Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 3, 618–643
DOI: 10.1214/09-AIHP206
© Association des Publications de l'Institut Henri Poincaré, 2010

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

The ODE method for some self-interacting diffusions on $\mathbb{R}^d$

Aline Kurtzmann

Nancy Universite, Institut Elie Cartan Nancy, B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex, France.
E-mail: aline.kurtzmann@iecn.u-nancy.fr

Received 23 January 2008; revised 20 May 2009; accepted 11 June 2009

Abstract. The aim of this paper is to study the long-term behavior of a class of self-interacting diffusion processes on $\mathbb{R}^d$. These are solutions to SDEs with a drift term depending on the actual position of the process and its normalized occupation measure μ_t. These processes have so far been studied on compact spaces by Benaïm, Ledoux and Raimond, using stochastic approximation methods. We extend these methods to $\mathbb{R}^d$, assuming a confinement potential satisfying some conditions. These hypotheses on the confinement potential are required since in general the process can be transient, and is thus very difficult to analyze. Finally, we illustrate our study with an example on $\mathbb{R}^2$.

Résumé. Le but de cet article est d'étudier le comportement asymptotique d'une classe de processus en auto-interaction sur $\mathbb{R}^d$. Ces processus de diffusion s'écrivent comme solution d'E.D.S. dont le terme de dérive dépend à la fois de la position actuelle du processus et de sa mesure empirique μ_t. Jusqu'à présent, Benaïm, Ledoux et Raimond ont mené l'étude de ce type de diffusions sur des espaces compacts, via des méthodes d'approximation stochastique. Nous étendons ces techniques à $\mathbb{R}^d$, en supposant l'existence d'un potentiel de confinement (vérifiant certaines conditions). Nous avons besoin de ces hypothèses sur le potentiel de confinement, car, en général, un tel processus peut être transient. Nous concluons cet article par un exemple sur $\mathbb{R}^2$.

MSC: 60K35; 37C50

Keywords: Self-interaction diffusion; Reinforced processes; Stochastic approximation

1. Introduction

This paper addresses the long-term behavior of a class of "self-interacting diffusion" processes $(X_t, t \geq 0)$ living on $\mathbb{R}^d$. These processes are time-continuous and non-Markov. They are solutions to a kind of diffusion SDEs, whose drift term depends on the whole past of the path through the occupation measure of the process. Due to their non-Markovianity, they often exhibit an interesting ergodic behavior.

1.1. *Previous results on self-interacting diffusions*

Time-continuous self-interacting processes, also named "reinforced processes," have already been studied in many contexts. Under the name of "Brownian polymers", Durrett and Rogers [10] first introduced them as a possible mathematical model for the evolution of a growing polymer. They are solutions of SDEs of the form

$$\mathrm{d}X_t = \mathrm{d}B_t + \mathrm{d}t \int_0^t \mathrm{d}s\, f(X_t - X_s),$$

where $(B_t; t \geq 0)$ is a standard Brownian motion on $\mathbb{R}^d$ and f is a given function. As the process $(X_t; t \geq 0)$ evolves in an environment changing with its past trajectory, this SDE defines a self-interacting diffusion, either self-repelling or self-attracting, depending on f.

Another modelisation, with dependence on the normalized occupation measure $(\mu_t, t \ge 0)$, has been considered by Benaïm et al. [5]. They introduced a process living in a compact smooth connected Riemannian manifold M without boundary:

$$\mathrm{d}X_t = \sum_{i=1}^N F_i(X_t) \circ \mathrm{d}B_t^i - \int_M \nabla_x W(X_t, y)\mu_t(\mathrm{d}y)\,\mathrm{d}t, \tag{1.1}$$

where W is a (smooth) interaction potential, $(B^1, \dots, B^N)$ is a standard Brownian motion on $\mathbb{R}^N$ and the symbol $\circ$ stands for the Stratonovich stochastic integration. The family of smooth vector fields $(F_i)_{1\le i\le N}$ comes from the Hörmander "sum of squares" decomposition of the Laplace–Beltrami operator $\Delta = \sum_{i=1}^N F_i^2$. The normalized occupation (or empirical) measure μ_t is defined by

$$\mu_t := \frac{r}{r+t}\mu + \frac{1}{r+t}\int_0^t \delta_{X_s}\,\mathrm{d}s, \tag{1.2}$$

where μ is an initial probability measure and r is a positive weight. In the compact-space case, they showed that the asymptotic behavior of μ_t can be related to the analysis of some deterministic dynamical flow defined on the space of the Borel probability measures. They went further in this study in [6] and gave sufficient conditions for the a.s. convergence of the empirical measure. When the interaction is symmetric, then μ_t converges a.s. to a local minimum of a nonlinear free energy functional (each local minimum having a positive probability to be chosen). All these results are summarized in a recent survey of Pemantle [18].

The present paper follows the same lead and extends the results of Benaïm et al. [5] in the non-compact setting. We present all results in the Euclidean space $\mathbb{R}^d$ for the sake of simplicity, but they can be extended to the case of a complete connected Riemannian manifold M without boundary with no further difficulty than the use of notations and a bit of geometry. One needs in particular to involve the Ricci curvature in the assumptions and work on the space $M\setminus \mathrm{cut}(o)$, where $\mathrm{cut}(o)$ is the cut locus of o (which has zero Lebesgue-measure).

1.2. *Statement of the problem*

Here we set the main definitions. Consider a confinement potential $V:\mathbb{R}^d \to \mathbb{R}_+$ and an interaction potential $W:\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$. For any bounded Borel measure μ, we consider the "convoled" function

$$W * \mu : \mathbb{R}^d \to \mathbb{R}, \quad W * \mu(x) := \int_{\mathbb{R}^d} W(x, y)\mu(\mathrm{d}y). \tag{1.3}$$

Our main object of interest is the self-interacting diffusion solution to

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}B_t - \big(\nabla V(X_t) + \nabla W * \mu_t(X_t)\big)\,\mathrm{d}t, \\ \mathrm{d}\mu_t = (\delta_{X_t} - \mu_t)\frac{\mathrm{d}t}{r+t}, \\ X_0 = x, \qquad \mu_0 = \mu, \end{cases} \tag{1.4}$$

where (B_t) is a d-dimensional Brownian motion. Our goal is to study the long-term behavior of $(\mu_t, t \ge 0)$. Let us recall that the main difference with the work [5] is that the state space is $\mathbb{R}^d$ and hence is not compact anymore. However, we are able to extend the results obtained in the compact case: the behavior of μ_t is closely related to the behavior of a deterministic flow. We will also give some sufficient conditions on the interaction potential in order to prove ergodic results for X.

Before stating the theorems proved in this paper, let us briefly describe the main results obtained so far on self-interacting diffusions in non-compact spaces. They concern the model of Durrett and Rogers, and can be classified in three categories. First, when f is real, non-negative and compactly supported, Cranston and Mountford [8] have solved a (partially proved) conjecture of Durrett and Rogers and shown that X_t/t converges a.s. Second, the attracting interaction on $\mathbb{R}$ (i.e. $xf(x) \le 0$ for all $x \in \mathbb{R}$) has been studied in the constant case by Cranston and Le Jan [7] and its generalization by Raimond [19] for the case $f(x) = -ax/\|x\|$, or by Herrmann and Roynette [11] for a local interaction. Under some conditions, it is proved that X_t converges a.s., whereas for a non-local interaction, it does not in general (but the paths are a.s. bounded for $f(x) = -\operatorname{sign}(x)\mathbb{1}_{|x|\geq a}$). The third category concerns a non-integrable repulsive f on $\mathbb{R}$ (i.e. $xf(x) \geq 0$ for all $x \in \mathbb{R}$) studied by Mountford and Tarrès [17] and solving a conjecture of Durrett and Rogers. They have proved that for $f(x) = x/(1+|x|^{1+\beta})$, with $0 < \beta < 1$, there exists a positive c such that with probability $1/2$, the symmetric process $t^{-2/(1+\beta)}X_t$ converges to c.

These previous works have in common that the drift may overcome the noise, so that the randomness of the process is "controlled". To illustrate that, let us mention, for the same model of Durrett and Rogers, the case of a compactly supported repulsive function, also conjectured in [10], which is still unsolved.

Conjecture 1 ([10]). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an odd function, of compact support. Then X_t/t converges a.s. to* 0.

Coming back to our process of interest, the role of the confinement potential is to similarly "control" the drift term of the diffusion. Indeed, for the process (1.4) with $V = 0$, the interaction potential is in general not strong enough for the process to be recurrent, and the behavior is then very difficult to analyze. In particular, it is hard to predict the relative importance of the drift term in the evolution.

1.2.1. *Technical assumptions on the potentials*

First, we denote the Euclidian scalar product by $(\cdot,\cdot)$ and by (H) the following set of hypotheses:

(i) (*regularity and positivity*) $V \in \mathcal{C}^2(\mathbb{R}^d)$, $W \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $V \geq 1$, $W \geq 0$;

(ii) (*growth*) there exists $C > 0$ such that for all $x, y \in \mathbb{R}^d$

$$\left|\nabla V(x) - \nabla V(y)\right| \leq C\big(|x-y| \wedge 1\big)\big(V(x) + V(y)\big); \tag{1.5}$$

(iii) (*domination*) there exists $\kappa \geq 1$ such that for all $x, y \in \mathbb{R}^d$,

$$W(x,y) \leq \kappa\big(V(x) + V(y)\big) \quad \text{and} \quad \left|\nabla^2_{xx} W(x,y)\right| + \left|\nabla_x W(x,y)\right| \leq \kappa\big(V(x) + W(x,y)\big), \tag{1.6}$$

$$\lim_{|x|\to\infty} \sup_{y\in\mathbb{R}^d} \frac{|\nabla V(x)|^2 + 2(\nabla V(x), \nabla_x W(x,y))}{V(x) + W(x,y)} = +\infty; \tag{1.7}$$

(iv) (*curvature*) there exist $\alpha, a > 0$, $\delta > 1$ and $M \in \mathbb{R}$ such that for all $x, y, \xi \in \mathbb{R}^d$,

$$\big(x, \nabla V(x) + \nabla_x W(x,y)\big) \geq a|x|^{2\delta} - \alpha \quad \text{and} \quad \big(\big(\nabla^2 V(x) + \nabla^2_{xx} W(x,y)\big)\xi, \xi\big) \geq M|\xi|^2. \tag{1.8}$$

Remark 1.1.

(1) *The most important conditions are the domination and the curvature.*

(2) *The growth condition* (1.5) *on V ensures that there exists $a > 0$ such that for all $x \in \mathbb{R}^d$,*

$$\Delta V(x) \leq aV(x). \tag{1.9}$$

(3) *The positivity and domination conditions on the interaction potential are not so hard to be satisfied, since the self-interacting process will be invariant by the gauge transform $W(x,y) \mapsto W(x,y) + \phi(y)$ for any function ϕ that does not grow faster than V.*

1.2.2. *Results*

We can now describe the behavior of μ_t.

Theorem 1.2. *Suppose* (H). *For any probability measure μ on $\mathbb{R}^d$, let $\Pi(\mu) := \mathrm{e}^{-2(V+W*\mu)}/Z(\mu)$, where $Z(\mu)$ is the normalization constant*:

(1) *$\mathbb{P}_{x,r,\mu}$-a.s., the ω-limit set* (*i.e., the accumulation points*) *of* $(\mu_t, t \geq 0)$ *is weakly compact, invariant by the flow generated by $\dot{\mu} = \Pi(\mu) - \mu$ and admits no other* (*sub-*)*attractor than itself.*

(2) *If* W *is symmetric, then* $\mathbb{P}_{x,r,\mu}$*-a.s., the* ω*-limit set of* $(\mu_t, t \geq 0)$ *is a connected subset of the set of fixed points of the application* $\mu \mapsto \Pi(\mu)$.

Even if the model studied could at a first glance seem restrictive (because of V), the drift term competes with the Brownian motion. The evolution is non-trivial and strongly depends on the drift.

Theorem 1.3. *Consider the diffusion* (1.4) *on* $\mathbb{R}^2$, *with* $V(x) = V(|x|)$ *and* $W(x, y) = (x, Ry)$, *where* R *is the rotation matrix of angle* θ. *For* $\rho \geq 0$, *define the probability measure* $\gamma(\mathrm{d}\rho) := \mathrm{e}^{-2V(\rho)}\,\mathrm{d}\rho/Z$. *Then one of the following holds*:

(1) *If* V *is such that* $\int_0^\infty \rho^2\gamma(\mathrm{d}\rho)\cos\theta > -1$, *then a.s.* $\mu_t \to \gamma$ (*weakly*);
(2) *Else*, *we get two different cases*:

- (a) *if* $\theta = \pi$, *then there exists a random measure* μ_∞ *such that a.s.* $\mu_t \to \mu_\infty$ (*weakly*),
- (b) *if* $\theta \neq \pi$, *then* μ_t *circles around.*

1.3. *Outline of contents*

As mentioned earlier, the main difficulty here stems from the non-compactness of the state space. The way to get around it is first to introduce, in Section 2, the V-norm (also named "dual weighted norm"), compatible with non-bounded functions, controlled by V. The family of measures $(\mu_t, t \geq 0)$ will then prove being (uniformly) bounded (for t large enough) for the dual V-norm in Section 5.1. Second, the dynamical system involved induces only a local semiflow and not necessarily a global one. The last important property is the following. Consider the Feller diffusion X^μ, corresponding to the SDE (1.4) where μ_t is fixed to μ. Its fundamental kernel (i.e. the inverse of the infinitesimal generator) is denoted by Q_μ. In order to study the ergodicity (in the limit-quotient sense) of X, one has to find a (uniform in μ) upper bound for the operator Q_μ. More precisely, we will prove the ultracontractivity of the semigroup in Section 4.1.1.

The organization of this paper is as follows. In the next section, we introduce some notations and prove the existence and uniqueness of X. Section 3 is devoted to the presentation of the main results and is divided in three parts. First, we recall the former results and ideas of Benaïm et al. [5]. Then, we state the tightness of $(\mu_t)_t$ and introduce the uniform estimates on the Feller semigroup. We finally end by describing the behavior of μ_t. Section 4 prepares the proofs of the main results by computing some useful estimates. First, we study in details the family of Markov semigroups, corresponding to X^μ, for which we prove the uniform ultracontractivity property and the regularity of the operators A_μ and Q_μ. After that, we analyze, in Section 4.2, the deterministic semiflow associated to the self-interacting diffusion and show its local existence. The proofs of the main results are given in Section 5, which heavily relies on the spectral analysis of Section 4.1. We first show the tightness of $(\mu_t)_t$. Then, Section 5.2 deals with the approximation of the normalized occupation measure $(\mu_t, t \geq 0)$ by a deterministic semiflow. In Section 5.3, we prove Theorem 1.2. Finally, Section 6 is devoted to the illustration in dimension $d = 2$ and the proof of Theorem 1.3.

2. Preliminaries and tools

2.1. *Some useful spaces and results*

In all the following, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ will be a filtered probability space satisfying the usual conditions.

2.1.1. *Spaces and topology*

We begin to introduce the weighted supremum norm (or V-norm)

$$\|f\|_V := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{V(x)}, \tag{2.1}$$

and the space of continuous V-bounded functions

$$\mathcal{C}^0(\mathbb{R}^d; V) := \{f \in \mathcal{C}^0(\mathbb{R}^d) \colon \|f\|_V < \infty\}. \tag{2.2}$$

Similarly let $\mathcal{C}^k(\mathbb{R}^d; V) := \mathcal{C}^k(\mathbb{R}^d) \cap \mathcal{C}^0(\mathbb{R}^d; V)$ for all $k \geq 1$.

We denote by $\mathcal{M}(\mathbb{R}^d)$ the space of signed (bounded) Borel measures on $\mathbb{R}^d$ and by $\mathcal{P}(\mathbb{R}^d)$ its subspace of probability measures. We will need the following measure space:

$$\mathcal{M}\big(\mathbb{R}^d; V\big) := \left\{\mu \in \mathcal{M}\big(\mathbb{R}^d\big); \int_{\mathbb{R}^d} V(y)|\mu|(\mathrm{d}y) < \infty\right\}, \tag{2.3}$$

where $|\mu|$ is the variation of μ: $|\mu| := \mu^+ + \mu^-$ with (μ^+, μ^-) the Hahn–Jordan decomposition of μ. This space will enable us to always check the integrability of V (and of W and its derivatives thanks to the domination condition (1.6)) with respect to the measures to be considered. For example, it contains the measure

$$\gamma(\mathrm{d}x) := \mathrm{e}^{-2V(x)}\,\mathrm{d}x. \tag{2.4}$$

We endow $\mathcal{M}(\mathbb{R}^d; V)$ with the following dual weighted supremum norm (or dual V-norm) defined by

$$\|\mu\|_V := \sup_{\varphi \in \mathcal{C}^0(\mathbb{R}^d; V); \|\varphi\|_V \leq 1} \left|\int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu\right|, \quad \mu \in \mathcal{M}\big(\mathbb{R}^d; V\big). \tag{2.5}$$

This norm naturally arises in the approach of ergodic results for time-continuous Markov processes by Meyn and Tweedie [16]. It makes $\mathcal{M}(\mathbb{R}^d; V)$ a Banach space. Since we will mainly consider probability measures in the following, we set $\mathcal{P}(\mathbb{R}^d; V) := \mathcal{M}(\mathbb{R}^d; V) \cap \mathcal{P}(\mathbb{R}^d)$. The strong topology on $\mathcal{P}(\mathbb{R}^d; V)$ is the trace topology of the one defined on $\mathcal{M}(\mathbb{R}^d; V)$. It makes $\mathcal{P}(\mathbb{R}^d; V)$ a complete metric space.

In order to study the dynamical system in Section 4.2, we need to endow the space $\mathcal{P}(\mathbb{R}^d; V)$ with two different topologies. When nothing else is stated, we will consider that it is equipped with the strong topology defined by the dual weighted supremum norm $\|\cdot\|_V$. But, as the reader will notice, we will frequently need to switch from the strong topology to the weak topology of convergence of measures. We adopt here a non-standard definition compatible with possibly unbounded functions (yet dominated by V). For any sequence of probability measures $(\mu_n, n \geq 1)$ and any probability measure μ (all belonging to $\mathcal{P}(\mathbb{R}^d; V)$), we define the weak convergence as

$$\mu_n \xrightarrow{w} \mu \quad \text{if and only if} \quad \int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu_n \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu \quad \forall \varphi \in \mathcal{C}^0\big(\mathbb{R}^d; V\big). \tag{2.6}$$

We point out that our definition of the weak convergence is stronger than the usual one. We recall that $\mathcal{P}(\mathbb{R}^d; V)$, equipped with the weak topology, is a metrizable space. Since $\mathcal{C}^0(\mathbb{R}^d; V)$ is separable, we exhibit a sequence $(f_k)_k$ dense in $\{f \in \mathcal{C}^0(\mathbb{R}^d; V); \|f\|_V \leq 1\}$, and set for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^d; V)$:

$$\mathrm{d}(\mu, \nu) := \sum_{k=1}^{\infty} 2^{-k}\big|\mu(f_k) - \nu(f_k)\big|. \tag{2.7}$$

Then the weak topology is the metric topology generated by d.

Finally, for any $\beta > 1$, we introduce the subspace

$$\mathcal{P}_\beta\big(\mathbb{R}^d; V\big) := \left\{\mu \in \mathcal{P}\big(\mathbb{R}^d; V\big); \int_{\mathbb{R}^d} V(y)\mu(\mathrm{d}y) \leq \beta\right\}. \tag{2.8}$$

Proposition 2.1. *Let $\beta > 1$. The set $\mathcal{P}_\beta(\mathbb{R}^d; V)$ is a weakly compact subset of $\mathcal{P}(\mathbb{R}^d; V)$.*

Proof. Straightforward. □

2.1.2. *Preliminary results*

Through this paper, we will use (many times) some easy results. First, to illustrate the need of the space $\mathcal{M}(\mathbb{R}^d; V)$, we state

Lemma 2.2. *For any $\mu \in \mathcal{M}(\mathbb{R}^d; V)$, the function $W * \mu$ belongs to $\mathcal{C}^2(\mathbb{R}^d; V)$ and*

$$\left|W * \mu(x)\right| \le 2\kappa \|\mu\|_V V(x).$$

There exists $D > 0$ such that for all $\mu \in \mathcal{M}(\mathbb{R}^d; V)$,

$$\left|\Delta\big(V(x) + W * \mu(x)\big)\right| \le D\big(V(x) + W * \mu(x)\big). \tag{2.9}$$

Proof. It results from the growth (1.5) and domination (1.6) conditions. □

Corollary 2.3. *Let $\beta > 1$. For $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, we get*

$$Z(\mu) = \int_{\mathbb{R}^d} \mathrm{e}^{-2W*\mu(x)} \gamma(\mathrm{d}x) \ge \mathrm{e}^{-2\kappa\beta} \int_{\mathbb{R}^d} \mathrm{e}^{-2\kappa V(x)} \gamma(\mathrm{d}x) \ge \int_{\mathbb{R}^d} \mathrm{e}^{-4\kappa\beta V(x)} \gamma(\mathrm{d}x). \tag{2.10}$$

The following function will also prove being very useful, as a Lyapunov function

$$\mathcal{E}_\mu(x) := V(x) + W * \mu(x). \tag{2.11}$$

Lemma 2.4. *Let $\beta > 1$. For any $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, we have the following upper bound*:

$$\mathcal{E}_\mu(x) \le 3\kappa\beta V(x). \tag{2.12}$$

Proof. It follows from the domination (1.6) condition. □

For any probability measure $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, let $(X_t^\mu, t \ge 0)$ be the Feller diffusion defined by the SDE

$$\mathrm{d}X_t^\mu = \mathrm{d}B_t - \big(\nabla V\big(X_t^\mu\big) + \nabla W * \mu\big(X_t^\mu\big)\big)\,\mathrm{d}t, \qquad X_0^\mu = x. \tag{2.13}$$

Suppose that X^μ a.s. never explodes. We denote by $(P_t^\mu; t \ge 0)$ the associated Markov semigroup. Its infinitesimal generator is then the differential operator A_μ defined on $\mathcal{C}^\infty(\mathbb{R}^d)$ by

$$A_\mu f := \frac{1}{2}\Delta f - (\nabla V + \nabla W * \mu, \nabla f). \tag{2.14}$$

We emphasize that X^μ is a positive-recurrent (reversible) diffusion. Denote by $\Pi(\mu) \in \mathcal{P}(\mathbb{R}^d; V)$ its unique invariant probability measure:

$$\Pi(\mu)(\mathrm{d}x) := \frac{\mathrm{e}^{-2W*\mu(x)}}{Z(\mu)} \gamma(\mathrm{d}x), \tag{2.15}$$

where $Z(\mu) := \int_{\mathbb{R}^d} \mathrm{e}^{-2W*\mu(x)} \gamma(\mathrm{d}x) < +\infty$ is the normalization constant.

Proposition 2.5. *The diffusion X_t^μ a.s. never explodes.*

Proof. It is enough to check with Itô's formula and (2.9) that $\mathcal{E}_\mu$, defined in (2.11), is a Lyapunov function: $A_\mu \mathcal{E}_\mu \le D\mathcal{E}_\mu$. As a by-product we get the naive estimate

$$\mathbb{E}\mathcal{E}_\mu\big(X_t^\mu\big) \le \mathcal{E}_\mu(x)\mathrm{e}^{Dt}. \tag{2.16}$$

□

The classical ergodic (limit-quotient) theorem is true for X^μ: a.s. we have, for all $f \in \mathcal{C}^0(\mathbb{R}^d; V)$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f\big(X_s^\mu\big)\,\mathrm{d}s = \Pi(\mu) f =: \int_{\mathbb{R}^d} f \,\mathrm{d}\Pi(\mu). \tag{2.17}$$

To end this part, for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, let $L^2(\Pi(\mu))$ be the Lebesgue space of Borel square-integrable functions with respect to the measure $\Pi(\mu)$. We remark that this space depends on μ, but we will consider mainly the subspace $\mathcal{C}^0(\mathbb{R}^d; V) \subset L^2(\Pi(\mu))$. We denote the inner product on this space by

$$(f, g)_\mu := \int_{\mathbb{R}^d} f(x)g(x)\Pi(\mu)(\mathrm{d}x)$$

and $\|\cdot\|_{2,\mu}$ is the associated norm. We introduce two operators: Q_μ (sometimes called the "fundamental kernel" as in Kontoyiannis and Meyn [12]) is the solution to Poisson's equation, that is the "inverse" of A_μ, defined for any function $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$ by

$$Q_\mu f := \int_0^\infty \left(P_t^\mu f - \Pi(\mu) f\right) \mathrm{d}t \tag{2.18}$$

and K_μ is the orthogonal projector defined by

$$K_\mu f := f - \Pi(\mu) f. \tag{2.19}$$

These operators are linked together by the Poisson equation: $\forall f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$,

$$A_\mu \circ Q_\mu(f) = Q_\mu \circ A_\mu(f) = -K_\mu f.$$

Remark 2.6. *The existence of $Q_\mu f$ comes from the uniform spectral gap obtained in Corollary 3.4.*

2.2. *The self-interacting diffusion*

We recall the self-interacting diffusion considered here:

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}B_t - \left(\nabla V(X_t) + \nabla W * \mu_t(X_t)\right)\mathrm{d}t, \qquad X_0 = x, \\ \mathrm{d}\mu_t = (\delta_{X_t} - \mu_t)\frac{\mathrm{d}t}{r+t}, \qquad \mu_0 = \mu. \end{cases}$$

Proposition 2.7. *For any $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and $r > 0$, there exists a unique global strong solution $(X_t, \mu_t, t \geq 0)$.*

Proof. Let us introduce the increasing sequence of stopping times, $\tau_0 = 0$, and

$$\tau_n := \inf\left\{t \geq \tau_{n-1}; \mathcal{E}_{\mu_t}(X_t) + \int_0^t \left|\nabla \mathcal{E}_{\mu_s}(X_s)\right|^2 \mathrm{d}s > n\right\}.$$

In order to show that the solution never explodes, we use again the Lyapunov functional $(x, \mu) \mapsto \mathcal{E}_\mu(x)$ defined in (2.11). As the process $(t, x) \mapsto \mathcal{E}_{\mu_t}(x)$ is of class $\mathcal{C}^2$ (in the space variable) and is a $\mathcal{C}^1$-semi-martingale (in the time variable), the generalized Itô formula (or Itô–Ventzell formula, see [13]), applied to $(t, x) \mapsto \mathcal{E}_{\mu_{t\wedge\tau_n}}(x)$ implies

$$\begin{aligned} \mathcal{E}_{\mu_{t\wedge\tau_n}}(X_{t\wedge\tau_n}) = \mathcal{E}_\mu(x) &+ \int_0^{t\wedge\tau_n} \left(\nabla \mathcal{E}_{\mu_s}(X_s), \mathrm{d}B_s\right) - \int_0^{t\wedge\tau_n} \left|\nabla \mathcal{E}_{\mu_s}(X_s)\right|^2 \mathrm{d}s \\ &+ \frac{1}{2}\int_0^{t\wedge\tau_n} \Delta \mathcal{E}_{\mu_s}(X_s)\,\mathrm{d}s + \int_0^{t\wedge\tau_n} \left(W(X_s, X_s) - W * \mu_s(X_s)\right)\frac{\mathrm{d}s}{r+s}. \end{aligned} \tag{2.20}$$

We note that $\int_0^{t\wedge\tau_n} (\nabla \mathcal{E}_{\mu_s}(X_s), \mathrm{d}B_s)$ is a true martingale. Letting $k = a + 2\kappa/r + D$, we then get, similarly to (2.9),

$$\mathbb{E}\mathcal{E}_{\mu_{t\wedge\tau_n}}(X_{t\wedge\tau_n}) \leq \mathcal{E}_\mu(x) + k\log(1+t)\int_0^t \mathbb{E}\mathcal{E}_{\mu_{s\wedge\tau_n}}(X_{s\wedge\tau_n})\,\mathrm{d}s.$$

So, Gronwall's lemma leads to the same kind of estimate as for X^μ:

$$\mathbb{E}V(X_{t\wedge\tau_n}) \le \mathbb{E}\mathcal{E}_{\mu_{t\wedge\tau_n}}(X_{t\wedge\tau_n}) \le \mathcal{E}_\mu(x)\mathrm{e}^{kt\log(1+t)}.$$

As $\lim_{|x|\to\infty} V(x) = \infty$, the process $(X_t, t \ge 0)$ does not explode in a finite time and there exists a global strong solution. □

3. Main results

3.1. *Former tools and general idea*

We remind how Benaïm et al. [5] handled the asymptotic behavior of μ_t in a compact space. Indeed, we sketch here the general idea and explain why the tools introduced in Section 2 arise quite naturally.

First, suppose that the empirical measure appearing in the drift is "frozen" to some fixed measure μ. We obtain the Feller diffusion X^μ, for which there exists a spectral gap. The associated semigroup $(P_t^\mu; t \ge 0)$ is exponentially V-uniformly ergodic:

$$\left\| P_t^\mu f - \Pi(\mu) f \right\|_V \le c(\mu)\|f\|_V \mathrm{e}^{-c(\mu)t}, \quad f \in \mathcal{C}^0\big(\mathbb{R}^d; V\big). \tag{3.1}$$

To get, as a by-product, the almost sure convergence of the empirical measure of X^μ (as defined in (2.17)), a standard technique is to consider the operator Q_μ defined by (2.18). Then, it is enough to apply Itô's formula to $Q_\mu f(X_t^\mu)$ and divide both members by t to get the desired result. Indeed, one has

$$Q_\mu f\big(X_t^\mu\big) = Q_\mu f(x) + \int_0^t \big(\nabla Q_\mu f\big(X_s^\mu\big), \mathrm{d}B_s\big) + \int_0^t A_\mu \circ Q_\mu f\big(X_s^\mu\big)\,\mathrm{d}s.$$

Thanks to some easy bounds on the semigroup (P_t^μ), one proves that the martingale term is negligible compared to t and then, one recognizes the last term since $A_\mu \circ Q_\mu f = \Pi(\mu)f - f$.

Now when μ_t changes in time, we still can write a convenient extended form of Itô's formula, which let appear the time derivative of $Q_{\mu_t} f$, but we need to improve the remainder of the argument. Intuitively, as for stochastic approximation processes, one expects the trajectories of μ_t to approximate the trajectories of a deterministic semiflow. This very last remark conveyed to Benaïm et al. [5] the idea to compare the asymptotic evolution of $(\mu_t; t \ge 0)$ with a semiflow.

Definition 3.1. *A continuous function $\xi : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d; V)$ is an asymptotic pseudotrajectory* (*or asymptotic pseudo-orbit*) *for the semiflow Φ if for all $T > 0$,*

$$\lim_{t\to+\infty} \sup_{0\le s\le T} \mathrm{d}\big(\xi_{t+s}, \Phi_s(\xi_t)\big) = 0. \tag{3.2}$$

The notion of asymptotic pseudotrajectory has been introduced by Benaïm and Hirsch [4]. It is particularly useful to analyze the long-term behavior of stochastic processes, considered as approximations of solutions of ordinary differential equation (the "ODE method"). In Section 5, we prove that the empirical measure is an asymptotic pseudotrajectory for the semiflow Φ induced by $\Pi(\mu) - \mu$.

3.2. *New tools*: *Tightness and ultracontractivity*

The paper of Benaïm et al. [5] crucially relies on the compactness of the manifold where the diffusion lives. It readily implies that the measure μ_t is close to $\Pi(\mu_t)$. On the contrary, if the state space is $\mathbb{R}^d$ and $V \equiv 0$, then X will escape from any compact set. Indeed, the confinement potential V forces the process $(\mu_t, t \ge 0)$ to remain in a (weakly) compact space of measures, for t large, and X is then positive-recurrent.

Theorem 3.2. $\mathbb{P}_{x,r,\mu}$*-a.s.*, $\beta := \sup\{\int V\,\mathrm{d}\mu_t; t \ge 0\} < +\infty$.

The proof is postponed to Section 5 and we emphasize that β is a random variable.

We also need some precise bounds on the family of semigroups $(P_t^\mu, t \geq 0)$ where $\mu \in \mathcal{P}(\mathbb{R}^d; V)$. A priori, it is not obvious that the semigroup (P_t^μ) admits a (uniform) spectral gap. Indeed, we will prove a stronger result: (P_t^μ) is uniformly bounded as an operator from $L^2(\Pi(\mu))$ to L^∞. Section 4.1 will be devoted to those uniform properties. In the following, define $\|P_t^\mu f\|_\infty := \operatorname{essup}_{x\in\mathbb{R}^d} |P_t^\mu f(x)|$.

Proposition 3.3. *The family of semigroups* $(P_t^\mu, t \geq 0, \mu \in \mathcal{P}(\mathbb{R}^d; V))$ *is uniformly ultracontractive*: *there exists* $c > 0$ *independent of* μ *such that for all* $1 \geq t > 0$ *and* $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, *we have*

$$\sup_{f\in\mathcal{C}^\infty(\mathbb{R}^d;V)\setminus\{0\}} \frac{\|P_t^\mu f\|_\infty}{\|f\|_{2,\mu}} \leq \exp\bigl(ct^{-\delta/(\delta-1)}\bigr). \tag{3.3}$$

The proof is postponed to Section 4.1.

Corollary 3.4. *The family of measures* $(\Pi(\mu), \mu \in \mathcal{P}(\mathbb{R}^d; V))$ *satisfies a uniform* (*in* μ) *logarithmic Sobolev inequality and admits a uniform spectral gap. So, there exists* $C > 0$, *independent of* μ, *such that for all* $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, *for all* $t \geq 0$:

$$\bigl\|P_t^\mu(K_\mu f)\bigr\|_{2,\mu} \leq \mathrm{e}^{-t/C}\|K_\mu f\|_{2,\mu}.$$

Proof. When a semigroup is ultracontractive, then it is hypercontractive. As being hypercontractive is equivalent to satisfy a logarithmic Sobolev inequality, we conclude (see, for instance, Bakry [1]). The given inequality is a consequence of the logarithmic Sobolev one. □

3.3. *The ω-limit set*

First, let us define an ω-limit set:

Definition 3.5. *For every continuous function* $\xi : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d; V)$, *the ω-limit set of* ξ, *denoted by* $\omega(\xi_t, t \geq 0)$, *is the set of limits of weak convergent sequences* $\xi(t_k)$, $t_k \uparrow \infty$, *that is*

$$\omega(\xi_t, t \geq 0) := \bigcap_{t\geq 0} \overline{\xi\bigl([t,\infty)\bigr)}, \tag{3.4}$$

where $\overline{\xi([t,\infty))}$ *stands for the closure of* $\xi([t,\infty))$ *according to the weak topology.*

Let $\Phi : \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d; V) \to \mathcal{P}(\mathbb{R}^d; V)$ be the semiflow generated by

$$\Phi_t(\mu) = \mathrm{e}^{-t}\mu + \mathrm{e}^{-t}\int_0^t \mathrm{e}^s \Pi\bigl(\Phi_s(\mu)\bigr)\,\mathrm{d}s, \qquad \Phi_0(\mu) = \mu. \tag{3.5}$$

We will prove the local existence of the semiflow in Section 4.2, and for W symmetric or bounded, we will show it never explodes. In other cases, we will assume the global existence of the semiflow. Section 5 is devoted to the study of μ_t. Indeed, the time-changed process $\mu_{h(t)}$ (and not μ_t) is an asymptotic pseudotrajectory for Φ, where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$h(t) := r\bigl(\mathrm{e}^t - 1\bigr). \tag{3.6}$$

This deterministic time-change h comes from the normalization of the occupation measure μ_t. The factor $(r+t)^{-1}$ disappears while considering

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_{h(t)} = \delta_{X_{h(t)}} - \mu_{h(t)}.$$

Theorem 3.6. *Under $\mathbb{P}_{x,r,\mu}$, the function $t \mapsto \mu_{h(t)}$ is almost surely an asymptotic pseudotrajectory for the semi-flow Φ.*

The proof is given in Section 5. This result enables us to describe the ω-limit set of $(\mu_t, t \geq 0)$:

Corollary 3.7. $\mathbb{P}_{x,r,\mu}$*-a.s., $\omega(\mu_t, t \geq 0)$ is weakly compact, invariant by Φ and the flow restricted to $\omega(\mu_t, t \geq 0)$ contains no attractor (other than itself). The convex hull of the image of Π contains $\omega(\mu_t, t \geq 0)$.*

In some cases, we state and prove a more precise description of $\omega(\mu_t, t \geq 0)$ in Section 5.

Theorem 3.8. *Assume that W is symmetric. Then, $\mathbb{P}_{x,r,\mu}$-a.s., $\omega(\mu_t, t \geq 0)$ is a connected subset of the fixed points of Π.*

Corollary 3.9. *Suppose that W is symmetric. If Π admits only finitely many fixed points, then $\mathbb{P}_{x,r,\mu}$-a.s., $(\mu_t; t \geq 0)$ converges to one of them.*

4. Estimates on the semigroups and dynamical system

4.1. *The family of semigroups*

In this part, we exhibit the ultracontractivity (implying the existence of a spectral gap) for the family of semigroups $(P_t^\mu, \mu \in \mathcal{P}(\mathbb{R}^d; V))$. Since we consider these semigroups altogether for all the measures $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, we will prove that the constants involved are uniform in μ. The need for ultracontractivity will impose some kind of boundedness on the convolution term in the SDE that cannot be easily removed. Finally, we compute several estimates preparing Section 5.

4.1.1. *Uniform ultracontractivity*

The notion of ultracontractivity and its relation to the analysis of Markov semigroups were first studied by Davies and Simon [9] and recently by Röckner and Wang [20] for more general diffusions. To prove that the family of semigroups $(P_t^\mu, t \geq 0, \mu \in \mathcal{P}(\mathbb{R}^d; V))$ is uniformly ultracontractive, we will rely on the following result of Röckner and Wang:

Lemma 4.1 ([20], Corollary 2.5). *Let $(P_t, t \geq 0)$ be a Markov semigroup, with infinitesimal generator $A := \frac{1}{2}\Delta - (\nabla U, \nabla)$, and $\nabla^2 U \geq -K$ with $K > 0$. Assume that there exists a continuous increasing map $\chi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \setminus \{0\}$ such that*:

(1) $\lim_{r\to\infty} \frac{\chi(r)}{r} = \infty$,
(2) *the mapping $g_\chi(r) := r\chi(m \log r)$ is convex on $[1, \infty)$ for any $m > 0$,*
(3) $A|x|^2 \leq b - \chi(|x|^2)$ *for some $b > 0$.*

Then P_t admits a unique invariant probability measure. If $\int_2^\infty \frac{\mathrm{d}r}{r\chi(m \log r)} < \infty$, $m > 0$, then P_t is ultracontractive.

If, moreover, $\chi(r) = \chi r^\delta$, with $\chi > 0, \delta > 1$, then there exists $c = c(b, \chi) > 0$ such that for all $t \in (0, 1]$, $\sup_{f \in \mathcal{C}^\infty(\mathbb{R}^d)\setminus\{0\}} \frac{\|P_t f\|_\infty}{\|f\|_2} \leq \exp(ct^{-\delta/(\delta-1)})$.

Proof of Proposition 3.3. First, we prove that there exist c_1, c_2 independent of μ such that $|P_t^\mu f(x)| \leq \mathrm{e}^{(c_1 + c_2|x|^2)/t}$ for all $t \in (0, 1)$. Let M be the constant involved in the curvature condition (1.8) and denote $m_t := \frac{M}{1-\mathrm{e}^{-2Mt}}$. By Wang [24], Lemma 2.1, it appears that for all $x, y \in \mathbb{R}^d$,

$$\left|P_t^\mu f(x)\right|^2 \leq P_t^\mu f^2(y) \exp\left\{m_t |x-y|^2\right\}.$$

As $\Pi_1(\mu) := \mathrm{e}^{-2(V+W*\mu)}/Z_1$, where $Z_1 := \int_{|y|\le 1} \mathrm{e}^{-2(V+W*\mu)(y)}\,\mathrm{d}y$, is an invariant measure for the process X^μ, we have that $\int_{|y|\le 1} \mathrm{e}^{-m_t|y|^2}\Pi_1(\mu)(\mathrm{d}y) \ge \mathrm{e}^{-m_t}$. So, we get

$$\int_{\mathbb{R}^d} \mathrm{e}^{-m_t|x-y|^2}\Pi_1(\mu)(\mathrm{d}y) \ge \mathrm{e}^{-2m_t(|x|^2+1)}.$$

It remains to choose $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$ such that $\Pi_1(\mu)f = 1$ to conclude that

$$\left|P_t^\mu f(x)\right|^2 \mathrm{e}^{-2m_t(|x|^2+1)} \le \left|P_t^\mu f(x)\right|^2 \int_{\mathbb{R}^d} \mathrm{e}^{-m_t|x-y|^2}\Pi_1(\mu)(\mathrm{d}y) \le 1.$$

Now, we apply Lemma 4.1 with $U := V + W * \mu$ to show that each $(P_t^\mu)_{t\ge 0}$ is ultracontractive. Indeed, the curvature condition (1.8) implies that there exist $a, b > 0$ such that for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$,

$$A_\mu|x|^2 = d - 2\big(x, \nabla V(x) + \nabla W * \mu(x)\big) \le b - a|x|^{2\delta}.$$

As $\chi(r) = r^\delta$ with $\delta > 1$, the constant c is uniform in μ. □

We are now able to derive some useful bounds on the operator Q_μ. As we need these bounds being uniform (in μ), and depending on x only through $V(x)$, the ultracontractivity is essential.

Proposition 4.2. *For all $\varepsilon > 0$, there exists a positive constant $K(\varepsilon)$ such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, $x \in \mathbb{R}^d$, $f \in \mathcal{C}^0(\mathbb{R}^d; V)$:*

$$\left|Q_\mu f(x)\right| \le \big(\varepsilon V(x) + K(\varepsilon)\big)\|f\|_V. \tag{4.1}$$

Proof. Let $t_0 \in (0, 1]$ (we will choose it precisely later). We have

$$\left|Q_\mu f(x)\right| \le \int_0^\infty \left|P_t^\mu(K_\mu f)(x)\right|\mathrm{d}t = \int_0^{t_0} \left|P_t^\mu(K_\mu f)(x)\right|\mathrm{d}t + \int_{t_0}^\infty \left|P_t^\mu(K_\mu f)(x)\right|\mathrm{d}t.$$

We begin to work with the second right-hand term. Using the composition property of the semigroup, the uniform ultracontractivity and uniform spectral gap, we have

$$\int_{t_0}^\infty \left|P_t^\mu(K_\mu f)(x)\right|\mathrm{d}t \le \exp\big(ct_0^{-\delta/(\delta-1)}\big)\int_0^\infty \mathrm{e}^{-t/C}\,\mathrm{d}t\,\|K_\mu f\|_{2,\mu}.$$

As K_μ is an orthogonal projector, $\|K_\mu f\|_{2,\mu} \le \|f\|_{2,\mu} \le (\int V^2\,\mathrm{d}\Pi(\mu))^{1/2}\|f\|_V$, and we get

$$\int_{t_0}^\infty \left|P_t^\mu(K_\mu f)(x)\right|\mathrm{d}t \le C\|f\|_V \exp\big(ct_0^{-\delta/(\delta-1)}\big)\left(\int V^2\,\mathrm{d}\Pi(\mu)\right)^{1/2}.$$

We now have to work with the first right-hand term. We get

$$\left|P_t^\mu f(x)\right| \le \|f\|_V P_t^\mu V(x) \le \|f\|_V \mathbb{E}\mathcal{E}_\mu\big(X_t^\mu\big) \le \mathcal{E}_\mu(x)\mathrm{e}^{Dt}\|f\|_V.$$

By Proposition 5.1, we get that $\int_0^t \mathbb{E}\mathcal{E}_\mu(X_s^\mu)\,\mathrm{d}s = \mathrm{O}(t)$ and so, we choose t_0 small enough such that $\int_0^{t_0} \mathbb{E}\mathcal{E}_\mu(X_s^\mu)\,\mathrm{d}s \le \varepsilon$ to conclude. □

Proposition 4.3. *For all $\varepsilon > 0$, there exists $K_1(\varepsilon) > 0$ such that for all $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, $x \in \mathbb{R}^d$, $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we have $Q_\mu f \in \mathcal{C}^1(\mathbb{R}^d)$ and*

$$\left|\nabla Q_\mu f(x)\right| \le \big(\varepsilon V(x) + K_1(\varepsilon)\big)\|f\|_V. \tag{4.2}$$

Proof. We introduce the operator $\Gamma_2^\mu(f) := |\nabla^2 f|^2 + (\nabla f, \nabla^2(V + W * \mu)\nabla f)$. Using the curvature condition (1.8), the Γ_2-criterion ($\Gamma_2^\mu(f) \geq M|\nabla f|^2$), due to Bakry and Emery [2], is satisfied and implies (see Ledoux [14], p. 22)

$$\left|\nabla P_t^\mu(K_\mu f)\right|^2 \leq \frac{M}{e^{2Mt} - 1}\left|P_t^\mu(K_\mu f)^2\right| \quad \forall f \in \mathcal{C}^\infty\big(\mathbb{R}^d; V\big), \forall t > 0. \tag{4.3}$$

Indeed, the previous inequality, together with the ultracontractivity, shows

$$\int_{t_0}^\infty \left|\nabla P_t^\mu(K_\mu f)(x)\right| \mathrm{d}t \leq 2C\sqrt{\frac{M}{e^{2Mt_0} - 1}} e^{ct_0^{-\delta/(\delta-1)}} \|f\|_V \left(\int V^4 \,\mathrm{d}\Pi(\mu)\right)^{1/4}.$$

Finally, one concludes by using again Proposition 5.1. □

4.1.2. *Regularity with respect to the measure μ*

We endow the space $\mathcal{P}(\mathbb{R}^d; V)$ with a structure of infinite-dimensional differentiable manifold. This structure is used only for differentiating functions defined on $\mathcal{P}(\mathbb{R}^d; V)$, which is also needed in the study of the semiflow in Section 4.2.

For any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ we consider the set $\mathcal{C}^k(\mu)$ ($k \geq 1$) of (germs of) curves defined on some neighborhood of zero $(-\varepsilon, \varepsilon)$ with values in $\mathcal{P}(\mathbb{R}^d; V)$, passing through μ at time zero and that are of class $\mathcal{C}^k$ when they are considered as functions with values in the Banach space $\mathcal{M}(\mathbb{R}^d; V)$. Now we say that a function $\phi : \mathcal{P}(\mathbb{R}^d; V) \to \mathbb{R}$ is of class $\mathcal{C}^k$ if for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and any curve $f \in \mathcal{C}^k(\mu)$ the real function $\phi \circ f$ is of class $\mathcal{C}^k$. This enables to define the differential of such a function ϕ. For any μ the tangent space at μ to $\mathcal{P}(\mathbb{R}^d; V)$ can be identified with the space $\mathcal{M}_0(\mathbb{R}^d; V)$ of zero-mass measures in $\mathcal{M}(\mathbb{R}^d; V)$, that is $\nu(\mathbb{R}^d) = 0$. The differential is then the linear operator

$$D\phi(\mu) \cdot \nu = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mu + t\nu)\bigg|_{t=0}, \quad \nu \in \mathcal{M}_0\big(\mathbb{R}^d; V\big). \tag{4.4}$$

The same definition applies to functions with values in a Banach space or even in $\mathcal{P}(\mathbb{R}^d; V)$. As an example, the maps $\mu \mapsto W * \mu(x)$ (for any point x) and Π (applying Lebesgue's theorem) are $\mathcal{C}^\infty$.

First, consider the Banach space $\mathcal{B}$ of bounded linear operators from $\mathcal{C}^\infty(\mathbb{R}^d; V) \subset L^2(\gamma)$, endowed with the norm $\|f\|_{2,\mu,1} := \|f\|_{2,\mu} + \|A_\mu f\|_{2,\mu}$, to the same space equipped with the standard quadratic norm. We endow $\mathcal{B}$ with the operator norm. Then, A_μ obviously belongs to the closed subset of $\mathcal{B}$ consisting in operators A such that $A1 = 0$.

Proposition 4.4. *The mappings $\mu \mapsto A_\mu$ and $\mu \mapsto K_\mu$ are $\mathcal{C}^\infty$. For any function $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, the application $\mu \mapsto Q_\mu f$ is $\mathcal{C}^\infty$ for the strong topology and the differentials are (for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$):*

$$D(A_\mu f) \cdot \nu = -(\nabla W * \nu, \nabla f),$$
$$D(K_\mu f) \cdot \nu = -\big(D\Pi(\mu) \cdot \nu\big)(f),$$
$$D(Q_\mu f) \cdot \nu = \big(D\Pi(\mu) \cdot \nu\big)(Q_\mu f) + Q_\mu(\nabla W * \nu, \nabla Q_\mu f).$$

Proof. Consider measures $\mu \in \mathcal{P}(\mathbb{R}^d; V)$. As $\mu \mapsto W * \mu$ and Π are $\mathcal{C}^\infty$, there is nothing to prove in case of A_μ or K_μ. To look at Q_μ, we need to consider the resolvent operator of P_t^μ:

$$R_\lambda^\mu := \int_0^\infty e^{-\lambda t} P_t^\mu \,\mathrm{d}t = (\lambda - A_\mu)^{-1} \quad \forall \lambda > 0. \tag{4.5}$$

For $\lambda > 0$, we define the approximation of Q_μ,

$$Q_\mu(\lambda) := \int_0^\infty e^{-\lambda t} P_t^\mu K_\mu \,\mathrm{d}t = K_\mu(\lambda - A_\mu)^{-1}. \tag{4.6}$$

As $\mu \mapsto K_\mu$ and $\mu \mapsto A_\mu$ are $\mathcal{C}^\infty$, the map $\mu \mapsto Q_\mu(\lambda) f$ is $\mathcal{C}^\infty$ by composition.

The uniform spectral gap shows the existence of $C, C_1 > 0$ such that

$$\left\| Q_\mu f - Q_\mu(\lambda) f \right\|_V \le \int_0^\infty \left(1 - \mathrm{e}^{-\lambda t}\right) \left\| P_t^\mu K_\mu f \right\|_V \mathrm{d}t \le \lambda C \| f \|_V \int_0^\infty t \mathrm{e}^{-tC_1} \mathrm{d}t.$$

Hence the convergence of $Q_\mu(\lambda)$ towards Q_μ is uniform with respect to μ. As a by-product, $\mu \mapsto Q_\mu f$ is continuous.

The differential of $Q_\mu(\lambda)$ is

$$DQ_\mu(\lambda) \cdot \nu = (DK_\mu \cdot \nu)(\lambda - A_\mu)^{-1} + K_\mu(\lambda - A_\mu)^{-1}(DA_\mu \cdot \nu)(\lambda - A_\mu)^{-1}.$$

Replacing DK_μ and DA_μ by their expressions, we will prove that each right-hand side term of the equality converges uniformly. For the first one, as $(D\Pi(\mu) \cdot \nu)((\lambda - A_\mu)^{-1} f) = (D\Pi(\mu) \cdot \nu)(K_\mu(\lambda - A_\mu)^{-1} f)$, we have uniformly

$$\lim_{\lambda \to 0} \big(D\Pi(\mu) \cdot \nu\big)\big((\lambda - A_\mu)^{-1} f\big) = \big(D\Pi(\mu) \cdot \nu\big)(Q_\mu f).$$

To prove the convergence of the second term, remark that

$$K_\mu(\lambda - A_\mu)^{-1}(\nabla W * \nu, \nabla)\big((\lambda - A_\mu)^{-1} f\big) = Q_\mu(\lambda)\big(\nabla W * \nu, \nabla Q_\mu(\lambda) f\big).$$

It remains now to show that $\nabla Q_\mu(\lambda) f$ converges (uniformly in μ) to $\nabla Q_\mu f$. By definition of $Q_\mu(\lambda)$, we find

$$\left|\nabla Q_\mu f - \nabla Q_\mu(\lambda) f\right| \le \int_0^\infty \left|\nabla\big(P_t^\mu K_\mu f\big)\right| \left(1 - \mathrm{e}^{-\lambda t}\right) \mathrm{d}t.$$

We use the inequality (4.3) to prove that this family of differentials converges uniformly with respect to μ; so $\mu \mapsto Q_\mu f$ is actually $\mathcal{C}^1$ with the stated differential. □

Remark 4.5. *Looking at the differential $D(Q_\mu f)$, we see that it is itself a $\mathcal{C}^1$ function of μ, so by induction one proves that $\mu \mapsto Q_\mu f$ is $\mathcal{C}^\infty$ and also that $\mu \mapsto P_t^\mu f$ is $\mathcal{C}^\infty$.*

Corollary 4.6. *For every $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we have the uniform inequality*

$$\left|(DQ_\mu \cdot \nu)(f)(x)\right| \le \big(\varepsilon V^2(x) + K_2(\varepsilon)\big) \| f \|_V \|\nu\|_V.$$

Proof. We easily get the inequality

$$\left|(DQ_\mu \cdot \nu)(f)(x)\right| \le \left|\big(D\Pi(\mu) \cdot \nu\big)(Q_\mu f)(x)\right| + \left|Q_\mu\big(\nabla W * \nu(x), \nabla Q_\mu f(x)\big)\right|.$$

If we consider the second right-hand term, we find (using Cauchy's inequality)

$$\begin{aligned}\left|Q_\mu\big(\nabla W * \nu(x), \nabla Q_\mu f(x)\big)\right| &\le \big(\varepsilon V^2(x) + K(\varepsilon)\big) \left\|\big(\nabla W * \nu, \nabla Q_\mu f\big)\right\|_{V^2} \\ &\le \big(\varepsilon V^2(x) + K'(\varepsilon)\big) \| f \|_V \|\nu\|_V.\end{aligned}$$

We work now with the other member of the inequality:

$$\begin{aligned}\left|\big(D\Pi(\mu) \cdot \nu\big)(Q_\mu f)\right| &\le 2 \int \left|Q_\mu f(x)\right| \left| W * \nu(x) - \int W * \nu \, \mathrm{d}\Pi(\mu) \right| \Pi(\mu)(\mathrm{d}x) \\ &\le C \| f \|_V \|\nu\|_V \int V^2(x) \Pi(\mu)(\mathrm{d}x) = C' \| f \|_V \|\nu\|_V.\end{aligned}$$

□

4.2. *The dynamical system*

Define the semiflow $\Phi : \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d; V) \to \mathcal{P}(\mathbb{R}^d; V)$ by

$$\Phi_t(\mu) = \mathrm{e}^{-t}\mu + \mathrm{e}^{-t} \int_0^t \mathrm{e}^s \Pi\big(\Phi_s(\mu)\big) \mathrm{d}s, \qquad \Phi_0(\mu) = \mu. \tag{4.7}$$

4.2.1. *Existence of the semiflow*

We first prove the local existence of the semiflow and then give sufficient conditions on the potentials for non-explosion. To show the local existence of a solution, since $\mathcal{P}(\mathbb{R}^d; V)$ is not a vector space, we will proceed directly by approximation. The following lemma is helpful in order to find a good security cylinder.

Lemma 4.7. *For any* $\beta > 1$, *the application* Π *restricted to* $\mathcal{P}_\beta(\mathbb{R}^d; V)$ *is bounded and Lipschitz.*

Proof. By equation (2.10), we have the following bound for $\Pi(\mu)$:

$$\|\Pi(\mu)\|_V \le \left(\int_{\mathbb{R}^d} \mathrm{e}^{-4\kappa\beta V(x)}\gamma(\mathrm{d}x)\right)^{-1}\int_{\mathbb{R}^d} V(x)\gamma(\mathrm{d}x) =: C_\beta. \tag{4.8}$$

Remind, that Π is $\mathcal{C}^\infty$ on $\mathcal{P}(\mathbb{R}^d; V)$ equipped with the strong topology. Its differential (at μ) is the continuous linear operator $D\Pi(\mu) : \mathcal{M}_0(\mathbb{R}^d; V) \to \mathcal{M}_0(\mathbb{R}^d; V)$ defined by

$$D\Pi(\mu)\cdot\nu(\mathrm{d}x) := -2\left(W*\nu(x) - \int_{\mathbb{R}^d} W*\nu(y)\Pi(\mu)(\mathrm{d}y)\right)\Pi(\mu)(\mathrm{d}x). \tag{4.9}$$

Fix $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$. Lemma 2.2 implies that

$$\|D\Pi(\mu)\cdot\nu\|_V \le 4\kappa(1 + C_\beta)\|\nu\|_V \int_{\mathbb{R}^d} V^2(x)\Pi(\mu)(\mathrm{d}x).$$

For $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$, the computation used for the bound of $\Pi(\mu)$ enables to control the last integral, hence we get a bound (call it C'_β) on the differential and Π is Lipschitz as stated. □

Proposition 4.8. *For all* $\mu \in \mathcal{P}(\mathbb{R}^d; V)$, *the Eq.* (4.7) *admits a local solution. This defines a* C^∞ *semiflow* Φ *for the strong topology.*

Proof. Let μ belong to $\mathcal{P}(\mathbb{R}^d; V)$ and choose $\beta > 2\|\mu\|_V$ (so that $\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$). We introduce the classic Picard approximation scheme

$$\begin{cases} \mu_t^{(0)} := \mu, \\ \mu_t^{(n)} := \mathrm{e}^{-t}\mu + \int_0^t \mathrm{e}^{s-t}\Pi\big(\mu_s^{(n-1)}\big)\,\mathrm{d}s. \end{cases}$$

We set ε small enough such that $\|\mu\|_V + (1 - \mathrm{e}^{-\varepsilon})C_\beta \le \beta$ and $\varepsilon C'_\beta < 1$ where both constants were defined in Lemma 4.7. Then, for all n, $\mu_t^{(n)}$ is defined and belongs to $\mathcal{P}_\beta(\mathbb{R}^d; V)$, which makes $[0, \varepsilon) \times \mathcal{P}_\beta(\mathbb{R}^d; V)$ a good security cylinder. We have, for $t < \varepsilon$,

$$\|\mu_t^{(n+1)} - \mu_t^{(n)}\|_V \le \big(1 - \mathrm{e}^{-\varepsilon}\big)C'_\beta \sup_{t<\varepsilon}\|\mu_t^{(n)} - \mu_t^{(n-1)}\|_V.$$

Now the series with general term $\sup_{t<\varepsilon}\|\mu_t^{(n+1)} - \mu_t^{(n)}\|_V$ converges and thus the sequence of functions $\mu^{(n)}$ is Cauchy for the topology of uniform convergence. Since $\mathcal{P}(\mathbb{R}^d; V)$ is complete, we have successfully built a solution on $[0, \varepsilon)$. As the map Π is $\mathcal{C}^\infty$ for the strong topology, every Picard approximation $\mu \mapsto \mu_t^{(n)}$ is $\mathcal{C}^\infty$ by induction, and it is enough to take the limit uniformly in μ on $\mathcal{P}_\beta(\mathbb{R}^d; V)$ to conclude that the semiflow is smooth. □

Definition 4.9. *A subset* A *of* $\mathcal{P}(\mathbb{R}^d; V)$ *is positively invariant* (*negatively invariant*, *invariant*) *for* Φ *provided* $\Phi_t(A) \subset A$ ($A \subset \Phi_t(A)$, $\Phi_t(A) = A$) *for all* $t \ge 0$.

For a symmetric W, we introduce the *free energy* (up to a multiplicative constant) for any $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ absolutely continuous with respect to Lebesgue's measure:

$$\mathcal{F}(\mu) := \int_{\mathbb{R}^d} \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\gamma}\right)\mathrm{d}\mu + \int_{\mathbb{R}^d\times\mathbb{R}^d} W(x, y)\mu(\mathrm{d}x)\mu(\mathrm{d}y), \tag{4.10}$$

and $\mathcal{F}(\mu) = +\infty$ if μ has a singular part with respect to Lebesgue. This functional is the sum of an entropy and an interacting energy term. Under some convexity, the competition between them implies the existence of a unique minimizer for $\mathcal{F}$ (see [23]).

Proposition 4.10. *Whenever that W is either symmetric or uniformly bounded in the second variable ($W(x, y) \leq \kappa V(x)$), the semiflow Φ does not explode.*

Proof. Suppose that $W(x, y)$ is bounded in y: $W(x, y) \leq \kappa V(x)$. Mimicking the proof of Lemma 4.7, we show that Π is globally bounded (call C the upper bound). This means that $\Phi_t(\mu)$ remains in the space $\mathcal{P}_C(\mathbb{R}^d; V)$, so it cannot explode.

Let us now assume that W is symmetric. We point out that the free energy $\mathcal{F}$ is not a Lyapunov function for the semiflow Φ because, in general, the measure $\Phi_t(\mu)$ is not absolutely continuous with respect to Lebesgue's measure and so, $\mathcal{F}(\Phi_t(\mu)) = \infty$. Consider the Lyapunov function $\mathcal{I}(\mu) := \mathcal{F}(\Pi(\mu))$, which can be viewed as $\mathcal{F}$ restricted to absolutely continuous probability measures, is a $\mathcal{C}^\infty$ function for the strong topology. We compute (thanks to the symmetry of W) for $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$

$$D\mathcal{F}(\mu) \cdot \nu = \int_{\mathbb{R}^d} \left[\log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\gamma}(x) \right) + 2W * \mu(x) \right] \mathrm{d}\nu(x). \tag{4.11}$$

But Π is $\mathcal{C}^\infty$ and its differential is given by (4.9). Computing the differential of $\mathcal{I}(\mu)$ by composition, we obtain

$$D\mathcal{I}(\mu) \cdot \nu = -4 \int_{\mathbb{R}^d} \big(W * \Pi(\mu) - W * \mu \big) \left(W * \nu - \int_{\mathbb{R}^d} W * \nu \, \mathrm{d}\Pi(\mu) \right) \mathrm{d}\Pi(\mu).$$

We choose $\nu = \Pi(\mu) - \mu$ and get

$$\frac{1}{4} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}\big(\Phi_t(\mu)\big) = -\int_{\mathbb{R}^d} (W * \nu)^2 \, \mathrm{d}\Pi(\mu) + \left(\int_{\mathbb{R}^d} W * \nu \, \mathrm{d}\Pi(\mu) \right)^2 \leq 0.$$

So, for all $c > 0$, the sets $\{\mu; \mathcal{I}(\mu) \leq c\}$ are positively invariant. As they are (weakly) compact, the semiflow cannot explode. $\square$

We have defined the smooth dynamical system Φ, with respect to the strong topology. But, in order to study the asymptotic behavior of $(\mu_t, t \geq 0)$, it is technically easier to work with the weak topology. Therefore, we will also consider the semiflow Φ with the weak topology:

Proposition 4.11. *Φ induces a continuous semiflow with respect to the weak topology.*

Proof. Since $\mu \mapsto W * \mu(x)$ is readily weakly continuous (by the domination condition again), Π is weakly continuous. Now, going back to the Picard approximation scheme, it results that $\mu \mapsto \mu_t^{(n)}$ is weakly continuous for every n and t. Passing to the limit, we conclude. $\square$

4.2.2. *The free energy*

We show how the free energy functional $\mathcal{F}$ helps to find the fixed points of Π. From now on, we restrict ourselves to the set of absolutely continuous measures.

Lemma 4.12. *Suppose that W is symmetric. Then the fixed points of Π are the minima of $\mathcal{F}$.*

Proof. Equation (4.11) readily implies that $D\mathcal{F}(\mu) \cdot \nu = 0$ for all $\nu \in \mathcal{M}_0(\mathbb{R}^d; V)$ if and only if $\mu = \Pi(\mu)$. So, the fixed points of Π are the critical points of $\mathcal{F}$. Indeed, $\mathcal{F}$ is a $\mathcal{C}^\infty$ functional, with second differential $D^2\mathcal{F}(\mu)$. Let $\nu_1, \nu_2 \in \mathcal{M}_0(\mathbb{R}^d; V)$. We have:

$$D^2\mathcal{F}(\mu) \cdot (\nu_1, \nu_2) = \int_{\mathbb{R}^d} \nu_1(x)\nu_2(x)\mu(x)^{-1}\gamma(x) \, \mathrm{d}x + 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x, y)\nu_1(\mathrm{d}x)\nu_2(\mathrm{d}y) \geq 0.$$

It then implies that $\mu = \Pi(\mu)$ is a minimum for $\mathcal{F}$. □

Corollary 4.13. *Suppose that W is symmetric and for all $y \in \mathbb{R}^d$, the function $x \mapsto V(x) + W(x, y)$ is strictly convex. Then, $\mathcal{F}$ admits a unique minimizer μ_∞ and this probability measure μ_∞ satisfies* $\lim_{t\to\infty} \Phi_t(\mu) = \mu_\infty$.

Proof. Under strict convexity, McCann [15] has proved that $\mathcal{F}$ admits a unique critical point μ_∞, which is a unique global minimum and also the unique fixed point of Π. □

Proposition 4.14. *Whenever that W is either symmetric or bounded in the second variable, then the set $\{\mu \in \mathcal{P}(\mathbb{R}^d; V); \Pi(\mu) = \mu\}$ is a nonempty compact (for the weak topology) subset of $\mathcal{P}(\mathbb{R}^d; V)$.*

Proof. Suppose first that W is bounded in y: $W(x, y) \le \kappa V(x)$ and let $\beta := \frac{\int_{\mathbb{R}^d} V(x)\gamma(\mathrm{d}x)}{\int_{\mathbb{R}^d} \mathrm{e}^{-2\kappa V(x)}\gamma(\mathrm{d}x)}$. By Lemma 4.7, Π maps (weakly) continuously the compact convex space $\mathcal{P}_\beta(\mathbb{R}^d; V)$ into itself. The Leray–Schauder fixed point theorem then ensures that the set $\{\mu \in \mathcal{P}_\beta(\mathbb{R}^d; V); \Pi(\mu) = \mu\}$ is nonempty.

Suppose now that W is symmetric. We use again the free energy $\mathcal{I} = \mathcal{F} \circ \Pi$. Let $m := \inf\{\mathcal{I}(\mu); \mu \in \mathcal{P}(\mathbb{R}^d; V)\}$. There exists a sequence of probability measures (μ_n) absolutely continuous with respect to Lebesgue's measure such that $m \le \mathcal{I}(\mu_n) \le m + 1/n$. As for any $c > 0$, the set $\{\mu; \mathcal{I}(\mu) \le c\}$ is compact, we extract a subsequence (μ_{n_k}) converging (weakly) to μ_∞. As $\mu \mapsto W * \mu$ and $\mu \mapsto \Pi(\mu)$ are continuous, $\mu \mapsto \mathcal{I}(\mu)$ is also weakly continuous and so $\mathcal{I}(\mu_\infty) = m$. Lemma 4.12 permits to conclude. □

5. Behavior of the occupation measure

5.1. *Tightness of* $(\mu_t, t \ge 0)$

Thanks to the potential V, we manage to obtain a weak form of compactness for the empirical measure, the tightness.

Proof of Theorem 3.2. Set $\phi(t) := \int_0^t V(X_s)\,\mathrm{d}s$. All we need to prove is that $\phi(t) = \mathrm{O}(t)$ a.s. We use again the Lyapunov functional $\mathcal{E}_\mu(x) = V(x) + W * \mu(x)$ and remind Itô's formula (2.20) for $\mathcal{E}_{\mu_t}(X_t)$. Moreover, Eq. (1.7) implies that for all $\varepsilon > 0$, there exists $\eta > 0$ such that for any $|x| \ge \eta$, we have that $V(x) + W(x, y) \le \varepsilon(|\nabla V(x)|^2 + 2\langle\nabla V(x), \nabla_x W(x, y)\rangle)$. So, for all $\varepsilon > 0$, there exists k_ε such that $\mathcal{E}_{\mu_s}(x) \le k_\varepsilon + \varepsilon|\nabla\mathcal{E}_{\mu_s}(x)|^2$. On one hand, if $\int_0^\infty |\nabla\mathcal{E}_{\mu_s}(X_s)|^2\,\mathrm{d}s < \infty$ a.s., then the strong LLN for martingales asserts that $\int_0^\infty \langle\nabla\mathcal{E}_{\mu_s}(X_s), \mathrm{d}B_s\rangle$ converges a.s. to M_∞ and the proof is then similar to the following. (Indeed the ergodic theorem implies that this case does not happen). On the other, if $\int_0^\infty |\nabla\mathcal{E}_{\mu_s}(X_s)|^2\,\mathrm{d}s = \infty$ a.s., then a.s. there exists $T(\omega)$ such that for all $t \ge T$, we have $\int_0^t \langle\nabla\mathcal{E}_{\mu_s}(X_s), \mathrm{d}B_s\rangle \le \frac{1}{2}\int_0^t |\nabla\mathcal{E}_{\mu_s}(X_s)|^2\,\mathrm{d}s$. So, we get the a.s. inequality for t (random) large enough:

$$\int_0^t \left|\nabla\mathcal{E}_{\mu_s}(X_s)\right|^2 \mathrm{d}s \le 2\mathcal{E}_\mu(x) + \int_0^t \Delta\mathcal{E}_{\mu_s}(X_s)\,\mathrm{d}s + \frac{2}{r}\int_0^t W(X_s, X_s)\,\mathrm{d}s.$$

The domination condition (1.6) leads to $W(X_s, X_s) \le 2\kappa V(X_s) \le 2\kappa(k_\varepsilon + \varepsilon|\nabla\mathcal{E}_{\mu_s}(X_s)|^2)$. Moreover, it also implies:

$$\Delta\mathcal{E}_{\mu_s} \le D\mathcal{E}_{\mu_s} \le D\big(k_\varepsilon + \varepsilon\left|\nabla\mathcal{E}_{\mu_s}(X_s)\right|^2\big).$$

So, putting all the pieces together, we get

$$\big(1 - (D + 4\kappa/r)\varepsilon\big)\int_0^t \left|\nabla\mathcal{E}_{\mu_s}(X_s)\right|^2 \mathrm{d}s \le 2\mathcal{E}_\mu(x) + (D + 4\kappa/r)k_\varepsilon t.$$

It remains to choose $\varepsilon = (D + 4\kappa/r)^{-1}/2$ and then we obtain the desired inequality: for some $C > 0$,

$$\phi(t) \le \int_0^t \mathcal{E}_{\mu_s}(X_s)\,\mathrm{d}s \le C(1 + t).$$

We finally conclude that $\beta(\omega) := \sup\{\mu_t(V); t \geq 0\} < +\infty$ a.s. □

Proposition 5.1. *For all $n \in \mathbb{N}$, we have that $\int_0^t \mathbb{E}_{x,r,\mu}(V^n(X_s))\,\mathrm{d}s = \mathrm{O}(t)$.*

Proof. We drop the subscripts x, r, μ in the following. We prove the result for the Lyapunov function $\mathcal{E}_\mu(x)$ instead of V. For $n = 1$, it suffices to adapt the previous proof to show that for all $t > 0$

$$\int_0^t \mathbb{E}\left|\nabla \mathcal{E}_{\mu_s}(X_s)\right|^2 \mathrm{d}s \leq \mathcal{E}_\mu(x) + \int_0^t \mathbb{E}\Delta \mathcal{E}_{\mu_s}(X_s)\,\mathrm{d}s + \frac{2}{r}\int_0^t \mathbb{E}W(X_s, X_s)\,\mathrm{d}s.$$

The result follows.

We conclude the general case $n \geq 1$ by induction. Indeed, we have for all $\epsilon > 0$:

$$\mathcal{E}_\mu^n(x) \leq k_\epsilon \mathcal{E}_\mu^{n-1}(x) + \epsilon \mathcal{E}_\mu^{n-1}(x)\left|\nabla \mathcal{E}_\mu(x)\right|^2.$$

Moreover, by Itô's formula, we also find for all $s < t$ that

$$\begin{aligned}
&\int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^{n-1}(X_u)\left|\nabla \mathcal{E}_{\mu_u}(X_u)\right|^2 \mathrm{d}u \\
&\quad\leq \int_s^t \frac{2\kappa}{r+u}\mathbb{E}\mathcal{E}_{\mu_u}^n(X_u)\,\mathrm{d}u + (n-1)\int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^{n-2}(X_u)\left|\nabla \mathcal{E}_{\mu_u}(X_u)\right|^2 \mathrm{d}u \\
&\qquad + k\int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^n(X_u)\,\mathrm{d}u + \int_s^t \frac{\kappa}{r+u}\mathbb{E}\left(\mathcal{E}_{\mu_u}^{n-1}(X_u)\int_0^u \mathcal{E}_{\mu_v}(X_v)\,\mathrm{d}v\right)\mathrm{d}u.
\end{aligned}$$

Young's inequality: $x^{n-1}y \leq \frac{n-1}{n}x^n + \frac{1}{n}y^n$, with $x = \mathcal{E}_{\mu_u}^{n-1}(X_u)$ and $y = \mathcal{E}_{\mu_v}(X_v)$, yields to the existence of $\alpha, A > 0$ such that

$$\int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^{n-1}(X_u)\left|\nabla \mathcal{E}_{\mu_u}(X_u)\right|^2 \mathrm{d}u \leq \alpha\int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^n(X_u)\,\mathrm{d}u + A\int_s^t \frac{\mathrm{d}u}{r+u}\int_0^u \mathbb{E}\mathcal{E}_{\mu_v}^n(X_v)\,\mathrm{d}v.$$

We thus obtain:

$$\begin{aligned}
&\int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^n(X_u)\,\mathrm{d}u \\
&\quad\leq k_\epsilon \int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^{n-1}(X_u)\,\mathrm{d}u + \epsilon\left(\alpha\int_s^t \mathbb{E}\mathcal{E}_{\mu_u}^n(X_u)\,\mathrm{d}u + A\int_s^t \frac{\mathrm{d}u}{r+u}\int_0^u \mathbb{E}\mathcal{E}_{\mu_v}^n(X_v)\,\mathrm{d}v\right) \\
&\quad\leq k(t-s) + \int_s^t \frac{\mathrm{d}u}{r+u}\int_0^u \mathbb{E}\mathcal{E}_{\mu_v}^n(X_v)\,\mathrm{d}v.
\end{aligned}$$

Let $x(t) := \int_0^t \mathbb{E}\mathcal{E}_{\mu_s}^n(X_s)\,\mathrm{d}s$. Solving the preceding inequality boil down to solve $\dot{x} \leq M + x/(r+t)$. The solution satisfies $x(t) = \mathrm{O}(t)$ and we finally conclude. □

Corollary 5.2. *For all $n \in \mathbb{N}$, we have that $\mathbb{E}_{x,r,\mu}(V^n(X_t)) = \mathrm{O}(t)$.*

5.2. *Asymptotic behavior*

Define the family of measures $\{\varepsilon_{t,t+s}; t \geq 0, s \geq 0\}$ by

$$\varepsilon_{t,t+s} := \int_t^{t+s}\left(\delta_{X_{h(u)}} - \Pi(\mu_{h(u)})\right)\mathrm{d}u. \tag{5.1}$$

This family will be essential for proving that $t \mapsto \mu_{h(t)}$ is an asymptotic pseudotrajectory for Φ.

Proposition 5.3. *For all $T > 0$ and all $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$, we have $\mathbb{P}_{x,r,\mu}$-a.s.,*

$$\lim_{t\to\infty} \sup_{0\le s\le T} |\varepsilon_{t,t+s} f| = 0.$$

Proof. First, we need the uniform estimates on the family of semigroups (P_t^μ) proved in Section 4.1. Let $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$. We begin to rewrite

$$\varepsilon_{t,t+s} f = \int_{h(t)}^{h(t+s)} A_{\mu_u} \circ Q_{\mu_u} f \frac{\mathrm{d}u}{r+u}.$$

We consider the $\mathcal{C}^2$-valued process $(t,x) \mapsto Q_{\mu_{h(t)}} f(x)$, which is of class $\mathcal{C}^2$ and a $\mathcal{C}^1$-semimartingale. Indeed, it is easy to see that $t \mapsto \mu_{h(t)}$ is a.s. a bounded variation process with values in $\mathcal{M}(\mathbb{R}^d; V)$ (since Proposition 4.3 shows that $\mu \mapsto Q_\mu f$ is also $\mathcal{C}^1$, the claim follows by composition). So, we apply the generalized Itô formula to $(t,x) \mapsto h(t)^{-1} Q_{\mu_{h(t)}} f(x)$ and decompose $\varepsilon_{t,t+s}$ in four parts:

$$\varepsilon_{t,t+s} f = \varepsilon_{t,t+s}^{(1)} f + \varepsilon_{t,t+s}^{(2)} f + \varepsilon_{t,t+s}^{(3)} f + \varepsilon_{t,t+s}^{(4)} f$$

with

$$\begin{aligned}
\varepsilon_{t,t+s}^{(1)} f &= -\frac{1}{h(t+s)} Q_{\mu_{h(t+s)}} f(X_{h(t+s)}) + \frac{1}{h(t)} Q_{\mu_{h(t)}} f(X_{h(t)}),\\
\varepsilon_{t,t+s}^{(2)} f &= -\int_{h(t)}^{h(t+s)} Q_{\mu_u} f(X_u) \frac{\mathrm{d}u}{(r+u)^2},\\
\varepsilon_{t,t+s}^{(3)} f &= \int_{h(t)}^{h(t+s)} \frac{\partial}{\partial u} Q_{\mu_u} f(X_u) \frac{\mathrm{d}u}{r+u},\\
\varepsilon_{t,t+s}^{(4)} f &= M_{h(t+s)}^f - M_{h(t)}^f,
\end{aligned}$$

where M_t^f is the local martingale $M_t^f := \int_0^t \nabla Q_{\mu_u} f(X_u) \frac{\mathrm{d}B_u}{r+u}$.

Before controlling each term separately, we remind the estimates of Propositions 4.2 and 4.3: $\forall \varepsilon > 0$, $f \in \mathcal{C}^\infty(\mathbb{R}^d; V)$,

$$\begin{aligned}
\left|Q_{\mu_{h(t)}} f(X_{h(t)})\right| &\le \|f\|_V \bigl(\varepsilon V(X_{h(t)}) + K(\varepsilon)\bigr),\\
\left|\nabla Q_{\mu_{h(t)}} f(X_{h(t)})\right| &\le \|f\|_V \bigl(\varepsilon V(X_{h(t)}) + K_1(\varepsilon)\bigr).
\end{aligned}$$

We also remind that $\int_0^t V(X_s)\,\mathrm{d}s = \mathrm{O}(t)$ a.s. and $\int_0^t \mathbb{E} V(X_s)\,\mathrm{d}s = \mathrm{O}(t)$. Now, we are able to find for all $\varepsilon > 0$:

$$\begin{aligned}
\left|\varepsilon_{t,t+s}^{(1)} f\right| &\le h(t)^{-1} \bigl(\left|Q_{\mu_{h(t+s)}} f(X_{h(t+s)})\right| + \left|Q_{\mu_{h(t)}} f(X_{h(t)})\right|\bigr)\\
&\le h(t)^{-1} \|f\|_V \bigl(\varepsilon\bigl(V(X_{h(t+s)}) + V(X_{h(t)})\bigr) + K_\varepsilon\bigr),
\end{aligned}$$

and so $\lim_{t\to\infty} \sup_{0\le s\le T} |\varepsilon_{t,t+s}^{(1)} f| \le \varepsilon \|f\|_V$ a.s. As the latter is true for all $\varepsilon > 0$, we deduce that a.s. $\lim_{t\to\infty} \sup_{0\le s\le T} |\varepsilon_{t,t+s}^{(1)} f|$ vanishes. Similarly, with $\varepsilon = 1$, there exists C_2 such that

$$\left|\varepsilon_{t,t+s}^{(2)} f\right| \le \int_{h(t)}^{h(t+s)} \bigl(V(X_u) + K\bigr) \frac{\mathrm{d}u}{(r+u)^2} \|f\|_V \le \frac{C_2 \|f\|_V}{h(t)^2} \int_{h(t)}^{h(t+s)} V(X_u)\,\mathrm{d}u,$$

and so $\sup_{0\le s\le T} |\varepsilon_{t,t+s}^{(2)} f| \le C_2 h(t)^{-1} \|f\|_V$ a.s.

By Markov's inequality and using the bound on the differential of Q_μ given in Corollary 4.6, we get:

$$\mathbb{P}\Big(\sup_{0\le s\le T}\big|\varepsilon^{(3)}_{t,t+s}f\big|\ge\delta\Big)\le\delta^{-2}\int_{h(t)}^{h(t+T)}\mathbb{E}\big|(DQ_{\mu_u}\cdot\dot{\mu}_u)(f)(X_u)\big|^2\frac{\mathrm{d}u}{r+u}$$
$$\le\frac{C}{\delta^2}\|f\|_V^2\int_{h(t)}^{h(t+T)}\mathbb{E}\big(V^6(X_u)\big)\frac{\mathrm{d}u}{(r+u)^3}.$$

As for all $\varepsilon>0$ and $n\in\mathbb{N}$ we have $\int_0^t\mathbb{E}V^n(X_s)\,\mathrm{d}s=\mathrm{O}(t)$, there exists $C_3>0$ such that

$$\mathbb{P}\Big(\sup_{0\le s\le T}\big|\varepsilon^{(3)}_{t,t+s}f\big|\ge\delta\Big)\le\frac{C_3}{\delta^2}h(t)^{-1}\|f\|_V^2.$$

Since the quadratic variation of $M^f_{h(t+s)}-M^f_{h(t)}$ is bounded by $\|f\|_V^2\int_{h(t)}^{h(t+T)}(\varepsilon V(X_u)+K_1(\varepsilon))^2\frac{\mathrm{d}u}{(r+u)^2}$, Burkholder–Davis–Gundy's inequality (BDG) implies

$$\mathbb{P}_{x,r,\mu}\Big(\sup_{s\in[0,T]}\big|\varepsilon^{(4)}_{t,t+s}f\big|\ge\delta\Big)\le\frac{C_4}{\delta^2}h(t)^{-1}\|f\|_V^2. \tag{5.2}$$

It only remains to prove that a.s.

$$\lim_{t\to\infty}\sup_{0\le s\le T}\big|\varepsilon^{(4)}_{t,t+s}f\big|=\lim_{t\to\infty}\sup_{0\le s\le T}\big|\varepsilon^{(3)}_{t,t+s}f\big|=0.$$

First, for all $\varepsilon>0$, we have by Doob's inequality together with BDG's inequality that

$$\mathbb{P}_{x,r,\mu}\Big(\sup_{n\le t<n+1}\sup_{s\in[0,T]}\big|\varepsilon^{(4)}_{t,t+s}f\big|\ge\delta\Big)\le\frac{C}{\delta^2}\|f\|_V^2\sup_{n\le t<n+1}h(t)^{-1}=\frac{C}{\delta^2}\|f\|_V^2h(n)^{-1}.$$

Since the series $\sum_n h(n)^{-1}$ converges, we conclude by Borel–Cantelli's lemma that a.s.

$$\lim_{n\to\infty}\sup_{n\le t<n+1}\sup_{0\le s\le T}\big|\varepsilon^{(4)}_{t,t+s}f\big|=0.$$

The same argument for $|\varepsilon^{(3)}_{t,t+s}f|$ permits to conclude. □

Lemma 5.4. *If for all $T>0$, all $f\in\mathcal{C}^\infty(\mathbb{R}^d;V)$, we have*

$$\lim_{t\to\infty}\sup_{0\le s\le T}|\varepsilon_{t,t+s}f|=0\quad a.s.,$$

then the time-changed process, given by $\mathbb{R}_+\to\mathcal{P}(\mathbb{R}^d;V)$, $t\mapsto\mu_{h(t)}$ is a.s. an asymptotic pseudotrajectory for Φ (for the weak topology of measures).

Proof. The family $(\mu_t,t\ge0)$ is a.s. tight and by Prokhorov's theorem (because $\mathcal{P}(\mathbb{R}^d;V)$ is a Polish space), it is equivalent to the relative compactness of $(\mu_t,t\ge0)$. Benaïm [3], Theorem 3.2, asserts that a continuous map $\nu:\mathbb{R}_+\to\mathcal{P}(\mathbb{R}^d;V)$ is an asymptotic pseudotrajectory for the semiflow Φ if and only if ν is (weakly) uniformly continuous and every limit point of $\{\nu(t+\cdot);t\ge0\}$ is an orbit of Φ. We first show that $\mu_{h(t)}$ is uniformly continuous. By definition of μ_t, we have

$$|\mu_{h(t+s)}f-\mu_{h(t)}f|\le\int_t^{t+s}\big(|\mu_{h(u)}f|+\big|f(X_{h(u)})\big|\big)\,\mathrm{d}u.$$

As asymptotically $\mu_t\in\mathcal{P}_\beta(\mathbb{R}^d;V)$ a.s., we get for all t large enough

$$|\mu_{h(t+s)}f-\mu_{h(t)}f|\le2\beta s\|f\|_V. \tag{5.3}$$

We put these estimates in (2.7) and the uniform continuity follows. As a.s. $\mu_{h(t)}$ belongs to a compact set (for t large enough), Ascoli's theorem implies that there exist an increasing sequence $(t_n)_n$ and $\tilde{\mu} \in \mathcal{P}(\mathbb{R}^d; V)$ such that $(\mu_{h(t_n+s)}, s \geq 0)$ converges weakly to $(\tilde{\mu}_s, s \geq 0)$. Then, we have $\mu_{h(t_n+s)} = \mu_{h(t_n)} + \varepsilon_{t_n,t_n+s} + \int_{t_n}^{t_n+s} (\Pi(\mu_{h(u)}) - \mu_{h(u)})\,\mathrm{d}u$. As $\mu_{h(t_n+\cdot)}$ converges weakly to $\tilde{\mu}$ and $\varepsilon_{t_n,t_n+\cdot}$ goes to 0, the limit $\tilde{\mu}$ satisfies (4.7).

Suppose that $\mu_{h(t)}$ is not an asymptotic pseudotrajectory for Φ. It means that:

$$\exists T > 0, \exists \varepsilon_0 > 0, \exists t_n \uparrow \infty, \exists s_n \in [0, T] \text{ such that } \quad d\big(\mu_{h(t_n+s_n)}, \Phi_{s_n}(\mu_{h(t_n)})\big) \geq \varepsilon_0.$$

It implies, denoting by s the limit of s_n and $\tilde{\mu}$ the limit of $\mu_{h(t_n)}$, that $d(\tilde{\mu}_s, \Phi_s(\tilde{\mu})) \geq \varepsilon_0$, which contradicts that $\tilde{\mu}$ is an orbit of Φ. $\square$

Remark 5.5. *Combine Proposition 5.3 with Lemma 5.4 to deduce Theorem 3.6: $\mathbb{P}_{x,r,\mu}$-a.s., the function $t \mapsto \mu_{h(t)}$ is an asymptotic pseudotrajectory for Φ.*

5.3. *Back to the dynamical system: A global attractor for the semiflow*

As explained in Section 4.2, we will consider from now on the semiflow Φ with the weak topology. A good candidate to be an attractor of the semiflow is the ω-limit set of (μ_t),

$$\omega(\mu_t, t \geq 0) := \bigcap_{t \geq 0} \overline{\{\mu_s; s \geq t\}} \tag{5.4}$$

which is (a.s.) weakly compact, since it is contained in $\mathcal{P}_\beta(\mathbb{R}^d; V)$ a.s.

We introduce here a crucial set to analyse the dynamical system Φ. Let

$$\operatorname{Im}(\Pi) := \big\{\Pi(\mu); \mu \in \mathcal{P}\big(\mathbb{R}^d; V\big)\big\}, \tag{5.5}$$

and denote its convex hull by $\widehat{\operatorname{Im}(\Pi)}$.

Proposition 5.6. *$\widehat{\operatorname{Im}(\Pi)}$ is a positively invariant set for the semiflow Φ and contains every negatively invariant bounded subset of $\mathcal{P}(\mathbb{R}^d; V)$.*

Proof. By Jensen's inequality applied to the convex combination $\Phi_t(\mu) = \mathrm{e}^{-t}\mu + \mathrm{e}^{-t}\int_0^t \mathrm{e}^s \Pi(\Phi_s(\mu))\,\mathrm{d}s$ and to the convex map $\mu \mapsto \mathrm{d}_V(\mu, \widehat{\operatorname{Im}(\Pi)})$, we show, for every $\mu \in \mathcal{P}(\mathbb{R}^d; V)$ and every $t \geq 0$, that

$$\mathrm{d}_V\big(\Phi_t(\mu), \widehat{\operatorname{Im}(\Pi)}\big) \leq \mathrm{e}^{-t}\mathrm{d}_V\big(\mu, \widehat{\operatorname{Im}(\Pi)}\big), \tag{5.6}$$

where $\mathrm{d}_V(\mu, X) := \inf\{\|\mu - \nu\|_V; \nu \in X\}$. So, for any negatively invariant bounded subset A of $\mathcal{P}(\mathbb{R}^d; V)$, we get for all $t \geq 0$: $\mathrm{d}_V(A, \widehat{\operatorname{Im}(\Pi)}) \leq \mathrm{d}_V(\Phi_t(A), \widehat{\operatorname{Im}(\Pi)}) \leq \mathrm{e}^{-t}\mathrm{d}_V(A, \widehat{\operatorname{Im}(\Pi)})$. $\square$

Now, we need to recall a short list of important definitions coming from the theory of Dynamical Systems.

Definition 5.7. (a) *A subset A of $\mathcal{P}(\mathbb{R}^d; V)$ is an attracting set* (*respectively attractor*) *for Φ provided*:

(1) *A is nonempty, weakly compact and positively invariant* (*respectively invariant*) *and*
(2) *A has a neighborhood $\mathcal{N} \subset \mathcal{P}(\mathbb{R}^d; V)$ such that* $\mathrm{d}(\Phi_t(\mu), A) \to 0$ *as $t \to +\infty$ uniformly in $\mu \in \mathcal{N}$.*

(b) *The basin of attraction of an attractor $K \subset A$ for $\Phi|A = (\Phi_t|A)_t$ is the positively invariant open set* (*in A*) *comprising all points whose orbits are asymptotically in K*:

$$B(K, \Phi|A) := \Big\{\mu \in A; \lim_{t \to \infty} \mathrm{d}\big(\Phi_t(\mu), K\big) = 0\Big\}.$$

(c) *A global attracting set* (*respectively global attractor*) *is an attracting set* (*respectively attractor*) *whose basin is the whole space $\mathcal{P}(\mathbb{R}^d; V)$.*

(d) *An attractor-free set is a nonempty compact invariant set A such that $\Phi|A$ has no attractor except A itself.*

Our aim is now to describe the limit set of μ_t and find a global attracting set for Φ. The natural candidate is the set $\omega(\mu_t, t \geq 0)$. First, we describe it dynamically.

Theorem 5.8. *The ω-limit set of $\{\mu_t, t \geq 0\}$ is $\mathbb{P}_{x,r,\mu}$-almost surely an attractor-free set of Φ.*

Proof. It results from Theorem 3.6 and [4]. $\square$

Corollary 5.9. $\mathbb{P}_{x,r,\mu}(\overline{\lim}_{t\to+\infty}|X_t| = +\infty) = 1$.

Proof. Let A be a open subset of $\mathbb{R}^d$ such that $\gamma(A) > 0$. Since the measure γ is diffusive, for all $\nu \in \widehat{\mathrm{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$, there exist $m, M > 0$ such that $m\gamma \leq \nu \leq M\gamma$. If we consider a sequence $(\nu_{t_n}, n \geq 0)$ in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d; V))$, the limits of its convergent subsequences will belong to $\widehat{\mathrm{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$, because $\omega(\mu_t, t \geq 0)$ is a.s. an attractor-free set of Φ. Thus, there exists a subsequence $(\nu_{t_{n_k}})$ such that a.s. $\nu_{t_{n_k}}$ converges (weakly) to ν: for any smooth function φ of compact support, we have that $\nu_{t_n}(\varphi)$ converges to $\nu(\varphi)$. If we consider a function φ such that $\varphi(x) = 1$ for $x \in A$ and $\varphi(x) = 0$ for $x \notin B$, $A \subset B$, we find that $\nu(\varphi) \geq \nu(A) > 0$. Thus

$$\nu(B) \geq \limsup \nu_t(\varphi) \geq \liminf \nu_t(\varphi) \geq \nu(A) \geq m\gamma(A).$$

So, $\int_0^{t_n} \delta_{X_s}(A)\,\mathrm{d}s$ is asymptotically equivalent to $t_n m\gamma(A)$, which in turn gives $\int_0^\infty \delta_{X_s}(A)\,\mathrm{d}s = \infty$ a.s. Then, for all $K > 0$, $\int_0^\infty \delta_{X_s}(\mathbb{R}^d \setminus \overline{B}_K)\,\mathrm{d}s = \infty$ a.s., where $\overline{B}_K$ is the closed ball of radius K. Finally,

$$\mathbb{P}_{x,r,\mu}\left(\bigcap_K \left\{\int_0^\infty \mathbb{1}_{\{|X_s| \geq K\}}\,\mathrm{d}s = \infty\right\}\right) = 1.$$

$\square$

Second, we consider the (nonempty) set $\widehat{\mathrm{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$.

Theorem 5.10. *The set $\widehat{\mathrm{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$ is a.s. a global attracting set for Φ.*

Proof. We begin to notice that $\widehat{\mathrm{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0)$ is weakly compact a.s. and by definition, it is also positively invariant. Let $\mu \in \omega(\mu_t, t \geq 0)$. Since $\omega(\mu_t, t \geq 0)$ is an attractor-free set for Φ, for all $s \geq 0$, we have $\Phi_s(\mu) \in \omega(\mu_t, t \geq 0)$. By Proposition 5.6, we know that $\lim_{s\to\infty} \mathrm{d}(\Phi_s(\mu), \widehat{\mathrm{Im}(\Pi)}) = 0$ (uniformly in μ). So, $\mathrm{d}(\Phi_s(\mu), \omega(\mu_t, t \geq 0) \cap \widehat{\mathrm{Im}(\Pi)})$ converges to 0 uniformly in μ. Using again Jensen's inequality, we show that the basin of attraction of Φ is the whole space. $\square$

Corollary 5.11. $\omega(\mu_t, t \geq 0)$ *is a.s. a subset of* $\widehat{\mathrm{Im}(\Pi)}$.

Proof. As $\omega(\mu_t, t \geq 0)$ is attractor-free, Theorem 5.10 implies that $\omega(\mu_t, t \geq 0)$ is the only attractor of Φ restricted to this set. So, $\widehat{\mathrm{Im}(\Pi)} \cap \omega(\mu_t, t \geq 0) = \omega(\mu_t, t \geq 0)$. $\square$

When W is symmetric, we can give a better description of $\omega(\mu_t, t \geq 0)$. Let begin with the following:

Theorem 5.12 (Tromba [22]). *Let $\mathcal{B}$ be a $\mathcal{C}^\infty$ Banach manifold, F a $\mathcal{C}^\infty$ vector field on $\mathcal{B}$ and $\mathcal{I}: \mathcal{B} \to \mathbb{R}$ a $\mathcal{C}^\infty$ function. Assume that:*

(1) *$D\mathcal{I}(\mu) = 0$ if and only if $F(\mu) = 0$;*
(2) *$F^{-1}(0)$ is compact;*
(3) *for each $\mu \in F^{-1}(0)$, $D\mathcal{I}(\mu)$ is a Fredholm operator.*

Then $\mathcal{I}(F^{-1}(0))$ has an empty interior.

Proposition 5.13 ([3], Proposition 6.4). *Let Λ be a compact invariant set for a semiflow Φ on a metric space E. Assume that there exists a continuous function $\mathcal{V} : E \to \mathbb{R}$ such that*:

(1) $\mathcal{V}(\Phi_t(x)) < \mathcal{V}(x)$ *for* $x \in E \backslash \Lambda$ *and* $t > 0$;
(2) $\mathcal{V}(\Phi_t(x)) = \mathcal{V}(x)$ *for* $x \in \Lambda$ *and* $t > 0$.

If $\mathcal{V}$ has an empty interior, then every attractor-free set A for Φ is contained in Λ. Furthermore, $\mathcal{V}$ restricted to A is constant.

Proof of Theorem 3.8. The fixed points of Π form a nonempty compact subset of $\mathcal{P}(\mathbb{R}^d; V)$ thanks to Proposition 4.14. Let $F(\mu) := \Pi(\mu) - \mu$. We already know that $F^{-1}(0)$ is compact for the weak topology. If we show that $\mathcal{I}(F^{-1}(0))$ has an empty interior, then the result is a consequence of Proposition 5.13 with the Lyapunov function $\mathcal{I} = \mathcal{F} \circ \Pi$. Let $\mu \in F^{-1}(0)$ and prove that $DF(\mu)$ is a Fredholm operator. Let $\nu \in \mathcal{P}_\beta(\mathbb{R}^d; V)$. By Lemma 2.2, there exists a constant $C(\beta)$ such that $\|DF(\mu) \cdot \nu\|_V \le C(\beta)\|\nu\|_V$. So, the set $\{DF(\mu) \cdot \nu; \|\nu\|_V \le 1\}$ is a.s. bounded. For $x, y \in \mathbb{R}^d$, we get

$$\begin{aligned}
\bigl|DF(\mu) \cdot \nu(x) - DF(\mu) \cdot \nu(y)\bigr| \le{}& 2\bigl|W * \nu(x)\Pi(\mu)(x) - W * \nu(y)\Pi(\mu)(y)\bigr| \\
&+ 2\int W * \nu \, \mathrm{d}\Pi(\nu)\bigl|\bigl(\Pi(\mu)(x) - \Pi(\mu)(y)\bigr)\bigr| \\
\le{}& M\bigl[\bigl|V(x) - V(y)\bigr| + \bigl|\mu(x) - \mu(y)\bigr| \\
&+ \|\mu\|_V\bigl(|x - y| + \bigl\|W(y, \cdot) - W(x, \cdot)\bigr\|_V\bigr)\bigr].
\end{aligned}$$

So, the map $DF(\mu) \cdot \nu(\|\nu\|_V \le 1)$ is equicontinuous and by Ascoli's theorem, we conclude that the preceding set is relatively compact in $\mathcal{C}^0(\mathbb{R}^d; V)$ and thus the operator $DF(\mu)$ is compact. Moreover, it is self-adjoint. It follows from the spectral theory of compact self-adjoint operators that DF has at most countably many real eigenvalues and the set of nonzero eigenvalues is either finite or can be ordered as $|\lambda_1| > |\lambda_2| > \cdots > 0$ with $\lim_{n\to\infty}\lambda_n = 0$. So, by Tromba, $\mathcal{I}(F^{-1}(0))$ has an empty interior. □

6. Illustration in dimension $d = 2$

When W is not symmetric, it can happen that no Lyapunov function exists and that the ω-limit set is a non-trivial orbit. Suppose for instance that (for $d = 2$) $W(x, y) = (x, Ry)$ where R is a rotation matrix and $V(x) = V(|x|) \ge a|x|^4 + b|x|^2 + 1$ (with $a, b \ge 1$). Note, that the measure $\gamma(\mathrm{d}x) = \mathrm{e}^{-2V(x)}\,\mathrm{d}x/Z$ is invariant by rotation. Then, one expects, depending on R and V, that either the unique invariant set for the semiflow is $\{\gamma\}$ and so a.s. μ_t converges to γ; or a.s. μ_t converges to a random measure, related to the critical points of the free energy; or $\omega(\mu_t, t \ge 0)$ is a periodic orbit related to γ. Remark that, equivalently considering $W(x, y) + \frac{1}{2}(b|x|^2 + |y|^2/b)$ or W, the set of conditions (H) is satisfied. Denote $p := \binom{1}{0}$.

Lemma 6.1 ([5], Lemma 4.6). *For all continuous $\varphi : \mathbb{R} \to \mathbb{R}$, for all $y \in \mathbb{S}^1$ we have*

$$\int_{\mathbb{R}^2} \bigl[\varphi\bigl((x, y)\bigr) - \varphi\bigl((x, p)\bigr)\bigr]\gamma(\mathrm{d}x) = \int_{\mathbb{R}^2} \varphi\bigl((x, y)\bigr)\bigl(x - (x, y)y\bigr)\gamma(\mathrm{d}x) = 0.$$

Proof. For all $y \in \mathbb{S}^1$, there exists $g \in \mathrm{O}(2)$ such that $y = gp$. We show the first equality by a change of variable in the integral (because $V(x) = V(|x|)$). Define $\phi(y) := \int_{\mathbb{R}^2} \varphi((x, y))(x - (x, y)y)\gamma(\mathrm{d}x)$. We have $(\phi(y), y) = 0$ and the rotation-invariance of γ implies for the antisymmetry matrix j, that $\phi(p) = j\phi(p)$. So, $\phi(p) = 0$ and thus $\phi(y) = 0$. □

For any $\mu \in \mathcal{P}(\mathbb{R}^2; V)$, define its mean by $\bar{\mu} := \int_{\mathbb{R}^2} x\mu(\mathrm{d}x)$. Let the probability measure

$$\bar{\Pi}(\bar{\mu})(\mathrm{d}x) := \frac{\mathrm{e}^{-2(x, R\bar{\mu})}}{Z(\bar{\mu})}\gamma(\mathrm{d}x). \tag{6.1}$$

Here, $\bar{\Pi}(\bar{\mu}) = \Pi(\mu)$. If we let $\overline{\Pi}(\mu) := \int_{\mathbb{R}^2} x\bar{\Pi}(\mu)(\mathrm{d}x)$, then $\bar{\Phi}_t(\mu)$ is readily the semiflow

$$\bar{\Phi}_t(\mu) = \mathrm{e}^{-t}\bar{\mu} + \mathrm{e}^{-t}\int_0^t \mathrm{e}^s \overline{\Pi}\big(\bar{\Phi}_s(\mu)\big)\,\mathrm{d}s, \qquad \bar{\Phi}_0(\mu) = \bar{\mu}. \tag{6.2}$$

Lemma 6.2. *Let $m = \rho v$ with $\rho \geq 0$ and $v \in \mathbb{S}^1$. Then, we get*

$$\overline{\Pi}(m) = \int_{\mathbb{R}^2} x\bar{\Pi}(m)(\mathrm{d}x) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\log\left(\int_{\mathbb{R}^2}\mathrm{e}^{-2\rho\langle x,v\rangle}\gamma(\mathrm{d}x)\right)Rv.$$

Proof. It follows from differentiating the function $\alpha \mapsto \log(\int_{\mathbb{R}^2}\mathrm{e}^{-2\alpha\langle x,v\rangle}\gamma(\mathrm{d}x))$ and Lemma 6.1. $\square$

Let $m = \rho v$ be the solution to $\dot{m} = \overline{\Pi}(m) - m$, with $\rho = |m|$ and $v \in \mathbb{S}^1$. Then, Lemma 6.2 implies that $\dot{v} = 0$. If we let $\alpha = 2\rho$, then α satisfies the one-dimensional ODE

$$\dot{\alpha} = J(\alpha) = -\alpha + 2\,\partial_\alpha \log\left(\int_{\mathbb{R}^2}\mathrm{e}^{-\alpha\langle x,Rp\rangle}\gamma(\mathrm{d}x)\right). \tag{6.3}$$

The problem expressed in polar coordinates becomes $J(\alpha) = -\alpha(1 - 2\frac{\tilde{H}(\alpha)}{H(\alpha)})$, where

$$H(\alpha) := \int_0^\infty \mathrm{d}\rho\,\gamma(\rho)\int_0^{2\pi}\mathrm{d}v\,\mathrm{e}^{-\alpha\rho\cos v},$$
$$\tilde{H}(\alpha) := \int_0^\infty \mathrm{d}\rho\,\gamma(\rho)\rho^2\int_0^{2\pi}\mathrm{d}v\,\sin^2 v\mathrm{e}^{-\alpha\rho\cos v}.$$

Remark 6.3. *The function $t \mapsto \int_0^{2\pi}\mathrm{e}^{-t\cos v}\,\mathrm{d}v$ is the Bessel function $I_0(t)$.*

6.1. *The case $R = -Id$*

Here, W is a symmetric function.

Proposition 6.4. *If $\int_0^\infty \rho^2\gamma(\rho)\,\mathrm{d}\rho \leq 1$, then 0 is the unique equilibrium of (6.3) and 0 is stable. Its basin of attraction is $\mathbb{R}_+$.*

If $\int_0^\infty \rho^2\gamma(\rho)\,\mathrm{d}\rho > 1$, then 0 is linearly unstable and there is another stable equilibrium α_1, whose basin of attraction is $\mathbb{R}_+^*$.

Proof. Remark, that J is $\mathcal{C}^\infty$. A computation yields to

$$J^{(3)}(\alpha) = 2\frac{H^{(4)}(\alpha)}{H(\alpha)} - 8\frac{H^{(3)}H'(\alpha)}{H^2(\alpha)} + 24\frac{H''(\alpha)}{H(\alpha)}\left(\frac{H'(\alpha)}{H(\alpha)}\right)^2 - 12\left(\frac{H'(\alpha)}{H(\alpha)}\right)^4.$$

The point is to determine the sign of $J^{(3)}$. This function corresponds to (twice) the kurtosis of the projection on the first coordinate of a random variable X (expressed in polar coordinates) having the law γ. As the graph of the symmetric part of the density function cuts exactly twice the graph of the corresponding Gaussian variable (with same mean and variance), the kurtosis of X is negative: $J^{(3)}(\alpha) < 0$ for $\alpha > 0$ and $J^{(3)}(0) = 0$. So, for all $\alpha \geq 0$, we have $J''(\alpha) \leq J''(0) = 0$. Similarly, we find

$$J'(\alpha) \leq J'(0) = -1 + \int_0^\infty \rho^2\gamma(\rho)\,\mathrm{d}\rho.$$

So, if $J'(0) \leq 0$, then J decreases and, as $J(0) = 0$, the first result follows. Else $J'(0) > 0$. As J' is monotonic and $\lim_{\alpha\to\infty} J'(\alpha) = -1$, by continuity of J', there exists $\alpha_0 > 0$ such that $J'(\alpha_0) = 0$. Moreover, we have

$\lim_{\alpha\to\infty} J(\alpha) = -\infty$. Finally, there exists a positive solution to $J(\alpha) = 0$ if and only if $\int_0^\infty \rho^2\gamma(\rho)\,\mathrm{d}\rho > 1$. In that case, 0 is unstable and there exists a stable equilibrium. □

The next result shows that we can reduce the problem in studying the semiflow generated by (6.2) and then deduce the same (asymptotic) statements for μ.

Lemma 6.5 ([5], Proposition 3.9–Corollary 3.10).

(1) *Let* $L \subset \mathcal{P}_\beta(\mathbb{R}^d; V)$ *be an attractor-free set for* Φ *and* $A \subset \mathcal{P}_\beta(\mathbb{R}^d; V)$ *an attractor for* Φ, *with basin of attraction* $B(A)$. *If* $L \cap B(A) \neq \varnothing$, *then* $L \subset A$.
(2) *Let* (E, d) *be a metric space*, $\bar{\Phi} : E \times \mathbb{R} \to E$ *a semiflow on* E *and* $G : \mathcal{P}_\beta(\mathbb{R}^d; V) \to E$ *a continuous function. Assume that* $G \circ \Phi_t = \bar{\Phi}_t \circ G$. *Then, almost surely* $G(\omega(\mu_t, t \geq 0))$ *is attractor-free for* $\bar{\Phi}$.

We can now state and prove the following:

Theorem 6.6. *Consider the self-interacting diffusion on* $\mathbb{R}^2$, *where* $W(x, y) = -(x, y)$. *Then, we have two different cases*:

(1) *If* $\int_0^\infty \mathrm{d}\rho\,\gamma(\rho)\rho^2 \leq 1$, *then a.s.* $\mu_t \xrightarrow{w} \gamma$;
(2) *If* $\int_0^\infty \mathrm{d}\rho\,\gamma(\rho)\rho^2 > 1$, *then there exists a random variable* $v \in \mathbb{S}^1$ *such that a.s.* $\mu_t \xrightarrow{w} \mu_\infty^v$ *with*

$$\mu_\infty^v(\mathrm{d}x) = \frac{\mathrm{e}^{\alpha_1(x,v)}}{Z_1}\gamma(\mathrm{d}x),$$

where α_1 *is the unique positive solution to* $J(\alpha) = -\alpha + 2\frac{H'(\alpha)}{H(\alpha)} = 0$.

Proof. Let $G : \mathcal{P}_\beta(\mathbb{R}^2; V) \to \mathbb{R}^2$ be the mapping defined by $G(\mu) = \bar{\mu}$. By Lemma 6.5, the limit set of $\bar{\mu}_t$ is a.s. attractor-free for $\bar{\Phi}$. If $\int_0^\infty \mathrm{d}\rho\,\gamma(\rho)\rho^2 \leq 1$, then 0 is a global attractor for the semiflow generated by $\bar{\Phi}$. So, each attractor-free set of $\bar{\Phi}$ reduces to 0, and a.s. $\bar{\mu}_t \to 0$ and $\omega(\mu_t, t \geq 0) \subset G^{-1}(0)$. The definitions of $\overline{\Pi}(\bar{\mu})$ and J imply that $G^{-1}(0)$ is invariant for Φ and, as $\Pi(\Phi_t|_{G^{-1}(0)}(\mu)) = \gamma$, we have

$$\Phi|_{G^{-1}(0)}(\mu) = \mathrm{e}^{-t}(\mu - \gamma) + \gamma.$$

So, γ is a global attractor for $\Phi|_{G^{-1}(0)}$ and each attractor-free set reduces to γ. By Theorem 5.8, we conclude that $\omega(\mu_t, t \geq 0) = \{\gamma\}$.

Suppose now that 0 is unstable for $\overline{\Pi} - Id$. For all $f \in \mathcal{C}^\infty(\mathbb{R}^2; V)$, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_{h(t)} f = -\mu_{h(t)} f + \Pi(\mu_{h(t)}) f + \frac{\mathrm{d}}{\mathrm{d}s}\varepsilon_{t,t+s}\bigg|_{s=0} f.$$

If we consider the projection map $P_i(x) = x_i$, then $\partial_t \bar{\mu}_{h(t)} = \overline{\Pi}(\bar{\mu}_{h(t)}) - \bar{\mu}_{h(t)}) + \eta_t$ where η_t is the random vector $\eta_t = \frac{\mathrm{d}}{\mathrm{d}s}\varepsilon_{t,t+s}|_{s=0}(P_1, P_2)^T$. As 0 is an unstable linear equilibrium for $\overline{\Pi} - Id$, by Tarrès [21] we get that $\mathbb{P}(\lim_{t\to\infty} \bar{\mu}_{h(t)} = 0) = 0$. Using Theorem 3.6, we obtain that $\lim_{t\to\infty} \sup_{0\leq s\leq T} |\bar{\mu}_{h(t+s)} - \bar{\Phi}_s(\bar{\mu}_{h(t)})| = 0$. Denote by α_1 the unique positive solution to $-\alpha + 2\frac{H'(\alpha)}{H(\alpha)} = 0$ and consider the $\bar{\Phi}$-invariant set $A := \{m = \rho v; \rho = \frac{\alpha_1}{2}, v \in \mathbb{S}^1\}$. By Lemma 6.5, the limit set of $\bar{\mu}_{h(t)}$ is attractor-free, so $\omega(\bar{\mu}_{h(t)}, t \geq 0)$ either reduces to $\{0\}$, or is included in A. But, as $\mathbb{P}(\lim_{t\to\infty}\bar{\mu}_{h(t)} = 0) = 0$, it is a.s. a subset of A. Finally, as $\dot{v} = 0$, we have $\bar{\Phi}_t|_A = Id|_A$ and so, $\bar{\mu}_{h(t)}$ is a Cauchy sequence in A. Then, there exists $v \in \mathbb{S}^1$ such that

$$\lim_{t\to\infty} |\bar{\mu}_{h(t)} - \alpha_1 v| = 0.$$

To conclude, on one side, $\omega(\mu_t, t \geq 0)$ is an attractor-free set for $\Phi|_{G^{-1}(\alpha_1 v)}$ and on the other side, the semiflow $\Phi|_{G^{-1}(\alpha_1 v)}$ admits μ_∞^v as a global attractor. This leads to $\omega(\mu_t, t \geq 0) = \mu_\infty^v$. □

6.2. *When R is a rotation*

We assume that R is the rotation matrix $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, with $0 \le \theta < 2\pi$. We emphasize that (unless $\theta = 0, \pi$) W is not a symmetric function. We state and prove a more precise version of Theorem 1.3.

Theorem 6.7. *Consider the self-interacting diffusion on $\mathbb{R}^2$ associated with $W(x, y) = (x, Ry)$. Then one of the following holds*:

(1) *If V is such that $\int_0^\infty \rho^2\gamma(\rho)\,\mathrm{d}\rho\,(\cos\theta) > -1$, then a.s. $\mu_t \xrightarrow{w} \gamma$*;
(2) *If V is such that $\int_0^\infty \rho^2\gamma(\rho)\,\mathrm{d}\rho\,(\cos\theta) \le -1$, then*:

(a) *if $\theta = \pi$, then there exists a random variable $v \in \mathbb{S}^1$ such that a.s. $\mu_t \xrightarrow{w} \mu_\infty^v$ with $\mu_\infty^v(\mathrm{d}x) = \frac{\mathrm{e}^{\alpha_1(x,v)}}{Z_1}\gamma(\mathrm{d}x)$, where α_1 is the unique positive solution to $-\alpha + 2\frac{H'(\alpha)}{H(\alpha)} = 0$*,

(b) *if $\theta \neq \pi$, then $\omega(\mu_t, t \ge 0) = \{\nu(\delta), 0 \le \delta < 2\pi\}$ a.s., where $\nu(\delta) = \frac{1}{\mathrm{e}^{T_\theta}-1}\int_0^{T_\theta} \mathrm{e}^s \mu_\infty^{v,\theta}\,\mathrm{d}s$, with $T_\theta = 2\pi(\tan\theta)^{-1}$ and $\mu_\infty^{v,\theta}$ is the unique positive solution to $-\alpha + 2\cos\theta\frac{H'(\alpha)}{H(\alpha)} = 0$.*

Proof. Let $v = gp$ with $g \in \mathrm{O}(2)$ and $m = \alpha v/2$. We remind the equations

$$\dot{\alpha} = -\alpha - 2\frac{H'(\alpha)}{H(\alpha)}(Rv, v); \qquad \dot{v} = -\frac{2}{\alpha}\frac{H'(\alpha)}{H(\alpha)}\big((Rv, v)v - Rv\big).$$

By definition of R and $v = \begin{pmatrix} -\sin\sigma \\ \cos\sigma \end{pmatrix}$, a simple computation yields to

$$\dot{\alpha} = -\alpha - \frac{2H'(\alpha)}{H(\alpha)}\cos\theta; \qquad \dot{\sigma} = -\frac{2H'(\alpha)}{\alpha H(\alpha)}\sin\theta. \tag{6.4}$$

We recall that $\frac{H'(\alpha)}{H(\alpha)} > 0$ for $\alpha > 0$. By Proposition 6.4, we have a bifurcation: if $\cos\theta\int_0^\infty \gamma(\mathrm{d}\rho)\rho^2 \ge 1$, then the set $\{(\sigma, \alpha); \dot{\alpha} = 0\}$ is a global attracting set for the semiflow generated by (6.4) and so a.s. $\mu_t \to \gamma$. Let α_θ be such that $\dot{\alpha}_\theta = 0$. If $\cos\theta\int_0^\infty \gamma(\mathrm{d}\rho)\rho^2 < 1$, then $\tilde{A} := \{(\sigma, \alpha); \alpha = \alpha_\theta\}$ is a global attracting set. On $\tilde{A}$, the dynamics is given by

$$\dot{\sigma} = -\frac{2H'(\alpha_\theta)}{\alpha_\theta H(\alpha_\theta)}\sin\theta = \tan\theta.$$

By Theorem 6.6, there exists a random variable σ_0 such that a.s.

$$\lim_{t\to\infty}\left|\bar{\mu}_{h(t)} - \frac{\alpha_\theta}{2}v(t\tan\theta + \sigma_0)\right| = 0. \tag{6.5}$$

At that point, we know the dynamics on the set $\tilde{A}$. But, we need to study the system defined on $\mathcal{M}(\mathbb{R}^2; V) \times \mathbb{R}^2$ by

$$\dot{m} = -m + \bar{\Pi}(m); \qquad \dot{\nu} = -\nu + \bar{\Pi}(m).$$

By Lemma 6.5, $\omega(\mu_t, t \ge 0) \times \tilde{A}$ is attractor-free for the preceding semiflow restricted to $\mathcal{P}(\mathbb{R}^2; V) \times \mathbb{R}^2$. The dynamics on $\omega(\mu_t, t \ge 0) \times \tilde{A}$ is given by

$$\dot{\sigma} = \tan\theta; \qquad \dot{\nu} = -\nu + f(\sigma) = -\nu + \mu_\infty^{v,\theta}. \tag{6.6}$$

As the set $\omega(\mu_t, t \ge 0) \times \tilde{A}$ is (weakly) compact and invariant in $\mathcal{P}(\mathbb{R}^2; V) \times \mathbb{R}^2$, we conclude similarly to [5], Theorem 4.11. □

Acknowledgments

This work has partially been supported by the Swiss National Foundation Grants 200020-112316/1 and PBNE2-119027. The author is glad to thank M. Benaïm for introducing her to the subject, P. Cattiaux for an introduction to the ultracontractivity property, V. Kleptsyn and P. Tarrès for helpful comments, and J.-F. Jouanin for valuable discussions. I also wish to thank anonymous referees for careful reading and comments, which improved this paper.

References

[1] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on Probability Theory and Statistics. Ecole de Prob. de St-Flour* 1–114. Springer, Berlin, 1994. MR1307413

[2] D. Bakry and M. Emery. Diffusions hypercontractives. In *Séminaire de Probabilités XIX* 177–206. *Lecture Notes in Math.* **1123**. Springer, Berlin, 1985. MR0889476

[3] M. Benaïm. Dynamics of stochastic approximation algorithms. In *Séminaire de Probabilités XXXIII* 1–68. *Lecture Notes in Math.* **1709**. Springer, Berlin, 1999. MR1767993

[4] M. Benaïm and M. W. Hirsch. Asymptotic pseudotrajectories and chain recurrent flows, with applications. *J. Dynam. Differential Equation* **8** (1996) 141–176. MR1388167

[5] M. Benaïm, M. Ledoux and O. Raimond. Self-interacting diffusions. *Probab. Theory Related Fields* **122** (2002) 1–41. MR1883716

[6] M. Benaïm and O. Raimond. Self-interacting diffusions III: Symmetric interactions. *Ann. Probab.* **33** (2005) 1716–1759. MR2165577

[7] M. Cranston and Y. Le Jan. Self-attracting diffusions: Two cases studies. *Math. Ann.* **303** (1995) 87–93. MR1348356

[8] M. Cranston and T. S. Mountford. The strong law of large numbers for a Brownian polymer. *Ann. Probab.* **2** (1996) 1300–1323. MR1411496

[9] E. B. Davies and B. Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59** (1984) 335–395. MR0766493

[10] R. T. Durrett and L. C. G. Rogers. Asymptotic behavior of Brownian polymers. *Probab. Theory Related Fields* **92** (1992) 337–349. MR1165516

[11] S. Herrmann and B. Roynette. Boundedness and convergence of some self-attracting diffusions. *Math. Ann.* **325** (2003) 81–96. MR1957265

[12] I. Kontoyiannis and S. P. Meyn. Spectral theory and limit theory for geometrically ergodic Markov processes. *Ann. Appl. Probab.* **13** (2003) 304–362. MR1952001

[13] H. Kunita. *Stochastic Flows and Stochastic Differential Equations*. *Cambridge Studies in Advanced Mathematics* **24**. Cambridge Univ. Press, Cambridge, 1990. MR1070361

[14] M. Ledoux. The geometry of Markov diffusion generators. *Ann. Fac. Sci. Toulouse Math. (6)* **IX** (2000) 305–366. MR1813804

[15] R. McCann. A convexity principle for interacting gases. *Adv. Math.* **128** (1997) 153–179. MR1451422

[16] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Springer, London, 1993. MR1287609

[17] T. S. Mountford and P. Tarrès. An asymptotic result for Brownian polymers. *Ann. Inst. H. Poincaré, Probab. Statist.* **44** (2008) 29–46.

[18] R. Pemantle. A survey of random processes with reinforcement. *Probab. Surv.* **4** (2007) 1–79. MR2282181

[19] O. Raimond. Self-attracting diffusions: Case of the constant interaction. *Probab. Theory Related Fields* **107** (1997) 177–196. MR1431218

[20] M. Röckner and F. Y. Wang. Supercontractivity and ultracontractivity for (nonsymmetric) diffusions semigroups on manifolds. *Forum Math.* **15** (2003) 893–921. MR2010284

[21] P. Tarrès. Pièges répulsifs. *C. R. Acad. Sci. Paris Sér. I Math.* **330** (2000) 125–130. MR1745175

[22] A. J. Tromba. The Morse–Sard–Brown theorem for functionals and the problem of Plateau. *Amer. J. Math.* **99** (1977) 1251–1256. MR0464285

[23] C. Villani. *Topics in Optimal Transportation*. *Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI, 2003. MR1964483

[24] F. Y. Wang. Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. *Probab. Theory Related Fields* **109** (1997) 417–424. MR1481127