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Identification of periodic and cyclic fractional stable motions[1]

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Abstract. We consider an important subclass of self-similar, non-Gaussian stable processes with stationary increments known as self-similar stable mixed moving averages. As previously shown by the authors, following the seminal approach of Jan Rosiński, these processes can be related to nonsingular flows through their minimal representations. Different types of flows give rise to different classes of self-similar mixed moving averages, and to corresponding general decompositions of these processes. Self-similar stable mixed moving averages related to dissipative flows have already been studied, as well as processes associated with identity flows which are the simplest type of conservative flows. The focus here is on self-similar stable mixed moving averages related to periodic and cyclic flows. Periodic flows are conservative flows such that each point in the space comes back to its initial position in finite time, either positive or null. The flow is cyclic if the return time is positive.

Self-similar mixed moving averages are called periodic, resp. cyclic, fractional stable motions if their minimal representations are generated by periodic, resp. cyclic, flows. In practice, however, minimal representations are not particularly easy to determine and, moreover, self-similar stable mixed moving averages are often defined by nonminimal representations. We therefore provide a way which is not based on flows, to detect whether these processes are periodic or cyclic even if their representations are nonminimal. These identification results lead naturally to a decomposition of self-similar stable mixed moving averages which includes the new classes of periodic and cyclic fractional stable motions, and hence is more refined than the one previously established.

Résumé. Nous considérons une sous-classe de l'ensemble des processus autosimilaires stables non gaussiens à accroissements stationnaires. C'est la sous-classe des processus à moyenne mobile mixte. Appliquant une méthodologie introduite par Jan Rosiński, nous avons établi précédemment une correspondance entre les représentations minimales de ces processus et des flots non singuliers. Les processus associés aux flots dissipatifs et ceux associés au flot "identité" (qui est un flot conservatif) ont déjà été caractérisés. Nous étudions ici les processus associés aux flots périodiques et cycliques. Un flot est "périodique" s'il ramène tout point de l'espace à sa position de départ en un temps fini, positif ou nul. Ce flot est "cyclique" si ce temps de retour est strictement positif. Les flots périodiques et cycliques sont des flots conservatifs.

Un processus autosimilaire stable à moyenne mobile mixte est appelé "périodique" (ou "cyclique") si sa representation minimale est associée à un flot périodique (ou cyclique). Il n'est toutefois pas toujours facile de déterminer la représentation minimale d'un processus, et, de plus, les processus autosimilaires sont souvent caractérisés par une représentation non minimale. C'est pouquoi nous offrons une méthode directe pour déterminer si ces processus sont périodiques (ou cycliques) sans devoir passer par l'intermédiaire des flots. Cette méthode fonctionne même si la representation des processus est non minimale.

Nous obtenons finalement une décomposition des processus autosimilaires stables à moyenne mobile mixte qui inclue les processus périodiques et cycliques. Cette décomposition est plus fine que celles connues auparavant.

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1. Introduction

Consider continuous-time stochastic processes $\{X(t)\}_{t\in\mathbb{R}}$ which have *stationary increments* and are *self-similar* with self-similarity parameter $H > 0$. Stationarity of the increments means that the processes $X(t+h) - X(h)$ and $X(t) - X(0)$ have the same finite-dimensional distributions for any fixed $h \in \mathbb{R}$. Self-similarity means that, for any fixed $c > 0$, the processes $X(ct)$ and $c^H X(t)$ have the same finite-dimensional distributions. The parameter $H > 0$ is called the self-similarity parameter. Self-similar stationary increments processes are of interest because their increments can be used as models for stationary, possibly strongly dependent time series.

It is known that, for $\alpha \in (0, 2)$, there are infinitely many non-Gaussian α-stable self-similar processes with stationary increments. In [6,7], the authors have started to classify an important subclass of such processes, called *self-similar mixed moving averages*, by relating them to "flows," an idea which has originated with Rosiński [12]. In this paper, we focus on self-similar mixed moving averages which are related to *periodic* and, more specifically, *cyclic* flows in the sense of [6,7]. We call such processes *periodic* and *cyclic fractional stable motions*. We show how, given a representation of the process, one can determine whether a general self-similar mixed moving average is a periodic or cyclic fractional stable motion. This leads to a decomposition of self-similar mixed moving averages which is more refined than that obtained in [6,7]. In a subsequent paper [10], we study the properties of periodic and cyclic fractional stable motions in greater detail, provide examples and show that periodic fractional stable motions have canonical representations.

The considered stable case $\alpha \in (0, 2)$ should be contrasted to the Gaussian case which is the stable case with $\alpha = 2$. Since Gaussian processes are determined by their covariance structure, it is easy to show that fractional Brownian motion is the only (up to a multiplicative constant and for fixed $H \in (0, 1)$) Gaussian H-self-similar process with stationary increments. See, for example, [3] Section 7 in [14] or two recent collections [2] and [11] of survey articles. In the non-Gaussian stable case, covariance functions do not exist and do not characterize stable processes, leading to infinitely many different (for fixed H and α) stable self-similar processes with stationary increments. We attempt to understand these processes by focusing on their large subclass consisting of self-similar mixed moving averages. One of our main objectives is to be able to say when two such given processes are, in fact, different. Stable self-similar mixed moving averages related to different classes of flows, for example, turn out to be different.

Section 2 contains definitions and additional information about the methodology and the results. The rest of the paper is described at the end of that section.

2. Self-similar mixed moving average processes and flows

We now recall relevant concepts, discuss previous related work and describe our results. Consider symmetric α-stable ($S\alpha S$, in short), $\alpha \in (0, 2)$, self-similar processes $\{X_\alpha(t)\}_{t\in\mathbb{R}}$ with a *mixed moving average* representation

$$\big\{X_\alpha(t)\big\}_{t\in\mathbb{R}} \overset{d}{=} \left\{\int_X \int_{\mathbb{R}} \big(G(x, t+u) - G(x, u)\big) M_\alpha(\mathrm{d}x, \mathrm{d}u)\right\}_{t\in\mathbb{R}}, \tag{2.1}$$

where $\overset{d}{=}$ stands for the equality in the sense of the finite-dimensional distributions. Here, $(X, \mathcal{X}, \mu)$ is a standard Lebesgue space, that is, $(X, \mathcal{X})$ is a measurable space with one-to-one, onto and bimeasurable correspondence to a Borel subset of a complete separable metric space, and μ is a σ-finite measure. M_α is a $S\alpha S$ random measure on $X \times \mathbb{R}$ with the control measure $\mu(\mathrm{d}x)\,\mathrm{d}u$ and

$$G : X \times \mathbb{R} \mapsto \mathbb{R}$$

is some measurable deterministic function. Saying that the process X_α is given by the representation (2.1) where M_α has the control measure $\mu(\mathrm{d}x)\,\mathrm{d}u$, is equivalent to having its characteristic function expressed as

$$E \exp\left\{\mathrm{i} \sum_{k=1}^{n} \theta_k X_\alpha(t_k)\right\} = \exp\left\{-\int_X \int_{\mathbb{R}} \left|\sum_{k=1}^{n} \theta_k G_{t_k}(x, u)\right|^\alpha \mu(\mathrm{d}x)\,\mathrm{d}u\right\}, \tag{2.2}$$

where

$$G_t(x,u) = G(x,t+u) - G(x,u), \quad x \in X, u \in \mathbb{R}, \tag{2.3}$$

and

$$\{G_t\}_{t\in\mathbb{R}} \subset L^\alpha\big(X \times \mathbb{R}, \mu(\mathrm{d}x)\,\mathrm{d}u\big).$$

The function $G_t(x,u)$, or sometimes the function G, is called a *kernel function* of the representation (2.1). For more information on $S\alpha S$ random measures, control measures and integral representations of the type (2.1), see for example [14]. Moreover, by setting $\xi = \sum_{k=1}^n \theta_k X_\alpha(t_k)$, relation (2.2) implies that $E\exp\{\mathrm{i}\theta\xi\} = \exp\{-\sigma^\alpha|\theta|^\alpha\}$ for some $\sigma \geq 0$ and all $\theta \in \mathbb{R}$. By definition, ξ is a $S\alpha S$ random variable and hence X_α is a $S\alpha S$ process as well. Note also that X is used in (2.1) to denote both the random process and the underlying Lebesgue space. To avoid confusion, the subindex α will always be added to X when a random process is meant.

The process X_α may have equivalent representations (in the sense of the finite-dimensional distributions), each involving a different function G. The so-called "minimal representations" are of particular interest. Minimal representations were introduced by Hardin [4] and subsequently developed by Rosiński [13]. See also Section 4 in [6], or Appendix B in [9]. The representation $\{G_t\}_{t\in\mathbb{R}}$ of (2.1) is *minimal* if (2.10) holds, and if for any nonsingular map $\Phi: X \times \mathbb{R} \to X \times \mathbb{R}$ such that, for any $t \in \mathbb{R}$,

$$G_t\big(\Phi(x,u)\big) = k(x,u)G_t(x,u) \quad \text{a.e. } \mu(\mathrm{d}x)\,\mathrm{d}u \tag{2.4}$$

with some $k(x,u) \neq 0$, we have $\Phi(x,u) = (x,u)$, that is, Φ is the identity map, a.e. $\mu(\mathrm{d}x)\,\mathrm{d}u$.

It follows from (2.2) that a mixed moving average X_α has always stationary increments. Additional assumptions have to be imposed on the function G for the process X_α to be also self-similar. These assumptions are stated in Definition 2.1 and are formulated in terms of flows and some additional functionals which we now define (see also [6]).

A (multiplicative) *flow* $\{\psi_c\}_{c>0}$ on $(X, \mathcal{X}, \mu)$ is a collection of deterministic measurable maps $\psi_c : X \to X$ satisfying

$$\psi_{c_1c_2}(x) = \psi_{c_1}\big(\psi_{c_2}(x)\big), \quad \text{for all } c_1, c_2 > 0, x \in X, \tag{2.5}$$

and $\psi_1(x) = x$ for all $x \in X$. The flow is *nonsingular* if each map ψ_c, $c > 0$, is nonsingular, that is, $\mu(A) = 0$ implies $\mu(\psi_c^{-1}(A)) = 0$. It is *measurable* if the map $\psi_c(x) : (0,\infty) \times X \to X$ is measurable.

A *cocycle* $\{b_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ taking values in $\{-1, 1\}$ is a measurable map

$$b_c(x) : (0,\infty) \times X \to \{-1, 1\}$$

satisfying

$$b_{c_1c_2}(x) = b_{c_1}(x)b_{c_2}\big(\psi_{c_1}(x)\big), \quad \text{for all } c_1, c_2 > 0,\ x \in X. \tag{2.6}$$

A *semi-additive functional* $\{g_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ is a measurable map

$$g_c(x) : (0,\infty) \times X \to \mathbb{R}$$

such that

$$g_{c_1c_2}(x) = c_2^{-1}g_{c_1}(x) + g_{c_2}\big(\psi_{c_1}(x)\big), \quad \text{for all } c_1, c_2 > 0,\ x \in X. \tag{2.7}$$

We use throughout the paper the useful notation

$$\kappa = H - \frac{1}{\alpha}. \tag{2.8}$$

The support of $\{f_t\}_{t\in\mathbb{R}} \subset L^0(S, \mathcal{S}, m)$, denoted $\mathrm{supp}\{f_t, t \in \mathbb{R}\}$, is a minimal (a.e.) set $A \in \mathcal{S}$ such that $m\{f_t(s) \neq 0, s \notin A\} = 0$ for every $t \in \mathbb{R}$.

Definition 2.1. *A* $S\alpha S$, $\alpha \in (0,2)$, *self-similar process* X_α *having a mixed moving average representation* (2.1) *is said to be* generated by a nonsingular measurable flow $\{\psi_c\}_{c>0}$ *on* $(X, \mathcal{X}, \mu)$ (*through the kernel function* G) *if*:

(i) *for all* $c > 0$,

$$c^{-\kappa} G(x, cu) = b_c(x) \left\{ \frac{\mathrm{d}(\mu \circ \psi_c)}{\mathrm{d}\mu}(x) \right\}^{1/\alpha} G\big(\psi_c(x), u + g_c(x)\big) + j_c(x) \quad a.e.\ \mu(\mathrm{d}x)\,\mathrm{d}u, \tag{2.9}$$

where $\{b_c\}_{c>0}$ *is a cocycle* (*for the flow* $\{\psi_c\}_{c>0}$) *taking values in* $\{-1, 1\}$, $\{g_c\}_{c>0}$ *is a semi-additive functional* (*for the flow* $\{\psi_c\}_{c>0}$) *and* $j_c(x)$ *is some function*, *and*

(ii)

$$\operatorname{supp}\big\{G(x, t+u) - G(x,u), t \in \mathbb{R}\big\} = X \times \mathbb{R} \qquad a.e.\ \mu(\mathrm{d}x)\,\mathrm{d}u. \tag{2.10}$$

Relation (2.10) is imposed in order to eliminate ambiguities stemming from taking too big a space X. Definition 2.1 can be found in [6]. Observe that it involves the kernel G and hence the representation (2.1) of X_α. Definition 2.1 is also closely related to self-similarity. By using (2.2) together with (2.5)–(2.7), it is easy to verify that a mixed moving average (2.1) with a function G satisfying (2.9) is self-similar (see [6]). On the other hand, by Theorems 4.1 and 4.2 in [6], any $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average is generated by a flow in the sense of Definition 2.1 with the kernel G in (2.9) associated with the minimal representation of the process.

By using the connection between processes and flows, we proved in [6] that $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving averages can be decomposed uniquely (in distribution) into two independent processes as

$$X_\alpha \stackrel{d}{=} X_\alpha^D + X_\alpha^C. \tag{2.11}$$

Here, X_α^D is a self-similar mixed moving average generated by a *dissipative flow*. Informally, the flow $\{\psi_c\}_{c>0}$ is dissipative when the points x and $\psi_c(x)$ move further apart as c approaches ∞ ($\ln c \to \infty$) or c approaches 0 ($\ln c \to -\infty$). An example of a dissipative flow is $\psi_c(x) = x + \ln c$, $x \in \mathbb{R}$. Self-similar mixed moving average processes generated by dissipative flows have a canonical representation (see Theorem 4.1 in [7]) and are studied in detail in [8], where they are called *dilated fractional stable motions*. By a *canonical representation*, we mean a representation (2.1) where the kernel function G has a particular, explicit form ensuring both self-similarity and stationarity of the increments of the process (2.1). Another example of canonical representation is (2.14) below where the kernel function G has the form (2.13).

The process X_α^C in (2.11) is a self-similar mixed moving average generated by a *conservative flow*. Conservative flows $\{\psi_c\}_{c>0}$ are such that the points x and $\psi_c(x)$ become arbitrarily close at infinitely many values of c. An example of a conservative flow is $\psi_c(x) = x\mathrm{e}^{\mathrm{i}\ln c}$, $|x| = 1$, $x \in \mathbb{C}$ since $\psi_c(x) = x$ every time that $\ln c$ is a multiple of 2π. Although this example is elementary, the general structure of conservative flows is complex and, in particular, more intricate than that of dissipative flows. Consequently, contrary to the processes generated by dissipative flows, there is no simple canonical representation of the self-similar mixed moving averages generated by conservative flows.

It is nevertheless possible to obtain a further decomposition of self-similar mixed moving averages generated by conservative flows. As shown in [7],

$$X_\alpha^C \stackrel{d}{=} X_\alpha^F + X_\alpha^{C\setminus F}, \tag{2.12}$$

where the decomposition is unique in distribution and has independent components. The processes X_α^F in the decomposition (2.12) are those self-similar mixed moving averages that have a canonical representation (2.1) with the kernel function

$$G(x,u) = \begin{cases} F_1(x)u_+^\kappa + F_2(x)u_-^\kappa, & \kappa \neq 0, \\ F_1(x)\ln|u| + F_2(x)1_{(0,\infty)}(u), & \kappa = 0, \end{cases} \tag{2.13}$$

where $u_+ = \max\{0, u\}$, $u_- = \max\{0, -u\}$ and $F_1, F_2 : Z \mapsto \mathbb{R}$ are some functions. Thus,

$$X_\alpha^F(t) \stackrel{d}{=} \begin{cases} \int_X \int_{\mathbb{R}} \big(F_1(x)\big((t+u)_+^\kappa - u_+^\kappa\big) + F_2(x)\big((t+u)_-^\kappa - u_-^\kappa\big)\big) M_\alpha(\mathrm{d}x, \mathrm{d}u), & \kappa \neq 0, \\ \int_X \int_{\mathbb{R}} \big(F_1(x)\ln\frac{|t+u|}{|u|} + F_2(x)1_{(-t,0)}(u)\big) M_\alpha(\mathrm{d}x, \mathrm{d}u), & \kappa = 0. \end{cases} \tag{2.14}$$

The processes (2.14) are called *mixed linear fractional stable motions* (mixed LFSM, in short) and are essentially generated by *identity flows* ('essentially' will become clear in the sequel). An identity flow is the simplest type of conservative flow, defined by $\psi_c(x) = x$ for all $c > 0$, and the upperscript F in X_α^F refers to the fact that the points x are *fixed* points under the flow. The processes $X_\alpha^{C\setminus F}$ in (2.12) are self-similar mixed moving averages generated by conservative flows but without the *mixed LFSM component* (2.14), that is, they cannot be represented in distribution by a sum of two independent processes, one of which is a nondegenerate mixed LFSM (2.14).

Our goal here is to obtain a more detailed decomposition of self-similar mixed moving averages. We will show that there are independent self-similar mixed moving averages X_α^L and $X_\alpha^{C\setminus P}$ such that

$$X_\alpha^{C\setminus F} \stackrel{d}{=} X_\alpha^L + X_\alpha^{C\setminus P} \tag{2.15}$$

and hence, in view of (2.12),

$$X_\alpha^C \stackrel{d}{=} X_\alpha^F + X_\alpha^L + X_\alpha^{C\setminus P} =: X_\alpha^P + X_\alpha^{C\setminus P}, \tag{2.16}$$

where the decompositions (2.15) and (2.16) are unique in distribution and have independent components. While the processes X_α^F are essentially generated by identity flows, the process $X_\alpha^P = X_\alpha^F + X_\alpha^L$ and the process X_α^L are essentially generated by *periodic* and *cyclic* flows, respectively.[1] Periodic flows are conservative flows such that any point in the space comes back to its initial position in a finite period of time. Identity flows are periodic flows with period zero. Cyclic flows are periodic flows with positive period. Cyclic flows are probably the simplest type of conservative flows after the identity flows.

These flows are defined as follows. Let $\{\psi_c\}_{c>0}$ be a measurable flow on a standard Lebesgue space $(X, \mathcal{X}, \mu)$. Consider the following subsets of X induced by the flow $\{\psi_c\}_{c>0}$:

$$P := \{x\colon \exists p = p(x) \neq 1\colon \psi_p(x) = x\}, \tag{2.17}$$

$$F := \{x\colon \psi_c(x) = x \text{ for all } c > 0\}, \tag{2.18}$$

$$L := P \setminus F. \tag{2.19}$$

Definition 2.2. *The elements of P, F, L are called the* periodic, fixed *and* cyclic *points of the flow $\{\psi_c\}_{c>0}$, respectively.*

Definition 2.3. *A measurable flow $\{\psi_c\}_{c>0}$ on $(X, \mathcal{X}, \mu)$ is* periodic *if $X = P$ μ-a.e., is* identity *if $X = F$ μ-a.e., and it is* cyclic *if $X = L$ μ-a.e.*

The processes X_α^P and X_α^L in (2.16) will be called, respectively, *periodic fractional stable motions* and *cyclic fractional stable motions*. We indicated above that the processes X_α^F, X_α^P and X_α^L are essentially determined by identity, periodic and cyclic flows, respectively. By 'essentially determined,' we mean that if the processes X_α^P and X_α^L are given by their minimal representations, then they are necessarily generated by periodic and cyclic flows, respectively, in the sense of Definition 2.1. This terminology is not restrictive in the case $\alpha \in (1, 2)$ because mixed moving averages always have minimal representations (2.1) by Theorem 4.2 in [6] and, according to Theorem 4.1 of that paper, self-similar mixed moving averages given by a minimal representation (2.1) are always generated by a unique flow in the sense of Definition 2.1.[2] More generally, when a self-similar mixed moving average X_α given by a minimal representation (2.1), is generated by the flow, the processes X_α^P and X_α^L in the decomposition (2.16) can be defined by replacing respectively the space X in the integral representation (2.1) by P and L, that is, the periodic and cyclic point sets of the generating flow.

[1] The letters D and C are associated with *D*issipative and *C*onservative flows, respectively. The letter F ("*F*ixed") is associated with identity flows, the letter P with *P*eriodic flows and the letter L with cyc*L*ic flows.

[2] When $\alpha \in (0, 1]$, we were able to prove Theorem 4.2 concerning existence of minimal representations for mixed moving averages only under additional assumptions on the process (see Remark following Theorem 4.2 in [6]). To keep the presentation simple, we do not introduce here these additional assumptions and hence suppose in this paper that $\alpha \in (1, 2)$, unless stated explicitly otherwise.

Why are we referring to minimal representations? If one makes no restrictions on the form of a representation (2.1), periodic and cyclic fractional stable motions can be generated by flows other than periodic and cyclic, and the components X_α^P and X_α^L in the decomposition (2.16) may not be related to the periodic and cyclic point sets of the underlying flow. An analogous phenomenon is also associated with the decomposition (2.12). Since we would like to work with an arbitrary (not necessarily minimal) representation (2.1), it is desirable to be able to recognize periodic and cyclic fractional stable motions without relying on minimal representations and flows. We shall therefore provide identification criteria based on the (possibly nonminimal) kernel function G in the representation (2.1) and not on flows. These criteria allow one to obtain the decompositions (2.15) and (2.16) when starting with an arbitrary (possibly nonminimal) representation (2.1).

Many ideas of this paper are adapted from [9], where we investigated stable *stationary* processes related to periodic and cyclic flows in the sense of [12]. Since these ideas appear in a simpler form in [9], we suggest that the reader refers to that paper for further clarifications and insight. The focus here is on stationary increments mixed moving averages which are self-similar. Their connection to flows is more involved and the results obtained in the stationary case cannot be readily applied. On the other hand, Definitions 2.2 and 2.3 can also be found in [9], as well as an alternative equivalent definition of a cyclic flow which will not be used here. By Lemma 2.1 in [9], the sets P, L appearing in Definition 2.2 are μ-measurable (measurable with respect to the measure μ) and the set F is (Borel) measurable.

Our presentation is also different from that of [9]. While in [9], we focused first on stationary stable processes having an *arbitrary* representation, we focus first here on periodic and cyclic fractional stable motions having a "minimal representation." It is convenient to work first with minimal representations because periodic and cyclic fractional motions with minimal representations can be directly related to periodic and cyclic flows. We then turn to self-similar mixed moving averages having an arbitrary, possibly nonminimal, representation. This approach sheds additional light on the various relations between stable processes and flows, and their corresponding decompositions in disjoint classes.

The paper is organized as follows. In Section 3, we establish the decompositions (2.15) and (2.16) using representations (2.1) that are minimal, and introduce periodic and cyclic fractional stable motions. Criteria to identify periodic and cyclic fractional stable motions through (possibly nonminimal) kernel functions G are provided in Sections 4 and 5. The decompositions (2.15) and (2.16) which are based on these criteria can be found in Section 6. In Section 7, we provide an example of a process $X_\alpha^{P\setminus C}$ of the "fourth kind" in the decomposition (2.15).

3. Periodic and cyclic fractional stable motions: the minimal case

By Theorem 4.2 in [6], any $S\alpha S$, $\alpha \in (1,2)$, mixed moving average X_α has an integral representation (2.1) which is minimal. By Theorem 4.1 in [6], a self-similar mixed moving average X_α given by a minimal representation (2.1) is generated by a unique flow $\{\psi_c\}_{c>0}$ in the sense of Definition 2.1.

By the Hopf decomposition (see [1,5]), the space X can be decomposed into two parts, D and C, invariant under the flow, D denoting the dissipative points of $\{\psi_c\}_{c>0}$ and C denoting the conservative points of $\{\psi_c\}_{c>0}$. Let D, C, F, L and P be then the dissipative, conservative, fixed, cyclic and periodic point sets of the flow $\{\psi_c\}_{c>0}$, respectively. Since

$$X = D + C = D + P + C \setminus P = D + F + L + C \setminus P,$$

we can write

$$X_\alpha \stackrel{d}{=} X_\alpha^D + X_\alpha^P + X_\alpha^{C\setminus P} = X_\alpha^D + X_\alpha^F + X_\alpha^L + X_\alpha^{C\setminus P}, \tag{3.1}$$

where

$$X_\alpha^P = X_\alpha^F + X_\alpha^L$$

and where, for a set $S \subset X$,

$$X_\alpha^S(t) = \int_S \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u). \tag{3.2}$$

Since by their definitions, the sets D, C, F, P and L are invariant under the flow, the processes X_α^D, X_α^F, X_α^L and $X_\alpha^{C\setminus P}$ are self-similar mixed moving averages. These processes are independent because the sets D, F, L and $C \setminus P$ are disjoint (see Theorem 3.5.3 in [14]). The processes X_α^S are generated by the flow ψ^S where ψ^S denotes the flow ψ restricted to a set S, which is invariant under the flow. Observe that ψ^D, ψ^F, ψ^L and ψ^P are dissipative, identity, cyclic and periodic flows, respectively, and that $\psi^{C\setminus P}$ is a conservative flow without periodic points, and, for example, the process X_α^D is generated by the dissipative flow ψ^D.

A self-similar mixed moving average may have another minimal representation (2.1) with a kernel function $\widetilde{G}$ on the space $\widetilde{X}$, and hence be generated by another flow $\{\widetilde{\psi}_c\}_{c>0}$. Partitioning $\widetilde{X}$ into the dissipative, fixed, cyclic and "other" conservative point sets of the flow $\{\widetilde{\psi}_c\}_{c>0}$ as above, leads to the decomposition

$$X_\alpha \stackrel{d}{=} \widetilde{X}_\alpha^D + \widetilde{X}_\alpha^F + \widetilde{X}_\alpha^L + \widetilde{X}_\alpha^{C\setminus P}. \tag{3.3}$$

We will say that the decomposition (3.1) obtained from a minimal representation (2.1) is *unique in distribution* if the distribution of its components does not depend on the minimal representation used in the decomposition. In other words, uniqueness in distribution holds if

$$X_\alpha^D \stackrel{d}{=} \widetilde{X}_\alpha^D, \quad X_\alpha^F \stackrel{d}{=} \widetilde{X}_\alpha^F, \quad X_\alpha^L \stackrel{d}{=} \widetilde{X}_\alpha^L, \quad X_\alpha^{C\setminus P} \stackrel{d}{=} \widetilde{X}_\alpha^{C\setminus P}, \tag{3.4}$$

where X_α^S and $\widetilde{X}_\alpha^S$ with $S = D, F, L$ and $C \setminus P$, are the components of the decompositions (3.1) and (3.3) obtained from two different minimal representations of the process.

Theorem 3.1. *Let $\alpha \in (1,2)$. The decomposition* (3.1) *obtained from a minimal representation* (2.1) *of a self-similar mixed moving average X_α is unique in distribution.*

Proof. Suppose that a self-similar mixed moving average X_α is given by two different minimal representations with the kernel functions G and $\widetilde{G}$, and the spaces (X,μ) and $(\widetilde{X},\widetilde{\mu})$, respectively. Suppose also that X_α is generated through these minimal representations by two different flows $\{\psi_c\}_{c>0}$ and $\{\widetilde{\psi}_c\}_{c>0}$ on the spaces X and $\widetilde{X}$, respectively. Let (3.1) and (3.3) be two decompositions of X_α obtained from these two minimal representations and the generating flows. Let also D, F, L, P, C and $\widetilde{D}, \widetilde{F}, \widetilde{L}, \widetilde{P}, \widetilde{C}$ be the dissipative, fixed, cyclic, periodic and conservative point sets of the flows $\{\psi_c\}_{c>0}$ and $\{\widetilde{\psi}_c\}_{c>0}$, respectively. We need to show that the equalities (3.4) hold.

By Theorem 4.3 and its proof in [6], the kernel functions G and $\widetilde{G}$, and the flows ψ and $\widetilde{\psi}$ are related in the following way. There is a map $\Phi : \widetilde{X} \mapsto X$ such that: (i) Φ is one-to-one, onto and bimeasurable (up to two sets of measure zero); (ii) $\widetilde{\mu} \circ \Phi$ and μ are mutually absolutely continuous; (iii) for all $c > 0$, $\psi_c \circ \Phi = \Phi \circ \widetilde{\psi}_c$ $\widetilde{\mu}$-a.e., and (iv) for all $t \in \mathbb{R}$,

$$\widetilde{G}_t(\widetilde{x},u) = b(\widetilde{x}) \left\{ \frac{\mathrm{d}(\mu \circ \Phi)}{\mathrm{d}\widetilde{\mu}}(\widetilde{x}) \right\}^{1/\alpha} G_t\big(\Phi(\widetilde{x}), u + g(\widetilde{x})\big) \quad \text{a.e. } \widetilde{\mu}(\mathrm{d}\widetilde{x})\,\mathrm{d}u, \tag{3.5}$$

where $b : \widetilde{X} \mapsto \{-1,1\}$ and $g : \widetilde{X} \mapsto \mathbb{R}$ are measurable functions.

Since D (C, resp.) can be expressed as

$$D(C, \text{resp.}) = \left\{ x \in X\colon \int_0^\infty f\big(\psi_c(x)\big) \frac{\mathrm{d}(\mu \circ \psi_c)}{\mathrm{d}\mu}(x) c^{-1}\,\mathrm{d}c < \infty\ (= \infty,\ \text{resp.}) \right\}, \quad \mu\text{-a.e.},$$

for any $f \in L^1(X,\mu)$, $f > 0$ a.e. (see, for example, (3.22) and (3.33) in [6] in the case of additive flows), we obtain by using the relations (ii) and (iii) above that

$$\Phi^{-1}(D) = \widetilde{D}, \qquad \Phi^{-1}(C) = \widetilde{C}, \quad \widetilde{\mu}\text{-a.e.} \tag{3.6}$$

By using relations (i)–(iii), we can deduce directly from (2.18) and (2.19) that

$$\Phi^{-1}(F) = \widetilde{F}, \qquad \Phi^{-1}(P) = \widetilde{P}, \qquad \Phi^{-1}(L) = \widetilde{L}, \quad \widetilde{\mu}\text{-a.e.} \tag{3.7}$$

and hence

$$\Phi^{-1}(C \setminus P) = \widetilde{C} \setminus \widetilde{P}, \quad \widetilde{\mu}\text{-a.e.} \tag{3.8}$$

The equalities (3.4) can now be obtained by using (3.5) together with (3.6)–(3.8). For example, the first equality in (3.4) follows by using (3.5) and (3.6) to show that

$$\begin{aligned}
&\int_{\widetilde{D}} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} \theta_k \bigl(\widetilde{G}(\widetilde{x}, t_k + u) - \widetilde{G}(\widetilde{x}, u)\bigr) \right|^{\alpha} \widetilde{\mu}(\mathrm{d}\widetilde{x})\, \mathrm{d}u \\
&\quad = \int_{\Phi^{-1}(D)} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} \theta_k \bigl(G\bigl(\Phi(\widetilde{x}), t_k + u + g(\widetilde{x})\bigr) - G\bigl(\Phi(\widetilde{x}), u + g(\widetilde{x})\bigr)\bigr) \right|^{\alpha} \frac{\mathrm{d}(\mu \circ \Phi)}{\mathrm{d}\widetilde{\mu}}(\widetilde{x}) \widetilde{\mu}(\mathrm{d}\widetilde{x})\, \mathrm{d}u \\
&\quad = \int_{\Phi^{-1}(D)} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} \theta_k \bigl(G\bigl(\Phi(\widetilde{x}), t_k + u\bigr) - G\bigl(\Phi(\widetilde{x}), u\bigr)\bigr) \right|^{\alpha} (\mu \circ \Phi)(\mathrm{d}\widetilde{x})\, \mathrm{d}u \\
&\quad = \int_{D} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} \theta_k \bigl(G(x, t_k + u) - G(x, u)\bigr) \right|^{\alpha} \mu(\mathrm{d}x)\, \mathrm{d}u,
\end{aligned}$$

where in the last equality, we used a change of variables. □

Since the decomposition (3.1) can be obtained through a minimal representation for any $S\alpha S$, $\alpha \in (1, 2)$, self-similar mixed moving average, and it is unique in distribution by Theorem 3.1, we may give the following definition.

Definition 3.1. *A $S\alpha S$, $\alpha \in (1, 2)$, self-similar mixed moving average X_α is called* periodic fractional stable motion (cyclic fractional stable motion, *resp.*) *if*

$$X_\alpha \stackrel{d}{=} X_\alpha^P, \qquad \bigl(X_\alpha \stackrel{d}{=} X_\alpha^L, \text{ resp.}\bigr),$$

where X_α^L and X_α^P are the two components in the decomposition (3.1) *of X_α obtained through a minimal representation.*

Notation. *Periodic and cyclic fractional stable motion will be abbreviated as PFSM and CFSM, respectively.*

An equivalent definition of periodic and cyclic fractional stable motions is as follows.

Proposition 3.1. *A $S\alpha S$, $\alpha \in (1, 2)$, self-similar mixed moving average is a periodic (cyclic, resp.) fractional stable motion if and only if the generating flow corresponding to its minimal representation is periodic (cyclic, resp.).*

Proof. By Definition 3.1, a self-similar mixed moving average X_α is a PFSM (CFSM, resp.) if and only if $X_\alpha =_d X_\alpha^P$ ($X_\alpha =_d X_\alpha^L$, resp.), where P (L, resp.) is the set of periodic (cyclic, resp.) points of the generating flow ψ corresponding to a minimal representation. It follows from (3.1) and (3.2) that $X_\alpha =_d X_\alpha^P$ ($X_\alpha =_d X_\alpha^L$, resp.) if and only if $X = P$ ($X = L$, resp.) μ-a.e. and hence, by Definition 2.3, if and only if the flow ψ is periodic (cyclic, resp.). □

Definition 3.1 and Proposition 3.1 use minimal representations. Minimal representations, however, are not very easy to determine in practice. It is therefore desirable to recognize a PFSM and a CFSM based on any, possibly nonminimal representation. Since many self-similar mixed moving averages given by nonminimal representations are generated by a flow in the sense of Definition 2.1, we could expect that the process is a PFSM (CFSM, resp.) if the generating flow is periodic (cyclic, resp.). This, however, is not the case in general. For example, if a PFSM or CFSM $X_\alpha(t) = \int_X \int_{\mathbb{R}} G_t(x, u) M_\alpha(\mathrm{d}x, \mathrm{d}u)$ is generated by a periodic or cyclic flow $\psi_c(x)$ on X, we can also represent the process X_α as $\int_Y \int_X \int_{\mathbb{R}} G_t(x, u) M_\alpha(\mathrm{d}y, \mathrm{d}x, \mathrm{d}u)$, where $G_t(x, u)$ does not depend on y and the control measure $\eta(\mathrm{d}y)$ of $M_\alpha(\mathrm{d}y, \mathrm{d}x, \mathrm{d}u)$ in the variable y is such that $\eta(Y) = 1$. If $\widetilde{\psi}_c(y)$ is a measure preserving flow on (Y, η), then the process X_α is also generated by the flow $\Phi_c(y,x) = (\widetilde{\psi}_c(y), \psi_c(x))$ on $Y \times X$. If, in addition, the flow $\widetilde{\psi}_c(y)$ is not periodic (and hence not cyclic), then the flow $\Phi_c(y,x)$ is neither periodic nor cyclic.

We will provide identification criteria for a PFSM and a CFSM which do not rely on either minimal representations or flows, and which are based instead on the structure of the kernel function G. An analogous approach was taken by Rosiński [12] to identify harmonizable processes, by Pipiras and Taqqu [7] to identify a mixed LFSM, and by Pipiras and Taqqu [9] to identify periodic and cyclic stable stationary processes.

4. Identification of periodic fractional stable motions: the nonminimal case

We first provide a criterion to identify periodic fractional stable motions without using flows or minimal representations. The criterion is based on the periodic fractional stable motion set which we define next. Let X_α be a self-similar mixed moving average (2.1) defined through a (possibly nonminimal) kernel function G.

Definition 4.1. *A* periodic fractional stable motion set (*PFSM set*, *in short*) *of a self-similar mixed moving average* X_α *given by* (2.1), *is defined as*

$$C_P = \big\{x \in X\colon \exists c = c(x) \neq 1\colon\ G(x,cu) = bG(x,u+a) + d \text{ a.e. } \mathrm{d}u \\ \text{for some } a = a(c,x), b = b(c,x) \neq 0, d = d(c,x) \in \mathbb{R}\big\}. \tag{4.1}$$

Proposition 4.1. *The relation in* (4.1) *can be expressed as*

$$G(x, cu+g) = b\,G(x,u+g) + d \tag{4.2}$$

for some $b \neq 0$, $c \neq 1$, $g, d \in \mathbb{R}$. *When* $b \neq 1$, *it can also be expressed as*

$$G(x,cu+g) + f = b\big(G(x,u+g) + f\big) \tag{4.3}$$

for some $b \neq 0$, $c \neq 1$, g, $f \in \mathbb{R}$.

Proof. Relation (4.2) follows by making the change of variables $u = v + a/(c-1)$ in $G(x,cu) = bG(x,u+a) + d$. When $b \neq 1$, by writing $d = bf - f$ with $f = d/(b-1)$ in (4.2), we get (4.3). □

Whereas the set of periodic points P is defined by (2.17) in terms of the flow $\{\psi_c\}_{c>0}$, the set C_P in (4.1) is defined in terms of the kernel G. Definition 4.1 states that there is a factor c such that the kernel G at time u is related to the kernel at time cu.

Lemma 4.1. *The PFSM set* C_P *in* (4.1) *is* μ*-measurable. Moreover, the functions* $c(x), a(x) = a(c(x),x)$, $b = b(c(x),x)$ *and* $d = d(c(x),x)$ *in* (4.1) *can be taken to be* μ*-measurable as well.*

Proof. We first show the measurability of C_L. Consider the set

$$A = \big\{(x,c,a,b,d)\colon\ G(x,cu) = b\,G(x,u+a) + d \text{ a.e. } \mathrm{d}u\big\}.$$

Since $A = \{F(x,c,a,b,d) = 0\}$, where the function

$$F(x,c,a,b,d) = \int_{\mathbb{R}} 1_{\{G(x,cu)=bG(x,u+a)+d\}}(x,c,a,b,d,u)\,\mathrm{d}u$$

is measurable by Fubini's theorem, we obtain that the set A is measurable. Observe that the set C_P is a projection of the set A on x, namely, that

$$C_P = \mathrm{proj}_X A := \{x\colon\ \exists c,a,b,d\colon\ (x,c,a,b,d) \in A\}.$$

Lemma 3.3 in [9] implies that the PFSM set C_P is μ-measurable and that the functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$ can be taken to be μ-measurable as well. □

In the next theorem, we characterize a PFSM in terms of the set C_P instead of using the set P which involves flows as is done in Definition 4.1 and Proposition 3.1. Flows and minimal representations, however, are used in the proof.

Theorem 4.1. *A* $S\alpha S$, $\alpha \in (1,2)$, *self-similar mixed moving average* X_α *given by* (2.1) *with* G *satisfying* (2.10) *is a PFSM if and only if* $C_P = X$ μ-a.e., *where* C_P *is the PFSM set defined in* (4.1).

Proof. Suppose first that X_α is a self-similar mixed moving average given by (2.1) with G satisfying (2.10) and such that $C_P = X$ μ-a.e. To show that X_α is a PFSM, we adapt the proof of Theorem 3.2 in [9]. The proof consists of 2 steps.

Step 1: We will show that without loss of generality, the representation (2.1) can be supposed to be minimal with $C_P = X$ μ-a.e. By Theorem 4.2 in [6], the process X_α has a minimal integral representation

$$\int_{\widetilde{X}}\int_{\mathbb{R}}\bigl(\widetilde{G}(\widetilde{x},t+u)-\widetilde{G}(\widetilde{x},u)\bigr)\widetilde{M}_\alpha(\mathrm{d}\widetilde{x},\mathrm{d}u), \tag{4.4}$$

where $(\widetilde{X},\widetilde{\mathcal{X}},\widetilde{\mu})$ is a standard Lebesgue space and $\widetilde{M}_\alpha(\mathrm{d}\widetilde{x},\mathrm{d}u)$ has control measure $\widetilde{\mu}(\mathrm{d}\widetilde{x})\,\mathrm{d}u$. Letting $\widetilde{C}_P$ be the periodic component set of X_α defined using the kernel function $\widetilde{G}$, we need to show that $\widetilde{C}_P = \widetilde{X}$ $\widetilde{\mu}$-a.e. By Corollary 5.1 in [6], there are measurable maps $\Phi_1 : X \mapsto \widetilde{X}$, $h : X \mapsto \mathbb{R}\setminus\{0\}$ and $\Phi_2, \Phi_3 : X \mapsto \mathbb{R}$ such that

$$G(x,u) = h(x)\widetilde{G}\bigl(\Phi_1(x), u+\Phi_2(x)\bigr) + \Phi_3(x) \tag{4.5}$$

a.e. $\mu(\mathrm{d}x)\,\mathrm{d}u$, and

$$\widetilde{\mu} = \mu_h \circ \Phi_1^{-1}, \tag{4.6}$$

where $\mu_h(\mathrm{d}x) = |h(x)|^\alpha \mu(\mathrm{d}x)$. If $x \in C_P$, then

$$G\bigl(x, c(x)u\bigr) = b(x)G\bigl(x, u+a(x)\bigr) + d(x) \quad \text{a.e. } \mathrm{d}u, \tag{4.7}$$

for some functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$. Hence, by using (4.5) and (4.7), we have for some functions F_1,F_2 and F_3, a.e. $\mu(\mathrm{d}x)$,

$$\begin{aligned}\widetilde{G}\bigl(\Phi_1(x), c(x)u+\Phi_2(x)\bigr) &= \bigl(h(x)\bigr)^{-1}G\bigl(x,c(x)u\bigr) + F_1(x)\\ &= \bigl(h(x)\bigr)^{-1}b(x)G\bigl(x,u+a(x)\bigr) + F_2(x)\\ &= b(x)\widetilde{G}\bigl(\Phi_1(x), u+a(x)+\Phi_2(x)\bigr) + F_3(x)\end{aligned}$$

a.e. $\mathrm{d}u$. This shows that $\Phi_1(x) \in \widetilde{C}_P$ and hence

$$C_P \subset \Phi_1^{-1}(\widetilde{C}_P), \quad \mu\text{-a.e.} \tag{4.8}$$

Since $C_P = X$ μ-a.e., we have $X = \Phi^{-1}(\widetilde{C}_P)$ μ-a.e. This implies $\widetilde{C}_P = \widetilde{X}$ $\widetilde{\mu}$-a.e., because if $\widetilde{\mu}(\widetilde{X}\setminus\widetilde{C}_P) > 0$, then by (4.6), we have $\mu(\Phi_1^{-1}(\widetilde{X}\setminus\widetilde{C}_P)) = \mu(\Phi_1^{-1}(\widetilde{X})\setminus X) = \mu(\emptyset) > 0$.

Remark. *The converse is shown in the same way*: *if* C_P *is not equal to* X μ*-a.e., then* $\Phi_1^{-1}(\widetilde{C}_P) \subset C_P$ μ*-a.e. Together with* (4.8), *this implies*

$$C_P = \Phi_1^{-1}(\widetilde{C}_P), \quad \mu\text{-a.e.} \tag{4.9}$$

The relation (4.9) *is used in the proof of the converse of this theorem and in the proof of Theorem* 6.1.

We may therefore suppose without loss of generality that the representation (2.1) is minimal and that $C_P = X$ μ-a.e. By Theorem 4.1 in [6], since the representation (2.1) is minimal, the process X_α is generated by a flow $\{\psi_c\}_{c>0}$ and related functionals $\{b_c\}_{c>0}$, $\{g_c\}_{c>0}$ and $\{j_c\}_{c>0}$ in the sense of Definition 2.1.

Step 2: To conclude the proof, it is enough to show, by Proposition 3.1, that the flow $\{\psi_c\}_{c>0}$ is periodic. The idea can informally be explained as follows. By using (2.9) and (4.1), we get that for $c = c(x) \neq 1$,

$$G\big(\psi_{c(x)}(x), u\big) = h(x)G\big(x, c(x)u + a(x)\big) + j(x) = k(x)G\big(x, u + b(x)\big) + l(x),$$

for some $a, b, h \neq 0, j, k \neq 0, l$. Then, for any $t \in \mathbb{R}$, $G_t(\Psi(x, u)) = k(x)G_t(x, u)$, where G_t is defined by (2.3), $\Psi(x, u) = (\psi_{c(x)}(x), u - b(x))$ and $k(x) \neq 0$. Since the representation $\{G_t\}_{t\in\mathbb{R}}$ is minimal, $\Psi(x, u) = (x, u)$ and therefore $\psi_{c(x)}(x) = x$ for $c(x) \neq 1$, showing that the flow $\{\psi_c\}_{c>0}$ is periodic. This argument is not rigorous because c depends on x and hence the relation (2.9) cannot be applied directly. The rigorous proof below shows how this technical difficulty can be overcome.

Consider the set

$$A = \big\{(x, c) \in X \times \big((0, \infty) \setminus \{1\}\big)\colon\ G(x, cu) = bG(x, u + a) + d \text{ a.e. } \mathrm{d}u \\ \text{for some } a = a(x, c), b = b(x, c) \neq 0, d = d(x, c) \in \mathbb{R}\big\}$$

and let

$$A_0 = A \cap \big\{(x, c) \in X \times \big((0, \infty) \setminus \{1\}\big)\colon\ G(x, cu) = hG\big(\psi_c(x), u + g\big) + j \text{ a.e. } \mathrm{d}u \\ \text{for some } h = h(x, c) \neq 0, g = g(x, c), j = j(x, c) \in \mathbb{R}\big\}.$$

Since G satisfies (2.9), we have $1_A(x, c) = 1_{A_0}(x, c)$ a.e. $\mu(\mathrm{d}x)$ for all $c > 0$ and hence, by Fubini's theorem (see also Lemma 3.1 in [6]), we have that $1_A(x, c) = 1_{A_0}(x, c)$ a.e. $\mu(\mathrm{d}x)\tau(\mathrm{d}c)$ or $A = A_0$ a.e. $\mu(\mathrm{d}x)\tau(\mathrm{d}c)$, where τ is any σ-finite measure on $(0, \infty)$. Setting

$$A_1 = A_0 \cap \big\{(x, c) \in X \times \big((0, \infty) \setminus \{1\}\big)\colon\ \psi_c(x) = x\big\}$$

we want to show that $A_1 = A_0$ a.e. $\mu(\mathrm{d}x)\tau(\mathrm{d}c)$ and to do so, it is enough to prove that

$$\psi_c(x) = x \quad \text{a.e. for } (x, c) \in A_0. \tag{4.10}$$

We proceed by contradiction. Suppose that (4.10) were not true. We can then find a fixed $c_0 \neq 1$ such that $\psi_{c_0}(x) \neq x$ a.e. on a set of positive measure for $(x, c_0) \in A_0$. Define first $\widetilde{\psi}(x) = \psi_{c_0}(x)$ for $(x, c_0) \in A_0$ and $\widetilde{\psi}(x) = x$ for $(x, c_0) \notin A_0$. Then,

$$G\big(\widetilde{\psi}(x), u + \widetilde{a}(x)\big) + \widetilde{c}(x) = \widetilde{b}(x)G(x, u) \tag{4.11}$$

a.e. $\mu(\mathrm{d}x)\,\mathrm{d}u$, for some measurable functions $\widetilde{a}$, $\widetilde{b} \neq 0$ and $\tilde{c}$. Indeed, relation (4.11) is clearly true for x such that $(x, c_0) \notin A_0$ since $\widetilde{\psi}(x) = x$. It is also true for $(x, c_0) \in A_0$ because it follows from the definition of A_0 that the relations $G(x, c_0u) = bG(x, u + a) + d$ and $G(x, c_0u) = hG(\psi_{c_0}(x), u + g) + j$ imply $G(\psi_{c_0}(x), u + \widetilde{a}) + \tilde{c} = \widetilde{b}G(x, u)$. Now define $\Psi(x, u) = (\widetilde{\psi}(x), u + \widetilde{a}(x))$. We obtain from (4.11) that, for all $t \in \mathbb{R}$,

$$G_t\big(\Psi(x, u)\big) = h(x)G_t(x, u) \quad \text{a.e. } \mu(\mathrm{d}x)\,\mathrm{d}u, \tag{4.12}$$

where $h(x) \neq 0$ and where we used the notation (2.3). Since $\widetilde{\psi}$ is nonsingular by construction, the map Ψ is nonsingular as well and, since $\psi(x) \neq x$ on a set of positive measure $\mu(\mathrm{d}x)$, we have $\Psi(x, u) \neq (x, u)$ (Ψ is not an identity map) on a set of positive measure $\mu(\mathrm{d}x)\,\mathrm{d}u$. This contradicts (2.4) and hence the minimality of the representation $\{G_t\}_{t\in\mathbb{R}}$. Hence, $A_1 = A_0$ a.e. $\mu(\mathrm{d}x)\tau(\mathrm{d}c)$ and since $A_0 = A$ a.e. $\mu(\mathrm{d}x)\tau(\mathrm{d}c)$ as well, we have

$$A = A_1 \quad \text{a.e. } \mu(\mathrm{d}x)\tau(\mathrm{d}c). \tag{4.13}$$

By Lemma 3.3 in [9], we can choose a μ-measurable function $c(x) \neq 1$ defined for $x \in \mathrm{proj}_X A_1$ such that $(x, c(x)) \in A_1$ and, in particular,

$$\psi_{c(x)}(x) = x. \tag{4.14}$$

By using (4.13), the definition of C_P and the assumption $C_P = X$ μ-a.e., we obtain that $\mathrm{proj}_X A_1 = \mathrm{proj}_X A = C_P = X$ μ-a.e., that is, (4.14) holds for μ-a.e. $x \in X$. Hence, $X = P$ μ-a.e., showing that the flow ψ_c is periodic.

To prove the converse, suppose that X_α given by (2.1) with a kernel G satisfying (2.10), is a PFSM. By Proposition 3.1, the minimal representation (4.4) of X_α is generated by a periodic flow $\{\widetilde{\psi}_c\}_{c>0}$. Let $\widetilde{P}$ be the set of the periodic points of the flow $\{\widetilde{\psi}_c\}_{c>0}$, and $\widetilde{C}_P$ be the PFSM set defined using the representation (4.4). Since the flow $\{\widetilde{\psi}_c\}_{c>0}$ is periodic, $\widetilde{P} = \widetilde{X}$ a.e. $\widetilde{\mu}(\mathrm{d}\widetilde{x})$. Since $\widetilde{P} \subset \widetilde{C}_P$ a.e. $\widetilde{\mu}(\mathrm{d}\widetilde{x})$ by Proposition 4.2, we have $\widetilde{C}_P = \widetilde{X}$ a.e. $\widetilde{\mu}(\mathrm{d}\widetilde{x})$. In addition, the following three equalities hold a.e. $\mu(\mathrm{d}x)$:

$$C_P = \Phi_1^{-1}(\widetilde{C}_P), \qquad \Phi_1^{-1}(\widetilde{C}_P) = \Phi_1^{-1}(\widetilde{X}) \quad \text{and} \quad \Phi_1^{-1}(\widetilde{X}) = X.$$

The first equality follows from (4.9), the second holds because the measures $\mu \circ \Phi_1^{-1}$ and $\widetilde{\mu}$ are absolutely continuous by (4.6) and hence $\widetilde{C}_X = \widetilde{X}$ a.e. $\widetilde{\mu}(\mathrm{d}\widetilde{x})$ implies $\mu(\Phi_1^{-1}(\widetilde{X} \setminus \widetilde{C}_P)) = 0$. The third equality follows from the definition of Φ_1. Stringing these equalities together one gets $C_P = X$ a.e. $\mu(\mathrm{d}x)$. □

The next result describes relations between the PFSM set C_P defined using a kernel function G, and the set of periodic points P of a flow related to the kernel G as in Definition 2.1. The first part of the result was used in the proof of Theorem 4.1.

Proposition 4.2. *Suppose that a $S\alpha S$, $\alpha \in (0, 2)$, self-similar mixed moving average X_α given by* (2.1), *is generated by a flow $\{\psi_c\}_{c>0}$. Let P be the set of periodic points* (2.17) *of the flow $\{\psi_c\}_{c>0}$ and C_P the PFSM set* (4.1) *defined using the kernel G of the representation* (2.1). *Then, we have*

$$P \subset C_P, \quad \mu\text{-a.e.} \tag{4.15}$$

If, moreover, the representation (2.1) *is minimal, we have*

$$P = C_P, \quad \mu\text{-a.e.} \tag{4.16}$$

Proof. We first prove (4.15). Let $\tau(\mathrm{d}c)$ denote any σ-finite measure on $(0, \infty)$. By Fubini's theorem (see also Lemma 3.1 in [6]), relation (2.9) implies that a.e. $\mu(\mathrm{d}x)\tau(\mathrm{d}c)$,

$$G(x, cu) = hG\bigl(\psi_c(x), u + g\bigr) + j \quad \text{a.e. } \mathrm{d}u,$$

for some $h = h(x, c) \neq 0$, $g = g(x, c)$ and $j = j(x, c)$. Hence, setting

$$\widetilde{P} := \bigl\{(x, c) \in X \times \bigl((0, \infty) \setminus \{1\}\bigr) \colon \psi_c(x) = x\bigr\},$$

we have a.e. $\mu(\mathrm{d}x)\tau(\mathrm{d}c)$,

$$\begin{aligned} \widetilde{P} &= \widetilde{P} \cap \bigl\{(x, c) \colon G(x, cu) = hG\bigl(\psi_c(x), u + g\bigr) + j \text{ a.e. } \mathrm{d}u \text{ for some } h \neq 0, g, j\bigr\} \\ &= \widetilde{P} \cap \bigl\{(x, c) \colon G(x, cu) = hG(x, u + g) + j \text{ a.e. } \mathrm{d}u \text{ for some } h \neq 0, g, j\bigr\}. \end{aligned} \tag{4.17}$$

Since $P = \mathrm{proj}_X \widetilde{P}$, relation (4.17) implies that a.e. $x \in P$ belongs to the set

$$\mathrm{proj}_X\bigl(\bigl\{(x, c) \colon G(x, cu) = hG(x, u + g) + j \text{ a.e. } \mathrm{d}u \text{ for some } h \neq 0, g, j\bigr\}\bigr),$$

that is, there is $c = c(x) \neq 1$ such that

$$G(x, cu) = hG(x, u + g) + j \quad \text{a.e. } \mathrm{d}u$$

for some $h \neq 0$, g, j. This shows that $P \subset C_P$ a.e. $\mu(\mathrm{d}x)$.

To prove (4.16), suppose that the representation (2.1) is minimal. It is enough to show that $C_P \subset P$ μ-a.e. Let $\{G_t|_{C_P}\}$ be the kernel G_t of (2.1) restricted to the set $C_P \times \mathbb{R}$. By Lemma 4.2, the set C_P is a.e. invariant under the flow $\{\psi_c\}$. Then, $\{G_t|_{C_P}\}$ is a representation of a self-similar mixed moving average. Since $\{G_t\}$ is minimal, so is the representation $\{G_t|_{C_P}\}$. It is obviously generated by the flow $\psi|_{C_P}$, the restriction of the flow ψ to the set C_P. By arguing as in Step 2 of the proof of Theorem 4.1, we therefore obtain that for a.e. $x \in C_P$, $\psi_{c(x)}(x) = x$ for some $c(x) \neq 1$. This shows that $C_P \subset P$ a.e. $\mu(\mathrm{d}x)$. □

The following lemma was used in the proof of Proposition 4.2 above.

Lemma 4.2. *If a $S\alpha S$, $\alpha \in (0,2)$, self-similar mixed moving average X_α given by a representation* (2.1) *is generated by a flow $\{\psi_c\}_{c>0}$, and C_P is the PFSM set defined by* (4.1), *then C_P is a.e. invariant under the flow $\{\psi_c\}_{c>0}$, that is, $\mu(C_P \triangle \psi_c^{-1}(C_P)) = 0$ for all $c > 0$.*

Proof. Since $\{\psi_c\}_{c>0}$ satisfies the group property (2.5), it is enough to show that $C_P \subset \psi_r^{-1}(C_P)$ μ-a.e. for any fixed $r > 0$. By (2.9), we have for any $c > 0$,

$$G\big(\psi_r(x), cu + a(x)\big) = b(c,x)G(x, cru) + j(c,x) \quad \text{a.e. } \mu(\mathrm{d}x)\,\mathrm{d}u, \tag{4.18}$$

for some $a, b \neq 0$, j (these depend on r but since r is fixed we do not indicate their dependence on r). By using Lemma 4.2 in [9] and arguing as in Step 2 of the proof of Theorem 4.1, we can choose a function $c(x) \neq 1$ such that, for a.e. $x \in C_P$,

$$G\big(x, c(x)ru\big) = b(x)G\big(x, ru + a(x)\big) + j(x) \quad \text{a.e. } \mathrm{d}u, \tag{4.19}$$

for some $a, b \neq 0$, j, and such that the relation (4.18) holds with c replaced by $c(x)$. By substituting (4.19) into (4.18) with $c = c(x)$ and then making a change of variables in u, we obtain that, for a.e. $x \in C_P$,

$$G\big(\psi_r(x), c(x)u + d(x)\big) = h(x)G(x, ru) + l(x) \quad \text{a.e. } \mathrm{d}u,$$

for some $d, h \neq 0, l$. Then, by using (2.9) and making a change of variables in u, we get that, for a.e. $x \in C_P$,

$$G\big(\psi_r(x), c(x)u\big) = k(x)G\big(\psi_r(x), u + p(x)\big) + q(x) \quad \text{a.e. } \mathrm{d}u,$$

for some $k \neq 0$, p, q. Hence, for a.e. $x \in C_P$, $\psi_r(x) \in C_P$ or $x \in \psi_r^{-1}(C_P)$, showing that $C_P \subset \psi_r^{-1}(C_P)$ μ-a.e. □

We now provide examples of PFSMs. These examples show, in particular, that the criterion of Theorem 4.1 is of practical use. Further examples of PFSMs can be found in [10].

Example 4.1. *Let $\alpha \in (0,2)$, $H \in (0,1)$ and $\kappa = H - 1/\alpha < 0$. By Section 8 of* [7], *the mixed moving average process*

$$\int_0^1 \int_{\mathbb{R}} \Big((t+u)_+^\kappa 1_{[0,1/2)}\big(\{x + \ln|t+u|\}\big) - u_+^\kappa 1_{[0,1/2)}\big(\{x + \ln|u|\}\big)\Big) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{4.20}$$

where M_α has the control measure $\mathrm{d}x\,\mathrm{d}u$ and $\{u\}$ stands for the fractional part of $u \in \mathbb{R}$, is well defined and self-similar. It has the representation (2.1) *with $X = [0,1)$ and*

$$G(x,u) = u_+^\kappa 1_{[0,1/2)}\big(\{x + \ln|u|\}\big), \quad x \in [0,1), u \in \mathbb{R}.$$

Since $G(x, cu) = c^\kappa G(x,u)$ for all $x \in [0,1)$, $u \in \mathbb{R}$ and $c > 0$, we deduce that $X = C_P$ for the process (4.20). *Hence, by Theorem* 4.1, *the process* (4.20) *is a PFSM when $\alpha \in (1,2)$.*

Example 4.2. *Let $\alpha \in (0,2)$, $H \in (0,1)$, $a, b \in \mathbb{R}$ and $\kappa = H - 1/\alpha \neq 0$. The mixed moving average process*

$$\int_{\mathbb{R}} \big(a\big((t+u)_+^\kappa - u_+^\kappa\big) + b\big((t+u)_-^\kappa - u_-^\kappa\big)\big) M_\alpha(\mathrm{d}u),$$

where M_α has the control measure du, *is well defined and self-similar. It is called linear fractional stable motion or LFSM* (*see* [14]), *and is a special case of mixed LFSMs* (2.14). *LFSM has the representation* (2.1) *with $X = \{1\}$, $\mu(\mathrm{d}x) = \delta_{\{1\}}(\mathrm{d}x)$* (*the point mass at* 1) *and*

$$G(1,u) = au_+^\kappa + bu_-^\kappa, \quad u \in \mathbb{R}.$$

Since $G(1,cu) = c^\kappa G(1,u)$ for all $c > 0, u \in \mathbb{R}$, we deduce that $X = C_P$ for LFSM. Hence, LFSM is also a PFSM when $\alpha \in (1,2)$. This should not be surprising since LFSM is associated with identity flows which are periodic flows with period zero.

5. Identification of cyclic fractional stable motions: the nonminimal case

We focused so far on periodic fractional stable motions. By using the set C_P we were able to identify them without requiring the representation to be minimal. We now want to do the same thing for cyclic fractional stable motions by introducing a corresponding set C_L. To do so, observe that:

Lemma 5.1. *A CFSM is a PFSM without a mixed LFSM component.*

Proof. This follows from (3.1) and the fact that X_α^F is a mixed LFSM (see (2.14)). □

We showed in [7] that a mixed LFSM can be identified through the *mixed LFSM set*

$$C_F = \big\{x \in X\colon\ G(x,u) = d(u+f)_+^\kappa + h(u+f)_-^\kappa + g \text{ a.e. } \mathrm{d}u \\ \text{for some reals } d = d(x), f = f(x), g = g(x), h = h(x)\big\}, \tag{5.1}$$

when $\kappa \neq 0$, and

$$C_F = \big\{x \in X\colon\ G(x,u) = d\ln|u+f| + h1_{(0,\infty)}(u+f) + g \text{ a.e. } \mathrm{d}u \\ \text{for some reals } d = d(x), f = f(x), g = g(x), h = h(x)\big\}, \tag{5.2}$$

when $\kappa = 0$. The following lemma shows that this set is a subset of C_P.

Lemma 5.2. *We have*

$$C_F \subset C_P. \tag{5.3}$$

Proof. Suppose that $\kappa \neq 0$. If $x \in C_F$, then $G(x,u) = d(u+f)_+^\kappa + h(u+f)_-^\kappa + g$ for some reals d, f, g, h and hence

$$G(x,cu) = c^\kappa\big(d\big(u + c^{-1}f\big)_+^\kappa + h\big(u+c^{-1}f\big)_-^\kappa + g\big) + \big(1-c^\kappa\big)g \\ = c^\kappa G\big(x, u + c^{-1}f\big) + \big(1-c^\kappa\big)g \tag{5.4}$$

for arbitrary c. This shows that $x \in C_P$ and hence that (5.3) holds. The proof in the case $\kappa = 0$ is similar. □

Since a CFSM is a PFSM without a mixed LFSM component, we expect that a CFSM can be identified through the set $C_L = C_P \setminus C_F$. We will show that this is indeed the case.

Definition 5.1. *A* cyclic fractional stable motion set (CFSM set, *in short*) *of a self-similar mixed moving average* X_α *given by* (2.1) *is defined by*

$$C_L := C_P \setminus C_F, \tag{5.5}$$

where C_P *is the PFSM set defined by* (4.1) *and* C_F *is the mixed LFSM set defined by* (5.1) *and* (5.2).

The following result shows that a CFSM can indeed be identified through the CFSM set.

Theorem 5.1. *A* $S\alpha S$, $\alpha \in (1,2)$, *self-similar mixed moving average* X_α *given by* (2.1) *with* G *satisfying* (2.10), *is a CFSM if and only if* $C_L = X$ μ*-a.e., where* C_L *is the CFSM set defined in* (5.5).

Proof. If X_α is a CFSM, then it is also a PFSM and hence, by Theorem 4.1, $C_P = X$ μ-a.e. By (5.5), $C_P = C_L + C_F$. Since X_α does not have a mixed LFSM component (Lemma 5.1), Propositions 7.1 and 7.2 in [7] imply that $C_F = \emptyset$ μ-a.e. Hence, $C_L = X$ μ-a.e. Conversely, if $C_L = X$ μ-a.e., then $C_P = X$ μ-a.e. and hence X_α is a PFSM. But $C_L = X$ μ-a.e. implies $C_F = \emptyset$ μ-a.e., that is, X_α does not have a mixed LFSM component. The PFSM X_α is therefore a CFSM. □

Observe that the mixed LFSM set C_F in (5.1) and (5.2) is expressed in a different way from the PFSM set (4.1). It can, however, be expressed in a similar way.

Proposition 5.1. *Let* $\alpha \in (1,2)$. *We have*

$$\begin{aligned} C_F = \big\{ & x \in X\colon \exists c_n = c_n(x) \to 1 (c_n \neq 1)\colon\ G(x, c_n u) = b_n\, G(x, u + a_n) + d_n \textit{ a.e. } \mathrm{d}u \\ & \textit{for some } a_n = a_n(c_n, x), b_n = b_n(c_n, x) \neq 0, d_n = d_n(c, x) \in \mathbb{R} \big\}, \quad \mu\textit{-a.e.} \end{aligned} \tag{5.6}$$

Proof. Consider the case $\kappa \neq 0$. Denote the set on the right-hand side of (5.6) by C_F^0. If $x \in C_F$, then for any $c \neq 1$, $G(x, cu) = c^\kappa G(x, u + c^{-1} f) + (1 - c^\kappa) g$ (see (5.4)) and hence $x \in C_F^0$ with any $c_n \to 1$ $(c_n \neq 1)$. This shows that $C_F \subset C_F^0$ in the case $\kappa \neq 0$. The proof in the case $\kappa = 0$ is similar.

To show that $C_F^0 \subset C_F$ μ-a.e., we adapt the proof of Proposition 5.1 in [9]. Let $\widetilde{G} : \widetilde{X} \times \mathbb{R} \mapsto \mathbb{R}$ be the kernel function of a minimal representation of the process X_α, and $\widetilde{C}_F$ and $\widetilde{C}_F^0$ be the sets defined in the same way as C_F and C_F^0 by using the kernel function $\widetilde{G}$. One can show as in the proof of Theorem 4.1 (see (4.9)) that $C_F^0 = \Phi_1^{-1}(\widetilde{C}_F^0)$ μ-a.e., where Φ_1 is the map appearing in (4.5) and (4.6). As shown in the proof of Proposition 7.1 of [7], $C_F = \Phi_1^{-1}(\widetilde{C}_F)$. By using (4.6), it is then enough to show that $\widetilde{C}_F^0 \subset \widetilde{C}_F$ $\widetilde{\mu}$-a.e., or equivalently, $C_F^0 \subset C_F$ μ-a.e. but where C_F^0 and C_F are defined by using the kernel function G corresponding to a minimal representation of X_α.

If the process X_α is given by a minimal representation involving a kernel G, then it is generated by a flow $\{\psi_c\}_{c>0}$ and related functionals (Theorem 4.1 in [6]). By Lemma 5.3, the set C_F^0 is a.e. invariant under the flow $\{\psi_c\}_{c>0}$. Then, the process

$$\int_{C_F^0} \int_{\mathbb{R}} \big(G(x, t+u) - G(x, u)\big) M_\alpha(\mathrm{d}x, \mathrm{d}u) \tag{5.7}$$

is a self-similar mixed moving average, the representation (5.7) is minimal and the process (5.7) is generated by the flow $\{\psi_c\}_{c>0}$ restricted to the set C_F^0. Arguing as in the proof of Theorem 4.1, one can show that, for a.e. $x \in C_F^0$,

$$\psi_{c_n(x)}(x) = x \quad \text{for } \ c_n(x) \to 1\big(c_n(x) \neq 1\big). \tag{5.8}$$

Relation (5.8) cannot hold for points which are not fixed. This follows by using the so-called "special representation" of a flow as in the end of the proof of Proposition 5.1, [9] (see relation (5.6) of that paper). Hence, for a.e. $x \in C_F^0$, $\psi_c(x) = x$ for all $c > 0$. Since

$$C_F = F = \big\{ x : \psi_c(x) = x \text{ for all } c > 0 \big\}, \quad \mu\text{-a.e.}$$

by Theorem 10.1 in [7], we obtain that $C_F^0 \subset C_F$ μ-a.e. □

The following lemma was used in the proof of Proposition 5.1. An a.e. invariant set is defined in Lemma 4.2.

Lemma 5.3. *If a $S\alpha S$, $\alpha \in (0,2)$, self-similar mixed moving average X_α given by a representation* (2.1) *is generated by a flow, and C_F^0 denotes the right-hand side of* (5.6), *then C_F^0 is a.e. invariant under the flow.*

Proof. Since the proof of this result is very similar to that of Lemma 4.2, we only outline it. Proceeding as in the proof of Lemma 4.2, we can choose functions $c_n(x) \to 1$ ($c_n(x) \neq 1$) such that, for a.e. $x \in C_F^0$, the relation (4.19) holds with $c(x)$ replaced by $c_n(x)$ (and with $a_n, b_n \neq 0$, j_n replacing $a, b \neq 0$, j) and the relation (4.18) holds with c replaced by $c_n(x)$. The conclusion follows as in the proof of Lemma 4.2. □

The new formulation (5.6) of C_F yields the following characterization of $C_L = C_P \setminus C_F$:

Corollary 5.1. *We have*

$$\begin{aligned} C_L = \Big\{ & x \in X\colon \exists c_0 = c_0(x) \neq 1, \nexists c_n = c_n(x) \to 1\ (c_n \neq 1)\colon \\ & G(x, c_n u) = b_n G(x, u + a_n) + d_n \text{ a.e. } \mathrm{d}u,\ n = 0, 1, 2, \ldots, \\ & \text{for some } a_n = a_n(c_n, x), b_n = b_n(c_n, x) \neq 0, d_n = d_n(c, x) \in \mathbb{R} \Big\}, \quad \mu\text{-a.e.} \end{aligned} \tag{5.9}$$

The next result is analogous to the second part of Proposition 4.2.

Proposition 5.2. *Suppose that a $S\alpha S$, $\alpha \in (0,2)$, self-similar mixed moving average X_α given by a minimal representation* (2.1), *is generated by a flow $\{\psi_c\}_{c>0}$. Then,*

$$L = C_L, \quad \mu\text{-a.e.}, \tag{5.10}$$

where L is the set of cyclic points (2.19) *of the flow $\{\psi_c\}_{c>0}$ and C_L is the CFSM set* (5.5) *defined using the kernel of a minimal representation* (2.1).

Proof. By Proposition 4.2 above and Theorem 10.1 in [7], we have $P = C_P$ μ-a.e. and $F = C_F$ μ-a.e., where P and F are the sets of the periodic and fixed points of the flow $\{\psi_c\}_{c>0}$, and C_P and C_F are the PFSM and the mixed LFSM sets. The equality (5.10) follows since $L = P \setminus F$ and $C_L = C_P \setminus C_F$. □ □

The PFSM considered in Example 4.1 is also a CFSM.

Example 5.1. *The self-similar mixed moving average* (4.20) *considered in Example* 4.1 *is a CFSM because it is a PFSM and, as can be seen by using* (5.2), $C_F = \emptyset$.

Example 5.2. *LFSM considered in Example* 4.2 *is not a CFSM. This is immediate from Lemma* 5.1 *since LFSM is also a mixed LFSM. It can also be deduced from* (5.1) *or* (5.6) *by noting that $C_F = X$ and hence $C_L = \emptyset$.*

6. Refined decomposition of self-similar mixed moving averages

Suppose that X_α is a $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average. By using its minimal representation, we showed in Section 3 that X_α admits a decomposition (3.1) which is unique in distribution and has independent components. We show here that the components of the decomposition (3.1) can be expressed in terms of a possibly nonminimal representation (2.1) of the process X_α.

Let G be the kernel function of a possibly nonminimal representation (2.1) of the process X_α. With the notation (2.3), let

$$D = \left\{ x \in X: \int_0^\infty \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u \, c^{-H\alpha} \left| G_c(x, cu) \right|^\alpha < \infty \right\}, \tag{6.1}$$

$$C = \left\{ x \in X: \int_0^\infty \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u \, c^{-H\alpha} \left| G_c(x, cu) \right|^\alpha = \infty \right\}. \tag{6.2}$$

Recall also the definitions (4.1), (5.1), (5.2) and (5.5) of the mixed LFSM, PFSM and CFSM sets defined by using the kernel function G.

Theorem 6.1. *Let X_α be a $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average given by a possibly nonminimal representation* (2.1). *Suppose that*

$$X_\alpha^D, \qquad X_\alpha^F, \qquad X_\alpha^L, \qquad X_\alpha^{C\setminus P}$$

are the four independent components in the unique decomposition (3.1) *of the process X_α obtained by using its minimal representation. Then,*

$$X_\alpha^D(t) \stackrel{d}{=} \int_D \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.3}$$

$$X_\alpha^F(t) \stackrel{d}{=} \int_{C_F} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.4}$$

$$X_\alpha^L(t) \stackrel{d}{=} \int_{C_L} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.5}$$

$$X_\alpha^{C\setminus P}(t) \stackrel{d}{=} \int_{C\setminus C_P} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.6}$$

where $\stackrel{d}{=}$ stands for the equality in the sense of the finite-dimensional distributions and the sets D, C, C_F, C_P and C_L are defined by (6.1), (6.2), (5.1), (5.2), (4.1) *and* (5.5), *respectively.*

Proof. The equalities (6.3) and (6.4) follow from Theorem 5.5 in [6] and Corollary 9.1 in [7], respectively. Consider now the equality (6.5). Let $\widetilde{G}$ be the kernel of a minimal representation (4.4) of the process X_α, and let also $\widetilde{C}_F$, $\widetilde{C}_P$ and $\widetilde{C}_L$ be the sets defined by (5.1), (5.2), (4.1) and (5.5), respectively, using the kernel function $\widetilde{G}$. Since $C_P = \Phi_1^{-1}(\widetilde{C}_P)$ μ-a.e. by (4.9) and $C_F = \Phi_1^{-1}(\widetilde{C}_F)$ μ-a.e. as shown in the proof of Proposition 7.1 in [7], we obtain that

$$C_L = C_P \setminus C_F = \Phi_1^{-1}(\widetilde{C}_P \setminus \widetilde{C}_F) = \Phi_1^{-1}(\widetilde{C}_L), \quad \mu\text{-a.e.}$$

Then, by using (4.5), (4.6) and a change of variables as at the end of the proof of Proposition 7.1 in [7], we get that

$$\int_{C_L} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u) \stackrel{d}{=} \int_{\widetilde{C}_L} \int_{\mathbb{R}} \widetilde{G}_t(\widetilde{x}, u) \widetilde{M}_\alpha(\mathrm{d}\widetilde{x}, \mathrm{d}u).$$

Since $\widetilde{G}$ is a kernel of a minimal representation, it is related to a flow in the sense of Definition 2.1. Let $\widetilde{L}$ be the set of the cyclic points of the flow corresponding to the kernel $\widetilde{G}$. Since $\widetilde{L} = \widetilde{C}_L$ μ-a.e. by Proposition 5.2, we get that

$$\int_{C_L} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u) \stackrel{d}{=} \int_{\widetilde{L}} \int_{\mathbb{R}} \widetilde{G}_t(\widetilde{x}, u) \widetilde{M}_\alpha(\mathrm{d}\widetilde{x}, \mathrm{d}u). \tag{6.7}$$

The process on the right-hand side of (6.7) has the distribution of X_α^L by the definition of X_α^L and the uniqueness result in Theorem 3.1.

To show the equality (6.6), observe that by Lemma 5.2 and Lemma 6.1, we have $C_F \subset C_P \subset C$. Since $C_P = C_F + C_L$, the sets C_F, C_L and $C \setminus C_P$ are disjoint, and $C_F + C_L + C \setminus C_P = C$. Hence, the processes on the right-hand side of (6.3)–(6.6) are independent. Since the processes on the left-hand side of (6.3)–(6.6) are also independent, since the sum of the processes on the left-hand side of (6.3)–(6.6) has the same distribution as the sum of the processes on the right-hand side of (6.3)–(6.6), and since we already showed that the equalities (6.3)–(6.5) hold, we conclude that the equality (6.6) holds as well. □

The following lemma was used in the proof of Theorem 6.1.

Lemma 6.1. *We have*

$$C_P \subset C, \tag{6.8}$$

where C_P is the PFSM set (4.1) *and C is defined by* (6.2).

Proof. If $x \in C_P$, then by (2.3) and (4.1),

$$\begin{aligned} G_{rc}(x, rcu) &= G\big(x, rc(1+u)\big) - G(x, rcu) \\ &= b\big(G\big(x, c(1+u)+a\big) - G(x, cu+a)\big) = bG_c(x, cu+a) \quad \text{a.e. } \mathrm{d}u, \end{aligned}$$

for any $c > 0$ and some $r = r(x) \neq 1$, $b = b(x) \neq 0$ and $a = a(x)$. Suppose without loss of generality that $r = r(x) > 1$. Then, by making changes of variables c to rc and u to $u - c^{-1}a$, we obtain that, for any $n \in \mathbb{Z}$,

$$\int_{r^n}^{r^{n+1}} \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u\, c^{-H\alpha} \big|G_c(x, cu)\big|^\alpha = r^{1-H\alpha} |b|^\alpha \int_{r^{n-1}}^{r^n} \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u\, c^{-H\alpha} \big|G_c(x, cu)\big|^\alpha$$

and hence

$$\int_{r^n}^{r^{n+1}} \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u\, c^{-H\alpha} \big|G_c(x, cu)\big|^\alpha = r^{(1-H\alpha)n} |b|^{\alpha n} \int_1^r \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u\, c^{-H\alpha} \big|G_c(x, cu)\big|^\alpha.$$

This yields that, for $x \in C_P$,

$$\begin{aligned} \int_0^\infty \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u\, c^{-H\alpha} \big|G_c(x, cu)\big|^\alpha &= \sum_{n=-\infty}^{\infty} \int_{r(x)^n}^{r(x)^{n+1}} \mathrm{d}c \int_{\mathbb{R}} \mathrm{d}u\, c^{-H\alpha} \big|G_c(x, cu)\big|^\alpha \\ &= \int_1^{r(x)} \mathrm{d}c \int_{\mathbb{R}} c^{-H\alpha} \big|G_c(x, cu)\big|^\alpha \,\mathrm{d}u \sum_{n=-\infty}^{\infty} r(x)^{(1-H\alpha)n} \big|b(x)\big|^{n\alpha} = \infty, \end{aligned}$$

since $\sum_{n=-\infty}^{0} r^{(1-H\alpha)n} |b|^{n\alpha} + \sum_{n=1}^{\infty} r^{(1-H\alpha)n} |b|^{n\alpha} = \infty$, which shows that $x \in C$. □

The following theorem is essentially a reformulation of Theorem 6.1 and some other previous results. It provides a decomposition of self-similar mixed moving averages which is more refined than those established in [6,7]. As in Section 3, we will say that a decomposition of a process X_α obtained from its representation (2.1) is unique in distribution if the distribution of its components does not depend on the representation (2.1). We will also say that a process does not have a PFSM component if it cannot be expressed as the sum of two independent processes where one process is a PFSM.

Theorem 6.2. *Let X_α be a $S\alpha S$, $\alpha \in (1,2)$, self-similar mixed moving average given by a possibly nonminimal representation* (2.1). *Then, the process X_α can be decomposed uniquely in distribution into four independent processes*

$$X_\alpha \stackrel{d}{=} X_\alpha^D + X_\alpha^F + X_\alpha^L + X_\alpha^{C\setminus P}, \tag{6.9}$$

where

$$X_\alpha^D(t) = \int_D \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.10}$$

$$X_\alpha^F(t) = \int_{C_F} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.11}$$

$$X_\alpha^L(t) = \int_{C_L} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.12}$$

$$X_\alpha^{C\setminus P}(t) = \int_{C\setminus C_P} \int_{\mathbb{R}} G_t(x,u) M_\alpha(\mathrm{d}x, \mathrm{d}u), \tag{6.13}$$

and the sets D, C, C_F, C_P *and* C_L *are defined by* (6.1), (6.2), (5.1), (5.2), (4.1) *and* (5.5), *respectively. Here*:

(i) *The process* X_α^D *has the canonical representation given in Theorem* 4.1 *of* [7], *and is generated by a dissipative flow.*

(ii) *The process* X_α^F *is a mixed LFSM and has the representation* (2.14).

(iii) *The process* X_α^L *is a CFSM, and the sum* $X_\alpha^P = X_\alpha^F + X_\alpha^L$ *is a PFSM.*

(iv) *The process* $X_\alpha^{C\setminus P}$ *is a self-similar mixed moving average without a PFSM component.*

If the process X_α *is generated by a flow* $\{\psi_c\}_{c>0}$ *then the sets* D *and* C *are identical* (*a.e.*) *to the dissipative and the conservative parts of the flow* $\{\psi_c\}_{c>0}$.

If, in addition, the representation of the process X_α *is minimal, then the sets* C_P, C_F *and* C_L *are the sets of the periodic, fixed and cyclic points of the flow* $\{\psi_c\}_{c>0}$, *respectively.*

Remark. *It is important to distinguish* (6.10)–(6.13) *from* (6.3)–(6.6). *Because of the relations* (6.4)–(6.6), *the processes* X_α^F, X_α^L *and* $X_\alpha^{C\setminus P}$ *defined through* (6.11)–(6.13) *are equal in finite-dimensional distributions with the corresponding processes* X_α^F, X_α^L *and* $X_\alpha^{C\setminus P}$ *defined through* (3.2). *They are not identical to them because we are integrating here with respect to the sets* C_F, C_L *and* $C \setminus C_P$ *which are defined in terms of the kernel* G *whereas in the integration in* (3.2), *one is integrating with respect to the sets* F, L *and* $C \setminus P$ *which are defined in terms of the flow* $\{\psi_c\}_{c>0}$. *We use the same notation for convenience. The abuse is small because one has equality in distribution and because* $C_F = F$, $C_L = L$ *and* $C_P = P$ *when working with minimal representations. In the case of the process* X_α^D *defined through* (6.3), *the notation is consistent because* D, *defined by* (6.1) *in terms of the kernel function* G, *is equal to the set of dissipative points of the flow* $\{\psi_c\}_{c>0}$ *for arbitrary, not necessarily minimal, representations* (*see Corollary* 5.2 *in* [6]).

Proof. The uniqueness of the decomposition (6.9) into four independent components follows by using Theorem 6.1 and the uniqueness result in Theorem 3.1. Parts (i) and (ii) follow from Theorem 9.1 in [6]. Part (iii) is a consequence of the equalities (6.4) and (6.5) in Theorem 6.1 and Definition 3.1. To show that the process $X_\alpha^{C\setminus P}$ does not have a PFSM component, we argue by contradiction. Suppose on the contrary that $X_\alpha^{C\setminus P}$ has a PFSM component, that is,

$$X_\alpha^{C\setminus P} \stackrel{d}{=} V + W,$$

where V and W are independent, and W is a PFSM. Let

$$G^{C\setminus P}\colon (C \setminus P) \times \mathbb{R} \mapsto \mathbb{R} \quad \text{and} \quad F : Y \times \mathbb{R} \mapsto \mathbb{R}$$

be the kernel functions in the representation of $X_\alpha^{C\setminus P}$ and W, respectively, where the integral representation of W is equipped with the control measure $\sigma(\mathrm{d}y)\,\mathrm{d}u$. By using Theorem 5.2 in [6], there are functions

$$\Phi_1 : Y \mapsto C \setminus C_P, \qquad h : Y \mapsto \mathbb{R} \setminus \{0\} \quad \text{and} \quad \Phi_2, \Phi_3 : Y \mapsto \mathbb{R}$$

such that

$$F(y,u) = h(y)G^{C\setminus P}\big(\Phi_1(y), u + \Phi_2(y)\big) + \Phi_3(y) \quad \text{a.e. } \sigma(\mathrm{d}y)\,\mathrm{d}u \tag{6.14}$$

or

$$G^{C\setminus P}\big(\Phi_1(y), u\big) = \big(h(y)\big)^{-1}F\big(y, u - \Phi_2(y)\big) - \big(h(y)\big)^{-1}\Phi_3(y) \quad \text{a.e. } \sigma(\mathrm{d}y)\,\mathrm{d}u. \tag{6.15}$$

Since F is the kernel function of a PFSM, it satisfies

$$F\big(y, c(y)u\big) = b(y)F\big(y, u + a(y)\big) + d(y) \quad \text{a.e. } \sigma(\mathrm{d}y)\,\mathrm{d}u, \tag{6.16}$$

for some $c(y) > 0$ ($c(y) \neq 1$), $b(y) \neq 0$, $a(y), d(y) \in \mathbb{R}$. Then, by replacing u by $c(y)u$ in (6.15) and by using (6.16) and (6.14), we get that

$$G^{C\setminus P}\big(\Phi_1(y), c(y)u\big) = B(y)G^{C\setminus P}\big(\Phi_1(y), u + A(y)\big) + D(y) \quad \text{a.e. } \sigma(\mathrm{d}y)\,\mathrm{d}u, \tag{6.17}$$

for some $B(y) \neq 0$, $A(y), D(y) \in \mathbb{R}$. Since $\sigma(\mathrm{d}y)$ is not a zero measure, relation (6.17) contradicts the fact that $\Phi_1(y) \in C \setminus C_P$ in view of the definitions of the set C_P.

The last two statements of the theorem follow from the proof of Theorem 5.3 in [6], Theorem 10.1 in [7] and Propositions 4.2 and 5.2. □

7. Example of a process of the "fourth" kind

We provide here examples of the "fourth" kind processes $X_\alpha^{C\setminus P}$ in the decomposition (6.9) which are related to $S\alpha S$ sub-Gaussian, more generally, sub-stable processes.

Let $\{W(t)\}_{t\in\mathbb{R}}$ be a stationary process which has càdlàg (that is, right continuous and with limits from the left) paths, satisfies $E|W(t)|^\alpha < \infty$,

$$E\big|W(t) - W(s)\big|^\alpha \le C|t-s|^{2p}, \quad s, t \in \mathbb{R}, \tag{7.1}$$

for some $p > 0$, $P(|W(t)| < c) < 1$ for all $c > 0$ and is ergodic. Let $\Omega = \{w\colon\ w(t),\ t \in \mathbb{R},\ \text{is càdlàg}\}$ be the space of càdlàg functions on $\mathbb{R}$. It is a complete metric space and hence a Lebesgue space. Let $P(\mathrm{d}w)$ be the probability measure corresponding to the process W.

Consider now the $S\alpha S$ stationary process

$$Y_\alpha^{(1)}(t) = \int_\Omega F(w,t)M_\alpha(\mathrm{d}w),$$

where $F(w,t) = w(t)$ and $M_\alpha(\mathrm{d}w)$ has the control measure $P(\mathrm{d}w)$. The process $Y_\alpha^{(1)}$ is well defined since $E|W(t)|^\alpha < \infty$. When the probability measure P corresponds to a Gaussian, more generally stable process, the process $Y_\alpha^{(1)}$ is called sub-Gaussian, more generally sub-stable (see [14]). The Lamperti transformation of the process $Y_\alpha^{(1)}$ leads to a $S\alpha S$ self-similar process

$$Y_\alpha^{(2)}(t) = \int_\Omega |t|^H F\big(w, \ln|t|\big)M_\alpha(\mathrm{d}w).$$

The process $Y_\alpha^{(2)}$ does not have stationary increments. We can transform it to a process with stationary increments by the following procedure. Let

$$Y_\alpha^{(3)}(t) = \int_\Omega\int_{\mathbb{R}} |t+u|^H F\big(w, \ln|t+u|\big)M_\alpha(\mathrm{d}w, \mathrm{d}u),$$

where $M_\alpha(\mathrm{d}w, \mathrm{d}u)$ has the control measure $P(\mathrm{d}w)\,\mathrm{d}u$. The process $Y_\alpha^{(3)}$ is self-similar and also stationary (in the sense of generalized processes). We can transform it to a self-similar stationary increments process through the usual "infrared correction" transformation $Y_\alpha^{(3)}(t) - Y_\alpha^{(3)}(0)$, that is,

$$X_\alpha(t) = \int_\Omega \int_{\mathbb{R}} \big(|t+u|^H F\big(w, \ln|t+u|\big) - |u|^H F\big(w, \ln|u|\big)\big) M_\alpha(\mathrm{d}w, \mathrm{d}u). \tag{7.2}$$

Observe that the process X_α is a self-similar mixed moving average by construction. By Lemma 7.1, it is well defined when $H < \min\{p, 1\}$. Moreover, the process X_α is generated by a conservative flow. Indeed, by setting $G(w, u) = |u|^\kappa F(w, \ln|u|)$, we have $c^{-\kappa} G(w, cu) = G(\psi_c(w), u)$, $c > 0$, where

$$\psi_z : w(z), \quad z \in \mathbb{R} \mapsto w(z + \ln c), z \in \mathbb{R},$$

is a measurable flow on Ω. Since the process $W(t)$, $t \in \mathbb{R}$, is stationary, the flow $\{\psi_c\}_{c>0}$ is measure preserving. It is conservative because the measure P on Ω is finite and therefore there can be no wandering set of positive measure. By Lemma 7.2, the PFSM set C_P associated with the kernel in the representation (7.2) is empty a.e. Hence, in view of Theorem 6.2, the process X_α is an example of the "fourth" kind process $X_\alpha^{C\setminus P}$ in the decomposition (6.9). We state this result in the following theorem.

Theorem 7.1. *The process X_α defined by* (7.2) *under the assumptions stated above, is an example of the process $X_\alpha^{C\setminus P}$ in the decomposition* (6.9).

The following auxiliary lemma shows that the process X_α in (7.2) is well defined.

Lemma 7.1. *The process X_α in* (7.2) *is well defined for $H \in (0, \min\{p, 1\})$ and $\alpha \in (0, 2)$ under the assumption* (7.1).

Proof. The result follows since, by using (7.1) and stationarity of W,

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}} \big||t+u|^\kappa F\big(w, \ln|t+u|\big) - |u|^\kappa F\big(w, \ln|u|\big)\big|^\alpha P(\mathrm{d}w)\,\mathrm{d}u \\
&= \int_{\mathbb{R}} E\big||t+u|^\kappa W\big(\ln|t+u|\big) - |u|^\kappa W\big(\ln|u|\big)\big|^\alpha \mathrm{d}u \\
&\le 2^\alpha \int_{\mathbb{R}} |t+u|^{\kappa\alpha} E\big|W\big(\ln|t+u|\big) - W\big(\ln|u|\big)\big|^\alpha \mathrm{d}u + 2^\alpha \int_{\mathbb{R}} E\big|W\big(\ln|u|\big)\big|^\alpha \big||t+u|^\kappa - |u|^\kappa\big|^\alpha \mathrm{d}u \\
&\le 2^\alpha C \int_{\mathbb{R}} |t+u|^{\kappa\alpha} \big|\ln|t+u| - \ln|u|\big|^{p\alpha} \mathrm{d}u + 2^\alpha C \int_{\mathbb{R}} \big||t+u|^\kappa - |u|^\kappa\big|^\alpha \mathrm{d}u < \infty,
\end{aligned}$$

when $\kappa\alpha - p\alpha + 1 = (H - 1/\alpha)\alpha - p\alpha + 1 = \alpha(H - p) < 0$ and $H < 1$. □

The following lemma was used to show that the process X_α defined by (7.2) does not have a PFSM component.

Lemma 7.2. *If C_P is the PFSM set* (4.1) *associated with the representation* (7.2) *of the process X_α, then $C_P = \emptyset$ a.e. $P(\mathrm{d}w)$.*

Proof. By the definition of the set C_P in (4.1), we have

$$C_P = \big\{w \in \Omega\colon \exists c \neq 1, a, b \neq 0, d\colon |cu|^\kappa w\big(\ln|cu|\big) = b|u+a|^\kappa w\big(\ln|u+a|\big) + d,\ \forall u\big\}, \tag{7.3}$$

where the "a.e. $\mathrm{d}u$" condition in (4.1) was replaced by the "$\forall u$" condition because the functions w are càdlàg. We may suppose without loss of generality that $c > 1$ in (7.3). (If $c < 1$, by making the change of variables $u + a = c^{-1}v$ and dividing both sides of the relation in (7.3) by b, we obtain the relation analogous to (7.3) where c is replaced by c^{-1}.) We shall consider the cases $\kappa > 0$ and $\kappa \le 0$ separately.

The case $\kappa > 0$: We first examine the case when $b \neq 1$ in (7.3). By using (4.3) in Proposition 4.1, we can express the equation in (7.3) as

$$|cu+g|^{\kappa} w\big(\ln|cu+g|\big) + f = b\big(|u+g|^{\kappa} w\big(\ln|u+g|\big) + f\big), \tag{7.4}$$

for some $c > 1$, $b \neq 0$, $f, g \in \mathbb{R}$. Setting

$$\widetilde{w}(v) = \mathrm{e}^{-\kappa v}\big(\big|\mathrm{e}^{v}+g\big|^{\kappa} w\big(\ln\big|\mathrm{e}^{v}+g\big|\big) + f\big) \tag{7.5}$$

and $\tilde{c} = \ln c > 0$, we have from (7.4) with $u = \mathrm{e}^{v}$ that

$$\widetilde{w}(v+\tilde{c}) = \widetilde{b}\widetilde{w}(v), \quad v \in \mathbb{R}, \tag{7.6}$$

where $\widetilde{b} = bc^{\kappa}$. Observe also that, by making the change of variables $v = \ln(\mathrm{e}^{u} - g)$ in (7.5) for large v, we have

$$w(u) = \mathrm{e}^{-\kappa u}\big(\big(\mathrm{e}^{u}-g\big)^{\kappa} \widetilde{w}\big(\ln\big(\mathrm{e}^{u}-g\big)\big) - f\big) \tag{7.7}$$

for large u.

If $|\widetilde{b}| \leq 1$ in (7.6), then $|\widetilde{w}(v)|$ is bounded for large v. Indeed, if $|\widetilde{b}| = 1$, then $|\widetilde{w}(v)|$ is periodic with period $\tilde{c}$ and, being càdlàg, it has to be bounded. If $|\widetilde{b}| < 1$, then $|\widetilde{w}(v)| \to 0$ as $v \to \infty$ because $|\widetilde{w}(v+n\tilde{c})| = |\widetilde{b}|^{n}|\widetilde{w}(v)|$ and $|\widetilde{b}|^{n} \to 0$ as $n \to \infty$. By (7.7), since $\kappa > 0$, we obtain that $|w(u)|$ is bounded for large u as well. By Lemma 7.3, (i), the P-probability of such w is zero.

Suppose now that $|\widetilde{b}| > 1$ in (7.6). We have either (i) $\widetilde{w}(v) = 0$ for $v \in [0, \tilde{c}]$, or (ii) $\inf\{|\widetilde{w}(v)|\colon v \in A\} > 0$ for $A \subset [0, \tilde{c}]$ of positive Lebesgue measure. In the case (i), (7.6) implies that $\widetilde{w}(v) = 0$ for all v and hence, by (7.7), $w(u) = -f\mathrm{e}^{-\kappa u}$ for large u. By Lemma 7.3, (i), the P-probability of such w is zero. Consider now the case (ii). Since $|\widetilde{b}| > 1$, we get that

$$\inf\big\{\big|\widetilde{w}(v)\big|\colon v \in A + n\tilde{c}\big\} \to \infty \quad \text{as } n \to \infty.$$

Using (7.5), since $\kappa > 0$ (and hence $f\mathrm{e}^{-\kappa u} \to 0$ as $u \to \infty$), this yields that

$$\inf\big\{\big|w\big(\ln\big(\mathrm{e}^{v}+g\big)\big)\big|\colon v \in A + n\tilde{c}\big\} \to \infty \quad \text{as } n \to \infty.$$

By Lemma 7.3, (iii), the P-probability of such w is zero.

If $b = 1$ in (7.3), by using (4.2) in Proposition 4.1, we get

$$|cu+g|^{\kappa} w\big(\ln|cu+g|\big) = |u+g|^{\kappa} w\big(\ln|u+g|\big) + d, \tag{7.8}$$

for some $c > 1$, $g, d \in \mathbb{R}$. Setting

$$\widetilde{w}(v) = \big|\mathrm{e}^{v}+g\big|^{\kappa} w\big(\ln\big|\mathrm{e}^{v}+g\big|\big) \tag{7.9}$$

and $\tilde{c} = \ln c > 0$, we deduce from (7.8) with $u = \mathrm{e}^{v}$ that

$$\widetilde{w}(v+\tilde{c}) = \widetilde{w}(v) + d, \quad v \in \mathbb{R}. \tag{7.10}$$

The function $\widetilde{w}$ is bounded on $[0, \tilde{c}]$ since it is càdlàg and in view of (7.10), we get

$$\big|\widetilde{w}(v)\big| \leq C|v|, \tag{7.11}$$

for large v and some constant $C = C(w) > 0$. Substituting (7.9) into (7.11), and since $\kappa > 0$, we get that $w(v) \to 0$ as $v \to \infty$. By Lemma 7.3, (i), the P-probability of such w is zero. Combining this with the analogous conclusion when $b \neq 1$ above, we deduce that $C_P = \emptyset$ a.e. $P(\mathrm{d}w)$ when $\kappa > 0$.

The case $\kappa \leq 0$: By using (4.2) in Proposition 4.1, we express the equation in (7.3) as

$$|cu+g|^{\kappa} w\big(\ln|cu+g|\big) = b|u+g|^{\kappa} w\big(\ln|u+g|\big) + d, \tag{7.12}$$

for some $c > 1, b \neq 0, d, g \in \mathbb{R}$. When $d = 0$, we can use here the argument in the case $\kappa > 0$ because the assumption $\kappa > 0$ was used above only to ensure that the term $\mathrm{e}^{-\kappa v} f$ in (7.5) is negligible for large v. Suppose then $d \neq 0$. We can rewrite (7.12) as

$$\widetilde{w}(v + \tilde{c}) = b\widetilde{w}(v) + d, \quad v \in \mathbb{R}, \tag{7.13}$$

where $\tilde{c} = \ln c > 0$ and

$$\widetilde{w}(v) = \left|\mathrm{e}^v + g\right|^\kappa w\left(\ln\left|\mathrm{e}^v + g\right|\right), \quad v \in \mathbb{R}. \tag{7.14}$$

It follows from (7.13) that

$$\widetilde{w}(v + n\tilde{c}) = \begin{cases} b^n \widetilde{w}(v) + d\frac{b^n - 1}{b-1}, & \text{if } b \neq 1, \\ \widetilde{w}(v) + dn, & \text{if } b = 1. \end{cases} \tag{7.15}$$

Observe also that, by making the change of variable $v = \ln(\mathrm{e}^u - g)$ in (7.14) for large v, we get

$$w(u) = \mathrm{e}^{-\kappa u} \widetilde{w}\left(\ln\left(\mathrm{e}^u - g\right)\right), \tag{7.16}$$

for large u. We now consider separately the cases $|b| < 1$, $|b| > 1$, $b = 1$ and $b = -1$.

(a) Consider first the case $|b| < 1$. By using (7.15), we have:

$$\sup_{v \in [0,\tilde{c})} \left|\widetilde{w}(v + n\tilde{c}) + \frac{d}{b-1}\right| = |b|^n \sup_{v \in [0,\tilde{c})} \left|\widetilde{w}(v) + \frac{d}{b-1}\right| \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\sup_{v \in [n\tilde{c},(n+1)\tilde{c})} \left|\widetilde{w}(v) + \frac{d}{b-1}\right| \to 0 \quad \text{as } n \to \infty,$$

and

$$\widetilde{w}(v) \to -\frac{d}{b-1} \quad \text{as } v \to \infty, \tag{7.17}$$

or equivalently, by using (7.16),

$$\mathrm{e}^{\kappa u} w(u) \to -\frac{d}{b-1} \neq 0 \quad \text{as } u \to \infty. \tag{7.18}$$

When $\kappa < 0$, relation (7.18) implies that $|w(u)| \geq \varepsilon \mathrm{e}^{-\kappa u}$ for large u and some constant $\varepsilon > 0$, that is, $|w(u)|$ is unbounded for large u. When $\kappa = 0$, we get that $|w(u)|$ is bounded for large u. By Lemma 7.3, (i) and (ii), the P-probability of such w in either case and hence those w satisfying (7.12) with $|b| < 1$ is zero.

(b) Consider now the case $|b| > 1$. Relation (7.15) can be expressed as

$$\widetilde{w}(v + n\tilde{c}) + \frac{d}{b-1} = b^n\left(\widetilde{w}(v) + \frac{d}{b-1}\right). \tag{7.19}$$

We have either (i) $\widetilde{w}(v) = -d/(b-1)$ for all v, or (ii) there is a set $A \subset [0, \tilde{c}]$ of positive Lebesgue measure such that

$$\inf\left\{\left|\widetilde{w}(v) + \frac{d}{b-1}\right| \colon v \in A\right\} > 0.$$

In the case (i), by using (7.16), we get that $w(u) = -d\mathrm{e}^{-\kappa u}/(b-1)$ for large u. The P-probability of such w is zero by Lemma 7.3, (i) and (ii). In the case (ii), by using (7.19) and since $|b| > 1$, we have

$$\inf\left\{\left|\widetilde{w}(v) + \frac{d}{b-1}\right| \colon v \in A + n\tilde{c}\right\} \to \infty \quad \text{as } n \to \infty$$

or, in view of (7.14) and $\kappa \le 0$,

$$\inf\{|w(\ln(e^v + g))|: v \in A + n\tilde{c}\} \to \infty \quad \text{as } n \to \infty.$$

The P-probability of such w is zero by Lemma 7.3, (iii).

(c) When $b = -1$ in (7.13), we have $b^{2n} = 1$ and $b^{2n} - 1 = 0$, and hence relation (7.15) implies that

$$\widetilde{w}(v + n2\tilde{c}) = \widetilde{w}(v), \quad v \in \mathbb{R}. \tag{7.20}$$

Consider first the case $\kappa < 0$. We have either (i) $\widetilde{w}(v) = 0$ for all v, or (ii) there is a set $A \subset [0, 2\tilde{c}]$ of positive Lebesgue measure such that

$$\inf\{|\widetilde{w}(v)|: v \in A\} > 0. \tag{7.21}$$

Arguing as in part (i) above, the P-probability of w satisfying this part (i) is zero. In the case (ii), relations (7.20) and (7.21) imply that

$$\inf\{|\widetilde{w}(v)|: v \in A + n2\tilde{c}\} = \inf\{|\widetilde{w}(v)|: v \in A\} > 0.$$

By using (7.14), and since $\kappa < 0$,

$$\inf\{|w(\ln(e^v + g))|: v \in A + n\tilde{c}\} \to \infty \quad \text{as } n \to \infty.$$

The P-probability of such w is zero by Lemma 7.3, (iii). Turning to the case $\kappa = 0$, relation (7.20) shows that $|\widetilde{w}(v)|$ is periodic and hence bounded, since it is càdlàg. By using (7.16), since $\kappa = 0$, $|w(u)|$ is bounded for large u as well. By Lemma 7.3, (i), the P-probability of such w and hence of those w satisfying (7.12) with $b = -1$ is zero.

(d) When $b = 1$ in (7.13), relation (7.15) becomes $\widetilde{w}(v + n\tilde{c}) = \widetilde{w}(v) + dn$, $v \in \mathbb{R}$. Consider the cases (i) $\widetilde{w}(v) = 0$ for $v \in [0, \tilde{c}]$, and (ii) there is a set $A \subset [0, \tilde{c}]$ of positive Lebesgue measure such that $\inf\{|\widetilde{w}(v)|: v \in A\} > 0$. Arguing as above, the P-probability of w satisfying (i) is zero. In the case (ii), since $|d|n \to \infty$ as $n \to \infty$, we get that $\inf\{|\widetilde{w}(v)|: v \in A + n\tilde{c}\} \to \infty$ as $n \to \infty$. By using (7.14), we get again that

$$\inf\{|w(\ln(e^v + g))|: v \in A + n\tilde{c}\} \to \infty \quad \text{as } n \to \infty.$$

The P-probability of such w is zero by Lemma 7.3, (iii). Combining the results for $|b| < 1$, $|b| > 1$, $b = -1$ and $b = 1$, we conclude that the P-probability of w satisfying (7.12) is zero. In other words, $C_P = \emptyset$ a.e. $P(dw)$ when $\kappa \le 0$ as well. □

The next result was used in the proof of Lemma 7.2. Consider a function $w : \mathbb{R} \mapsto \mathbb{R}$. We say that the function $|w(u)|$, $u \in \mathbb{R}$, is *ultimately unbounded* if there is a set $A = A(w) \subset [0, C]$ of positive Lebesgue measure with a fixed constant C such that

$$\inf\{|w(u)|: u \in A + nC\} \to \infty \quad \text{as } n \to \infty.$$

We say that $|w(u)|$, $u \in \mathbb{R}$, is bounded for large u if there is $N = N(w)$ such that $|w(u)| \le N$ for large enough u. Denote

$$A_1 = \{w: |w(u)| \text{ is bounded for large } u\},$$
$$A_2 = \{w: |w(u)| \text{ is ultimately unbounded}\},$$
$$A_3 = \{w: |w(\ln(e^u + g))| \text{ is ultimately unbounded}\},$$

where $g = g(w) \in \mathbb{R}$.

Lemma 7.3. *Under the assumptions on the process W (and hence on the corresponding probability P) stated in the beginning of the section and with the sets A_1, A_2, A_3 defined above, we have*

(i) $P(A_1) = 0$, (ii) $P(A_2) = 0$ *and* (iii) $P(A_3) = 0$.

Proof. To show (i), observe that $P(A_1) \le \sum_{n=1}^{\infty} P(B_n)$, where $B_n = \{w\colon |w(u)| < n \text{ for large } u\}$. It is enough to show that $P(B_n) = 0$ for $n \ge 1$. When $w \in B_n$, we have

$$\frac{1}{T}\int_0^T 1_{\{|w(u)|<n\}}\,\mathrm{d}u \to 1,$$

as $T \to \infty$. But by ergodicity and the assumption $P(|w(0)| < c) < 1$ for any $c > 0$, we have

$$\frac{1}{T}\int_0^T 1_{\{|w(u)|<n\}}\,\mathrm{d}u \to P\bigl(\bigl|w(0)\bigr| < n\bigr) < 1 \quad \text{a.e. } P(\mathrm{d}w).$$

This implies that $P(B_n) = 0$.

We now show (ii). Let $w \in A_2$, and A and C be the set and the constant appearing in the definition of ultimate unboundedness of $|w|$. Observe that $\int_0^T |w(u)|^\alpha\,\mathrm{d}u = \sum_{k=1}^{K}\int_{(k-1)C}^{kC} |w(u)|^\alpha\,\mathrm{d}u$ when $T = KC$, and $\int_{(k-1)C}^{kC} |w(u)|^\alpha\,\mathrm{d}u \ge Leb(A)(\inf\{|w(u)|\colon u \in A + (k-1)C\})^\alpha \to \infty$ as $k \to \infty$, by the ultimate unboundedness of w. Then, for $w \in A_2$, we have

$$\frac{1}{T}\int_0^T \bigl|w(u)\bigr|^\alpha\,\mathrm{d}u \to \infty. \tag{7.22}$$

However, by ergodicity and the assumption $E|w(0)|^\alpha < \infty$, we have

$$\frac{1}{T}\int_0^T \bigl|w(u)\bigr|^\alpha\,\mathrm{d}u \to E\bigl|w(0)\bigr|^\alpha < \infty \quad \text{a.e. } P(\mathrm{d}w). \tag{7.23}$$

This implies that $P(A_2) = 0$.

Consider now part (iii). When $w \in A_3$ and $u = u_0$ is large enough, we have

$$\frac{1}{T}\int_{u_0}^T \bigl|w\bigl(\ln\bigl(\mathrm{e}^u + g\bigr)\bigr)\bigr|^\alpha\,\mathrm{d}u \to \infty.$$

Making the change of variables $\ln(\mathrm{e}^u + g) = v$, we obtain that

$$\frac{1}{T}\int_{\ln(\mathrm{e}^{u_0}+g)}^{\ln(\mathrm{e}^T+g)} \bigl|w(v)\bigr|^\alpha \frac{\mathrm{e}^v}{\mathrm{e}^v - g}\,\mathrm{d}v \to \infty.$$

It is easy to see that this implies (7.22) when $w \in A_3$. In view of (7.23), we get $P(A_3) = 0$. □

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