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Limits of determinantal processes near a tacnode

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Abstract. We study a Markov process on a system of interlacing particles. At large times the particles fill a domain that depends on a parameter $\varepsilon > 0$. The domain has two cusps, one pointing up and one pointing down. In the limit $\varepsilon \downarrow 0$ the cusps touch, thus forming a tacnode. The main result of the paper is a derivation of the local correlation kernel around the tacnode in the transition regime $\varepsilon \downarrow 0$. We also prove that the local process interpolates between the Pearcey process and the GUE minor process.

Résumé. Nous étudions un processus de Markov sur un système de particules entrelacées. Lorsque le temps t est grand, les particules remplissent un domaine dépendant d'un paramètre $\varepsilon > 0$. Ce domaine possède deux points de rebroussement, dont l'un pointe vers le haut et l'autre vers le bas. À la limite $\varepsilon \downarrow 0$, les deux points de rebroussement sont tangents, formant ainsi un tacnode. Le résultat principal de cet article est un calcul du noyau de corrélation locale autour du point tacnodal pendant le régime de transition $\varepsilon \downarrow 0$. Nous démontrons aussi que le processus local interpole entre le processus de Pearcey et le processus des mineurs du GUE.

MSC: 82C22; 60K35; 60G55

Keywords: Determinantal point processes; Random growth; GUE minor process; Pearcey process

1. Introduction

In a recent paper [5] the authors introduced a Markov process on a system of interlacing particles. This model contains many parameters, creating a rich pool of interesting particular examples, and at the same time it has an integrable structure that allows for explicit computations. It is the purpose of this paper, to study a special case of this model. The interest of this model lies in the fact that for large time the particles will fill a domain that has a tacnode on the boundary. A somewhat similar situation also occurs in the case of non-intersecting Brownian paths with multiple sources and sinks [1]. The local process at the tacnode in this model is not yet understood, although a conjecture is given in [1]. The integrability of our model allows us to compute the local process around the tacnode. This is the main result of this paper.

We consider an evolution on particles that are placed on the grid

$$\mathcal{G} = \left\{ (x, m) \mid m = 1, 2, \ldots, x \in \mathbb{Z} + \frac{m+1}{2} \right\}. \tag{1.1}$$

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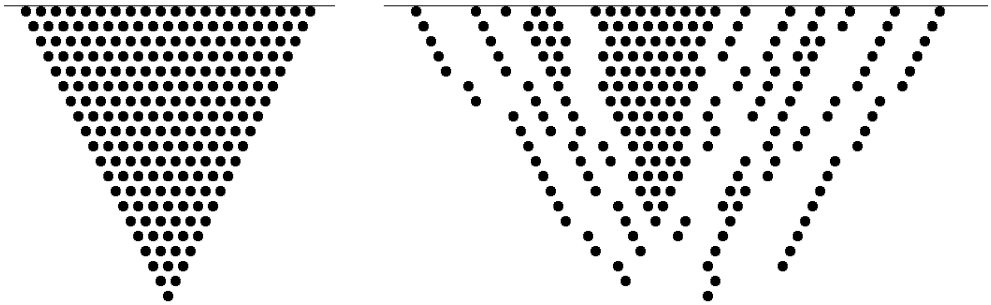

(a) (b)

Fig. 1. (a) The initial condition and (b) an example of a point configuration after some time.

Hence, if $(x, m) \in \mathcal{G}$, then x takes integer values for odd values of m and half-integer values for even values of m. At each horizontal m-section we put m particles and denote their horizontal coordinates by x_k^m for $k = 1, \dots, m$. The evolution is such that at each time the system of particles satisfies the interlacing condition

$$x_{k-1}^m < x_{k-1}^{m-1} < x_k^m, \quad k = 2, \dots, m, m = 2, \dots. \tag{1.2}$$

At time $t = 0$ we put the particles at positions $x_k^m = -(m+1)/2 + k$ as shown in Fig. 1.

The evolution of the particles is as follows: each particle has two independent exponential clocks, a left and a right clock, respectively. If the right (left) clock rings, the particle attempts to jump to the right (left) by one. But in doing so it is forced to respect the interlacing condition according to the following two rules: if the right (resp., left) clock of the particle at x_k^m rings, then

(1) If $x_k^m = x_k^{m-1} - 1/2$ (or $x_k^m = x_{k-1}^{m-1} + 1/2$ in case the left clock rings) then it remains put.
(2) Otherwise it jumps to the right by one and so do all particles x_{k+l}^{m+l} with $x_{k+l}^{m+l} = x_k^m + l/2$ for $l = 1, 2, \dots$ (in case the left clock rings all particles x_k^{m+l} with $x_k^{m+l} = x_k^m - l/2$ jump to the left for $l = 1, 2, \dots$).

Hence a particle that wants to jump is blocked by particles with lower m-index, but it pushes particles at a higher m-index.

Next we specify the rate of the exponential clocks. For odd m particles jump to the right with rate $\varepsilon^{-1} > 0$ and they jump to the left with rate ε. For even m the particles jump to the right with rate ε and to the left with rate ε^{-1}. We are interested in the case where ε is small. Hence the particles for odd m predominantly try to jump to the right. For even m they mostly jump to the left. However, they are still subject to the interlacing condition. Due to this condition, some particles that want to jump to the left (or right) are blocked, and even pushed to the right (or left) by particles at a lower level that want to travel right (or left).

If we let time evolve we will see mainly two clouds of particles travelling to the left and to the right respectively. See Fig. 1 for typical point configuration after some time. For large time we have the macroscopic picture as shown in Fig. 2. With high probability the particles will be distributed in a domain $\mathcal{D}$ contained in the upper half plane. This domain consist of two parts $\mathcal{D}_1$ and $\mathcal{D}_2$. In $\mathcal{D}_2$ the particles are still densely pact as in the initial configuration (which means the particles did not have the chance to jump yet). In $\mathcal{D}_1$ the particles have a density strictly less than one. The boundary of $\mathcal{D}_1$ is a smooth curve except for two cusp points. The cusps touch in the limit $\varepsilon \downarrow 0$.

After rescaling the time parameter, the process has a well-defined limit for $\varepsilon \downarrow 0$. In this limit the particles come in pairs. Indeed, by the interlacing condition some particles are blocked and even pushed by a particle at one level below that jumps in the reverse direction, thus forming a pair. The pairs are slanted to the right in the right half plane and slanted to the left in the left half plane, see also Fig. 1. The process for these pairs can be described in the following way: the process decouples in the sense that we have two independent processes, one in the upper left quadrant and the other in the upper right quadrant. The process in the right quadrant is equivalent to the process where particles can only jump to the right. The process at the left is just its reflected version (hence the particle only

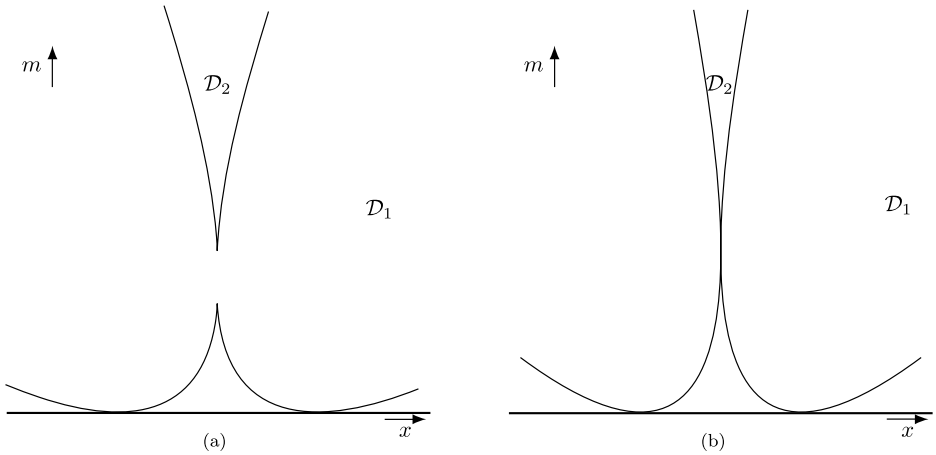

Fig. 2. The typical shape of the limiting domains $\mathcal{D}_1$ and $\mathcal{D}_2$. The case $\varepsilon > 0$ at the left and $\varepsilon = 0$ at the right. In $\mathcal{D}_1$ the particle are still in the initial configuration. Outside $\mathcal{D}_1$ and $\mathcal{D}_2$ there are no particles.

jump to the left). This process is analyzed in [5] and from their results we recover that the limiting domain has a tacnode.

By standard arguments we can compute the limiting mean density of the particles in all cases. This settles the macroscopic behavior for the particles at large time. At the local scale we retrieve the well-known universality classes.

First consider $\varepsilon > 0$. If we zoom in at a point away from the boundary, then we find that the local correlations are governed by one of the extensions to the discrete sine process that falls into the class as introduced in [3]. If we zoom in at a point at the boundary, but not the cusps points, then we obtain the Airy process (see [19] and [10] for a review). The local correlations near the cusps are determined by the Pearcey process [2,8,9,18,22]. Since the proofs of these results follow from standard computations, they will be omitted. We do not need these statements for our main results.

In the case $\varepsilon = 0$ we have a decoupled system. In addition to a discrete sine process and the Airy process we also obtain the local correlations around the tacnode, which are described by a process that is directly related to the GUE minor process (as we will prove).

The main result of the paper is to derive the process around the tacnode in the transition regime $\varepsilon \downarrow 0$. The main result is that we obtain a process that has not appeared in the literature (to the best of our knowledge). Naturally, it interpolates between the Pearcey process and a process related to the GUE minor process.

In Section 2 we will state our main results and prove them in Section 3.

2. Statement of results

We start with some definitions. Let $\mathcal{X}$ be a discrete set. A point process on $\mathcal{X}$ is a probability measure on $2^{\mathcal{X}}$. A point process is completely determined by its correlation functions

$$\rho(X) = \mathrm{Prob}\big\{Y \in 2^{\mathcal{X}} | X \subset Y\big\}. \tag{2.1}$$

A point process is called determinantal, if there exists a kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ such that

$$\rho(X) = \det\big[K(x, y)\big]_{x,y \in X}. \tag{2.2}$$

For more details on determinantal point processes we refer to [4,11,12,14,15,20,21]. A determinantal point process is completely determined by its kernel.

Now let us return to the evolution on the interlacing particle system as described in the introduction. By stopping the process at time t, we get a random collection of points on the grid $\mathcal{G}$. Hence, the Markov process at time t defines a point process on $\mathcal{G}$. In [5], the authors proved that this is in fact a determinantal point process on $\mathcal{G}$ with kernel K

given by

$$\begin{aligned}
&K(x_1,m_1,x_2,m_2)\\
&\quad=-\frac{\chi_{m_1<m_2}}{2\pi\mathrm{i}}\oint_{\Gamma_0}(1-\varepsilon w)^{[m_1/2]-[m_2/2]}(1-\varepsilon/w)^{[(m_1+1)/2]-[(m_2+1)/2]}w^{[x_1]-[x_2]}\frac{\mathrm{d}w}{w}\\
&\qquad+\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\oint_{\Gamma_{\varepsilon,\varepsilon^{-1}}}\mathrm{d}z\,\frac{\mathrm{e}^{t(w+1/w)}(1-\varepsilon w)^{[m_1/2]}(1-\varepsilon/w)^{[(m_1+1)/2]}w^{[x_1]}}{\mathrm{e}^{t(z+1/z)}(1-\varepsilon z)^{[m_2/2]}(1-\varepsilon/z)^{[(m_2+1)/2]}z^{[x_2]}}\frac{1}{z(w-z)},
\end{aligned}\tag{2.3}$$

where $[x]$ is the largest integer less then x. Here Γ_0 is a contour that encircles the essential singularity 0 but not the poles ε and ε^{-1}. The contour $\Gamma_{\varepsilon,\varepsilon^{-1}}$ encircles ε and ε^{-1}. Both Γ_0 and $\Gamma_{\varepsilon,\varepsilon^{-1}}$ have anti-clockwise orientation and do not intersect each other. Finally,

$$\chi_{m_1<m_2}=\begin{cases}1, & \text{if } m_1<m_2,\\ 0, & \text{otherwise.}\end{cases}\tag{2.4}$$

To be precise, the variables we use are different from [5]. We choose a symmetric picture since we have particles jumping both left and right, whereas the particular model that was analyzed in detail in [5] has particles jumping to the right only. Now (2.3) is obtained by taking the kernel in [5], Corollary 2.26, and substituting $y=[x]-[(m+1)/2)]$ and setting the α_l's for even values of l to ε, and the other ones to ε^{-1}. And finally, a conjugation by $(-\varepsilon)^{[(m_2+1)/2]-[(m_1+1)/2)]}$ which does not effect the determinants in the correlation functions.

Our first result is that for large time we obtain the limiting situation as described in the Introduction and shown in Fig. 2.

Theorem 2.1. *Let* $\mathbb{H}=\{z\in\mathbb{C}|\operatorname{Im}x>0\}$ *and* $F:\mathbb{H}\to\mathbb{C}$ *given by*

$$F(z)=\tau\big(z+z^{-1}\big)+\frac{\mu}{2}\log\big(1+\varepsilon^2-\varepsilon\big(z+z^{-1}\big)\big)-\xi\log z.\tag{2.5}$$

Define $\mathcal{D}_1$ *by*

$$\mathcal{D}_1=\big\{(\xi,\mu)\in\mathbb{R}\times\mathbb{R}_+|\exists z\in\mathbb{H}\ F'(z)=0\big\}.\tag{2.6}$$

The boundary ∂D_1 *has two cusp points located at*

$$\big(0,\varepsilon+\varepsilon^{-1}\pm 2\big).\tag{2.7}$$

Set

$$\begin{cases}t=\tau L,\\ m=[\mu L],\\ x=[\xi L],\end{cases}\tag{2.8}$$

then the limiting mean density is given by

$$\lim_{L\to\infty}K(x,m,x,m)=\begin{cases}1, & (\xi,\mu)\in\mathcal{D}_2,\\ \frac{1}{\pi}\arg z(\xi,\mu), & (\xi,\mu)\in\mathcal{D}_1,\\ 0, & \textit{otherwise},\end{cases}\tag{2.9}$$

where $z(\xi,\mu)$ *is the unique solution in the upper half plane of the equation* $F'(z)=0$.

The proof of this result follows from standard steepest descent analysis on the double integral formula for the kernel. Because the proof is standard, we will omit it in this paper. We do not use Theorem 2.1 in the sequel. See [5] for a proof of a similar statement in a comparable situation or [16] for an exposition of the steepest descent technique on double integral formulas.

Remark 2.2. *The boundary $\partial\mathcal{D}_1$ can be explicitly computed. Indeed, it is clear that*

$$\partial\mathcal{D}_1 = \big\{(\xi,\mu)\in\mathbb{R}\times\mathbb{R}_+ | \exists z\in\mathbb{R}\ F'(z)=0\wedge F''(z)=0\big\}. \tag{2.10}$$

Now for each $z\in\mathbb{R}$ we have that

$$\begin{cases} F'(z)=0, \\ F''(z)=0 \end{cases} \tag{2.11}$$

is system of equations that is linear in ξ and μ. By solving this system we find (ξ,μ) as a function of $z\in\mathbb{R}$. It turns out that for each $z\in\mathbb{R}\setminus\{0,\varepsilon,\varepsilon^{-1}\}$ the corresponding (ξ,μ) satisfy $\xi\in\mathbb{R}$ and $\mu\geq 0$ so that (ξ,μ) is a point on $\partial\mathcal{D}_1$. Therefore, the closure of the image of $\mathbb{R}\setminus\{0,\varepsilon,\varepsilon^{-1}\}$ under this map $z\mapsto(\xi,\mu)$ gives the boundary. The cusp pointing down corresponds to $z=-1$ and the cusp pointing up to $z=1$. The boundary touches the x-axis at $z=\varepsilon,\varepsilon^{-1}$.

From Theorem 2.1 it follows that we can achieve the situation of a tacnode in the following way. For every fixed ε the cusp points on the boundary differ by 4. The location tends to infinity when we take the limit $\varepsilon\downarrow 0$. By rescaling μ with ε the cusp points have a limit as $\varepsilon\downarrow 0$ and the gap between the cusp points vanishes, resulting in a tacnode.

From Theorem 2.1 and (2.3) we expect to arrive at a process around the tacnode when we scale

$$\begin{cases} t=\epsilon L, \\ m_j=\big[L^2(1+\mu_j/L)\big], \\ \varepsilon=\epsilon/L. \end{cases} \tag{2.12}$$

However, it is less clear how to describe the process that arises at the cusp. As shown in Fig. 3, there will be long vertical strings of particles. In fact, the length of these strings will be of order L and hence the density of the particles at this scale will diverge. Therefore we do not obtain a point process in the limit if we consider the process on the particles.

There are several ways of constructing a meaningful process in the limit. Perhaps the most straightforward approach is the following: instead of allowing μ_j to be a free variable, we choose $N\in\mathbb{N}$ and fix $\mu_j\in\mathbb{R}$ for $j=1,\dots,N$ with $\mu_j\neq\mu_k$ if $j\neq k$ and cut the process at the section $m_j=[L^2(1+\mu_j/L)]$. For simplicity, we will also shift the particles on any even m_j-section to the left by a half. In this way, we obtain a determinantal point process on $\mathbb{Z}\times(1,\dots,N)$ for each L with correlation functions

$$\mathrm{Prob}\big(\{\text{particle at } (x_k,j_k)\in\mathbb{Z}\times(1,\dots,N) \mid k=1,\dots,n\}\big)=\det\big(K(x_k,m_{j_k},x_l,m_{j_l})\big)_{k,l=1}^n \tag{2.13}$$

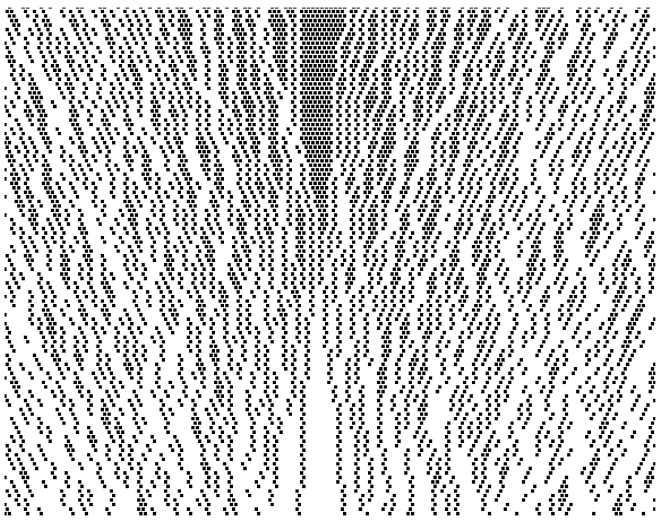

Fig. 3. A close up picture of the process, for $\varepsilon=0$, around the point where the cusps touch. Note that the particle come in pairs that are slanted to the right in the right half and to the left in the left half. Around the tacnode, there are forming long vertical strings of particles.

for all $n \in \mathbb{N}$.

Now to obtain the limiting process as $L \to \infty$ for the limiting process on $\mathbb{Z} \times (1, \dots, N)$, it suffices to compute the pointwise limit for the kernel K. The limit is given in the next theorem, which is the main result of the present paper.

Theorem 2.3. *With t, m_j and ε as in (2.12) we have that*

$$\lim_{L\to\infty} K(x_1, m_1, x_2, m_2) = \mathcal{K}^\epsilon(x_1, \mu_1, x_2, \mu_2) \tag{2.14}$$

for $(x_j, \mu_j) \in \mathbb{Z} \times \mathbb{R}$, where

$$\begin{aligned}\mathcal{K}^\epsilon(x_1, \mu_1, x_2, \mu_2) = &-\frac{\chi_{\mu_1<\mu_2}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{e}^{\epsilon(\mu_2-\mu_1)(w+1/w)} w^{x_1-x_2-1}\,\mathrm{d}w \\ &+ \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} \int_{\Sigma\cup\widetilde{\Sigma}} \frac{\mathrm{e}^{\epsilon\mu_2(z+1/z)+(\epsilon^2/2)(z+1/z)^2} w^{x_1}}{\mathrm{e}^{\epsilon\mu_1(w+1/w)+(\epsilon^2/2)(w+1/w)^2} z^{x_2}} \frac{\mathrm{d}z\,\mathrm{d}w}{z(w-z)}.\end{aligned} \tag{2.15}$$

The contours of integration and their orientation are as indicated in Fig. 4. More precisely, Γ_0 is a contour encircling the origin with counter clockwise orientation. The contour Σ is a contour connecting $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ that does not intersect Γ_0 and stays in the right-half plane. The contour $\widetilde{\Sigma}$ is a contour that does not intersect Γ_0, lies in the right-half plane and passes the origin tangential to the imaginary axis.

To the best of our knowledge, the kernel K^ϵ has not appeared in the existing literature yet.

Note that in the definition of the kernel we may take $\widetilde{\Sigma} = \Sigma^{-1}$. Moreover, there is some freedom to deform the contour Σ as long as it lies eventually inside the sector $S_\delta := \{z \mid \pi/4 + \delta < |\arg z| < 3\pi/4 - \delta\}$ as $|z| \to \infty$ for some $\delta > 0$. Similarly, we can deform $\widetilde{\Sigma}$ as long as it lies inside the sector S_δ as $|z| \to 0$.

It is not difficult to show that after inserting the new parameters given in (2.12) in the integrands in (2.3) and taking the pointwise limit as $L \to \infty$, one obtains the integrands as given in the right-hand side of (2.15). However, there is an important technical issue that needs to be taken care of. Note that the integrand in the double integral contains poles at ε and ε^{-1}, and also an essential singularity at 0. By taking the limit, the pole approaches the essential singularity at the origin, which complicates the contour deformation in the analysis.

A different way of creating a point process is the following. Instead of considering the location of the particles, one could consider the statistics of the upper endpoints of the vertical strings of particles. We will restrict our process only to the odd m-sections. Then (x, m) is defined to be a upper endpoint if there is a particle at (x, m) but there is no particle at $(x, m+2)$. The upper endpoints form a point process with correlation functions

$$\begin{aligned}&\tilde{\rho}_N\big((x_1, m_1), \dots, (x_N, m_N)\big) \\ &\quad = \mathrm{Prob}\big(\text{particle at } (x_j, m_j) \text{ and no particle at } (x_j, m_j+2) \mid j = 1, \dots, N\big)\end{aligned} \tag{2.16}$$

for $(x_j, m_j) \in \mathbb{Z} \times (2\mathbb{N} - 1)$ and $N \in \mathbb{N}$.

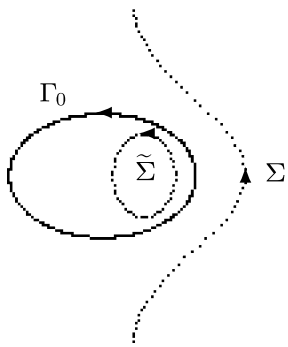

Fig. 4. The contours of integration in the kernel $\mathcal{K}^\epsilon$.

Theorem 2.4. *With $t = \epsilon L$ and m_j the closest odd integer to $L^2(1+\mu_j/L)$, we have that*

$$\lim_{L\to\infty} L^N \tilde{\rho}_N\big((x_1,m_1),\dots,(x_N,m_N)\big) = \epsilon^N \det\big(\mathcal{K}^\epsilon(x_i-1,\mu_i,x_j,\mu_j)+\mathcal{K}^\epsilon(x_i+1,\mu_i,x_j,\mu_j)\big)_{i,j=1}^N \tag{2.17}$$

for $(x_j,\mu_j)\in\mathbb{Z}\times\mathbb{R}$ for $j=1,\dots,N$.

Remark 2.5. *Another argument for the fact that the lengths of the vertical strings of particles must be of order L, is that one can prove that the process restricted to two horizontal sections that are close, is just constructed of two copies of the process restricted to one of these horizontal sections. To be precise, let $\mu_1\in\mathbb{R}$ and take $m_1=[L^2(1+\mu_1/L)]$. For each L let m_2 be such that $m_1-m_2=\mathrm{o}(L)$ as $L\to\infty$. For simplicity, we assume that $m_2\ge m_1$. Then it is not difficult to prove that*

$$\begin{aligned}
&\lim_{L\to\infty} K(x_1,m_1,x_2,m_1)=\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_1),\\
&\lim_{L\to\infty} K(x_1,m_2,x_2,m_2)=\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_1),\\
&\lim_{L\to\infty} K(x_1,m_2,x_2,m_1)=\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_1),\\
&\lim_{L\to\infty} K(x_1,m_1,x_2,m_2)=-\delta_{x_1,x_2}+\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_1)
\end{aligned} \tag{2.18}$$

for all $x_1,x_2\in\mathbb{Z}$. Hence, for $x_1,\dots,x_l,y_1,\dots,y_k$ we have

$$\begin{aligned}
&\lim_{L\to\infty}\mathrm{Prob}\big(\textit{particles at } (x_1,m_1),\dots,(x_l,m_1),(y_1,m_2),\dots,(y_k,m_2)\big)\\
&\quad=\det\begin{pmatrix}(\mathcal{K}^\epsilon(x_i,\mu_1,x_j,\mu_1))_{i,j=1}^l & (\mathcal{K}^\epsilon(x_i,\mu_1,y_j,\mu_1)-\delta_{x_i,x_j})_{i=1,j=1}^{l,k}\\ (\mathcal{K}^\epsilon(y_i,\mu_1,x_j,\mu_1))_{i=1,j=1}^{k,l} & (\mathcal{K}^\epsilon(y_i,\mu_1,y_j,\mu_1))_{i,j=1}^k\end{pmatrix}.
\end{aligned} \tag{2.19}$$

This implies that (in the limit $L\to\infty$) the process on the line m_2 is just a copy of the process on the line m_1.

We will now derive some properties of the kernel $\mathcal{K}^\epsilon$. The following proposition shows the symmetry in the kernel.

Proposition 2.6. *We have that*:

(1) $\mathcal{K}^\epsilon(-x_1,\mu_1,-x_2,\mu_2)=\mathcal{K}^\epsilon(x_1-1,\mu_1,x_2-1,\mu_2)$,
(2) $(-1)^{x_1-x_2}\mathcal{K}^\epsilon(x_1,-\mu_1,x_2,-\mu_2)=\delta_{x_1,x_2}\delta_{m_1,m_2}-\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_2)$

for all $(x_j,\mu_j)\in\mathbb{Z}\times\mathbb{R}$.

The first property shows that our point process is invariant with respect to the transform $x\mapsto -1-x$.

To interpret the second symmetry property in this proposition, we note that if $\mathcal{P}$ is a determinantal point process on a discrete set $\mathcal{X}$ with kernel K, then we have that $1-K$ is the kernel of the determinantal point process $\mathcal{P}'$ defined by $\mathcal{P}'(X)=\mathcal{P}(\mathcal{X}\setminus X)$, for $X\subset\mathcal{X}$. This process is sometimes referred to as the complementary process. It is obtained by replacing particles with holes and vice versa. For more details on this particle hole involution we refer to the Appendix of [7].

In the final results of this paper we investigate the limiting behavior of the kernel as $\epsilon\downarrow 0$ and $\epsilon\to\infty$. We start with the first case.

Theorem 2.7. *For $x_1,x_2<0$ we have that*

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\epsilon^{x_1-x_2}\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_2) &= -\chi_{\mu_1<\mu_2}\chi_{x_1\le x_2}\frac{(\mu_2-\mu_1)^{x_2-x_1}}{(x_2-x_1)!}\\
&\quad+\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\int_\Sigma \frac{\mathrm{e}^{\mu_2 z+(1/2)z^2}}{\mathrm{e}^{\mu_1 w+(1/2)w^2}}\frac{w^{x_1}}{z^{x_2+1}}\frac{\mathrm{d}z\,\mathrm{d}w}{w-z},
\end{aligned} \tag{2.20}$$

and for $x_1, x_2 \geq 0$.

$$\lim_{\epsilon\downarrow 0} \epsilon^{x_2-x_1}\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_2) = -\chi_{\mu_1<\mu_2}\chi_{x_2\leq x_1}\frac{(\mu_2-\mu_1)^{x_1-x_2}}{(x_1-x_2)!} + \frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0}\int_\Sigma \frac{\mathrm{e}^{\mu_2 z+(1/2)z^2}}{\mathrm{e}^{\mu_1 w+(1/2)w^2}}\frac{z^{x_2}}{w^{x_1+1}}\frac{\mathrm{d}z\,\mathrm{d}w}{w-z}. \tag{2.21}$$

Moreover, the limit of the kernel vanishes if neither of the two conditions on x_1 *and* x_2 *are satisfied.*

The one-dimensional version $\mu_1 = \mu_2$ of this kernel has appeared before in the literature, see [6]. By expanding the term $(w-z)^{-1}$ we can express the double integral as a sum of products of Hermite polynomials.

By inserting (2.21) (or (2.20)) in (2.17) we find the limiting process for the endpoints in the right (or left) half plane. It turns out that in each of the half planes one of the kernels in the determinant tends to zero as $\epsilon \downarrow 0$. In fact, the limiting kernel is the kernel corresponding to the GUE minor process, see for example [13,17]. Indeed, in the case $x_1, x_2 \geq 0$ we rewrite (2.17) as

$$\begin{aligned}&\lim_{L\to\infty} L^N \tilde{\rho}_N\big((x_1,m_1),\dots,(x_N,m_N)\big)\\ &\quad = \det\big(\epsilon^{x_j+1-x_i}\mathcal{K}^\epsilon(x_i-1,\mu_i,x_j,\mu_j) + \epsilon^2\epsilon^{x_j-1-x_j}K^\epsilon(x_i+1,\mu_i,x_j,\mu_j)\big)_{i,j=1}^N,\end{aligned} \tag{2.22}$$

and then by (2.21) we find

$$\lim_{\epsilon\downarrow 0}\lim_{L\to\infty} L^N \tilde{\rho}_N\big((x_1,m_1),\dots,(x_N,m_N)\big) = \det\big(K_{\mathrm{GUE}}(x_i,\mu_i,x_j,\mu_j)\big)_{i,j=1}^N, \tag{2.23}$$

where

$$K_{\mathrm{GUE}}(x_i,\mu_i,x_j,\mu_j) = -\chi_{\mu_1<\mu_2}\chi_{x_2<x_1}\frac{(\mu_2-\mu_1)^{x_2-x_1-1}}{(x_2-x_1-1)!} + \frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0}\int_\Sigma \frac{\mathrm{e}^{\mu_2 z+(1/2)z^2}}{\mathrm{e}^{\mu_1 w+(1/2)w^2}}\frac{z^{x_2}}{w^{x_1}}\frac{\mathrm{d}z\,\mathrm{d}w}{w-z}. \tag{2.24}$$

The case $x_1, x_2 < 0$ is similar. By comparing with [13], Definition 1.2, we see that the kernel K_{GUE} describes the GUE minor process.

Remark 2.8. *Based upon the relation between the construction of the two processes just above and below Theorem* 2.3, *we see how the process defined by the kernel at the right-hand side of* (2.21) *can be constructed out of the GUE minor process explicitly: pick a point configuration random from the GUE minor process and denote the points by* $(x, y_x^l) \in \mathbb{N}\times\mathbb{R}$. *At each vertical* x *section we draw* x *vertical line segments. Each segment has* y_x^l *as a lower endpoint and* y_{x-1}^{l-1} *as the upper endpoint. Here we define* $y_x^0 = \infty$. *See also Fig.* 5. *We next fix* N *different real numbers* $\mu_j \in \mathbb{R}$. *Then we define a point process in* $\mathbb{N}\times\{1,\dots,N\}$ *by*

$$\mathrm{Prob}\big\{\text{particle at } (x_i, j_i), i = 1,\dots,k)\big\} = \mathrm{Prob}\big\{\text{Each } (x_i,\mu_{j_i}) \text{ is on a line segment}\big\}. \tag{2.25}$$

The conclusion is that the new process is in fact a determinantal point process with kernel as in (2.21).

The second situation that can be obtained is that for $\epsilon \to \infty$, in which the cusps should separate. Hence we expect to obtain the Pearcey process when we take the simultaneous limit $\mu_j \to -\infty$ (or $\mu_j \to +\infty$). The following theorem states that we indeed find the Pearcey process at the lowest of the two cusps.

Theorem 2.9. *Set*

$$\begin{cases}\epsilon = M,\\ \mu_j = -M(1-\nu_j/2M),\\ x = \big[\xi_j M^{1/2}\big].\end{cases} \tag{2.26}$$

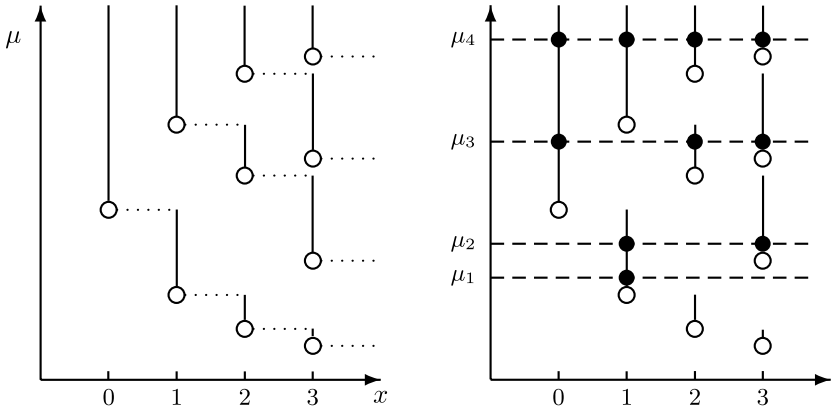

Fig. 5. From the GUE minor process to the point process with kernel given in (2.21). The open circles represent the point (x, y_x^l) from $l = 0, \dots, x$ and $x = 0, 1, \dots$. In the left picture we draw the vertical lines. The dotted lines are only auxillary. In the right picture, we draw the lines associated to the choice of the μ_j. The solid circles are the intersection point of the dashed horizontal and solid vertical line. The solid circles describe the process with kernel (2.21).

Then

$$\lim_{M\to\infty} M^{1/2} \mathrm{e}^{2M(\nu_1-\nu_2)} \mathcal{K}^{\epsilon}(x_1, \mu_1, x_2, \mu_2) = -\frac{\chi_{\nu_1<\nu_2}}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}w\, \mathrm{e}^{(\nu_2-\nu_1)w^2-(\xi_2-\xi_1)w}$$
$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \int_{\mathcal{C}} \frac{\mathrm{e}^{(1/2)z^4+\nu_2 z^2-\xi_2 z}}{\mathrm{e}^{(1/2)w^4+\nu_1 w^2-\xi_1 w}} \frac{\mathrm{d}w\, \mathrm{d}z}{w-z}. \tag{2.27}$$

The contour $\mathcal{C}$ consists of four rays, from $\pm \mathrm{e}^{\pi \mathrm{i}/4}\infty$ to 0 and from 0 to $\pm \mathrm{e}^{3\pi \mathrm{i}/4}\infty$ (see also Fig. 6).

The kernel given (2.27) is known in the literature as the extended Pearcey kernel that describes the Pearcey process, see [2,8,9,18,22] for more details.

To conclude this section, note that if we combine Theorem 2.9 with the second symmetry property in Proposition 2.6 then we see that the complementary process near the top cusp locally converges to the Pearcey process, as expected.

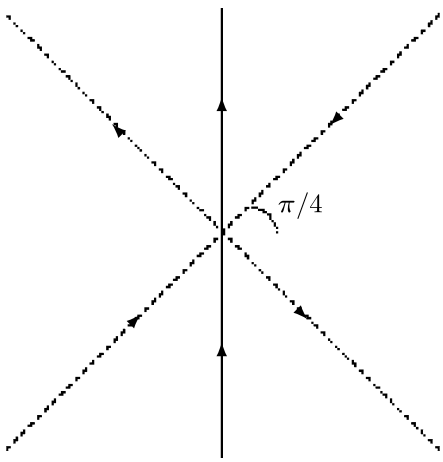

Fig. 6. The contours of integration for the Pearcey kernel.

3. Proofs

In this section we prove our results.

3.1. *Proof of Theorem 2.3*

Let us first introduce some notation. Write the kernel in (2.3) as

$$K(x_1,m_1,x_2,m_2) = -\frac{\chi_{m_1<m_2}}{2\pi \mathrm{i}}\oint_{\Gamma_0}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(w)}\frac{\mathrm{d}w}{w} + \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\oint_{\Gamma_{\varepsilon,\varepsilon^{-1}}}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)}\frac{1}{z(w-z)}\,\mathrm{d}z\,\mathrm{d}w, \tag{3.1}$$

where

$$G_{t,m,x}(w) = \mathrm{e}^{t(w+1/w)}(1-\varepsilon w)^{[m_1/2]}(1-\varepsilon/w)^{[(m_1+1)/2]}w^{[x]}. \tag{3.2}$$

Let us for a moment ignore the contours of integration and take the limit $L\to\infty$ for the integrand. It is not difficult to see that we have the following pointwise limit

$$\lim_{L\to\infty}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)} = \frac{\mathrm{e}^{\epsilon\mu_2(z+1/z)+\epsilon^2(z+1/z)^2}}{\mathrm{e}^{\epsilon\mu_2(w+1/w)+\epsilon^2(w+1/w)^2}}\frac{w^{[x_1]}}{z^{[x_2]}} \tag{3.3}$$

for $z,w\in\mathbb{C}\setminus\{0\}$. Hence the difficulty in the proof lies in the fact that we have to control the contours of integration (which clearly depend on ε and hence L) while taking the limit.

As a first step we prove the following result.

Lemma 3.1. *With G as in (3.2) and K as in (2.3) we have that*

$$\begin{aligned}&K(x_1,m_1,x_2,m_2)\\ &\quad= -\frac{\chi_{m_1<m_2}}{2\pi\mathrm{i}}\oint_{\Gamma_{0,\varepsilon}}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(w)}\frac{\mathrm{d}w}{w} + \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_{0,\varepsilon}}\oint_{\Gamma_\varepsilon\cup\Gamma_{\varepsilon^{-1}}}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)}\frac{1}{z(w-z)}\,\mathrm{d}z\,\mathrm{d}w.\end{aligned} \tag{3.4}$$

Here Γ_ε and $\Gamma_{\varepsilon^{-1}}$ are two contours encircling the poles ε and ε^{-1}, respectively, but no other poles. The contour $\Gamma_{0,\varepsilon}$ is a contour encircling the origin and the contour Γ_ε but not $\Gamma_{\varepsilon^{-1}}$. The contours are taken so that they do not intersect and have counter clockwise orientation. See also Fig. 7.

Proof. First split the contour $\Gamma_{\varepsilon,\varepsilon^{-1}}$ into two small contours Γ_ε around ε and $\Gamma_{\varepsilon^{-1}}$ around ε^{-1}. Then

$$\begin{aligned}K(x_1,m_1,x_2,m_2) &= -\frac{\chi_{m_1<m_2}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(w)}\frac{\mathrm{d}w}{w} + \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\oint_{\Gamma_\varepsilon}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)}\frac{1}{z(w-z)}\,\mathrm{d}z\,\mathrm{d}w\\ &\quad+\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\oint_{\Gamma_{\varepsilon^{-1}}}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)}\frac{1}{z(w-z)}\,\mathrm{d}z\,\mathrm{d}w.\end{aligned} \tag{3.5}$$

Now in the first double integral in the right-hand side we deform Γ_0 so that it encircles Γ_ε. This deformation will be denoted by $\Gamma_{0,\varepsilon}$. Note that the contour $\Gamma_{0,\varepsilon}$ goes thus around the poles 0 and ε. While deforming the contour we pick up a residue at $w=z$. Hence we obtain an extra single integral over the contour Γ_ε

$$\begin{aligned}&K(x_1,m_1,x_2,m_2)\\ &\quad= -\frac{\chi_{m_1<m_2}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(w)}\frac{\mathrm{d}w}{w} - \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_\varepsilon}\frac{G_{t,m_1,x_1}(z)}{G_{t,m_2,x_2}(z)}\frac{\mathrm{d}z}{z}\\ &\qquad+\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_{0,\varepsilon}}\oint_{\Gamma_\varepsilon}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)}\frac{1}{z(w-z)}\,\mathrm{d}z\,\mathrm{d}w + \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\oint_{\Gamma_{\varepsilon^{-1}}}\frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)}\frac{1}{z(w-z)}\,\mathrm{d}z\,\mathrm{d}w.\end{aligned} \tag{3.6}$$

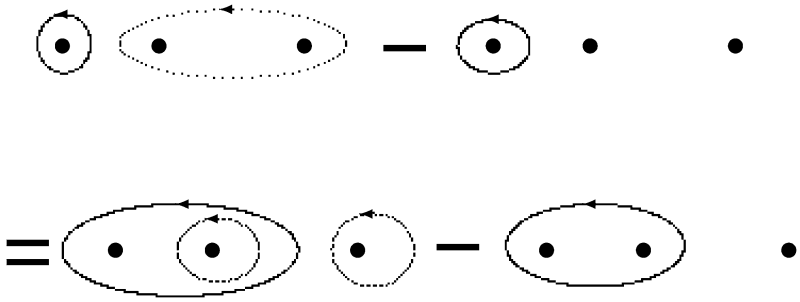

Fig. 7. Deforming the contours as in Lemma 3.1.

Now the extra single integral has the same integrand as the first single integral. The integration is now over a contour encircling the pole at ε (and no other pole). However, this pole is not present in the case $m_1 < m_2$ and then the integral vanishes. Therefore we can glue the integrals over Γ_0 and Γ_ε together and obtain

$$K(x_1, m_1, x_2, m_2) = -\frac{\chi_{m_1<m_2}}{2\pi \mathrm{i}} \oint_{\Gamma_{0,\varepsilon}} \frac{G_{t,m_1,x_1}(z)}{G_{t,m_2,x_2}(z)} \frac{\mathrm{d}z}{z} + \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_{0,\varepsilon}} \oint_{\Gamma_\varepsilon} \frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)} \frac{1}{z(w-z)} \,\mathrm{d}z\,\mathrm{d}w$$
$$+ \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} \oint_{\Gamma_{\varepsilon^{-1}}} \frac{G_{t,m_1,x_1}(w)}{G_{t,m_2,x_2}(z)} \frac{1}{z(w-z)} \,\mathrm{d}z\,\mathrm{d}w. \tag{3.7}$$

Finally, note that the integrand has no pole at $w = \varepsilon$ so in the second double integral we can safely deform Γ_0 to be $\Gamma_{0,\varepsilon}$. This proves the claim. □

Now the proof of Theorem 2.3 follows by simply taking the limit in the integrands and correctly choosing the contours Γ_ε and $\Gamma_{\varepsilon^{-1}}$.

Proof of Theorem 2.3. We take the contour $\Gamma_{0,\varepsilon}$ to be some fixed contour encircling the origin independent of ε, say the unit circle for simplicity. Due to the behavior of $G_{t,m_2,x_2}(z)$ for $z \to \infty$ we can deform $\Gamma_{\varepsilon^{-1}}$ to any simple contour Σ that connects ∞ to ∞ crossing the real axis once at a point between 1 and ε^{-1} and contained in the sector

$$\{z \in \mathbb{C} \mid -\pi/2 + \delta < \arg z < \pi/2 - \delta\} \tag{3.8}$$

for some fixed $\delta > 0$. In fact we choose Σ to be a contour so that it eventually falls inside the sector

$$\{z \in \mathbb{C} \mid -\pi/2 + \delta < \arg z < -\pi/4 - \delta,\ \pi/4 + \delta < \arg z < \pi/2 - \delta\} \tag{3.9}$$

for some $\delta > 0$.

The behavior of $G_{t,m_2,x_2}(z)$ for $z \to 0$ is to a large extent similar to the behavior near ∞, which can be seen by performing the transform $z \mapsto 1/z$. We can (and do) deform the contour Γ_ε to be Σ^{-1}. Note that in this way, we have deformed the contours, so that they go through the essential singularities of the integrand.

Now that we have chosen the contours, that are clearly independent of ε, we can compute the limits of the integrand, which is given in (3.3). Due to the choice of the sector in which Σ eventually falls inside, the convergence is uniform in $z \in \Sigma \cup \Sigma^{-1}$ and $w \in \Gamma_0$. Finally, we can replace Σ^{-1} by any deformation $\widetilde{\Sigma}$ that lies inside the sector

$$\left\{z \in \mathbb{C} \mid \pi/4 + \delta \le |\arg z| \le 3\pi/4 - \delta\right\}, \tag{3.10}$$

as $z \to 0$. This proves the statement. □

3.2. *Proof of Theorem 2.4*

We start by noting that by expanding a factor $(1-\varepsilon/w)(1+\varepsilon w)$ in the integrals for the kernel K in (2.3) we get for odd n, m

$$K(x,n+2,y,m) = \big(1+\varepsilon^2\big)K(x,n,y,m) - \varepsilon\big(K(x+1,n,y,m)+K(x-1,n,y,m)\big) + \frac{\delta_{n,m-2}}{2\pi\mathrm{i}}\oint_{\Gamma_0}(1-\varepsilon w)(1-\varepsilon/w)w^{x-y-1}\,\mathrm{d}w. \tag{3.11}$$

There is an additional single integral because of the fact that $\chi_{n+2<m} = \chi_{n<m} - \delta_{n,m-1} - \delta_{n,m-2}$. Now since n, m are both odd $\delta_{n,m-1}$ vanishes trivially and only $\delta_{n,m-2}$ remains.

Similarly,

$$K(x,n,y,m-2) = \big(1+\varepsilon^2\big)K(x,n,y,m) - \varepsilon\big(K(x,n,y+1,m)+K(x,n,y-1,m)\big) + \frac{\delta_{n,m-2}}{2\pi\mathrm{i}}\oint_{\Gamma_0}(1-\varepsilon w)(1-\varepsilon/w)w^{x-y-1}\,\mathrm{d}w. \tag{3.12}$$

Therefore

$$\begin{aligned}
&-K(x_i,m_i+2,x_j,m_j)+K(x_i,m_i,x_j,m_j)\\
&\quad=\varepsilon\big(K(x_i+1,m_i,x_j,m_j)+K(x_i-1,m_i,x_j,m_j)+\delta_{m_i,m_j-2}(\delta_{x_i+1,x_j}+\delta_{x_i-1,x_j})\big)\\
&\qquad-\delta_{m_i,m_j-2}\delta_{x_i,x_j}+\mathcal{O}\big(\varepsilon^2\big).
\end{aligned} \tag{3.13}$$

Now note that if we have $m_i \neq m_j$ then $m_i - m_j$ is of order L, so that we can ignore the term δ_{m_i,m_j-2}, so that

$$-K(x_i,m_i+2,x_j,m_j)+K(x_i,m_i,x_j,m_j)=\varepsilon\big(K(x_i+1,m_i,x_j,m_j)+K(x_i-1,m_i,x_j,m_j)\big) \tag{3.14}$$

and

$$\begin{aligned}
&-K(x_i,m_i+2,x_j,m_j+2)+K(x_i,m_i,x_j,m_j+2)\\
&\quad=\varepsilon\big(K(x_i+1,m_i,x_j,m_j+2)+K(x_i-1,m_i,x_j,m_j+2)+\delta_{m_i,m_j}(\delta_{x_i+1,x_j}+\delta_{x_i-1,x_j})\big)\\
&\qquad-\delta_{m_i,m_j}\delta_{x_i,x_j}+\mathcal{O}\big(\varepsilon^2\big).
\end{aligned} \tag{3.15}$$

By subtracting (3.15) from (3.14) and inserting (3.12) we obtain

$$\begin{aligned}
&-K(x_i,m_i+2,x_j,m_j)+K(x_i,m_i,x_j,m_j)+K(x_i,m_i+2,x_j,m_j+2)-K(x_i,m_i,x_j,m_j+2)\\
&\quad=\delta_{m_i,m_j}\delta_{x_i,x_j}+\mathcal{O}\big(\varepsilon^2\big).
\end{aligned} \tag{3.16}$$

By the complementation principle (see [7])

$$\begin{aligned}
&\tilde{\rho}_N\big((x_1,m_1),\dots,(x_N,m_N)\big)\\
&\quad=\det\begin{pmatrix} K(x_i,m_i,x_j,m_j) & -K(x_i,m_i+2,x_j,m_j)\\ K(x_i,m_i,x_j,m_j+2) & \delta_{x_i,x_j}\delta_{m_i,m_j}-K(x_i,m_i+2,x_j,m_j+2)\end{pmatrix}.
\end{aligned} \tag{3.17}$$

By adding the first column to the second and afterward subtracting the second row from the first we obtain that the determinant equals

$$\det\begin{pmatrix} K(x_i,m_i,x_j,m_j)-K(x_i,m_i,x_j,m_j+2) & \begin{array}{c}-K(x_i,m_i+2,x_j,m_j)+K(x_i,m_i,x_j,m_j)\\ -\delta_{x_i,x_j}\delta_{m_i,m_j}+K(x_i,m_i+2,x_j,m_j+2)\\ -K(x_i,m_i,x_j,m_j+2)\end{array}\\ K(x_i,m_i,x_j,m_j+2) & \begin{array}{c}\delta_{x_i,x_j}\delta_{m_i,m_j}-K(x_i,m_i+2,x_j,m_j+2)\\ +K(x_i,m_i,x_j,m_j+2)\end{array}\end{pmatrix}. \tag{3.18}$$

Now using, (3.12) in the upper left block, (3.16) in the upper right block, and (3.15) and (3.12) in the lower right block, this determinant has the form

$$\begin{aligned}&\tilde{\rho}_N\big((x_1,m_1),\dots,(x_N,m_N)\big)\\&\quad=\det\begin{pmatrix}1+\mathcal{O}(\varepsilon^2) & \mathcal{O}(\varepsilon^2)\\ \mathcal{O}(1) & \varepsilon(K(x_i+1,m_i,x_j,m_j)+K(x_i-1,m_i,x_j,m_j))\end{pmatrix},\end{aligned}\tag{3.19}$$

from which the proposition easily follows.

3.3. *Proof of Proposition* 2.6

(1) The statement easily follows by the transform $z\mapsto 1/z$ and $w\mapsto 1/w$ in the integral representation of $K^\epsilon(-x_1,\mu_1,-x_2,\mu_2)$.

(2) The second property follows by the transformation $z\mapsto -z$ and $w\mapsto -w$ in the integrals and a deformation of Σ. In the definition of the kernel K^ϵ we take Σ on the right of Γ_0. However, we can also take Σ to be on the left at the cost of an extra integral as shown in Fig. 8. The second double integral at the right is easy to compute since the integral over z encircles the pole $z=w$ only and hence can be computed by the Residue Theorem. The result of this is that we can rewrite the kernel in the following way

$$K^\epsilon(x_1,\mu_1,x_2,\mu_2)=\delta_{x_1,x_2}\delta_{\mu_1,\mu_2}+\chi_{\mu_2<\mu_1}\int_{\Gamma_0}-\oint_{\Gamma_0}\int_{-(\Sigma\cup\widetilde{\Sigma})}.\tag{3.20}$$

Now by inserting $\mu_j=-\mu_j$ and the transformation $w\mapsto -w$ and $z\mapsto -z$ we arrive at the statement.

3.4. *Proof of Theorem* 2.7

By the first symmetry property in Proposition 2.6 it suffices to consider the case $x_2<0$ only. Using the transform $z\mapsto z/\epsilon$ and $w\mapsto w/\epsilon$ in (2.15) we obtain

$$\begin{aligned}\epsilon^{x_1-x_2}\mathcal{K}^\epsilon(x_1,\mu_1,x_2,\mu_2)=&-\frac{\chi_{\mu_1<\mu_2}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{e}^{(\mu_2-\mu_1)(w+\epsilon^2/w)}w^{x_1-x_2-1}\,\mathrm{d}w\\&+\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\int_{\Sigma\cup\widetilde{\Sigma}}\frac{\mathrm{e}^{\mu_2(z+\epsilon^2/z)+(1/2)(z+\epsilon^2/z)^2}w^{x_1}}{\mathrm{e}^{\mu_1(w+\epsilon^2/w)+(1/2)(w+\epsilon^2/w)^2}z^{x_2}}\frac{\mathrm{d}z\,\mathrm{d}w}{z(w-z)}.\end{aligned}\tag{3.21}$$

Note that in the new variables the contours of integration should also be scaled. However, by analyticity we can deform them back to the original contours again.

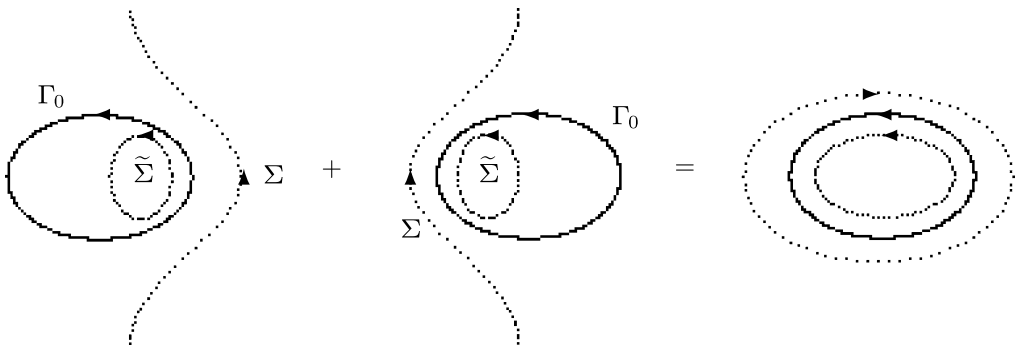

Fig. 8. In the definition of $\mathcal{K}^\epsilon$ we can switch the contour Σ to $-\Sigma$ to obtain (3.20).

Setting $\epsilon = 0$ at the right-hand side gives

$$\lim_{\epsilon\downarrow 0} \epsilon^{x_1-x_2}\mathcal{K}^{\epsilon}(x_1,\mu_1,x_2,\mu_2) = -\frac{\chi_{\mu_1<\mu_2}}{2\pi\mathrm{i}}\oint_{\Gamma_0} \mathrm{e}^{(\mu_2-\mu_1)w} w^{x_1-x_2-1}\,\mathrm{d}w + \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\int_{\Sigma\cup\widetilde{\Sigma}} \frac{\mathrm{e}^{\mu_2 z+(1/2)z^2} w^{x_1}}{\mathrm{e}^{\mu_1 w+(1/2)w^2} z^{x_2}} \frac{\mathrm{d}z\,\mathrm{d}w}{z(w-z)}. \tag{3.22}$$

The single integral can be easily computed. As for the double integral, we note that since $x_2 < 0$ we have that the integral over $\widetilde{\Sigma}$ vanishes and hence

$$\lim_{\epsilon\downarrow 0} \epsilon^{x_2-x_1}\mathcal{K}^{\epsilon}(x_1,\mu_1,x_2,\mu_2) = -\frac{\chi_{\mu_1<\mu_2}}{2\pi\mathrm{i}}\oint_{\Gamma_0} \mathrm{e}^{(\mu_2-\mu_1)w} w^{x_1-x_2-1}\,\mathrm{d}w + \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\int_{\Sigma} \frac{\mathrm{e}^{\mu_2 z+(1/2)z^2} w^{x_1}}{\mathrm{e}^{\mu_1 w+(1/2)w^2} z^{x_2}} \frac{\mathrm{d}z\,\mathrm{d}w}{z(w-z)}. \tag{3.23}$$

The single integral can easily be computed. This proves (2.20).

Finally, note that if $x_1 \geq 0$ and $x_2 < 0$ then both integrals vanish since there is no pole for $w = 0$.

3.5. *Proof of Theorem 2.9*

The double integral in (2.15) in the new parameters given by (2.26) reads

$$\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\int_{\Sigma\cup\widetilde{\Sigma}}\mathrm{d}z\,\frac{\mathrm{e}^{(M^2/2)(z-1)^4/z^2+M\nu_2(z+1/z)-M^{1/2}\xi_2\ln z}}{\mathrm{e}^{(M^2/2)(w-1)^4/w^2+M\nu_1(w+1/w)-M^{1/2}\xi_1\ln w}}\frac{1}{z(w-z)}. \tag{3.24}$$

For large M the main contribution comes from the terms

$$\frac{(z-1)^4}{z^2} \quad \text{and} \quad \frac{(w-1)^4}{w^2}. \tag{3.25}$$

This term has a critical point at $z = 1$ (resp., $w = 1$) of order three. Therefore, there are four paths of steepest descent leaving from $z = 1$ and four paths of steepest ascend, as shown in Fig. 9. In fact, the paths of steepest ascent leaving from $w = 1$ are the unit circle and the real line. We now deform Γ_0 to be the unit circle. The contour Σ is deformed so that it passes through $z = 1$ and follows the path of steep descent outside the unit circle. By symmetry, we take $\widetilde{\Sigma} = \Sigma^{-1}$. Then locally around $z = 1$ and $w = 1$ the contours Γ_0, Σ and $\widetilde{\Sigma}$ can be deformed to the contours for the Pearcey kernel as shown in Fig. 6.

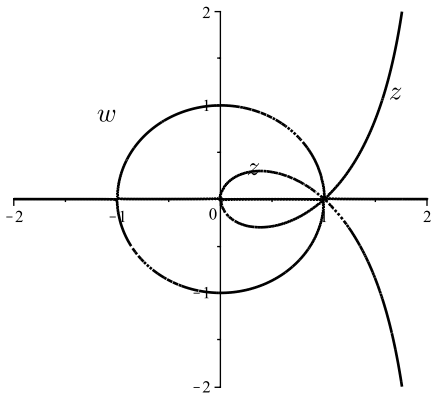

Fig. 9. The line $\{w \mid \mathrm{Im}((w-1)^4/w^2) = 0\}$ which are the contours of steepest descent/ascent leaving from $w = 1$.

By standard steepest descent arguments one can now show that the dominant contribution comes from a neighborhood around $z = 1$ and $w = 1$. More precisely, after introducing the local variables

$$\begin{cases} w = 1 + \tilde{w}/M^{1/2}, \\ z = 1 + \tilde{z}/M^{1/2}, \end{cases} \tag{3.26}$$

it is not difficult to prove that

$$\begin{aligned} &\frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \int_{\Sigma \cup \tilde{\Sigma}} \mathrm{d}z \, \frac{\mathrm{e}^{(M^2/2)(z-1)^4/z^2 + M\nu_2(z+1/z) - M^{1/2}\xi_2 \ln z}}{\mathrm{e}^{(M^2/2)(w-1)^4/w^2 + M\nu_1(w+1/w) - M^{1/2}\xi_1 \ln w}} \frac{1}{z(w-z)} \\ &\quad = \frac{\mathrm{e}^{2M(\nu_2 - \mu_1)}}{M^{1/2}(2\pi \mathrm{i})^2} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tilde{z} \int_{\mathcal{C}} \mathrm{d}\tilde{w} \, \frac{\mathrm{e}^{(1/2)\tilde{z}^4 + \nu_2 \tilde{z}^2 - \xi_2 \tilde{z}}}{\mathrm{e}^{(1/2)\tilde{w}^4 + \nu_1 \tilde{w}^2 - \xi_1 \tilde{w}}} \frac{1}{\tilde{w} - \tilde{z}} \big(1 + \mathcal{O}(M^{-1/2})\big), \end{aligned} \tag{3.27}$$

as $M \to \infty$.

Now consider the single integral in (2.15), which in the new parameters (2.26) reads

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \, \mathrm{e}^{M(\nu_2 - \nu_1)(w + 1/w)} w^{M^{1/2}(\xi_1 - \xi_2) - 1}. \tag{3.28}$$

The dominant term in this integral is

$$\mathrm{e}^{M(\nu_2 - \nu_1)(w + 1/w)}. \tag{3.29}$$

A simple computation shows that $w + 1/w$ has two saddle points $w = \pm 1$, both of order one. Since $\nu_2 > \nu_1$ (otherwise the single integral is not present) we have that $w = +1$ is the dominant saddle point. This means that, the main contribution comes from the part of the contour close to $w = 1$. Hence, if we introduce the new local variable

$$w = 1 + \tilde{w}/M^{1/2}, \tag{3.30}$$

then by standard arguments one can prove that

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \, \mathrm{e}^{M(\nu_2 - \nu_1)(w + 1/w)} w^{M^{1/2}(\xi_1 - \xi_2) - 1} \\ &\quad = \frac{\mathrm{e}^{2M(\nu_2 - \nu_1)}}{M^{1/2} 2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tilde{w} \, \mathrm{e}^{(\nu_2 - \nu_1)\tilde{w}^2 - (\xi_1 - \xi_2)\tilde{w}} \big(1 + \mathcal{O}(M^{-1/2})\big). \end{aligned} \tag{3.31}$$

By inserting (3.31) and (3.27) in (2.15) and taking the limit $M \to \infty$ we obtain (2.27). This proves Theorem 2.9.

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