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ON THE CONVEX HULL OF PROJECTIVE PLANES *

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Abstract. We study the finite projective planes with linear programming models. We give a complete description of the convex hull of the finite projective planes of order 2. We give some integer linear programming models whose solution are, either a finite projective (or affine) plane of order n, or a $(n + 2)$-arc.

Keywords. Convex hull, finite projective plane.

Mathematics Subject Classification. 5299, 0599

1. Introduction

A finite projective plane P_n is defined by two sets of $n^2 + n + 1$ elements called respectively set of points of P_n and set of lines of P_n satisfying the following properties:

1. a line of P_n is a subset of $n + 1$ points of P_n;
2. each point of P_n belongs to $n + 1$ lines of P_n;
3. any pair of points of P_n belong to exactly one line of P_n;
4. there exist four points of P_n such that no three of them belong to the same line of P_n.

A finite projective plane of order n is a symmetric balanced incomplete block design $(n^2 + n + 1, n + 1, 1)$. The smallest P_n has order 2 and it is the unique Steiner triple system of order 7.

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For any finite field of order n a projective plane can be constructed. In the other cases, the Bruck-Ryser theorem [5] eliminates some orders, and the order 10 has been eliminated by Lam [11].

2. The polyhedron of the finite projective planes of order 2

Let T be the set of triples of the set $\{1, 2, 3, 4, 5, 6, 7\}$. For a finite projective plane each pair of points $\{u, v\}$ is contained in a line (triple) $t \in T$, thus we have: $\sum_{t\in T, t\supset\{u,v\}} x_t = 1$. These 21 equations are the unique equations satisfied by any characteristic vector $x \in \mathbb{R}^{35}$ of a finite projective plane P_2 of order 2.

We will identify a finite projective plane P_2, with its characteristic vector x defined on the set of triples. If the triple i belongs to the plane P_2, $x_i = 1$, else $x_i = 0$.

Let us note A the $\{0/1\}$-matrix whose entries are the coefficients of the above 21 equations, and $\mathbf{1} = (1, 1, \ldots, 1)$. The rows of A are linearly independent. We will show in the next section that any incidence matrix of the pairs belonging to the p-subsets, of a set with E $(E \geq p + 2)$ elements has a maximal rank.

Thus the finite projective planes of order 2 are the solutions of:

$$D = \{x \in \mathbb{R}^{35} | Ax = \mathbf{1}, x \in \{0, 1\}^{35}\}.$$

We will study $\mathcal{P} = conv(D)$. We call the space $\mathbb{R}^{35}$ the *natural* space of $\mathcal{P}$. Let $C \subset T$ be such that the matrix A_C is non-singular, the projection P' of $\mathcal{P}$ on $\mathbb{R}^{T\setminus C}$ is full-dimensionned, this space is **a** *proper* space of $\mathcal{P}$. If C do not contains a projective plane, the projection P' does not contain the origin of the system of ccordinates which implies that any projective plane of order 2 contains at least one line which is indexed by the set of columns $T \setminus C$.

Let $\mathcal{Q} \supset P$, the set of solutions of:

$$D' = \{x \in \mathbb{R}^{35} | Ax = \mathbf{1}, x \geq 0\}.$$

The integer vertices of $\mathcal{Q}$ are in bijection with the vertices of $\mathcal{P}$. To characterize the convex hull $\mathcal{P}$ we will use a software like cdd [3], lrs [2] or Porta [7], which can switch between the V-representation (convex hull of extreme points) and the H-representation (intersection of hyper-planes) of a polyhedron. However, dimension 35 seams to be intractable for these softwares so we need to use a proper space of $\mathcal{P}$, but we will try to go back to the natural space, in order to have a more significant representation of $\mathcal{P}$.

Let $M \subset A$, $M(21 \times 21)$ be an invertible matrix, the columns of which are indexed by $C \subset T$. Then we can write:

$$\mathcal{P} = \{x_C = M^{-1}\mathbf{1} - M^{-1}A_{T\setminus I}x_{T\setminus C}, \quad x = (x_C, x_{T\setminus C}) \in \{0, 1\}^{35}\}. \qquad (1)$$

There are 30 projective planes of order 2, who are easy to enumerate. We can give to the software either the 30 projections of the vertices of $\mathcal{P}$, or the 35 inequalities defining the polyhedron $\mathcal{Q}' \supset \mathcal{Q}$:

$$\mathcal{Q}' = \{M^{-1}A_{T\setminus C}x_{T\setminus C} \leq M^{-1}\mathbf{1}, \quad x_{T\setminus C} \geq 0\}.$$

In this last case we will have to choose the (30) integer vertices that we will again give to our software in order to get the representation of $\mathcal{P}'$ the projection of $\mathcal{P}$ in this $\mathbb{R}^{14}$. One of the interest of this last method that we dont need to enumerate the projective planes, the H-representation software will do the enumeration.

We can observe that there are 155 facet-defining inequalities which split into 2 classes:

(C1) 35 inequalities *tight* (satisfied at equality) for 24 planes. They correspond to the non negativity constrains.
(C2) 120 inequalities tight for 14 planes.

Using (1) we can reconstruct $\mathcal{P}$ in its natural space. However, the representation in the natural space is not unique. However we have the following nice property:

Remark 1. Any of the 120 inequalities which represent the convex hull of the finite projective planes of order 2, can be written in $\mathbb{R}^{35}$ like: $\sum_{t\in Y} x_t \geq 1$, where $Y \subset T$ is the set of triples of any two disjoint projective planes of order 2.

In this way we can represent all facets of $\mathcal{P}$. Let us notice that:

Remark 2. There are no more than 2 disjoint projective planes of order 2.

The result claimed in Remark 1 was obtained after an analysis of the (reconstructed) characteristic tight vectors of a given facet. We have thus observed that there were 14 triples containing exactly one triple of the tight characteristic tight vectors.

This result looks mainly as a negative one. It expresses the convex hull of the finite projective planes of order 2 in terms of the finite projective planes of order 2. If he is generalized, he will say nothing on the existence of such a plane. Of course, solving the corresponding integer programming model will give such an answer.

3. The linear variety of finite projective planes of order n

Let C be the set of the $(n+1)$-uples of a set of n^2+n+1 elements, and L the set of pairs of the same set. Let A_{LC} be the incidence matrix of the pairs belonging to each $(n+1)$ subset, thus $A_{lc} = 1$ if the pair l belongs to the n-uple c, else $A_{lc} = 0$. The projective planes of order n, if such planes exist, are represented by the integer solutions of the following system:

$$P_n = \{x_C \in \mathbb{R}^C | A_{LC}x_C = \mathbf{1}\}. \tag{2}$$

We will show that A_{LC} has maximal rank, and more generally, if $M = \{1, \dots, m\}$ is a set with m elements, and B is the incidence matrix of the pairs of M in the p-uples of M, then we have:

Proposition 1. *The rank of B is C_m^2, the number of its rows.*

Proof. Let us consider first the case when $m = p + 2$. In this case we can index the p-uple $c \subset M$ by the pair $c' = M \setminus c$. Thus B_{LL} is a square matrix with rows and columns indexed by the same set L of pairs of M. This matrix is symmetric. B_{ij}, the element indexed in row by the pair i and in column by the pair j which corresponds to the p-uple $M \setminus j$, equal 1 if the pairs i and j are disjoint and 0 else. Let call J_{LL} the matrix with all entries equal to 1 and I_{LL} the identity matrix. Let n_0 be the number of common elements of two rows of B_{LL} indexed by two disjoint pairs, n_1 the one of two rows indexed by two pairs having one element in common and n_2 the number of elements of one row. We have:

$$\begin{aligned} B_{LL}^2 &= (n_2 - n_1)I_{LL} + n_1 J_{LL} + (n_1 - n_0)B_{LL}, \\ B_{LL}J_{LL} &= n_0 J_{LL}, \\ B_{LL}I_{LL} &= B_{LL}. \end{aligned} \tag{3}$$

Expressing the inverse B'_{LL} of B_{LL} as a linear combination of these three matrices, $B'_{LL} = \alpha B_{LL} + \beta J_{LL} + \gamma I_{LL}$, we can calculate the three coefficients α, β and γ, proving thus the existence of B'_{LL}.

We will now study the case when $m > p + 2$ with a proof in the spirit of the results given in this paper.

For doing so let us consider the transposed matrix B_{LC}^t of B_{LC}. This matrix can be seen as the edge-incidence matrix of the p-cliques of the complete graph K_m. The rows of this matrix are the characteristic vectors of the p-cliques, these characteristic vectors satisfy obviously the equality $\sum_{l \in L} x_l = \frac{p(p-1)}{2}$. We will show that there is no other equality satisfied by all the cliques.

Suppose that it is not true and that all the cliques satisfy also:

$$\sum_{l \in L} \alpha_l x_l = \beta.$$

We can choose a set V of $k+2$ vertices from K_m and then we consider the k-cliques C with vertices in V; let E be the edge set of C. We note B_{LL} the corresponding edge-incidence matrix and as before B_{LL} is invertible, thus the coefficients α_e for $e \in E$ are all equal to $\frac{\beta}{|E|}$. Using the equation, we can fixe one α_e to 0, thus all the α_e are equal to 0 and also β. Any procedure which allows us to cover the edge set of K_m by an ordered sequence $C_1, C_2, \ldots$ of k-cliques, beginning by C, such that for $i \geq 1$, C_i and C_{i+1} have an edge (at least one) in common, allows to show that the coefficients α_e of the edges of each consecutive k-clique are also zero. This shows that such an equation doesn't exist.

We can also proof this result, in the general case, as in the first part. We can consider, for $n > k + 2$, the product $B_{LC}B_{LC}^t$ instead of B_{LL}, where B_{LC}^t is the transposed matrix of B_{LC}. □

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References

[1] O. Anglada and J.F. Maurras, Enveloppe convexe des hyperplans d'un espace affine fini, avec Olivier Anglada. *RAIRO-Oper. Res.* **37** (2003) 213–219.
[2] D. Avis, http://cgm.cs.mcgill.ca/ avis/C/lrs.html.
[3] D. Avis and K. Fukuda, A pivoting algorithm for convex hulls and vertex enumeration of arrankements and polyhedra. *Discrete Comput. Geom.* **8** (1992) 295–313.
[4] I. Bárány and Pór, 0-1 polytopes with many facets, *Adv. Math.* **161** (2001) 209–228.
[5] R.H. Bruck and H.J. Ryser, The nonexistence of certain finite projective planes. *Can. J. Math.* **1** (1949) 88–93.
[6] F.C. Bussemaker and J.J. Seidel, *Symmetric Hadamard matrices of order 36*. Report 70-WSK-02, TH Eindhoven, July (1970).
[7] T. Christof, `www.zib.de/Optimization/Software/porta`.
[8] K. Fukuda, `http://cs.mcgill.ca/ fukuda/soft/cdd`.
[9] P.B. Gibbons, *Computing Techniques for the Construction and Analysis of Block Designs,* Techn. Report N° 92, Dept. of Computer Science, University of Toronto (1976).
[10] T.R. Kirkman, On a problem in combinations. *Camb. Dublin Math. J.* **2** (1847) 191–204.
[11] C.W.H. Lam, The Search for a Finite Projective Plane of Order 10. *Am. Math. Mon.* **98** (1991) 305–318.
[12] M. Limbos, Projective embeddings of small Steiner triple systems. *Ann. Discrete Math.* **7** (1980) 151–173.
[13] R.A Mathon, K.T. Phelps and A. Rosa, Small Steiner triple systems and their properties. *Ars Combinatoria* **15** (1983) 3–110.
[14] J.F. Maurras, An exemple of dual polytopes in the unit hypercube. *Ann. Discrete Math.* **1** (1977) 391–392.
[15] J.F. Maurras, The Line Polytope of a finite Affine Plane. *Discrete Math.* **115** (1993) 283–286.
[16] T.S. Motzkin, H. Raiffa, G.L. Thompson and R.M. Thrall, The double description method, in H.W. Kuhn and A.W. Tucker, Eds., *Contributions to theory of games*, Vol. **2**, Princeton University Press, Princeton (1953).
[17] H.S. White, F.N. Cole and L.D. Cummings, Complete classification of the triad systems on fifteen elements. *Mem. Nat. Acad. Sci. U.S.A.* **14**, 2nd memoir (1919) 1–89.