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Limiting curlicue measures for theta sums

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Abstract. We consider the ensemble of curves $\{\gamma_{\alpha,N}\colon\ \alpha\in(0,1], N\in\mathbb{N}\}$ obtained by linearly interpolating the values of the normalized theta sum $N^{-1/2}\sum_{n=0}^{N'-1}\exp(\pi i n^2\alpha)$, $0\le N'<N$. We prove the existence of limiting finite-dimensional distributions for such curves as $N\to\infty$, when α is distributed according to any probability measure λ, absolutely continuous w.r.t. the Lebesgue measure on $[0,1]$. Our Main Theorem generalizes a result by Marklof [*Duke Math. J.* **97** (1999) 127–153] and Jurkat and van Horne [*Duke Math. J.* **48** (1981) 873–885, *Michigan Math. J.* **29** (1982) 65–77]. Our proof relies on the analysis of the geometric structure of such curves, which exhibit spiral-like patterns (*curlicues*) at different scales. We exploit a renormalization procedure constructed by means of the continued fraction expansion of α with even partial quotients and a renewal-type limit theorem for the denominators of such continued fraction expansions.

Résumé. Nous considérons l'ensemble des courbes $\{\gamma_{\alpha,N}\colon\ \alpha\in(0,1], N\in\mathbb{N}\}$ obtenues en interpolant les valeurs des sommes thêta normalisées $N^{-1/2}\sum_{n=0}^{N'-1}\exp(\pi i n^2\alpha)$, $0\le N'<N$. Nous démontrons l'existence de la limite des distributions fini-dimensionnelles de telles courbes quand $N\to\infty$, où α est distribué selon une quelconque mesure de probabilité λ, absolument continue par rapport à la mesure de Lebesgue sur $[0,1]$. Notre théorème principal généralise un resultat de Marklof [*Duke Math. J.* **97** (1999) 127–153] et de Jurkat et van Horne [*Duke Math. J.* **48** (1981) 873–885, *Michigan Math. J.* **29** (1982) 65–77]. Notre démonstration se base sur l'analyse des structures géomètriques de telles courbes, qui présentent des motifs à spirale (*curlicues*) à différentes échelles. Nous exploitons une procédure de renormalisation construite par le developpement de α en fractions continues avec quotients partiels pairs et un théorème de renouvellement pour les dénominateurs de tels developpements en fractions continues.

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1. Introduction

Given $a\in\mathbb{R}$ and $N\in\mathbb{N}$ consider the theta sum

$$\mathcal{S}_a(N):=\sum_{n=0}^{N-1}\exp\bigl(\pi i n^2 a\bigr)\in\mathbb{C}. \tag{1}$$

For arbitrary $L\ge 0$ let us define it as

$$\mathcal{S}_a(L):=\sum_{n=0}^{\lfloor L\rfloor-1}\exp\bigl(\pi i n^2 a\bigr)+\{L\}\exp\bigl(\pi i\lfloor L\rfloor^2 a\bigr)\in\mathbb{C},$$

where $\lfloor\cdot\rfloor$ denotes the floor function and $\{\cdot\}$ the fractional part. One has $\mathcal{S}_{a+2}(N)=\mathcal{S}_a(N)$, $\mathcal{S}_{-a}(N)=\overline{\mathcal{S}_a(N)}$ and $\int_{-1}^{1}|\mathcal{S}_a(N)|^2\,\mathrm{d}a=N$. It is convenient to restrict ourselves to $a\in(-1,1]\setminus\{0\}$ and consider $\alpha=|a|\in(0,1]$ and to study $\mathcal{S}_\alpha(L)$, see Section 2.2.

Our goal is to study the curves generated by theta sums, i.e.

$$\gamma=\gamma_{\alpha,N}:[0,1]\to\mathbb{C}\simeq\mathbb{R}^2,\qquad t\mapsto\frac{\mathcal{S}_\alpha(tN)}{\sqrt{N}}$$

as $N\to\infty$. Such curves are piecewise linear, of length $\sqrt{N}$ (being made of N segments of length $N^{-1/2}$). In particular we are interested in the ensemble of curves $\{\gamma_{\alpha,N}\}_{\alpha\in(0,1]}$ as $N\to\infty$ when α is distributed according to some probability measure on $[0,1]$.

As illustrated in Fig. 1, these curves exhibit a geometric multi-scale structure, including spiral-like fragments (*curlicues*). For a discussion on the geometry of $t\mapsto\mathcal{S}_\alpha(tN)$ (and more general curves defined using exponential sums) in connection with uniform distribution modulo 1, see Dekking and Mendès France [7]. For the study of other geometric and thermodynamical properties of such curves, see Mendès France [18,19] and Moore and van der Poorten [20].

Denote by $\mathcal{B}^k$ the Borel σ-algebra on $\mathbb{C}^k$ and let us fix a probability measure λ, absolutely continuous w.r.t. the Lebesgue measure on $[0,1]$.

Theorem 1.1 (Main theorem). *For every $k\in\mathbb{N}$, for every $t_1,\dots,t_k\in[0,1]$, $0\le t_1<t_2<\cdots<t_k\le 1$, there exists a probability measure $\mathrm{P}^{(k)}_{t_1,\dots,t_k}$ on $\mathbb{C}^k$ such that for every open, nice $A\in\mathcal{B}^k$,*

$$\lim_{N\to\infty}\lambda\Big(\Big\{\alpha\in(0,1]:\ \big(\gamma_{\alpha,N}(t_j)\big)_{j=1}^{k}\in A\Big\}\Big)=\mathrm{P}^{(k)}_{t_1,\dots,t_k}(A).\tag{2}$$

The measure $\mathrm{P}^{(k)}_{t_1,\dots,t_k}$ is called curlicue measure *associated with the moments of time $t_1,\dots,t_k$.*

We shall define later what we mean by “nice” and prove that many interesting sets are indeed nice. For example, if $B_z(\rho):=\{w\in\mathbb{C}:\ |z-w|<\rho\}$, then for every $(z_1,\dots,z_k)\in\mathbb{C}^k$, the set $A=B_{z_1}(\rho_1)\times\cdots\times B_{z_k}(\rho_k)\subseteq\mathbb{C}^k$ is nice for all $(\rho_1,\dots,\rho_k)\in\mathbb{R}^k_{>0}$, except possibly for a countable set.

Our main theorem generalizes a result by Marklof [17] (corresponding to $k=1$, $t_1=1$ and $\lambda=$ the Lebesgue measure), which in particular implies the following theorem by Jurkat and van Horne [12,13].

Theorem 1.2 (Jurkat and van Horne). *There exists a function $\Psi(a,b)$ such that for all (except for countably many) $a,b\in\mathbb{R}$,*

$$\lim_{N\to\infty}\Big|\Big\{\alpha:\ a<N^{-1/2}\big|\mathcal{S}_\alpha(N)\big|<b\Big\}\Big|=\Psi(a,b).$$

Let us remark that Marklof's approach uses the equidistribution of long, closed horocycles in the unit tangent bundle of a suitably constructed non-compact hyperbolic manifold of finite volume. Moreover, the explicit asymptotics for the moments of $N^{-1/2}|\mathcal{S}_a(N)|$ (along with central limit theorems [12–14]) were found by Jurkat and van Horne and generalized by Marklof [17] in the case of more general theta sums using Eisenstein series. In particular it is known that the above distribution function Ψ is not Gaussian. In the present paper we only show existence of the limiting measures $\mathrm{P}^{(k)}_{t_1,\dots,t_k}$. It is in principle possible to derive quantitative informations on the decay of their moments from our method too, but we shall not dwell on this. For a preliminary discussion of the present work, see Sinai [28].

Remark 1.3. *Consider the probability space $([0,1],\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$ and λ is as above. We look at $\gamma_{\alpha,N}$ as a random function, i.e. as a measurable map*

$$\gamma_{\cdot,N}:\big([0,1],\mathcal{B},\lambda\big)\to\big(\mathcal{C}\big([0,1],\mathbb{C}\big),\mathcal{B}_\mathcal{C}\big),$$

where $\mathcal{B}_\mathcal{C}$ is the Borel σ-algebra on $\mathcal{C}([0,1],\mathbb{C})$ coming from the topology of uniform convergence. Let P_N be the corresponding induced probability measure on $\mathcal{C}([0,1],\mathbb{C})$, $\mathrm{P}_N(A):=\lambda(\gamma^{-1}_{\cdot,N}(A))$, where $A\in\mathcal{B}_\mathcal{C}$. For $0\le t_1<t_2<\cdots<t_k\le 1$, let $\pi_{t_1,\dots,t_k}:\mathcal{C}([0,1],\mathbb{C})\to\mathbb{C}^k$ be the natural projection defined as $\pi_{t_1,\dots,t_k}(\gamma):=(\gamma(t_1),\dots,\gamma(t_k))$.

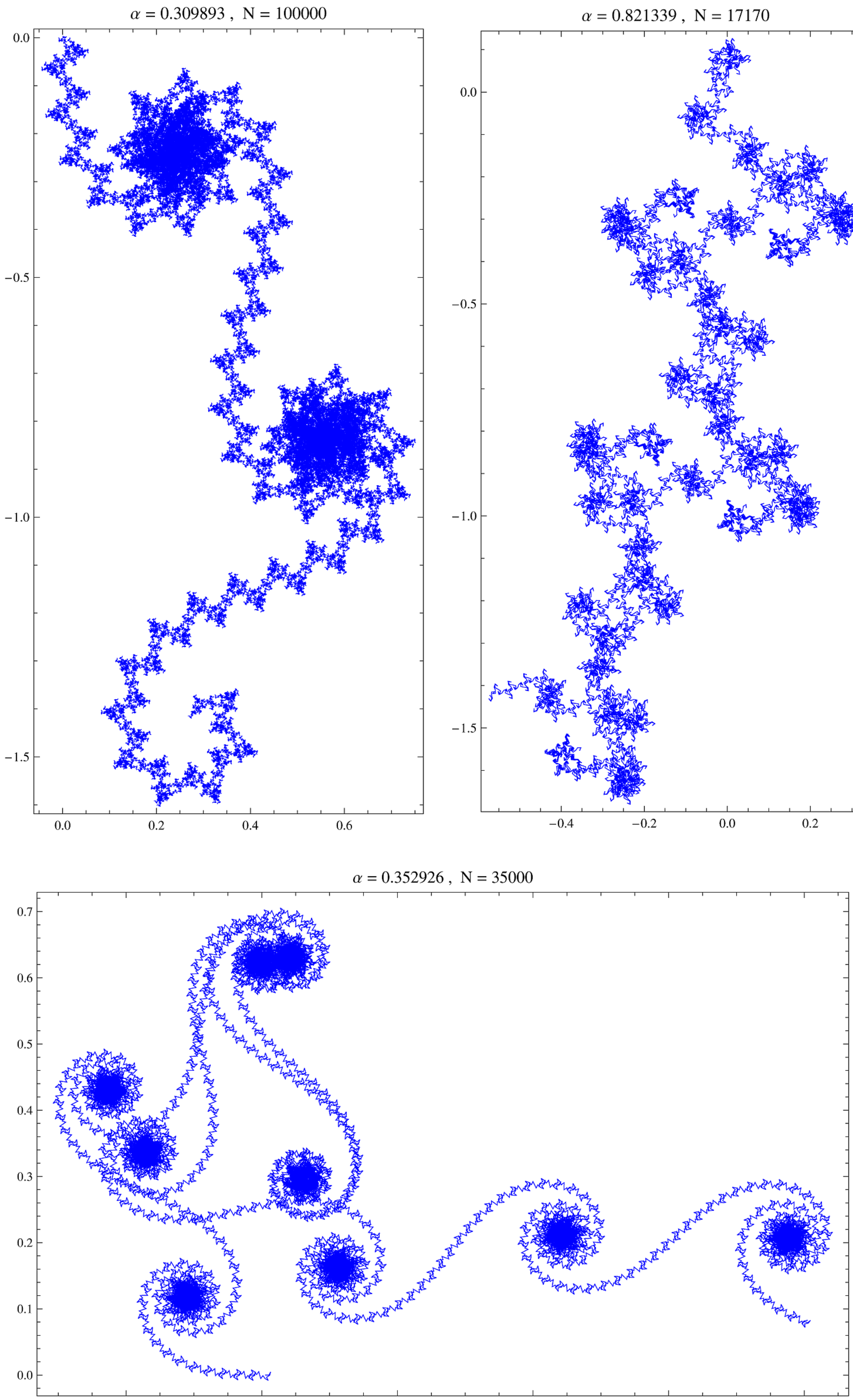

Fig. 1. Three curves of the form $t \mapsto \gamma_{\alpha,N}(t)$.

Theorem 1.1 can be rephrased as follows: for every $k \in \mathbb{N}$ and for every $0 \le t_1 < \cdots < t_k \le 1$

$$\mathrm{P}_N \pi_{t_1,\dots,t_k}^{-1} \quad \Longrightarrow \quad \mathrm{P}_{t_1,\dots,t_k}^{(k)} \quad \text{as } N \to \infty,$$

where "$\Rightarrow$" denotes weak convergence of probability measures. In other words, we prove weak convergence of finite-dimensional distributions of P_N as $N \to \infty$.

Remark 1.4. *By construction, the measures $\mathrm{P}_{t_1,\dots,t_k}^{(k)}$ automatically satisfy Kolmogorov's consistency conditions and hence there exists a probability measure $\tilde{\mathrm{P}}$ on the σ-algebra generated by finite-dimensional cylinders $\mathcal{B}_{\mathrm{fdc}} \subset \mathcal{B}_{\mathcal{C}}$ so that $\tilde{\mathrm{P}}\pi_{t_1,\dots,t_k}^{-1} = \mathrm{P}_{t_1,\dots,t_k}^{(k)}$.*

Remark 1.5 (Scaling property of the limiting measures). *Notice that*

$$\gamma_{\alpha,N}(\tau t) = N^{-1/2}\mathcal{S}_\alpha(\tau t N) = \tau^{1/2}\gamma_{\alpha,\tau N}(t).$$

Thus, the limiting probability measures $\mathrm{P}_{t_1,\dots,t_k}^{(k)}$ satisfy the following scaling property: for every $\tau \in (0,1]$

$$\mathrm{P}_{\tau t_1,\dots,\tau t_k}^{(k)}(A) = \mathrm{P}_{t_1,\dots,t_k}^{(k)}\big(\tau^{-1/2}A\big).$$

In particular, for example, $\mathrm{P}_t^{(1)}(A) = \mathrm{P}_1^{(1)}(t^{-1/2}A)$.

Remark 1.6. *Our results are of* probabilistic *nature, since we look at the measure of α's for which some event happens. Let us stress the fact that the growth of $|\mathcal{S}_\alpha(N)|$ for* specific *or* generic *α has also been thoroughly studied. For instance, Hardy and Littlewood [11] proved that if α is of bounded-type, then $|\mathcal{S}_\alpha(N)| \le C\sqrt{N}$ for some constant C. To the best of our knowledge, the most refined result in this direction is due to Flaminio and Forni [10]. A particular case of their results on equidistribution of nilflows reads as follows. For every increasing function $b : (1,\infty) \to (0,\infty)$ such that $\int_1^\infty t^{-1}b^{-4}(t)\,\mathrm{d}t < \infty$, there exists a full measure set $\mathcal{G}_b$ such that for every $\alpha \in \mathcal{G}_b$, every $\beta \in \mathbb{R}$ the following holds: for every $s > \frac{5}{2}$, there exists a constant $C = C(s,\alpha)$ such that for every $f \in W^s$, 2-periodic,*

$$\left| \sum_{n=0}^{N-1} f\big(\alpha n^2 + \beta\big) - N \int_{-1}^{1} f(x)\,\mathrm{d}x \right| \le C\sqrt{N} b(N) \|f\|_s,$$

where W^s denotes the Sobolev space and $\|\cdot\|_s$ is the corresponding Sobolev norm. This generalizes the work of Fiedler, Jurkat and Körner [9] where $f(x) = \mathrm{e}^{\pi \mathrm{i} x}$ and $\beta = 0$.

The paper is organized as follows. In Section 2 we discuss the geometric multi-scale structure of the curve $t \mapsto \gamma_{\alpha,N}(t)$ and we deal with the first step of the renormalization procedure which allows us to move from a scale to the next one. Moreover, we describe the connection of the renormalization map T with the continued fraction expansion of α with even partial quotients and we consider an "accelerated" version of it, i.e. the associated jump transformation R. For the corresponding accelerated continued fraction expansions we prove some estimates on the growth of the entries. In Section 3 we iterate the renormalization procedure and we approximate the curve $\gamma_{\alpha,N}$ by a curve $\gamma_{\alpha,N}^J$ in which only the J largest scales are present. Furthermore, we write $(\gamma_{\alpha,N}(t_j))_{j=1}^k \in \mathbb{C}^k$ as a function of certain random variables defined in terms of the renewal time $\hat{n}_N := \min\{n \in \mathbb{N}\colon \hat{q}_n > N\}$, where $\{\hat{q}_n\}_{n\in\mathbb{N}}$ is the subsequence of denominators of the convergents of α corresponding to the map R. In Section 4 we use a renewal-type limit theorem (proven in the Appendix) to show the existence of the limit for finite-dimensional distributions for the approximating curve $\gamma_{\alpha,N}^J$ as $N \to \infty$. Estimates from Section 3 allow us to take the limit as $J \to \infty$ and prove the existence of finite-dimensional distributions for $\gamma_{\alpha,N}$ as $N \to \infty$. We also discuss the notion of *nice* sets and give a sufficient condition for a set to be nice.

2. Renormalization of curlicues

In this section we recall some known facts concerning the geometry of the curves $\gamma_{\alpha,N}$. In particular we discuss the presence/absence of spiral-like fragments and at different scales using a renormalization procedure. The renormalization map T is connected with a particular class of continued fraction expansions. From a metrical point of view, this classical renormalization is very ineffective, because of the intermittent behavior of the map T (which preserves an infinite, ergodic measure). It is therefore very natural to study an "accelerated version" of T (preserving an ergodic probability measure) and the corresponding continued fraction expansion.

2.1. *Geometric structure at level zero*

In order to investigate the presence/absence of spiraling geometric structures at the smallest scale we introduce the *local discrete radius of curvature*, following Coutsias and Kazarinoff [5,6]. Set $\mathcal{T}_N := \{\frac{m}{N}, 0 \le m \le N\}$ and let $\tau_n := \frac{n}{N} \in \mathcal{T}_N \setminus \{0,1\}$, so that $\gamma(\tau_n) = \gamma_{\alpha,N}(\tau_n) = N^{-1/2}\mathcal{S}_\alpha(n)$. Define $\rho_{\alpha,N}(\tau_n)$ as the radius of the circle passing through the three points $\gamma(\tau_{n-1})$, $\gamma(\tau_n)$ and $\gamma(\tau_{n+1})$. A simple computation shows that $\rho_{\alpha,N}(\tau_n) = \frac{1}{2\sqrt{N}}|\csc(\frac{\pi\alpha(2n-1)}{2})|$ and for arbitrary $t \in [0,1]$ we set

$$\rho(t) = \rho_{\alpha,N}(t) := \frac{1}{2\sqrt{N}}\left|\csc\left(\frac{\pi\alpha(2tN-1)}{2}\right)\right| \in \overline{\mathbb{R}}.$$

The function $t \mapsto \rho_{\alpha,N}(t)$ is $\frac{1}{\alpha N}$-periodic; it has vertical asymptotes at $\tau_k^{(\text{flat})} = \tau_k^{(\text{flat})}(\alpha,N) := \frac{k}{\alpha N} + \frac{1}{2N}$ and local minima at $\tau_k^{(\text{curl})} = \tau_k^{(\text{curl})}(\alpha,N) := \frac{2k+1}{2\alpha N} + \frac{1}{2N}$, $k \in \mathbb{Z}$, where $\rho_{\alpha,N}(\tau_k^{(\text{curl})}) = \frac{1}{2\sqrt{N}}$. We partition the interval $[0,1]$ into subintervals as follows:

$$[0,1] = \bigsqcup_{k=0}^{k^*+1} I_k^{(0)},$$

where $k^* = k^*_{\alpha,N} := \lfloor \alpha N - \frac{\alpha+1}{2} \rfloor$ and

$$I_k^{(0)} = I_{k;\alpha,N}^{(0)} := \begin{cases} \left[0, \tau_0^{(\text{curl})}\right) & \text{if } k = 0, \\ \left[\tau_{k-1}^{(\text{curl})}, \tau_k^{(\text{curl})}\right) & \text{if } 1 \le k \le k^*, \\ \left[\tau_{k^*}^{(\text{curl})}, 1\right] & \text{if } k = k^*+1. \end{cases}$$

By construction, the lengths of the above intervals are $|I_k^{(0)}| = \frac{1}{\alpha N}$ for $1 \le k \le k^*$, $|I_0^{(0)}| = \frac{1}{2N}$ and $0 \le |I_{k^*+1}^{(0)}| = 1 - \frac{1}{2N} - \frac{k^*}{\alpha N} < \frac{1}{\alpha N}$. The number of $\mathcal{T}_N$-rationals inside each subinterval is of order $\frac{1}{\alpha}$ and explicitly given by

$$\#\left(I_k^{(0)} \cap \mathcal{T}_N\right) = \begin{cases} \left\lceil \frac{1}{2\alpha} + \frac{1}{2} \right\rceil & \text{if } k = 0, \\ \left\lceil \frac{2k+1}{2\alpha} + \frac{1}{2} \right\rceil - \left\lceil \frac{2k-1}{2\alpha} + \frac{1}{2} \right\rceil & \text{if } 1 \le k \le k^*, \\ N + 1 - \left\lceil \frac{2k^*+1}{2\alpha} + \frac{1}{2} \right\rceil & \text{if } k = k^*+1. \end{cases}$$

The whole curve $\gamma_{\alpha,N}([0,1])$ can be recovered by means of the values of the function ρ at the rationals in $\mathcal{T}_N$. Suppose we know the values of $\gamma(\tau_0), \gamma(\tau_1), \ldots, \gamma(\tau_{n-1}), \gamma(\tau_n)$ and the radius $\rho(\frac{n}{N})$. Then the point $\gamma(\tau_{n+1})$ should be placed at the intersection of the circle of radius $N^{-1/2}$ centered at $\gamma(\tau_n)$ and one of the two circles of radius $\rho(\frac{n}{N})$ passing through $\gamma(\tau_{n-1})$ and $\gamma(\tau_n)$ in order to get a counterclockwise oriented triple $(\gamma(\tau_{n-1}), \gamma(\tau_n), \gamma(\tau_{n+1}))$ when $\frac{n}{N} \in [\tau_{k-1}^{(\text{curl})}, \tau_k^{(\text{flat})})$ (resp., clockwise when $\frac{n}{N} \in [\tau_k^{(\text{flat})}, \tau_k^{(\text{curl})})$). For arbitrary $t \in [0,1]$ the curve $\gamma(t)$ is defined by linear interpolation.

For small values of α, each subinterval $I_k^{(0)}$, $1 \le k \le k^*$, contains approximately $\frac{1}{\alpha}$ integer multiples of $\frac{1}{N}$ and the curlicue structure is easily understood: those n's for which $\rho(\tau_n)$ is large correspond to straight-like parts of $\gamma([0,1])$, while the points close to the minima of ρ give the spiraling fragments (*curlicues*). For $\alpha \sim 1$ the curlicues disappear. See Fig. 2. We shall see in Section 2.2 how these curlicues appear at different scales though.

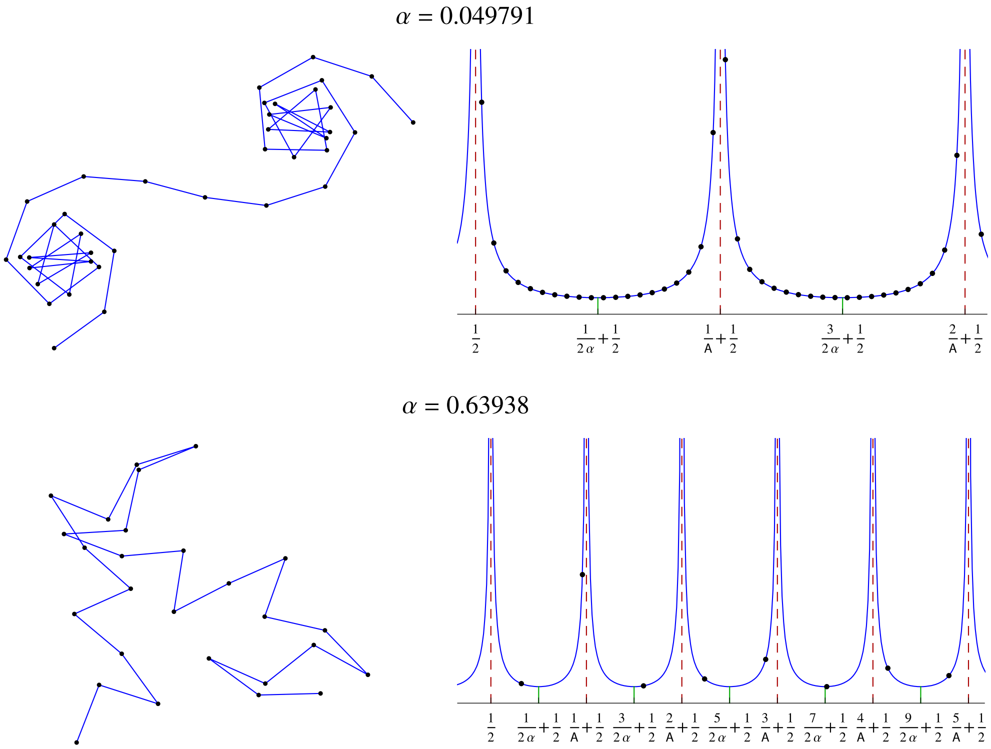

Fig. 2. Geometric patterns at level zero (left) and the function $\rho_{\alpha,N}$ (right).

2.2. *Approximate and exact renormalization formulae*

Let us introduce the map $U\colon (-1,1]\setminus\{0\} \to (-1,1]\setminus\{0\}$ where $U(t) := -\frac{1}{t} \pmod 2$. The graph of U has countably many smooth branches. Each interval $(\frac{1}{2k+1}, \frac{1}{2k-1}]$ is mapped in a one-to-one way onto $(-1,1]$ via $t \mapsto -\frac{1}{t} + 2k$.

For $a \in (-1,1]\setminus\{0\}$ and $N \in \mathbb{N}$ one has the Approximate Renormalization Formula (ARF)

$$\left|\mathcal{S}_a(N) - \mathrm{e}^{(\pi/4)\mathrm{i}}|a|^{-1/2}\mathcal{S}_{a_1}\left(\lfloor N_1 \rfloor\right)\right| \le C_1|a|^{-1/2} + C_2, \tag{3}$$

where $a_1 = U(a)$, $N_1 = |a|N$ and $C_1, C_2 > 0$ are absolute constants which do not depend on N. This result was established by Hardy and Littlewood [11], Mordell [21], Wilton [32] and Coutsias and Kazarinoff [6], the constants C_1, C_2 being always improved.

Let us explain the ARF (3) geometrically. Recall that the curve $t \mapsto \gamma_{a,N}(t)$ contains $k^*_{|a|,N} \simeq N_1$ intervals of the form $[\tau_{k-1}^{(\mathrm{curl})}, \tau_k^{(\mathrm{curl})})$ at level zero. By (3), the curve $t \mapsto \sqrt{N}\gamma_{a,N}(t)$ can be approximated (up to scaling by $|a|^{-1/2}$ and rotating by $\pi/4$) by $t \mapsto \sqrt{N_1}\gamma_{a_1,N_1}(t)$. In other words, replace each interval of the form $I_k^{(0)}$, $1 \le k \le k^*$, for $\gamma_{a,N}(t)$ by a $\mathcal{T}_{N_1}$-rational point in $\gamma_{a_1,N_1}(t)$. The renormalization map can be seen as a "coarsening" transformation, which deletes of the geometric structure at level zero. Beside the above-mentioned references, we also want to mention the work by Berry and Goldberg [3], in which typical and untypical behaviors of $\{\mathcal{S}_\alpha(N')\}_{N'=1}^{N}$ are studied with the help of a renormalization procedure.

Coutsias and Kazarinoff [6] also proved a stronger version of (3):

$$\left|\mathcal{S}_a(N) - \mathrm{e}^{(\pi/4)\mathrm{i}}|a|^{-1/2}\mathcal{S}_{a_1}(n)\right| \le C_3\left|\frac{|a|N - n}{a}\right| \le C_4$$

for some $C_3, C_4 > 0$, where $n \in \mathbb{N}$ is arbitrary and $N = \langle n/|a| \rangle$ is a function of n, $\langle\cdot\rangle$ denoting the nearest-integer function.

In our analysis we shall focus on (3), which can be extended to $\mathcal{S}_a(L)$ for arbitrary $L \geq 0$:

$$\left|\mathcal{S}_a(L) - \mathrm{e}^{(\pi/4)\mathrm{i}}|a|^{-1/2}\mathcal{S}_{a_1}(L_1)\right| \leq C_5|a|^{-1/2} + C_6, \tag{4}$$

where $a_1 = U(a)$, $L_1 = |a|L$, $C_5 = C_1 + 2$ and $C_6 = C_2 + 1$.

Since the function U is odd w.r.t. the origin and $\mathcal{S}_{-a}(N) = \overline{\mathcal{S}_a(N)}$, it is natural to consider $\alpha = |a| \in (0,1]$ and keep track of $|U(\alpha)|$ and $\mathrm{sgn}(U(\alpha))$ separately. Define $\eta(\alpha) := \mathrm{sgn}(U(\alpha))$, $\xi(\alpha) := -\eta(\alpha)$ and introduce a new map $T:(0,1] \to (0,1]$, $T := |U|_{(0,1]}|$. More explicitly, let us partition the interval $(0,1]$ into subintervals $B(k,\xi)$, $k \in \mathbb{N}$, $\xi = \pm 1$, where $B(k,-1) := (\frac{1}{2k}, \frac{1}{2k-1}]$ and $B(k,+1) := (\frac{1}{2k+1}, \frac{1}{2k}]$. The map T can be represented accordingly as

$$T(\alpha) = \xi \cdot \left(\frac{1}{\alpha} - 2k\right), \quad \alpha \in B(k,\xi), k \in \mathbb{N}, \xi \in \{\pm 1\}.$$

We shall deal with this map, first introduced by Schweiger [24,25], in Section 2.3 in connection with the even continued fraction expansion of α. Moreover, for every complex-valued function F set

$$F^{(\eta)} := \begin{cases} F & \text{if } \eta = +1, \\ \overline{F} & \text{if } \eta = -1. \end{cases}$$

With this notations we can define the remainder terms of (3) and (4) for $\alpha \in (0,1]$ as follows:

$$\Lambda(\alpha, N) := \mathcal{S}_\alpha(N) - \mathrm{e}^{(\pi/4)\mathrm{i}}\alpha^{-1/2}\mathcal{S}_{\alpha_1}^{(\eta_1)}\big(\lfloor N_1 \rfloor\big), \quad N \in \mathbb{N}, \tag{5}$$

$$\Gamma(\alpha, L) := \mathcal{S}_\alpha(L) - \mathrm{e}^{(\pi/4)\mathrm{i}}\alpha^{-1/2}\mathcal{S}_{\alpha_1}^{(\eta_1)}(L_1), \quad L \in \mathbb{R}, \tag{6}$$

where $\alpha_1 = T(\alpha)$, $\eta_1 = \eta(\alpha)$, $N_1 = \alpha N$ and $L_1 = \alpha L$.

Later, we shall use the fact that $\Gamma(\alpha, L)$ is a continuous function of $(\alpha, L) \in (0,1] \times \mathbb{R}_{\geq 0}$ (one can actually prove that it has piecewise $\mathcal{C}^\infty$ partial derivatives). An explicit formula for $\Lambda(\alpha, N)$, $N \in \mathbb{N}$, has been provided by Fedotov and Klopp [8] in terms of a special function $\mathcal{F}_\alpha : \mathbb{C} \to \mathbb{C}$ as follows. For $\alpha \in (0,1]$ and $w \in \mathbb{C}$ set

$$\mathcal{F}_\alpha(w) := \int_{\Gamma_w} \frac{\exp(\pi \mathrm{i} z^2/\alpha)}{\exp(2\pi\mathrm{i}(z-w)) - 1}\,\mathrm{d}z, \tag{7}$$

where Γ_w is the contour given by

$$\mathbb{R} \ni t \mapsto \Gamma_w(t) = \begin{cases} w + t + \mathrm{i}t & \text{if } |t| \geq \varepsilon, \\ w + \varepsilon\exp\big(\pi\mathrm{i}\big(\frac{t}{2\varepsilon} - \frac{1}{4}\big)\big) & \text{if } |t| < \varepsilon, \end{cases}$$

and $\varepsilon = \varepsilon(\alpha, w)$ is smaller than the distance between w and the other poles of the integrand in (7). We have the following theorem.

Theorem 2.1 (Exact renormalization formula [8]). *For every $0 < \alpha \leq 1$ and every $N \in \mathbb{N}$ we have*

$$\overline{\Lambda(\alpha, N)} = \mathrm{e}^{-(\pi/4)\mathrm{i}}\alpha^{-1/2}\big[\mathrm{e}^{-\pi\mathrm{i}\alpha N^2}\mathcal{F}_\alpha\big(\{N_1\}\big) - \mathcal{F}_\alpha(0)\big], \tag{8}$$

where $N_1 = \alpha N$.

In order to write $\Gamma(\alpha, L)$ in terms of $\Lambda(\alpha, \lfloor L \rfloor)$, we notice that $\alpha L = \lfloor \alpha \lfloor L \rfloor \rfloor + H(\alpha, L)$, where $H(\alpha, L) := \alpha\{L\} + \{\alpha\lfloor L \rfloor\} \in [0,2)$. Moreover, if $H(\alpha, L) \in [0,1)$ then $\lfloor \alpha L \rfloor = \lfloor \alpha \lfloor L \rfloor \rfloor$, while if $H(\alpha, L) \in [1,2)$ then $\lfloor \alpha L \rfloor = \lfloor \alpha \lfloor L \rfloor \rfloor + 1$. Now, a simple computation shows that for every $\alpha \in (0,1]$ and every $L \geq 0$

$$\Gamma(\alpha, L) = \Lambda\big(\alpha, \lfloor L \rfloor\big) + G_1(\alpha, L) - \mathrm{e}^{(\pi/4)\mathrm{i}}\alpha^{-1/2}G_2(\alpha, L), \tag{9}$$

where $G_1(\alpha, L) := \{L\}\mathrm{e}^{\pi\mathrm{i}\lfloor L \rfloor^2\alpha}$ and

$$G_2(\alpha, L) := \begin{cases} H(\alpha, L)\mathrm{e}^{\pi\mathrm{i}\lfloor \alpha L \rfloor^2\alpha_1} & \text{if } H(\alpha, L) \in [0,1), \\ \mathrm{e}^{\pi\mathrm{i}(\lfloor \alpha L \rfloor - 1)^2\alpha_1} + \big(H(\alpha, L) - 1\big)\mathrm{e}^{\pi\mathrm{i}\lfloor \alpha L \rfloor^2\alpha_1} & \text{if } H(\alpha, L) \in [1,2). \end{cases}$$

Remark 2.2. *Applying the stationary phase method to the integrals in* (8) *and* (9) *as in* [8] *one can obtain the approximate renormalization estimates* (3) *and* (4) (*possibly with different constants* C_1, C_2, C_5, C_6).

We want to describe $\mathcal{S}_\alpha(tN)$ for $N \in \mathbb{N}$ and $t \in [0,1]$. In this case (4) and (9) can be rewritten as

$$\mathcal{S}_\alpha(tN) = \mathrm{e}^{(\pi/4)\mathrm{i}}\alpha^{-1/2}\mathcal{S}_{\alpha_1}^{(\eta_1)}(t\alpha N) + \Gamma(\alpha, tN), \tag{10}$$

$$\Gamma(\alpha, tN) = \Lambda\big(\alpha, \lfloor tN \rfloor\big) + G_1(\alpha, tN) + \mathrm{e}^{(\pi/4)\mathrm{i}}\alpha^{-1/2}G_2(\alpha, tN). \tag{11}$$

2.3. *Continued fractions with even partial quotients*

In this section we discuss the relation between the map T and expansions in continued fractions with even partial quotients. Consider the following *ECF-expansion* for $\alpha \in (0,1]$:

$$\alpha = \frac{1}{2k_1 + \xi_1/(2k_2 + \xi_2/(2k_3 + \cdots))} =: \big[\big[(k_1,\xi_1),(k_2,\xi_2),(k_3,\xi_3),\ldots\big]\big], \tag{12}$$

where $k_j \in \mathbb{N}$ and $\xi_j \in \{\pm 1\}$, $j \in \mathbb{N}$. ECF-expansions have been introduced by Schweiger [24,25] and studied by Kraaikamp–Lopes [16]. Since $1 = [[(1,-1),(1,-1),\ldots]]$, it is easy to see that every $\alpha \in (0,1] \setminus \mathbb{Q}$ has an infinite expansion with no $(1,-1)$-tail.

Using the notations introduced in Section 2.2 we notice that if $\alpha \in B(k,\xi)$, then $\alpha = \frac{1}{2k+\xi T(\alpha)}$. Therefore,

$$\begin{aligned} &\text{for } \alpha = \big[\big[(k_1,\xi_1),(k_2,\xi_2),(k_3,\xi_3),\ldots\big]\big] \in B(k_1,\xi_1),\\ &\quad T^n(\alpha) = \big[\big[(k_{n+1},\xi_{n+1}),(k_{n+2},\xi_{n+2}),\ldots\big]\big] \in B(k_{n+1},\xi_{n+1}), \end{aligned} \tag{13}$$

i.e. T acts as a shift on the space $\Omega^{\mathbb{N}}$, where $\Omega := \mathbb{N} \times \{\pm 1\}$. Despite its similarities with the Gauss map in the context of Euclidean continued fractions, the map T has an indifferent fixed point at $\alpha = 1$ and we have the following theorem.

Theorem 2.3 (Schweiger [24]). *The map* $T:(0,1] \to (0,1]$ *has a* σ*-finite, infinite, ergodic invariant measure* μ_T *which is absolutely continuous w.r.t. the Lebesgue measure on* $(0,1]$. *Its density is* $\varphi_T(\alpha) := \frac{\mathrm{d}\mu_T(\alpha)}{\mathrm{d}\alpha} = \frac{1}{\alpha+1} - \frac{1}{\alpha-1}$.

One of the consequences of this fact is the anomalous growth of Birkhoff sums for integrable functions. Given $f \in L^1((0,1],\mu_T)$, $f \geq 0$ μ_T-almost everywhere, let $\mu_T(f) = \int_0^1 f(\alpha)\,\mathrm{d}\mu_T(\alpha)$ and denote by $\mathrm{S}_n^T(f)$ the ergodic sum $\sum_{j=0}^{n-1} f \circ T^j$. Since $\mu_T((0,1]) = \infty$, the Birkhoff Ergodic theorem implies that $\frac{1}{n}\mathrm{S}_n^T(f) \to 0$ almost everywhere as $n \to \infty$. According to the Hopf's Ergodic theorem there exists a sequence of measurable functions $\{a_n(\alpha)\}_{n\in\mathbb{N}}$ such that $\frac{1}{a_n(\alpha)}\mathrm{S}_n^T(f)(\alpha) \to \mu_T(f)$ for almost every $\alpha \in (0,1]$ as $n \to \infty$. The question *"Can the sequence* $a_n(\alpha)$ *be chosen independently of* α*?"* is answered negatively by Aaronson's theorem ([2], Theorem 2.4.2), according to which for almost every $\alpha \in (0,1]$ and for every sequence of constants $\{a_n\}_{n\in\mathbb{N}}$ either $\liminf_{n\to\infty} \frac{1}{a_n}\mathrm{S}_n^T(f)(\alpha) = 0$ or $\frac{1}{a_{n_k}}\mathrm{S}_{n_k}^T(f)(\alpha) \to \infty$ along some subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ as $k \to \infty$. However, for weaker types of convergence such a sequence of constants can indeed be found. The following theorem establishes $a_n = \frac{n}{\log n}$ and provides convergence in probability:

Theorem 2.4 (Weak law of large numbers for *T*). *For every probability measure* P *on* $(0,1]$, *absolutely continuous w.r.t.* μ_T, *for every* $f \in L^1(\mu_T)$ *and for every* $\varepsilon > 0$,

$$\mathrm{P}\left(\left|\frac{\mathrm{S}_n^T(f)}{n/\log n} - \mu_T(f)\right| \geq \varepsilon\right) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Remark 2.5. *The proof of Theorem* 2.4 *follows from standard techniques in infinite ergodic theory. See Aaronson* [1] *and* [2], *Chapter* 4. *The same rate* $\frac{n}{\log n}$ *rate for the growth of Birkhoff sums for integrable observables over ergodic transformations preserving an infinite measure appears in several examples, e.g. the Farey map. A recent interesting example comes from the study of linear flows over regular* n*-gons, see Smillie and Ulcigrai* [30].

Let us come back to ECF-expansions. For $\alpha = [[(k_1, \xi_1), (k_2, \xi_2), \ldots]]$ the convergents have the form

$$\frac{p_n}{q_n} = \frac{1}{2k_1 + \xi_1/(2k_2 + \xi_2/(2k_3 + \cdots + \xi_{n-2}/(2k_{n-1} + \xi_{n-1}/(2k_n))))}$$
$$= \big[\big[(k_1, \xi_1), (k_2, \xi_2), \ldots, (k_n, *)\big]\big], \quad (p_n, q_n) = 1,$$

where "$*$" denotes any $\xi_n = \pm 1$. They satisfy the following recurrent relations:

$$p_n = 2k_n p_{n-1} + \xi_{n-1} p_{n-2}, \qquad q_n = 2k_n q_{n-1} + \xi_{n-1} q_{n-2}, \tag{14}$$

with $q_{-1} = p_0 = 0$, $p_{-1} = q_0 = \xi_0 = 1$. Moreover, we have

$$p_{n+1} q_n - p_n q_{n+1} = (-1)^n \prod_{j=0}^{n} \xi_j. \tag{15}$$

The proof of (15) follows from (14) and can be recovered *mutatis mutandis* from the proof of the analogous result for Euclidean continued fractions. See, e.g., [23].

Set $\alpha_0 := \alpha$ and $\alpha_n := T^n(\alpha)$. In Section 3, we shall deal with the product $\alpha_0 \alpha_1 \cdots \alpha_{n-1}$. As in the case of Euclidean continued fractions, this product can be written in terms of the denominators of the convergents; however the formula involves the ξ_n as well: for $n \in \mathbb{N}$,

$$(\alpha_0 \cdots \alpha_{n-1})^{-1} = q_n \left(1 + \xi_n \alpha_n \frac{q_{n-1}}{q_n}\right). \tag{16}$$

Notice that, considering $f(\alpha) = -\log \alpha$, Theorem 2.4 reads as follows: for every $\varepsilon > 0$ and every probability measure P on $(0, 1]$, absolutely continuous w.r.t. μ_T,

$$P\left(\left|\frac{-\log(\alpha_0 \cdots \alpha_{n-1})}{n/\log n} - \frac{\pi^2}{4}\right| \geq \varepsilon\right) \to 0 \quad \text{as } n \to \infty.$$

In other words, the product along the T-orbit of α decays subexponentially in probability.

2.4. *The jump transformation R*

In order to overcome the issues connected with the infinite invariant measure for T, it is convenient to introduce an "accelerated" version of T, namely its associated *jump transformation* (see [26]) $R : (0, 1] \to (0, 1]$. Define the *first passage time* to the interval $(0, \frac{1}{2}]$ as $\tau : (0, 1] \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as $\tau(\alpha) := \min\{j \geq 0\colon\ T^j(\alpha) \in B(1, -1)^c = (0, \frac{1}{2}]\}$ and the jump transformation w.r.t. $(0, \frac{1}{2}]$ as $R(\alpha) := T^{\tau(\alpha)+1}(\alpha)$. Let us remark that this construction is very natural. For instance, if we consider the jump transformation associated to the Farey map w.r.t. the interval $(\frac{1}{2}, 1]$ we get precisely the celebrated Gauss map. Another example is given by the Zorich map, obtained by accelerating the Rauzy map, in the context of interval exchange transformations.

The map R was extensively studied in [4]. It is a Markov, uniformly expanding map with bounded distortion and has an invariant probability measure μ_R which is absolutely continuous w.r.t. the Lebesgue measure on $[0, 1]$. The density of μ_R is given by $\varphi_R(\alpha) := \frac{\mathrm{d}\mu_R(\alpha)}{\mathrm{d}\alpha} = \frac{1}{\log 3}(\frac{1}{3-\alpha} + \frac{1}{1+\alpha})$. For a different acceleration of T in connection with the geometry of theta sums, see Berry and Goldberg [3].

We want to describe a symbolic coding for R. Let us restrict ourselves to $\alpha \in (0, 1] \setminus \mathbb{Q}$ and identify $(0, 1] \setminus \mathbb{Q}$ with the subset $\dot{\Omega}^N \subset \Omega^{\mathbb{N}}$ of infinite sequences with no $(1, -1)$-tail. Let $\bar{\omega} = (1, -1)$. Given $\alpha = [[\omega_1, \omega_2, \omega_3, \ldots]] \in \dot{\Omega}^{\mathbb{N}}$ we have $\tau = \tau(\alpha) = \min\{j \geq 0\colon\ \omega_{j+1} \neq \bar{\omega}\}$ and $R(\alpha) = [[\omega_{\tau+2}, \omega_{\tau+3}, \omega_{\tau+4}, \ldots]] \in \dot{\Omega}^{\mathbb{N}}$. Setting $\Omega^* := \Omega \setminus \{\bar{\omega}\}$, $\Sigma := \mathbb{N}_0 \times \Omega^*$ and denoting by $\sigma = (h, \omega) \in \Sigma$ the Ω-word $(\bar{\omega}, \ldots, \bar{\omega}, \omega)$ of length $h + 1$ for which $\omega \in \Omega^*$, we can identify $\dot{\Omega}^{\mathbb{N}}$ and $\Sigma^{\mathbb{N}}$ and the map R acts naturally as a shift over this space.

For brevity, we denote $m^{\pm} = 0 \cdot m^{\pm} = (0, (m, \pm 1)) \in \Sigma$ and $h \cdot m^{\pm} = (h, (m, \pm 1)) \in \Sigma$. For $\alpha = (h_1 \cdot m_1^{\pm}, h_2 \cdot m_2^{\pm}, \ldots) \in \Sigma^{\mathbb{N}}$ define $\nu_0 := 1$, $\nu_n = \nu_n(\alpha) = h_1 + \cdots + h_n + n + 1$ and let $\hat{q}_n = \hat{q}_n(\alpha) := q_{\nu_n(\alpha)}(\alpha)$ be the denominator of the nth R-convergent of α. We shall refer to $\{\hat{q}_n\}_{n \in \mathbb{N}}$ as *R-denominators* and to $(h_j \cdot m_j^{\pm})$ as *Σ-entries*.

In [4] the following estimates were proven:

Lemma 2.6.

(i) *For every* $\alpha \in (0,1]$, $\hat{q}_n \geq 3^{n/3}$.
(ii) *For Lebesgue-almost every* $\alpha \in (0,1]$ *and sufficiently large* n, $\hat{q}_n \leq \mathrm{e}^{C_7 n}$, *where* $C_7 > 0$ *is some constant*.

In Section 3, we will need the following renewal-type limit theorem.

Theorem 2.7. *Let* $L > 0$ *and* $\hat{n}_L = \hat{n}_L(\alpha) = \min\{n \in \mathbb{N}\colon \hat{q}_n > L\}$. *Fix* $N_1, N_2 \in \mathbb{N}$. *The ratios* $\frac{\hat{q}_{\hat{n}_L - 1}}{L}$ *and* $\frac{\hat{q}_{\hat{n}_L}}{L}$ *and the entries* $\sigma_{\hat{n}_L + j}$, $-N_1 < j \leq N_2$ *have a joint limiting probability distribution w.r.t. the measure* λ *as* $L \to \infty$.

In other words, there exists a probability measure $\mathrm{Q}^{(0)} = \mathrm{Q}^{(0)}_{N_1,N_2}$ *on the space* $(0,1] \times (1,\infty) \times \Sigma^{N_1+N_2}$ *such that for every* $0 \leq a < b \leq 1 \leq c < d$ *and every* (N_1+N_2)*-tuple* $\underline{\vartheta} = \{\vartheta_j\}_{j=-N_1+1}^{N_2} \in \Sigma^{N_1+N_2}$ *we have*

$$\lim_{L\to\infty} \lambda\left(\left\{\alpha\colon a < \frac{\hat{q}_{\hat{n}_L-1}}{L} < b, c < \frac{\hat{q}_{\hat{n}_L}}{L} < d, \sigma_{\hat{n}_L+j} = \vartheta_j, -N_1 < j \leq N_2\right\}\right)$$
$$= \mathrm{Q}^{(0)}\big((a,b)\times(c,d)\times\{\underline{\vartheta}\}\big). \tag{17}$$

Theorem 2.7 is more general than the one given in [4] (Theorem 1.6 therein) because it also includes the R-denominator $\hat{q}_{\hat{n}_L-1}$ preceding the renewal time $\hat{n}_L$ and the limiting distribution obtained for general absolutely continuous measure λ (instead of simply μ_R). However, it is a special case of Theorem 4.1 (whose proof is sketched in the Appendix). Let us just mention that it relies on the mixing property of a suitably defined special flow over the natural extension $\hat{R}$ of R. The same strategy was used before by Sinai and Ulcigrai [29] in the proof of the analogous statement for Euclidean continued fractions. Another remarkable result in this direction is due to Ustinov [31] who provides an explicit expression and an approximation, with an error term of order $\mathcal{O}(\frac{\log L}{L})$, for their limiting distribution function.

2.5. *Estimates of the growth of* Σ*-entries*

In this section we prove a number of estimates for the growth of Σ-entries. The analogous results for Euclidean continued fraction expansions are well known, but in our case the proofs are more involved.

Recall that $\alpha = (h_1 \cdot m_1^{\zeta_1}, h_2 \cdot m_2^{\zeta_2}, \ldots) \in \Sigma^{\mathbb{N}}$. Let us fix a sequence $\underline{\sigma} = \{\sigma_j\}_{j\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$. For every n and every $s \cdot t^{\zeta} \in \Sigma$, set

$$J_n = J_n(\underline{\sigma}) := \big\{\alpha : h_j \cdot m_j^{\zeta_j} = \sigma_j, j = 1,\ldots,n\big\} \quad \text{and}$$
$$J_{n+1}\big[s\cdot t^{\zeta}\big] = J_{n+1}(\underline{\sigma})\big[s\cdot t^{\zeta}\big] := \big\{\alpha \in J_n\colon h_{n+1}\cdot m_{n+1}^{\zeta_{n+1}} = s\cdot t^{\zeta}\big\} \subset J_n.$$

Lemma 2.8. *Let* J_n *and* $J_{n+1}[s\cdot t^{\zeta}]$ *be as above. Then*

$$\frac{1}{30(s+1)^2t^2} \leq \frac{|J_{n+1}[s\cdot t^{\zeta}]|}{|J_n|} \leq \frac{6}{(s+1)^2t^2}. \tag{18}$$

Proof. This proof follows closely the one given by Khinchin concerning Euclidean continued fraction (see [15], Chapter 12). Let us introduce the convergents p_j/q_j, $j = 1,\ldots,\nu_n - 1$ associated to $(\sigma_1,\ldots,\sigma_n)$. The endpoints of the interval J_n can be written as

$$\frac{p_{\nu_n-1}}{q_{\nu_n-1}} \quad \text{and} \quad \frac{p_{\nu_n-1} - \zeta_n p_{\nu_n-2}}{q_{\nu_n-1} - \zeta_n q_{\nu_n-2}}.$$

Applying the recurrent relations (14) $s+1$ times we define the convergents p_j/q_j, $j = 1,\ldots,\nu_n + s = \nu_{n+1} - 1$ corresponding to $(\sigma_1,\ldots,\sigma_n, s\cdot t^{\zeta})$. The endpoints of the interval $J_{n+1}[s\cdot t^{\zeta}]$ are

$$\frac{p_{\nu_{n+1}-1}}{q_{\nu_{n+1}-1}} \quad \text{and} \quad \frac{p_{\nu_{n+1}-1} - \zeta p_{\nu_{n+1}-2}}{q_{\nu_{n+1}-1} - \zeta q_{\nu_{n+1}-2}},$$

where $q_{\nu_{n+1}-2} = (s+1)q_{\nu_n-1} + s\zeta_n q_{\nu_n-2}$ and $q_{\nu_{n+1}-1} = (2t(s+1)-s)q_{\nu_n-1} + (2ts-s+1)\zeta_n q_{\nu_n-2}$ (the values of the corresponding numerators are unimportant). Using the formula (15) and setting $x = \frac{q_{\nu_n-2}}{q_{\nu_n-1}}$ we obtain

$$\begin{aligned}\frac{|J_{n+1}[s\cdot t^{\zeta}]|}{|J_n|} &= \frac{q_{\nu_n-1}(q_{\nu_n-1}+\zeta_n q_{\nu_n-2})}{q_{\nu_{n+1}-1}(q_{\nu_{n+1}-1}+\zeta q_{\nu_{n+1}-2})} \\ &= \frac{1}{(s+1)^2t^2}\frac{(1+\zeta_n x)}{(2-\frac{s}{(s+1)t}+\zeta_n x\frac{2st-s+1}{(s+1)t})(2-\frac{s}{(s+1)t}+\zeta_n x\frac{2st-s+1+\zeta s}{(s+1)t}+\frac{\zeta}{t})} \\ &= \frac{1}{(s+1)^2t^2}\frac{\mathrm{A}}{\mathrm{BC}}, \end{aligned} \tag{19}$$

where A, B and C correspond to the terms in parentheses. We distinguish two main cases: (i) $\zeta_n = +1$ and (ii) $\zeta_n = -1$:

(i) If $\zeta_n = +1$, then $0 \le x \le 1$ and we get

$$1 \le \mathrm{A} \le 2, \qquad 1 \le \mathrm{B} \le 4, \qquad 1 \le \mathrm{C} \le 5. \tag{20}$$

The above estimates for A and B are elementary; the one for C is obtained discussing the cases $\zeta = +1$ ($\Rightarrow t \ge 1$) and $\zeta = -1$ ($\Rightarrow t \ge 2$) separately and is also elementary.

(ii) If $\zeta_n = -1$, then $m_n \ge 2$ and by (14) $0 \le x \le \frac{1}{3}$. We get

$$\frac{2}{3} \le \mathrm{A} \le 1, \qquad \frac{2}{3} \le \mathrm{B} \le 2, \qquad \frac{1}{2} \le \mathrm{C} \le 3. \tag{21}$$

Now, (19), (20) and (21) give

$$\frac{1}{30(s+1)^2t^2} = \frac{1}{(s+1)^2t^2}\frac{2/3}{4\cdot 5} \le \frac{|J_{n+1}[s\cdot t^{\zeta}]|}{|J_n|} \le \frac{1}{(s+1)^2t^2}\frac{2}{2/3\cdot 1/2} = \frac{6}{(s+1)^2t^2}. \qquad \square$$

The next lemma estimates the Lebesgue measure of the set of α for which the Σ-entries $h_j \cdot m_j^{\zeta_j}$ satisfy the inequalities $h_j \le H_j - 1$, $j = 1, \ldots, n$, where $\{H_j\}_{j=1}^n$ is an arbitrary sequence.

Lemma 2.9. *Let* $\underline{H} = (H_1, \ldots, H_n) \in \mathbb{N}^n$ *and set* $Y(\underline{H}) := \{\alpha\colon h_1 + 1 < H_1, \ldots, h_n + 1 < H_n\}$. *Then*

$$|Y(\underline{H})| \ge \left(1 - \frac{1}{H_1}\right)\prod_{j=2}^{n}\lambda_{H_j}, \tag{22}$$

where $\lambda_H =: 1 - \frac{4\pi^2}{H}$.

Proof. For $\underline{\sigma} \in \Sigma^n$ and $\underline{H} \in \mathbb{N}^n$, let us define the set

$$W_{j,n}^{(\underline{\sigma},\underline{H})} := \{\alpha\colon h_i \cdot m_i^{\zeta_i} = \sigma_i, i = 1, \ldots, j, h_l < H_l - 1, l = j+1, \ldots, n\}.$$

Notice that $W_{n,n}^{(\underline{\sigma},\underline{H})} = J_n(\sigma_1, \ldots, \sigma_n)$ and does not depend on $\underline{H}$. Moreover, $Y(H_1, \ldots, H_n) = W_{0,n}^{(\underline{\sigma},\underline{H})}$. Consider the following estimate obtained from the second inequality of (18): for $S \in \mathbb{N}$

$$\sum_{\substack{s \ge S-1\\ t^{\zeta} \in \Omega^*}} |J_{n+1}[s\cdot t^{\zeta}]| \le 12|J_n| \sum_{\substack{s \ge S-1,\\ t \in \mathbb{N}}} \frac{1}{(s+1)^2t^2} \le \frac{4\pi^2|J_n|}{S}. \tag{23}$$

Now (23) yields

$$\big|W^{(\sigma,\underline{H})}_{n-1,n}\big| = \sum_{\substack{h_n<H_n-1,\\ m_n^{\zeta_n}\in\Omega^*}} \big|J_n\big(\sigma_1,\ldots,\sigma_{n-1},h_n\cdot m_n^{\zeta_n}\big)\big| = |J_{n-1}| - \sum_{\substack{h_n\geq H_n-1,\\ m_n^{\zeta_n}\in\Omega^*}} \big|J_n\big[h_n\cdot m_n^{\zeta_n}\big]\big|$$

$$\geq |J_{n-1}|\left(1-\frac{4\pi^2}{H_n}\right) = \lambda_{H_n}\big|W^{(\sigma,\underline{H})}_{n-1,n-1}\big|, \tag{24}$$

where $\lambda_{H_n} = (1-\frac{4\pi^2}{H_n})$. Considering the sum for $h_{n-1} < H_{n-1}-1, m_{n-1}^{\zeta_{n-1}} \in \Omega^*$ in (24) we get

$$\big|W^{(\sigma,\underline{H})}_{n-2,n}\big| \geq \lambda_{H_n}\big|W^{(\sigma,\underline{H})}_{n-2,n-1}\big| \geq \lambda_{H_n}\cdot\lambda_{H_{n-1}}\big|W^{(\sigma,\underline{H})}_{n-2,n-2}\big|. \tag{25}$$

Iterating (25) we come to

$$\big|W^{(\sigma,\underline{H})}_{1,n}\big| \geq \prod_{j=2}^{n}\lambda_{H_j}\big|W^{(\sigma,\underline{H})}_{1,1}\big| = \prod_{j=2}^{n}\lambda_{H_j}\big|J_1\big(h_1\cdot m_1^{\zeta_1}\big)\big|$$

and summing over $h_1 < H_1 - 1, m_1^{\zeta_1}\in\Omega^*$ we get the desired estimate (22):

$$\big|Y(\underline{H})\big| = \big|W^{(\sigma,\underline{H})}_{0,n}\big| \geq \left(1-\frac{1}{H_1}\right)\prod_{j=2}^{n}\lambda_{H_j}.$$

□

Now we provide an estimate which will be useful later. Let us fix a sequence $\underline{\sigma} = \{\sigma_j\}_{j\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ and let J_n and $J_{n+1}[s\cdot t^{\xi}]$ be as before. Moreover, set

$$J'_{n-1} = J_{n-1}(\underline{\sigma}) := \big\{\alpha\colon\ h_j\cdot m_j^{\zeta_j} = \sigma_j,\ j=2,\ldots,n\big\} \quad \text{and}$$

$$J'_n\big[s\cdot t^{\zeta}\big] = J'_n(\underline{\sigma})\big[s\cdot t^{\zeta}\big] := \big\{\alpha\in J'_n\colon\ h_{n+1}\cdot m_{n+1}^{\zeta_{n+1}} = s\cdot t^{\zeta}\big\} \subset J'_{n-1}.$$

Lemma 2.10.

$$\left|\frac{|J_{n+1}[s\cdot t^{\zeta}]|}{|J_n|}\cdot\frac{|J'_{n-1}|}{|J'_n[s\cdot t^{\zeta}]|} - 1\right| \leq C_8 \mathrm{e}^{-C_9 n}$$

for some constants $C_8, C_9 > 0$.

Proof. Let us observe that $RJ'_{n-1}(\underline{\sigma}) = J_{n-1}(\underline{\sigma}')$ and $RJ'_n(\underline{\sigma})[s\cdot t^{\zeta}] = J_n(\underline{\sigma}')[s\cdot t^{\zeta}]$, where $\underline{\sigma}' = \{\sigma'_j\}_{j\in\mathbb{N}}$ and $\sigma'_j = \sigma_{j+1}$. We have

$$\big|J'_{n-1}\big| = \int_{J'_{n-1}} \mathbf{1}\,\mathrm{d}x = \int_{J_{n-1}(\underline{\sigma}')} \mathcal{P}(\mathbf{1})(x)\,\mathrm{d}x, \qquad \big|J'_n\big[s\cdot t^{\zeta}\big]\big| = \int_{J'_n[s\cdot t^{\zeta}]} \mathbf{1}\,\mathrm{d}x = \int_{J_n(\underline{\sigma}')[s\cdot t^{\zeta}]} \mathcal{P}(\mathbf{1})(x)\,\mathrm{d}x,$$

where $\mathcal{P}$ is the Perron–Frobenius operator associated to R. The density $\mathcal{P}(\mathbf{1})(x)$ is computed as follows. The cylinders of rank one are of the form

$$J_1\big(h\cdot m^+\big) = \left(\frac{1+2mh}{1+2m(h+1)}, 1+\frac{1-2m}{2m(h+1)-h}\right],$$

$$J_1\big(h\cdot m^-\big) = \left(1+\frac{1-2m}{2m(h+1)-h}, \frac{1+2h(m-1)}{2m(h+1)-2h-1}\right],$$

and $R|_{J_1(h\cdot m^{\pm})}(x)=\mp 2m\pm\frac{1+h(x-1)}{h(x-1)+x}$. Therefore, $(R|_{J_1(h\cdot m^{\pm})})'(y)=\mp(h-(h+1)y)^{-2}$ and $(R|_{J_1(h\cdot m^{\pm})})^{-1}(x)=\frac{2hm-h+1\pm hx}{2hm+2m-h\pm(h+1)x}=: y_{h\cdot m^{\pm}}$. We get

$$\begin{aligned}\mathcal{P}(\mathbf{1})(x)&=\sum_{y\in R^{-1}(x)}\left|R'(y)\right|^{-1}=\sum_{h\cdot m^{\zeta}\in\Sigma}\big(h-(h+1)y_{h\cdot m^{\zeta}}\big)^2\\&=\sum_{h\geq 0}\bigg(\sum_{m\geq 1}\frac{1}{(2hm+2m-h+(h+1)x)^2}+\sum_{m\geq 2}\frac{1}{(2hm+2m-h-(h+1)x)^2}\bigg)\\&=\sum_{h\geq 0}\frac{1}{4(h+1)^2}\bigg(\psi^{(1)}\Big(\frac{h+2+(h+1)x}{2h+2}\Big)+\psi^{(1)}\Big(\frac{3h+4-(h+1)x}{2h+2}\Big)\bigg),\end{aligned}$$

where $\psi^{(1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}\frac{\Gamma'(x)}{\Gamma(x)}$ is the derivative of the digamma function. Notice that the function $\mathcal{P}(\mathbf{1})$ is differentiable and strictly decreasing on $[0,1]$; moreover,

$$\mathcal{P}(\mathbf{1})'(0)\simeq -0.88575>-1\quad\text{and}\quad\mathcal{P}(\mathbf{1})'(1)=0. \tag{26}$$

By the mean value theorem

$$\left|J'_{n-1}\right|=\mathcal{P}(\mathbf{1})(x_1)\cdot\left|J_{n-1}(\underline{\sigma}')\right|\quad\text{and}\quad\left|J'_n\big[s\cdot t^{\zeta}\big]\right|=\mathcal{P}(\mathbf{1})(x_2)\cdot\left|J_n\big(\underline{\sigma}'\big)\big[s\cdot t^{\zeta}\big]\right| \tag{27}$$

for some $x_1\in J_{n-1}(\underline{\sigma}')$ and $x_2\in J_n(\underline{\sigma}')[s\cdot t^{\zeta}]$.

Let $\{p_j/q_j\}_{j\in\mathbb{N}}$ and $\{p'_j/q'_j\}_{j\in\mathbb{N}}$ be the sequences of T-convergents corresponding to $\underline{\sigma}$ and $\underline{\sigma}'$ respectively. Set $x=\frac{q_{\nu_n-2}}{q_{\nu_n-1}}$ and $x'=\frac{q'_{\nu_{n-1}-2}}{q'_{\nu_{n-1}-1}}$. The ECF-expansions of x and x' coincide up to the $(n-1)$st R-digit (see [4], Lemma A.1) and therefore, by Lemma 2.6(i), we have $|x-x'|\leq 3^{(1-n)/3}$. Now, by (27) and (19), we get

$$\begin{aligned}&\frac{|J_{n+1}[s\cdot t^{\zeta}]|}{|J_n|}\cdot\frac{|J'_{n-1}|}{|J'_n[s\cdot t^{\zeta}]|}\\&\quad=\frac{(1+\zeta_n x)(2-\frac{s}{(s+1)t}+\zeta_n x'\frac{2st-s+1}{(s+1)t})(2-\frac{s}{(s+1)t}+\zeta_n x'\frac{2st-s+1+\zeta s}{(s+1)t}+\frac{\zeta}{t})\mathcal{P}(\mathbf{1})(x_1)}{(1+\zeta_n x')(2-\frac{s}{(s+1)t}+\zeta_n x\frac{2st-s+1}{(s+1)t})(2-\frac{s}{(s+1)t}+\zeta_n x\frac{2st-s+1+\zeta s}{(s+1)t}+\frac{\zeta}{t})\mathcal{P}(\mathbf{1})(x_2)}.\end{aligned} \tag{28}$$

Noticing that $\zeta_n x\geq -\frac{1}{3}$ one can show that

$$\left|\frac{(1+\zeta_n x)}{(1+\zeta_n x')}-1\right|,\left|\frac{(2-\frac{s}{(s+1)t}+\zeta_n x'\frac{2st-s+1}{(s+1)t})}{(2-\frac{s}{(s+1)t}+\zeta_n x\frac{2st-s+1}{(s+1)t})}-1\right|,\left|\frac{(2-\frac{s}{(s+1)t}+\zeta_n x'\frac{2st-s+1+\zeta s}{(s+1)t}+\frac{\zeta}{t})}{(2-\frac{s}{(s+1)t}+\zeta_n x\frac{2st-s+1+\zeta s}{(s+1)t}+\frac{\zeta}{t})}-1\right|\leq\frac{3^{(4-n)/3}}{2}.$$

Let us now consider the term $\frac{\mathcal{P}(\mathbf{1})(x_1)}{\mathcal{P}(\mathbf{1})(x_2)}$. To get estimates of it from above and below we can replace x_1 and x_1 with appropriate endpoints of $J_{n-1}(\underline{\sigma}')$ and $J_n(\underline{\sigma}')[s\cdot t^{\zeta}]$. Since $J_n(\underline{\sigma}')[s\cdot t^{\zeta}]\subset J_{n-1}(\underline{\sigma}')$, those four endpoints can be ordered in four different ways. Let us discuss only one of those cases, the others being similar. Let the endpoints $y_1=\frac{p'_{\nu_{n-1}-1}}{q'_{\nu_{n-1}-1}}$, $y_2=\frac{p'_{\nu_{n-1}-1}-\zeta_n p'_{\nu_{n-1}-2}}{q'_{\nu_{n-1}-1}-\zeta_n q'_{\nu_{n-1}-2}}$, $z_1=\frac{p'_{\nu_n-1}}{q'_{\nu_n-1}}$, $z_2=\frac{p'_{\nu_n-1}-\zeta p'_{\nu_n-2}}{q'_{\nu_n-1}-\zeta q'_{\nu_n-2}}$ be arranged as follows: $0<y_1<z_1<z_2<y_2<1$. Then, since the function $\mathcal{P}(\mathbf{1})$ is decreasing, $y_1\leq x_1\leq y_2$ and $z_1\leq x_2\leq z_2$, we get

$$\frac{\mathcal{P}(\mathbf{1})(x_1)}{\mathcal{P}(\mathbf{1})(x_2)}\leq 1+\frac{\mathcal{P}(\mathbf{1})(z_2)-\mathcal{P}(\mathbf{1})(y_1)}{\mathcal{P}(\mathbf{1})(y_1)}.$$

Let us use (26), the fact that z_2 and y_1 have the same R-expansion up to the $(n-1)$st digit, (18) and the fact that $\mathcal{P}(\mathbf{1})(1)\simeq 0.90238$:

$$\frac{|\mathcal{P}(\mathbf{1})(z_2)-\mathcal{P}(\mathbf{1})(y_1)|}{\mathcal{P}(\mathbf{1})(y_1)}\leq\frac{|z_2-y_1|}{\mathcal{P}(\mathbf{1})(y_1)}\leq\frac{C_{10}3^{(1-n)/3}}{(s+1)t\mathcal{P}(\mathbf{1})(y_1)}\leq C_{11}3^{(1-n)/3}$$

for some constants $C_{10}, C_{11} > 0$. Thus we get the desired estimate:

$$\left| \frac{|J_{n+1}[s \cdot t^{\zeta}]|}{|J_n|} \cdot \frac{|J'_{n-1}|}{|J'_n[s \cdot t^{\zeta}]|} - 1 \right| \le C_8 \mathrm{e}^{-C_9 n}$$

for some $C_8, C_9 > 0$. □

3. Iterated renormalization of $\gamma_{\alpha,N}$

In Section 2.2 we discussed the renormalization of $\gamma_{\alpha,N}$, i.e. the procedure which "erases" the geometric structure at smallest scale in the curve $\gamma_{\alpha,N}$. Now we want to iterate the renormalization formula (10). In order to do this, we consider $\alpha_0 := \alpha$, $\alpha_l := T^l(\alpha_0)$ (as in Section 2.3), $N_0 := N$, $N_l := \alpha_{l-1} N_{l-1}$, $\eta_0 := 1$ and $\eta_l := \eta(\alpha_{l-1})$, $l \in \mathbb{N}$. Define also $\kappa_0 := 0$, $\kappa_l := \kappa_l(\alpha) := 1 + \eta_1 + \eta_1\eta_2 + \cdots + \eta_1\eta_2 \cdots \eta_{l-1}$. With these notations, iterating (10) r times we get

$$\begin{aligned}
\mathcal{S}_\alpha(tN) &= (\alpha_0 \cdots \alpha_{r-1})^{-1/2} \Bigg(\exp\left\{\kappa_r \frac{\pi}{4} \mathrm{i}\right\} \mathcal{S}_{\alpha_r}^{(\eta_1 \cdots \eta_r)}(tN_r) \\
&\quad + \exp\left\{\kappa_{r-1} \frac{\pi}{4} \mathrm{i}\right\} \alpha_{r-1}^{1/2} \Gamma^{(\eta_1 \cdots \eta_{r-1})}(\alpha_{r-1}, tN_{r-1}) \\
&\quad + \exp\left\{\kappa_{r-2} \frac{\pi}{4} \mathrm{i}\right\} (\alpha_{r-2}\alpha_{r-1})^{1/2} \Gamma^{(\eta_1 \cdots \eta_{r-2})}(\alpha_{r-2}, tN_{r-2}) + \cdots \\
&\quad + \exp\left\{\kappa_{r-j} \frac{\pi}{4} \mathrm{i}\right\} (\alpha_{r-j} \cdots \alpha_{r-1})^{1/2} \Gamma^{(\eta_1 \cdots \eta_{r-j})}(\alpha_{r-j}, tN_{r-j}) + \cdots \\
&\quad + \exp\left\{\kappa_0 \frac{\pi}{4} \mathrm{i}\right\} (\alpha_0 \cdots \alpha_{r-1})^{1/2} \Gamma^{(1)}(\alpha_0, tN_0) \Bigg) \\
&= (\alpha_0 \cdots \alpha_{r-1})^{-1/2} \Bigg(\exp\left\{\kappa_r \frac{\pi}{4} \mathrm{i}\right\} \mathcal{S}_{\alpha_r}^{(\eta_1 \cdots \eta_r)}(tN_r) \\
&\quad + \sum_{j=1}^{r} \exp\left\{\kappa_{r-j} \frac{\pi}{4} \mathrm{i}\right\} (\alpha_{r-j} \cdots \alpha_{r-1})^{1/2} \Gamma^{(\eta_1 \cdots \eta_{r-j})}(\alpha_{r-j}, tN_{r-j}) \Bigg).
\end{aligned} \tag{29}$$

Our next step is to choose r as a function of N and α so that $N_r = \alpha_0 \cdots \alpha_{r-1} N$ is $\mathcal{O}(1)$, that is $(\alpha_0 \cdots \alpha_{r-1})^{-1/2} = \mathcal{O}(\sqrt{N})$. We make use of the relation (16) and we define r in terms of the R-denominator corresponding to the renewal time $\hat{n}_N$. For $\alpha = (h_1 \cdot m_1^{\pm}, h_2 \cdot m_2^{\pm}, \ldots) \in \Sigma^{\mathbb{N}}$, set $r = r(\alpha, N) := \nu_{\hat{n}_N} - 1 = h_1 + \cdots + h_{\hat{n}_N} + \hat{n}_N$, where $\hat{n}_N = \min\{n \in \mathbb{N}\colon \hat{q}_n > N\}$ as in Theorem 2.7. Define $\alpha_0 \cdots \alpha_{r(\alpha,N)-1} N = N_{r(\alpha,N)} =: \Theta_\alpha(N)$. We have the following proposition.

Proposition 3.1. *$\Theta_\alpha(N)$ has a limiting probability distribution on $(0, \infty)$ w.r.t. λ as $N \to \infty$. In other words: there exists a probability measure $\mathrm{Q}^{(1)}$ on $(0, \infty)$ such that for every $0 < a < b$ we have*

$$\lim_{N \to \infty} \lambda\big(\{\alpha\colon a < \Theta_\alpha(N) < b\}\big) = \mathrm{Q}^{(1)}\big((a, b)\big).$$

Proof. Our goal is to write $\Theta_\alpha(N)$ as a function of $\hat{q}_{\hat{n}_N - 1}/N$, $\hat{q}_{\hat{n}_N}/N$ and a finite number of Σ-entries of α preceding and/or following the renewal time $\hat{n}_N$. By (16) we have

$$\Theta_\alpha(N) = \alpha_0 \cdots \alpha_{\nu_{\hat{n}_N} - 2} N = \left(\frac{q_{\nu_{\hat{n}_N} - 1}}{N} + \xi_{\nu_{\hat{n}_N} - 1} \cdot \alpha_{\nu_{\hat{n}_N} - 1} \cdot \frac{q_{\nu_{\hat{n}_N} - 2}}{N} \right)^{-1}. \tag{30}$$

In order to write $q_{\nu_{\hat n_N}-1}$ and $q_{\nu_{\hat n_N}-2}$ in terms of $\hat q_{\hat n_N}=q_{\nu_{\hat n_N}}$ and $\hat q_{\hat n_N-1}=q_{\nu_{\hat n_N-1}}$ we use the recurrent relation (14) for the ECF-denominators, getting the $h_{\hat n}\times h_{\hat n}$ linear system

$$\begin{bmatrix} 2k_{\nu_{\hat n}} & \xi_{\nu_{\hat n}-1} & & & & \\ -1 & 2k_{\nu_{\hat n}-1} & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & -1 & \ddots & -1 & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}\cdot\begin{bmatrix} q_{\nu_{\hat n}-1}\\ q_{\nu_{\hat n}-2}\\ \vdots\\ q_{\nu_{\hat n}-j}\\ \vdots\\ q_{\nu_{\hat n}-(h_{\hat n}-1)}\\ q_{\nu_{\hat n}-h_{\hat n}}\end{bmatrix}=\begin{bmatrix}\hat q_{\hat n}\\ 0\\ \vdots\\ 0\\ \vdots\\ 0\\ \hat q_{\hat n-1}\end{bmatrix}, \tag{31}$$

where $\hat n=\hat n_N$. The quantities $k_{\nu_{\hat n}-1}^{\xi_{\nu_{\hat n}-1}}=m_{\hat n}^{\zeta_{\hat n}}\in\Omega^*$ and $k_{\nu_{\hat n}}\in\mathbb{N}$, along with the size $h_{\hat n}$ of the linear system, depend only on the two Σ-entries $(h_{\hat n}\cdot m_{\hat n}^{\zeta_{\hat n}},h_{\hat n+1}\cdot m_{\hat n+1}^{\zeta_{\hat n+1}})\in\Sigma^2$. We are interested in the first two entries of the solution of (31). One can check that

$$\begin{aligned} q_{\nu_{\hat n}-1}&=\frac{((2h_{\hat n}-2)k_{\nu_{\hat n}-1}-(h_{\hat n}-2))\hat q_{\hat n}-\xi_{\nu_{\hat n}-1}\hat q_{\hat n-1}}{(4h_{\hat n}-4)k_{\nu_{\hat n}}k_{\nu_{\hat n}-1}-(2h_{\hat n}-4)k_{\nu_{\hat n}}+(n-1)\xi_{\nu_{\hat n}-1}}\quad\text{and}\\ q_{\nu_{\hat n}-2}&=\frac{(h_{\hat n}-1)\hat q_{\hat n}+2k_{\nu_{\hat n}}\hat q_{\hat n-1}}{(4h_{\hat n}-4)k_{\nu_{\hat n}}k_{\nu_{\hat n}-1}-(2h_{\hat n}-4)k_{\nu_{\hat n}}+(n-1)\xi_{\nu_{\hat n}-1}}.\end{aligned} \tag{32}$$

Therefore (30) and (32) show that $\Theta_\alpha(N)$ is a function of $\hat q_{\hat n_N-1}/N\in(0,1]$, $\hat q_{\hat n_N}/N\in(1,\infty)$, $(h_{\hat n_N}\cdot m_{\hat n_N}^{\zeta_{\hat n_N}},h_{\hat n_N+1}\cdot m_{\hat n_N+1}^{\zeta_{\hat n_N+1}})\in\Sigma^2$ and $\alpha_{\nu_{\hat n_N}-1}=R^{\hat n_N}(\alpha)$. Now, by Theorem 2.7, we obtain the existence of a limiting distribution as $N\to\infty$, w.r.t. λ. □

3.1. *Approximation of $\gamma_{\alpha,N}$*

In this section we construct an approximation for the curve $\gamma_{\alpha,N}$. The approximating curve $\gamma_{\alpha,N}^J$ will contain only the J largest geometric scales (corresponding to J iterations of the jump transformation R). Having specified our choice for r, we can also regroup the $\nu_{\hat n_N}$ terms in (29) involving Γ's into $\hat n_N$ terms as follows:

$$\Delta_l(t)=\Delta_l(t;\alpha,N):=\sum_{j=2}^{h_l+2}\exp\left\{\kappa_{\nu_l-j}\frac{\pi}{4}\mathrm{i}\right\}\left((\alpha)_{\nu_l-j}^{\nu_l-2}\right)^{1/2}\Gamma^{(\eta_1\cdots\eta_{\nu_l-j})}(\alpha_{\nu_l-j},tN_{\nu_l-j}) \tag{33}$$

for $l=1,\dots,\hat n_N$, where $(\alpha)_{l_1}^{l_2}:=\alpha_{l_1}\cdots\alpha_{l_2}$ if $l_1\le l_2$ and $(\alpha)_{l_1}^{l_2}:=1$ if $l_1>l_2$. Also recall that $\nu_{l-1}=\nu_l-h_l-1$. Formula (29) becomes now

$$\begin{aligned}\gamma_{\alpha,N}(t)&=\frac{\mathcal{S}_\alpha(tN)}{\sqrt{N}}\\ &=\Theta_\alpha^{-1/2}(N)\left(\exp\left\{\kappa_{\nu_{\hat n}-1}\frac{\pi}{4}\mathrm{i}\right\}\mathcal{S}_{\alpha_{\nu_{\hat n}-1}}^{(\eta_1\cdots\eta_{\nu_{\hat n}-1})}\left(t\Theta_\alpha(N)\right)+\sum_{j=0}^{\hat n-1}\left((\alpha)_{\nu_{\hat n-j}-1}^{\nu_{\hat n}-2}\right)^{1/2}\Delta_{\hat n-j}(t)\right),\end{aligned} \tag{34}$$

where $\hat n=\hat n_N$. The following lemma proves that, on a set of arbitrarily large measure, the product $((\alpha)_{\nu_{\hat n-j}-1}^{\nu_{\hat n}-2})^{1/2}\times\Delta_{\hat n-j}(t)$ decays sufficiently fast as j grows. One can assume that $\hat n$ is large enough so that $\hat n-j\ge1$. This is the case because later we shall let $N\to\infty$ and hence $\hat n_N\to\infty$.

Lemma 3.2. *For all sufficiently large J*

$$\lambda\left(\left\{\alpha\colon\left|\left((\alpha)_{\nu_{\hat n-j}-1}^{\nu_{\hat n}-2}\right)^{1/2}\Delta_{\hat n-j}(t)\right|\le C_{12}\mathrm{e}^{-C_{13}j},\ j=J,\dots,\hat n-1\right\}\right)\ge1-\delta_1(J), \tag{35}$$

where $C_{12}, C_{13} > 0$ are some constants and $\delta_1(J) \to 0$ as $J \to \infty$.

Proof. Notice that, by (4), for every $l = 2, \dots, h_{\hat{n}-j} + 2$,

$$\left|\alpha_{\nu_{\hat{n}-j}-l}^{1/2} \Gamma(\alpha_{\nu_{\hat{n}-j}-l}, tN_{\nu_{\hat{n}-j}-l})\right| \le C_5 + C_6 \alpha_{\nu_{\hat{n}-j}-l}^{1/2} \le C_{12},$$

where $C_{12} := C_5 + C_6$. Now, by (33),

$$\left|\Delta_{\hat{n}-j}(t)\right| \le \sum_{l=2}^{h_{\hat{n}-j}+2} \left((\alpha)_{\nu_{\hat{n}-j}-l}^{\nu_{\hat{n}-j}-2}\right)^{1/2} \Gamma(\alpha_{\nu_{\hat{n}-j}-l}, tN_{\nu_{\hat{n}-j}-l}) \le C_{12}(h_{\hat{n}-j} + 1).$$

By construction of the jump transformation R, exactly one of the factors in $(\alpha)_{\nu_{\hat{n}-j}-1}^{\nu_{\hat{n}-j+1}-2}$ is less then $\frac{1}{2}$. Therefore for every $j = 1, \dots, \hat{n} - 1$

$$\left|\left((\alpha)_{\nu_{\hat{n}-j}-1}^{\nu_{\hat{n}}-2}\right)^{1/2} \Delta_{\hat{n}-j}(t)\right| \le C_{12} 2^{-(1/2)j}(h_{\hat{n}-j} + 1).$$

Thus it is enough to show that, for all sufficiently large $J \in N$ and $\hat{n}$,

$$\left|\left\{\alpha\colon h_{\hat{n}-j} \le \mathrm{e}^{C_{14} j}, j = J, \dots, \hat{n} - 1\right\}\right| \ge 1 - \delta_2(J), \tag{36}$$

where $0 < C_{14} < \frac{\log 2}{2} \simeq 0.346574$ and $\delta_2(J) \to 0$ as $J \to \infty$. By Lemma 2.9, setting $\underline{H} = (\mathrm{e}^{C_{14}(\hat{n}-1)} + 1, \mathrm{e}^{C_{14}(\hat{n}-2)} + 1, \dots, \mathrm{e}^{C_{14}J} + 1) \in \mathbb{N}^{\hat{n}-J}$, we get

$$\begin{aligned}
&\left|\left\{\alpha\colon h_{\hat{n}-j} \le \mathrm{e}^{C_{14} j}, j = J, \dots, \hat{n} - 1\right\}\right| \\
&\quad = \left|Y(\underline{H})\right| \ge \left(1 - \frac{1}{\mathrm{e}^{C_{14}(\hat{n}-1)} + 1}\right) \prod_{j=J}^{\hat{n}-2} \left(1 - \frac{4\pi^2}{\mathrm{e}^{C_{14} j} + 1}\right) \ge \prod_{j=J}^{\infty} \left(1 - 4\pi^2 \mathrm{e}^{-C_{14} j}\right) =: \delta_2(J).
\end{aligned}$$

The estimate (36) is thus proven, along with our initial statement (35) setting $C_{13} := \frac{\log 2}{2} - C_{14}$. □

For $J \in \mathbb{N}$ define the curve associated to the truncated renormalized sum as

$$t \mapsto \gamma_{\alpha,N}^J(t) := \Theta_\alpha^{-1/2}(N) \left(\mathrm{e}^{\kappa_{\nu_{\hat{n}}-1}(\pi/4)\mathrm{i}} \mathcal{S}_{\alpha_{\nu_{\hat{n}}-1}}^{(\eta_1 \cdots \eta_{\nu_{\hat{n}}-1})}\left(t\Theta_\alpha(N)\right) + \sum_{j=0}^{J-1} \left((\alpha)_{\nu_{\hat{n}-j}-1}^{\nu_{\hat{n}}-2}\right)^{1/2} \Delta_{\hat{n}-j}(t)\right). \tag{37}$$

The number J corresponds to the number of scales one considers in approximating the curve $\gamma_{\alpha,N}$, starting from the largest scale. The following lemma shows that $\gamma_{\alpha,N}$ is exponentially well approximated by $\gamma_{\alpha,N}^J$ for a set of α's whose measure tends to 1 as J increases.

Lemma 3.3. *For all sufficiently large J and N*

$$\lambda\left(\left\{\left|\gamma_{\alpha,N}(t) - \gamma_{\alpha,N}^J(t)\right| \le \mathrm{e}^{-C_{15} J}\right\}\right) \ge 1 - \delta_3(J) \tag{38}$$

for every $t \in [0, 1]$, where $C_{15} > 0$ is some constant and $\delta_3(J) \to 0$ as $J \to \infty$.

Proof. Since by Proposition 3.1 $\Theta_\alpha(N)$ has a limiting distribution on $(0, \infty)$ as $N \to \infty$, so $\Theta_\alpha^{-1/2}(N)$ does. Then, for sufficiently large N, we have

$$\lambda\left(\left\{\alpha\colon \Theta_\alpha^{-1/2}(N) \le J\right\}\right) \ge 1 - \delta_4(J),$$

where $\delta_4(J)\to 0$ as $J\to\infty$. On the other hand, by Lemma 3.2, for sufficiently large J and N,

$$\left|\gamma_{\alpha,N}(t)-\gamma^J_{\alpha,N}(t)\right| = \Theta_\alpha^{-1/2}(N)\left|\sum_{j=J+1}^{\hat n-1}\left((\alpha)^{\nu_{\hat n}-2}_{\nu_{\hat n-j}-1}\right)^{1/2}\Delta_{\hat n-j}(t)\right| \le C_{12}\Theta_\alpha^{-1/2}(N)\sum_{j=J}^{\hat n-1}\mathrm{e}^{-C_{13}j}$$
$$= C_{12}\Theta_\alpha^{-1/2}(N)\frac{\mathrm{e}^{-C_{13}(J-1)}-\mathrm{e}^{-C_{13}(\hat n-1)}}{\mathrm{e}^{C_{13}}-1} \le \frac{C_{12}\mathrm{e}^{C_{13}}}{\mathrm{e}^{C_{13}}-1}\Theta_\alpha^{-1/2}(N)\mathrm{e}^{-C_{13}J}$$

holds for every $t\in[0,1]$ on a set of μ_R-measure bigger than $1-\delta_1(J)$. Therefore

$$\left|\gamma_{\alpha,N}(t)-\gamma^J_{\alpha,N}(t)\right| \le \frac{C_{12}\mathrm{e}^{C_{13}}J}{\mathrm{e}^{C_{13}}-1}\mathrm{e}^{-C_{13}J} \le \mathrm{e}^{-C_{15}J}$$

for some constant $C_{15}>0$ on a set of μ_R-measure bigger than $1-\delta_1(J)-\delta_4(J)$. The lemma is thus proven setting $\delta_3(J) := \delta_1(J)+\delta_4(J)$. □

3.2. *Rewriting of $\gamma^J_{\alpha,N}$ in terms of renewal variables*

Now we can study the curve $\gamma^J_{\alpha,N}(t)$. Our goal is to rewrite it in terms of $\Theta_\alpha(N)$, $\alpha_{\nu_{\hat n}-1}$ and a finite number of Σ-entries preceding the renewal time. We will also need two additional functions, $K^8_\alpha(N)$ and $E_\alpha(N)$ to take into account phase terms and conjugations coming from the renormalization procedure.

For $\alpha=(h_1\cdot m_1^{\zeta_1}, h_2\cdot m_2^{\zeta_2},\dots)\in\Sigma^{\mathbb N}$ we have an explicit expression for η_l, $l=1,\dots,\nu_{\hat n}-1$:

$$\eta_1=\dots=\eta_{h_1}=1,\ \eta_{\nu_1-1}=-\zeta_1,$$
$$\eta_{\nu_1}=\dots=\eta_{\nu_1+h_2}=1,\ \eta_{\nu_2-1}=-\zeta_2,$$
$$\vdots$$
$$\eta_{\nu_{\hat n-1}}=\dots=\eta_{\nu_{\hat n-1}+h_{\hat n}}=1,\ \eta_{\nu_{\hat n}-1}=-\zeta_{\hat n}.$$

Thus

$$\eta_1\cdots\eta_{\nu_l-1}=\prod_{s=1}^{l}(-\zeta_s)\quad\text{and}\tag{39}$$

$$\kappa_{\nu_l}=1+(h_1-\zeta_1)+(-\zeta_1)(h_2-\zeta_2)+(-\zeta_1)(-\zeta_2)(h_3-\zeta_3)+\cdots$$
$$+(-\zeta_1)\cdots(-\zeta_{l-1})(h_l-\zeta_l)=1+\sum_{j=1}^{l}(h_j-\zeta_j)\prod_{s=1}^{j-1}(-\zeta_s).\tag{40}$$

The following lemma gives an explicit formula for the partial products along the T-orbit of α which appear in (37).

Lemma 3.4. *Let $\alpha=(h_1\cdot m_1^{\zeta_1}, h_2\cdot m_2^{\zeta_2},\dots)\in\Sigma^{\mathbb N}$. Set $\beta_j := \alpha_{\nu_{\hat n-j}-2}$. Then*

$$B_{s,j}=B_{s,j}(\alpha):=(\alpha)^{\nu_{\hat n-j}-2}_{\nu_{\hat n-j}-s}=\frac{\beta_j}{(s-1)-(s-2)\beta_j},\tag{41}$$

$$D_j=D_j(\alpha):=(\alpha)^{\nu_{\hat n}-2}_{\nu_{\hat n-j}-1}=\prod_{u=0}^{j-1}\frac{\beta_u}{1+h_{\hat n-u}(1-\beta_u)}.\tag{42}$$

Proof. Both identities follow, after telescopic cancellations, from

$$\alpha_{\nu_{\hat n-j}-s}=\frac{(s-2)-(s-3)\beta_j}{(s-1)-(s-2)\beta_j}.\tag{43}$$

□

Notice that β_j is a function of $R^{\hat{n}_N}(\alpha)$ and j ($j \le J$) Σ-entries preceding the renewal time $\hat{n}_N$. With the above notation (37) becomes

$$\gamma^J_{\alpha,N}(t) = \Theta_\alpha(N)^{-1/2}\left(\exp\left\{\kappa_{\nu_{\hat{n}}-1}\frac{\pi}{4}\mathrm{i}\right\}\mathcal{S}^{(\eta_1\cdots\eta_{\nu_{\hat{n}}-1})}_{\alpha_{\nu_{\hat{n}}-1}}\big(t\Theta_\alpha(N)\big)\right.$$
$$\left.+\sum_{j=0}^{J-1} D_j^{1/2}\sum_{s=2}^{h_{\hat{n}-j}+2}\exp\left\{\kappa_{\nu_{\hat{n}-j}-s}\frac{\pi}{4}\mathrm{i}\right\}B^{1/2}_{s,j}\Gamma^{(\eta_1\cdots\eta_{\nu_{\hat{n}-j}-s})}(\alpha_{\nu_{\hat{n}-j}-s}, tN_{\nu_{\hat{n}-j}-s})\right). \tag{44}$$

We want to collect a phase term of the form $\exp\{\kappa_{\nu_{\hat{n}-J}-1}\frac{\pi}{4}\mathrm{i}\}$ and the corresponding "conjugation" index $(\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})$. To do this, using (39) and (40), we introduce the quantities Ψ_J, Υ_J, $\mathcal{E}_J$ and $\mathcal{E}^j_J$, depending only on a finite number of Σ-entries of α preceding the renewal time $\hat{n}_N$:

$$\begin{aligned}
&(\kappa_{\nu_{\hat{n}}-1}-\kappa_{\nu_{\hat{n}-J}-1})(\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})\\
&\quad=(\kappa_{\nu_{\hat{n}}}-\kappa_{\nu_{\hat{n}-J}}-\eta_1\cdots\eta_{\nu_{\hat{n}}-1}+\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})(\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})\\
&\quad=\sum_{u=\hat{n}-J+1}^{\hat{n}}(h_u-\zeta_u)\prod_{v=\hat{n}-J+1}^{u-1}(-\zeta_v)-\prod_{v=\hat{n}-J+1}^{\hat{n}}(-\zeta_v)+1\\
&\quad=:\Psi_J=\Psi_J\big(h_l\cdot m_l^{\zeta_l}, l=\hat{n}-J+1,\dots,\hat{n}\big),\\
&(\kappa_{\nu_{\hat{n}-j}-s}-\kappa_{\nu_{\hat{n}-J}-1})(\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})\\
&\quad=\big(\kappa_{\nu_{\hat{n}-j-1}}+(h_{\hat{n}-j}-s+1)(\eta_1\cdots\eta_{\nu_{\hat{n}-j-1}-1})-\kappa_{\nu_{\hat{n}-J}}+(\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})\big)(\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})\\
&\quad=\sum_{u=\hat{n}-J+1}^{\hat{n}-j-1}(h_u-\zeta_u)\prod_{v=\hat{n}-J+1}^{u-1}(-\zeta_v)+(h_{\hat{n}-j}-s+1)\prod_{v=\hat{n}-J+1}^{\hat{n}-j-1}(-\zeta_v)+1\\
&\quad=:\Upsilon_{s,J}=\Upsilon_{s,J}\big(h_l\cdot m_l^{\zeta_l}, l=\hat{n}-J+1,\dots,\hat{n}-j\big),\\
&\mathcal{E}_J:=\eta_{\nu_{\hat{n}-J}}\cdots\eta_{\nu_{\hat{n}}-1}=\prod_{v=\hat{n}-J+1}^{\hat{n}}(-\zeta_v),\qquad \mathcal{E}^j_J:=\eta_{\nu_{\hat{n}-J}}\cdots\eta_{\nu_{\hat{n}-j}-s}=\prod_{v=\hat{n}-J+1}^{\hat{n}-j-1}(-\zeta_v).
\end{aligned}$$

Now (44) becomes

$$\gamma^J_{\alpha,N}(t)=\exp\left\{\kappa_{\nu_{\hat{n}-J}-1}\frac{\pi}{4}\mathrm{i}\right\}\Theta_\alpha(N)^{-1/2}\left(\exp\left\{\Psi_J\frac{\pi}{4}\mathrm{i}\right\}\mathcal{S}^{(\mathcal{E}_J)}_{R^{\hat{n}}(\alpha)}\big(t\Theta_\alpha(N)\big)\right.$$
$$\left.+\sum_{j=0}^{J-1}D_j^{1/2}\sum_{s=2}^{h_{\hat{n}-j}+2}\exp\left\{\Upsilon_{s,J}\frac{\pi}{4}\mathrm{i}\right\}B^{1/2}_{s,j}\Gamma^{(\mathcal{E}^j_J)}(\alpha_{\nu_{\hat{n}-j}-s}, tN_{\nu_{\hat{n}-j}-s})\right)^{(\eta_1\cdots\eta_{\nu_{\hat{n}-J}-1})}. \tag{45}$$

On the other hand, we also introduce the functions $E_\alpha(N)$ and $K_\alpha(N)$, depending on the entire trajectory of α under the jump transformation R until the renewal time $\hat{n}_N$ (exactly as $\Theta_\alpha(N)$ does):

$$E_\alpha(N):=\eta_1\cdots\eta_{\nu_{\hat{n}}-1}=\prod_{v=1}^{\hat{n}}(-\zeta_v),\qquad K_\alpha(N):=\kappa_{\nu_{\hat{n}}}=\sum_{u=1}^{\hat{n}}(h_u-\zeta_u)\prod_{v=1}^{u-1}(-\zeta_v).$$

Using (39) and (41)–(43), let us recall that $\alpha_{\nu_{\hat n-j}-s}$ is a function of β_j and s; moreover, notice that

$$\eta_1\cdots\eta_{\nu_{\hat n-J}-1}=\mathcal{E}_J\cdot E_\alpha(N)\quad\text{and}$$
$$N_{\nu_{\hat n-j}-s}=\alpha_0\cdots\alpha_{\nu_{\hat n-j}-s-1}\cdot N=\frac{\Theta_\alpha(N)}{(\alpha)_{\nu_{\hat n-j}-s}^{\nu_{\hat n-j}-2}\cdot(\alpha)_{\nu_{\hat n-j}-1}^{\nu_{\hat n}-2}}=\frac{\Theta_\alpha(N)}{B_{s,j}\cdot D_j}$$

are functions of $\Theta_\alpha(N)$, $E_\alpha(N)$, $R^{\hat n_N}(\alpha)$ and a finite number of Σ-entries of α preceding the renewal time $\hat n_N$. Furthermore, by (30) and (32), $\Theta_\alpha(N)$ is a function of $\hat q_{\hat n-1}/N$, $\hat q_{\hat n}/N$, $R^{\hat n_N(\alpha)}$ and the two Σ-entries $(h_{\hat n_N}\cdot m_{\hat n_N}^{\zeta_{\hat n_N}}, h_{\hat n_N+1}\cdot m_{\hat n_N+1}^{\zeta_{\hat n_N+1}})$.

In addition to this, since $\kappa_{\nu_{\hat n-J}-1}$ appears in the phase term of (45) as multiplier of $\frac{\pi}{4}\mathrm{i}$ it is also natural to consider its values modulo 8. Defining $K_\alpha^8(N):=K_\alpha(N)\ (\mathrm{mod}\ 8)$, we have

$$\kappa_{\nu_{\hat n-J}-1}\equiv K_\alpha^8(N)-E_\alpha(N)\sum_{u=\hat n-J+1}^{\hat n}(h_u-\zeta_u)\mathcal{E}_{\hat n-u+1}\ (\mathrm{mod}\ 8).$$

Therefore, we can rewrite (45) as

$$\gamma_{\alpha,N}^J(t)=\mathrm{F}_1\left(t,R^{\hat n_N}(\alpha),\frac{\hat q_{\hat n_N-1}}{N},\frac{\hat q_{\hat n_N}}{N},K_\alpha^8(N),E_\alpha(N),\left\{h_l\cdot m_l^{\zeta_l},\hat n_N-J\le l\le\hat n_N\right\}\right),\tag{46}$$

where F_1 is a complex-valued, measurable function of its arguments. Notice that the formulae (8) and (11) enter into the definition of F_1, but we shall not use them directly.

Let us recall that Theorem 2.7 (which is a special case of Theorem 4.1 and generalizes Theorem 1.6 in [4]) already establishes the existence of a limiting probability distribution for $\hat q_{\hat n_N-1}/N$ and $\hat q_{\hat n_N}/N$, jointly with any finite number of Σ-entries preceding (and/or following) the renewal time as $N\to\infty$, w.r.t. the measure λ.

In the next section we study the quantities $K_\alpha^8(N)\in\{0,1,\dots,7\}$ and $E_\alpha(N)\in\{\pm1\}$ in (46) and our Main Renewal-Type Limit Theorem 4.1 will allow us to include them in the statement about the existence of a joint liming probability distribution. This fact is non-trivial since $K_\alpha^8(N)$ and $E_\alpha(N)$ depend on the entire trajectory of α under R until the renewal time $\hat n_N$.

3.3. *Limiting distribution for phase and conjugation terms*

Let $x_n:=\eta_1\cdots\eta_{\nu_n-1}=\prod_{s=1}^n(-\zeta_s)$ and $y_n:=\kappa_{\nu_n}-1=\sum_{s=1}^n(h_s-\zeta_s)\prod_{u=1}^{s-1}(-\zeta_u)\ (\mathrm{mod}\ 8)$. We want to prove that $(x_n,y_n)\in\{\pm1\}\times\{0,1,\dots,7\}=:\Xi$ have a joint limiting distribution as $n\to\infty$. We will follow the strategy used by Sinai [27], Chapter 12, to see how the dynamics creates conditional probability distributions and these distributions define uniquely a limiting probability measure.

Let us consider the natural extension $\hat R:\Sigma^{\mathbb Z}\to\Sigma^{\mathbb Z}$ of R. For $\sigma\in\Sigma^{\mathbb Z}$, denote by $\sigma^-=(\dots,\sigma_{-2},\sigma_{-1},\sigma_0)$ and $\sigma^+=(\sigma_1,\sigma_2,\dots)$ and identify the pair (σ^+,σ^-) with a point in the rectangle $(0,1]\times(-1/3,1]\setminus\mathbb{Q}^2$ as discussed in [4]. One should notice that the "past" is identified with the y-axis and the "future" with the x-axis. Let us consider cylinders in $\Sigma^{\mathbb Z}$ of the form $J^{(m+1)}_{\sigma_{-n-m},\dots,\sigma_{-n-1},\sigma_{-n}}$, $n\ge0$, i.e. depending only on the past. Such cylinders J are identified with rectangles $(0,1]\times I$, where I is an interval in the y-direction, and by $|J|$ we mean the 1-dimensional Lebesgue measure of I.

Lemma 3.5. *For every $\sigma^-\in\Sigma^{\mathbb N}$, the limit*

$$\mu(\sigma_0|\sigma_{-1},\sigma_{-2},\dots):=\lim_{n\to\infty}\frac{|J^{(n+1)}_{\sigma_{-n},\dots,\sigma_{-1},\sigma_0}|}{|J^{(n)}_{\sigma_{-n},\dots,\sigma_{-1}}|}$$

exists and satisfies the following conditions:

$$\mu(\sigma_0|\sigma_{-1},\dots)\geq C_{16},$$
$$\sum_{\sigma_0\in\Sigma}\mu(\sigma_0|\sigma_{-1},\dots)=1,$$
$$\left|\frac{\mu(\sigma_0|\sigma_{-1},\dots,\sigma_{-s},\sigma'_{-s-1},\sigma'_{-s-2},\dots)}{\mu(\sigma_0|\sigma_{-1},\dots,\sigma_{-s},\sigma_{-s-1},\sigma_{-s-2},\dots)}-1\right|\leq C_{17}\mathrm{e}^{-C_{18}s} \tag{47}$$

for some constants $C_{16},C_{17},C_{18}>0$.

Proof. Let $l_n=|J^{(n+1)}_{\sigma_{-n},\dots,\sigma_{-1},\sigma_0}|/|J^{(n)}_{\sigma_{-n},\dots,\sigma_{-1}}|$. By Lemma 2.10 we have

$$\left|\frac{l_{n+1}}{l_n}-1\right|=\left|\frac{|J^{(n+2)}_{\sigma_{-n-1},\dots,\sigma_{-1},\sigma_0}|}{|J^{(n+1)}_{\sigma_{-n-1},\dots,\sigma_{-1}}|}\cdot\frac{|J^{(n)}_{\sigma_{-n},\dots,\sigma_{-1}}|}{|J^{(n+1)}_{\sigma_{-n},\dots,\sigma_{-1},\sigma_0}|}-1\right|\leq C_8\mathrm{e}^{-C_9 n}.$$

This implies the existence of the limit $\lim_{n\to\infty}l_n$ and also the desired properties. □

Since we are working with the natural extension of R, setting $z_n:=h_n-\zeta_n \pmod 8$, the quantities $(\zeta_n,z_n)\in\Xi$ are defined for every $n\in\mathbb{Z}$. Now we want to define conditional probability distributions $\mu_0((\zeta_0,z_0)|(\zeta_{-1},z_{-1}),(\zeta_{-2},z_{-2}),\dots)$ over $\Xi^{\mathbb{Z}}$. Let us fix a sequence $\underline{\sigma}^{(0)}=\{\sigma_j^{(0)}\}\in\Sigma^{\mathbb{Z}}$ and for every $n\in\mathbb{N}$ consider

$$\begin{aligned}
&\mu_0^{(0)}\big((\zeta_0,z_0)|(\zeta_{-1},z_{-1}),(\zeta_{-2},z_{-2}),\dots,(\zeta_{-n},z_{-n})\big)\\
&\quad=\frac{\mu_0^{(0)}((\zeta_{-n},z_{-n}),\dots,(\zeta_{-1},z_{-1}),(\zeta_0,z_0))}{\mu_0^{(0)}((\zeta_{-n},z_{-n}),\dots,(\zeta_{-1},z_{-1}))}\\
&\quad:=\frac{\sum_{\sigma_0,\sigma_{-1},\dots,\sigma_{-n}}\mu(\sigma_{-n},\dots,\sigma_{-1},\sigma_0)}{\sum_{\sigma_{-1},\dots,\sigma_{-n}}\mu(\sigma_{-n},\dots,\sigma_{-1})}\\
&\quad=\frac{\sum_{\sigma_0,\sigma_{-1},\dots,\sigma_{-n}}\prod_{s=0}^n\mu(\sigma_{-s}|\sigma_{-s-1},\dots,\sigma_{-n},\sigma^{(0)}_{-n-1},\sigma^{(0)}_{-n-2},\dots)}{\sum_{\sigma_{-1},\dots,\sigma_{-n}}\prod_{s=1}^n\mu(\sigma_{-s}|\sigma_{-s-1},\dots,\sigma_{-n},\sigma^{(0)}_{-n-1},\sigma^{(0)}_{-n-2},\dots)},
\end{aligned}\tag{48}$$

where the sums are taken over all possible $\sigma_0,\sigma_{-1},\dots,\sigma_{-n}\in\Sigma$ which are compatible with the values of $(\zeta_{-n},z_{-n}),\dots,(\zeta_{-1},z_{-1}),(\zeta_0,z_0)$.

Lemma 3.6. *The limit*

$$\mu_0\big((\zeta_0,z_0)|(\zeta_{-1},z_{-1}),(\zeta_{-2},z_{-2}),\dots\big):=\lim_{n\to\infty}\mu_0^{(0)}\big((\zeta_0,z_0)|(\zeta_{-1},z_{-1}),(\zeta_{-2},z_{-2}),\dots,(\zeta_{-n},z_{-n})\big)$$

exists and does not depend on $\underline{\sigma}^{(0)}$.

Proof. The Markov process $\{\dots,\sigma_{-n},\dots,\sigma_{-1},\sigma_0\}$ has a countable state-space but, by (18), it satisfies a Doeblin condition. Therefore, it can be exponentially well approximated by a process with finite (but sufficiently large) state-space. To this end, let us introduce also $\mu^{(0)}_{0,L}$ as in (48), with the additional constraint that $\sigma_{-j}=h_{-j}\cdot m_{-j}^{\zeta_{-j}}$, satisfy the inequalities $h,m\leq L$ for $0\leq j\leq n$. The sums in the corresponding numerator and denominator are thereby finite and contain at most $(2L^2-L-1)^{n+1}$ and $(2L^2-L-1)^n$ terms respectively. In order to prove that $\mu^{(0)}_{0,L}((\zeta_0,z_0)|(\zeta_{-1},z_{-1}),(\zeta_{-2},z_{-2}),\dots,(\zeta_{-n},z_{-n}))$ has a limit as $n\to\infty$ we shall perform a second approximation of the process $\{\sigma_j\}$ by a finite Markov chain with memory of order $\sqrt{n}$.

We partition the integers $1,\dots,n$ into fragments with $\lfloor\sqrt{n}\rfloor$ elements. Notice that $0\le n-\lfloor\sqrt{n}\rfloor^2\le 2\lfloor\sqrt{n}\rfloor$ and define

$$\mathrm{sq}(n)=\begin{cases}\lfloor\sqrt{n}\rfloor-1 & \text{if } 0\le n-\lfloor\sqrt{n}\rfloor^2<\lfloor\sqrt{n}\rfloor,\\ \lfloor\sqrt{n}\rfloor & \text{if } \lfloor\sqrt{n}\rfloor\le n-\lfloor\sqrt{n}\rfloor^2<2\lfloor\sqrt{n}\rfloor,\\ \lfloor\sqrt{n}\rfloor+1 & \text{if } n-\lfloor\sqrt{n}\rfloor^2=2\lfloor\sqrt{n}\rfloor.\end{cases}$$

The product in the denominator of $\mu^{(0)}_{0,L}$ becomes

$$\begin{aligned}&\prod_{s=1}^{n}\mu\big(\sigma_{-s}|\sigma_{-s-1},\dots,\sigma_{-n},\sigma^{(0)}_{-n-1},\sigma^{(0)}_{-n-2},\dots\big)\\ &\quad=\prod_{j=1}^{\mathrm{sq}(n)}\mu\big(\sigma_{-(j-1)\lfloor\sqrt{n}\rfloor-1},\dots,\sigma_{-j\lfloor\sqrt{n}\rfloor}|\sigma_{-j\lfloor\sqrt{n}\rfloor-1},\dots,\sigma_{-(j+1)\lfloor\sqrt{n}\rfloor},\dots,\sigma_{-n},\sigma^{(0)}_{-n-1},\dots\big)\\ &\qquad\cdot\mu\big(\sigma_{-\mathrm{sq}(n)\lfloor\sqrt{n}\rfloor-1},\dots,\sigma_{-(\mathrm{sq}(n)+1)\lfloor\sqrt{n}\rfloor}|\sigma_{-(\mathrm{sq}(n)+1)\lfloor\sqrt{n}\rfloor-1},\dots,\sigma_{-n},\sigma^{(0)}_{-n-1},\dots\big) &&(49)\\ &\qquad\cdot\mu\big(\sigma_{-(\mathrm{sq}(n)+1)\lfloor\sqrt{n}\rfloor-1},\dots,\sigma_{-n}|\sigma^{(0)}_{-n-1},\sigma^{(0)}_{-n-2},\dots\big) &&(50)\\ &\quad=\left(\prod_{j=1}^{\mathrm{sq}(n)}\mu(\hat\sigma_{-j}|\hat\sigma_{-j-1})\delta_j\right)\cdot\tilde\mu^{(1)}\cdot\tilde\mu^{(0)},\end{aligned}$$

where

$$\begin{aligned}&\hat\sigma_{-j}=(\sigma_{-(j-1)\lfloor\sqrt{n}\rfloor-1},\dots,\sigma_{-j\lfloor\sqrt{n}\rfloor})\in\Sigma^{\lfloor\sqrt{n}\rfloor},\\ &\delta_j=\frac{\mu(\hat\sigma_{-j}|\hat\sigma_{-j-1},\sigma_{-(j+1)\lfloor\sqrt{n}\rfloor-1},\dots)}{\mu(\hat\sigma_{-j}|\hat\sigma_{-j-1})},\end{aligned}\tag{51}$$

and $\tilde\mu^{(1)}$, $\tilde\mu^{(0)}$ correspond the factors in (49) and (50), respectively. Notice that for $n-\lfloor\sqrt{n}\rfloor^2=k\lfloor\sqrt{n}\rfloor$, $k=0,1,2$, the factor $\tilde\mu^{(0)}$ disappears and $\tilde\mu^{(1)}=\mu(\sigma_{-\mathrm{sq}(n)\lfloor\sqrt{n}\rfloor-1},\dots,\sigma_{-n}|\sigma^{(0)}_{-n-1},\dots)$. We claim that

$$|\delta_j-1|\le C_{19}\sqrt{n}\mathrm{e}^{-C_{20}\sqrt{n}}.\tag{52}$$

In fact, the correction factor δ_j can be written as

$$\delta_j=\prod_{s=(j-1)\lfloor\sqrt{n}\rfloor+1}^{j\lfloor\sqrt{n}\rfloor}\frac{\mu(\sigma_{-s}|\sigma_{-s-1},\dots,\sigma_{-j\lfloor\sqrt{n}\rfloor},\hat\sigma_{-j-1},\sigma_{-(j+1)\lfloor\sqrt{n}\rfloor-1},\dots)}{\mu(\sigma_{-s}|\sigma_{-s-1},\dots,\sigma_{-j\lfloor\sqrt{n}\rfloor},\hat\sigma_{-j-1})}\tag{53}$$

and, by (47), each factor in (53), is $(C_{17}\mathrm{e}^{-C_{18}\sqrt{n}})$-close to 1. Therefore, for some constants $C_{21},C_{22}>0$, $|\log\delta_j|\le C_{21}\sqrt{n}\cdot\mathrm{e}^{-C_{22}\sqrt{n}}$ and we get (52) for some $C_{19},C_{20}>0$. The factors $\tilde\mu^{(0)}$ and $\tilde\mu^{(1)}$ can be approximated in the same way, by truncating the length of the condition after $\lfloor\sqrt{n}\rfloor$ digits. Denoting by $\delta^{(l)}=\frac{\tilde\mu^{(l)}}{\hat\mu^{(l)}}$, $l=0,1$, the correction terms as in (51), one gets $|\delta^{(l)}-1|\le C_{22}\sqrt{n}\mathrm{e}^{-C_{23}\sqrt{n}}$ for $l=0,1$ and for some $C_{22},C_{23}>0$.

Therefore $\mu^{(0)}_{0,L}((\zeta_0,z_0)|(\zeta_{-1},z_{-1}),(\zeta_{-2},z_{-2}),\dots,(\zeta_{-n},z_{-n}))$ is exponentially well approximated by

$$\frac{\sum_{\sigma_0,\sigma_{-1},\dots,\sigma_{-n}}\mu(\sigma_0|\sigma_{-1})\prod_{j=1}^{\mathrm{sq}(n)}\mu(\hat\sigma_{-j}|\hat\sigma_{-j-1})\cdot\hat\mu^{(1)}\hat\mu^{(0)}}{\sum_{\sigma_{-1},\dots,\sigma_{-n}}\prod_{j=1}^{\mathrm{sq}(n)}\mu(\hat\sigma_{-j}|\hat\sigma_{-j-1})\cdot\hat\mu^{(1)}\hat\mu^{(0)}},$$

which can be understood as the expectation of $\mu(\sigma_0|\sigma_{-1})$ with respect to the measure for the finite Markov chain $\{\dots,\hat\sigma_{-n},\dots,\hat\sigma_{-1}\}$. Recall that the phase-space of such Markov chain is $\{h\cdot m^{\zeta}\in\Sigma\colon h,m\le L\}^{\lfloor\sqrt{n}\rfloor}$, which has

$(2L^2 - L - 1)^{\lfloor\sqrt{n}\rfloor}$ elements. This Markov chain is ergodic because, by the symbolic coding of the map R, every sequence of elements of Σ is allowed. By the ergodic theorem for Markov chains and the Doeblin condition we get the existence of the limit

$$\mu_0^{(0)}\big((\zeta_0, z_0)|(\zeta_{-1}, z_{-1}), (\zeta_{-2}, z_{-2}), \ldots\big) = \lim_{n\to\infty} \lim_{L\to\infty} \mu_{0,L}^{(0)}\big((\zeta_0, z_0)|(\zeta_{-1}, z_{-1}), (\zeta_{-2}, z_{-2}), \ldots, (\zeta_{-n}, z_{-n})\big).$$

Moreover, by (47), the conditional probability distributions $\mu_0^{(0)}((\zeta_0, z_0)|(\zeta_{-1}, z_{-1}), \ldots)$ do not depend on the sequence $\underline{\sigma}^{(0)}$ and will be denoted simply by $\mu_0((\zeta_0, z_0)|(\zeta_{-1}, z_{-1}), \ldots)$. □

Now, let us fix an arbitrary sequence $\{(\zeta_j^{(0)}, z_j^{(0)})\}_{j\in\mathbb{Z}} \in \Xi^{\mathbb{Z}}$. For each $s \in \mathbb{Z}$ consider the measure $\lambda_s^{(0)}$ defined on $\Xi^{\mathbb{Z}}$ using Lemma 3.6 as follows:

$$\lambda_s^{(0)}\big\{\big(\zeta_{s-n}^{(0)}, z_{s-n}^{(0)}\big), \ldots, \big(\zeta_{s-1}^{(0)}, z_{s-1}^{(0)}\big)\big\} := 1 \quad \text{for every } n \in \mathbb{N},$$

$$\lambda_s^{(0)}\big\{(\zeta_s, z_s), (\zeta_{s+1}, z_{s+1}), \ldots, (\zeta_{s+t}, z_{s+t})\big\} := \prod_{l=s}^{s+t} \mu_0\big((\zeta_l, z_l)|(\zeta_{l-1}, z_{l-1}), \ldots, (\zeta_s, z_s), \big(\zeta_{s-1}^{(0)}, z_{s-1}^{(0)}\big), \big(\zeta_{s-2}^{(0)}, z_{s-2}^{(0)}\big), \ldots\big)$$

for every $t \geq 0$. Since $\Xi^{\mathbb{Z}}$ is compact, the space of all probability measures on it is weakly compact and therefore there exists a subsequence $\{-s_j\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} s_j = \infty$ and $\lambda_{-s_j}^{(0)} \Longrightarrow \lambda^{(0)}$ as $j \to \infty$. One can show (see [27], Chapter 12, Theorem 2 and Lemma 2) that

$$\lim_{n\to\infty} \lambda^{(0)}\big((\zeta_s, z_s)|(\zeta_{s-1}, z_{s-1}), \ldots, (\zeta_{s-n}, z_{s-n})\big) = \mu_0\big((\zeta_s, z_s)|(\zeta_{s-1}, z_{s-1}), (\zeta_{s-2}, z_{s-2}), \ldots\big)$$

and such $\lambda^{(0)}$ is shift-invariant and unique.

Let us now prove the existence of the limiting probability distribution for the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$. Observe that

$$x_1 = -\zeta_1, \qquad x_n = x_{n-1} \cdot (-\zeta_n),$$

$$y_1 = z_1, \qquad y_n = y_{n-1} + z_n \cdot x_{n-1}.$$

Lemma 3.7. *For every $(X, Y) \in \Xi$ the limit*

$$\lim_{n\to\infty} \lambda^{(0)} \begin{pmatrix} x_n = X \\ y_n = Y \end{pmatrix}$$

exists.

Proof. Using the above relations we get

$$\begin{aligned}\lambda^{(0)} \begin{pmatrix} x_n = X \\ y_n = Y \end{pmatrix} &= \sum_{\substack{X_{n-1},\ldots,X_1 \\ Y_{n-1},\ldots,Y_1}} \prod_{j=1}^{n-1} \lambda^{(0)} \left(\begin{matrix} x_{j+1} = X_{j+1} \\ y_{j+1} = Y_{j+1} \end{matrix} \,\middle|\, \begin{matrix} x_j = X_j \\ y_j = Y_j \end{matrix} \right) \cdot \lambda^{(0)} \begin{pmatrix} x_1 = X_1 \\ y_1 = Y_1 \end{pmatrix} \\ &= \sum_{\substack{X_{n-1},\ldots,X_1 \\ Y_{n-1},\ldots,Y_1}} \prod_{j=1}^{n-1} \lambda^{(0)}\big((\zeta_{j+1}, z_{j+1}) = Z_{j+1}|(\zeta_j, z_j) = Z_j\big) \cdot \lambda^{(0)}\big((\zeta_1, z_1) = Z_1\big),\end{aligned} \tag{54}$$

where $(X_n, Y_n) := (X, Y), (X_{n-1}, Y_{n-1}), \ldots, (X_1, Y_1) \in \Xi$ and $Z_j \in \Xi$ are defined as

$$Z_1 := (-X_1, Y_1), \qquad Z_j := \big(-X_{j-1}X_j, X_{j-1}(Y_j - Y_{j-1}) \pmod 8\big), \quad j \geq 2. \tag{55}$$

Notice that, by (55), the sum over all $X_1, \dots, X_{n-1}, Y_1, \dots, Y_{n-1}$ in (54) can be replaced by the sum over all possible $Z_1, \dots, Z_{n-1} \in \Xi$.

Let us denote by $p_{Z,W} := \lambda^{(0)}((\zeta_{j+1}, z_{j+1}) = W | (\zeta_j, z_j) = Z)$, the transition probabilities for $Z, W \in \Xi$, by $\Pi := (p_{Z,W})_{Z,W\in\Xi}$ the corresponding $2^4 \times 2^4$ stochastic matrix and by $\underline{\pi} := (\lambda^{(0)}((\zeta_1, z_1) = Z))_{Z\in\Xi}$ the initial probability distribution. Thus, we can write (54) as

$$\lambda^{(0)} \begin{pmatrix} x_n = X \\ y_n = Y \end{pmatrix} = \left(\Pi^n \underline{\pi}\right)_Z, \tag{56}$$

where $Z = (-X_{j-1}X_j, X_{j-1}(Y_j - Y_{j-1}) \pmod 8)$. The stochastic matrix Π has positive entries and therefore $\lambda^{(0)} \left(\begin{smallmatrix} x_n = X \\ y_n = Y \end{smallmatrix}\right)$ has a limit for every $(X, Y) \in \Xi$ as $n \to \infty$. □

Let J be as in the previous section. It represents a finite number of Σ-entries preceding the renewal time $\hat{n}_N$ defining the approximating curve $t \mapsto \gamma^J_{\alpha,N}(t)$. We can rewrite $E_\alpha(N)$ and $K^8_\alpha(N)$ as follows:

$$E_\alpha(N) = x_{\hat{n}_N - J} \cdot \mathcal{E}_J,$$
$$K^8_\alpha(N) = \left[1 + y_{\hat{n}_N - J} + x_{\hat{n}_N - J} \cdot \sum_{u=\hat{n}_N - J + 1}^{\hat{n}_N} (h_u - \zeta_u)\mathcal{E}_J^{\hat{n}_N - u}\right]_8, \tag{57}$$
$$\left(E_\alpha(N), K^8_\alpha(N)\right) = \mathrm{F}_2\left((x_{\hat{n}_N - J}, y_{\hat{n}_N - J}), \left\{h_l \cdot m_l^{\zeta_l}, \hat{n}_N - J < l \le \hat{n}_N\right\}\right),$$

where $\mathrm{F}_2 : \Xi \times \Sigma^J \to \Xi$.

4. Existence of limiting finite-dimensional distributions

In this section we prove the existence of limiting finite-dimensional distribution for $\gamma^J_{\alpha,N}$ as $N \to \infty$, w.r.t. λ. Thereafter, we extend the result to $\gamma_{\alpha,N}$. We also discuss the notion of *nice* set and we give a sufficient condition for a set $A \subset \mathbb{C}^k$ to be nice.

For every $t \in [0, 1]$, by (46) and (57), we can write

$$\gamma^J_{\alpha,N}(t) = \mathrm{F}\left(t; R^{\hat{n}_N}(\alpha), \frac{\hat{q}_{\hat{n}_N - 1}}{N}, \frac{\hat{q}_{\hat{n}_N}}{N}, (x_{\hat{n}_N - J}, y_{\hat{n}_N - J}), \{\sigma_l\}_{l=\hat{n}_N - J}^{\hat{n}_N}\right),$$

where $\mathrm{F} = \mathrm{F}^{(1)} : [0, 1] \times (0, 1] \times (0, 1] \times (1, \infty) \times \Xi \times \Sigma^J \to \mathbb{C}$ is a measurable function of its arguments. Similarly, for every $0 \le t_1 < t_2 < \cdots < t_k \le 1$, setting $\underline{\gamma}^J_{\alpha,N}(t_1, \dots, t_k) := (\gamma^J_{\alpha,N}(t_1), \dots, \gamma^J_{\alpha,N}(t_k))$, we have

$$\underline{\gamma}^J_{\alpha,N}(t_1, \dots, t_k) = \mathrm{F}^{(k)}\left((t_1, \dots, t_k); R^{\hat{n}_N}(\alpha), \frac{\hat{q}_{\hat{n}_N - 1}}{N}, \frac{\hat{q}_{\hat{n}_N}}{N}, (x_{\hat{n}_N - J}, y_{\hat{n}_N - J}), \{\sigma_l\}_{l=\hat{n}_N - J}^{\hat{n}_N}\right),$$

where $\mathrm{F}^{(k)} : [0, 1]^k \times (0, 1] \times (0, 1] \times (1, \infty) \times \Xi \times \Sigma^J \to \mathbb{C}^k$.

The following Renewal-Type Limit theorem is the core of the proof of the existence of finite-dimensional distributions for $\gamma^J_{\alpha,N}$ as $N \to \infty$. It is a generalization of Theorem 1.6 in [4] and its proof will be sketched in the Appendix. Let us just mention that it relies on the mixing property of the special flow built over the natural extension of R, under the a suitably chosen roof function.

Theorem 4.1 (Main Renewal-Type Limit theorem). *Fix $N_1, N_2 \in \mathbb{N}$. The quantities $\frac{\hat{q}_{\hat{n}_N - 1}}{N}$, $\frac{\hat{q}_{\hat{n}_N}}{N}$, $\{\sigma_{\hat{n}_N + l}\}_{l=-N_1+1}^{N_2}$, $(x_{\hat{n}_N - N_1}, y_{\hat{n}_N - N_1})$ have a joint limiting probability distribution w.r.t. the measure λ as $N \to \infty$.*

In other words: there exists a probability measure $\mathrm{Q} = \mathrm{Q}_{N_1,N_2}$ on the space $(0, 1] \times (1, \infty) \times \Sigma^{N_1+N_2} \times \Xi$ such that for every $a_1, b_1, a_2, b_2 \in \mathbb{R}$, $0 < a_1 < b_1 < 1 < a_2 < b_2$, for every $\underline{c} = (c_l)_{l=-N_1+1}^{N_2} \in \Sigma^{N_1+N_2}$ and for every

$(x,y)\in\Xi$, we have

$$\lambda\left(\left\{\alpha\colon a_1<\frac{\hat q_{\hat n_N-1}}{N}<b_1, a_2<\frac{\hat q_{\hat n_N}}{N}<b_2, (\sigma_{\hat n_N+l})_{l=-N_1+1}^{N_2}=\underline{c}, \begin{pmatrix}x_{\hat n_N-N_1}\\ y_{\hat n_N-N_1}\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}\right\}\right)$$
$$\longrightarrow \mathrm{Q}\big((a_1,b_1)\times(a_2,b_2)\times\{\underline{c}\}\times\big\{(x,y)\big\}\big)\qquad \text{as } N\to\infty. \tag{58}$$

Remark 4.2. *Let us also mention that the proof of Theorem 4.1 provides an explicit formula for $\mathrm{Q}((a_1,b_1)\times(a_2,b_2)\times\{\underline{c}\}\times\{(x,y)\})$, based on a geometrical construction. Moreover, if we fix $\underline{c}\in\Sigma^{N_1+N_2}$ and $(x,y)\in\Xi$, then the measure on $(0,1]\times(1,\infty)$ defined as $\mathrm{Q}_{N_1,N_2;\underline{c},(x,y)}(E):=\mathrm{Q}_{N_1,N_2}(E\times\{\underline{c}\}\times\{(x,y)\})$ is equivalent to the Lebesgue measure on $(0,1]\times(1,\infty)$.*

Notice that the limiting probability distribution of $R^{\hat n_N}(\alpha)=(\sigma_{\hat n_N+1},\sigma_{\hat n_N+2},\ldots)\in\Sigma^{\mathbb{N}}$ can be obtained by providing a limiting probability distribution for any fixed number of Σ-entries after the renewal time $\hat n_N$, i.e. $\sigma_{\hat n_N+1},\ldots,\sigma_{\hat n_N+N_2}$, $N_2\in\mathbb{N}$. We immediately get the following corollary.

Corollary 4.3. *Fix $J\in\mathbb{N}$. The quantities $R^{\hat n_N}$, $\frac{\hat q_{\hat n_N-1}}{N}$, $\frac{\hat q_{\hat n_N}}{N}$, $(x_{\hat n_N-J},y_{\hat n_N-J})$, $\{\sigma_l\}_{l=\hat n_N-J}^{\hat n_N}$ have a joint limiting probability distribution on $(0,1]\times(0,1]\times(1,\infty)\times\Xi\times\Sigma^{J+1}$ as $N\to\infty$, with respect to the measure λ on $[0,1]$.*

Let us denote the limiting probability measure by $\mathrm{Q}^{(J)}$. For every $(x,y)\in\Xi$ and $\underline{\sigma}\in\Sigma^{J+1}$ the measure on $(0,1]^2\times(1,\infty)$ defined as $\mathrm{Q}^{(J)}_{(x,y),\underline{\sigma}}(E):=\mathrm{Q}^{(J)}(E\times\{(x,y)\}\times\{\underline{\sigma}\})$ is equivalent to the Lebesgue measure on $(0,1]^2\times(1,\infty)$. This fact is a consequence of Remark 4.2.

Remark 4.4. *Fix $(t_1,\ldots,t_k)\in[0,1]^k$, $J\in\mathbb{N}$, $(x,y)\in\Xi$ and $\underline{\sigma}\in\Sigma^{J+1}$. Denoting $(u,v,w)=(R^{\hat n_N}(\alpha),\frac{\hat q_{\hat n_N-1}}{N},\frac{\hat q_{\hat n_N}}{N})$, we can rewrite the functions in Lemma 3.4 as*

$$\beta_j=\beta_j(u)=\frac{a_j^{(1)}+b_j^{(1)}u}{c_j^{(1)}+d_j^{(1)}u},\qquad B_{s,j}=B_{s,j}(u)=\frac{a_{s,j}^{(2)}+b_{s,j}^{(2)}u}{c_{s,j}^{(2)}+d_{s,j}^{(2)}u},\qquad D_j=D_j(u)=\prod_{l=0}^{j-1}\frac{a_l^{(3)}+b_l^{(3)}u}{c_l^{(3)}+d_l^{(3)}u}$$

for some constants $a_j^{(1)},b_j^{(1)},c_j^{(1)},d_j^{(1)},a_{s,j}^{(2)},b_{s,j}^{(2)},c_{s,j}^{(2)},d_{s,j}^{(2)},a_l^{(3)},b_l^{(3)},c_l^{(3)},d_l^{(3)}$ (determined by $\underline{\sigma}$). Notice that the functions β_j, $B_{s,j}$ and D_j take values in $(0,1]$ and, despite their rational structure, they are $\mathcal{C}^\infty$ functions of $u\in(0,1]$. Moreover, $\alpha_{\nu_{\hat n_N-1}}=\alpha_{\nu_{\hat n_N-1}}(u)=\frac{a^{(4)}+b^{(4)}u}{c^{(4)}+d^{(4)}u}\in(0,1]$, by (30) and (32),

$$\begin{aligned}\Theta_\alpha(N)=:\theta(u,v,w)&=\big(a^{(5)}v+b^{(5)}w+c^{(5)}\alpha_{\nu_{\hat n_N-1}}\big(d^{(5)}v+e^{(5)}w\big)\big)^{-1}\\ &=\frac{c^{(4)}+d^{(4)}u}{(a^{(5)}v+b^{(5)}w)(c^{(4)}+d^{(4)}u)+c^{(5)}(a^{(4)}+b^{(4)}u)(d^{(5)}v+e^{(5)}w)}\in(0,\infty)\end{aligned}$$

is also a $\mathcal{C}^\infty$ function of (u,v,w), where $a^{(4)},b^{(4)},c^{(4)},d^{(4)},a^{(5)},b^{(5)},c^{(5)},d^{(5)},e^{(5)}$ are some constants (determined by $\underline{\sigma}$). For $\underline{t}=(t_1,\ldots,t_k)$, set

$$f_{\underline{t}}^{(J)}:=\mathrm{F}^{(k)}\big((t_1,\ldots,t_k),\cdot\big):(0,1]^2\times(1,\infty)\times\Xi\times\Sigma^{J+1}\to\mathbb{C}^k.$$

Finally, $\alpha_{\nu_{\hat n_N-j}}=:A_j(u)=\frac{a_j^{(6)}+b_j^{(6)}u}{c_j^{(6)}+d_j^{(6)}u}\in(0,1]$ for some constants $a_j^{(6)},b_j^{(6)},c_j^{(6)},d_j^{(6)}$ and

$$f_{\underline{t};(x,y),\underline{\sigma}}^{(J)}:=\mathrm{F}^{(k)}\big((t_1,\ldots,t_k);\cdot,(x,y),\underline{\sigma}\big)=f_{\underline{t}}^{(J)}\big(\cdot,(x,y),\underline{\sigma}\big):(0,1]^2\times(1,\infty)\to\mathbb{C}^k$$

reads as

$$f^{(J)}_{\underline{t};(x,y),\underline{\sigma}}(u,v,w) = \Bigg(C^{(1)}\theta(u,v,w)^{-1/2}\Bigg[C^{(2)}\mathcal{S}_u^{(C^{(3)})}\big(t_l\theta(u,v,w)\big) + \sum_{j=0}^{J-1} D_j(u)^{1/2} \sum_{s=2}^{C_j^{(4)}+2} C_s^{(5)} B_{s,j}(u)^{1/2}\Gamma\bigg(A_j(u), t_l\frac{\theta(u,v,w)}{B_{s,j}(u)D_j(u)}\bigg)\Bigg]^{(C^{(6)})}\Bigg)_{l=1}^{k},$$

where $C^{(1)}, C^{(2)}, C_s^{(5)} \in \mathbb{C}$, $C^{(3)}, C^{(6)} \in \{\pm 1\}$ *and* $C_j^{(4)} \in \mathbb{N}$ *are constants determined by* $(x,y) \in \Xi$ *and* $\underline{\sigma} \in \Sigma^{J+1}$. *Notice that* $f^{(J)}_{\underline{t};(x,y),\underline{\sigma}} : (0,1]^2 \times (1,\infty) \to \mathbb{C}^k$ *a continuous function* (*with piecewise* $\mathcal{C}^\infty$ *partial derivatives*) *of* (u,v,w).

4.1. *Nice sets*

We say that $A \in \mathcal{B}^k$ is $(t_1,\dots,t_k)$-*nice* (or simply *nice*) if for every $J \in \mathbb{N}$, for every $(x,y) \in \Xi$ and every $\underline{\sigma} \in \Sigma^{J+1}$, $\partial((f^{(J)}_{\underline{t};(x,y),\underline{\sigma}})^{-1}(A))$ has zero Lebesgue measure in $(0,1]^2 \times (0,\infty)$.

Notice that if $A = A_1 \times \dots \times A_k$, where $A_l \in \mathcal{B}^1$ and A_l is t_l-nice for $l = 1,\dots,k$, than A is $(t_1,\dots,t_k)$-nice. The following lemma gives a sufficient condition for $A \in \mathcal{B}^1$ to be t-nice, analogous to Lemma 5.1 in [17].

Lemma 4.5. *Let* $A \in \mathcal{B}^1$ *be an open convex set*, $0 \in A$, *with smooth boundary. Let* $A(w,\rho) := \{\rho z + w\colon z \in A\}$. *Fix* $t \in [0,1]$ *and* $w \in \mathbb{C}$. *Then, except for countably many* ρ, $A(w,\rho)$ *is* t-*nice.*

Proof. Let $t \in [0,1]$ be fixed. For every $J \in \mathbb{N}$, every $(x,y) \in \Xi$ and every $\underline{\sigma} \in \Sigma^{J+1}$ the set $(0,1]^2 \times (1,\infty)$ has finite $\mathrm{Q}^{(J)}_{(x,y),\underline{\sigma}}$-measure, say $q^{(J)}_{(x,y),\underline{\sigma}} > 0$. Since $f^{(J)}_{t;(x,y),\underline{\sigma}}$ is measurable, the measure of the set $\mathcal{X}(\rho) = \{(u,v,w) \in (0,1]^2 \times (1,\infty)\colon f^{(J)}_{t;(x,y),\underline{\sigma}}(u,v,w) \in A(w,\rho)\}$ tends to $q^{(J)}_{(x,y),\underline{\sigma}}$ as $\rho \to \infty$. Since $A(w,\rho)$ is convex for every ρ, the sets $\mathcal{I}(\rho) = \{(u,v,w) \in (0,1]^2 \times (1,\infty)\colon f_t^{(J)} \in \partial A(w,\rho)\}$ are disjoint for different values of ρ. Therefore, there can be only countably many ρ for which $\mathcal{I}(\rho)$ has positive $\mathrm{Q}^{(J)}_{(x,y),\underline{\sigma}}$ (and thus Lebesgue) measure. Since $f^{(J)}_{t;(x,y),\underline{\sigma}}$ is continuous, the boundary of $\mathcal{X}(\rho)$ is contained in $\mathcal{I}(\rho)$, concluding thus the proof. □

4.2. *Limiting finite-dimensional distributions for* $\gamma^J_{\alpha,N}$ *and* $\gamma_{\alpha,N}$

The main consequence of our Main Renewal-Type Limit Theorem 4.1 is the following proposition.

Proposition 4.6 (Limiting finite-dimensional distributions for $\gamma^J_{\alpha,N}$). *For every* $k \in \mathbb{N}$ *and every* $0 \le t_1 < t_2 < \dots < t_k \le 1$ *there exists a probability measure* $\mathrm{P}^{(J,k)}_{t_1,\dots,t_k}$ *on* $\mathbb{C}^k$ *such that for every open,* $(t_1,\dots,t_k)$-*nice set* $A \in \mathcal{B}^k$, *we have*

$$\lim_{N\to\infty} \lambda\big(\big\{\alpha \in (0,1]\colon \underline{\gamma}^J_{\alpha,N}(t_1,\dots,t_k) \in A\big\}\big) = \mathrm{P}^{(J,k)}_{t_1,\dots,t_k}(A). \tag{59}$$

Moreover, if $\{A^{(j)}\}_{j\in\mathbb{N}}$, $A^{(j)} \in \mathcal{B}^k$, *is a decreasing sequence of open,* $(t_1,\dots,t_k)$-*nice sets such that* $\mathrm{Leb}(A^{(j)}) \to 0$, *then* $\lim_{j\to\infty} \mathrm{P}^{(J,k)}_{t_1,\dots,t_k}(A^{(j)}) = 0$.

Proof. Since $A \in \mathcal{B}^k$ is open and $(t_1,\dots,t_k)$-nice, the set $\{\alpha \in (0,1]\colon \underline{\gamma}^J_{\alpha,N}(t_1,\dots,t_k) \in A\}$ can be written as

$$\left\{\alpha\colon \left(R^{\hat{n}_N}, \frac{\hat{q}_{\hat{n}_N-1}}{N}, \frac{\hat{q}_{\hat{n}_N}}{N}, (x_{\hat{n}_N-J}, y_{\hat{n}_N-J}), \{\sigma_l\}_{l=\hat{n}_N-J}^{\hat{n}_N}\right) \in \big(f_{\underline{t}}^{(J)}\big)^{-1}(A)\right\} \tag{60}$$

and

$$\big(f_{\underline{t}}^{(J)}\big)^{-1}(A) = \bigsqcup_{\substack{(x,y)\in\Xi,\\ \underline{\sigma}\in\Sigma^{J+1}}} B_{(x,y),\underline{\sigma}} \times \big\{(x,y)\big\} \times \{\underline{\sigma}\} = \bigsqcup_{\substack{(x,y)\in\Xi,\\ \underline{\sigma}\in\Sigma^{J+1}, l\in\mathbb{N},\\ B_{(x,y),\underline{\sigma}}\neq\varnothing}} R^{(l)}_{(x,y),\underline{\sigma}} \times \big\{(x,y)\big\} \times \{\underline{\sigma}\},$$

where $B_{(x,y),\underline{\sigma}} = B_{(x,y),\underline{\sigma}}(A) := (f^{(J)}_{\underline{t};(x,y),\underline{\sigma}})^{-1}(A)$ are open (possibly empty) subsets of $(0,1]^2\times(1,\infty)$ with boundaries of measure zero and $R^{(l)}_{(x,y),\underline{\sigma}} = R^{(l)}_{(x,y),\underline{\sigma}}(A) \subseteq (0,1]^2\times(1,\infty)$ are parallelepipeds of the form $(a_0,b_0)\times(a_1,b_1)\times(a_2,b_2)$ (the endpoints in each coordinate can be either included or not for different values of (x,y) and $\underline{\sigma}$) and a_0,b_0,a_1,b_1,a_2,b_2, depend on (x,y), $\underline{\sigma}$ and l. Thus the set in (60) is a disjoint union of sets of the form[1]

$$\left\{\alpha\colon\, a_0 < R^{\hat{n}_N} < b_0,\, a_1 < \frac{\hat{q}_{\hat{n}_N-1}}{N} < b_1,\, a_2 < \frac{\hat{q}_{\hat{n}_N}}{N} < b_2,\, (x_{\hat{n}_N-J}, y_{\hat{n}_N-J}) = (x,y),\, \{\sigma_l\}_{l=\hat{n}_N-J}^{\hat{n}_N} = \underline{\sigma}\right\}$$

whose λ-measures converge to $\mathrm{Q}^{(J)}(R^{(l)}_{(x,y),\underline{\sigma}}\times\{(x,y)\}\times\{\underline{\sigma}\})$ as $N\to\infty$ by Corollary 4.3. This concludes the proof of Proposition 4.6 setting

$$\mathrm{P}^{(J,k)}_{t_1,\dots,t_k}(A) := \sum_{\substack{(x,y)\in\Xi,\\ \underline{\sigma}\in\Sigma^{J+1}, l\in\mathbb{N},\\ B_{(x,y),\underline{\sigma}}\neq\varnothing}} \mathrm{Q}^{(J)}\big(R^{(l)}_{(x,y),\underline{\sigma}}(A)\big).$$

□

Now, for fixed k and $t_1,\dots,t_k$ we want to consider the limit of $\mathrm{P}^{(J,k)}_{t_1,\dots,t_k}(A)$ as $J\to\infty$. We have the following lemma.

Lemma 4.7. *For every $k\in\mathbb{N}$, every $0\le t_1<t_2<\cdots<t_k\le 1$ and every open, $(t_1,\dots,t_k)$-nice set $A\in\mathcal{B}^k$, the limit $\lim_{J\to\infty}\mathrm{P}^{(J,k)}_{t_1,\dots,t_k}(A)$ exists. It will be denoted by $\mathrm{P}^{(k)}_{t_1,\dots,t_k}(A)$.*

Proof. For simplicity, write $X_N^J(\alpha) = \underline{\gamma}^J_{\alpha,N}(t_1,\dots,t_k)$, $X_N(\alpha) = \underline{\gamma}_{\alpha,N}(t_1,\dots,t_k)$ and $\mathrm{P}^J = \mathrm{P}^{(J,k)}_{t_1,\dots,t_k}$. Moreover, for $z=(z_1,\dots,z_k)\in\mathbb{C}^k$ set $|z| := |z_1|+\cdots+|z_k|$. Assume, by contradiction, that the sequence $\{\mathrm{P}^J\}_{J\in\mathbb{N}}$ does not have a limit as $J\to\infty$. In this case there exist $\varepsilon>0$ and a subsequence $\mathcal{J}=\{J_l\}_{l\in\mathbb{N}}$ such that $|\mathrm{P}^{J'}(A)-\mathrm{P}^{J''}(A)|>\varepsilon$ for every $J',J''\in\mathcal{J}$. By definition of $\mathrm{P}^{J'}(A)$ and $\mathrm{P}^{J''}(A)$ we have that for every $\delta_5>0$ and for sufficiently large N,

$$\big|\lambda\big\{X_N^{J'}\in A\big\} - \lambda\big\{X_N^{J''}\in A\big\}\big| \ge 1-\delta_5. \tag{61}$$

On the other hand, by Lemma 3.3, we know that

$$\lambda\big\{\big|X_N - X_N^J\big| \le k\mathrm{e}^{-C_{15}J}\big\} \ge 1-\delta_3(J) \tag{62}$$

and $\delta_3(J)\to 0$ as $J\to\infty$. Now (62) implies that

$$\lambda\big\{\big|X_N^{J'} - X_N^{J''}\big| \le k\big(\mathrm{e}^{-C_{15}J'}+\mathrm{e}^{-C_{15}J''}\big)\big\} \ge 1-\delta_3\big(J'\big)-\delta_3\big(J''\big)$$

and thus

$$\begin{aligned}
&\big|\lambda\big\{X_N^{J'}\in A\big\} - \lambda\big\{X_N^{J''}\in A\big\}\big| \\
&\quad\le \big|\lambda\big\{X_N^{J'}\in A, \big|X_N^{J'}-X_N^{J''}\big| \le k\big(\mathrm{e}^{-C_{15}J'}+\mathrm{e}^{-C_{15}J''}\big)\big\} - \lambda\big\{X_N^{J''}\in A\big\}\big| + \delta_3\big(J'\big)+\delta_3\big(J''\big) \\
&\quad\le \big|\lambda\big\{X_N^{J''}\in A'\big\} - \lambda\big\{X_N^{J''}\in A\big\}\big| + \delta_3\big(J'\big)+\delta_3\big(J''\big),
\end{aligned} \tag{63}$$

[1] Strict inequalities are replaced by "≤" when the endpoints are included.

where $A' = \{z \in \mathbb{C}^k\colon |z-w| \le k(\mathrm{e}^{-C_{15}J'} + \mathrm{e}^{-C_{15}J''}),\ w \in A\}$. Now, by taking sufficiently large $J', J'' \in \mathcal{J}$ and using the last part of Proposition 4.6, (63) gives

$$\left|\lambda\left\{X_N^{J'} \in A\right\} - \lambda\left\{X_N^{J''} \in A\right\}\right| \le \lambda\left\{X_N^{J''} \in A' \setminus A\right\} + \delta_3\left(J'\right) + \delta_3\left(J''\right) \le \varepsilon/3,$$

contradicting thus (61) if we choose $\delta_5 = \varepsilon/2$. □

Now we can prove our Main theorem.

Proof of Theorem 1.1. So far, by Lemma 4.7, we know that

$$\lim_{J\to\infty}\lim_{N\to\infty}\lambda\left\{\alpha\colon \underline{\gamma}^J_{\alpha,N}(t_1,\ldots,t_k) \in A\right\} = \mathrm{P}^{(k)}_{t_1,\ldots,t_k}(A).$$

Roughly speaking, we want to interchange the order of the two limits. Let us use the same notations of the proof of Lemma 4.7 and, in addition, set $Y_N^J(\alpha) := X_N(\alpha) - X_N^J(\alpha)$ and $\mathrm{P} := \mathrm{P}^{(k)}_{t_1,\ldots,t_k}$. By (62) we have

$$\lambda\{X_N \in A\} \le \lambda\left\{X_N^J + Y_N^J \in A, \left|Y_N^J\right| \le k\mathrm{e}^{-C_{15}J}\right\} + \delta_3(J) \le \lambda\left\{X_N^J \in A'\right\} + \delta_3(J), \tag{64}$$

where $A' = \{z \in \mathbb{C}^k\colon |z-w| \le k\mathrm{e}^{-C_{15}J},\ w \in A\}$ and $\delta_3(J) \to 0$ as $J \to \infty$. Now, by Proposition 4.6 and Lemma 4.7, (64) becomes

$$\lambda\{X_N \in A\} \le P^J(A) + \delta_6(N) + \delta_3(J) = P(A) + \delta_7(J) + \delta_6(N) + \delta_3(J), \tag{65}$$

where $\delta_6(N) \to 0$ as $N \to \infty$ and $\delta_7(J) \to 0$ as $J \to \infty$. On the other hand, in a similar way we get

$$\begin{aligned}\lambda\{X_N \in A\} &\ge \lambda\left\{X_N^J + Y_N^J \in A, \left|Y_N^J\right| \le k\mathrm{e}^{C_{15}J}\right\} \ge \lambda\left\{X_N^J \in A''\right\} \ge P^J\left(A''\right) + \delta_8(N)\\ &= P(A) + \delta_9(J) + \delta_8(N),\end{aligned} \tag{66}$$

where $A'' = \{z \in A\colon |z-w| \le k\mathrm{e}^{-C_{15}J},\ w \in A^c\}$, $\delta_8(N) \to 0$ as $N \to \infty$ and $\delta_9(J) \to 0$ as $J \to \infty$. Now, taking $\lim_{N\to\infty}\lim_{J\to\infty}$, in (64) and (66), we obtain $\lim_{N\to\infty}\lambda\{X_N \in A\} = P(A)$, i.e. (2) as desired. □

Remark 4.8. *Considering, as in Remark 1.3, our reference probability space* $([0,1], \mathcal{B}, \lambda)$,

$$\gamma_{\cdot,N}, \gamma^J_{\cdot,N} \colon \left([0,1], \mathcal{B}, \lambda\right) \to \left(\mathcal{C}\left([0,1], \mathbb{C}\right), \mathcal{B}_{\mathcal{C}}\right)$$

are two random function. Let P_N *and* P_N^J *the corresponding induced probability measures on* $\mathcal{C}([0,1],\mathbb{C})$. *Now Proposition 4.6, Lemma 4.7 and Theorem 1.1 read as follows: for every* $k \in \mathbb{N}$ *and for every* $0 \le t_1 < \cdots < t_k \le 1$,

$$\mathrm{P}_N^J\pi^{-1}_{t_1,\ldots,t_k} \xrightarrow[\text{Prop. 4.6}]{N\to\infty} \mathrm{P}^{(J,k)}_{t_1,\ldots,t_k} \xrightarrow[\text{Lem. 4.7}]{J\to\infty} \mathrm{P}^{(k)}_{t_1,\ldots,t_k} \xleftarrow[\text{Thm. 1.1}]{N\to\infty} \mathrm{P}_N\pi^{-1}_{t_1,\ldots,t_k}.$$

Appendix: Proof of Theorem 4.1

This appendix is devoted to the explanation of the proof of Theorem 4.1. This theorem is a generalization of Theorem 1.6 in [4] and therefore we shall indicate how to modify its proof. Let us first recall some notation from [4].

Let $\hat{R}\colon \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ the natural extension of R as in Section 3.3 and let $\mu_{\hat{R}}$ be the natural invariant measure induced by μ_R. Set $D(\hat{R}) := \Sigma^{\mathbb{Z}}$. For $\psi \in L^1(D(\hat{R}))$ set $D_\Phi = \{(\hat{\sigma}, z)\colon \hat{\sigma} \in D(\hat{R}), 0 \le z \le \psi(\hat{\sigma})\}$, let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the special flow on D_Φ and let $\mu_\Phi = \mu_{\hat{R}} \times \mathrm{Leb}$, where Leb is the Lebesgue measure in the z-direction. This flow is mixing,[2] i.e. $\lim_{t\to\infty}\mu_\Phi(A \cap \Phi_{-t}(B)) = \mu(A)\mu(B)$ for every Borel subsets $A, B \subset D_\Phi$ (see Proposition 3.4 in [4]). We shall

[2]The flow $\{\Phi_t\}_t$ is actually proven to be a K-flow.

use the following relation between the special flow Φ_t and the (non-normalized) Birkhoff sum of ψ under $\hat{R}$. Setting $\mathrm{S}_r^{\hat{R}}(\psi)(\hat\sigma) := \sum_{j=0}^{r-1} \psi(\hat{R}^j(\hat\sigma))$ and $r(\hat\sigma, t) := \min\{r \in \mathbb{N}\colon \mathrm{S}_r^{\hat{R}}(\psi)(\hat\sigma) > t\}$ we get for $t \in \mathbb{R}^+$

$$\Phi_t(\hat\sigma, 0) = \big(\hat{R}^{r(\hat\sigma,t)-1}(\hat\sigma), t - \mathrm{S}^{\hat{R}}_{r(\hat\sigma,t)-1}(\psi)(\hat\sigma)\big).$$

Fix a cylinder $\mathcal{C}$ and set $g_{\mathcal{C}} := \sup_{\hat\sigma\in\mathcal{C}} g(\hat\sigma)$, where $g : D(\hat{R}) \to \mathbb{R}^+$ is a function defined so that

$$\log \hat{q}_n(\hat\sigma) = \mathrm{S}_n^{\hat{R}}(\psi)(\hat\sigma) + g(\hat\sigma) + \varepsilon_n(\hat\sigma), \qquad \sup_{\hat\sigma\in D(\hat{R})} \big|\varepsilon_n(\hat\sigma)\big| \le C_{23} 3^{-n/3} \tag{67}$$

for some constant $C_{23} > 0$. If $|g(\hat\sigma) - g_{\mathcal{C}}| \le \varepsilon/2$ on $\mathcal{C}$ (this is always possible, by considering a sufficiently small cylinder $\mathcal{C}$), then one can choose a time $\mathrm{T} = \mathrm{T}(N, \mathcal{C}) = \log N - g_{\mathcal{C}}$ so that $\hat{n}_N(\hat\sigma) = r(\hat\sigma, \mathrm{T})$ holds on $\mathcal{C} \setminus U$, where $U = U(\mathcal{C}) \subset \mathcal{C}$, $\mu_{\hat{R}}(U) \le 7\varepsilon\mu_{\hat{R}}(C)$. Given two functions $F_1, F_2 : D(\hat{R}) \to \mathbb{R}$ we define

$$D_\Phi(F_1, F_2) := \big\{(\hat\sigma, z) \in D_\Phi\colon\ \psi(\hat\sigma) - F_2(\hat\sigma) < z < \psi(\hat\sigma) - F_1(\hat\sigma)\big\}.$$

Notice that for some values of $F_1(\hat\sigma)$, $F_2(\hat\sigma)$ (e.g., when they are negative) the corresponding sets of z's can be empty.

Sketch of proof of Theorem 4.1. The condition $(\sigma_{\hat{n}_N+l})_{l=-N_1+1}^{N_2} = \underline{c}$ in (58) can be rewritten as $\hat{R}^{\hat{n}_N(\hat\sigma)-1}(\hat\sigma) \in \mathcal{C}^{(\underline{c})}_{N_1,N_2}$, where $\mathcal{C}^{(\underline{c})}_{N_1,N_2}$ is a cylinder determined by N_1, N_2 and $\underline{c}$. We claim that

$$\begin{aligned}\lim_{N\to\infty} \lambda\Bigg(\bigg\{\alpha \in (0,1]\colon\ a_1 < \frac{\hat{q}_{\hat{n}_N-1}}{N} < b_1, a_2 < \frac{\hat{q}_{\hat{n}_N}}{N} < b_2, \hat{R}^{\hat{n}_N(\hat\sigma)-1}(\hat\sigma) \in \mathcal{C}^{(\underline{c})}_{N_1,N_2}, \begin{pmatrix} x_{\hat{n}_N-N_1} \\ y_{\hat{n}_N-N_1}\end{pmatrix} = \begin{pmatrix} x \\ y\end{pmatrix}\bigg\}\Bigg) \\ = p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi(a_1, b_1, a_2, b_2, \underline{c})\big),\end{aligned} \tag{68}$$

where $p_{x,y,\underline{c}}$ is a real number between 0 and 1 (we shall define it later in this proof), $\bar{D}_\Phi(a_1, b_1, a_2, b_2) := D_\Phi(\log a_1 + \psi\circ\hat{R}^{-1}, \log b_1 + \psi\circ\hat{R}^{-1}) \cap D_\Phi(\log a_2, \log b_2) \cap p^{-1}\mathcal{C}^{(\underline{c})}_{N_1,N_2}$ (see Fig. 3) and $p : D_\Phi \to D(\hat{R})$ is the vertical projection onto the base. Set

$$A_{\mathcal{C}} := \bigg\{\hat\sigma \in \mathcal{C}\colon\ a_1 < \frac{\hat{q}_{\hat{n}_N-1}}{N} < b_1, a_2 < \frac{\hat{q}_{\hat{n}_N}}{N} < b_2, \hat{R}^{\hat{n}_N(\hat\sigma)-1}(\hat\sigma) \in \mathcal{C}^{(\underline{c})}_{N_1,N_2}, \begin{pmatrix} x_{\hat{n}_N-N_1} \\ y_{\hat{n}_N-N_1}\end{pmatrix} = \begin{pmatrix} x \\ y\end{pmatrix}\bigg\}.$$

Consider $\varepsilon > 0$. One can find a finite collection of cylinders $\mathfrak{C}_\varepsilon$ for which (58) can be 10ε-approximated by $\sum_{\mathcal{C}\in\mathfrak{C}_\varepsilon} \mu_{\hat{R}}(A_{\mathcal{C}\setminus U})$, where $U = U(\mathcal{C})$ is as above.

Let $\hat\lambda$ be an absolutely continuous measure on the $\Sigma^{\mathbb{Z}} = (0,1] \times (-1/3, 1] \setminus \mathbb{Q}^2$ that projects onto λ on $\Sigma^{\mathbb{N}} = (0,1] \setminus \mathbb{Q}$, i.e. for every interval $I \subset (0,1]$ we have $\hat\lambda(I \times (-1/3, 1]) = \lambda(I)$. If, for instance, $\lambda = \mu_R$, then we can take $\hat\lambda = \mu_{\hat{R}}$.

In order to show (68), noticing that $A_{\mathcal{C}}$ depends on N, it is enough to prove that, for sufficiently large N,

$$\left|\frac{\hat\lambda(A_{\mathcal{C}\setminus U})}{\hat\lambda(\mathcal{C}\setminus U)} - p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi(a_1, b_1, a_2, b_2, \underline{c})\big)\right| \le C_{24}\varepsilon$$

for some $C_{24} > 0$. Since $\hat\lambda$ is absolutely continuous w.r.t. $\mu_{\hat{R}}$, it is enough to show, for sufficiently large N and sufficiently small cylinders $\mathcal{C}$, that

$$\left|\frac{\mu_{\hat{R}}(A_{\mathcal{C}\setminus U})}{\mu_{\hat{R}}(\mathcal{C}\setminus U)} - p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi(a_1, b_1, a_2, b_2, \underline{c})\big)\right| \le C_{24}\varepsilon. \tag{69}$$

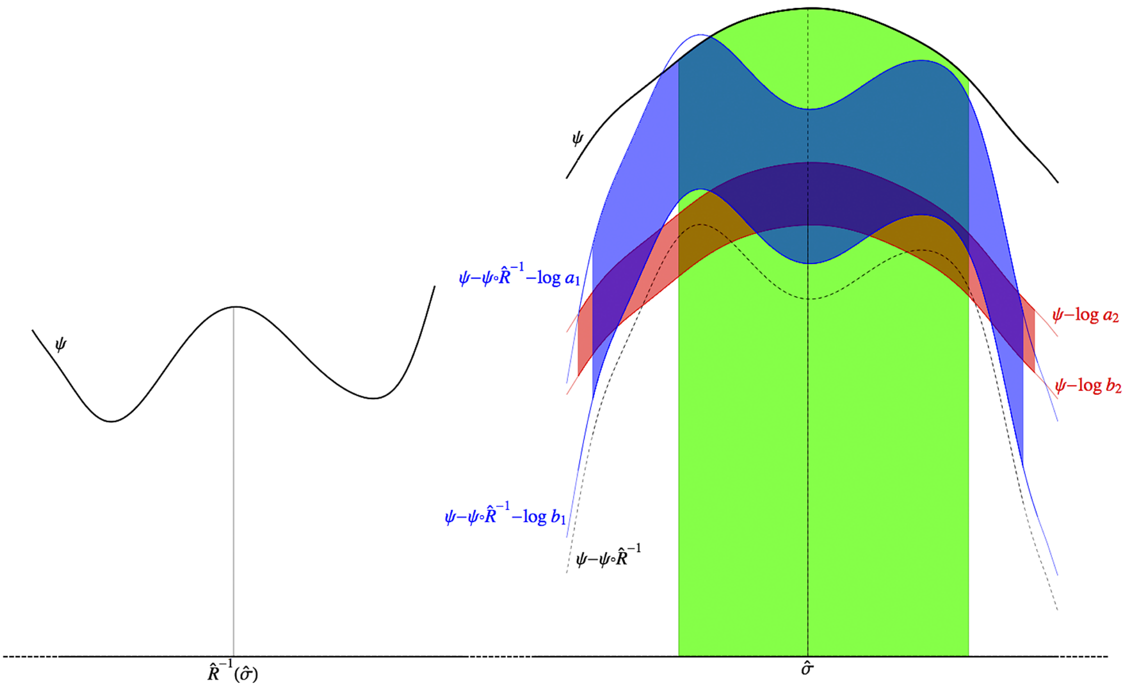

Fig. 3. The region $\bar{D}_{\Phi}(a_1, b_1, a_2, b_2, \underline{c})$ described in the proof of Theorem 4.1 is the intersection of the three shaded regions: $D_{\Phi}(\log a_1 + \psi \circ \hat{R}^{-1}, \log b_1 + \psi \circ \hat{R}^{-1})$, $D_{\Phi}(\log a_2, \log b_2)$ and $p^{-1}\mathcal{C}^{(\underline{c})}_{N_1,N_2}$.

If N is sufficiently large we get

$$\left\{\hat{\sigma} \in \mathcal{C} \setminus U\colon a_1 < \frac{\hat{q}_{\hat{n}_N-1}}{N} < b_1, a_2 < \frac{\hat{q}_{\hat{n}_N}}{N} < b_2\right\}$$
$$= \left\{\hat{\sigma} \in \mathcal{C} \setminus U\colon \log a_1 < \mathrm{S}^{\hat{R}}_{r(\hat{\sigma},\mathrm{T})-1}(\psi)(\hat{\sigma}) - \mathrm{T} + \varepsilon_{N,\mathcal{C}}(\hat{\sigma}) < \log b_1\right\}$$
$$\cap \left\{\hat{\sigma} \in \mathcal{C} \setminus U\colon \log a_2 < \mathrm{S}^{\hat{R}}_{r(\hat{\sigma},\mathrm{T})}(\psi)(\hat{\sigma}) - \mathrm{T} + \varepsilon'_{N,\mathcal{C}}(\hat{\sigma}) < \log b_2\right\},$$

where $\varepsilon_{N,\mathcal{C}}(\hat{\sigma}) := \varepsilon_{\hat{n}_N(\hat{\sigma})-1}(\hat{\sigma}) - g_{\mathcal{C}} + g(\hat{\omega})$, $\varepsilon'_{N,\mathcal{C}}(\hat{\sigma}) := \varepsilon_{\hat{n}_N(\hat{\sigma})}(\hat{\sigma}) - g_{\mathcal{C}} + g(\hat{\omega})$ and $\varepsilon_{\hat{n}_N(\hat{\sigma})-1}$, $\varepsilon_{\hat{n}_N(\hat{\sigma})}$ are defined in (67). One can show that $\sup_{\hat{\sigma}\in\mathcal{C}\setminus U} |\varepsilon_{N,\mathcal{C}}(\hat{\sigma})| + \sup_{\hat{\sigma}\in\mathcal{C}\setminus U} |\varepsilon'_{N,\mathcal{C}}(\hat{\sigma})| \le C_{25}\varepsilon$ for some $C_{25} > 0$. Notice that $v := \mathrm{S}^{\hat{R}}_{r(\hat{\sigma},\mathrm{T})}(\psi)(\hat{\sigma}) - \mathrm{T}$ is the vertical distance from $\Phi_T(\hat{\sigma}, 0)$ and the roof function $\psi(\hat{R}^{\hat{n}_N(\hat{\sigma})-1}(\hat{\sigma}))$ and therefore $\mathrm{S}^{\hat{R}}_{r(\hat{\sigma},\mathrm{T})-1}(\psi)(\hat{\sigma}) - \mathrm{T} = v - \psi(\hat{R}^{\hat{n}_N(\hat{\sigma})-2}(\hat{\sigma}))$. Using the vertical projection $p\colon D_{\Phi} \to D(\hat{R})$ we write the condition $\hat{R}^{\hat{n}_N(\hat{\sigma})-1}(\hat{\sigma}) \in \mathcal{C}^{(\underline{c})}_{N_1,N_2}$ as $p(\Phi_T(\hat{\sigma}, 0)) \in \mathcal{C}^{(\underline{c})}_{N_1,N_2}$ and setting $B_N(x, y) := \{\hat{\sigma} \in D(\hat{R})\colon x_{\hat{n}_N(\hat{\sigma})-N_1}(\hat{\sigma}) = x, y_{\hat{n}_N(\hat{\sigma})-N_1}(\hat{\sigma}) = y\}$ we get

$$A_{\mathcal{C}\setminus U} \times \{0\} \subseteq \big((\mathcal{C} \setminus U) \times \{0\}\big) \cap \big(B_N(x, y) \times \{0\}\big)$$
$$\cap\, \Phi_{-T}\big(D_{\Phi}\big(\log a_1 + \psi \circ \hat{R}^{-1} - C_{25}\varepsilon, \log b_1 + \psi \circ \hat{R}^{-1} + C_{25}\varepsilon\big)$$
$$\cap\, D_{\Phi}(\log a_2 - C_{25}\varepsilon, \log b_2 + C_{25}) \cap p^{-1}\mathcal{C}^{(\underline{c})}_{N_1,N_2}\big)$$

and

$$A_{\mathcal{C}\setminus U} \times \{0\} \supseteq \big((\mathcal{C} \setminus U) \times \{0\}\big) \cap \big(B_N(x, y) \times \{0\}\big)$$
$$\cap\, \Phi_{-T}\big(D_{\Phi}\big(\log a_1 + \psi \circ \hat{R}^{-1} + C_{25}\varepsilon, \log b_1 + \psi \circ \hat{R}^{-1} - C_{25}\varepsilon\big)$$
$$\cap\, D_{\Phi}(\log a_2 + C_{25}\varepsilon, \log b_2 - C_{25}) \cap p^{-1}\mathcal{C}^{(\underline{c})}_{N_1,N_2}\big).$$

For sufficiently small δ, $0<\delta<\varepsilon$, one can show that

$$A_{\mathcal{C}\setminus U}\times[0,\delta)\subseteq\Phi_{-T}\big(\big(D_\Phi\big(\log a_1+\psi\circ\hat{R}^{-1}-C_{25}\varepsilon-\delta,\log b_1+\psi\circ\hat{R}^{-1}+C_{25}\varepsilon\big)\\ \cap D_\Phi(\log a_2-C_{25}\varepsilon-\delta,\log b_2+C_{25}\varepsilon)\cap p^{-1}\mathcal{C}^{(\underline{c})}_{N_1,N_2}\big)\cup D^\delta_\Phi\big),$$

where $D^\delta_\Phi:=D(\hat{R})\times[0,\delta)$. Thus, recalling that $\mathrm{T}=\mathrm{T}(N)=\log N-g_{\mathcal{C}}$ and setting $W^+_N(\varepsilon,\delta):=\Phi_{-T}(\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c})\cup D^\delta_\Phi)$, where $\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c}):=(D_\Phi(\log a_1+\psi\circ\hat{R}^{-1}-C_{26}\varepsilon,\log b_1+\psi\circ\hat{R}^{-1}+C_{25}\varepsilon)\cap D_\Phi(\log a_2-C_{26}\varepsilon,\log b_2+C_{25}\varepsilon)\cap p^{-1}\mathcal{C}^{(\underline{c})}_{N_1,N_2})$ and $C_{26}=C_{25}+1$, we obtain

$$\delta\cdot\mu_{\hat{R}}(A_{\mathcal{C}\setminus U})\le\mu_\Phi\big(\big((\mathcal{C}\setminus U)\times[0,\delta)\big)\cap\big(B_N(x,y)\times[0,\delta)\big)\cap W^+_N(\varepsilon,\delta)\big). \tag{70}$$

Our goal is to show that, for sufficiently large N, one can $C_{27}\varepsilon$-approximate (for some constant $C_{27}>0$) the left-hand side of (70) with the product of the μ_Φ-measures of the three sets $(\mathcal{C}\setminus U)\times[0,\delta)$, $B_N(x,y)\times[0,\delta)$ and $W^+_N(\varepsilon,\delta)$. First, we can replace $B_N(x,y)\times[0,\delta)$ in (70) by $B'_N(x,y):=B_N(x,y)\times\{(\hat{\sigma},z)\in D_\Phi\colon 0\le z\le\psi(\hat{\sigma})\}$ and write $D_N=D_N(x,y,a_1,b_1,a_2,b_2,\underline{c},\varepsilon,\delta):=B'_N\cap W^+_N(\varepsilon,\delta)=\Phi_{-\mathrm{T}(N)}(E_N)$, where $E_N:=\Phi_{\mathrm{T}(N)}(B'_N)\cap\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c})$.

Let us recall the following classical result by Rényi [22]: let $(\Omega,\mathfrak{B},P)$ be a probability space and let $G,H_N\in\mathfrak{B}$, $N\in\mathbb{N}$, then

$$\lim_{N\to\infty}P(G\cap H_N)\to P(A)\cdot\beta\quad\text{iff}\quad\lim_{N\to\infty}P(H_k\cap H_N)=P(H_k)\cdot\beta\quad\text{for each }k\in\mathbb{N}_0, \tag{71}$$

where $H_0=\Omega$. In our case $\Omega=D_\Phi$, $P=\mu_\Phi$, $A=(\mathcal{C}\setminus U)\times[0,\delta)$ and $H_N=D_N$. We can compute $P(H_k\cap H_N)$ for fixed k as follows

$$\mu_\Phi(D_k\cap D_N)=\mu_\Phi\big(\Phi_{-\mathrm{T}(k)}\big(E_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}(E_N)\big)\big)=\mu_\Phi\big(E_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}(E_N)\big). \tag{72}$$

For every $k\in\mathbb{N}$ we can write E_k as a disjoint union of

$$E^{(\overline{n},\underline{\theta})}_k:=\big\{(\hat{\sigma},y)\in D_\Phi\colon\hat{\sigma}=\hat{R}^{\hat{n}_k(\hat{\sigma}')-N_1}\big(\hat{\sigma}'\big),\hat{n}_k\big(\hat{\sigma}'\big)=\overline{n},\big(\hat{\sigma}'_j\big)^{\overline{n}-N_1}_{j=1}=\underline{\theta}\big\}\cap\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c}),$$

where $\overline{n}\in\mathbb{N}$ and $\underline{\theta}\in\Sigma^{\overline{n}-N_1}$ is such that $x_{\overline{n}-N_1}(\underline{\theta})=x$ and $y_{\overline{n}-N_1}(\underline{\theta})=y$ and we can write (72) as

$$\mu_\Phi\big(E_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}(E_N)\big)=\sum_{\overline{n},\underline{\theta}}\mu_\Phi\big(E^{(\overline{n},\underline{\theta})}_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}(E_N)\big). \tag{73}$$

Each term in the series above is now written as a product

$$\mu_\Phi\big(\Phi_{\mathrm{T}(k)}\big(B'_N\big)|E^{(\overline{n},\underline{\theta})}_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}\big(\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c})\big)\big) \tag{74}$$

$$\cdot\,\mu_\Phi\big(E^{(\overline{n},\underline{\theta})}_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}\big(\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c})\big)\big). \tag{75}$$

We apply the mixing property of the special flow $\{\Phi_t\}$ to the factor (75), getting

$$\mu_\Phi\big(E^{(\overline{n},\underline{\theta})}_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}\big(\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c})\big)\big)\longrightarrow\mu_\Phi\big(E^{(\overline{n},\underline{\theta})}_k\big)\mu_\Phi\big(\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c})\big).$$

as $N\to\infty$. We claim that the factor (74) also has a limit:

$$\lim_{N\to\infty}\mu_\Phi\big(\Phi_{\mathrm{T}(k)}\big(B'_N(x,y)\big)|E^{(\overline{n},\underline{\theta})}_k\cap\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}\big(\bar{D}^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c})\big)\big)=:p_{x,y,\underline{c}}. \tag{76}$$

In order to see this one can analyze geometrically the action of the special flow as follows. The set $E^{(\overline{n},\underline{\theta})}_k$ is fixed and involves a finite number of entries of $\hat{\sigma}^-$ in the base $D(\hat{R})$ and some region in the z-direction. In the $D(\hat{R})$ component, the set $\Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}(D^{\varepsilon,+}_\Phi(a_1,b_1,a_2,b_2,\underline{c}))$ corresponds to setting to $\underline{c}$ the coordinates at from $(\hat{\sigma}_j)^{\hat{n}_N-\overline{n}+N_2}_{j=\hat{n}_N-\overline{n}-N_1+1}$, i.e. in a neighborhood (of fixed size) of the renewal time $\hat{n}_N$. In the z-direction it gives a region which, by mixing, spreads according to the invariant measure μ_Φ as $N \to \infty$. Since the set $\Phi_{\mathrm{T}(k)}(B'_N(x,y))$ gives no restrictions in the z-direction, it is enough to establish the existence of the limit (76) for the projection of the sets onto the base $D(\hat{R})$. In the base, however, the limit follows from the Markov-like property of the process $\{(x_n, y_n)\}_{n\in\mathbb{N}} \in \Xi^{\mathbb{N}}$ (namely extending (56) to conditional probability distributions). Now taking the limit in (73) we get

$$\lim_{N\to\infty} \mu_\Phi\big(E_k \cap \Phi_{-(\mathrm{T}(N)-\mathrm{T}(k))}(E_N)\big) = p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi^{\varepsilon,+}(a_1,b_1,a_2,b_2,\underline{c})\big) \sum_{\overline{n},\underline{\theta}} \mu_\Phi\big(E_k^{(\overline{n},\underline{\theta})}\big)$$
$$= p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi^{\varepsilon,+}(a_1,b_1,a_2,b_2,\underline{c})\big) \cdot \mu_\Phi(E_k),$$

i.e. the rightmost part of (71) with $\beta = p_{x,y,\underline{c}} \cdot \mu_\Phi(\bar{D}_\Phi^{\varepsilon,+}(a_1,b_1,a_2,b_2,\underline{c}))$. Thus we proved that

$$\begin{aligned}
&\lim_{N\to\infty} \mu_\Phi\big(\big((\mathcal{C}\setminus U)\times[0,\delta)\big) \cap \big(B_N(x,y)\times[0,\delta)\big) \cap W_N^+(\varepsilon,\delta)\big) \\
&\quad = \mu_\Phi\big((\mathcal{C}\setminus U)\times[0,\delta)\big) \cdot p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi^{\varepsilon,+}(a_1,b_1,a_2,b_2,\underline{c})\big) \\
&\quad = \delta \cdot \mu_{\hat{R}}(\mathcal{C}\setminus U) \cdot p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi^{\varepsilon,+}(a_1,b_1,a_2,b_2,\underline{c})\big).
\end{aligned} \tag{77}$$

Now (70) and (77) imply that, for sufficiently large N,

$$\delta \cdot \mu_{\hat{R}}(A_{\mathcal{C}\setminus U}) \le \delta \cdot \mu_{\hat{R}}(\mathcal{C}\setminus U) \cdot \big(p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi^{\varepsilon,+}(a_1,b_1,a_2,b_2,\underline{c})\big) + C_{27}\varepsilon\big) \tag{78}$$

for some $C_{27} > 0$. Proceeding as in [4] (Lemma 3.8 therein) one can show that, for sufficiently small δ,

$$\big((\mathcal{C}\setminus U)\times[0,\delta)\big) \cap \Phi_{-\mathrm{T}}\big(\bar{D}_\Phi^{\varepsilon,-}(a_1,b_1,a_2,b_2,\underline{c}) \setminus D_\Phi^\delta\big) \subseteq A_{\mathcal{C}\setminus U} \times [0,\delta),$$

where $\bar{D}_\Phi^{\varepsilon,-} = D_\Phi(\log a_1 + \psi\circ\hat{R}^{-1} + C_{28}\varepsilon, \log b_1 + \psi\circ\hat{R}^{-1} - C_{29}\varepsilon) \cap D_\Phi(\log a_2 + C_{28}\varepsilon, \log b_2 - C_{29}\varepsilon) \cap p^{-1}\mathcal{C}_{N_1,N_2}^{(\underline{c})}$, for some $C_{28}, C_{29} > 0$. Using the mixing property of the flow $\{\Phi_t\}_t$ as above we get, for sufficiently large N,

$$\delta \cdot \mu_{\hat{R}}(A_{\mathcal{C}\setminus U}) \ge \delta \cdot \mu_{\hat{R}}(\mathcal{C}\setminus U) \cdot \big(p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi^{\varepsilon,-}(a_1,b_1,a_2,b_2,\underline{c})\big) - C_{30}\varepsilon\big) \tag{79}$$

for some $C_{30} > 0$. Moreover, by Fubini's theorem, for some $C_{31} > 0$,

$$\big|\mu_\Phi\big(\bar{D}_\Phi^{\varepsilon,\pm}(a_1,b_1,a_2,b_2,\underline{c})\big) - p_{x,y,\underline{c}} \cdot \mu_\Phi\big(\bar{D}_\Phi(a_1,b_1,a_2,b_2,\underline{c})\big)\big| \le C_{31}\varepsilon. \tag{80}$$

Finally, by (78)–(80) we get (69) for some $C_{24} > 0$ and this completes the proof of Theorem 4.1. □

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