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Hiding a constant drift

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Dedicated to Marc Yor at the occasion of his 60th birthday

Abstract. The following question is due to Marc Yor: Let B be a Brownian motion and $S_t = t + B_t$. Can we define an $\mathcal{F}^B$-predictable process H such that the resulting stochastic integral $(H \cdot S)$ is a Brownian motion (without drift) in its own filtration, i.e. an $\mathcal{F}^{(H\cdot S)}$-Brownian motion?

In this paper we show that by dropping the requirement of $\mathcal{F}^B$-predictability of H we can give a positive answer to this question. In other words, we are able to show that there is a weak solution to Yor's question. The original question, i.e., existence of a strong solution, remains open.

Résumé. La question suivante a été posée par Marc Yor: Soit B un mouvement Brownien et $S_t = t + B_t$. Peut-on définir un processus H qui est $\mathcal{F}^B$-prévisible tel que l'intégrale stochastique $(H \cdot S)$ soit un mouvement Brownien (sans drift) pour sa propre filtration $\mathcal{F}^{(H\cdot S)}$?

Dans cet article nous fournissons une réponse affirmative en relâchant la condition que H soit $\mathcal{F}^B$-prévisible. Autrement dit, nous montrons qu'il existe une solution faible pour cette question de Yor. La question originale (c'est à dire, l'existence d'une solution forte) reste ouverte.

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Keywords: Brownian motion with drift; Stochastic integral; Enlargement of filtration

1. Introduction

1.1. *Main result*

In this paper, we investigate the following question, due to Marc Yor: let B be a Brownian motion and $S_t = t + B_t$. Can we define an $\mathcal{F}^B$-predictable process H such that the resulting stochastic integral $(H \cdot S)$ is a Brownian motion (without drift) in its own filtration, i.e. an $\mathcal{F}^{H\cdot S}$-Brownian motion? Our main result here is the following theorem.

Theorem 1. *Let $(W_t)_{t\ge 0}$ be a real-valued standard Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$ and denote by $(\mathcal{F}_t^W)_{t\ge 0}$ its (right continuous, saturated) natural filtration.*

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Fix $\mu \in \mathbb{R}$. Then there is an $(\mathcal{F}_t^W)_{t\geq 0}$-Brownian motion $(B_t)_{t\geq 0}$ as well as an $(\mathcal{F}_t^W)_{t\geq 0}$-predictable, $\{-1,+1\}$-valued process H such that the stochastic integral $H \cdot S$ is a Brownian motion in its own filtration $(\mathcal{F}_t^{H\cdot S})_{t\geq 0}$, where $S_t = B_t + \mu t$.

Theorem 1 improves upon the result in a previous paper [9], where it was proved that for a given Brownian motion B, a constant $\mu \in \mathbb{R}$ and a threshold $\delta > 0$ one can define $(\mu_t)_{t\geq 0}$ and $(H_t)_{t\geq 0}$ both $(\mathcal{F}_t^B)_{t\geq 0}$-predictable such that $|\mu - \mu_t| \leq \delta$ and $\beta_t = \int_0^t H_s(\mathrm{d}B_s + \mu_s\,\mathrm{d}s)$ is a Brownian motion in its own filtration. In other words any constant drift can be uniformly approximated by "strongly hidable" random drifts.

Roughly speaking, Theorem 1 gives a positive answer to Yor's question, provided one replaces the filtration generated by $(B_t)_{t\geq 0}$ by the larger filtration $(\mathcal{F}_t^W)_{t\geq 0}$ generated by $(W_t)_{t\geq 0}$. However, Yor's original question remains open and is left for further research.

Laurent Serlet in [11] also deals with the problem of creation and deletion of drift. His approach, based on excursion theory and the notion of contour process focused on somewhat different problems.

Remark. *Following the advice of the anonymous referee, we remark that in addition to the assertions of Theorem 1, the following fact also holds true: the filtration $(\mathcal{F}_t^W)_{t\geq 0}$ is weakly generated by the Brownian motion $(B_t)_{t\geq 0}$, see [5], Definition 6.2 and Remark 6.1. In other words, every $(\mathcal{F}_t^W)_{t\geq 0}$-martingale $(M_t)_{t\geq 0}$ can be represented as a stochastic integral $M = c + K \cdot B$, where $c \in \mathbb{R}$ is a constant and K is a predictable process with respect to $(\mathcal{F}_t^W)_{t\geq 0}$.*

This follows from the Itô representation theorem applied to the Brownian motion $(W_t)_{t\geq 0}$ in its natural filtration $(\mathcal{F}_t^W)_{t\geq 0}$. By Theorem 1 the process B is a Brownian motion in this filtration so that there is an $(\mathcal{F}_t^W)_{t\geq 0}$-predictable process $L = (L_t)_{t\geq 0}$ such that $B = L \cdot W$. Clearly L takes values in $\{-1,+1\}$ for almost all $(\omega, t) \in \Omega \times \mathbb{R}_+$ with respect to the product of $\mathbf{P}$ and the Lebesgue measure on $\mathbb{R}_+$. Then, obviously $W = L \cdot B$ is also true.

Given an arbitrary martingale M in the filtration $(\mathcal{F}_t^W)_{t\geq 0}$ we may again apply Itô's representation theorem to obtain a $(\mathcal{F}_t^W)_{t\geq 0}$-predictable process K' such that $M = c + K' \cdot W$ with some $c \in \mathbb{R}$. Then, $K = K'L$ gives the representation $M = c + K \cdot B$.

1.2. *Generalizations*

Here is the rationale of our paper: One can easily check that for Theorem 1 we have to define H such that $\mathbf{E}(H_s|\mathcal{F}_s^{H\cdot S}) = 0$ for almost all $s \geq 0$. Our construction initially uses a larger filtration than the natural filtration of $(W_t)_{t\geq 0}$. We start with a probability space on which, apart from the Brownian motion W, there is an independent random variable U uniformly distributed on $]0,1[$. The construction of B_t and $\beta_t = (H \cdot S)_t$ will be such that at each moment t the value of H_t depends on whether U is larger or smaller than the conditional median of U, given the sample path $(\beta_s)_{0\leq s\leq t}$. This idea of construction has been already used in [9]. With this "median rule" we achieve that for all t the random variable H_t is independent of $(\beta_s)_{0\leq s\leq t}$. The difficulty is of course the existence of such H, B and β as they are strongly related.

As a byproduct of our investigations we also get the following result which is interesting in its own right.

Theorem 2. *Let $(\beta_t)_{t\geq 0}$ be a real-valued standard Brownian motion based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$.*

Fix $\mu \in \mathbb{R}_+$. Then there is an enlargement $(\mathcal{F}_t')_{t\geq 0}$ of $(\mathcal{F}_t)_{t\geq 0}$ such that β is an $(\mathcal{F}_t')_{t\geq 0}$-semimartingale of the form $\mathrm{d}\beta_t = \mathrm{d}\beta_t' + \nu_t\,\mathrm{d}t$ where β' is an $(\mathcal{F}_t')_{t\geq 0}$-Brownian motion and $|\nu_t| = \mu$ for all t.

The enlargement $(\mathcal{F}_t')_{t\geq 0}$ in Theorem 2 is an initial enlargement, that is we actually prove that there is a random variable U uniformly distributed on $]0,1[$, such that $\mathcal{F}_t' = \bigcap_{s>t}(\mathcal{F}_s \vee \sigma(U))$. We still point out, following a suggestion of the referee, that it is also possible to define $(\mathcal{F}_t')_{t\geq 0}$ such that it is the filtration generated by some Brownian motion $(W_t)_{t\geq 0}$.

The statement of Theorem 2 can be generalized, by replacing the Brownian motion β with a continuous local martingale M and the constant μ by a predictable process $(\mu_t)_{t\geq 0}$ that can be integrated with respect to M, i.e. $\int_0^t \mu_s^2\,\mathrm{d}\langle M\rangle_s < \infty$ almost surely for all $t \geq 0$. We use the notation $L(M)$ of [8] for the family of predictable processes, that can be integrated with respect to the continuous local martingale $(M_t)_{t\geq 0}$.

Theorem 3. *Let $(M_t)_{t\geq 0}$ be a continuous local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. We assume that on $(\Omega, \mathcal{F}, \mathbf{P})$ there is a random variable U, uniformly distributed on $]0,1[$ and independent from $\mathcal{F}_\infty$.*

Fix an $(\mathcal{F}_t)_{t\geq 0}$-predictable, non-negative process $(\mu_t)_{t\geq 0}$ in $L(M)$. Then there is an enlargement $(\mathcal{F}'_t)_{t\geq 0}$ of the filtration $(\mathcal{F}_t)_{t\geq 0}$ such that, M is an $(\mathcal{F}'_t)_{t\geq 0}$-semimartingale of the form

$$\mathrm{d}M_t = \mathrm{d}M'_t + \nu_t \,\mathrm{d}\langle M\rangle_t, \tag{1}$$

where M' is an $(\mathcal{F}'_t)_{t\geq 0}$ local martingale and $|\nu_t| = \mu_t$ for all t.

It turns out from the proof that if

$$\int_0^\infty \mu_s^2 \,\mathrm{d}\langle M\rangle_s = \infty \quad \text{almost surely}$$

then, we can do the construction in such a way that $\mathcal{F}'_t \subset \mathcal{F}_\infty$ for all t, i.e. U is not needed in this case, showing that Theorem 3 is indeed a generalization of Theorem 2.

1.3. *Outline of the paper*

The structure of the paper is as follows. First, we present the discussion of the discrete analogue of the problem. Then, we solve the equation formally derived from the discrete case and show that as in the discrete time case this gives a solution using the "median" strategy. It turns out that the extra randomness U used throughout our construction is encoded in the sample paths of β. Using this observation we prove Theorem 2. Theorem 3 then follows easily. Finally, we prove Theorem 1 completely. In the Appendix we present minor, but useful technical results.

2. The discrete case

This section only serves as motivation for the subsequent continuous time case.

Fix $\mu \in \mathbb{R}$ and $\Delta t > 0$ such that $|\mu(\Delta t)^{1/2}| < 1$. We consider a biased random walk $S = (S_{t_i})_{i=0}^\infty$ on a fine grid $(t_i)_{i=0}^\infty$, with $t_i = i\Delta t$. We suppose that the increments $(\Delta S_{t_i})_{i=0}^\infty = (S_{t_{i+1}} - S_{t_i})$ are an i.i.d. sequence with

$$\mathbf{P}\big(\Delta S_{t_i} = +(\Delta t)^{1/2}\big) = \frac{1+\mu(\Delta t)^{1/2}}{2},$$
$$\mathbf{P}\big(\Delta S_{t_i} = -(\Delta t)^{1/2}\big) = \frac{1-\mu(\Delta t)^{1/2}}{2}.$$

The process S is a discrete analog to $B_t + \mu t$, the Brownian motion with drift μ, which will be considered below, because

$$\mathbf{E}(\Delta S_{t_i}) = \mu\Delta t, \qquad \mathbf{E}\big(\Delta S_{t_i}^2\big) = \Delta t.$$

In addition, let U be a uniformly distributed $]0,1[$-valued random variable independent of S. The filtration $(\mathcal{F}_{t_i})_{i=0}^\infty$ is defined as the smallest one such that U is $\mathcal{F}_0$-measurable and $(S_{t_i})_{i=0}^\infty$ is adapted to $(\mathcal{F}_{t_i})_{i=0}^\infty$.

We shall construct inductively a predictable, $\{-1,+1\}$-valued process $(H_{t_i})_{i=1}^\infty$ such that $((H\cdot S)_{t_i})_{i=0}^\infty$ is an unbiased random walk (i.e., it is a martingale) in its own filtration.

To do so we construct a $]0,1[$-valued process $(D_{t_i}(U))_{i=0}^\infty$ adapted to $(\mathcal{F}_{t_i})_{i=0}^\infty$ inductively by letting $D_{t_0}(U) = U$ and

$$\Delta D_{t_i}(U) = \min\big(D_{t_i}(U), \big(1 - D_{t_i}(U)\big)\big)\mu \operatorname{sign}\left(\frac{1}{2} - D_{t_i}(U)\right)\Delta S_{t_i}, \quad i \geq 0, \tag{2}$$

where $\Delta D_{t_i}(U)$ denotes the increment $D_{t_{i+1}}(U) - D_{t_i}(U)$.

The process $(H_{t_i})_{i=1}^{\infty}$ is derived from $(D_{t_i}(U))_{i=0}^{\infty}$ by

$$H_{t_{i+1}} = \operatorname{sign}\left(D_{t_i}(U) - \frac{1}{2}\right), \quad i \geq 0. \tag{3}$$

Here is the idea behind this construction. At the first step $i = 0$, formula (3) yields $H_{t_1} = \operatorname{sign}(U - \frac{1}{2})$, i.e., we flip a coin, independent of S, whether H_{t_1} equals $+1$ or -1. So that obviously $\mathbf{E}((H \cdot S)_{t_1}) = \mathbf{E}(H_{t_1} S_{t_1}) = 0$. Now we may observe the outcome $(H \cdot S)_{t_1} = H_{t_1} S_{t_1}$ which takes the value $+(\Delta t)^{1/2}$ or $-(\Delta t)^{1/2}$. Applying Bayes' rule, this information updates the conditional distribution of U: conditionally on the event $\{H_{t_1} S_{t_1} = +(\Delta t)^{1/2}\}$ we obtain for the conditional distribution function $D_{t_1}(x)$,

$$D_{t_1}(x) = \begin{cases} \left(1 - \mu(\Delta t)^{1/2}\right)x & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \left(1 - \mu(\Delta t)^{1/2}\right)\frac{1}{2} + \left(1 + \mu(\Delta t)^{1/2}\right)\left(x - \frac{1}{2}\right) & \text{for } \frac{1}{2} \leq x \leq 1, \end{cases}$$

and an analogous expression conditionally on the event $\{H_{t_1} S_{t_1} = -(\Delta t)^{1/2}\}$. The random variable $D_{t_1}(U)$ equals the conditional distribution function $D_{t_1}(\cdot)$ at the random point U. Hence it makes sense to define

$$H_{t_2} = \operatorname{sign}\left(D_{t_1}(U) - \frac{1}{2}\right)$$

as the preceding argument shows that $\mathbf{P}(H_{t_2} = 1 | \mathcal{G}_{t_1}) = \mathbf{P}(H_{t_2} = -1 | \mathcal{G}_{t_1}) = \frac{1}{2}$, almost surely, where $\mathcal{G}_{t_1}$ denotes the sigma-algebra generated by $H_{t_1} S_{t_1}$.

Now we may continue inductively and some elementary calculations show that we thus obtain the updating rule for the process $(D_{t_i}(U))_{i=0}^{\infty}$ given by (2). We summarize these facts in the subsequent statement.

Proposition 4. *Defining the process $(D_{t_i}(U))_{i=0}^{\infty}$ and $(H_{t_i})_{i=1}^{\infty}$ as above and denoting by $(\mathcal{G}_{t_i})_{i=0}^{\infty}$ the filtration generated by $((H \cdot S)_{t_i})_{i=0}^{\infty}$, where $(H \cdot S)_{t_i} = \sum_{u=1}^{i} H_{t_u} \Delta S_{t_{u-1}}$, we have, for almost all $\omega \in \Omega$,*

$$D_{t_i}(U)(\omega) = \mathbf{P}(U \leq x | \mathcal{G}_{t_i})(\omega), \tag{4}$$

where $x = U(\omega)$. In particular we have

$$\mathbf{E}(H_{t_i} | \mathcal{G}_{t_{i-1}}) = 0, \quad a.s., \text{ for } i \geq 1,$$

so that $\mathbf{E}(H_{t_i} \Delta S_{t_{i-1}} | \mathcal{G}_{t_{i-1}}) = 0$, hence $(H \cdot S_{t_i})_{i=0}^{\infty}$ is a martingale in its own filtration $(\mathcal{G}_{t_i})_{i=0}^{\infty}$.

Remark 5. *A word of warning seems to be in order: it is not clear*[3] *how to define* sign(0) *in the formulas* (2) *and* (3), *and the above arguments do not apply to the case where this occurs. However, this is not really a problem in the present discrete time setting as this case only appears on a null set of Ω and therefore can be safely ignored by inserting the words "almost surely."*

On the other hand, in the continuous time case, this problem will become crucial and is discussed later at the end of Section 3.1.

3. Continuous time

We shall now try to pass to the continuous time limit of the above random walk construction. We prove the next statement, which is nothing else but the statement of Theorem 1, written once again, for ease of the reader, but modulo the addition of an auxiliary uniform variable U.

[3]The anonymous referee observed that, following the tradition of P. A. Meyer, it sometimes is advantageous to define $\operatorname{sign}(0) = -1$; using this convention, e.g., in the definition of local times, one thus obtains càdlàg versions of $(L_t^a)_{a \in \mathbb{R}, t \geq 0}$.

Proposition 6. *Let $(W_t)_{t\geq 0}$ be a real-valued standard Brownian motion based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$, i.e., $(W_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$ Brownian motion. Assume that there is an $\mathcal{F}_0$-measurable U uniformly distributed on $]0,1[$.*

Fix $\mu \in \mathbb{R}$. Then there is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(B_t)_{t\geq 0}$ as well as an $(\mathcal{F}_t)_{t\geq 0}$-predictable, $\{-1,+1\}$-valued process H such that the stochastic integral $\beta = H \cdot S$ is a Brownian motion in its own filtration, where $S_t = B_t + \mu t$. Moreover, the processes B, H and $H \cdot S$ are adapted to $(\mathcal{F}_t^{W,U})_{t\geq 0}$.

We split the proof into two parts. First, we look for a solution of the system of equations formally derived from the discrete time case. This means that, for a given Brownian motion B and an independent random variable U, we want to solve

$$\mathrm{d}D_t = -\mu \min\bigl(D_t, (1-D_t)\bigr) \operatorname{sign}\left(D_t - \frac{1}{2}\right) \mathrm{d}S_t, \quad D_0 = U, \tag{5}$$

where

$$S_t = B_t + \mu t. \tag{6}$$

We will also use the notation

$$\beta_t = \int_0^t \operatorname{sign}\left(D_u - \frac{1}{2}\right) \mathrm{d}S_u. \tag{7}$$

We shall see that the process $(D_t)_{t\geq 0}$ satisfying (5) cannot be adapted to the filtration $(\mathcal{F}_t^{U,B})_{t\geq 0}$. We only can find a weak solution. In the setting of Proposition 6 we can derive the processes D, B, β from the given data W and U such that (5) and (7) hold true as Itô-integrals in $(\mathcal{F}_t)_{t\geq 0}$, see Corollary 8 below.

Secondly, we show that if the processes B, D and β are related to each other according to this system of equations, then they provide a solution to Yor's question in the weak sense as formulated in Proposition 6 (see Lemma 10 below).

3.1. *Heuristic description*

Before turning to the proof, it is worth to have a closer look at the equations. The delicate term on the right-hand side of (5) is $\operatorname{sign}(D_t - \frac{1}{2})$, similarly as in Tanaka's equation

$$\mathrm{d}X_t = \operatorname{sign}(X_t)\, \mathrm{d}B_t. \tag{8}$$

It is well known that there is no solution $(X_t)_{t\geq 0}$ to (8) adapted to the filtration generated by $(B_t)_{t\geq 0}$; rather one has to enlarge the filtration, i.e. introduce additional sources of randomness, in order to obtain a weak solution to (8).

A similar problem appears in (5) when the process $(D_t)_{t\geq 0}$ hits the value $\frac{1}{2}$. This happens for the first time at

$$\tau = \inf\left\{t > 0\colon\ D_t = \frac{1}{2}\right\},$$

which is a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. For $0 \leq t \leq \tau$ the SDE (5) clearly has a strong solution with respect to the filtration $(\mathcal{F}_t^{B,U})_{t\geq 0}$ (hence, a fortiori, with respect to $(\mathcal{F}_t^{W,U})_{t\geq 0}$), which is explicitly given by the formula

$$D_t = U \exp\left\{\mu S_t - \frac{\mu^2}{2} t\right\} = \exp\left\{\ln(U) + \mu\left(B_t + \frac{\mu}{2} t\right)\right\}, \quad 0 \leq t \leq \tau, \tag{9}$$

for $U \in]0, \frac{1}{2}[$, and by the formula

$$1 - D_t = (1-U) \exp\left\{\mu S_t - \frac{\mu^2}{2} t\right\} = \exp\left\{\ln(1-U) + \mu\left(B_t + \frac{\mu}{2} t\right)\right\}, \quad 0 \leq t \leq \tau, \tag{10}$$

for $U \in]\frac{1}{2}, 1[$.

A unified way of writing (9) and (10) is

$$1-|1-2D_t|=\exp\left\{\ln\bigl(1-|1-2U|\bigr)+\mu\left(B_t+\frac{1}{2}\mu t\right)\right\}\quad \text{for } 0\le t\le\tau. \tag{11}$$

Using this identity we can express τ with the help of B and U, as

$$\tau=\inf\left\{t>0\colon\ B_t+\frac{1}{2}\mu t=-\frac{1}{\mu}\ln\bigl(1-|1-2U|\bigr)\right\}, \tag{12}$$

so τ is a stopping time with respect to $(\mathcal{F}_t^{B,U})_{t\ge 0}$.

We now give a heuristic and intuitive description of the present construction of a global solution to the SDE (5).

We motivate the construction by recalling the solution to Tanaka's equation

$$\mathrm{d}X_t=\operatorname{sign}(X_t)\,\mathrm{d}B_t,\qquad X_0=0, \tag{13}$$

where $(B_t)_{t\ge 0}$ is a standard Brownian motion starting at $B_0=0$. It is well known how to construct and interpret a weak solution $(X_t)_{t\ge 0}$. We summarize the construction in a heuristic way: we decompose each trajectory $(B_t(\omega))_{t\ge 0}$ into its running minimum $(M_t(\omega))_{t\ge 0}$ and its positive excursions $(E_t(\omega))_{t\ge 0}$, i.e.

$$M_t(\omega)=\inf_{0\le s\le t}B_s(\omega),\qquad E_t(\omega)=B_t(\omega)-M_t(\omega).$$

We may decompose $(E_t(\omega))_{t\ge 0}$ into its excursions. More precisely, there are sequences of random times $(\sigma_n)_{n=1}^\infty$ and $(\tau_n)_{n=1}^\infty$ taking values in $[0,\infty]$ such that $[\![\sigma_n,\tau_n]\!]$ is a.s. a sequence of disjoint intervals of $\mathbb{R}_+$, whose union has full Lebesgue measure and such that $E_{\sigma_n}=E_{\tau_n}=0$ while $E_t>0$, for $t\in]\!]\sigma_n,\tau_n[\![$. (For details we refer the reader to the classical book of Itô and McKean [4], Section 2.9, see also [6].)

Choose an i.i.d. sequence $(\varepsilon_n)_{n=1}^\infty$ of symmetric random signs independent of $(B_t)_{t\ge 0}$ and define

$$X_t(\omega)=\sum_{n=1}^\infty\varepsilon_n(\omega)E_t(\omega)\chi_{[\![\sigma_n,\tau_n]\!]}(t).$$

The filtration $(\mathcal{F}_t^X)_{t\ge 0}$ generated by $(X_t)_{t\ge 0}$ then contains the filtration $(\mathcal{F}_t^B)_{t\ge 0}$ generated by $(B_t)_{t\ge 0}$, and $(X_t)_{t\ge 0}$ as well as $(B_t)_{t\ge 0}$ are Brownian motions in the filtration $(\mathcal{F}_t^X)_{t\ge 0}$, satisfying the stochastic differential equation (13).

Summing up informally, we obtain the trajectories $(X_t)_{t\ge 0}$ from the trajectories $(B_t)_{t\ge 0}$ by flipping coins and pasting together the excursions of $(B_t(\omega))_{t\ge 0}$ multiplied with the corresponding random signs $\varepsilon_n(\omega)$ to obtain the trajectories $(X_t(\omega))_{t\ge 0}$.

This may be rephrased as follows: the process $H_t=\sum_{n=0}^\infty\varepsilon_n\chi_{]\!]\sigma_n,\tau_n]\!]}(t)$ is predictable in $(\mathcal{F}_t^X)_{t\ge 0}$ and we have $B=H\cdot X$ as well as $X=H\cdot B$, holding true with respect to this filtration.

When comparing the situation of Tanaka's equation (13) with the present equation (5), let us start with the trivial observation that when changing the sign in Tanaka's equation, i.e.

$$\mathrm{d}X_t=-\operatorname{sign}(X_t)\,\mathrm{d}B_t,\qquad X_0=0,$$

we have to replace the running minimum in the above construction by the running maximum and the positive excursions by the negative ones.

Now we pass to our present situation. We start with the process $S_t=B_t+\mu t$ where $(B_t)_{t\ge 0}$ is a given Brownian motion defined on $((\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge 0}),\mathbf{P})$. We need, as additional stochastic input, a $\mathcal{F}_0$-measurable random variable U, uniformly distributed on $]0,1[$, as well as an $\mathcal{F}_0$-measurable i.i.d. sequence $(\varepsilon_n)_{n=1}^\infty$ of random signs such that U and $(\varepsilon)_{n=1}^\infty$ are independent.

We now decompose the process $B_t+\frac{1}{2}\mu t$ into its running maximum $(M_t)_{t\ge 0}$ and its negative excursions $(E_t)_{t\ge 0}$ from the running maximum $(M_t)_{t\ge 0}$,

$$M_t=\sup_{0\le s\le t}\left(B_s+\frac{1}{2}\mu s\right),\qquad E_t=\left(B_t+\frac{1}{2}\mu t\right)-M_t.$$

Again we enumerate these excursions by random times $(\sigma_n, \tau_n)_{n=1}^{\infty}$ as above.

Let

$$H_t = \begin{cases} \operatorname{sign}\bigl(U - \frac{1}{2}\bigr) & \text{for } 0 \leq \tau, \\ \varepsilon_n & \text{for } t > \tau \text{ and } t \in \rrbracket\sigma_n, \tau_n\rrbracket. \end{cases} \tag{14}$$

Denoting by $(\mathcal{F}_t^{B,H})_{t\geq 0}$ the filtration generated by $(B_t)_{t\geq 0}$ and $(H_t)_{t\geq 0}$ we find that in this filtration B is a Brownian motion and H is a predictable process, so that we may well define the process

$$\beta_t = \int_0^t H_u \, \mathrm{d}(B_u + \mu u). \tag{15}$$

We shall show that $(\beta_t)_{t\geq 0}$ is a Brownian motion in its own filtration $(\mathcal{F}_t^{\beta})_{t\geq 0}$ which is contained in $(\mathcal{F}_t^{B,H})_{t\geq 0}$.

Before doing so we interpret intuitively the construction given by (14) and (15). Let the trajectory $(B_t(\omega))_{t\geq 0}$ as well as the random number $U(\omega)$ be given. The trajectory $(\beta_t)_{t\geq 0}$ is initially given by either the trajectory $(B_t + \mu t)_{t\geq 0}$ or $-(B_t + \mu t)_{t\geq 0}$, depending on the sign of $U - \frac{1}{2}$, namely up to time τ, which is defined in (12).

Note that at time τ the process $B_t + \frac{1}{2}\mu t$ attains for the first time the level $-\frac{1}{\mu}\ln(1 - |1 - 2U|)$, whence, in particular, $B_\tau + \frac{1}{2}\mu\tau = M_\tau$ almost surely.

After time τ we multiply the excursions $(E_t \chi_{\llbracket\sigma_n,\tau_n\rrbracket}(t))_{n=1}^{\infty}$ of the process $B_t + \frac{1}{2}\mu t$ with the random signs $(\varepsilon_n)_{n=1}^{\infty}$ and paste them together in order to obtain the trajectory of $(\beta_t)_{t\geq 0}$. This result is a variant of the construction in Tanaka's case above: we have, for $t \in \llbracket\sigma_n, \tau_n\rrbracket$, where $\sigma_n \geq \tau$,

$$\beta_t - \beta_{\sigma_n} = \varepsilon_n\left(E_t + \frac{\mu}{2}(t - \sigma_n)\right) = \varepsilon_n\bigl(B_t - B_{\sigma_n} + \mu(t - \sigma_n)\bigr) \quad \text{for } t \in \llbracket\sigma_n, \tau_n\rrbracket.$$

In addition, the trajectories of $(\beta_t)_{t\geq 0}$ have to be continuous; this—in conjunction with the above equation—uniquely determines $(\beta_t)_{t\geq 0}$ pathwise.

Let us try to explain why this construction of β indeed yields a martingale in its own filtration $(\mathcal{F}_t^{\beta})_{t\geq 0}$. We consider the distribution function $(D_t(x))$ of U conditionally on $(\mathcal{F}_t^{\beta})_{t\geq 0}$, where $0 \leq x \leq 1$. Fix the random element $\omega \in \Omega$ and suppose that $U(\omega) = y \in [0, 1]$. To fix ideas let us suppose that $y < \frac{1}{2}$. Then, for $0 \leq t \leq \tau$, the Bayesian updating for the conditional distribution function

$$D_t(x) = \mathbf{P}\bigl(U \leq x \,|\, (\beta_s)_{0\leq s\leq t}\bigr)$$

is such that for $x = y$ the value of $D_t(y)$ is given by (9). Hence it is less than $\frac{1}{2}$, for $0 \leq t \leq \tau$, and hits the value $\frac{1}{2}$ for the first time at $t = \tau$. According to our "median rule" $H_t = \operatorname{sign}(D_t - \frac{1}{2})$, this is the critical moment to change the sign of H. The good interpretation of sign(0) in the present context is to flip a coin whether sign(0) equals $+1$ or -1. If the process does "not start a negative excursion" at this moment, we have to flip a new coin at an infinitesimal time unit later and to continue to do so until "eventually a negative excursion $E_t \chi_{\llbracket\sigma_n,\tau_n\rrbracket}(t)$ starts." (The preceding heuristic phrase should be interpreted by visualising the process $(B_t + \frac{1}{2}\mu t)_{t\geq 0}$ as a normalized random walk on a very fine grid.) During this interval $\llbracket\sigma_n, \tau_n\rrbracket$ the sign of H_t is determined by the coin flip ε_n (which, intuitively speaking, was done at time σ_n, the "last moment before the excursion $E_t \chi_{(\llbracket\sigma_n,\tau_n\rrbracket)}$ started"). During this random interval the Bayesian updating rule for the conditional distribution function $D_t(\cdot)$ of U evaluated at $y = U(\omega)$ yields an excursion of $D_t(y)$ from the value $\frac{1}{2}$, to the right or left, depending on the sign of ε_n. Precisely at time τ_n the value of $D_t(y)$ is back to $\frac{1}{2}$.

The preceding heuristics should motivate that we again use the idea of the "median rule" for H, similarly as in [9]; to do so we need, apart from the initial random input U, also the random variables $(\varepsilon_n)_{n=1}^{\infty}$. The random sign ε_n is interpreted as a coin flipping at the random times σ_n. These random times σ_n fail to be stopping times in the filtration $(\mathcal{F}_t^{B,U})_{t\geq 0}$. This is the reason why we cannot define the strategy H in such a way that it is an $(\mathcal{F}_t^{B,U})_{t\geq 0}$-predictable process; rather we have to pass to an enlargement of the filtration and find a weak solution for H, which eventually leads to Theorems 1 and 2.

3.2. *Proofs*

We now turn to Eq. (5). First we transform it into an equation which is more tractable. So assume that B is a Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and $(D_t)_{t\geq 0}$, $(S_t)_{t\geq 0}$ are $(\mathcal{F}_t)_{t\geq 0}$ adapted processes satisfying (5) and (6). Then,

$$\frac{\mathrm{d}D_t}{1/2-|D_t-1/2|}=-\mu\operatorname{sign}\left(D_t-\frac{1}{2}\right)\mathrm{d}S_t. \tag{16}$$

In order to integrate the left-hand side we consider $f(D_t)$ where

$$f(x)=-\operatorname{sign}\left(x-\frac{1}{2}\right)\ln\bigl(1-|2x-1|\bigr)\quad\text{for } x\in(0,1) \tag{17}$$

is the quantile function of the symmetrized exponential law with parameter 1. The function f is continuously differentiable and the second derivative exists except at $\frac{1}{2}$, where it has right and left limits:

$$f'(x)=\left(\frac{1}{2}-\left|x-\frac{1}{2}\right|\right)^{-1},\qquad f''(x)=\operatorname{sign}\left(x-\frac{1}{2}\right)\left(\frac{1}{2}-\left|x-\frac{1}{2}\right|\right)^{-2}.$$

The inverse of f, the distribution function of the symmetrized exponential law, has the same differentiability properties as f, i.e., it is continuously differentiable and the second derivative exists except at 0, where it has right and left limits. The discontinuities of f'' and $(f^{-1})''$ are harmless for the application of Itô's formula, which, together with

$$\operatorname{sign}\bigl(f(x)\bigr)=\operatorname{sign}\left(x-\frac{1}{2}\right)=\frac{f''(x)}{(f'(x))^2},$$

yields the next statement:

Proposition 7. *Let $(X_t)_{t\geq 0}$ and $(D_t)_{t\geq 0}$ be semimartingales in the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then, the differential equation*

$$\mathrm{d}D_t=\min(D_t,1-D_t)\,\mathrm{d}X_t,\quad \textit{with } 0<D_0<1,$$

is satisfied if and only if $Z_t=f(D_t)$ is the solution of

$$\mathrm{d}Z_t=\mathrm{d}X_t+\frac{1}{2}\operatorname{sign}(Z_t)\,\mathrm{d}\langle X\rangle_t,\quad Z_0=f(D_0).$$

When we consider Eq. (5), then $\mathrm{d}X_t=-\mu\operatorname{sign}(D_t-\frac{1}{2})\,\mathrm{d}S_t$ so we have to solve

$$\begin{aligned}\mathrm{d}Z_t&=-\mu\operatorname{sign}(Z_t)\,\mathrm{d}S_t+\frac{1}{2}\mu^2\operatorname{sign}(Z_t)\,\mathrm{d}t\\&=-\mu\operatorname{sign}(Z_t)\,\mathrm{d}B_t-\frac{1}{2}\mu^2\operatorname{sign}(Z_t)\,\mathrm{d}t,\quad Z_0=f(U).\end{aligned} \tag{18}$$

The way we solve this equation depends on the initial data. In the heuristic description, when we started with a given B and U, we used that these two objects determine $|Z_t|$, by the formula

$$\begin{aligned}|Z_t|&=|Z_0|-\mu B_t-\frac{1}{2}\mu^2t+L_t^0(Z)\\&=|Z_0|-\mu B_t-\frac{1}{2}\mu^2t+\max_{s\leq t}\left\{\left(\mu B_s+\frac{1}{2}\mu^2s-|Z_0|\right)\vee 0\right\}.\end{aligned}$$

So we have to unfold this process with the help of independent random signs to obtain Z and hence D. Such an "unfolding" result can be found in [7].

In Proposition 6 we would like to find B and D such that B is a Brownian motion, and (5) holds within a filtration generated by W and U. So we follow another approach here. If we set $W_t = \int_0^t \operatorname{sign}(Z_s)\,\mathrm{d}B_s$, then (18) reads as follows

$$\mathrm{d}Z_t = -\mu\,\mathrm{d}W_t - \frac{1}{2}\mu^2 \operatorname{sign}(Z_t)\,\mathrm{d}t, \quad Z_0 = f(U). \tag{19}$$

Note, that the discontinuous function $\operatorname{sign}(\cdot)$ now is in the drift term rather than in the diffusion term. Existence and pathwise uniqueness of the solution of (19) follows from Theorem 3.5(i) [10], Chapter IX. So, we can take $(Z_t)_{t\geq 0}$ as the solution of (19) and then, we define $B_t = \int_0^t \operatorname{sign}(Z_s)\,\mathrm{d}W_s$. It is clear that $(B_t)_{t\geq 0}$ is a Brownian motion in $(\mathcal{F}_t)_{t\geq 0}$, adapted to $(\mathcal{F}_t^{U,W})_{t\geq 0}$. We define S and X as before

$$S_t = B_t + \mu t, \qquad X_t = -\mu \int_0^t \operatorname{sign}(Z_u)\,\mathrm{d}S_u.$$

Thus, with this notation,

$$\mathrm{d}Z_t = \mathrm{d}X_t + \frac{1}{2}\operatorname{sign}(Z_s)\,\mathrm{d}\langle X\rangle_s.$$

So application of Proposition 7 gives the following corollary.

Corollary 8. *Under the assumptions of Proposition 6, there is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(B_t)_{t\geq 0}$ and an adapted process $(D_t)_{t\geq 0}$ such that (5) holds true as an Itô integral in $(\mathcal{F}_t)_{t\geq 0}$. Moreover, $(B_t)_{t\geq 0}$ and $(D_t)_{t\geq 0}$ are adapted to $(\mathcal{F}_t^{W,U})_{t\geq 0}$.*

Remark 9. *The SDE (19) defines a Markov process, which is ergodic if $\mu \neq 0$, sometimes it is called the "bang–bang" process. The scale function s of Z is given by the formula $s(x) = \operatorname{sign}(x)(\mathrm{e}^{|x|} - 1)$, and the speed measure on the natural scale has a density $p(x) = (s' \circ s^{-1})^{-2}$. Then, the density of the invariant distribution on the original scale is proportional to $p(s(x))s'(x) = 1/s'(x) = \mathrm{e}^{-|x|}$, i.e. the symmetrized exponential distribution with parameter 1.*

Since Z_0 has symmetrized exponential law with parameter 1, the solution of (19) we used to define D is a stationary Markov process. This means that the law of D_t is the same for all t, i.e., D_t is uniformly distributed on $]0, 1[$. The byproduct of the next lemma is that this is even true for the conditional law of D_t given $\mathcal{F}_t^{\beta}$. In fact, item (iii) below asserts that conditional distribution function $(\hat{D}_t(x, \omega))_{0\leq x\leq 1}$ of the random variable U evaluated at $x = U(\omega)$ equals $D_t(\omega)$, i.e., $\hat{D}_t(U(\omega), \omega) = D_t(\omega)$ for almost all ω. This is also crucial in the construction, because it means that we indeed apply here the median strategy. The process $H_t = \operatorname{sign}(D_t - \frac{1}{2})$ is plus or minus one if the value of U is above or below the conditional median, respectively.

Lemma 10. *Assume that B is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion, U is an $\mathcal{F}_0$ measurable random variable uniformly distributed on $]0, 1[$, and D, S, β are $(\mathcal{F}_t)_{t\geq 0}$-adapted processes satisfying (5), (6) and (7). Then, β is a Brownian motion in its own filtration.*

Moreover, the process $\hat{D}_t(x) = \mathbf{P}(U \leq x|\mathcal{F}_t^{\beta})$, depending on the two parameters $x \in [0, 1]$ and $t \geq 0$ has a version which:

(i) *is the unique solution of the parametric family of equations*

$$\mathrm{d}\hat{D}_t(x) = -\mu \min\bigl(\hat{D}_t(x), 1 - \hat{D}_t(x)\bigr)\,\mathrm{d}\beta_t, \quad \hat{D}_0(x) = x;$$

(ii) *is continuous in both parameters*;

(iii) *satisfies $D_t = \hat{D}_t(U)$.*

Hence the conditional law of D_t given $\mathcal{F}_t^{\beta}$ is uniform on $]0, 1[$ for each t.

Proof. To prove that β is a Brownian motion in it own filtration $(\mathcal{F}_t^{\beta})_{t\geq 0}$, it is enough to show that for each $T > 0$, $(\beta_t)_{t\in[0,T]}$ has the correct law.

On $\mathcal{F}_T$ we can define a new measure $\mathbf{Q}$ by the Cameron–Martin formula $\mathrm{d}\mathbf{Q} = \exp\{-\mu B_T - \mu^2 T/2\}\,\mathrm{d}\mathbf{P}$. Under $\mathbf{Q}$ the process $(S_t)_{t\in[0,T]}$ is a Brownian motion and therefore $(\beta_t)_{t\in[0,T]}$ is so too, by (7). Hence it is enough to prove that on $\mathcal{F}_T^\beta$ the measures $\mathbf{P}$ and $\mathbf{Q}$ coincide, as this ensures that the law of β under $\mathbf{P}$ is the law of a Brownian motion. So it is enough to show that

$$\mathbf{E}_{\mathbf{Q}}\left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\middle|\mathcal{F}_T^\beta\right) = 1. \tag{20}$$

Here $\mathrm{d}\mathbf{P}/\mathrm{d}\mathbf{Q} = \exp\{\mu B_T + \mu^2 T/2\} = \exp\{\mu S_T - \mu^2 T/2\}$.

Now let us consider the parametric SDE:

$$\mathrm{d}\bar{D}_t(x) = -\mu \min\bigl(\bar{D}_t(x), 1 - \bar{D}_t(x)\bigr)\,\mathrm{d}\beta_t, \quad \bar{D}_0(x) = x. \tag{21}$$

We use here the symbol $\bar{D}$ to emphasize the—a priori—difference between the solution of the parametric SDE and the conditional distribution function $\hat{D}_t(x)$.

We consider this equation under $\mathbf{Q}$ and for the finite time horizon $[0, T]$. The diffusion coefficient $\sigma(y) = -\mu\min(y, 1-y)$ on right-hand side of (21) is Lipschitz continuous and satisfies the linear growth condition. Hence, this equation has a unique strong solution (adapted to the filtration $(\mathcal{F}_t^\beta)_{t\geq 0}$). Moreover, (21) defines a martingale (under $\mathbf{Q}$) for each x. Combination of Doob's moment inequality with the continuity lemma of Kolmogorov gives that the resulting process has a version which is almost surely continuous in both variable x and t. This is a special case of a more general statement, see [8], Chapter V, Theorem 37. We use this version in what follows.

We can treat the mapping $x \mapsto \bar{D}_t(x)$, with t, ω fixed as a stochastic flow. Theorem 46 of Chapter V [8], p. 318, states that, on an almost sure event the mapping $x \mapsto \bar{D}_t(x)$ is a homeomorphism of $\mathbb{R}$ for all t.

In fact, we shall prove in Lemma 16 below the stronger result that the flow is absolutely continuous almost surely, and its derivative satisfies the variational equation

$$Y_t(x) = \partial_x \bar{D}_t(x), \qquad \mathrm{d}Y_t(x) = \mu\,\mathrm{sign}\left(\bar{D}_t(x) - \frac{1}{2}\right) Y_t(x)\,\mathrm{d}\beta_t, \quad Y_0(x) = 1. \tag{22}$$

Observe, that in our case $Y_t(x)$ is the stochastic exponential of $\mu\bar{S}_t(x)$, where $\bar{S}_t(x)$ is a Brownian motion (under $\mathbf{Q}$) defined by $\bar{S}_t(x) = \int_0^t \mathrm{sign}(\bar{D}_s(x) - \frac{1}{2})\,\mathrm{d}\beta_s$.

Since $\bar{D}_T(0) = 0$ and $\bar{D}_T(1) = 1$, the integral of the derivative on [0, 1] gives one, i.e.

$$1 = \bar{D}_T(1) - \bar{D}_T(0) = \int_0^1 \mathrm{e}^{\mu\bar{S}_T(x) - \mu^2 T/2}\,\mathrm{d}x. \tag{23}$$

Note, that $(\beta_t)_{0\leq t\leq T}$ is a Brownian motion under $\mathbf{Q}$ in the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ and U is $\mathcal{F}_0$ measurable. Hence U is independent from $\mathcal{F}_T^\beta$ under $\mathbf{Q}$, and the mapping $\omega \mapsto (\omega, U(\omega))$ is measure preserving from $(\Omega, \mathcal{F}, \mathbf{Q})$ to the product space $(\Omega \times [0,1], \mathcal{F}_T^\beta \times \mathcal{B}, \mathbf{Q}')$ where $\mathrm{d}\mathbf{Q}' = \mathrm{d}\mathbf{Q}|_{\mathcal{F}_T^\beta} \times \mathrm{d}x$. This implies that the continuous process $\bar{D}(U)$ and D are indistinguishable as well as $\bar{S}(U)$ and S. Using this measure preserving transformation (23) reads as follows

$$1 = \bar{D}_T(1) - \bar{D}_T(0) = \int_0^1 \mathrm{e}^{\mu\bar{S}_T(x) - \mu^2 T/2}\,\mathrm{d}x = \mathbf{E}_{\mathbf{Q}}\bigl(\mathrm{e}^{\mu S_T - \mu^2 T/2}\big|\mathcal{F}_T^\beta\bigr).$$

This proves (20).

In the same way we can write

$$\bar{D}_T(x) = \bar{D}_T(x) - \bar{D}_T(0) = \int_0^x \mathrm{e}^{\mu\bar{S}_T(y) - \mu^2 T/2}\,\mathrm{d}y = \mathbf{E}_{\mathbf{Q}}\bigl(\chi_{(U\leq x)}\mathrm{e}^{\mu S_T - \mu^2 T/2}\big|\mathcal{F}_T^\beta\bigr) = \mathbf{P}\bigl(U \leq x|\mathcal{F}_T^\beta\bigr) = \hat{D}_T(x),$$

where we have used (20) and the Bayes formula.

We also obtain that $\hat{D}_T$, the conditional distribution function of U given $\mathcal{F}_T^\beta$ is continuous almost surely. To see that $D_T = \hat{D}_T(U)$ is uniformly distributed even given $\mathcal{F}_T^\beta$, we recall the simple fact that law of $F(X)$ is uniform provided that F is the distribution function of X and F is continuous. The proof is complete. $\square$

Remark 11. *Fix $x \in (0,1)$. Then, $\hat{D}_t(x)$ is a closed martingale under $\mathbf{P}$ in the filtration $(\mathcal{F}_t^\beta)_{t\geq 0}$, i.e. $\hat{D}_t(x) = \mathbf{E}(\chi_{(U\leq x)}|\mathcal{F}_t^\beta)$. It is well known that a closed martingale is convergent and its limit is $\mathbf{P}(U \leq x|\mathcal{F}_\infty^\beta)$. In the next lemma we show that U is $\mathcal{F}_\infty^\beta$ measurable, which implies that $\hat{D}_t(x) \to \chi_{(U\leq x)}$ almost surely as $t\to\infty$. This is an interesting feature of the process, as on the other hand, $\hat{D}_t(U) = D_t$ does not converge at all, instead it is a stationary Markov process with non-degenerate invariant distribution. The resolution of this seemingly paradoxical result is that, for each fixed $x \in]0,1[$, $\hat{D}_t(x)$ equals to $D_t = \hat{D}_t(U)$ on the null set $\{U = x\}$, and on this event the limiting relation above is not really meaningful, as $\mathbf{P}(U \leq x|\mathcal{F}_t^\beta) = \mathbf{P}(U < x|\mathcal{F}_t^\beta)$ and also $\chi_{(U\leq x)} = \chi_{(U<x)}$ almost surely. So we use the part of $\hat{D}_t(x)$ where the above almost sure convergence result tells us nothing about the behavior of its sample path, and rather, for almost all $x \in]0,1[$, the sample path of $\hat{D}_t(x)$ on $\{U = x\}$ is divergent.*

Proposition 12. *Assume that B is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion, U is an $\mathcal{F}_0$ measurable random variable uniformly distributed on $]0,1[$, and D, S, β are $(\mathcal{F}_t)_{t\geq 0}$-adapted processes satisfying (5), (6) and (7). Then, U is $\mathcal{F}_\infty^\beta$ measurable.*

Proof. We know from Lemma 10 that β is a Brownian motion in its own filtration, and that $\hat{D}_t(x) = \mathbf{P}(U \leq x|\mathcal{F}_t^\beta)$ solves Eq. (21) for $x \in]0,1[$. By Proposition 7 the process $\bar{Z}_t(x) = f(\hat{D}_t(x))$ satisfies

$$\mathrm{d}\bar{Z}_t(x) = -\mu\,\mathrm{d}\beta_t + \frac{1}{2}\mu^2 \operatorname{sign}\bigl(\bar{Z}_t(x)\bigr)\,\mathrm{d}t, \qquad \bar{Z}_0(x) = f(x).$$

The main difference between this equation and (19) used in Corollary 8 is that in this case we have drift which is against stability. This implies that $\bar{Z}_t(x)$ tends to $+\infty$ or $-\infty$ almost surely as t goes to infinity. Therefore $\lim_{t\to\infty} \hat{D}_t(x) = \mathbf{P}(U \leq x|\mathcal{F}_\infty^\beta) \in \{0,1\}$ almost surely for each x. That is $\mathbf{P}(U \leq x|\mathcal{F}_\infty^\beta)$ is an indicator and it is easy to see that it equals $\chi_{(U\leq x)}$ almost surely. Hence U is $\mathcal{F}_\infty$ measurable by the completeness of the latter. □

Remark 13. *The proof of the previous proposition shows us how to "decode" the value of U from the sample path of the Brownian motion β. We use this observation in the proof of Theorem 2.*

Before proving Theorem 2, we state a version of Girsanov's theorem which will be useful in the proof of Theorem 3 too.

Lemma 14. *Let $(M_t)_{t\geq 0}$ be a continuous local martingale in the filtration $(\mathcal{F}_t)_{t\geq 0}$ and U a random variable uniformly distributed on $]0,1[$. We denote by $\mathcal{F}'_t$ the σ-algebra $\mathcal{F}_t \vee \sigma(U)$.*

Assume that, for each $t \geq 0$, the conditional distribution of U, given $\mathcal{F}_t$, has a positive density, denoted by $Y_t(x)$, with the following properties:

- (i) *$(x,t,\omega) \mapsto Y_t(x,\omega)$ is $\mathcal{B}\times\mathcal{P}$ measurable, where $\mathcal{B}$ stands for the family of Borel subsets of $[0,1]$ and $\mathcal{P}$ is the predictable σ-algebra (with respect to the filtration $\mathcal{F}$).*
- (ii) *$Y_t(x)$ is a continuous local martingale for each $x \in [0,1]$.*

Then,

$$M'_t = M_t - \int_0^t \frac{1}{Y_s(x)}\,\mathrm{d}\bigl\langle Y(x), M\bigr\rangle_s \bigg|_{x=U}$$

is a local martingale in $(\mathcal{F}'_t)_{t\geq 0}$.

Proof. Let $T > 0$ be fixed. Beside $\mathbf{P}|_{\mathcal{F}'_T}$ we define another probability measure $\mathbf{Q}$ on $\mathcal{F}'_T = \mathcal{F}_T \vee \sigma(U)$, such that $\mathbf{P} \sim \mathbf{Q}$, and then we apply Girsanov's theorem.

For $t \in [0,T]$ we define $\mathbf{Q}_t$ on $\mathcal{F}_t \vee \sigma(U)$ by the formula

$$\mathrm{d}\mathbf{Q}_t = \frac{1}{Y_t(U)}\,\mathrm{d}\mathbf{P}\bigg|_{\mathcal{F}_t\vee\sigma(U)}$$

and let $\mathbf{Q}=\mathbf{Q}_T$. For $x\in[0,1]$ and $A\in\mathcal{F}_t$, we have that

$$\begin{aligned}\mathbf{Q}_t\big((U<x)\cap A\big)&=\mathbf{E_P}\bigg(\chi_{((U<x)\cap A)}\frac{1}{Y_t(U)}\bigg)=\mathbf{E_P}\bigg(\chi_A\mathbf{E_P}\bigg(\chi_{(U<x)}\frac{1}{Y_t(U)}\bigg|\mathcal{F}_t\bigg)\bigg)\\&=\mathbf{E_P}\bigg(\chi_A\int_0^x\frac{1}{Y_t(y)}Y_t(y)\,\mathrm{d}y\bigg)=x\mathbf{P}(A).\end{aligned}\tag{24}$$

This shows that $\mathbf{Q}_t$ is a probability measure ($x=1$, $A=\Omega$) and that U is independent from $\mathcal{F}_t$ under $\mathbf{Q}_t$, moreover, $\mathbf{Q}_t|_{\mathcal{F}_t}=\mathbf{P}|_{\mathcal{F}_t}$. By the independence of U and $\mathcal{F}_T$ under $\mathbf{Q}$ we can also conclude that $(\mathcal{F}'_t)_{0\le t<T}$ is right continuous, and obviously saturated by $\mathcal{F}'_0\supset\mathcal{F}_0$, i.e., it fulfills the usual conditions (it is tacitly assumed for $(\mathcal{F}_t)_{t\ge0}$).

It also follows from (24), that for $t<T$ and $A\in\mathcal{F}_t$ we have that $\mathbf{Q}((U<x)\cap A)=\mathbf{Q}_t((U<x)\cap A)$, so

$$\mathbf{Q}|_{\mathcal{F}_t\vee\sigma(U)}=\mathbf{Q}_t.\tag{25}$$

This proves that $(Y_t^{-1}(U))_{t\ge0}$ is a martingale in $(\mathcal{F}'_t)_{t\ge0}$ under $\mathbf{P}$ and also that $Y_t(U)=\mathbf{E_Q}(\mathrm{d}\mathbf{P}/\mathrm{d}\mathbf{Q}|\mathcal{F}'_t)$, i.e., $(Y_t(U))_{0\le t\le T}$ is a martingale in $(\mathcal{F}'_t)_{t\in[0,T]}$ under $\mathbf{Q}$. Observe that $Y_t(U)$ is a continuous process.

Under $\mathbf{Q}$ the process $(M_t)_{t\in[0,T]}$ is a continuous local martingale in $(\mathcal{F}'_t)_{t\in[0,T]}$, by the independence of U and $\mathcal{F}_T$, so application of Girsanov theorem proves that

$$M_t-\int_0^t\frac{1}{Y_s(U)}\,\mathrm{d}\big\langle Y(U),M\big\rangle_s,\quad 0\le t\le T,$$

is a local martingale in $\mathcal{F}'$ under $\mathbf{P}$. Here $\langle Y(U),M\rangle$ is the compensator of $(Y_t(U)M_t)_{t\in[0,T]}$ under $\mathbf{Q}$. It is easy to see using the independence of U and $\mathcal{F}_T$ under $\mathbf{Q}$ that $\langle Y(U),M\rangle=\langle Y(x),M\rangle|_{x=U}$ and the statement follows. □

Proof of Theorem 2. By hypothesis, β is an $(\mathcal{F}_t)_{t\ge0}$-Brownian motion. We define the parametric process $\bar{D}_t(x)$ by (21). We can assume that $(x,t)\mapsto\bar{D}_t(x)$ is almost surely continuous. Define

$$U=\inf\Big\{x\in(0,1)\cap\mathbb{Q}\colon\lim_{t\to\infty}\bar{D}_t(x)=1\Big\}.$$

Recall that on an almost sure event $\bar{D}_t(x)$ is increasing in x and its limit for fixed x exists as $t\to\infty$. The limit is either $+1$ or 0. From this it is clear that the indicator of the event $\{U\le x\}$ is the same as $\lim_{t\to\infty}\bar{D}_t(x)$.

For each fixed x, the process $\bar{D}(x)$ is a bounded martingale, so

$$\mathbf{P}(U\le x|\mathcal{F}_s)=\mathbf{E}\Big(\lim_{t\to\infty}\bar{D}_t(x)\big|\mathcal{F}_s\Big)=\bar{D}_s(x).$$

By Lemma 16 of the Appendix, U has almost surely a conditional density $Y_t(x)$ satisfying

$$\mathrm{d}Y_t(x)=\mu\,\mathrm{sign}\bigg(D_t(x)-\frac{1}{2}\bigg)Y_t(x)\,\mathrm{d}\beta_t,\quad Y_0(x)=1.$$

Now we can apply Lemma 14 to $M=\beta$ and we obtain that in the filtration $\mathcal{F}'_t=\mathcal{F}_t\vee\sigma(U)$

$$\beta_t-\mu\int_0^t\mathrm{sign}\bigg(\bar{D}_u(U)-\frac{1}{2}\bigg)\,\mathrm{d}u$$

is a local martingale. By Lévy's characterization it is an $\mathcal{F}'$-Brownian motion, proving Theorem 2. □

Proof of Theorem 3. We modify the equation defining $\bar{D}$ using the process $(\mu_t)_{t\ge0}$, rather than the constant μ, i.e.

$$\mathrm{d}\bar{D}_t(x)=-\mu_t\min\big(\bar{D}_t(x),1-\bar{D}_t(x)\big)\,\mathrm{d}M_t,\quad \bar{D}_0(x)=x.\tag{26}$$

As in the case of constant μ we can see, that this SDE has a unique solution, adapted to the filtration of $\mu\cdot M$, which can be chosen to be continuous in both variables and, for any fixed t, increasing in x.

Since $\bar{D}_t(1) = 1$ and $\bar{D}_t(0) = 0$ for all t we can see, that for $x \in]0, 1[$ the martingale $\bar{D}(x)$ is bounded, therefore convergent. For each $x \in]0, 1[$ the limit $\lim_{t\to\infty} \bar{D}_t(x)$ exists almost surely, so we can define

$$\bar{D}_\infty(x) = \sup\Bigl\{\lim_{t\to\infty} \bar{D}_t(y):\ y \in \mathbb{Q}, y \le x\Bigr\}, \quad x \in]0, 1[.$$

The function $x \mapsto \bar{D}_\infty(x)$ is a (random) distribution function on $]0, 1[$. We denote by q its quantile function, i.e., $q(s) = \inf\{x:\ \bar{D}_\infty(x) > s\}$.

Now we define a random variable with $V = q(U)$. Observe that

$$\mathbf{P}(V < x|\mathcal{F}_\infty) = \mathbf{P}\bigl(U \le \bar{D}_\infty(x)|\mathcal{F}_\infty\bigr) = \bar{D}_\infty(x).$$

Since $\bar{D}(x)$ is a martingale for each fixed x we also have that

$$\mathbf{P}(V < x|\mathcal{F}_s) = \mathbf{E}\bigl(\mathbf{P}(V < x|\mathcal{F}_\infty)|\mathcal{F}_s\bigr) = \bar{D}_s(x).$$

This is clear for $x \in \mathbb{Q} \cap]0, 1[$ from the definitions, and extends to all $x \in]0, 1[$ by the continuity of both $x \mapsto \bar{D}_s(x)$ and $x \mapsto \mathbf{P}(V < x|\mathcal{F}_s)$. It follows that V is uniformly distributed on $]0, 1[$ as

$$\mathbf{P}(V < x) = \mathbf{E}\bigl(\mathbf{P}(V < x|\mathcal{F}_0)\bigr) = \bar{D}_0(x) = x.$$

Hence Theorem 3 follows from the combination of Lemmas 16 and 14 in the same way as in the proof of Theorem 2. □

3.3. *Proof of Theorem 1 based on Proposition 6*

The proof is based on a simple idea analogous to Tsirelson's ingenious construction of an SDE with a weak but no strong solution ([12], compare also [3]).

We take an increasing sequence $\{t_k:\ k \in \mathbb{Z}\}$ such that $\lim_{k\to-\infty} t_k = 0$ and $\lim_{k\to\infty} t_k = \infty$. This determines a partition of the half line $]0, \infty[$. On each interval $[t_k, t_{k+1}]$ we form the Lévy transform of W, this yields a Brownian motion $W^{(k)}$ on $[t_k, t_{k+1}]$ and an independent random variable U_k (from the signs of the excursions, see Proposition 15 of the Appendix) which is uniformly distributed on $]0, 1[$. Then, we use Proposition 6 on each such subinterval with $W^{(k)}$ and U_{k-1} the uniform variable from the previous interval. This gives on each subinterval the piece of β, B and H. Finally, we join these pieces to obtain the whole sample path of β, B and H. Since these pieces are independent β will be a Brownian motion in its own filtration and by similar reasons B a Brownian motion in the filtration of W.

To give some details we use the following notations.

$$\left.\begin{aligned} \mathcal{G}_t^{(k)} &= \mathcal{F}_{t+t_k}^W \\ W_t^{(k)} &= \int_0^t \operatorname{sign}(W_{s+t_k} - W_{t_k})\,\mathrm{d}W_{s+t_k} \end{aligned}\right\} \quad \text{for } 0 \le t \le t_{k+1} - t_k.$$

As described in the outline of the proof, U_k is a random variable, uniformly distributed on $]0, 1[$. It is formed from the random signs appearing in excursions of $(W_{t+t_k} - W_{t_k})_{t\in[0,t_{k+1}-t_k]}$, i.e. it is measurable to the σ-algebra $\mathcal{F}_{t_{k+1}}^W$. It follows that $\{(U_{k-1}, W^{(k)}):\ k \le 0\}$ form an independent sequence and U_{k-1} is independent from $W^{(k)}$. So we can apply Proposition 6 to $(W_t^{(k)})_{t\in[0,t_{k+1}-t_k]}$ as W, $(\mathcal{G}_t^{(k)})_{t\in[0,t_{k+1}-t_k]}$ as $\mathcal{F}$ and U_{k-1} as U. This way we obtain $(B_t^{(k)})_{t\in[0,t_{k+1}-t_k]}$ a Brownian motion in $(\mathcal{G}_t^{(k)})_{t\in[0,t_{k+1}-t_k]}$, a random sign process $(H_t^{(k)})_{t\in[0,t_{k+1}-t_k]}$.

Next, we can define B and H as

$$B_t = \sum_{k\in\mathbb{Z}} B^{(k)}_{((t\wedge t_{k+1})-t_k)\vee 0}, \qquad H_t = \begin{cases} 1 & \text{if } t = 0, \\ H^{(k)}_{t-t_k} & \text{if } t_k \le t < t_{k+1}. \end{cases}$$

Observe that the series defining B_t is almost surely convergent, as the summands are independent and the partial sums are bounded in L^2. It is easy to see that it is Brownian motion in $(\mathcal{F}_t^W)_{t\ge 0}$ as $B^{(k)}$ is a Brownian motion in $\mathcal{G}^{(k)}$.

Finally, put $S_t = B_t + \mu t$. Then with obvious notation

$$\beta_t = (H \cdot S)_t = \sum_{k\in\mathbb{Z}} \big(H^{(k)} \cdot S^{(k)}\big)_{((t\wedge t_{k+1})-t_k)\vee 0} = \sum_{k\in\mathbb{Z}} \beta^{(k)}_{((t\wedge t_{k+1})-t_k)\vee 0}. \tag{27}$$

To prove that β is a Brownian motion in its own filtration only the law of β has to be considered. By Proposition 6 $(\beta^{(k)}_t)_{t\in[0,t_{k+1}-t_k]}$ is a Brownian motion in its own filtration. It is defined from $W^{(k)}$, U_{k-1} so again the summands in (27) are independent. It now follows that β is a Brownian motion in its own filtration.

Appendix

We recall here a reformulation of a statement that was used in the proof of Theorem 1. It is in the spirit of the paper [1], where Gilat's theorem was analyzed.

Proposition 15. *Let $(W_t)_{t\ge 0}$ be a Brownian motion and $(B_t)_{t\ge 0}$ its Lévy transform*:

$$B_t = \int_0^t \operatorname{sign}(W_s)\,\mathrm{d}W_s.$$

Fix $T > 0$. Then, there is an $\mathcal{F}^W_T$ measurable random variable U uniformly distributed on $[0, 1]$ *and independent from* $\mathcal{F}^B_\infty$.

Proof. Let $\mathfrak{z} = \{t \in [0, T]\colon\ W_t = 0\}$. Since W is a Brownian motion the random set $[0, T] \setminus \mathfrak{z}$ is an open subset of $[0, T]$ having infinitely many connected components almost surely. We can take an $\mathcal{F}^{|W|}_T$ measurable enumeration of these components, i.e., there is a sequence of $\mathcal{F}^{|W|}_T$ measurable random times $(\sigma_n)_{n\ge 1}$ and $(\tau_n)_{n\ge 1}$ such that $\sigma_n < \tau_n$ and

$$[0, T] \setminus \mathfrak{z} = \bigcup_{n\ge 1}]\sigma_n, \tau_n[, \quad \text{almost surely.}$$

Note, that these random times are not stopping times, but $\mathcal{F}^{|W|}_T$ measurable variables. However, by a classical result, that can be traced back to Paul Lévy, we have that the random signs $\varepsilon_n = \operatorname{sign}(W_{(\sigma_n+\tau_n)/2})$ form an i.i.d. sequence of fair coin-tossing, i.e. $\mathbf{P}(\varepsilon_n = \pm 1) = \frac{1}{2}$, and independent from $\mathcal{F}^{|W|}_\infty = \mathcal{F}^B_\infty$. For proof, see [4], Section 2.9. So the choice

$$U = \sum_{n\ge 1} \varepsilon_n 2^{-n}$$

proves the statement. □

A.1. *Variational formula*

During the proof of Lemma 10 we met the following situation. There is given a function $f(x) = x \wedge (1 - x)$ and we considered the parametric SDE

$$\mathrm{d}\bar{D}_t(x) = f\big(\bar{D}_t(x)\big)\,\mathrm{d}\beta_t, \quad \bar{D}_0(x) = x.$$

It is well known that the solution of this parametric equation has a version which is continuous in both variables, and that on an almost sure event the mapping $x \mapsto \bar{D}_t(x)$ is a homeomorphism for all t. For a continuously differentiable f, with bounded derivative this solution is even differentiable in x and

$$Y_t(x) = \partial_x \bar{D}_t(x), \qquad \mathrm{d}Y_t(x) = Y_t(x) f'\big(\bar{D}_t(x)\big)\,\mathrm{d}\beta_t, \quad Y_0(x) = 1.$$

We extend this variational formula to our case relatively easily, since we consider a one-dimensional equation. In one dimension we only have to prove that the flow $x \mapsto \bar{D}_t(x)$ is absolutely continuous for all t almost surely. Indeed, assuming that $\bar{D}_t(y) - \bar{D}_t(x) = \int_x^y Y_t(u)\,\mathrm{d}u$ one can derive using the stochastic version of the Fubini theorem, see [8], Chapter IV, Theorem 65, that

$$\int_x^y Y_t(u)\,\mathrm{d}u = \int_x^y \left(1 + \int_0^t f'\bigl(\bar{D}_s(u)\bigr) Y_s(u)\,\mathrm{d}\beta_u\right)\mathrm{d}u.$$

Since this is true for all x, y we obtain that for almost all u

$$Y_t(u) = 1 + \int_0^t f'\bigl(\bar{D}_s(u)\bigr) Y_s(u)\,\mathrm{d}\beta_s,$$

i.e. the stochastic exponential of $f'(\bar{D}(u)) \cdot \beta$ is the Radon–Nikodym derivative at u of the flow $x \mapsto \bar{D}_t(x)$ for all t. To apply the Fubini theorem we also need that $\int_0^t \int_x^y Y_s^2(u)\,\mathrm{d}u\,\mathrm{d}s < \infty$ almost surely for each t and x, y.

So for our purposes it is enough to show the following lemma.

Lemma 16. *Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded function, $f(x) = \int_0^x g(y)\,\mathrm{d}y$ the integral function of g and β a Brownian motion. Consider the solution of the parametric equation*

$$\mathrm{d}\bar{D}_t(x) = f\bigl(\bar{D}_t(x)\bigr)\,\mathrm{d}\beta_t, \quad \bar{D}_0(x) = x,$$

which is continuous in both variables x, t. Then, on an almost sure event, the mapping $x \mapsto \bar{D}_t(x)$ is absolutely continuous for each fixed t. Moreover, for each $T > 0$ and $x < y$

$$\int_0^T \int_x^y \left(\frac{\mathrm{d}\bar{D}_t(u)}{\mathrm{d}u}\right)^2 \mathrm{d}u\,\mathrm{d}t < \infty, \quad \textit{almost surely.}$$

Proof. We recall the following simple fact from analysis. A continuous function $\varphi : [0, 1] \to \mathbb{R}$ is the integral of a square integrable function if and only if there is a constant C such that

$$s_n = \sum_{k=1}^{2^n} 2^n \bigl(\varphi\bigl(k/2^n\bigr) - \varphi\bigl((k-1)/2^n\bigr)\bigr)^2 \le C \quad \text{for all } n. \tag{A.1}$$

In this case $\int_0^1 (\mathrm{d}\varphi(x)/\mathrm{d}x)^2\,\mathrm{d}x \le C$.

We shall have established the lemma if we show that (A.1) holds for $\varphi(x) = \bar{D}_t(a + x) - \bar{D}_t(a)$ $(x \in [0, 1])$ with any fixed $a \in \mathbb{R}$, locally uniformly in t with probability one. We simply write s_n for $s_n(a, t)$. Since $s_n \le s_{n+1}$, by the convexity of the function x^2, it is enough to show that $\sup_n \mathbf{E}(\sup_{t \le T} s_n) < \infty$.

We can simply estimate $\mathbf{E}(\sup_{t \le T} s_n)$. To this end put $Z_t = \bar{D}_t(b) - \bar{D}_t(a)$ and write

$$Z_t = (b - a) + \int_0^t Z_s q_s\,\mathrm{d}\beta_s = (b - a) + \int_0^t f\bigl(\bar{D}_s(b)\bigr) - f\bigl(\bar{D}_s(a)\bigr)\,\mathrm{d}\beta_s,$$

where $q_s = (f(\bar{D}_s(b)) - f(\bar{D}_s(a)))/Z_s$ is a uniformly bounded quotient process. The upper bound is denoted by L. We can use Doob's inequality to obtain that

$$\mathbf{E}\Bigl(\sup_{t \le T} Z_t^2\Bigr) \le 4\left((b - a)^2 + L^2 \int_0^T \mathbf{E}\bigl(Z_s^2\bigr)\,\mathrm{d}s\right).$$

Let us divide by $(b - a)$

$$\mathbf{E}\left(\sup_{t \le T} \frac{Z_t^2}{b - a}\right) \le 4(b - a) + 4L^2 \int_0^T \mathbf{E}\left(\sup_{u \le s} \frac{Z_u^2}{b - a}\right)\mathrm{d}s.$$

So by Gronwall lemma

$$\mathbf{E}\left(\sup_{t\le T}\frac{Z_t^2}{b-a}\right)\le 4(b-a)\mathrm{e}^{4L^2T}$$

showing that $\mathbf{E}(\sup_{t\le T} s_n)\le 4\mathrm{e}^{4L^2T}$. The proof is complete. □

By stopping, the above argument extends to continuous local martingales in place of the Brownian motion β, and also with trivial modification to continuous semimartingales. Similar results, with different means and purposes, were found by Bass and Burdzy, see Theorem 3.9 in [2].

An interesting corollary is the following observation.

Corollary 17. *Let $(M_t)_{t\ge 0}$ be a continuous local martingale in the filtration $(\mathcal{F}_t)_{t\ge 0}$. Then, there is a predictable process $(H_t)_{t\ge 0}$ taking values in $\{-1,1\}$, such that with $M'=H\cdot M$ the stochastic exponential of M' is a true martingale.*

Proof. Let $\bar{D}$ be the solution of the parametric equation

$$\mathrm{d}\bar{D}_t(x)=-\min\bigl(\bar{D}_t(x),1-\bar{D}_t(x)\bigr)\,\mathrm{d}M_t,\qquad \bar{D}_0(x)=x,$$

which is continuous in both variables, cf. [8], Chapter V, Theorem 37. By Lemma 16, the random function $x\mapsto \bar{D}_t(x)$ is absolutely continuous almost surely and

$$\bar{D}_t(x)=\int_0^x Y_t(y)\,\mathrm{d}y,\qquad\text{where}$$

$$\mathrm{d}Y_t(x)=\operatorname{sign}\left(\bar{D}_t(x)-\frac{1}{2}\right)Y_t(x)\,\mathrm{d}M_t,\qquad Y_0(x)=1.$$

Let $H_t(x)=\operatorname{sign}(\bar{D}_t(x)-\frac{1}{2})$ and $M'(x)=H(x)\cdot M$. With this notation $Y(x)=\exp\{M'(x)-\frac{1}{2}\langle M'(x)\rangle\}$ is the stochastic exponential of $M'(x)$.

We show that $Y(x)$ is a martingale for almost all $x\in\,]0,1[$, which proves the claim. To see that $Y(x)$ is martingale it is enough that $\mathbf{E}(Y_n(x))=1$ for all n, since $Y(x)$ is a stochastic exponential of a continuous local martingale. We have that

$$1=\mathbf{E}(1)=\mathbf{E}\left(\int_0^1 Y_n(x)\,\mathrm{d}x\right)=\int_0^1\mathbf{E}\bigl(Y_n(x)\bigr)\,\mathrm{d}x.$$

Since $\mathbf{E}(Y_n(x))\le 1$, it implies that $\mathbf{E}(Y_n(x))=1$ for almost all x and the statement follows. □

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