a constant $K \geq 0$ such that

$$
|\sigma(x)|+|b(x)|+\left|h_{0}(x)\right|+\left|h_{1}(x)\right| \leq K(1+|x|), \quad x \in \mathbb{R} .
$$

(4b) There exists a non-decreasing and concave function $z \mapsto r(z)$ on $\mathbb{R}_{+}$such that $\int_{0+} r(z)^{-1} \mathrm{~d} z=\infty$ and

$$
|b(x)-b(y)|+\left|h_{1}(x)-h_{1}(y)\right| \leq r(|x-y|), \quad x, y \in \mathbb{R} .
$$

(4c) There is a constant $p>0$ and a non-decreasing function $z \mapsto \rho(z)$ on $\mathbb{R}_{+}$such that $\int_{0+} \rho(z)^{-2} \mathrm{~d} z=\infty$ and

$$
|\sigma(x)-\sigma(y)|+\left|h_{0}(x)-h_{0}(y)\right|^{1 / 2 p} \leq \rho(|x-y|), \quad x, y \in \mathbb{R} .
$$

(4d) $\sigma(0)=h_{0}(0)=0, b(0) \geq 0$, and $h_{1}(x) \geq 0$ for $x \in \mathbb{R}_{+}$.
(4e) There is a constant $K \geq 0$ such that

$$
b(x)+h_{1}(x) \leq K(1+x), \quad x \geq 0 .
$$

Theorem 4.1. (i) If conditions (4a-c) are satisfied with $p>1-1 / \alpha_{0}$, then for any given $x(0) \in \mathbb{R}$ there is a pathwise unique strong solution to

$$
\begin{equation*}
\mathrm{d} x(t)=\sigma(x(t)) \mathrm{d} B(t)+h_{0}(x(t-)) \mathrm{d} L_{0}(t)+b(x(t)) \mathrm{d} t+h_{1}(x(t-)) \mathrm{d} L_{1}(t) . \tag{4.1}
\end{equation*}
$$

(ii) If conditions (4b-e) are satisfied with $p>1-1 / \alpha_{0}$, then for any given $x(0) \in \mathbb{R}_{+}$there is a pathwise unique non-negative strong solution to (4.1).

Proof. By Lévy-Itô decompositions, the Lévy processes have the following representations

$$
\begin{aligned}
& L_{0}(t)=\int_{0}^{t} \int_{0}^{1} z \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} z)-\int_{0}^{t} \mathrm{~d} s \int_{1}^{\infty} z \nu_{0}(\mathrm{~d} z)+\int_{0}^{t} \int_{1}^{\infty} z N_{1}(\mathrm{~d} s, \mathrm{~d} z,\{0\}), \\
& L_{1}(t)=\int_{0}^{t} \int_{0}^{\infty} z N_{1}(\mathrm{~d} s, \mathrm{~d} z,\{1\}),
\end{aligned}
$$

where $N_{0}(\mathrm{~d} s, \mathrm{~d} z)$ and $N_{1}(\mathrm{~d} s, \mathrm{~d} z, \mathrm{~d} u)$ are Poisson random measures with intensities

$$
1_{\{z \leq 1\}} \mathrm{d} s v_{0}(\mathrm{~d} z) \quad \text { and } \quad \mathrm{d} s\left[1_{\{z>1\}} v_{0}(\mathrm{~d} z) \delta_{0}(\mathrm{~d} u)+\nu_{1}(\mathrm{~d} z) \delta_{1}(\mathrm{~d} u)\right],
$$

respectively, and $\tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} z)$ is the compensated measure of $N_{0}(\mathrm{~d} s, \mathrm{~d} z)$. Here $N_{0}(\mathrm{~d} s, \mathrm{~d} z)$ and $N_{1}(\mathrm{~d} s, \mathrm{~d} z, \mathrm{~d} u)$ are independent and they are independent of $\{B(t)\}$. By applying Theorem 2.2 with

$$
U_{0}=(0,1], \quad U_{1}=[(1, \infty) \times\{0\}] \cup[(0, \infty) \times\{1\}] \quad \text { and } \quad U_{2}=(0,1] \times\{1\},
$$

we see that there is a pathwise unique strong solution to

$$
\begin{aligned}
x(t)= & x(0)+\int_{0}^{t} \sigma(x(s)) \mathrm{d} B(s)+\int_{0}^{t} \int_{0}^{1} h_{0}(x(s-)) z \tilde{N}_{0}(\mathrm{~d} s, \mathrm{~d} z) \\
& +\int_{0}^{t}\left(b(x(s))-h_{0}(x(s)) \int_{1}^{\infty} z v_{0}(\mathrm{~d} z)\right) \mathrm{d} s \\
& +\int_{0}^{t} \int_{U_{1}} g_{1}(x(s-), z, u) N_{1}(\mathrm{~d} s, \mathrm{~d} z, \mathrm{~d} u),
\end{aligned}
$$

where

$$
g_{1}(x, z, u)=h_{0}(x) z 1_{\{z>1, u=0\}}+h_{1}(x) z 1_{\{u=1\}} .
$$

However, this is just another form of Eq. (4.1) and hence part (i) of the theorem follows. The proof of part (ii) is similar.

Theorem 4.2. Suppose that $\{B(t)\},\left\{L_{0}(t)\right\}$ and $\left\{L_{1}(t)\right\}$ are given as the above with $\nu_{0}(\mathrm{~d} z)=z^{-1-\alpha} \mathrm{d} z$ for $1<\alpha<2$. Then we have:
(i) If conditions ( $4 \mathrm{a}-\mathrm{c}$ ) are satisfied with $p \geq 1-1 / \alpha$, then for any given $x(0) \in \mathbb{R}$ there is a pathwise unique strong solution to (4.1).
(ii) If conditions (4b-e) are satisfied with $p \geq 1-1 / \alpha$, then for any given $x(0) \in \mathbb{R}_{+}$there is a pathwise unique non-negative strong solution to (4.1).

Proof. Let $\left\{a_{k}\right\}$, $\left\{\phi_{k}\right\}$ and $\left\{\psi_{k}\right\}$ be defined as before Lemma 3.2 with $\rho_{m}=\rho$. Then we can easily apply Lemma 3.2 to get that $\left\{\phi_{k}\right\}$ satisfies properties (i)-(iii) in Proposition 3.1. Moreover, using again Lemma 3.2 with $\mu_{0}(\mathrm{~d} u)=$ $u^{-1-\alpha} \mathrm{d} u, p_{m}=p=(\alpha-1) / \alpha, \rho_{m}=\rho$ and $f_{m}(u)=u$ we can rewrite (3.3) as

$$
\begin{aligned}
& \int_{0}^{\infty} D_{l_{0}(x, y, u)} \phi_{k}(x-y) \mu_{0}(\mathrm{~d} u) \\
& \quad \leq k^{-1} \rho(|x-y|)^{4 p-2} \int_{0}^{h} u^{1-\alpha} \mathrm{d} u+\rho(|x-y|)^{2 p} \int_{h}^{\infty} u^{-\alpha} \mathrm{d} u \\
& \quad=k^{-1}(2-\alpha)^{-1} \rho(|x-y|)^{4(\alpha-1) / \alpha-2} h^{2-\alpha}+(\alpha-1)^{-1} \rho(|x-y|)^{2(\alpha-1) / \alpha} h^{1-\alpha} .
\end{aligned}
$$

Take $h=\rho(|x-y|)^{2 / \alpha} v_{k}$, where $v_{k}$ is a sequence such that $v_{k} \rightarrow \infty$ and $v_{k}^{2-\alpha} k^{-1} \rightarrow 0$. Then one can check that

$$
\int_{0}^{\infty} D_{l_{0}(x, y, u)} \phi_{k}(x-y) \mu_{0}(\mathrm{~d} u) \leq k^{-1}(2-\alpha)^{-1} v_{k}^{2-\alpha}+(\alpha-1)^{-1} v_{k}^{1-\alpha},
$$

which tends to zero as $k \rightarrow \infty$. Now since all the properties in Proposition 3.1 are satisfied we get the pathwise uniqueness for (4.1). The existence of the solution follows by a modification of the proof of Theorem 2.2. That gives part (i) of the theorem. The proof of part (ii) can be given in a similar way.

Corollary 4.3. Let $a \geq 0, b \geq 0, c \geq 0,1 \leq r \leq 2,1<\alpha<2, q \geq 1$ and $\beta$ be constants. Suppose that $\{B(t)\},\left\{L_{0}(t)\right\}$ and $\left\{L_{1}(t)\right\}$ are given as the above with $\nu_{0}(\mathrm{~d} z)=z^{-1-\alpha} \mathrm{d} z$. If $1 / q+1 / \alpha \geq 1$, then for any given $x(0) \in \mathbb{R}_{+}$there is a pathwise unique strong solution to

$$
\begin{equation*}
\mathrm{d} x(t)=\sqrt[r]{a|x(t)|} \mathrm{d} B(t)+\operatorname{sign}(x(t-)) \sqrt[q]{c|x(t-)|} \mathrm{d} L_{0}(t)+(\beta x(t)+b) \mathrm{d} t+\mathrm{d} L_{1}(t) \tag{4.2}
\end{equation*}
$$

and this solution is non-negative.
Proof. One can choose $\rho(z)=\sqrt[r]{z}$ and $p=1 / q$ in (4c) and hence by Theorem 4.2, there is a pathwise unique strong solution to (4.2) which is non-negative.

In the special case where $r=2$ and $q=\alpha$, the solution of (4.2) is a continuous state branching process with immigration and the strong existence and uniqueness for (4.2) were obtained in [4].

## Remark 4.4. Theorem 1.1 follows immediately from Theorem 4.2.

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[^0]:    ${ }^{1}$ Supported by NSFC (10525103 and 10721091) and CJSP.
    ${ }^{2}$ Supported by Israel Science Foundation Grant No.1162/06.

