Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 1, 250–278
DOI: 10.1214/09-AIHP318

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Shape transition under excess self-intersections for transient random walk

Amine Asselah

Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est, UMR CNRS 8050, France. E-mail: amine.asselah@univ-paris12.fr

Received 3 May 2008; revised 10 February 2009; accepted 13 March 2009

Abstract. We reveal a shape transition for a transient simple random walk forced to realize an excess q-norm of the local times, as the parameter q crosses the value $q_c(d)=\frac{d}{d-2}$. Also, as an application of our approach, we establish a central limit theorem for the q-norm of the local times in dimension 4 or more.

Résumé. Nous décrivons un phénomène de transition de forme d'une marche aléatoire transiente forcée à réaliser une grande valeur de la norme-q du temps local, lorsque le paramètre q traverse la valeur critique $q_c(d)=\frac{d}{d-2}$. Comme application de notre approche, nous établissons un théorème de la limite centrale pour la norme-q du temps local en dimension 4 et plus.

MSC: 60K35; 82C22; 60J25

Keywords: Self-intersection local times; Large deviations; Random walk

1. Introduction

We consider a simple random walk $\{S(n), n\in\mathbb{N}\}$ on $\mathbb{Z}^d$, starting at the origin. For any set A, we denote by $\mathbb{1}_A$ the indicator of A, and consider the local times of the walk $\{l_n(z), z\in\mathbb{Z}^d\}$ given by

$$l_n(z)=\mathbb{1}_{\{S(0)=z\}}+\cdots+\mathbb{1}_{\{S(n-1)=z\}}. \tag{1.1}$$

For a real $q>1$, we form the sum of the qth power of the local times

$$\|l_n\|_q^q=\sum_{z\in\mathbb{Z}^d} l_n(z)^q. \tag{1.2}$$

When q is integer, $\|l_n\|_q^q$ can be written in terms of the q-fold self-intersection local times of a random walk. For instance, when $q=2$

$$\|l_n\|_2^2=n+2\sum_{0\le i<j<n}\mathbb{1}_{\{S(i)=S(j)\}}.$$

For q positive real, we still call $\|l_n\|_q^q$ the q-fold self-intersection local times.

In dimension three and more, Becker and König [6] have shown that there are positive constants, say $\kappa(q,d)$, such that almost surely

$$\lim_{n\to\infty}\frac{\|l_n\|_q^q}{n}=\kappa(q,d). \tag{1.3}$$

Here, we are concerned with estimating the deviations of $\|l_n\|_q^q$ away from its mean. That is, if P denotes the law of the walk started at 0, we give estimates for

$$P\big(\|l_n\|_q^q - E\big[\|l_n\|_q^q\big] \ge \xi n\big) \tag{1.4}$$

for ξ positive, and n going to infinity.

There is a rich literature concerning the two-fold self-intersection local times. The reason being that $\|l_n\|_2$ is a natural object in quantum-field theory (see [1,14,22], for instance), as well as in the statistical physics of polymers (see [8,9,13], for instance). However $\|l_n\|_q$ for $q \in \mathbb{R} \setminus \mathbb{N}$ has no such direct links with physics. It comes up naturally in studying large and moderate deviations for *random walk in random sceneries* (see [5] and [15]).

Now, in the large deviations results for the two-fold self-intersection of a transient random walk (see [2,3,5,11]) two strategies have a distinguished role:

- Strategy A: the walk visits of the order of $(\xi n)^{1/q}$-times, finitely many sites in a ball of bounded radius. For a transient walk, the number of visits of a bounded domain is bounded by a geometric variable. Thus, strategy A costs of the order of $\exp(-\mathrm{O}((n\xi)^{1/d}))$, where we use the notation $y_n = \mathrm{O}(x_n)$ for two positive sequences $\{x_n, y_n, n \in \mathbb{N}\}$, to mean that there is $K > 0$ such that $0 \le y_n \le K x_n$.
- Strategy B: the walk visits of the order of $\xi^{1/(q-1)}$-times, about $n/\xi^{1/(q-1)}$ sites. Presumably, the walk stays, a time n, in a ball of volume $n/\xi^{1/(q-1)}$. The cost of staying a time n within a ball of radius $r_n \ll \sqrt{n}$ is about $\exp(-\mathrm{O}(n/r_n^2))$, so that strategy B costs of the order of $\exp(-\mathrm{O}(n^{1-2/d}\xi^{2/(d(q-1))}))$.

When $q = 2$, [2,5] have shown that strategy A is adopted in $d \ge 5$, whereas [3] (see also Chapter 8.4 of [11]) suggests that strategy B is adopted in $d = 3$.

To summarize in words our main finding, assume $d \ge 3$, fix $\xi > 0$ and look at typical paths realizing $\{\|l_n\|_q^q - E[\|l_n\|_q^q] \ge \xi n\}$. As we increase q, we step on a value, $q_c(d)$, above which our large deviation event is realized by strategy A, and below which it is realized by strategy B. The critical value $q_c(d) = \frac{d}{d-2}$ is obtained as we equal the costs of strategies A and B.

Note that $q_c(d)$ is a well known number: if q is integer, then q independent simple random walks, on $\mathbb{Z}^d$, intersect infinitely often if and only if $q < q_c(d)$ (see, for instance, [18], Proposition 7.1 and [17], Section 4.1).

Let us now describe, in mathematical terms, this shape transition. The first theorem deals with the *sub-critical regime* $q < q_c(d)$.

Theorem 1.1. *Assume dimension $d \ge 3$. Then, for q and d such that $1 < q < \frac{d}{d-2}$, there are constants $c_1^{\pm}(q,d) > 0$ such that for $\xi \ge 1$, and n large enough*

$$\begin{aligned}\exp\big(-c_1^-(q,d)\xi^{(2/d)(1/(q-1))}n^{1-2/d}\big) &\le P\big(\|l_n\|_q^q - E\big[\|l_n\|_q^q\big] \ge \xi n\big) \\ &\le \exp\big(-c_1^+(q,d)\xi^{2/d(1/(q-1))}n^{1-2/d}\big).\end{aligned} \tag{1.5}$$

Moreover, in this regime the sites visited more than some large constant do not contribute to realizing the excess self-intersection. In other words,

$$\limsup_{A\to\infty}\limsup_{n\to\infty}\frac{1}{n^{1-2/d}}\log P\left(\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{l_n(z)>A\}}l_n(z)^q \ge \xi n\right) = -\infty. \tag{1.6}$$

Our second theorem deals with the *super-critical regime* $q > q_c(d)$.

Theorem 1.2. *Assume dimension $d \ge 3$. For q and d such that $q > \frac{d}{d-2}$, there are constants $c_2^{\pm}(q,d) > 0$ such that for $\xi \ge 1$, and n large enough*

$$\exp\big(-c_2^-(q,d)(\xi n)^{1/q}\big) \le P\big(\|l_n\|_q^q - E\big[\|l_n\|_q^q\big] \ge \xi n\big) \le \exp\big(-c_2^+(q,d)(\xi n)^{1/q}\big). \tag{1.7}$$

Moreover, the sites visited much less than $n^{1/q}$ do not contribute to realizing the excess self-intersection. In other words,

$$\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^{1/q}}\log P\left(\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{l_n(z)<\varepsilon n^{1/q}\}}l_n(z)^q\ge E\big[\|l_n\|_q^q\big]+\xi n\right)=-\infty. \tag{1.8}$$

Remark 1.3. *In Theorems 1.1 and 1.2, we could take ξ to grow with n. The only (necessary) bound on ξ_n comes from the bound $\|l_n\|_q\le n$ which imposes that $\xi_n\le n^{q-1}$. The proofs are written with general $\xi_n\ge 1$.*

The next result deals with the contribution of some level sets of the local times to deviation on a much larger scale than the mean, and can be obtained by the same approach yielding Theorem 1.2. We include it in this form since it can be of independent interest, while showing the possibilities offered by our approach. Also, it generalizes Lemma 1.8 of [5].

Lemma 1.4. *Assume $d\ge 3$ and $q\ge q_c(d)$. Choose $a,b>0$ such that $1<a<1+b(q-1)$. Then, for any $\varepsilon>0$, and n large enough*

$$P\left(\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{l_n(z)<n^b\}}l_n(z)^q\ge n^a\right)\le \mathrm{e}^{-n^{\zeta(q,a,b)-\varepsilon}}\quad \textit{with } \zeta(q,a,b)=b+\frac{1}{q_c(d)}(a-qb). \tag{1.9}$$

Remark 1.5. *Our approach is not suited to studying small ξ_n for reasons explained later in Remark 1.7. However, when $1>\xi_n\ge n^{-\delta}$, for some positive δ small enough, our approach yields a constant c_1 such that for $q<q_c(d)$*

$$P\big(\|l_n\|_q^q-E\big[\|l_n\|_q^q\big]\ge\xi_n n\big)\le\exp\big(-c_1\xi_n^{2/d(q/(q-1))}n^{1-2/d}\big). \tag{1.10}$$

When $q>q_c(d)$, we have a constant c_2 such that

$$P\big(\|l_n\|_q^q-E\big[\|l_n\|_q^q\big]\ge\xi_n n\big)\le\exp\big(-c_2\xi_n^{1/q+2/d}n^{1/q}\big). \tag{1.11}$$

We believe that the powers of ξ_n in (1.10) and (1.11) are not optimal. However, (1.10) and (1.11) are useful in deriving a central limit theorem stated in Theorem 1.9.

Our initial goal was to improve the main result of [3], which states that in dimension 3, there is $\underline{\chi}>0$ and $\varepsilon>0$ such that for $\xi>0$, and n large

$$P\left(\sum_{z\in\mathbb{Z}^3}\mathbb{1}_{\{l_n(z)>\log(n)^{\underline{\chi}}\}}l_n^2(z)>n\xi\right)\le\exp\big(-n^{1/3}\log(n)^{\varepsilon}\big). \tag{1.12}$$

Note that (1.6) improves (1.12). One reason to study $\|l_n\|_q$ for $q>2$, is that the upper bound (1.5) for $q>2$, yields (1.6) at once. More precisely, for $q<q_c(d)$, choose q' with $q<q'<q_c(d)$, and for any $A>0$, the obvious inequality

$$\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{l_n(z)>A\}}l_n^q(z)\le\frac{\|l_n\|_{q'}^{q'}}{A^{q'-q}}, \tag{1.13}$$

implies that

$$P\left(\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{l_n(z)>A\}}l_n^q(z)\ge n\xi\right)\le P\big(\|l_n\|_{q'}^{q'}\ge A^{q'-q}n\xi\big).$$

For A and n large enough, $A^{q'-q} n\xi \geq 2E[\|l_n\|_{q'}^{q'}]$. Thus, if we set $\beta = \frac{2}{d}\frac{q'-q}{q'-1} > 0$, then from (1.5), we have a constant $c_1(d, q')$ such that

$$P\left(\sum_{z\in\mathbb{Z}^d} \mathbb{1}_{\{l_n(z)>A\}} l_n^q(z) \geq n\xi\right) \leq \exp\bigl(-c_1\bigl(d, q'\bigr)\xi^{(2/d)(1/(q'-1))} A^\beta n^{1-2/d}\bigr). \tag{1.14}$$

Thus, in order to improve (1.12) in $d = 3$, we were left with studying q-fold self-intersections with $2 < q < 3 = q_c(3)$.

In most works on two-fold self-intersection, a starting point, which we trace back to the work of Westwater [21] and Le Gall [19], is a decomposition of $\|l_n\|_2^2$ in terms of *intersection local times* of two independent random walks starting at the origin. However, such a decomposition is restricted to q-fold self-intersection local times with $q \in \mathbb{N}$: When $q = 2$ and $d = 3$ (in the *sub-critical regime*) Le Gall's decomposition is a first step in obtaining, in [11], a moderate and large deviations principles. When $q = 3$ and $d \geq 4$ (in the *super-critical regime*), [15] uses a type of Le Gall's decomposition to obtain moderate and large deviations estimates.

Here, our starting point is an *approximate decomposition* obtained by slicing $\|l_n\|_q^q$ over level sets of the local times, for any real $q > 1$. This is based on the following simple inequality. Let $\{b_n, n \in \mathbb{N}\}$ be a subdivision of $[1, \infty)$, and let l_1 and l_2 be positive integers (which we think of as the local times of a given site in each half time-period). Then, for $q > 1$, we have the upper bound

$$(l_1 + l_2)^q \leq l_1^q + l_2^q + 2^q \sum_{i=0}^{\infty} b_{i+1}^{q-2} \mathbb{1}_{\{b_i \leq \max(l_1,l_2) < b_{i+1}\}} l_1 \times l_2, \tag{1.15}$$

as well as the obvious lower bound: $(l_1 + l_2)^q \geq l_1^q + l_2^q$. The desirable feature of (1.15) is that on its right-hand side, the qth power of l_1 and l_2 comes without penalty, whereas the term $l_1 \times l_2$ yields an intersection local times. Thus, (1.15) leads to the following result which plays here the role of Le Gall's decomposition of [19].

Proposition 1.6. *For any integers n and l, with $2^l < n$, let $\{n_i, i = 1, \ldots, 2^l\}$ be positive integers summing up to n. Let $\{l_.^{(i)}, i = 1, \ldots, 2^l\}$ be the local times of 2^l independent random walks starting at 0. If $\{b_i, i \in \mathbb{N}\}$ is a subdivision of $[1, n]$, then,*

$$S_q^{(l)} \leq \|l_n\|_q^q \leq S_q^{(l)} + \sum_{j=1}^{l} \mathcal{I}_j, \quad \textit{where } S_q^{(l)} \stackrel{law}{=} \sum_{i=1}^{2^l} \bigl\|l_{n_i}^{(i)}\bigr\|_q^q, \tag{1.16}$$

and, for $j = 1, \ldots, l$, and $m_k = n_{(k-1)2^{l-j}+1} + \cdots + n_{k2^{l-j}}$ for $k = 1, \ldots, 2^j$

$$\mathcal{I}_j \stackrel{law}{=} \sum_{k=1}^{2^{j-1}} \sum_i 2^q b_{i+1}^{q-1} \Biggl(\sum_{z: b_i \leq l_{m_{2k}}^{(2k)}(z) < b_{i+1}} l_{m_{2k-1}}^{(2k-1)}(z) + \sum_{z: b_i \leq l_{m_{2k-1}}^{(2k-1)}(z) < b_{i+1}} l_{m_{2k}}^{(2k)}(z) \Biggr). \tag{1.17}$$

Remark 1.7. *We first note some natural limitations in using the approximate decomposition (1.16). When we deal with $\{\|l_n\|_q^q - E[\|l_n\|_q^q] \geq \xi_n n\}$ for small ξ_n, we need to bound the difference between $E[\|l_n\|_q^q]$ and the expectation of the upper bound in (1.16). When, we take l such that $2^l \sim n^{1-\delta_0}$, then this difference turns out to be of order smaller than $n^{1-\delta_0/2}$, allowing us to write*

$$\bigl\{\|l_n\|_q^q - E\bigl[\|l_n\|_q^q\bigr] \geq \xi_n n\bigr\} \subset \left\{S_q^{(l)} - E\bigl[S_q^{(l)}\bigr] \geq \frac{\xi_n}{2} n\right\} \cup \left\{\sum_{j=1}^{l} \mathcal{I}_j - E[\mathcal{I}_j] \geq \frac{\xi_n}{2} n - n^{1-\delta_0/2}\right\}. \tag{1.18}$$

(1.18) *requires that $\xi_n \geq n^{-\delta_0/2}$.*

Proposition 1.6 is our initial step in the proof of Theorems 1.1 and 1.2, and leads to a central limit theorem (CLT) for $\|l_n\|_q^q$ in dimension 4 or more, as well as a characterization of the variance of $\|l_n\|_q^q$.

Before stating results concerning the typical behavior of $\|l_n\|_q^q$, we give a heuristic discussion of the proof of Theorem 1.1 assuming Proposition 1.6. More precisely, we wish to sketch the reasons why the *approximate decomposition* (1.16), reduces large deviations for the q-norm of the local times, to large deviations for a sum of independent geometric variables.

Consider a choice of l such that 2^l is close to n in (1.16), and $n_i \sim n/2^l$. Then, $S_q^{(l)}$ is a sum of about n independent terms, each one bounded by its time-span, $n/2^l$, to the power q. Recall that the probability of deviating from the mean, for a sum of n independent and essentially bounded variables, is of order $\exp(-\mathrm{O}(n))$ (see Lemma 2.4 for a precise statement). We can therefore neglect the contribution of $S_q^{(l)}$ to the excess q-norm, though the mean of $S_q^{(l)}$ is close to $E[\|l_n\|_q^q]$, as easily seen from Lemma 1.8 below. Now, for a fixed j in $\{1,\dots,l\}$, let $m = n/2^j$, and note that $\mathcal{I}_j$ is a finite sum of independent terms distributed as $b^{q-1} l_m(\tilde{\mathcal{D}}(b))$, where $\tilde{\mathcal{D}}(b)$ is the set $\{z\colon \tilde{l}_m(z) \sim b\}$, with $\tilde{l}_m$ an independent copy of l_m, and b spans a subdivision of $[1,n]$. Since we consider transient random walks, $l_m(\tilde{\mathcal{D}}(b))$ is bounded by a geometric variable (when fixing $\tilde{\mathcal{D}}(b)$, as shown in Lemma 1.2 of [4]). At this point, one normalizes $l_m(\tilde{\mathcal{D}}(b))$. If P_0 and $\tilde{P}_0$ are the law of the two independent copies of the (transient) walk started at 0, define

$$X = \frac{l_m(\tilde{\mathcal{D}}(b))}{E_0[l_m(\tilde{\mathcal{D}}(b))]}, \quad \text{so that for some } \kappa > 0, \qquad \tilde{P}_0 \otimes P_0(X > t) \le \mathrm{e}^{-\kappa t}. \tag{1.19}$$

Now, it is well known that for any m, $E_0[l_m(\tilde{\mathcal{D}}(b))] \le C|\tilde{\mathcal{D}}(b)|^{2/d}$, (see Lemma A.2). Thus, an estimate of the large deviation probability requires an estimate on the volume of level sets of the local times. Now, in obtaining a bound on the volume of $\tilde{\mathcal{D}}(b)$, assume for simplicity that we only have two types of b: that is, we distinguish *often visited sites*, say sites visited n^x-times with x close to $1/q$, whose level sets are part of what we call *top levels*, and say the once-visited sites, whose level sets are part of what we call *bottom levels*. The *bottom levels* are the easiest to treat (see Section 3.2 and Lemma 3.1). Indeed, we use essentially that $|\tilde{\mathcal{D}}(b)| \le n$ for $b \sim 1$, so that we expect (when we restrict $\mathcal{I}_j$ only to *bottom levels* and using $X_k = X$ in law)

$$P\big(\mathcal{I}_j - E[\mathcal{I}_j] \ge n\xi\big) \sim P\left(\sum_{k=1}^{2^j} X_k - E[X_k] \ge \frac{n\xi}{n^{2/d}}\right) \sim \exp\big(-\mathrm{O}\big(n^{1-2/d}\big)\big).$$

The *top levels*, treated in Section 3.3, require a type of *bootstrap argument*, using that if $\tilde{\mathcal{D}}(b)$ has a large volume, it implies that the q-norm of the local time $\tilde{l}_m$ is large. The bootstrap argument yields Corollary 2.2. It allows us to normalize properly our geometric-like random variables $b^{q-1} l_m(\tilde{\mathcal{D}}(b))$.

We turn now to the typical behavior of the q-norm. Chen has provided in [10] asymptotics for the variance of $\|l_n\|_2^2$ in $d \ge 3$. He shows that (i) in $d=3$, $\mathrm{var}(\|l_n\|_2^2) \sim \lambda_3 n \log(n)$, and (ii) in $d \ge 4$, $\mathrm{var}(\|l_n\|_2^2) \sim \lambda_d n$, where λ_d are constants expressed in terms of the Green's function of the walk. Following ideas of Jain and Pruitt [16], and of Le Gall and Rosen [20], Chen obtains a CLT in dimension 3 or more for $\|l_n\|_2^2$. Finally, Becker and König in [6] have shown that for q integer, (i) in $d=3$, $\mathrm{var}(\|l_n\|_q^q) \le n^{3/2}$, (ii) in $d=4$, $\mathrm{var}(\|l_n\|_q^q) \le n\log(n)$, and (iii) in $d \ge 5$, $\mathrm{var}(\|l_n\|_q^q) \le c_d n$. Our result deals with the general case ($q > 1$ real), where no representation of $\|l_n\|_q^q$ is possible in terms of multiple time-intersections. We transform Lindeberg's condition into a *large deviation* event for $\|l_{T_n}\|_2^2$ on the scale of time of the CLT, that is $T_n \approx \sqrt{n}$.

We start with an estimate for the expectation of $\|l_n\|_q^q$, of the same type as Theorem 1 of Dvoretzky and Erdös [12] for the range of a transient random walk. Thus, if γ_d is the probability of never returning to its original position, it is shown in [12] that for positive constants c_d, when R_n is the set of visited sites before time n,

$$\big|E\big[|R_n|\big] - n\gamma_d\big| \le c_d \psi_d(n), \quad \text{with } \psi_d(n) = \begin{cases} n^{1/2} & \text{for } d=3, \\ \log(n) & \text{for } d=4, \\ 1 & \text{for } d \ge 5, \end{cases} \tag{1.20}$$

Jain and Pruitt [16] obtain the asymptotics $\mathrm{var}(|R_n|) \sim a\log(n)n$ for some $a > 0$ in $d=3$, and $\mathrm{var}(|R_n|) \sim c_d' n$ in $d > 3$, for some positive constants c_d'. The corresponding CLT (in $d \ge 3$) was shown by Jain and Pruitt [16] for the simple random walk, and by Le Gall and Rosen [20] for stable random walks. Note that the limiting law is Gaussian, in $d \ge 3$ but fails to be so in $d=2$, as shown by Le Gall in [18].

Lemma 1.8. *Assume that $d \geq 3$ and $q > 1$. There is a constant C_d, such that*

$$0 \leq \kappa(q,d)n - E\big[\|l_n\|_q^q\big] \leq C_d \psi_d(n), \quad \text{with } \kappa(q,d) = \gamma_d E\big[l_\infty(0)^q\big]. \tag{1.21}$$

Also, if $d = 3$, then, there is a constant c_3 such that

$$\operatorname{var}\big(\|l_n\|_q^q\big) \leq c_3 \log(n)^2 n. \tag{1.22}$$

If $d \geq 4$, then there are positive constants $v(q,d)$ and $c(q,d)$, such that

$$\left|\frac{\operatorname{var}(\|l_n\|_q^q)}{n} - v(q,d)\right| \leq c(q,d)\frac{\log(n)}{\sqrt{n}}. \tag{1.23}$$

Finally, we have the following central limit theorem.

Theorem 1.9. *If Z is a standard normal variable, then*

$$\frac{\|l_n\|_q^q - n\kappa(q,d)}{\sqrt{n v(q,d)}} \xrightarrow{law} Z. \tag{1.24}$$

A challenging open question is to understand the strategy which realizes $\{\|l_n\|_q^q - E[\|l_n\|_q^q] \geq \xi n\}$, right at the critical value $q = q_c(d) = \frac{d}{d-2}$.

The paper is organized as follows. The approximate decomposition of $\|l_n\|_q^q$ is given in Section 2.1. The sub-critical regime is studied in Section 3: The upper bound in (1.5) is proved in Section 3.5, and the lower bound is given in Section 3.4. The super-critical regime is studied in Section 4. Theorem 1.2 is proved in Section 4. The proof of (1.8) is given in Section 4.1. The proof of Lemma 1.4 is given in Section 4.2. Lemma 1.8, as well as the CLT are proved in Section 5. In the Appendix, we recall Lemma 5.1 of [3], and improve Lemma 5.3 of [3], used to control intersection local times-type quantities.

2. General considerations ($q > 1$)

In this section, we deal with the general case $q > 1$. In Section 2.1, we develop a approximation of $\|l_n\|_q^q$ as sums of two types of independent variables:

1. Intersection local times of independent walks.
2. Self-intersection local times, on a much shorter time-period.

In Section 2.2, we treat the sums of self-intersection local times.

2.1. *Approximate decomposition for $\|l_n\|_q^q$*

Before we prove Proposition 1.6, we present a useful corollary which requires more notations.

For integers n and l, with $2^l < n$, we recall the "almost" dyadic decomposition of n of Remark 2.1 of [5]. We divide n into 2^l integers $n_1^{(l)}, \dots, n_{2^l}^{(l)}$ with $n = n_1^{(l)} + \dots + n_{2^l}^{(l)}$ and

$$\max_i\big(n_i^{(l)}\big) - \min_i\big(n_i^{(l)}\big) \leq 1, \qquad \frac{n}{2^l} - 1 \leq n_i^{(l)} \leq \frac{n}{2^l} + 1 \quad \text{and} \quad n_k^{(l-1)} = n_{2k-1}^{(l)} + n_{2k}^{(l)}. \tag{2.1}$$

We run 2^l independent random walks starting at the origin. The ith walk runs for a time-period $[0, n_i^{(l)}[$, and we denote by $l_i^{(l)} : \mathbb{Z}^d \to \mathbb{N}$ its local times during time-period $[0, n_i^{(l)}[$. Also, we introduce, for $k = 1, \dots, 2^l$, the following sets

$$\mathcal{D}_{k,i}^{(l)} = \big\{z \in \mathbb{Z}^d :\ b_i \leq l_k^{(l)}(z) < b_{i+1}\big\}. \tag{2.2}$$

Now, for any $M > 0$, let $\{b_i, i \in \mathbb{N}\}$ be a subdivision of $[1, M]$, and denote by $\Theta_M(x) = x\mathbb{1}_{\{x \leq M\}}$.

Remark 2.1. *We could restrict the sum over* $\mathbb{Z}^d$ *which enters* $\|l_n\|_q^q$ *in* (1.16) *over* $\{z\colon l_n(z) \le M\}$ *for any positive* M. *The proof of Proposition* 1.6 *yields, for any* $\{b_i, i \in \mathbb{N}\}$ *subdivision of* $[1, M]$,

$$\sum_{z\in\mathbb{Z}^d} \Theta_M\big(l_n(z)\big)^q \le \sum_{k=1}^{2^l}\sum_{z\in\mathbb{Z}^d} \Theta_M\big(l_k^{(l)}(z)\big)^q + \sum_{j=1}^{l} \mathcal{I}_j. \tag{2.3}$$

The only difference with (1.16) *is the subdivision which enters into the definition of* $\mathcal{I}_j$. *The proof of Proposition* 1.6 *is written in view of* (2.3) (*see the key step* (2.12)).

As a corollary of (2.3), we obtain the following result.

Corollary 2.2. *For any* $M > 0$, *let* $\{b_i, i \in \mathbb{N}\}$ *be a subdivision of* $[1, M]$. *For any integers* n *and* L, *with* $2^L < n$, *and for any sequence of positive numbers* $\{m_n, \varepsilon_n, n \in \mathbb{N}\}$, *we have*

$$P\big(\|\Theta_M(l_n)\|_q^q \ge m_n + \varepsilon_n\big) \le 2^{L+1} P\left(\sum_{j=1}^{2^L} \|\Theta_M\big(l_j^{(L)}\big)\|_q^q \ge m_n\right) + \sum_{h=1}^{L} 2^h P\left(\sum_{l=h}^{L} \mathbb{1}_{\{\mathcal{G}_1^{(l)}\cap\mathcal{G}_2^{(l)}\}}\mathcal{I}_l \ge \varepsilon_n\right), \tag{2.4}$$

where for $l \le L$, $k = 1, \dots, 2^l$, *and* $i \in \mathbb{N}$

$$\mathcal{G}_{k,i}^{(l)} = \left\{\big|\mathcal{D}_{k,i}^{(l)}\big| \le \frac{m_n + \varepsilon_n}{b_i^q}\right\}, \qquad \mathcal{G}_1^{(l)} = \bigcap_{k=1}^{2^l}\bigcap_i \mathcal{G}_{2k,i}^{(l)} \quad \text{and} \quad \mathcal{G}_2^{(l)} = \bigcap_{k=1}^{2^l}\bigcap_i \mathcal{G}_{2k-1,i}^{(l)}. \tag{2.5}$$

Remark 2.3. *The symbols,* ε_n *and* m_n, *are suggestive of the fact that when* L *is large enough, the sum of* 2^L *independent* q*-fold self-intersections, that we called* $S_q^{(L)}$, *stays close to its mean, which is also close to the mean of* $\|l_n\|_q^q$. *This is shown in Section* 2.2. *So,* m_n *stands for mean, and* ε_n *stands for excess. To estimate how small can* ε_n *be, we now compute the expectation of* $\sum_{l=1}^{L}\mathcal{I}_l$. *We use Lemma* A.2 *in the worse case, that is in dimension* 3, *to obtain for some constants* c_1, c_2 *and* c_3

$$\begin{aligned} E\left[\sum_{l=1}^{L}\mathcal{I}_l\right] &= \sum_{l=1}^{L} 2^l \sum_{i\in\mathbb{N}} 2^q (b_{i+1})^{q-1} C_d \psi_d\left(\frac{n}{2^l}\right) \mathrm{e}^{-\kappa_d b_i} \\ &\le c_1\sqrt{n}\sum_{l=1}^{L} 2^{l/2} \sum_{i\in\mathbb{N}} (b_{i+1})^{q-1}\mathrm{e}^{-\kappa_d b_i} \\ &\le c_2\sqrt{2^L n}\sum_{i\in\mathbb{N}} (b_{i+1})^{q-1}\mathrm{e}^{-\kappa_d b_i}. \end{aligned} \tag{2.6}$$

First, we need to choose a subdivision $\{b_i, i \in \mathbb{N}\}$ *such that the last sum in* (2.6) *is convergent. Secondly, the right-hand side of* (2.4) *is small if* $\sum_{l=h}^{L} E[\mathbb{1}\{\mathcal{G}_1^{(l)} \cap \mathcal{G}_2^{(l)}\}\mathcal{I}_l] \ll \varepsilon_n$. *From* (2.6), *we see that* ε_n *can be chosen small if* $2^L \ll n$. *On the other hand, we see in Section* 2.2, *that* L *has to be large enough, for the probability of* $\{S_q^{(L)} \ge m_n\}$ *to be negligible, when* $m_n = E[\|l_n\|_q^q] + n\varepsilon$. *Remark* 2.5 *shows that a choice of* L *such that* $2^L > n^{1-\delta_0}$ *with* $q\delta_0 < \frac{2}{d}$, *fulfills both requirements.*

Proof of Proposition 1.6. The proof proceeds by induction on $l \ge 1$. It is however easy to see that proving the case $l = 1$ requires the same arguments as going from $l - 1$ to l. We focus on the first step $l = 1$, and omit the easy passage from $l - 1$ to l.

For any $x \in [0, 1]$, and $q \ge 1$, we have

$$1 + x^q \le (1 + x)^q \le 1 + x^q + 2^q x. \tag{2.7}$$

Thus, for any nonnegative integers l_1, l_2 with $0 \le l_1, l_2 \le M$, we have from (2.7)

$$l_1^q + l_2^q \le (l_1 + l_2)^q \le l_1^q + l_2^q + 2^q M^{q-2} l_1 l_2. \tag{2.8}$$

Now, for any $M > 0$, let $\{b_i, i \in \mathbb{N}\}$ be a subdivision of $[1, M]$, and recall that $\Theta_M(x) = x\mathbb{1}_{\{x \le M\}}$. For any nonnegative integers l_1, l_2

$$\big(\Theta_M(l_1 + l_2)\big)^q \le \big(\Theta_M(l_1)\big)^q + \big(\Theta_M(l_2)\big)^q + 2^q \sum_{i=1}^{n} b_{i+1}^{q-2} \mathbb{1}\big\{b_i \le \max(l_1, l_2) < b_{i+1}\big\} l_1 \times l_2. \tag{2.9}$$

Indeed, $l_1 + l_2 \le M$ and $l_1, l_2 \ge 0$, imply (i) $l_1 \le M$ and $l_2 \le M$, and (ii) for some $i_0 > 0$, $\max(l_1, l_2) \in [b_{i_0}, b_{i_0+1}[$. Then, from (2.8)

$$\big(\Theta_M(l_1 + l_2)\big)^q \le \Theta_M(l_1)^q + \Theta_M(l_2)^q + 2^q b_{i_0+1}^{q-2} l_1 \times l_2.$$

For any integer n, we consider the local time l_n, which we denote as $l_{[0,n[}(z)$ to emphasize the time period. For any integer n_1 with $0 < n_1 < n$, set $n_2 = n - n_1$, and from the increments of our initial random walk, say $\{Y_n, n \in \mathbb{N}\}$, we build two independent random walks with local times

$$l_{]0,k]}^{(1,1)}(z) = \mathbb{1}\{Y_{n_1} = z\} + \cdots + \mathbb{1}\{Y_{n_1} + \cdots + Y_{n_1-k+1} = z\}$$

and

$$l_{[0,k[}^{(1,2)}(z) = \mathbb{1}\{0 = z\} + \mathbb{1}\{-Y_{n_1+1} = z\} + \cdots + \mathbb{1}\{-Y_{n_1} - \cdots - Y_{n_1+k-1} = z\}. \tag{2.10}$$

It is obvious that on $\{S(n_1) = y\}$,

$$l_n(z) = l_{]0,n_1]}^{(1,1)}(y - z) + l_{[0,n_2[}^{(1,2)}(y - z). \tag{2.11}$$

If we denote $\bar{l}^{(1)}(z) = \max(l_{]0,n_1]}^{(1,1)}(z), l_{[0,n_2[}^{(1,2)}(z))$, and sum (2.11) over $z \in \mathbb{Z}^d$, we obtain

$$\begin{aligned}
\sum_z \Theta_M\big(l_n(z)\big)^q &\le \sum_z \Theta_M\big(l_{]0,n_1]}^{(1,1)}\big(S(n_1) - z\big)\big)^q + \sum_z \Theta_M\big(l_{[0,n_2[}^{(1,2)}\big(S(n_1) - z\big)\big)^q \\
&\quad + 2^q \sum_{z \in \mathbb{Z}^d} \sum_{i=1}^{n} b_{i+1}^{q-2} \mathbb{1}_{\{b_i \le \bar{l}^{(1)}(S(n_1)-z) < b_{i+1}\}} l_{]0,n_1]}^{(1,1)}\big(S(n_1) - z\big) \times l_{[0,n_2[}^{(1,2)}\big(S(n_1) - z\big) \\
&\le \sum_{z \in \mathbb{Z}^d} \Theta_M\big(l_{]0,n_1]}^{(1,1)}(z)\big)^q + \sum_{z \in \mathbb{Z}^d} \Theta_M\big(l_{[0,n_2[}^{(1,2)}(z)\big)^q \\
&\quad + 2^q \sum_{z \in \mathbb{Z}^d} \sum_{i=1}^{n} b_{i+1}^{q-2} \mathbb{1}_{\{b_i \le \bar{l}^{(1)}(z) < b_{i+1}\}} l_{]0,n_1]}^{(1,1)}(z) \times l_{[0,n_2[}^{(1,2)}(z).
\end{aligned} \tag{2.12}$$

Now, we rewrite (2.12) in a concise form as

$$\big\|\Theta_M(l_n)\big\|_q^q \le \big\|\Theta_M\big(l_{]0,n_1]}^{(1,1)}\big)\big\|_q^q + \big\|\Theta_M\big(l_{[0,n_2[)}^{(1,2)}\big)\big\|_q^q + \tilde{\mathcal{I}}_1(n_1, n_2), \tag{2.13}$$

where the term dealing with intersection times of independent strands is

$$\tilde{\mathcal{I}}_1(n_1, n_2) = 2^q \sum_{z \in \mathbb{Z}^d} \sum_{i=1}^{n} b_{i+1}^{q-2} \mathbb{1}_{\{b_i \le \bar{l}^{(1)}(z) < b_{i+1}\}} l_{]0,n_1]}^{(1,1)}(z) \times l_{[0,n_2[}^{(1,2)}(z). \tag{2.14}$$

To prove the upper bound in (1.16) for $l=1$, it suffices to set $M=n$ (so that this truncation disappears), and to show that $\tilde{\mathcal{I}}_1(n_1,n_2)\le\mathcal{I}_1$ given in (1.17). This latter inequality follows first by noting the obvious inclusion

$$\left\{z\colon\ b_i\le\max\left(l^{(1,1)}_{]0,n_1]}(z),l^{(1,2)}_{[0,n_2[}(z)\right)<b_{i+1}\right\}\subset\left\{z\colon\ b_i\le l^{(1,1)}_{]0,n_1]}(z)<b_{i+1}\right\}\cup\left\{z\colon\ b_i\le l^{(1,2)}_{[0,n_2[}(z)<b_{i+1}\right\},$$

and secondly, using that

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d}\mathbb{1}_{\{b_i\le\tilde{l}^{(1)}(z)<b_{i+1}\}}l^{(1,1)}_{]0,n_1]}(z)\times l^{(1,2)}_{[0,n_2[}(z)&\le\sum_{z\in\mathbb{Z}^d}\mathbb{1}\left\{b_i\le l^{(1,1)}_{]0,n_1]}(z)<b_{i+1}\right\}b_{i+1}l^{(1,2)}_{[0,n_2[}(z)\\&\quad+\sum_{z\in\mathbb{Z}^d}\mathbb{1}\left\{b_i\le l^{(1,2)}_{[0,n_2[}(z)<b_{i+1}\right\}b_{i+1}l^{(1,1)}_{]0,n_1]}(z).\end{aligned}\tag{2.15}$$

As we iterate the approximate decomposition l-times, we obtain the upper bound in (1.16), or more generally the bound (2.3).

The lower bound in (1.16) is an obvious corollary of the inequality $(l_1+l_2)^q\ge l_1^q+l_2^q$, valid for $q\ge1$ and l_1,l_2 nonnegative. □

Proof of Corollary 2.2. We write the case $M=n$, that is the case with no truncation. The case with truncation is obtained as we replace $l_n(z)$ by $\Theta_M(l_n(z))$ wherever it appears. Assume that we stop the induction in Proposition 1.6 at some step L (typically $2^L=n^{1-\delta_0}$ and δ_0 small). For any sequence of positive numbers ε_n,m_n, we have from (1.16),

$$P\left(\|l_n\|_q^q\ge m_n+\varepsilon_n\right)\le P\left(S_q^{(L)}\ge m_n\right)+P\left(\sum_{l=1}^{L}\mathcal{I}_l\ge\varepsilon_n\right),\quad\text{where }S_q^{(l)}=\sum_{k=1}^{2^l}\left\|l_k^{(l)}\right\|_q^q.\tag{2.16}$$

We introduce, as in [3,5], a bootstrap control on the volume of $D^{(l)}_{k,i}$. Consider $\mathcal{G}^{(l)}_{k,i}$ given in (2.5). On the complement of $\mathcal{G}^{(l)}=\mathcal{G}_1^{(l)}\cap\mathcal{G}_2^{(l)}$, there is k_0,i_0 such that $|D^{(l)}_{k_0,i_0}|>(m_n+\varepsilon_n)/b_{i_0}^q$ so that

$$S_q^{(l)}\ge\sum_{z\in D^{(l)}_{k_0,i_0}}\left(l^{(l)}_{k_0}(z)\right)^q\ge\frac{m_n+\varepsilon_n}{b_{i_0}^q}b_{i_0}^q=m_n+\varepsilon_n.\tag{2.17}$$

Writing $S_q^{(0)}=\|l_n\|_q^q$, we write a more suggestive relation

$$P\left(S_q^{(0)}\ge m_n+\varepsilon_n\right)\le P\left(S_q^{(L)}\ge m_n\right)+P\left(\sum_{l=1}^{L}\mathbb{1}_{\{\mathcal{G}^{(l)}\}}\mathcal{I}_l\ge\varepsilon_n\right)+\sum_{l=1}^{L}P\left(S_q^{(l)}\ge m_n+\varepsilon_n\right).\tag{2.18}$$

Starting the approximation with $S_q^{(l)}$ with $l<L$, we obtain similarly

$$P\left(S_q^{(l)}\ge m_n+\varepsilon_n\right)\le P\left(S_q^{(L)}\ge m_n\right)+P\left(\sum_{j=l+1}^{L}\mathbb{1}_{\{\mathcal{G}^{(j)}\}}\mathcal{I}_j\ge\varepsilon_n\right)+\sum_{j=l+1}^{L}P\left(S_q^{(j)}\ge m_n+\varepsilon_n\right).\tag{2.19}$$

Assume now that for $j>l$, and $j<L$ we have

$$P\left(S_q^{(j)}\ge m_n+\varepsilon_n\right)\le 2^{L-j+1}P\left(S_q^{(L)}\ge m_n\right)+\sum_{h=j+1}^{L}2^{h-j-1}P\left(\sum_{i=h}^{L}\mathbb{1}_{\{\mathcal{G}^{(i)}\}}\mathcal{I}_i\ge\varepsilon_n\right).\tag{2.20}$$

Note that (2.20) is true for $j = L-1$. Then, using the hypothesis (2.19) in (2.20), we obtain

$$P\big(S_q^{(l)} \geq m_n + \varepsilon_n\big) \leq \sum_{j=l}^{L} 2^{L-j} P\big(S_q^{(L)} \geq m_n\big) + \sum_{j=l+1}^{L} \sum_{h=j+1}^{L} 2^{h-j-1} P\left(\sum_{i=h}^{L} \mathbb{1}_{\{\mathcal{G}^{(i)}\}} \mathcal{I}_i \geq \varepsilon_n\right)$$

$$\leq 2^{L-l+1} P\big(S_q^{(L)} \geq m_n\big) + \sum_{h=l+1}^{L} 2^h \left(\sum_{l<j<h} 2^{-j}\right) P\left(\sum_{i=h}^{L} \mathbb{1}_{\{\mathcal{G}^{(i)}\}} \mathcal{I}_i \geq \varepsilon_n\right). \tag{2.21}$$

By way of induction, (2.21) yields (2.4). □

2.2. *On large sums of q-fold self-intersection*

In this section, we consider the contribution of the term $S_q^{(l)}$, which appears in (1.16), in making $\{\|l_n\|_q^q - E[\|l_n\|_q^q] \geq \xi n\}$. Recall that $S_q^{(l)}$ is a sum of independent copies of q-fold self-intersection over times of order $n/2^l$. We first choose l large enough, and then use the boundedness of the q-fold self-intersection.

Fix δ_0 such that $0 < \delta_0 < \frac{2}{qd}$. Let L be an integer so that $2^L \leq n^{1-\delta_0} < 2^{L+1}$. Note the obvious bound

$$\max_{k \leq 2^L} \|l_k^{(L)}\|_q^q \leq \max_{k \leq 2^L} \big(n_k^{(L)}\big)^q \leq \left(\frac{n}{2^L} + 1\right)^q \leq 2^{q+1} n^{q\delta_0}. \tag{2.22}$$

The main result, in this section, reads as follows.

Lemma 2.4. *Fix $\delta \geq 0$, with either* (i) *dimension is* 3 *and $\delta < \delta_0/2$, or* (ii) *dimension is* 4 *or more and $\delta < \delta_0$. Let $\xi_n \geq n^{-\delta}$. Then, for n large enough*

$$P\big(\big|S_q^{(L)} - E\big[\|l_n\|_q^q\big]\big| > \xi_n n\big) \leq \exp\left(-\frac{\xi_n}{2} n^{1-\delta_0 q}\right). \tag{2.23}$$

Remark 2.5. *Let us consider now the regimes of Theorems 1.1 and 1.2:*

- *When $q < q_c(d)$, the speed exponent in (1.5) is $1 - \frac{2}{d}$. Thus, the right-hand side of (2.23) with $\xi_n = \xi$ is negligible when $1 - q\delta_0 > 1 - \frac{2}{d}$, so that we need $q\delta_0 < 2/d$.*
- *When $q > q_c(d)$, the speed exponent in (1.5) is $\frac{1}{q}$. It is enough to have again $q\delta_0 < 2/d$.*

Proof of Lemma 2.4. First, we write

$$S_q^{(L)} - E\big[\|l_n\|_q^q\big] = \sum_{k=1}^{2^L} Z(k) + R_1, \quad \text{with } Z(k) = \big\|l_k^{(L)}\big\|_q^q - E\big[\big\|l_k^{(L)}\big\|_q^q\big] \tag{2.24}$$

and

$$R_1 = \sum_{k=1}^{2^L} \big(E\big[\big\|l_k^{(L)}\big\|_q^q\big] - \kappa(q,d) n_k^{(L)}\big) - \big(E\big[\|l_n\|_q^q\big] - \kappa(q,d)n\big).$$

Using Lemma 1.8 in $d \geq 3$, we have a constants c_d such that

$$|R_1| \leq c_d \psi_d(n) + c_d \sum_{k=1}^{2^L} \psi_d\big(n_k^{(L)}\big) \leq c_d \left(\psi_d(n) + 2^{L+1} \psi_d\left(\frac{n}{2^L}\right)\right). \tag{2.25}$$

Thus, for $\xi_n \ge n^{-\delta}$ and (i) $0 \le \delta < \delta_0/2$ in $d = 3$, or (ii) $0 \le \delta < \delta_0$ in $d > 3$, we have

$$P\big(\big|S_q^{(L)} - E\big[\|l_n\|_q^q\big]\big| \ge \xi_n n\big) \le P\left(\left|\sum_{k=1}^{2^L} Z(k)\right| \ge \frac{\xi_n}{2} n\right). \tag{2.26}$$

We note that $|x| = \max(x, -x)$, and use Chebyshev's exponential inequality. For $\lambda \in [0, 1]$,

$$\begin{aligned} P\left(\pm\sum_{k=1}^{2^L} Z(k) \ge \frac{\xi_n}{2} n\right) &\le \exp\left(-\lambda\frac{\xi_n}{2}\left(\frac{2^L}{n}\right)^q n\right)\big(E\big[\mathrm{e}^{\pm\lambda(2^L/n)^q Z(k)}\big]\big)^{2^L} \\ &\le \exp\left(-\lambda\frac{\xi_n}{2}\left(\frac{2^L}{n}\right)^q n\right)\left(1 + \lambda^2\left(\frac{2^L}{n}\right)^{2q} \operatorname{var}\big(Z(k)\big)\right)^{2^L} \\ &\le \exp\left(-\lambda\frac{\xi_n}{2}\left(\frac{2^L}{n}\right)^q n + \lambda^2 2^L\left(\frac{2^L}{n}\right)^{2q} \operatorname{var}\big(Z(k)\big)\right). \end{aligned} \tag{2.27}$$

We used the uniform bound (2.22) on $|Z(k)|$ in the second inequality, and the fact that for $x \le 1$, we have $\mathrm{e}^x \le 1 + x + x^2$. We recall that the bound (1.22) holds in dimension 3 and more, and reads $\operatorname{var}(Z(k)) \le \frac{n}{2^L}\log^2(\frac{n}{2^L})$. Thus, (2.27) is useful if

$$\frac{\xi_n}{2}\left(\frac{2^L}{n}\right)^q n \ge 2\lambda 2^L \frac{n}{2^L}\log^2\left(\frac{n}{2^L}\right)\left(\frac{2^L}{n}\right)^{2q} \iff \xi_n \ge 4\lambda\log^2\left(\frac{n}{2^L}\right)\left(\frac{2^L}{n}\right)^q. \tag{2.28}$$

Since $\xi_n \ge n^{-\delta}$, (2.28) is implied if $\delta_0 q > \delta$, which holds if conditions (i) or (ii) of Lemma 2.4 are assumed.

In case (i) or (ii), we choose $\lambda = 1$, and take n large enough so that (2.28) holds. We then obtain (2.23). □

3. The sub-critical regime

We consider here the case $q < \frac{d}{d-2}$. The main result of this section is the upper bound of (1.5). Indeed, we have shown in the Introduction (in (1.14)) that (1.5) implies (1.6). Finally, the easy lower bound in (1.5) is proved in Section 3.4.

We have divided the proof into many sections. Our starting point is (1.16). With the notation of Section 2.1, we set, with $2\varepsilon_0 < 1$,

$$m_n = E\big[\|l_n\|_q^q\big] + n\varepsilon_0\xi_n \quad \text{and} \quad \varepsilon_n = n\xi_n(1 - \varepsilon_0).$$

In Section 3.1, we choose an appropriate subdivision of $[1, n]$. When $q < q_c(d)$, strategy B described in the Introduction suggests to divide the set of visited sites into those visited about $\xi_n^{1/(q-1)}$-times, and the remaining *too often visited sites*. The contribution of the former sites to $\sum \mathcal{I}_l$ in (1.16), is called the bottom-level term, and is treated in Section 3.2. The contribution of the latter sites is called the top-level term, and is treated in Section 3.3.

3.1. *A choice of a subdivision*

We first choose the largest α_0 such that

$$\alpha_0^{q-1}\sum_{l=0}^{\infty}\left(\frac{1}{2^{q-1}}\right)^l \le \frac{1}{16}, \tag{3.1}$$

and, for some positive integer j_0, $\alpha_0\xi_n^\gamma = 2^{j_0}$. Note that α_0 is bounded by 1, though j_0 grows with ξ_n. Recall that $\gamma = \frac{1}{q-1}$, and consider for $i = -j_0, \ldots, M_n$

$$b_i = \xi_n^\gamma \beta_i, \quad \text{with } \beta_i = \alpha_0 2^i, \tag{3.2}$$

where M_n is such that β_{M_n} is of order $n^{1/q_c(d)}$. We divide the intersection local times according to whether $l_k^{(l)}(z) \geq \alpha_0 \xi_n^\gamma$ (which yields what we call a top-level term), or $l_k^{(l)}(z) < \alpha_0 \xi_n^\gamma$ (which yields what we call a bottom-level term). Introduce, for $l \leq L$

$$\mathcal{C}_n^\uparrow(l) = \sum_{i\geq 0} \sum_{k=1}^{2^l} 2^q \big(\xi_n^\gamma \beta_{i+1}\big)^{q-1} \big(\mathbb{1}_{\mathcal{G}_{2k,i}^{(l)}} l_{2k-1}^{(l)}\big(\mathcal{D}_{2k,i}^{(l)}\big) + \mathbb{1}_{\mathcal{G}_{2k-1,i}^{(l)}} l_{2k}^{(l)}\big(\mathcal{D}_{2k-1,i}^{(l)}\big)\big), \tag{3.3}$$

where for a subset A of $\mathbb{Z}^d$, $l_k^{(l)}(A) = \sum_{z\in A} l_k^{(l)}(z)$ and,

$$\mathcal{C}_n^\downarrow(l) = \sum_{-j_0\leq j<0} \sum_{k=1}^{2^l} 2^q \big(\xi_n^\gamma \beta_{j+1}\big)^{q-1} \big(l_{2k-1}^{(l)}\big(\mathcal{D}_{2k,j}^{(l)}\big) + l_{2k}^{(l)}\big(\mathcal{D}_{2k-1,j}^{(l)}\big)\big). \tag{3.4}$$

Note that if for any α_0 satisfying (3.1), we have $\alpha_0 \xi_n^\gamma < 1$, then there will be no term $\mathcal{C}_n^\downarrow(l)$. Note also that in both $\mathcal{C}_n^\uparrow(l)$ and $\mathcal{C}_n^\downarrow(l)$, the sum over k is over independent variables. We call $\mathcal{C}_n^\uparrow(l)$ the top-level term, and $\mathcal{C}_n^\downarrow(l)$ the bottom-level term.

We choose now L such that $2^L = n^{1-\delta_0}$, and inequality (2.6) of Remark 2.3 gives us that for some constant c_3

$$\sum_{\dagger\in\{\uparrow,\downarrow\}} \sum_{h=1}^{L} E\big[\mathcal{C}^\dagger(h)\big] \leq c_3 n^{1-\delta_0/2}. \tag{3.5}$$

We denote by $\bar{\mathcal{C}}^\dagger(h) = \mathcal{C}^\dagger(h) - E[\mathcal{C}^\dagger(h)]$. Finally, $\xi_n n \geq 4c_3 n^{1-\delta_0/2}$ implies that $\xi_n \geq 4c_3 n^{-\delta_0/2}$. Inequality (2.4) yields

$$P\big(\|l_n\|_q^q - E\big[\|l_n\|_q^q\big] \geq n\xi_n\big) \leq 2^L P\big(S_q^{(L)} - E\big[\|l_n\|_q^q\big] \geq n\varepsilon_0\big) + \sum_{h=1}^{L-1} 2^{h-1} \sum_{\dagger\in\{\uparrow,\downarrow\}} P\bigg(\bar{\mathcal{C}}^\dagger(h) \geq \frac{n\xi_n}{8}\bigg). \tag{3.6}$$

Note that from Lemma 2.4, we have

$$P\big(S_q^{(L)} - E\big[\|l_n\|_q^q\big] \geq \varepsilon_0 \xi_n n\big) \leq \exp\bigg(-\frac{\varepsilon_0 \xi_n}{2} n^{1-\delta_0 q}\bigg). \tag{3.7}$$

(3.7) shows that the contribution of $S_q^{(L)}$ to an excess self-intersection local times is negligible when $1 - \delta_0 q > 1 - \frac{2}{d}$, that is when $q\delta_0 < \frac{2}{d}$. It remains to show that the other terms in (3.6) are of the right order.

3.2. *The bottom-level terms*

Note that the bottom-level sets $\mathcal{C}_n^\downarrow$ depend on ξ_n. Also, from (3.7), we need only consider generation $l < L$ with $2^L = n^{1-\delta_0}$ for $q\delta_0 < \frac{2}{d}$. We establish in this section, the following result.

Lemma 3.1. *Assume $d \geq 3$ and $q > 1$. There is a constant $C > 0$ such that for any $h \in \{1, \ldots, L\}$, and $1 \leq \xi_n$*

$$P\Bigg(\sum_{l=h}^{L} \bar{\mathcal{C}}_n^\downarrow(l) \geq \frac{\xi_n n}{8}\Bigg) \leq \exp\big(-C \xi_n^{(2/d)(1/(q-1))} n^{1-2/d}\big). \tag{3.8}$$

Remark 3.2. *Recall that $\alpha_0 \leq 1$, and that if $\xi_n < 1$, then the terms $\{\mathcal{C}_n^\downarrow(l)\}$ vanish.*

Proof of Lemma 3.1. We first show that we can restrict the sum over i in the definition of $\mathcal{C}_n^{\downarrow}(l)$ in (3.4), to $0 > i \geq -l$. We make use of the obvious fact that for any generation l, the total time over which run the local times of the 2^l strands is n. In other words,

$$\sum_{k=1}^{2^l}\sum_{z\in\mathbb{Z}^d} l_k^{(l)}(z) = \sum_{k=1}^{2^l} n_k^{(l)} = n.$$

We consider now $\mathcal{C}_n^{\downarrow}(l)$ given in (3.4), and divide it into $\mathcal{C}^{\mathrm{I}}(l)$, where the sum over i spans $\{-1,\dots,-l\}$, and $\mathcal{C}^{\mathrm{II}}(l)$ for the remaining terms. In case $j_0 > l$, then $\mathcal{C}^{\mathrm{II}}(l)$ vanishes. Note that for any $h \leq L$,

$$\begin{aligned}\sum_{l=h}^{L}\mathcal{C}^{\mathrm{II}}(l) &\leq \sum_{l=h}^{L}\sum_{k=1}^{2^l}\sum_{i<-l} 2^q\left(\frac{\alpha_0\xi_n^{\gamma}}{2^l}\right)^{q-1}\left(l_{2k}^{(l)}\left(\mathcal{D}_{2k-1,i}^{(l)}\right)+l_{2k-1}^{(l)}\left(\mathcal{D}_{2k,i}^{(l)}\right)\right)\\ &\leq 2^q\sum_{l=h}^{L}\left(\frac{\alpha_0\xi_n^{\gamma}}{2^l}\right)^{q-1}\sum_{k=1}^{2^l}\sum_{z\in\mathbb{Z}^d} l_k^{(l)}(z) \leq n\xi_n\alpha_0^{q-1}\sum_{l\geq 0}\left(\frac{1}{2^{q-1}}\right)^l < \frac{n\xi_n}{16}.\end{aligned} \tag{3.9}$$

We have used the condition (3.1) to obtain the last line in (3.9).

Now, we use that

$$P\left(\sum_{l=h}^{L}\bar{\mathcal{C}}_n^{\downarrow}(l) \geq \frac{n\xi_n}{8}\right) \leq P\left(\sum_{l=h}^{L}\bar{\mathcal{C}}^{\mathrm{I}}(l) \geq \frac{n\xi_n}{16}\right) + P\left(\sum_{l=h}^{L}\bar{\mathcal{C}}^{\mathrm{II}}(l) \geq \frac{n\xi_n}{16}\right).$$

Thus, in view of (3.9), the choice of (3.1) implies that for any h, $P(\sum_{l=h}^{L}\bar{\mathcal{C}}^{\mathrm{II}}(l) \geq \frac{n\xi_n}{16}) = 0$.

We proceed now with estimating $\{\bar{\mathcal{C}}^{\mathrm{I}}(l)\}$. We do so in three steps. In Step 1, we perform the transformation of the terms of $\mathcal{C}_n^{\downarrow}(l)$ into independent variables X_k distributed as geometric variables. Then, Lemma A.1 provides the following bound. For $1/2 > \delta > 0$,

$$P\left(\sum_{k=1}^{2^l}\bar{X}_k \geq x_n\right) \leq \mathrm{e}^{-\delta x_n/4}, \quad \text{if } 4c_u 2^l \max\left(E\left[X_k^2\right]^{1-\delta}, E\left[X_k^2\right]\right) \leq x_n. \tag{3.10}$$

Step 2 establishes the condition in the right-hand side of (3.10). Finally, Step 3 compares x_n with the desired rate of decay.

Step 1. Note that the volume $|\mathcal{D}_{k,i}^{(l)}|$ times the minimal amount of time spent on sites of $\mathcal{D}_{k,i}^{(l)}$, that is b_i, is bounded by the total time left for a strand of random walk at generation l, so that

$$\left|\mathcal{D}_{k,i}^{(l)}\right| \times b_i \leq \frac{n}{2^l}. \tag{3.11}$$

Now, for fixed l and $0 > i \geq -l$, and for $k = 1,\dots,2^l$, we define, following (1.19),

$$X_k := \left(b_i\frac{2^l}{n}\right)^{2/d} l_{2k-1}^{(l)}\left(\mathcal{D}_{2k,i}^{(l)}\right) \quad \text{and} \quad X_k' = \left(b_i\frac{2^l}{n}\right)^{2/d} l_{2k}^{(l)}\left(\mathcal{D}_{2k-1,i}^{(l)}\right). \tag{3.12}$$

We set $\bar{X}_k = X_k - E[X_k]$ and $\bar{X}_k' = X_k' - E[X_k']$, and using (3.4), we rewrite

$$\left\{\sum_{l=h}^{L}\bar{\mathcal{C}}^{\mathrm{I}}(l) \geq \frac{n\xi_n}{16}\right\} = \left\{\sum_{l=h}^{L}\sum_{i=-l}^{-1}\sum_{k=1}^{2^l} 2^q b_{i+1}^{q-1}\left(\frac{n}{2^l b_i}\right)^{2/d}\left(\bar{X}_k + \bar{X}_k'\right) \geq \frac{n\xi_n}{16}\right\}. \tag{3.13}$$

We recall an obvious general bound (for any countable set of indices $\mathcal{A}$)

$$P\Big(\sum_{a\in\mathcal{A}} X_a \ge \sum_{a\in\mathcal{A}} \alpha_a\Big) \le \sum_{a\in\mathcal{A}} P(X_a \ge \alpha_a). \tag{3.14}$$

Now, (3.14) applied to the expression of the right-hand side of (3.13), yields

$$P\Big(\sum_{l=h}^{L} \bar{\mathcal{C}}^{\mathrm{I}}(l) \ge \frac{\xi_n n}{16}\Big) \le 2\sum_{l=1}^{L-1}\sum_{i=-l}^{-1} P\Big(\sum_{k=1}^{2^l} \bar{X}_k \ge x_{n,l,i}\Big), \tag{3.15}$$

with, for $0 > i \ge -l$, and $\varepsilon > 0$ (using $\sum_{i\le 0} 2^{\varepsilon i} = \sum_{l\ge 0} 2^{-\varepsilon l} = (1-2^{-\varepsilon})^{-1}$)

$$x_{n,l,i} = c_1 \frac{1}{b_{i+1}^{q-1}}\Big(\frac{b_i 2^l}{n}\Big)^{2/d} 2^{-\varepsilon(l-i)} n\xi_n \quad \text{and} \quad c_1 = \frac{(1-2^{-\varepsilon})^2}{32\times 2^q}.$$

The factor 2 appearing in the right-hand side of (3.15) comes from noting that $\bar{X}'_k$ has the same law, as $\bar{X}_k$.

Note that for any $k = 1,\dots,2^l$, $P(X_k > t) \le P(X > t)$, where $X = l_m(\tilde{\mathcal{D}})/|\tilde{\mathcal{D}}|^{2/d}$ with $m = n/2^l$, and $\tilde{\mathcal{D}}$ is a certain level set of local times $\tilde{l}_m$ independent from l_m. If P_0 and $\tilde{P}_0$ are the law of the two independent local times, then Lemma 1.2 of [4] yields

$$P(X_k > t) \le \tilde{E}_0\big[P_0\big(l_m(\tilde{\mathcal{D}}) > |\tilde{\mathcal{D}}|^{2/d} t\big)\big] \le \tilde{E}_0\Big[\exp\Big(-\kappa \frac{|\tilde{\mathcal{D}}|^{2/d} t}{|\tilde{\mathcal{D}}|^{2/d}}\Big)\Big] \le \mathrm{e}^{-\kappa t}. \tag{3.16}$$

Step 2. First, by Lemma A.2, we have

$$E\big[X_k^2\big] \le C'_d \psi_d^2\Big(\frac{n}{2^l}\Big)\Big(\frac{b_i 2^l}{n}\Big)^{4/d} \mathrm{e}^{-\kappa_d b_i}. \tag{3.17}$$

When we recall that $b_{i+1} = 2b_i$ (see (3.2)), (3.10) requires that for some constant K, and $0 < \delta < 1/2$

$$\Big(\psi_d^2\Big(\frac{n}{2^l}\Big)\Big(\frac{b_i 2^l}{n}\Big)^{4/d} \mathrm{e}^{-\kappa_d b_i}\Big)^{1-\delta} \le K \frac{1}{b_i^{q-1}}\Big(\frac{b_i 2^l}{n}\Big)^{2/d} 2^{-\varepsilon(l-i)} n\xi_n. \tag{3.18}$$

We use now that $\psi_d^2(k) \le k$ (see (A.3)), and (3.18) follows as soon as

$$b_i^{q-1+(2/d)(1-2\delta)} \mathrm{e}^{-(1-\delta)\kappa_d b_i} \le \Big(\frac{n}{2^l}\Big)^{(2/d)(1-2\delta)} 2^{-\varepsilon(l-i)} \xi_n. \tag{3.19}$$

Recall that $b_i \ge 1$, and the left-hand side of (3.19) is bounded from above and below by constants, when δ is small enough. Since $2^l \le n^{1-\delta_0}$ with $q\delta_0 < \frac{2}{d}$, we have that (3.19) holds with as soon as $\xi_n \ge 1$ (and choosing $\delta < 1/2$ and ε small enough). In particular, nothing prevents ξ_n to be as large as possible here.

Step 3. Lemma 3.1 is proved if we show that for some constant $K > 0$, and any i and l,

$$\xi_n^{(2/d)\gamma} n^{1-2/d} \le K x_{n,l,i} \quad \Big(\text{recall that } \gamma = \frac{1}{q-1}\Big). \tag{3.20}$$

Condition (3.20) is the most critical to check. It requires (recalling that $0 > i \ge -l$)

$$2^{-i(q-1)}\Big(\frac{\alpha_0 \xi_n^{\gamma} 2^{l+i}}{n}\Big)^{2/d} 2^{-\varepsilon(l-i)} n \ge \xi_n^{(2/d)\gamma} n^{1-2/d} \quad \Longleftarrow \quad (l+i)\frac{2}{d} - i(q-1) > \varepsilon(l-i), \tag{3.21}$$

which holds if $2\varepsilon < \min(q-1, \frac{2}{d})$. $\square$

3.3. *The top-level terms*

Lemma 3.3. *Assume $d \geq 3$ and $1 < q < q_c(d)$. There is a constant $C > 0$, and $\delta > 0$, such that for any $h \in \{1, \dots, L\}$ and $\xi_n \geq n^{-\delta}$*

$$P\left(\sum_{l=h}^{L} \bar{\mathcal{C}}_n^{\uparrow}(l) \geq \frac{\xi_n n}{8}\right) \leq \exp\bigl(-C\xi_n^{(2/d)(1/(q-1))} \min\bigl(1, \xi_n^{2/d}\bigr) n^{1-2/d}\bigr). \tag{3.22}$$

Proof. Following [3], we take two sequences of positive numbers $\{a_i, i = 1, \dots, M_n\}$, and for each i $\{p_l^{(i)}, l = h, \dots, L-1\}$ (to be made explicit later) with

$$\sum_{i=1}^{M_n} a_i = 1, \quad \text{and} \quad \text{for each } i \quad \sum_{l=h}^{L-1} p_l^{(i)} = 1. \tag{3.23}$$

For any $h \leq L$, we have

$$\begin{aligned} & P\left(\sum_{l=h}^{L} \bar{\mathcal{C}}_n^{\uparrow}(l) \geq \frac{\xi_n n}{8}\right) \\ & \quad \leq 2\sum_{l=1}^{L}\sum_{i\geq 1} P\left(\sum_{k=1}^{2^l} \mathbb{1}_{\mathcal{G}_{2k,i}^{(l)}} l_{2k-1}^{(l)}\bigl(\mathcal{D}_{2k,i}^{(l)}\bigr) - E\bigl[\mathbb{1}_{\mathcal{G}_{2k,i}^{(l)}} l_{2k}^{(l)}\bigl(\mathcal{D}_{2k-1,i}^{(l)}\bigr)\bigr] \geq \frac{n}{8 \cdot 2^q \beta_{i+1}^{q-1}} p_l^{(i)} a_i\right). \end{aligned} \tag{3.24}$$

We proceed as in the proof of Lemma 3.1 with Steps 1–3.

Step 1. We first bound $|\mathcal{D}_{2k,i}^{(l)}|$. Note that on $\mathcal{G}_{2k,i}^{(l)}$ for any k, i, we have

$$\bigl|\mathcal{D}_{2k,i}^{(l)}\bigr| \leq \min\left(\frac{n/2^l}{\beta_i \xi_n^{\gamma}}, \frac{n(2\kappa(q,d) + \xi_n)}{\beta_i^q \xi_n^{\gamma+1}}\right) \leq \frac{n}{\beta_i \xi_n^{\gamma} \min(1, \xi_n)} \min\left(\frac{1}{2^l}, \frac{2\kappa(q,d)+1}{\beta_i^{q-1}}\right). \tag{3.25}$$

We used in (3.25) that

$$\frac{2\kappa(q,d) + \xi_n}{\xi_n} \leq \bigl(2\kappa(q,d)+1\bigr)\frac{\max(\xi_n, 1)}{\xi_n} = \frac{2\kappa(q,d)+1}{\min(\xi_n, 1)}.$$

In order to use Lemma A.1, we need to normalize $l_{2k}^{(l)}(\mathcal{D}_{2k-1,i}^{(l)})$ with a constant smaller than $|\mathcal{D}_{2k-1,i}^{(l)}|^{-2/d}$. We choose, for l and i fixed,

$$\zeta_i^{(l)} = \left(\frac{\beta_i \xi_n^{\gamma} \min(1, \xi_n)}{n}\right)^{2/d} \begin{cases} \bigl(2\kappa(q,d)+1\bigr)^{-2/d} \beta_i^{(2/d)(q-1)} & \text{for } l \leq l_i^*, \\ 2^{(1/d)l} & \text{for } l > l_i^*, \end{cases} \tag{3.26}$$

with l_i^* is such that $2^{l_i^*} = (2\kappa(q,d)+1)^{-2}\beta_i^{2(q-1)}$. As in (3.16), we set

$$X_k = \zeta_i^{(l)} \mathbb{1}_{\{\mathcal{G}_{2k,i}^{(l)}\}} l_{2k-1}^{(l)}\bigl(\mathcal{D}_{2k,i}^{(l)}\bigr) \quad \text{and} \quad P(X_k > t) \leq \exp(-\kappa t).$$

Using (3.25), and the notation $\bar{X}_k$ for $X_k - E[X_k]$, we have

$$P\left(\sum_{l=h}^{L} \bar{\mathcal{C}}_n^{\uparrow}(l) \geq \frac{\xi_n n}{8}\right) \leq 2\sum_{l=h}^{L}\sum_{i\geq 1} P\left(\sum_{k=1}^{2^l} \bar{X}_k \geq x_{n,l,i}\right) \quad \text{with } x_{n,l,i} = \frac{n\zeta_i^{(l)}}{16(2^q+1)\beta_{i+1}^{q-1}} a_i p_l^{(i)}. \tag{3.27}$$

Step 2. We establish now a condition equivalent to (3.10).

$$2^{l(1+\delta_0(2/d))} E\bigl[X_k^2\bigr] \leq K\xi_n^{(2/d)\gamma} \min\bigl(1, \xi_n^{2/d}\bigr) n^{1-2/d} \tag{3.28}$$

for some constant K. Thus, when $l \le l_i^*$, and for some constant C

$$\begin{aligned}2^{l(1+\delta_0(2/d))}E\big[X_k^2\big] &\le 2^l 2^{l\delta_0(2/d)}\left(\beta_i^q \frac{\xi_n^\gamma}{n}\right)^{4/d}\min\big(1,\xi_n^{4/d}\big)C_d\psi_d^2\left(\frac{n}{2^l}\right)\mathrm{e}^{-\kappa_d\xi_n^\gamma\beta_i}\\ &\le C_d\min\big(1,\xi_n^{4/d}\big)\frac{n^{1-2/d}}{\xi_n^{4/d}}\left(\frac{2^l}{n}\psi_d^2\left(\frac{n}{2^l}\right)\right)\left(\frac{2^{l\delta_0}}{n}\right)^{2/d}\sup_{x>0}\big\{x^{4q/d}\exp(-\kappa_d x)\big\}\\ &\le Cn^{1-2/d-(2/d)(1-\delta_0(1-\delta_0))}. \qquad (3.29)\end{aligned}$$

In this case, (3.28) holds if for some constant C

$$\xi_n^\gamma \min(1,\xi_n) \ge \frac{C}{n^{(1-\delta_0(1-\delta_0))}}. \qquad (3.30)$$

(3.30) holds when $\xi_n \ge n^{-\delta}$ for $\delta > 0$ small enough. When $l > l_i^*$, for a constant C'

$$\begin{aligned}2^{l(1+\delta_0(2/d))}E\big[X_k^2\big] &\le 2^l 2^{l\delta_0(2/d)}\left(\beta_i \frac{\xi_n^\gamma}{n}\right)^{4/d}\min\big(1,\xi_n^{4/d}\big)C_d\psi_d^2\left(\frac{n}{2^l}\right)\mathrm{e}^{-\kappa_d\xi_n^\gamma\beta_i}2^{(2/d)l}\\ &\le C_d n^{1-2/d}\min\big(1,\xi_n^{4/d}\big)\left(\frac{2^l}{n}\psi_d^2\left(\frac{n}{2^l}\right)\right)\left(\frac{2^{l(1+\delta_0)}}{n}\right)^{2/d}\sup_{x>0}\big\{x^{4/d}\exp(-\kappa_d x)\big\}\\ &\le C'\min\big(1,\xi_n^{4/d}\big)n^{1-2/d-(2/d)\delta_0^2}. \qquad (3.31)\end{aligned}$$

When $\xi_n \ge 1$, (3.28) follows from (3.29) and (3.31). When $\xi_n < 1$, we need in addition that $\xi_n^{\gamma-1} \ge n^{-\delta_0^2}$.

Step 3. We show that we can choose $p_l^{(i)}$ and a_i such that for any i, l (and n large enough), there is a constant c, independent on i, l and n, and

$$\frac{n\zeta_i^{(l)}}{\beta_{i+1}^{q-1}}p_l^{(i)}a_i \ge cn^{1-2/d}\xi_n^{(2/d)\gamma}\min\big(1,\xi_n^{2/d}\big). \qquad (3.32)$$

It is possible to choose a normalizing constant a_0 (which depends only on q), such that for $i = 1,\dots,M_n$, $\sum_{i\ge 1} a_i \le 1$, where

$$a_i = a_0\left(\frac{\beta_{i+1}^{q-1}}{\beta_i^{(2/d)q}}\right)^{1/2} = a_0 2^{(q-1)/2}2^{-\alpha i}, \quad \text{with } \alpha := \frac{1}{2}\left(1-\frac{q}{q_c(d)}\right). \qquad (3.33)$$

Indeed, the condition $q < q_c(d)$ implies that α is positive, and the series in (3.33) is convergent.

Now, for a fixed $i \ge 1$, we turn to the choice of $\{p_l^{(i)}, l \ge 1\}$. We will choose later two constants p^* and $\bar{p}$ such that for $l < l_i^*$,

$$p_l^{(i)} = p^* 2^{-\alpha i}, \qquad (3.34)$$

whereas for $l > l_i^*$,

$$\begin{aligned}p_l^{(i)} = \bar{p}\frac{\beta_{i+1}^{q-1}}{\beta_i^{2/d}}\frac{2^{\alpha i}}{2^{l/d}} &\le \bar{p}\frac{\beta_{i+1}^{q-1}}{\beta_i^{2/d}}\left(\frac{\beta_i^{(2/d)q}}{\beta_{i+1}^{(q-1)}}\frac{1}{2^{l_i^*/d}}\right)^{1/2}\frac{2^{(q-1)/2}}{2^{l/(2d)}}\\ &\le \bar{p}\left(\frac{\beta_{i+1}^{q-1}}{\beta_i^{(2/d)q}}\frac{2^{q-1}}{2^{(l-l_i^*)/d}}\right)^{1/2} \le \bar{p}\frac{2^{(q-1)/2}}{2^{(l-l_i^*)/(2d)}}.\end{aligned}$$

We proceed now to normalize $\{p_l^{(i)}, l \geq 1\}$. We need to choose p^* and $\bar{p}$ such that for each i, $\sum_l p_l^{(i)} \leq 1$. Recall that there is c_1 such that $l_i^* \leq c_1 i$. Now, note that

$$\begin{aligned}\sum_l p_l^{(i)} &\leq p^* l_i^* 2^{-\alpha i} + \bar{p} 2^{(q-1)/2} \sum_{l > l_i^*} 2^{-(l-l_i^*)/(2d)} \\ &\leq p^* c_1 i 2^{-\alpha i} + \bar{p} 2^{(q-1)/2} \sum_{l>0} \frac{1}{2^{l/(2d)}} \\ &\leq c_1 p^* \sup_{x>0}\{x 2^{-\alpha x}\} + \bar{p} \frac{2^{(q-1)/2}}{2^{1/(2d)} - 1}.\end{aligned} \tag{3.35}$$

It is important to see that p^* and $\bar{p}$ can be chosen independently of i. Now, we check (3.32). For $l < l_i^*$,

$$\frac{n\zeta_i^{(l)}}{\beta_{i+1}^{q-1}} p_l^{(i)} a_i = a_0 p^* \frac{n\zeta_i^{(l)}}{\beta_i^{2q/d}} = a_0 p^* \xi_n^{(2/d)\gamma} \min\bigl(1, \xi_n^{2/d}\bigr) n^{1-2/d}. \tag{3.36}$$

For $l \leq l_i^*$,

$$\frac{n\zeta_i^{(l)}}{\beta_{i+1}^{q-1}} p_l^{(i)} a_i = a_0 \bar{p} 2^{(q-1)/2} \frac{n\zeta_i^{(l)}}{2^{l/d} \beta_i^{2/d}} = a_0 \bar{p} 2^{(q-1)/2} \xi_n^{(2/d)\gamma} \min\bigl(1, \xi_n^{2/d}\bigr) n^{1-2/d}. \tag{3.37}$$

This concludes the proof of Lemma 3.3. □

3.4. *The lower bound in* (1.5)

As in inequalities (80) and (81) of [5], the lower bound follows from Hölder's inequality. Indeed, it is immediate that $\|l_n\|_q^q / n \geq (n/|R_n|)^{q-1}$, where R_n is the set of visited sites up to time n. Thus, when n is large enough

$$|R_n| \leq \frac{n}{(2\kappa(q,d) + \xi_n)^\gamma} \quad \Longrightarrow \quad \|l_n\|_q^q \geq n\bigl(2\kappa(q,d) + \xi_n\bigr) \geq E\bigl[\|l_n\|_q^q\bigr] + \xi_n n.$$

Now, forcing the walk to stay in a ball $B(0, r_n)$ centered at the origin, and of radius r_n satisfying $r_n^d = n/(2\kappa(q,d) + \xi_n)^\gamma$ implies that $|R_n| \leq n/(2\kappa(q,d) + \xi_n)^\gamma$. The cost of this constraint is $\exp(-c\frac{n}{r_n^2})$, which yields the lower bound in (1.5), when we recall that $\xi_n \geq 1$.

3.5. *Proof of Theorem* 1.1

We collect the estimates of the previous sections in order to prove (1.5) allowing ξ to depend on n, as in Remark 1.5.

When $\xi_n \geq 1$, using the decomposition (3.6), with the estimates (3.7), (3.8) and (3.22), we obtain the upper bound in (1.6). The lower bound in (1.6) follows from Section 3.4.

When $\xi_n < 1$, then Lemma 2.4 imposes that $\xi_n \geq n^{-\delta}$ with $0 \leq \delta < \delta_0/2$, whereas Lemma 3.3 holds for some positive δ. Thus, we conclude that Remark 1.5 holds with (1.10). Note that a lower bound is missing in this case.

4. The super-critical regime

We consider here $q > q_c(d) = \frac{d}{d-2}$. The main result of this section is to show that only sites of $\{z\colon l_n(z) \geq (n\xi_n)^{1/q}/A\}$ (for some $A > 0$), contribute to realize the excess self-intersection, at a cost given in (1.7).

The proof of Theorem 1.2 relies on the following estimates. For any ε with $0<\varepsilon<1/q$, and any δ, with $0<\delta<1/3$, and two constants $A>A_0$, we write

$$
\begin{aligned}
P\big(\|l_n\|_q^q-E\big[\|l_n\|_q^q\big]\ge\xi_n n\big)\le{}& P\Big(\sum_z \mathbb{1}\{z\colon l_n(z)<\xi_n^{1/q}n^{1/q-\varepsilon}\}l_n^q(z)-E\big[\|l_n\|_q^q\big]\ge n\delta\xi_n\Big)\\
&+P\Big(\sum_z \mathbb{1}\Big\{\xi_n^{1/q}n^{1/q-\varepsilon}<l_n(z)\le\frac{(\xi_n n)^{1/q}}{A}\Big\}l_n^q(z)\ge n\delta\xi_n\Big)\\
&+P\Big(\sum_z \mathbb{1}\Big\{\frac{(\xi_n n)^{1/q}}{A}<l_n(z)\le\frac{(\xi_n n)^{1/q}}{A_0}\Big\}l_n^q(z)\ge n\xi_n(1-3\delta)\Big)\\
&+P\Big(\sum_z \mathbb{1}\Big\{l_n(z)>\frac{(\xi_n n)^{1/q}}{A_0}\Big\}l_n^q(z)\ge n\xi_n\delta\Big).
\end{aligned}
\tag{4.1}
$$

In Section 4.1, we show that the contribution of $\{z\colon l_n(z)<\xi_n^{1/q}n^{1/q-\varepsilon}\}$, for any $\varepsilon>0$, is negligible. More precisely, we establish that there is $\varepsilon'>0$ such that for any $\delta>0$, and n large enough

$$
P\Big(\sum_{z\in\mathbb{Z}^d}\mathbb{1}\{z\colon l_n(z)<\xi_n^{1/q}n^{1/q-\varepsilon}\}l_n^q(z)-E\big[\|l_n\|_q^q\big]\ge n\delta\xi_n\Big)\le\exp\big(-\xi_n^{1/q}n^{1/q+\varepsilon'}\big).
\tag{4.2}
$$

The proof of (4.2) is similar to that of Theorem 1.1.

In Section 4.3, we show the following lemma.

Lemma 4.1. *Assume $d\ge3$ and $q>q_c(d)$. There is constants A_0 and κ_d, such that for $\varepsilon>0$, and any $\xi_n>0$, and any integer n,*

$$
P\Big(\sum_{z\in\mathbb{Z}^d}\mathbb{1}\Big\{\xi_n^{1/q}n^{1/q-\varepsilon}<l_n(z)\le\frac{(\xi_n n)^{1/q}}{A_0}\Big\}l_n(z)^q\ge n\xi_n\Big)\le\exp\big(-\kappa_d\xi_n^{1/q}n^{1/q}\big).
\tag{4.3}
$$

Furthermore, there is a constant $C>0$ such that for $\delta>0$, and $A>A_0$

$$
P\Big(\sum_{z\in\mathbb{Z}^d}\mathbb{1}\Big\{\xi_n^{1/q}n^{1/q-\varepsilon}<l_n(z)\le\frac{(\xi_n n)^{1/q}}{A}\Big\}l_n^q(z)\ge n\delta\xi_n\Big)\le\exp\big(-CA\delta^{1-2/d}n^{1/q}\big).
\tag{4.4}
$$

Finally, since we have a transient random walk, it is obvious that for $c>0$,

$$
P\Big(\sum_z \mathbb{1}\Big\{l_n(z)\ge\frac{(\xi_n n)^{1/q}}{A_0}\Big\}l_n^q(z)\ge n\xi_n\delta\Big)\le P\Big(\exists z\colon l_n(z)\ge\frac{(\xi_n n)^{1/q}}{A_0}\Big)\le n\mathrm{e}^{-c(\xi_n n)^{1/q}/A_0}.
$$

The lower bound comes from requiring that the origin is visited $(n\xi_n)^{1/q}$ times.

4.1. *The contribution of* $\{z\colon l_n(z)<\xi_n^{1/q}n^{1/q-\varepsilon}\}$

The first step is to perform a approximation of $\|l_n\|_q^q$ over $\{z\colon l_n(z)<\xi_n^{1/q}n^{1/q-\varepsilon}\}$ as in Section 2. This is explained in Remark 2.1.

To allow for the possibility of ξ_n to depend on n, we need to trace the occurrences of ξ_n, and in this respect, it is useful to modify the subdivision chosen in (3.2). We choose again α_0 as in (3.1), and for $i\ge-j_0$ we keep $\beta_i=\alpha_0 2^i$, and

$$
\forall i<0\quad b_i=\xi_n^{1/(q-1)}\beta_i\quad\text{and}\quad\forall i\ge0\quad b_i=\xi_n^{1/q}\beta_i.
\tag{4.5}
$$

Recall that when $\xi_n < 1$, then $\mathcal{D}_{k,i}^{(l)} = \varnothing$ for $i < 0$, and $\mathcal{C}_n^{\downarrow}$ vanishes. However, when $\xi_n \ge 1$, for each k and l, there is an overlap between $\mathcal{D}_{k,-1}^{(l)}$ and $\mathcal{D}_{k,0}^{(l)}$ since $\xi_n^{1/q} \le \xi_n^{1/(q-1)}$.

For a small $\varepsilon > 0$, the subdivision $\{b_i\}$ covers $[1, \xi_n^{1/q} n^{1/q-\varepsilon}]$. As in the proof of Theorem 1.1, we start with (3.6). We first treat $\mathcal{C}_n^{\uparrow}(l)$.

Lemma 4.2. *Assume $d \ge 3$, and $q > q_c(d)$. We consider a sequence $\{\xi_n, n \in \mathbb{N}\}$ such that for some $\delta > 0$ small enough $\xi_n \ge n^{-\delta}$. There is a constant $\varepsilon' > 0$, such that for any $h \in \{1, \dots, L\}$ and for n large enough*

$$P\left(\sum_{l=h}^{L} \bar{\mathcal{C}}_n^{\uparrow}(l) \ge \frac{\xi_n n}{8}\right) \le \exp\left(-\xi_n^{1/q} \min\left(1, \xi_n^{2/d}\right) n^{1/q+\varepsilon'}\right). \tag{4.6}$$

When $q = q_c(d)$, then for any $h \in \{1, \dots, L\}$, and n large enough

$$P\left(\sum_{l=h}^{L} \bar{\mathcal{C}}_n^{\uparrow}(l) \ge \frac{\xi_n n}{8}\right) \le \exp\left(-\xi_n^{1/q} \min\left(1, \xi_n^{2/d}\right) n^{1/q-\varepsilon'}\right). \tag{4.7}$$

Remark 4.3. *When $1 > \xi_n \ge n^{-\delta}$ with δ small, then the terms $\{\mathcal{C}_n^{\downarrow}(l), l \le L\}$ vanish, whereas $S_q^{(L)}$ is negligible. Indeed, according to Lemma 2.4, it suffices to show that*

$$\xi_n n^{1-q\delta_0} \ge \xi_n^{1/q+2/d} n^{1/q+\varepsilon'},$$

which holds when $\xi_n > 1/\sqrt{n}$ (which we always assume).

When $\xi_n \ge 1$, and for a choice of $\delta_0 < 2/(dq)$, we have $\xi_n n^{1-q\delta_0} \ge (\xi_n n)^{1/q}$ so that by (2.23), we can neglect $S_q^{(L)}$. Also, recall that we can assume $\xi_n \le n^{q-1}$ (see Remark 1.3). This latter bound is equivalent to

$$\xi_n^{(2/d)(1/(q-1))} n^{1-2/d} \ge (\xi_n n)^{1/q}.$$

Now, (3.8) of Lemma 3.1 allows us to neglect the contribution of $\{\mathcal{C}_n^{\downarrow}(l), l \le L\}$.

Proof of Lemma 4.2. We proceed with Steps 1–3 as in the proofs of Lemmas 3.1 and 3.3.

Step 1. The first difference with the proof of Theorem 1.1, is the choice of the subdivision $\{b_i\}$ of (4.5). Note that the bound on $|\mathcal{D}_{k,i}^{(l)}|$ of (3.25) becomes

$$\left|\mathcal{D}_{k,i}^{(l)}\right| \le \left(2\kappa(q,d) + 1\right) \frac{n}{\min(1, \xi_n)\beta_i^q}.$$

This implies a new definition for $\zeta_i^{(l)}$. Also, note that the choice (3.33) for a_i is not possible since $\alpha < 0$ in this case. Thus, we set for $i \in \mathbb{N}$, and $\delta > 0$ to be chosen later,

$$a_i = \left(1 - 2^{-\delta}\right) 2^{-\delta i}, \qquad p_l^{(i)} = \left(1 - 2^{-\delta}\right) 2^{-\delta l} \quad \text{and} \quad \zeta_i^{(l)} = \left(\frac{\beta_i^q}{n} \min(1, \xi_n)\right)^{2/d}. \tag{4.8}$$

Accordingly, inequality (3.27) holds, but with

$$\begin{aligned} x_{n,l,i} &= \frac{n \xi_n^{1/q} \min(1, \xi_n^{2/d}) \zeta_i^{(l)}}{16(2^q + 1)\beta_{i+1}^{q-1}} p_l^{(i)} a_i \\ &= c 2^{-\delta(i+l)} \beta_i^{q(2/d)-(q-1)} \xi_n^{1/q} \min\left(1, \xi_n^{2/d}\right) n^{1-2/d} \\ &= c 2^{-\delta(i+l)} \xi_n^{1/q} \min\left(1, \xi_n^{2/d}\right) n^{1/q_c(d)} \beta_i^{1-q/q_c(d)}. \end{aligned} \tag{4.9}$$

Note that $q > q_c(d)$ implies that $x_{n,l,i}$ is small when β_i is large.

Step 2. We establish that for $\delta > 0$ small $2^{l(1+\delta)} E[X_k^2] \le x_{n,l,i}$. This latter inequality is equivalent to

$$2^{l(1+\delta)} \zeta_i^{(l)} \psi_d^2\left(\frac{n}{2^l}\right) \mathrm{e}^{-\kappa_d \xi_n^{1/q} \beta_i} \le c \frac{n \xi_n^{1/q}}{\beta_i^{q-1}} 2^{-\delta(i+l)}, \tag{4.10}$$

which is equivalent to

$$\left(\frac{2^l}{n} \psi_d^2\left(\frac{n}{2^l}\right)\right) 2^{\delta(i+2l)} \beta_i^{q2/d+(q-1)} \mathrm{e}^{-\kappa_d \xi_n^{1/q} \beta_i} \le c n^{2/d} \xi_n^{1/q} \min\left(1, \xi_n^{2/d}\right). \tag{4.11}$$

Since $\psi_d^2(k) \le k$ when $d \ge 3$, (4.11) holds for any β_i, $\delta > 0$ small enough, and $\xi_n \ge n^{-\delta}$.

Step 3. We distinguish the cases $q > q_c(d)$ and $q = q_c(d)$.

When $q > q_c(d)$, then we need to show that

$$x_{n,l,i} \ge \xi_n^{1/q} \min\left(1, \xi_n^{2/d}\right) n^{1/q+\varepsilon'}$$

using (4.9), this is equivalent to

$$n^{1/q_c(d)} \beta_i^{1-q/q_c(d)} 2^{-\delta(i+l)} \ge c n^{1/q+\varepsilon'}. \tag{4.12}$$

So (4.12) holds if for some $\varepsilon' > 0$

$$2^{\delta(i+l)} \beta_i^{q/q_c(d)-1} \le n^{1/q_c(d)-1/q-\varepsilon'}. \tag{4.13}$$

Since $\beta_i \le n^{1/q-\varepsilon}$, (4.13) holds for δ and ε' both small enough.

When $q = q_c(d)$, we need to show that

$$x_{n,l,i} \ge \xi_n^{1/q} \min\left(1, \xi_n^{2/d}\right) n^{1/q_c(d)-\varepsilon'}.$$

This is obvious as soon as $n^{\varepsilon'} \ge 2^{\delta(l+i)}$, which holds for $\varepsilon' > 0$, when δ is small enough. □

4.2. *Proof of Lemma* 1.4

Since the proof of Lemma 1.4 is similar to the proof given in Section 4.1, we do not give all details, but only focus on the differences. When dealing with $\{\|\Theta_{n^b}(l_n)\|_q^q \ge n^a\}$, with $a > 1$, our starting point is inequality (2.4) of Corollary 2.2 with $M = n^b$. We choose $m_n = E[\|l_n\|_q^q] + \varepsilon n^a$, for $\varepsilon < 1/2$, and $\varepsilon_n = (1-\varepsilon) n^a$. We use the sets $\{\mathcal{D}_{k,i}^{(l)}, i \in \mathbb{N}\}$ of Section 4.1. However, $\{b_i, i \in \mathbb{N}\}$ only cover $[1, n^b]$, and ξ_n of (3.2) is set to 1. This latter choice implies that there is no term $\mathcal{C}_n^{\downarrow}(l)$. Note that the bootstrap bound of (2.5) defining $\mathcal{G}_{k,i}^{(l)}$ is here $\{|\mathcal{D}_{k,i}^{(l)}| \le n^a / \beta_i^q\}$.

Now, we proceed as in the proof of Lemma 1.4. To see that $S_q^{(L)}$ has a negligible contribution, note that for $a > 1$ and any $\varepsilon > 0$, (2.23) implies that

$$P\left(S_q^{(L)}(n) - E\left[\|l_n\|_q^q\right] \ge \varepsilon n^a\right) \le \mathrm{e}^{-\varepsilon n^{a-q\delta_0}}.$$

Since $q\delta_0 < 2/d$, it is enough (and easy) to check that for $a > 1$ and $q \ge q_c(d)$

$$a - \frac{2}{d} > \left(1 - \frac{2}{d}\right) a - \left(\frac{q}{q_c(d)} - 1\right) b.$$

The main differences with the proof of Section 4.1, is $\zeta_i^{(l)}$ and $x_{n,l,i}$ which read here

$$\zeta_i^{(l)} = \frac{\beta_i^{(2/d)q}}{n^{(2/d)a}} \quad \text{and} \quad x_{n,l,i} = \frac{n^a \zeta_i^{(l)}}{2(2^q+1)\beta_{i+1}^{q-1}} p_l^{(i)} a_i. \tag{4.14}$$

Step 2 (similar to (4.10)) is easy to check here, and we omit to do it.

To check Step 3, i.e. the condition corresponding to (3.20), we recall the definition of a_i and $p_l^{(i)}$ given in (4.8), and use that $\beta_i \le n^b$ (and $q \ge q_c(d)$), to obtain

$$x_{n,l,i} = c2^{-\delta(i+l)} n^{a(1-2/d)} \beta_i^{1-q/q_c(d)} \ge c2^{-\delta(i+l)} n^{a(1-2/d)-b(q/q_c(d)-1)}. \tag{4.15}$$

In conclusion, we obtain for any $\varepsilon > 0$, and $\delta > 0$ small enough

$$P\left(\sum_{l=h}^{L} \bar{\mathcal{C}}_n^{\uparrow}(l) \ge n^a\right) \le \exp\bigl(-n^{\zeta(q,a,b)-\varepsilon}\bigr) \quad \text{with } \zeta(q,a,b) = a\left(1-\frac{2}{d}\right) - b\left(\frac{q}{q_c(d)} - 1\right). \tag{4.16}$$

4.3. *The contribution of* $\{z\colon \xi_n^{1/q} n^{1/q-\varepsilon} < l_n(z)\}$

In this section, we prove Lemma 4.1. We deal with sites whose local times is close to $n^{1/q}$. We follow now the proof of Lemma 3.1 of [2]. Let $\{\alpha_i, i = 1, \dots, M\}$ be a subdivision of $[\frac{1}{q} - \varepsilon, \frac{1}{q}]$, to be chosen later. We justify later in the proof, the choice of

$$A_0 = \exp\left(2\left(\sqrt{\frac{q}{q_c(d)}} - 1\right)\right). \tag{4.17}$$

Also, let $\{p_i, i = 0, \dots, M\}$ be positive number summing up to 1, and define for $i < M$, and $A \ge A_0$

$$\mathcal{D}_i = \bigl\{z\colon \xi_n^{1/q} n^{\alpha_i} \le l_n(z) < \xi_n^{1/q} n^{\alpha_{i+1}}\bigr\} \quad \text{and} \quad \alpha_M = \frac{1}{q} - \frac{\log(A)}{\log(n)}. \tag{4.18}$$

Now, as in (3.5) of [2] (see also Lemma 3.1 of [5]), we have for any $\delta > 0$

$$P\left(\sum_{z\in\bigcup\mathcal{D}_i} l_n^q(z) \ge n\delta\xi_n\right) \le \sup_{0\le i<M} \bigl\{C_i(n)\exp\bigl(-\kappa_d \xi_n^{1/q} \delta^{1-2/d} n^{\zeta_i} p_i^{1-2/d}\bigr)\bigr\}, \tag{4.19}$$

with an innocuous combinatorial term $C_i(n)$ independent on ξ_n. For $0 \le i < M$,

$$\zeta_i = \alpha_i + \left(1 - \frac{2}{d}\right)(1 - q\alpha_{i+1}) = \frac{1}{q} + \frac{q}{q_c}\left(\frac{1}{q} - \alpha_{i+1}\right) - \left(\frac{1}{q} - \alpha_i\right). \tag{4.20}$$

Set $a = \sqrt{q/q_c} > 1$, and for $i < M$

$$\frac{1}{q} - \alpha_i = a\left(\frac{1}{q} - \alpha_{i+1}\right), \quad \text{so that} \quad \frac{1}{q} - \alpha_i = a^{M-i}\left(\frac{1}{q} - \alpha_M\right) = \frac{a^{M-i}\log(A)}{\log(n)}. \tag{4.21}$$

Now, M is chosen such that $\alpha_0 = \frac{1}{q} - \varepsilon$, that is $\varepsilon\log(n) = a^M\log(A)$. Also, we have $\zeta_i = \frac{1}{q} + (a-1)(\frac{1}{q} - \alpha_i)$, and we choose (with a normalizing constant $\bar{p}$ depending only on a)

$$\left(\frac{p_i}{\bar{p}}\right)^{1-2/d} = \mathrm{e}^{-(a-1)a^{M-i}} \quad \text{and} \quad n^{\zeta_i} p_i^{1-2/d} = n^{1/q}\bar{p}^{1-2/d}\mathrm{e}^{((\log(A)-(a-1))a^{M-i})}. \tag{4.22}$$

We need to choose $\log(A_0) > (a-1)$, and our arbitrary choice of (4.17) achieves this goal. Thus, the smallest value of $n^{\zeta_i} p_i^{1-2/d}$ is $n^{1/q}\bar{p}A\exp(1-a)$. When we choose $A = A_0$, and $\delta = 1$, we obtain (4.3), whereas when we choose $A > A_0$, and $\delta < 1$, we reach (4.4).

5. About the CLT

It will be convenient to use, in this section, the notation $\mathcal{L}_q(n) = \|l_n\|_q^q$.

5.1. *Expectation estimates*

Proof of Lemma 1.8. Let n_1 and n be two integers with $n_1 \le n$, and let $n_2 = n - n_1$. Taking expectation in (2.13) yields

$$E\big[S_q^{(1)}\big] \le E\big[\mathcal{L}_q(n)\big] \le E\big[S_q^{(1)}\big] + E\big[\mathcal{I}_1(n_1, n_2)\big]. \tag{5.1}$$

We choose the subdivision $\{b_i, i \in \mathbb{N}\}$ with $b_i = i$, and compute $E[\mathcal{I}_1(n_1, n_2)]$. Now, using inequality (A.4) of Lemma A.2, as well as (2.1) we have constants c_d such that, when calling $l_{n_1}^{(1)} = l_{]0,n_1]}^{(1,1)}$ and $l_{n_2}^{(2)} = l_{[0,n_2[}^{(1,2)}$, and using that the local time of a site increases with the length of the time-period,

$$\begin{aligned} E\big[\mathcal{I}_1(n_1, n_2)\big] &\le 2^q \sum_{z\in\mathbb{Z}^d}\sum_{i\ge 1} b_{i+1}^{q-1}\big(l_{n_1}^{(1)}\big(\big\{z\colon l_{n_2}^{(2)}(z) \ge b_i\big\}\big) + l_{n_2}^{(2)}\big(\big\{z\colon l_{n_1}^{(1)}(z) \ge b_i\big\}\big)\big) \\ &\le C_d\psi_d\big(\max(n_1, n_2)\big)\sum_{i\ge 1}(i+1)^{q-1}\mathrm{e}^{-\kappa_d i} \le c_d\psi_d\big(\max(n_1, n_2)\big). \end{aligned} \tag{5.2}$$

Thus, if we call $a(n) = E[\mathcal{L}_q(n)]$, and use (5.1) and (5.2)

$$a(n_1) + a(n_2) \le a(n) \le a(n_1) + a(n_2) + c_d\psi_d\big(\max(n_1, n_2)\big). \tag{5.3}$$

We fix an integer n, and for any k (going to infinity), we perform its euclidean division $k = m_k n + r_k$ with $0 \le r_k < n$, and obtain from (5.3)

$$m_k a(n) \le m_k a(n) + a(r_k) \le a(m_k n + r_k) \le a(m_k n) + a(r_k) + c_d\psi_d(m_k n). \tag{5.4}$$

Now, we can use the almost dyadic decomposition of m_k, so that if $L(m_k)$ denote the integer part of $\log_2(m_k) + 1$, we have

$$\begin{aligned} a(m_k n) &\le a\big(m_1^{(1)}n\big) + a\big(m_2^{(1)}n\big) + c_d\big(\psi_d\big(m_1^{(1)}n\big) + \psi_d\big(m_2^{(1)}n\big)\big) \\ &\le m_k a(n) + c_d\sum_{l=1}^{L(m_k)}\sum_{j=1}^{2^l}\psi_d\big(m_j^{(l)}n\big) \\ &\le m_k a(n) + 2c_d\sum_{l=1}^{L(m_k)} 2^l\psi_d\left(\frac{m_k}{2^l}n\right) \le m_k a(n) + 4c_d\psi_d(n)m_k. \end{aligned} \tag{5.5}$$

The last line of (5.5) is obtained after a simple computation that we omit. Thus, we are left with

$$\frac{nm_k}{nm_k + r_k}\frac{a(n)}{n} \le \frac{a(k)}{k} \le \frac{nm_k}{nm_k + r_k}\frac{a(n)}{n} + \frac{a(r_k)}{k} + \frac{4c_d\psi_d(n)m_k}{m_k n + r_k}. \tag{5.6}$$

Now, we take first the limit $k = m_k n + r_k$ to infinity while n is fixed. We obtain

$$\frac{a(n)}{n} \le \liminf\frac{a(k)}{k} \le \limsup\frac{a(k)}{k} \le \frac{a(n)}{n} + \frac{4c_d\psi_d(n)}{n}. \tag{5.7}$$

Then, we take n to infinity to obtain the existence of a limit for $a(k)/k$, say $\kappa(q,d)$. Looking at (5.7) with an identification of the limit, we have, for any n

$$E\big[\mathcal{L}_q(n)\big] \le n\kappa(q,d) \le E\big[\mathcal{L}_q(n)\big] + 4c_d\psi_d(n).$$

and this is (1.21). □

5.2. *Variance estimates*

We estimate now the variance of $\mathcal{L}_q(n)$, and prove (1.22) and (1.23) of Theorem 1.9.

Step 1. We show first that (1.22) holds in any dimension greater or equal to 3. To estimate the variance of $\mathcal{L}_q(n)$, we use the following simple fact. If X, Y, Z are random variables, and $\varepsilon > 0$, then

$$Y \le X \le Y + Z \quad \Longrightarrow \quad \mathrm{var}(X) \le (1+\varepsilon)\,\mathrm{var}(Y) + \left(1 + \frac{1}{\varepsilon}\right) E\big[Z^2\big]. \tag{5.8}$$

Indeed, we have $|X - E[Y]| \le |Y - E[Y]| + Z$ (note that $Z \ge 0$), and

$$\mathrm{var}(X) = \inf_c E\big[(X-c)^2\big] \le E\big[\big(X - E[Y]\big)^2\big] \le (1+\varepsilon) E\big[\big(Y - E[Y]\big)^2\big] + \left(1 + \frac{1}{\varepsilon}\right) E\big[Z^2\big].$$

Thus, we have from (2.13) and (5.8)

$$S_1 \le \mathcal{L}_q(n) \le S_1 + \mathcal{I}_1(n_1, n_2) \quad \Longrightarrow \quad \mathrm{var}\big(\mathcal{L}_q(n)\big) \le (1+\varepsilon)\,\mathrm{var}(S_1) + \left(1 + \frac{1}{\varepsilon}\right) E\big[\mathcal{I}_1^2(n_1, n_2)\big]. \tag{5.9}$$

Similarly as in (5.2), we have a constant C_d such that

$$E\big[\mathcal{I}_1^2(n_1, n_2)\big] \le C_d \psi_d^2\big(\max(n_1, n_2)\big) \le C_d \psi_d^2(n), \tag{5.10}$$

where we only used that ψ_d is increasing. Thus,

$$\mathrm{var}\big(\mathcal{L}_q(n)\big) \le (1+\varepsilon)\big(\mathrm{var}\big(\mathcal{L}_q(n_1)\big) + \mathrm{var}\big(\mathcal{L}_q(n_2)\big)\big) + \left(1 + \frac{1}{\varepsilon}\right) C_d \psi_d^2(n). \tag{5.11}$$

Now, when we choose the almost dyadic decomposition of Section 3, (2.1) and using induction, we have

$$\mathrm{var}\big(\mathcal{L}_q(n)\big) \le (1+\varepsilon)^L \left(\sum_{k=1}^{2^L} \mathrm{var}\big(\mathcal{L}_q\big(n_k^{(L)}\big)\big) \right) + \left(1 + \frac{1}{\varepsilon}\right) C'_d \sum_{k=1}^{2^L} (1+\varepsilon)^{k-1} 2^{k-1} \psi_d^2\left(\frac{n}{2^{k-1}}\right). \tag{5.12}$$

Recall that $\psi_d^2(k) \le k$ for $d \ge 3$. Thus, when reaching $L = \lfloor \log_2(n) \rfloor$, $\mathrm{var}(\mathcal{L}_q(n_k^{(L)}))$ are of order 1, and there is a constant C, such that

$$\mathrm{var}\big(\mathcal{L}_q(n)\big) \le C(1+\varepsilon)^L 2^L + C'_d \left(1 + \frac{1}{\varepsilon}\right) \frac{(1+\varepsilon)^L}{\varepsilon} n. \tag{5.13}$$

Choosing $\varepsilon = 1/L$, we obtain (1.22) in $d \ge 3$.

Step 2. We consider now $d \ge 4$. We show that there is a constant C_d such that

$$\mathrm{var}\big(\mathcal{L}_q(n)\big) \le C_d n. \tag{5.14}$$

We go back to (5.11) and optimize over ε to obtain

$$\begin{aligned}\mathrm{var}\big(\mathcal{L}_q(n)\big) \le{}& \big(\mathrm{var}\big(\mathcal{L}_q(n_1)\big)\big) + \mathrm{var}\big(\mathcal{L}_q(n_2)\big) + C'_d \psi_d^2\big(\max(n_1, n_2)\big) \\ &+ 2\Big(\big(\mathrm{var}\big(\mathcal{L}_q(n_1)\big) + \mathrm{var}\big(\mathcal{L}_q(n_2)\big)\big) C'_d \psi_d^2\big(\max(n_1, n_2)\big)\Big)^{1/2}.\end{aligned} \tag{5.15}$$

Now, choose first $n = 2^m$, and $n_1 = n_2 = 2^{m-1}$, and set $a_k = \mathrm{var}(\mathcal{L}_q(2^k))2^{-k}$. Then, using (1.22) to estimate the cross-product in (5.15), we have

$$a_m \le a_{m-1} + r_m, \quad \text{with } r_m = \frac{C'_d \psi_d^2(2^m)}{2^m} + 2\left(\frac{C'_d c_d m^2 \psi_d^2(2^m)}{2^m}\right)^{1/2}. \tag{5.16}$$

When $d \geq 4$, $\psi_d^2(2^m) \leq Cm^2$ and $\{r_m, m \in \mathbb{N}\}$ defines a convergent series. Thus,

$$a_m \leq a_0 + \sum_{k=1}^{m} r_k \leq c_d := a_0 + \sum_{k=1}^{\infty} r_k \quad \Longrightarrow \quad \operatorname{var}\big(\mathcal{L}_q\big(2^m\big)\big) \leq c_d 2^m. \tag{5.17}$$

Now, write any integer n in terms of its binary decomposition $n = 2^{m_1} + \cdots + 2^{m_k}$, with $0 \leq m_1 < m_2 < \cdots < m_k$. We call now $n_1 = 2^{m_k}$, and $n_2 = n - n_1$, and note that $n_1 \geq n_2$. In $d \geq 4$, we use the bound $\psi_d(k) \leq \log(k)$ in (5.15), and the estimate (1.22) in bounding the term $\operatorname{var}(\mathcal{L}_q(n_1)) + \operatorname{var}(\mathcal{L}_q(n_2))$ which appears in the square root in (5.15). Thus, we obtain that there exists a constant c independent of n such that

$$\operatorname{var}\big(\mathcal{L}_q(n)\big) \leq \operatorname{var}\big(\mathcal{L}_q(n_1)\big) + \operatorname{var}\big(\mathcal{L}_q(n_2)\big) + cm_k^2\sqrt{2^{m_k}}. \tag{5.18}$$

By iterating (5.18), we obtain using (5.17)

$$\operatorname{var}\big(\mathcal{L}_q(n)\big) \leq \sum_{j=1}^{k} \operatorname{var}\big(\mathcal{L}_q\big(2^{m_j}\big)\big) + c\sum_{j=1}^{k} m_j^2\sqrt{2^{m_j}} \leq c_d \sum_{j=1}^{k} 2^{m_j} + c\sum_{j=1}^{k} \frac{m_j^2}{\sqrt{2^{m_j}}} 2^{m_j} \leq (c_d + cc_3)n, \tag{5.19}$$

where c_3 is a constant such that for any m, $m \leq c_3\sqrt{2^m}$.

Step 3. We show now how to obtain (1.23). Note first that using similar arguments as those leading to (5.9) and (5.15), we have

$$\big(\operatorname{var}\big(\mathcal{L}_q(n_1)\big) + \operatorname{var}\big(\mathcal{L}_q(n_2)\big)\big) \leq \operatorname{var}\big(\mathcal{L}_q(n)\big) + C'_d\psi_d^2\left(\frac{n}{2}\right) + 2\left(\operatorname{var}\big(\mathcal{L}_q(n)\big)C'_d\psi_d^2\left(\frac{n}{2}\right)\right)^{1/2}. \tag{5.20}$$

Thus, using (1.22) and (5.20), there is $c_1 > 0$ such that for any integer j,

$$\big|\operatorname{var}\big(\mathcal{L}_q\big(2^j\big)\big) - 2\operatorname{var}\big(\mathcal{L}_q\big(2^{j-1}\big)\big)\big| \leq c_1 j\sqrt{2^j}. \tag{5.21}$$

Now, we consider m, l, i integers, such that $2^m = 2^l 2^i$, and consider for $j = 1, \ldots, l$ the system of inequalities obtained from (5.21)

$$\big|2^j \operatorname{var}\big(\mathcal{L}_q\big(2^{i+l-j}\big)\big) - 2^{j-1}\operatorname{var}\big(\mathcal{L}_q\big(2^{i+l-j+1}\big)\big)\big| \leq c_1(i + l - j + 1)2^{j-1}\sqrt{2^{i+l-j+1}}. \tag{5.22}$$

By summing (5.22) for $j = 1, \ldots, l$, and using the triangle inequality, we obtain

$$\big|2^l \operatorname{var}\big(\mathcal{L}_q\big(2^i\big)\big) - \operatorname{var}\big(\mathcal{L}_q\big(2^m\big)\big)\big| \leq c_1\sqrt{2^{i+l}} \sum_{j=1}^{l} (i + l - j + 1)\sqrt{2^{j-1}}. \tag{5.23}$$

By dividing both sides of (5.23) by 2^m, we have a constant c_2 such that

$$\left|\frac{\operatorname{var}(\mathcal{L}_q(2^i))}{2^i} - \frac{\operatorname{var}(\mathcal{L}_q(2^m))}{2^m}\right| \leq \frac{c_2 i\sqrt{2^l}}{\sqrt{2^{i+l}}}. \tag{5.24}$$

In (5.24), we take first the limit l to infinity (recall that $2^m = 2^l 2^i$), then i to infinity to conclude that there exists

$$\lim_{n\to\infty} \operatorname{var}\big(\mathcal{L}_q\big(2^n\big)\big)/2^n = v(q,d) \quad \text{and} \quad \left|\frac{\operatorname{var}(\mathcal{L}_q(2^n))}{2^n} - v(q,d)\right| \leq \frac{c_2 n}{\sqrt{2^n}}. \tag{5.25}$$

It is easy to conclude (1.23). Indeed, for any integer n, consider its dyadic decomposition, say $n = 2^{m_1} + \cdots + 2^{m_k}$, and note that using (5.20) and Step 2, we can improve (5.6) into

$$\left|\operatorname{var}\big(\mathcal{L}_q(n)\big) - \sum_{j=1}^{k} \operatorname{var}\big(\mathcal{L}_q\big(2^{m_j}\big)\big)\right| \leq c_1 \sum_{j=1}^{k} m_j\sqrt{2^{m_j}}, \tag{5.26}$$

and (5.25) allows us to conclude.

5.3. *The central limit theorem*

The aim of this section is to prove (1.24). We use the notations of Section 3. We fix $\delta_1 > 0$ small, and let L_n be the integer part of $\log_2(\sqrt{n}n^{-\delta_1})$. Note that this choice $2^{L_n} \sim \sqrt{n}/n^{\delta_1}$ is different from the choice of Section 2.2 where $2^L \sim n^{1-\delta_0}$ for δ_0 smaller that $2/(dq)$.

If we define $R(n) = \mathcal{L}_q(n) - S_q^{(L_n)}$, then (1.16) yields

$$0 \le R(n) \le \sum_{l=1}^{2^{L_n}} \mathcal{I}_l. \tag{5.27}$$

By subtracting to $\mathcal{L}_q(n)$ its average, we obtain

$$\mathcal{L}_q(n) - E\big[\mathcal{L}_q(n)\big] = \sum_{k=1}^{2^{L_n}} Z_k^{(L_n)} + R(n) - E\big[R(n)\big], \tag{5.28}$$

with $Z_k^{(L_n)} = \mathcal{L}_q^{(k)}(n_k^{(L_n)}) - E[\mathcal{L}_q^{(k)}(n_k^{(L_n)})]$. As a first step, we show that $R(n)/\sqrt{n}$ vanishes in law. More precisely, we show that

$$\lim_{n\to\infty} \frac{E[R(n)]}{\sqrt{n}} = 0. \tag{5.29}$$

Then, as a second step, we invoke the CLT for triangular arrays (see, for instance, [7], p. 310), since we deal with independent random variables $\{Z_k^{(L_n)}, k = 1, \dots, 2^{L_n}\}$. The CLT states that for a standard normal variable Z

$$\frac{\sum_{k=1}^{2^{L_n}} Z_k^{(L_n)}}{\sqrt{\sum_{k=1}^{2^{L_n}} \operatorname{var}(Z_k^{(L_n)})}} \xrightarrow{\text{law}} Z, \tag{5.30}$$

provided that Lindeberg's condition holds. This latter condition reads in our context

$$\lim_{n\to\infty} \sup_{k\le 2^{L_n}} \frac{E[\mathbb{1}_{\{|Z_k^{(L_n)}|>\varepsilon\sqrt{n}\}}(Z_k^{(L_n)})^2]}{E[(Z_k^{(L_n)})^2]} = 0. \tag{5.31}$$

Assuming (5.29) and (5.31) hold, we rely on Lemma 1.8 to replace $E[\mathcal{L}_q(n)]$ by $n\kappa(q,d)$ at a negligible cost, and rely on Theorem 1.9 to replace the $\sum_k \operatorname{var}(Z_k^{(L_n)})$ by $nv(q,d)$. Indeed, note that by (1.23)

$$\big|\operatorname{var}\big(Z_k^{(L_n)}\big) - n_k^{(L_n)} v(q,d)\big| \le c(q,d)\log\big(n_k^{(L_n)}\big)\sqrt{n_k^{(L_n)}}, \tag{5.32}$$

so that by summing over $k = 1, \dots, 2^{L_n}$,

$$\left|\sum_{k=1}^{2^{L_n}} \operatorname{var}\big(Z_k^{(L_n)}\big) - nv(q,d)\right| \le c(q,d)2^{L_n}\sqrt{\frac{n}{2^{L_n}}}\log\left(\frac{n}{2^{L_n}}\right) \le c(q,d)n^{3/4}\frac{\log(\sqrt{n}n^{\delta_1})}{\sqrt{n^{\delta_1}}}. \tag{5.33}$$

Step 1. We estimate the expectation of $R(n)$. From (1.17) and Lemma A.2, with $b_i = i$,

$$E[\mathcal{I}_l] \le \sum_{k=1}^{2^l}\sum_{i\ge 0} 2^q(i+1)^{q-1}\mathrm{e}^{-\kappa_d i}C_d\psi_d\big(n_k^{(l)}\big) \le C_d' 2^l \log\left(\frac{n}{2^l}\right). \tag{5.34}$$

Thus, $E[R(n)] \le C' 2^{L_n} \log(n) \le C' \frac{\log(n)\sqrt{n}}{n^{\delta_1}}$ and $\lim_{n\to\infty} E[\frac{R(n)}{\sqrt{n}}] = 0$.

Step 2. To check Lindeberg's condition, we start with estimating $P(|Z_k^{(L_n)}| \ge \varepsilon\sqrt{n})$. To simplify notation, we set $n_k = n_k^{(L_n)}$, and we note that

$$P\big(\big|Z_k^{(L_n)}\big| \ge \varepsilon\sqrt{n}\big) = P\big(\big|\mathcal{L}_q(n_k) - E\big[\mathcal{L}_q(n_k)\big]\big| \ge \xi_{n_k} n_k\big) \quad \text{and} \quad \xi_{n_k} = \frac{\varepsilon\sqrt{n}}{n_k} \ge \frac{\varepsilon}{2n_k^{\delta}}, \tag{5.35}$$

with $\delta = \frac{2\delta_1}{1+2\delta_1}$. Thus, Lindeberg's condition is written as a large deviation for $\mathcal{L}_q(n_k)$. Note that n_k is almost the scale of the CLT. We now use Remark 1.5, and Lemma A.3 of the Appendix. We apply (1.10), (1.11) of Remark 1.5, and (A.5) and (A.6) of Lemma A.3, to obtain for arbitrarily small ε' and δ

$$\begin{aligned} P\big(\big|Z_k^{(L_n)}\big| \ge \varepsilon\sqrt{n}\big) &\le P\big(Z_k^{(L_n)} \ge \varepsilon\sqrt{n}\big) + P\big(Z_k^{(L_n)} \le -\varepsilon\sqrt{n}\big) \\ &\le \exp\left(-C\left(\frac{2\varepsilon}{n_k^{\delta}}\right)^{\max(1/q,(2/d)\gamma)+2/d} n_k^{\min(1/q_c(d),1/q)-\varepsilon'}\right) + \mathrm{e}^{-(\varepsilon/4)n_k^{1-q\delta_0-\delta}}. \end{aligned} \tag{5.36}$$

Inequality (5.36) with the uniform bound $|Z_k^{(L_n)}| \le n^{q(\delta+1/2)}$, and the lower bound on $\mathrm{var}(Z_k^{(L_n)})$ in (5.32), imply that Lindeberg's condition (5.31) holds.

Appendix

In this section, we recall and improve some key estimates for dealing with large deviation for intersection local times. First, we recall a special form of Lemma 5.1 of [3].

Lemma A.1 (Lemma 5.1 of [3]). *Assume* $\{Y_1, \dots, Y_n\}$ *are positive and independent. Furthermore, assume that there is a constant* $C > 0$ *such that for any* $i \in \{1, \dots, n\}$

$$\forall t > 0 \quad P(Y_i > t) \le C \exp(-t). \tag{A.1}$$

Then, for some $c_u > 0$, *and any* $0 < \delta < 1$, *we have for any integer* n

$$P\left(\sum_{i=1}^{n}\big(Y_i - E[Y_i]\big) \ge x_n\right) \le \exp\left(c_u \delta^{2(1-\delta)} n \max_i\big(E\big[Y_i^2\big], E\big[Y_i^2\big]^{1-\delta}\big) - \frac{\delta}{2}x_n\right). \tag{A.2}$$

Secondly, we improve Lemma 5.3 of [3] into inequalities we believe are optimal. Consider two independent random walks $\{S(n), \tilde{S}(n), n \in \mathbb{N}\}$, and for an integer k, denote $\tilde{D}_n(k) := \{z \in \mathbb{Z}^d\colon \tilde{l}_n(z) > k\}$. We recall that if l_n is the local times and A a subset of $\mathbb{Z}^d$, then $l_n(A) = \sum_{z\in A} l_n(z)$.

Lemma A.2. *Assume dimension* $d \ge 3$. *There are constants* C_d, C'_d, κ_d *such that*

$$E\big[l_n\big(\tilde{D}_n(k)\big)\big] \le C_d \mathrm{e}^{-\kappa_d k}\psi_d(n), \quad \textit{with } \psi_d(n) = \begin{cases} n^{1/2} & \textit{for } d = 3, \\ \log(n) & \textit{for } d = 4, \\ 1 & \textit{for } d \ge 5, \end{cases} \tag{A.3}$$

and,

$$E\big[l_n\big(\tilde{D}_n(k)\big)^2\big] \le C'_d \mathrm{e}^{-\kappa_d k}\psi_d(n)^2. \tag{A.4}$$

Finally, we prove the following lemma. This result is not optimal, but suffices for our purpose.

Lemma A.3. *Assume* $d \ge 3$, *and take* $1 > \xi_n \ge n^{-\delta}$ *for* $\delta \le \delta_0/3$ *small enough.*

(i) *When $q \geq q_c(d)$, then for any $\varepsilon > 0$,*

$$P\big(\mathcal{L}_q(n) - E\big[\mathcal{L}_q(n)\big] \geq \xi_n n\big) \leq \exp\big(-C\xi_n^{1/q+2/d} n^{1/q-\varepsilon}\big). \tag{A.5}$$

(ii) *For any $q > 1$,*

$$P\big(\mathcal{L}_q(n) - E\big[\mathcal{L}_q(n)\big] \leq -\xi_n n\big) \leq \exp\left(-\frac{\xi_n}{2} n^{1-q\delta_0}\right). \tag{A.6}$$

A.1. *Proof of Lemma A.2*

To emphazise the starting point, we denote by P_z the law of the random walk started at site $z \in \mathbb{Z}^d$. We let $H_z = \inf\{n \geq 0\colon\ S(n) = z\}$, and use Theorem 3.2.3 of Lawler [17].

$$\sum_{z\in\mathbb{Z}^d} P_0(H_z \leq n)^2 \leq \sum_{z\in\mathbb{Z}^d} \left(\sum_{k=0}^{n} P_0\big(S(k) = z\big)\right)^2 \leq C_d \psi_d(n). \tag{A.7}$$

Now call $P_0(l_\infty(0) > 1) = \mathrm{e}^{-\kappa_d} < 1$, the return probability, and

$$E_0\big[l_\infty(0)\big] = \frac{1}{1-\mathrm{e}^{-\kappa_d}} \quad \text{and} \quad E_0\big[l_\infty(0)^2\big] = \frac{1+\mathrm{e}^{-\kappa_d}}{(1-\mathrm{e}^{-\kappa_d})^2}.$$

It is easy to see that for any $z \in \mathbb{Z}^d$

$$E_0\big[l_n(z)\big] \leq P_0(H_z \leq n) E_0\big[l_\infty(0)\big] \quad \text{and} \quad E_0\big[l_n^2(z)\big] \leq P_0(H_z \leq n) E_0\big[l_\infty^2(0)\big].$$

Similarly,

$$P_0\big(l_n(z) > k\big) \leq P_0(H_z \leq n) P_z\big(l_\infty(z) > k\big) = \mathrm{e}^{-\kappa_d k} P_0(H_z \leq n).$$

Thus, there is C_d such that

$$E\big[l_n\big(\tilde{D}_n(k)\big)\big] = \sum_{z\in\mathbb{Z}^d} E_0\big[l_n(z)\big] P_0\big(l_n(z) > k\big) \leq \mathrm{e}^{-\kappa_d k} E_0\big[l_\infty(0)\big] \sum_{z\in\mathbb{Z}^d} P_0(H_z \leq n)^2 \leq C_d \mathrm{e}^{-\kappa_d k} \psi_d(n). \tag{A.8}$$

Now, we expand the square of $l_n(\tilde{D}_n(k))$

$$\begin{aligned} l_n\big(\tilde{D}_n(k)\big)^2 &= \left(\sum_{z\in\mathbb{Z}^d} l_n(z) \mathbb{1}\big\{\tilde{l}_n(z) > k\big\}\right)^2 \\ &= \sum_z l_n(z)^2 \mathbb{1}\big\{\tilde{l}_n(z) > k\big\} + \sum_{z\neq z'} l_n(z) l_n\big(z'\big) \mathbb{1}\big\{\tilde{l}_n(z) > k, \tilde{l}_n\big(z'\big) > k\big\}. \end{aligned} \tag{A.9}$$

After taking the expectation of $l_n(\tilde{D}_n(k))^2$

$$\begin{aligned} E\big[l_n\big(\tilde{D}_n(k)\big)^2\big] &= \sum_z E_0\big[l_n(z)^2\big] P_0\big(l_n(z) > k\big) + \sum_{z\neq z'} E_0\big[l_n(z) l_n\big(z'\big)\big] P_0\big(l_n(z) \wedge l_n\big(z'\big) > k\big) \\ &\leq E_0\big[l_n(0)^2\big] \mathrm{e}^{-\kappa_d k} \sum_z P_0(H_z \leq n)^2 \\ &\quad + \sum_{z\neq z'} E_0\big[l_n(z) l_n\big(z'\big)\big] P_0\big(l_n(z) \wedge l_n\big(z'\big) > k\big). \end{aligned} \tag{A.10}$$

Now, in the last term in (A.10), we distinguish which of z or z' is hit first.

$$\begin{aligned} P_0(l_n(z)\wedge l_n(z')>k) &\le P_0(H_z<H_{z'},l_n(z')>k)+P_0(H_{z'}<H_z,l_n(z)>k)\\ &\le P_0(H_z\le n)P_z(l_n(z')>k)+P_0(H_{z'}\le n)P_{z'}(l_n(z)>k)\\ &\le \mathrm{e}^{-\kappa_d k}\big(P_0(H_z\le n)P_z(H_{z'}\le n)+P_0(H_{z'}\le n)P_{z'}(H_z\le n)\big). \end{aligned} \tag{A.11}$$

We treat now the term $E_0[l_n(z)l_n(z')]$. We have

$$\begin{aligned} E_0[l_n(z)l_n(z')] &= \sum_{k<k'\le n}\big(P_0(S(k)=z)P_z(S(k'-k)=z')+P_0(S(k)=z')P_{z'}(S(k'-k)=z)\big)\\ &\le E_0[l_n(z)]E_z[l_n(z')]+E_0[l_n(z')]E_{z'}[l_n(z)]\\ &\le E_0[l_\infty(0)]^2\big(P_0(H_z\le n)P_z(H_{z'}\le n)+P_0(H_{z'}\le n)P_{z'}(H_z\le n)\big). \end{aligned} \tag{A.12}$$

Thus, with the help of (A.11) and (A.12), (A.10) reads

$$\begin{aligned} E\big[l_n(\tilde D_n(k))^2\big] &\le E_0[l_n(0)^2]\mathrm{e}^{-\kappa_d k}\sum_z P_0(H_z\le n)^2\\ &\quad + E_0[l_\infty(0)]^2\mathrm{e}^{-\kappa_d k}\sum_{z\ne z'}\big(P_0(H_z\le n)P_z(H_{z'}\le n)+P_0(H_{z'}\le n)P_{z'}(H_z\le n)\big)^2\\ &\le E_0[l_n(0)^2]\mathrm{e}^{-\kappa_d k}\sum_z P_0(H_z\le n)^2\\ &\quad + 2E_0[l_\infty(0)]^2\mathrm{e}^{-\kappa_d k}\sum_{z\ne z'}P_0(H_z\le n)^2P_z(H_{z'}\le n)^2+P_0(H_{z'}\le n)^2P_{z'}(H_z\le n)^2. \end{aligned} \tag{A.13}$$

Now, we use translation invariance and (A.7)

$$\sum_{z\ne z'}P_0(H_z\le n)^2P_z(H_{z'}\le n)^2\le\Big(\sum_z P_0(H_z\le n)^2\Big)^2\le C_d^2\psi_d(n)^2.$$

The result (A.4) follows at once.

A.2. *Proof of Lemma A.3*

The proof of (i) follows from (4.7) of Lemma 4.2, and Remark 4.3 which deals with the contribution of $\{z\colon l_n(z)<\xi_n^{1/q}n^{1/q-\varepsilon}\}$. Using that for a transient walk, the local time of a site is bounded by a geometric variable, we have for a small $\delta>0$ and a constant $c>0$,

$$P\Big(\sum_z \mathbb{1}\{l_n(z)\ge\xi_n^{1/q}n^{1/q-\varepsilon}\}l_n^q(z)\ge n\xi_n\delta\Big)\le P\big(\exists z\colon l_n(z)\ge\xi_n^{1/q}n^{1/q-\varepsilon}\big)\le n\mathrm{e}^{-c\xi_n^{1/q}n^{1/q-\varepsilon}}.$$

Point (ii) follows from the lower bound in (1.16): $\mathcal{L}_q(n)\ge S_q^{(L)}$. Indeed, we choose $\delta=\delta_0/3$, (with $\delta_0<2/(dq)$) and L such that $2^L\sim n^{1-\delta_0}$. Then, we first have

$$\mathcal{L}_q(n)-E[\mathcal{L}_q(n)]\le-\xi_n n\quad\Longrightarrow\quad S_q^{(L)}-E[\mathcal{L}_q(n)]\le-\xi_n n.$$

Now, Lemma 2.4 gives us

$$P\big(S_q^{(L)}-E[\mathcal{L}_q(n)]\ge-\xi_n n\big)\le\exp\Big(-\frac{\xi_n}{2}n^{1-q\delta_0}\Big). \tag{A.14}$$

Acknowledgement

We would like to thank an anonymous referee for his questions, suggestions and comments which led to correcting a mistake in the first version, as well as an improved exposition.

References

[1] M. Aizenman. The intersection of Brownian paths as a case study of a renormalization group method for quantum field theory. *Comm. Math. Phys.* **97** (1985) 91–110. MR0782960
[2] A. Asselah. Large deviations principle for the self-intersection local times for simple random walk in dimension 5 or more. Preprint, 2007. Available at: arXiv:0707.0813.
[3] A. Asselah. Large deviations for the Self-intersection local times for simple random walk in dimension $d=3$. *Probab. Theory Related Fields* **141** (2008) 19–45. MR2372964
[4] A. Asselah and F. Castell. A note on random walk in random scenery. *Ann. Inst. H. Poincaré* **43** (2007) 163–173. MR2303117
[5] A. Asselah and F. Castell. Random walk in random scenery and self-intersection local times in dimensions $d \geq 5$. *Probab. Theory Related Fields* **138** (2007) 1–32. MR2288063
[6] M. Becker and W. König. Moments and distribution of the local times of a transient random walk on $\mathbb{Z}^d$. *J. Theoret. Probab.* **22** (2009) 365–374.
[7] P. Billingsley. *Probability and Measure*. Wiley, New York, 1979. MR0534323
[8] E. Bolthausen and U. Schmock. On self-attracting d-dimensional random walks. *Ann. Probab.* **25** (1997) 531–572. MR1434118
[9] D. C. Brydges and G. Slade. The diffusive phase of a model of self-interacting walks. *Probab. Theory Related Fields* **103** (1995) 285–315. MR1358079
[10] X. Chen. Limit laws for the energy of a charged polymer. *Ann. Inst. H. Poincaré Probab. Statist.* **44** (2008) 638–672. MR2446292
[11] X. Chen. Random walk intersections: Large deviations and some related topics. 2009. To appear.
[12] A. Dvoretzky and P. Erdös. Some problems on random walk in space. In *Proc. Berkeley Symposium 1951* 353–367. Univ. California Press, Berkeley, 1951. MR0047272
[13] S. Edwards. The statistical mechanics of polymers with excluded volume. *Proc. Phys. Sci* **85** (1965) 613–624. MR0183442
[14] G. Felder and J. Frölich. Intersection properties of simple random walks: A renormalization group approach. *Comm. Math. Phys.* **97** (1985) 111–124. MR0782961
[15] K. Fleischmann, P. Mörters and V. Wachtel. Moderate deviations for random walk in random scenery. *Stochastic Process. Appl.* **118** (2008) 1768–1802. MR2454464
[16] N. C. Jain and W. E. Pruitt. The range of transient random walk. *J. Anal. Math.* **24** (1971) 369–393. MR0283890
[17] G. Lawler. *Intersection of Random Walks. Probability and Its Applications*. Birkhäuser, Boston, MA, 1991. MR1117680
[18] J.-F. Le Gall. Propriétés d'intersection des marches aléatoires. *Comm. Math. Phys.* **104** (1985) 471–507. MR0840748
[19] J.-F. Le Gall. Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan. *Séminaire de probabilités de Strasbourg* **19** (1985) 314–331. MR0889492
[20] J.-F. Le Gall and J. Rosen. The range of stable random walks. *Ann. Probab.* **19** (1991) 650–705. MR1106281
[21] M. J. Westwater. On Edwards' model for long polymer chains. *Comm. Math. Phys.* **72** (1980) 131–174. MR0573702
[22] S. R. S. Varadhan. Appendix to Euclidean quantum field theory by K.Symanzik. In *Local Quantum Field Theory*. R. Jost (Ed.). Academic Press, New York, 1966.