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On the global maximum of the solution to a stochastic heat equation with compact-support initial data[1]

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Abstract. Consider a stochastic heat equation $\partial_t u = \kappa\, \partial^2_{xx} u + \sigma(u)\dot{w}$ for a space–time white noise $\dot{w}$ and a constant $\kappa > 0$. Under some suitable conditions on the initial function u_0 and σ, we show that the quantities

$$\limsup_{t\to\infty} t^{-1} \sup_{x\in\mathbf{R}} \ln \mathrm{E}\big(|u_t(x)|^2\big) \quad \text{and} \quad \limsup_{t\to\infty} t^{-1} \ln \mathrm{E}\Big(\sup_{x\in\mathbf{R}} |u_t(x)|^2\Big)$$

are equal, as well as bounded away from zero and infinity by explicit multiples of $1/\kappa$. Our proof works by demonstrating quantitatively that the peaks of the stochastic process $x \mapsto u_t(x)$ are highly concentrated for infinitely-many large values of t. In the special case of the parabolic Anderson model – where $\sigma(u) = \lambda u$ for some $\lambda > 0$ – this "peaking" is a way to make precise the notion of physical intermittency.

Résumé. Nous considérons l'équation de la chaleur stochastique $\partial_t u = \kappa \partial^2_{xx} u + \sigma(u)\dot{w}$ avec un bruit blanc spatio-temporel $\dot{w}$ et une constante $\kappa > 0$. Sous des conditions adéquates sur la condition initiale u_0 et sur σ, nous montrons que les quantités

$$\limsup_{t\to\infty} t^{-1} \sup_{x\in\mathbf{R}} \ln \mathrm{E}\big(|u_t(x)|^2\big) \quad \text{et} \quad \limsup_{t\to\infty} t^{-1} \ln \mathrm{E}\Big(\sup_{x\in\mathbf{R}} |u_t(x)|^2\Big)$$

sont égales. Par ailleurs, nous les bornons inférieurement et supérieurement par des constantes strictement positives et finies dépendant explicitement de $1/\kappa$. Nos démonstrations reposent sur la preuve quantitative de la forte concentration des pics du processus $x \mapsto u_t(x)$ pour de grandes valeurs de t infiniment nombreuses. Dans le cas particulier du modèle d'Anderson parabolique-où $\sigma(u) = \lambda u$ pour un $\lambda > 0$ – ce phénomène de pics est une façon de préciser la notion physique d'intermittence.

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1. Introduction

We consider the stochastic heat equation,

$$\frac{\partial u_t(x)}{\partial t} = \kappa \frac{\partial^2 u_t(x)}{\partial x^2} + \sigma\big(u_t(x)\big)\dot{w}(t,x) \quad \text{for } t > 0 \text{ and } x \in \mathbf{R}, \tag{1.1}$$

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where $\kappa > 0$ is fixed, $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous with $\sigma(0) = 0$, $\dot{w}$ denotes space–time white noise, and the initial data $u_0 : \mathbf{R} \to \mathbf{R}$ is nonrandom. There are several areas to which (1.1) has deep and natural connections; perhaps chief among them are the stochastic Burgers' equation [10] and the celebrated KPZ equation of statistical mechanics [11,12]; see also [13], Chapter 9.

It is well known that (1.1) has an almost-surely unique, adapted and continuous solution $\{u_t(x)\}_{t\geq 0, x\in\mathbf{R}}$ ([5], Theorem 6.4, p. 26). In addition, the condition that $\sigma(0) = 0$ implies that if $u_0 \in L^2(\mathbf{R})$, then $u_t \in L^2(\mathbf{R})$ a.s. for all $t \geq 0$; see Dalang and Mueller [6]. Note that our conditions on σ ensure that

$$\left|\sigma(u)\right| \leq \mathrm{Lip}_\sigma |u| \quad \text{for all } u \in \mathbf{R}, \tag{1.2}$$

where

$$\mathrm{Lip}_\sigma := \sup_{-\infty < x < x' < \infty} \left| \frac{\sigma(x) - \sigma(x')}{x - x'} \right|. \tag{1.3}$$

Our goal is to establish the following general growth estimate.

Theorem 1.1. *Suppose there exists* $\mathrm{L}_\sigma \in (0, \infty)$ *such that* $|\sigma(u)| \geq \mathrm{L}_\sigma |u|$ *for all* $u \in \mathbf{R}$. *Suppose also that* $u_0 \not\equiv 0$ *is Hölder-continuous of order* $\geq 1/2$, *nonnegative, and supported in* $[-K, K]$ *for some finite* $K > 0$. *Then,* (1.1) *has an almost-surely unique, continuous and adapted solution* $\{u_t(x)\}_{t\geq 0, x\in\mathbf{R}}$ *such that* $u_t \in L^2(\mathbf{R})$ *a.s. for all* $t \geq 0$, *and*

$$\frac{\mathrm{L}_\sigma^4}{8\kappa} \leq \limsup_{t\to\infty} t^{-1} \sup_{x\in\mathbf{R}} \ln \mathrm{E}\left(\left|u_t(x)\right|^2\right) = \limsup_{t\to\infty} t^{-1} \ln \mathrm{E}\left(\sup_{x\in\mathbf{R}} \left|u_t(x)\right|^2\right) \leq \frac{\mathrm{Lip}_\sigma^4}{8\kappa}.$$

Because of Mueller's comparison principle [14] (see also [7,16]), the nonnegativity of u_0 implies that $\sup_{t,x} \mathrm{E}(|u_t(x)|) = \sup_{t,x} \mathrm{E}(u_t(x))$, and this quantity has to be finite because u_0 is bounded; confer with (1.5). Consequently,

$$\sup_{x\in\mathbf{R}} \left\|u_t(x)\right\|_{L^1(\mathrm{P})} \ll \sup_{x\in\mathbf{R}} \left\|u_t(x)\right\|_{L^2(\mathrm{P})} \quad \text{as } t \to \infty. \tag{1.4}$$

When $\mathrm{Lip}_\sigma = \mathrm{L}_\sigma$, (1.1) becomes the well-studied parabolic Anderson model [1,3]. And (1.4) makes precise the physical notion that the solution to (1.1) concentrates near "very high peaks" [1,3,11,12].

In order to explain the idea behind our proof, we introduce the following.

Definition 1.2. *We say that a continuous random field* $f := \{f(t, x)\}_{t\geq 0, x\in\mathbf{R}}$ *has* effectively-compact support [*in the spatial variable* x] *if there exists a nonrandom measurable function* $p : \mathbf{R}_+ \to \mathbf{R}_+$ *of at-most polynomial growth such that*:

(a) $\limsup_{t\to\infty} t^{-1} \ln \int_{|x|\leq p(t)} \mathrm{E}(|f(t, x)|^2)\,\mathrm{d}x > 0$ *and*
(b) $\limsup_{t\to\infty} t^{-1} \ln \int_{|x|> p(t)} \mathrm{E}(|f(t, x)|^2)\,\mathrm{d}x < 0$.

We might refer to the function p *as the* radius of effective support *of* f.

One of the ideas here is to use Mueller's comparison principle [14] to compare $\sup_{x\in\mathbf{R}} |u_t(x)|$ with the $L^2(\mathbf{R})$-norm of $x \mapsto u_t(x)$, which is easier to analyze. We carry these steps out in Lemma 3.3. We also appeal to the fact that the compact-support property of u_0 implies that $u_t(x)$ has an effectively-compact support [Proposition 3.7]. This can be interpreted as a kind of optimal regularity theorem. However, these matters need to be handled delicately, as "effectively compact" cannot be replaced by "compact"; see Mueller [14].

Our method for establishing an effectively-compact support property is motivated strongly by ideas of Mueller and Perkins [15]. In the cases that $u_t(x)$ denotes the density of some particles at x at time t, our effectively-compact support property implies that most of the particles accumulate on a very small set. This method might appeal to the reader who is interested in mathematical descriptions of physical intermittency.

Throughout this paper we use the mild formulation of the solution, in accordance with Walsh [17]. That is, u is the a.s.-unique adapted solution to

$$u_t(x) = (p_t * u_0)(x) + \int_0^t \int_{-\infty}^{\infty} p_{t-s}(y-x)\sigma\big(u_s(y)\big) w(\mathrm{d}s\,\mathrm{d}y), \tag{1.5}$$

where $p_\tau(z) := (4\kappa\tau\pi)^{-1/2}\exp(-z^2/(4\kappa\tau))$ denotes the heat kernel corresponding to the operator $\kappa\,\partial^2/\partial x^2$, and the stochastic integral is understood in the sense of Walsh [17]. Some times we write $\|X\|_p$ in place of $\{\mathrm{E}(|X|^p)\}^{1/p}$.

2. A preliminary result

As mentioned in the Introduction, the strategy behind our proof of Theorem 1.1 is to relate the global maximum of the solution to a "closed-form quantity" that resembles $\sup_x |u_t(x)|$ for large values of t. That closed-form quantity turns out to be the $L^2(\mathbf{R})$-norm of $x \mapsto u_t(x)$. Our next result analyses the growth of the mentioned closed-form quantity. We related it to $\sup_x |u_t(x)|$ in the next section. The methods of this section follow closely the classical ideas of Choquet and Deny [4] that were developed in a determinstic setting.

Theorem 2.1. *Suppose $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous, $\sigma(0) = 0$, and there exists $\mathrm{L}_\sigma \in (0,\infty)$ such that $\mathrm{L}_\sigma|u| \le |\sigma(u)|$ for all $u \in \mathbf{R}$. If $u_0 \in L^2(\mathbf{R})$ and $u_0 \not\equiv 0$, then (1.1) has an almost-surely unique, continuous and adapted solution $\{u_t(x)\}_{t\ge 0, x\in\mathbf{R}}$ such that $u_t \in L^2(\mathbf{R})$ a.s. for all $t \ge 0$, and*

$$\frac{\mathrm{L}_\sigma^4}{8\kappa} \le \limsup_{t\to\infty} t^{-1} \ln \mathrm{E}\big(\|u_t\|_{L^2(\mathbf{R})}^2\big) \le \frac{\mathrm{Lip}_\sigma^4}{8\kappa}. \tag{2.1}$$

Proof. It suffices to establish (2.1). Note that

$$\mathrm{E}\big(|u_t(x)|^2\big) = \big|(p_t * u_0)(x)\big|^2 + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{E}\big(\big|\sigma\big(u_s(y)\big)\big|^2\big) \cdot \big|p_{t-s}(y-x)\big|^2. \tag{2.2}$$

Because $|\sigma(u)| \ge \mathrm{L}_\sigma|u|$,

$$\begin{aligned}\mathrm{E}\big(\|u_t\|_{L^2(\mathbf{R})}^2\big) &= \|p_t * u_0\|_{L^2(\mathbf{R})}^2 + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{E}\big(\big|\sigma\big(u_s(y)\big)\big|^2\big) \cdot \|p_{t-s}\|_{L^2(\mathbf{R})}^2 \\ &\ge \|p_t * u_0\|_{L^2(\mathbf{R})}^2 + \mathrm{L}_\sigma^2 \cdot \int_0^t \mathrm{E}\big(\|u_s\|_{L^2(\mathbf{R})}^2\big) \cdot \|p_{t-s}\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s.\end{aligned} \tag{2.3}$$

We can multiply the preceding by $\exp(-\lambda t)$ throughout and integrate $[\mathrm{d}t]$ to find that if

$$U(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{E}\big(\|u_t\|_{L^2(\mathbf{R})}^2\big)\,\mathrm{d}t, \tag{2.4}$$

then

$$U(\lambda) \ge \int_0^\infty \mathrm{e}^{-\lambda t}\|p_t * u_0\|_{L^2(\mathbf{R})}^2\,\mathrm{d}t + \mathrm{L}_\sigma^2 \cdot U(\lambda) \cdot \int_0^\infty \mathrm{e}^{-\lambda t}\|p_t\|_{L^2(\mathbf{R})}^2\,\mathrm{d}t. \tag{2.5}$$

According to Plancherel's theorem, the following holds for all finite Borel measures μ on $\mathbf{R}$:

$$\|p_t * \mu\|_{L^2(\mathbf{R})}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|\hat\mu(\xi)\big|^2 \mathrm{e}^{-2\kappa t\xi^2}\,\mathrm{d}\xi. \tag{2.6}$$

Therefore, Tonelli's theorem ensures that

$$\int_0^\infty \mathrm{e}^{-\lambda t}\|p_t * \mu\|_{L^2(\mathbf{R})}^2\,\mathrm{d}t = \frac{1}{2\pi}\int_0^\infty \frac{|\hat\mu(\xi)|^2}{\lambda + 2\kappa\xi^2}\,\mathrm{d}\xi. \tag{2.7}$$

We apply this identity twice in (2.3): Once with $\mu := \delta_0$; and once with $\mathrm{d}\mu/\mathrm{d}x := u_0$. This leads us to the following.

$$U(\lambda) \geq \frac{1}{2\pi}\int_0^\infty \frac{|\hat{u}_0(\xi)|^2}{\lambda + 2\kappa\xi^2}\,\mathrm{d}\xi + \mathrm{L}_\sigma^2 \cdot U(\lambda)\cdot \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}\xi}{\lambda+2\kappa\xi^2}$$
$$= \frac{1}{2\pi}\int_0^\infty \frac{|\hat{u}_0(\xi)|^2}{\lambda + 2\kappa\xi^2}\,\mathrm{d}\xi + \mathrm{L}_\sigma^2 \cdot U(\lambda)\cdot \frac{1}{2\sqrt{2\kappa\lambda}}. \tag{2.8}$$

Since $u_0 \not\equiv 0$, the first [Fourier] integral is strictly positive. Consequently, the above recursive relation shows that $U(\lambda) = \infty$ if $\lambda \leq \mathrm{L}_\sigma^4/(8\kappa)$. This and a real-variable argument together imply the first inequality in (2.1). Indeed, we follow the argument in [8] in this way: Suppose, to the contrary, that the first inequality in (2.1) failed. This means that for all $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $t > t_0$,

$$\mathrm{E}\big(\|u_t\|^2_{L^2(\mathbf{R})}\big) \leq \exp\left(t\left\{\frac{\mathrm{L}_\sigma^4}{8\kappa} - \varepsilon\right\}\right). \tag{2.9}$$

We multiply this by $\exp(-\lambda t)$ and integrate $[\mathrm{d}t]$ to deduce that $U(\lambda) < \infty$ for all $\lambda > \mathrm{L}_\sigma^4/(8\kappa) - \varepsilon$. And this contradicts the earlier finding that $U(\lambda) = \infty$ for all $\lambda \leq \mathrm{L}_\sigma^4/(8\kappa)$.

For the other bound we use a Picard-iteration argument in order to obtain an a priori estimate. Let $u_t^{(0)}(x) := u_0(x)$ and iteratively define

$$u_t^{(n+1)}(x) := (p_t * u_0)(x) + \int_0^t \int_{-\infty}^\infty p_{t-s}(y-x)\sigma\big(u_s^{(n)}(y)\big)w(\mathrm{d}s\,\mathrm{d}y). \tag{2.10}$$

Since $\|p_t * u_0\|_{L^2(\mathbf{R})} \leq \|u_0\|_{L^2(\mathbf{R})}$ and $|\sigma(u)| \leq \mathrm{Lip}_\sigma|u|$, Hölder's inequality yields

$$\mathrm{E}\big(\big\|u_t^{(n+1)}\big\|^2_{L^2(\mathbf{R})}\big) \leq \|u_0\|^2_{L^2(\mathbf{R})} + \mathrm{Lip}_\sigma^2 \cdot \int_0^t \mathrm{E}\big(\big\|u_s^{(n)}\big\|^2_{L^2(\mathbf{R})}\big)\cdot \|p_{t-s}\|^2_{L^2(\mathbf{R})}\,\mathrm{d}s. \tag{2.11}$$

Therefore, if we set

$$M^{(k)}(\lambda) := \sup_{t\geq 0}\big[\mathrm{e}^{-\lambda t}\mathrm{E}\big(\big\|u_t^{(k)}\big\|^2_{L^2(\mathbf{R})}\big)\big], \tag{2.12}$$

then it follows that

$$M^{(n+1)}(\lambda) \leq \|u_0\|^2_{L^2(\mathbf{R})} + \mathrm{Lip}_\sigma^2 \cdot M^{(n)}(\lambda)\cdot \int_0^\infty \mathrm{e}^{-\lambda(t-s)}\|p_{t-s}\|^2_{L^2(\mathbf{R})}\,\mathrm{d}s$$
$$= \|u_0\|^2_{L^2(\mathbf{R})} + \frac{\mathrm{Lip}_\sigma^2}{2\sqrt{2\kappa\lambda}}M^{(n)}(\lambda). \tag{2.13}$$

Thus, in particular, $\sup_{n\geq 0} M^{(n)}(\lambda) < \infty$ if $\lambda > \mathrm{Lip}_\sigma^4/(8\kappa)$. We can argue similarly to show also that if $\lambda > \mathrm{Lip}_\sigma^4/(8\kappa)$, then

$$\sum_n \sup_{t\geq 0}\big[\mathrm{e}^{-\lambda t}\mathrm{E}\big(\big\|u_t^{(n+1)} - u_t^{(n)}\big\|^2_{L^2(\mathbf{R})}\big)\big]^{1/2} < \infty. \tag{2.14}$$

In particular, uniqueness shows that if $\lambda > \mathrm{Lip}_\sigma^4/(8\kappa)$, then

$$\lim_{n\to\infty}\sup_{t\geq 0}\big[\mathrm{e}^{-\lambda t}\mathrm{E}\big(\big\|u_t^{(n)} - u_t\big\|^2_{L^2(\mathbf{R})}\big)\big] = 0. \tag{2.15}$$

Consequently, if $\lambda > \mathrm{Lip}_\sigma^4/(8\kappa)$, then

$$\sup_{t\geq 0}\big[\mathrm{e}^{-\lambda t}\mathrm{E}\big(\|u_t\|^2_{L^2(\mathbf{R})}\big)\big] = \lim_{n\to\infty} M^{(n)}(\lambda) \leq \sup_{k\geq 0} M^{(k)}(\lambda) < \infty. \tag{2.16}$$

The second inequality of (2.1) follows readily from this bound. □

3. Proof of Theorem 1.1

Our proof of Theorem 1.1 hinges on a number of steps, which we develop separately. First we recall the following.

Proposition 3.1 (Theorem 2.1 and Example 2.9 of [8]). *If u_0 is bounded and measurable, then $u_t(x) \in L^p(\mathrm{P})$ for all $p \in [1,\infty)$. Moreover, $\overline{\gamma}(p) < \infty$ for all $p \in [1,\infty)$ and $\overline{\gamma}(2) \le \mathrm{Lip}_\sigma^4/(8\kappa)$, where*

$$\overline{\gamma}(p) := \limsup_{t\to\infty} t^{-1} \sup_{x\in\mathbf{R}} \ln \mathrm{E}\big(\big|u_t(x)\big|^p\big) < \infty. \tag{3.1}$$

[Note that, in the preceding, the supremum is outside the expectation.]

Next, we record a simple though crucial property of the function $\overline{\gamma}$.

Remark 3.2. *Suppose X is a nonnegative random variable with finite moments of all orders. By Hölder's inequality, $p \mapsto \ln \mathrm{E}(X^p)$ is convex on $[1,\infty)$. It follows that $\overline{\gamma}$ is convex – in particular continuous – on $[1,\infty)$.*

Now we begin our analysis, in earnest, by deriving an upper bound on the $L^k(\mathrm{P})$-norm of the solution $u_t(x)$ that includes simultaneously a sharp decay rate in x and a sharp explosion rate in t.

Lemma 3.3. *Suppose that $u_0 \not\equiv 0$, and u_0 is supported in $[-K,K]$ for some finite constant $K > 0$. Then, for all real numbers $k \in [1,\infty)$ and $p \in (1,\infty)$,*

$$\limsup_{t\to\infty} t^{-1} \sup_{x\in\mathbf{R}} \left(\frac{x^2}{4t^2} + \frac{k+1-(1/p)}{k} \ln \mathrm{E}\big(\big|u_t(x)\big|^k\big)\right) \le \frac{\overline{\gamma}(kp)}{p}. \tag{3.2}$$

Proof. According to Mueller's comparison principle ([14]; more specifically, see [5], Theorem 5.1, p. 130; see also [7,16]), the solution to (1.1) has the following nonnegativity property: Because $u_0 \ge 0$ then outside a single null set, $u_t \ge 0$ for all $t \ge 0$. Since $u_t(x) \in L^2(\mathrm{P})$ [e.g., by Proposition 3.1], the stochastic integral in (1.5) is a martingale-measure stochastic integral in $L^2(\mathrm{P})$ [say], and consequently has mean zero. And therefore,

$$\big\|u_t(x)\big\|_1 = (p_t * u_0)(x) = \frac{1}{\sqrt{4\kappa\pi t}} \int_{-K}^{K} \mathrm{e}^{-(x-y)^2/(4\kappa t)} u_0(y)\,\mathrm{d}y. \tag{3.3}$$

Because $(x-y)^2 \ge (x^2/2) - K^2$,

$$\big\|u_t(x)\big\|_1 \le \mathrm{const}\cdot \mathrm{e}^{-x^2/(4t)} \quad \text{for all } x \in \mathbf{R} \text{ and } t \ge 1. \tag{3.4}$$

The constant appearing in the above display depends on K. Next we note that for every $\theta \in (0,\infty)$,

$$\begin{aligned} \mathrm{E}\big(\big|u_t(x)\big|^k\big) &\le \theta^k + \mathrm{E}\big(\big|u_t(x)\big|^k; u_t(x) \ge \theta\big) \\ &\le \theta^k + \big(\mathrm{E}\big(\big|u_t(x)\big|^{kp}\big)\big)^{1/p} \cdot \big(\mathrm{P}\big\{u_t(x) > \theta\big\}\big)^{1-(1/p)}. \end{aligned} \tag{3.5}$$

Proposition 3.1 implies that

$$\sup_{x\in\mathbf{R}} \big(\mathrm{E}\big(\big|u_t(x)\big|^{kp}\big)\big)^{1/p} \le \exp\left(t\,\frac{\overline{\gamma}(kp) + \mathrm{o}(1)}{p}\right), \tag{3.6}$$

where $\mathrm{o}(1) \to 0$ as $t \to \infty$. Also, we can apply (3.4) together with the Chebyshev inequality to find that

$$\big(\mathrm{P}\big\{u_t(x) > \theta\big\}\big)^{1-(1/p)} \le \mathrm{const}\cdot\theta^{-1+(1/p)} \exp\left(-\frac{x^2}{4t}\cdot\left[1-\frac{1}{p}\right]\right). \tag{3.7}$$

In light of (3.6) and (3.7), we can deduce that the following from (3.5):

$$\mathrm{E}\left(\left|u_t(x)\right|^k\right) \le \inf_{\theta>0}\left(\theta^k + \alpha\theta^{-1+(1/p)}\right), \tag{3.8}$$

where

$$\alpha := \exp\left(-\frac{x^2}{4t}\cdot\left[1-\frac{1}{p}\right] + t\,\frac{\overline{\gamma}(kp)+\mathrm{o}(1)}{p}\right). \tag{3.9}$$

Some calculus shows that the function $g(\theta) := (\theta^k + \alpha\theta^{-1+1/p})\mathbf{1}_{(0,\infty)}(\theta)$ attains its minimum at $\theta := ((p-1)/kp)^{p/(kp+p-1)}$. Consequently,

$$\mathrm{E}\left(\left|u_t(x)\right|^k\right) \le \alpha^{kp/(kp+p-1)}\left(\frac{p-1}{kp}\right)^{kp/(kp+p-1)}\cdot\left(\frac{1-p-kp}{1-p}\right). \tag{3.10}$$

We now divide both sides of the above display by $\alpha^{kp/(kp+p-1)}$ and take the appropriate limit to obtain the result. □

Our next lemma is a basic estimate of continuity in the variable x. It is not entirely standard as it holds uniformly for all times $t \ge 0$. We emphasize that the constant p is assumed to be an integer. We will deal with this shortcoming subsequently.

Lemma 3.4. *Suppose that the initial function u_0 is Hölder continuous of order $\ge 1/2$. Then, for all integers $p \ge 1$ and $\beta > \overline{\gamma}(2p)$ there exists a constant $A_{p,\beta} \in (0,\infty)$ such that the following holds: Simultaneously for all $t \ge 0$,*

$$\sup_{j\in\mathbf{Z}}\sup_{j\le x<x'\le j+1}\left\|\frac{u_t(x)-u_t(x')}{|x-x'|^{1/2}}\right\|_{2p} \le A_{p,\beta}\mathrm{e}^{\beta t/(2p)}. \tag{3.11}$$

Proof. Burkholder's inequality [2] and Minkowski's inequality together imply that

$$\begin{aligned}
\left\|u_t(x)-u_t(x')\right\|_{2p} &\le \left|(p_t*u_0)(x)-(p_t*u_0)(x')\right| \\
&\quad + z_{2p}\left\|\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\left|\sigma\left(u_s(y)\right)\right|^2\cdot\left|p_{t-s}(y-x)-p_{t-s}(y-x')\right|^2\right\|_p^{1/2} \\
&\le \left|(p_t*u_0)(x)-(p_t*u_0)(x')\right| \\
&\quad + z'_{2p}\left\|\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\left|u_s(y)\right|^2\cdot\left|p_{t-s}(y-x)-p_{t-s}(y-x')\right|^2\right\|_p^{1/2},
\end{aligned} \tag{3.12}$$

where z_p is a positive and finite constant that depend only on p, and $z'_p := z_p\,\mathrm{Lip}_\sigma$.

On one hand,

$$\begin{aligned}
\sup_{t\ge0}\sup_{|x-x'|\le\delta}\left|(p_t*u_0)(x)-(p_t*u_0)(x')\right| &\le \sup_{|a-b|\le\delta}\left|u_0(a)-u_0(b)\right| \\
&\le \mathrm{const}\cdot\delta^{1/2}.
\end{aligned} \tag{3.13}$$

On the other hand, the generalized Hölder inequality suggests that if $p \ge 1$ is an integer, then for all $s_1,\ldots,s_p \ge 0$ and $y_1,\ldots,y_p \in \mathbf{R}$,

$$\mathrm{E}\left(\prod_{j=1}^p\left|u_{s_j}(y_j)\right|^2\right) \le \prod_{j=1}^p\left\|u_{s_j}(y_j)\right\|_{2p}^2. \tag{3.14}$$

[It might help to recall that the generalized Hölder inequality states that $\mathrm{E}(\zeta_1\cdots\zeta_p)\le\prod_{j=1}^p\|\zeta_j\|_p$ for all nonnegative random variables $\zeta_1,\dots,\zeta_p$.] Therefore,

$$\left\|\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\left|u_s(y)\right|^2\cdot\left|p_{t-s}(y-x)-p_{t-s}\big(y-x'\big)\right|^2\right\|_p \le \int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\left\|u_s(y)\right\|_{2p}^2\cdot\left|p_{t-s}(y-x)-p_{t-s}\big(y-x'\big)\right|^2. \tag{3.15}$$

[Write the pth power of the left-hand side as the expectation of a product and apply (3.14).]

A proof by contradiction shows that Proposition 3.1 gives the following [see [8] for more details]:

$$c_\beta:=\sup_{s\ge0}\sup_{y\in\mathbf{R}}\left[\mathrm{e}^{-\beta s}\mathrm{E}\left(\left|u_s(y)\right|^{2p}\right)\right]<\infty\quad\text{for all }\beta>\overline{\gamma}(2p). \tag{3.16}$$

We omit the details, but state instead that the argument is quite similar to the real-variable method that was employed earlier, in the paragraph that preceeds (2.9).

Consequently,

$$\begin{aligned}&\left\|\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\left|u_s(y)\right|^2\cdot\left|p_{t-s}(y-x)-p_{t-s}\big(y-x'\big)\right|^2\right\|_p\\&\quad\le c_\beta^{1/p}\cdot\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{\beta s/p}\cdot\left|p_{t-s}(y-x)-p_{t-s}\big(y-x'\big)\right|^2\\&\quad\le c_\beta^{1/p}\mathrm{e}^{\beta t/p}\cdot\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\beta s/p}\int_{-\infty}^{\infty}\mathrm{d}y\,\left|p_s(y-x)-p_s\big(y-x'\big)\right|^2.\end{aligned} \tag{3.17}$$

Since $\hat{p}_s(\xi)=\exp(-\kappa s\xi^2)$, Plancherel's theorem tells us that the right-hand side of the preceding inequality is equal to

$$\begin{aligned}&\frac{c_\beta^{1/p}\mathrm{e}^{\beta t/p}}{\pi}\cdot\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\beta s/p}\int_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{e}^{-2\kappa s\xi^2}\left[1-\cos\big(\xi\big(x-x'\big)\big)\right]\\&\quad=\frac{2c_\beta^{1/p}\mathrm{e}^{\beta t/p}}{\pi}\cdot\int_0^\infty\frac{[1-\cos(\xi(x-x'))]}{(\beta/p)+2\kappa\xi^2}\,\mathrm{d}\xi.\end{aligned} \tag{3.18}$$

Because $1-\cos\theta\le\min(1,\theta^2)$, a direct estimation of the integral leads to the following bound:

$$\left\|\int_0^t \mathrm{d}s\int_{-\infty}^{\infty}\mathrm{d}y\,\left|u_s(y)\right|^2\cdot\left|p_{t-s}(y-x)-p_{t-s}\big(y-x'\big)\right|^2\right\|_p \le \text{const}\cdot\mathrm{e}^{\beta t/p}\cdot\left|x-x'\right|, \tag{3.19}$$

where the implied constant depends only on p, κ, and β. This, (3.13), and (3.12) together imply the lemma. □

The preceding lemma holds for all *integers* $p\ge1$. In the following, we improve it [at a slight cost] to the case that $p\in(1,2)$ is a real number.

Lemma 3.5. *Suppose the conditions of Lemma 3.4 are met. Then for all $p\in(1,2)$ and $\delta\in(0,1)$ there exists a constant $B_{p,\delta}\in(0,\infty)$ such that the following holds: Simultaneously for all $t\ge0$ and $x,x'\in\mathbf{R}$ with $|x-x'|\le1$,*

$$\mathrm{E}\left(\left|u_t(x)-u_t\big(x'\big)\right|^{2p}\right)\le B_{p,\delta}\cdot\left|x-x'\right|^p\cdot\mathrm{e}^{(1+\delta)\lambda_p t}, \tag{3.20}$$

where

$$\lambda_p := (2-p)\overline{\gamma}(2) + (p-1)\overline{\gamma}(4). \tag{3.21}$$

Proof. We start by writing $\mathrm{E}(|u_t(x) - u_t(x')|^{2p})$ as

$$\mathrm{E}\big(\big|u_t(x) - u_t\big(x'\big)\big|^{2(2-p)}\big|u_t(x) - u_t\big(x'\big)\big|^{4(p-1)}\big). \tag{3.22}$$

We can apply Hölder's inequality to conclude that for all $p \in (1,2)$, $t \geq 0$, and $x, x' \in \mathbf{R}$,

$$\mathrm{E}\big(\big|u_t(x) - u_t\big(x'\big)\big|^{2p}\big) \leq \big[\mathrm{E}\big(\big|u_t(x) - u_t\big(x'\big)\big|^2\big)\big]^{2-p}\big[\mathrm{E}\big(\big|u_t(x) - u_t\big(x'\big)\big|^4\big)\big]^{p-1}. \tag{3.23}$$

We now use Lemma 3.4 to obtain the following:

$$\big[\mathrm{E}\big(\big|u_t(x) - u_t\big(x'\big)\big|^2\big)\big]^{2-p} \leq \big|x - x'\big|^{(2-p)} A_{1,\beta_1}^{2(2-p)} \mathrm{e}^{\beta_1(2-p)t} \tag{3.24}$$

and

$$\big[\mathrm{E}\big(\big|u_t(x) - u_t\big(x'\big)\big|^4\big)\big]^{p-1} \leq \big|x - x'\big|^{2(p-1)} A_{2,\beta_2}^{4(p-1)} \mathrm{e}^{\beta_2(p-1)t}, \tag{3.25}$$

where $A_{1,\beta_1}, A_{2,\beta_2} \in (0,\infty)$ and $\beta_1 > \bar{\gamma}(2)$ and $\beta_2 > \bar{\gamma}(4)$ are fixed and finite constants. The proof now follows by combining the above and choosing β_1 and β_2 such that $(1+\delta)\bar{\gamma}(2) > \beta_1 > \bar{\gamma}(2)$ and $(1+\delta)\bar{\gamma}(4) > \beta_2 > \bar{\gamma}(4)$. □

The preceding lemma allows for a uniform modulus of continuity estimate, which we record next.

Lemma 3.6. *Suppose the conditions of Lemma 3.4 are met. Then for all $p \in (1,2)$ and $\varepsilon, \delta \in (0,1)$ there exists $C_{p,\varepsilon,\delta} \in (0,\infty)$ such that simultaneously for all $t \geq 0$,*

$$\sup_{j\in\mathbf{Z}}\bigg\|\sup_{j\leq x<x'\leq j+1}\frac{|u_t(x) - u_t(x')|^2}{|x - x'|^{1-\varepsilon}}\bigg\|_p \leq C_{p,\varepsilon,\delta}\cdot \mathrm{e}^{(1+\delta)\lambda_p t}, \tag{3.26}$$

where λ_p was defined in (3.21).

Proof. The proof consists of an application of the Kolmogorov continuity theorem. Recall that the spatial dimension is 1 and we are choosing a continuous version of the solution $(t,x) \mapsto u_t(x)$. Since $p > 1$ in Lemma 3.5, we can use a suitable version of Kolmogorov continuity theorem, for example Theorem 4.3 of reference [5], p. 10, to obtain the result. The stated dependence of the constant, $C_{p,\varepsilon,\delta}$ is consequence of the explicit form of inequality (3.20) and the proof of Theorem 4.3 in [5]. □

Before we begin our proof of Theorem 1.1, we prove that under some condition the $L^2(\mathrm{P})$-norm of the solution has an effectively-compact support.

Proposition 3.7. *If the conditions of Theorem 1.1 are met, then there exists a finite and positive constant m such that $u_t(x)$ has an effectively-compact support with radius of effective support $p(t) = mt$.*

Proof. We begin by noting that for all $m, t > 0$,

$$\int_{|x|>mt}\big|u_t(x)\big|^2\,\mathrm{d}x \leq \int_{|x|>mt} u_t(x)\,\mathrm{d}x + \int_{\substack{|x|>mt\\ u_t(x)\geq 1}}\big|u_t(x)\big|^2\,\mathrm{d}x. \tag{3.27}$$

Therefore,

$$\mathrm{E}\bigg(\int_{|x|>mt}\big|u_t(x)\big|^2\,\mathrm{d}x\bigg) \leq \int_{|x|>mt}(p_t * u_0)(x)\,\mathrm{d}x + \int_{|x|>mt}\mathrm{E}\big(\big|u_t(x)\big|^2; u_t(x) \geq 1\big)\,\mathrm{d}x. \tag{3.28}$$

Since u_0 has compact support, (3.4) implies that

$$\int_{|x|>mt} (p_t * u_0)(x)\,\mathrm{d}x = \mathrm{O}\big(\mathrm{e}^{-m^2t/2}\big) \quad \text{as } t \to \infty. \tag{3.29}$$

Next we estimate the final integral in (3.28).

Thanks to (3.4) and Chebyshev's inequality,

$$\mathrm{P}\big\{u_t(x) \geq 1\big\} \leq \text{const} \cdot \mathrm{e}^{-x^2/(4t)}, \tag{3.30}$$

uniformly for all $x \in \mathbf{R}$ and $t \geq 1$. Also, from Proposition 3.1, there exists a constant $b \in (0, \infty)$ such that

$$\sup_{x\in\mathbf{R}} \mathrm{E}\big(\big|u_t(x)\big|^4\big) \leq b\mathrm{e}^{bt/4} \quad \text{for all } t \geq 1. \tag{3.31}$$

Using the preceding two inequalities, the right-hand side of inequality (3.28) reduces to

$$\begin{aligned}
&\mathrm{E}\bigg(\int_{|x|>mt} \big|u_t(x)\big|^2\,\mathrm{d}x\bigg)\\
&\quad\leq \mathrm{O}\big(\mathrm{e}^{-m^2t/2}\big) + \text{const} \cdot \int_{|x|>mt} \sqrt{\mathrm{E}\big(\big|u_t(x)\big|^4\big)}\mathrm{e}^{-x^2/(8t)}\,\mathrm{d}x\\
&\quad\leq \mathrm{O}\big(\mathrm{e}^{-m^2t/2}\big) + \text{const} \cdot b^{1/2}\mathrm{e}^{bt/8} \cdot \int_{|x|>mt} \mathrm{e}^{-x^2/(8t)}\,\mathrm{d}x.
\end{aligned} \tag{3.32}$$

We now choose and fix $m > \sqrt{b}$ to obtain from the preceding that

$$\limsup_{t\to\infty} t^{-1} \ln \mathrm{E}\bigg(\int_{|x|>mt} \big|u_t(x)\big|^2\,\mathrm{d}x\bigg) < 0. \tag{3.33}$$

This implies part (b) of Definition 1.2 with $p(t) = mt$. We now prove the remaining part of Definition 1.2. From Theorem 2.1 and the preceding, we obtain for infinitely-many values of $t \to \infty$:

$$\begin{aligned}
\exp\bigg(\bigg[\frac{\mathrm{L}_\sigma^4}{8\kappa} + \mathrm{o}(1)\bigg]t\bigg) &\leq \mathrm{E}\bigg(\int_{-\infty}^{\infty} \big|u_t(x)\big|^2\,\mathrm{d}x\bigg)\\
&= \mathrm{E}\bigg(\int_{-mt}^{mt} \big|u_t(x)\big|^2\,\mathrm{d}x\bigg) + \mathrm{o}(1).
\end{aligned} \tag{3.34}$$

This finishes the proof. □

We will need the following elementary real-variable lemma from the theory of slowly-varying functions. It is without doubt well known; we include a derivation for the sake of completeness only.

Lemma 3.8. *For every $q, \eta \in (0, \infty)$,*

$$\int_{\mathrm{e}}^{\infty} \exp\bigg(-\frac{q(\ln x)^{\eta+1}}{t}\bigg)\,\mathrm{d}x = \mathrm{O}\big(t^{1/\eta} \exp\big\{(t/q)^{1/\eta}\big\}\big) \tag{3.35}$$

as $t \to \infty$.

Proof. The proof uses some standard tricks. First we write the integral as

$$\int_{\mathrm{e}}^{\infty} \mathrm{e}^{-q(\ln x)^{\eta+1}/t}\,\mathrm{d}x = \int_1^{\infty} \mathrm{e}^{-qz^{\eta+1}/t}\mathrm{e}^z\,\mathrm{d}z. \tag{3.36}$$

Next we change variables [$w := z/\theta$], for an arbitrary $\theta > 0$, and find that

$$\int_{\mathrm{e}}^{\infty} \mathrm{e}^{-q(\ln x)^{\eta+1}/t}\,\mathrm{d}x = \theta \int_{1/\theta}^{\infty} \exp\left(-\frac{q\theta^{\eta+1}}{t} w^{\eta+1} + \theta w\right)\mathrm{d}w. \tag{3.37}$$

Upon choosing $\theta := (t/q)^{1/\eta}$, we obtain

$$-\frac{q\theta^{\eta+1}}{t} w^{\eta+1} + \theta w = \left(\frac{t}{q}\right)^{1/\eta}\bigl(w - w^{\eta+1}\bigr),$$

and this yields

$$\int_{\mathrm{e}}^{\infty} \mathrm{e}^{-q(\ln x)^{\eta+1}/t}\,\mathrm{d}x = (t/q)^{1/\eta} \int_{(q/t)^{1/\eta}}^{\infty} \mathrm{e}^{(t/q)^{1/\eta}\cdot(w-w^{\eta+1})}\,\mathrm{d}w. \tag{3.38}$$

Therefore, for t sufficiently large, we split the integral on the right-hand side of the previous display as follows:

$$\int_{\mathrm{e}}^{\infty} \mathrm{e}^{-q(\ln x)^{\eta+1}/t}\,\mathrm{d}x = (t/q)^{1/\eta}(I_1 + I_2), \tag{3.39}$$

where

$$\begin{aligned} I_1 &:= \int_{(q/t)^{1/\eta}}^{1} \exp\bigl((t/q)^{1/\eta}\cdot\bigl(w - w^{\eta+1}\bigr)\bigr)\,\mathrm{d}w, \\ I_2 &:= \int_{1}^{\infty} \exp\bigl(-(t/q)^{1/\eta}\cdot w\bigl(w^{\eta}-1\bigr)\bigr)\,\mathrm{d}w. \end{aligned} \tag{3.40}$$

Clearly,

$$I_2 \le 1 + \int_{2}^{\infty} \exp\bigl(-\bigl(2^{\eta}-1\bigr)(t/q)^{1/\eta}\cdot w\bigr)\,\mathrm{d}w = \mathrm{O}(1). \tag{3.41}$$

The lemma follows because the integrand of I_1 is at most $\exp((t/q)^{1/\eta})$. □

We are now ready to establish Theorem 1.1.

Proof of Theorem 1.1. The proof of the first inequality of the theorem is a continuation of the proof Proposition 3.7. Indeed, from (3.34), we obtain

$$\begin{aligned} \exp\left(\left[\frac{\mathrm{L}_\sigma^4}{8\kappa} + \mathrm{o}(1)\right]t\right) &\le \mathrm{E}\left(\int_{-\infty}^{\infty} \bigl|u_t(x)\bigr|^2\,\mathrm{d}x\right) \\ &\le \mathrm{E}\left(\int_{-mt}^{mt} \bigl|u_t(x)\bigr|^2\,\mathrm{d}x\right) + \mathrm{o}(1) \\ &\le 2mt \cdot \sup_{x\in\mathbf{R}} \mathrm{E}\bigl(\bigl|u_t(x)\bigr|^2\bigr) + \mathrm{o}(1), \end{aligned} \tag{3.42}$$

valid for $t \to \infty$. We obtain first inequality of the theorem after taking the appropriate limit.

Next we prove the second inequality of the theorem by first observing that for every $j \ge 1$, every increasing sequence of real numbers $\{a_j\}_{j=1}^{\infty}$ with $\sup_{j\ge 1}(a_{j+1}-a_j) \le 1$, $p \in (1,2)$, $\varepsilon \in (0,1)$, and $t \ge 0$,

$$\begin{aligned}\sup_{a_j \le x \le a_{j+1}} \left|u_t(x)\right|^{2p} &= \sup_{a_j \le x \le a_{j+1}} \left|u_t(a_j) + u_t(x) - u_t(a_j)\right|^{2p} \\ &\le 2^{2p-1}\Big(\left|u_t(a_j)\right|^{2p} + \sup_{a_j \le x \le a_{j+1}} \left|u_t(x) - u_t(a_j)\right|^{2p}\Big) \\ &\le 2^{2p-1}\big(\left|u_t(a_j)\right|^{2p} + (a_{j+1}-a_j)^{p(1-\varepsilon)}\Omega_j^p\big), \end{aligned} \tag{3.43}$$

where

$$\Omega_j := \sup_{a_j \le x < x' \le a_{j+1}} \frac{|u_t(x) - u_t(x')|^2}{|x-x'|^{1-\varepsilon}}. \tag{3.44}$$

Consequently,

$$\mathrm{E}\Big(\sup_{a_j \le x \le a_{j+1}} \left|u_t(x)\right|^{2p}\Big) \le 2^{2p-1}\big(\mathrm{E}\big(\left|u_t(a_j)\right|^{2p}\big) + (a_{j+1}-a_j)^{p(1-\varepsilon)}\mathrm{E}\big(\Omega_j^p\big)\big). \tag{3.45}$$

We use inequality (3.2) of Lemma 3.3 with $k := 2p$ and $x := a_j$ to find that

$$\mathrm{E}\big(\left|u_t(a_j)\right|^{2p}\big) \le \mathrm{const}\cdot\exp\bigg(\beta_p \cdot \bigg[t\frac{\overline{\gamma}(2p^2) + \mathrm{o}(1)}{p} - \frac{a_j^2}{4t^2}\bigg]\bigg), \tag{3.46}$$

where

$$\beta_p := \frac{p}{p+1-(1/p)}, \tag{3.47}$$

the implied constant does not depend on j or t, and $\mathrm{o}(1) \to 0$ as $t \to \infty$, uniformly for all j. Also, Lemma 3.6 implies that

$$\sup_{j \ge 1}\mathrm{E}\big(\Omega_j^p\big) \le C_{p,\varepsilon,\delta}\cdot \mathrm{e}^{p(1+\delta)\lambda_p t}, \tag{3.48}$$

where δ is an arbitrarily-small positive constant, which we will choose and fix appropriately later on. We can combine the preceding inequalities to deduce that

$$\mathrm{E}\Big(\sup_{a_j \le x \le a_{j+1}} \left|u_t(x)\right|^{2p}\Big) \le \mathrm{const}\cdot \mathrm{e}^{-\beta_p a_j^2/(4t^2)}\cdot \mathrm{e}^{\beta_p t(\overline{\gamma}(2p^2)+\mathrm{o}(1))/p} + \mathrm{const}\cdot (a_{j+1}-a_j)^{p(1-\varepsilon)}\mathrm{e}^{p(1+\delta)\lambda_p t}. \tag{3.49}$$

Choose and fix an integer $\nu \ge 1$. We apply the preceding with $p(1-\varepsilon) > 1$; we also choose the a_l's so that $a_1 := 0$, $0 \le a_{j+1} - a_j \le 1$ for all $j \ge 1$, and $a_j := (\log j)^\nu$ for all j sufficiently large. Because $a_{j+1} - a_j = \mathrm{O}((\ln j)^\nu/j)$ as $j \to \infty$,

$$\sum_{j=1}^{\infty}(a_{j+1}-a_j)^{p(1-\varepsilon)} < \infty. \tag{3.50}$$

Also, for all $J > 1 + \mathrm{e}$, sufficiently large,

$$\begin{aligned}\sum_{j=J}^{\infty}\mathrm{e}^{-\beta_p a_j^2/(4t)} &\le \int_{J-1}^{\infty}\mathrm{e}^{-\beta_p(\ln x)^{2\nu}/(4t^2)}\,\mathrm{d}x \\ &= \mathrm{O}\big(t^{1/(2\nu-1)}\mathrm{e}^{(4t^2/\beta_p)^{1/(2\nu-1)}}\big) \quad (t \to \infty), \end{aligned} \tag{3.51}$$

where we have used Lemma 3.8 for the last equality. We can choose $\nu := \frac{1}{2}(\delta^{-1}+1)$ so that $1/(2\nu-1)=\delta$. We can combine these terms to deduce the following:

$$\begin{aligned}\mathrm{E}\Big(\sup_{x\geq a_J}\big|u_t(x)\big|^{2p}\Big) &\leq \sum_{j=J}^{\infty}\mathrm{E}\Big(\sup_{a_j\leq x\leq a_{j+1}}\big|u_t(x)\big|^{2p}\Big)\\ &= \mathrm{O}\big(t^{\delta}\mathrm{e}^{(4t^2/\beta_p)^{\delta}+\beta_p t(\overline{\gamma}(2p^2)+\mathrm{o}(1))/p}+\mathrm{e}^{p(1+\delta)\lambda_p t}\big).\end{aligned} \tag{3.52}$$

A similar – though slightly simpler – argument can be used to derive the very same upper bound for the quantity $\mathrm{E}(\sup_{0\leq x<a_J}|u_t(x)|^{2p})$. We now use symmetry and let $\delta\downarrow 0$,

$$\limsup_{t\to\infty} t^{-1}\ln\mathrm{E}\Big(\sup_{x\in\mathbf{R}}\big|u_t(x)\big|^{2p}\Big)\leq \max\left\{\frac{\beta_p\overline{\gamma}(2p^2)}{p}, p\lambda_p\right\}. \tag{3.53}$$

Let us substitute the evaluation of β_p in terms of p to find that

$$\limsup_{t\to\infty} t^{-1}\ln\mathrm{E}\Big(\sup_{x\in\mathbf{R}}\big|u_t(x)\big|^{2p}\Big)\leq \max\left\{\frac{\overline{\gamma}(2p^2)}{p+1-(1/p)}, p\lambda_p\right\}. \tag{3.54}$$

This and Jensen's inequality together prove that

$$\limsup_{t\to\infty} t^{-1}\ln\mathrm{E}\Big(\sup_{x\in\mathbf{R}}\big|u_t(x)\big|^{2}\Big)\leq \frac{1}{p}\max\left\{\frac{\overline{\gamma}(2p^2)}{p+1-(1/p)}, p\lambda_p\right\}, \tag{3.55}$$

and this valid for all $p\in(1,2)$. As $p\downarrow 1$, $\lambda_p\to\overline{\gamma}(2)$. Moreover, $\overline{\gamma}(2p^2)\to\overline{\gamma}(2)$ because $\overline{\gamma}$ is convex and hence continuous on $[1,\infty)$ [Remark 3.2]. It follows that

$$\limsup_{t\to\infty} t^{-1}\ln\mathrm{E}\Big(\sup_{x\in\mathbf{R}}\big|u_t(x)\big|^{2}\Big)\leq \overline{\gamma}(2), \tag{3.56}$$

and this is $\leq \mathrm{Lip}_\sigma^4/(8\kappa)$ by Proposition 3.1. The latter proposition implies the theorem because $\mathrm{E}(\sup_x|u_t(x)|^2)\geq \sup_x\mathrm{E}(|u_t(x)|^2)$.

□

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