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A NUMERICAL FEASIBLE INTERIOR POINT METHOD FOR LINEAR SEMIDEFINITE PROGRAMS

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Abstract. This paper presents a feasible primal algorithm for linear semidefinite programming. The algorithm starts with a strictly feasible solution, but in case where no such a solution is known, an application of the algorithm to an associate problem allows to obtain one. Finally, we present some numerical experiments which show that the algorithm works properly.

Keywords. Linear programming, semidefinite programming, interior point methods.

Mathematics Subject Classification. 90C51, 90C22, 90C05.

1. Introduction

The aim of this paper is to present a feasible interior point method for the linear semidefinite program:

$$z^* = \min_X \left[\langle C, X\rangle \ : \ X \in K,\ \langle A_i, X\rangle = b_i \ \text{ for } i = 1,\dots,m \right]. \qquad (SDP)$$

Here $b \in \mathbb{R}^m$, K denotes the cone of positive semidefinite matrices in the linear space of $n \times n$ symmetric matrices E. The matrices C and $A_i,\ i = 1,\dots,m$, are given and belong to E. The inner product on E of two matrices A and B is the trace of their product, *i.e.*, $\langle A, B\rangle = \mathrm{tr}(AB) = \sum_{i,j} a_{ij} b_{ij}$. It is known that the interior of K, denoted by int(K), is the set of positive definite matrices of E.

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In references [5, 11], the reader will find a description of a few applications of linear semidefinite programming, in particular, max-cut problems in a graph, graph-bisection problems, the search of a largest clique in a graph, min-max eigenvalue problems.

Linear semidefinite programming presents a great similarity with linear programming: the objective function is linear, the constraint functions are affine, the difference consists in the positive cone. Indeed, testing the positive semidefiniteness of an $n \times n$ matrix X requires to check $\langle Xh, h\rangle \geq 0$ for all h with norm 1, an infinity of linear constraints, while testing that a vector belongs to the nonnegative orthant of an Euclidean space involves, by definition, a finite number of linear constraints. The duality schemes present also similarities, one main difference is that for the strong duality result in linear semidefinite programming the primal and the dual problems have to be strictly feasible instead of simply feasible in classical linear programming. Besides, on an algorithmical point of view, interior point methods used in linear programming can be easily extended to linear semidefinite programming. Most of the algorithms [2, 3, 5, 6] are extensions of path following or related methods in linear programming to SDP programming using a Newton descent direction.

Our algorithm is close to the projective algorithm of Alizadeh [1] (for more informations on projective methods, see [4, 7, 8]). As in the Alizadeh algorithm, the descent direction is obtained by the projection on a linear subspace, but in our presentation the computations appear to be simpler, in particular our algorithm does not make use of a potential function. This simplicity has a price since potential functions are commonly used in interior point methods to prove the theoretical convergence of algorithms. Still, the numerical experiments show the good behaviour of our algorithm.

As many interior point methods, the algorithm needs the knowledge of an initial strictly feasible solution. In case no such a solution is available, a first application of the algorithm allows to get one.

Now, we make precise the notation used in the paper. We have already defined the sets $E, K, \mathrm{int(K)}$ and the scalar product $\langle\ ,\ \rangle$ in E. The identity matrix of E is denoted by I. Given $A \in E$ its norm is

$$\|A\| = \sqrt{\langle A, A\rangle} = (\sum_{i=1}^{n} |\lambda_i|^2)^{\frac{1}{2}},$$

where $\lambda_i,\ i = 1, 2, \ldots, n$ are the eigenvalues of A.

2. The duality in semidefinite programming

Let us consider the problems:

$$z^* = \min_X [\ \langle C, X\rangle \quad : \quad X \in K,\ \langle A_i, X\rangle = b_i \ \text{ for } i = 1, \ldots, m] \qquad (SDP)$$

and

$$m_d = \max_w [\ b^t w \quad : \quad (C - \sum_{i=1}^{m} w_i A_i) \in K]. \qquad (DSDP)$$

A matrix X is said to be a *strictly feasible solution* of (SDP) if it belongs to int(K) and $\langle A_i, X\rangle = b_i$, for $i = 1, \ldots, m$, a vector $w \in \mathbb{R}^m$ is said to be a *strictly feasible solution* of $(DSDP)$ if the matrix $(C - \sum_{i=1}^{m} w_i A_i) \in$ int(K). The weak duality result says that we have always $m_d \leq z^*$. The strong duality result says that if both (SDP) and $(DSDP)$ have strictly feasible solutions, then $m_d = z^*$ and both problems have optimal solutions. Furthermore, for such optimal solutions X and w, the following complementarity slackness condition holds

$$\langle X, C - \sum_{i=1}^{m} w_i A_i\rangle = 0.$$

Also, the sets of optimal solution of (SDP) and $(DSDP)$ are closed convex and bounded.

More information on semidefinite programming and its duality can be found in references [9–11].

3. Description of the algorithm

Throughout the paper, we made the following assumptions.

(1) $C \neq 0$ and the constraints $\langle A_i, X\rangle = b_i$ for $i = 1, \ldots, m$ are not redundant.

(2) Both problems (SDP) and $(DSDP)$ have strictly feasible solutions.

In this section, we describe the passage from an iterate X_k to the next one X_{k+1}. The problem of finding an initial feasible solution X_0 will be considered in another section.

At the beginning of step k, the current X_k is a strictly feasible solution of (SDP). The Cholesky factorization of the positive definite matrix X_k gives a lower triangular matrix L_k with positive diagonal entries such that $L_k L_k^t = X_k$. Then, we define the projective transformation

$$T_k(X) = (Y, \alpha),$$

where

$$Y = \alpha L_k^{-1} X L_k^{-t} \quad , \quad \alpha = \frac{(n+1)}{1 + \langle X_k^{-1}, X\rangle}\cdot$$

The transformation T_k is one to one from K to $\widetilde{K}$ where

$$\widetilde{K} = \{(Y, \alpha) \in K \times (0, +\infty) \, : \, \langle I, Y\rangle + \alpha = n + 1\}.$$

The inverse transformation T_k^{-1} is such that

$$T_k^{-1}(Y,\alpha) = \frac{1}{\alpha} L_k Y L_k^t.$$

For simplicity, we introduce the matrices:

(1) $C_k = L_k^t C L_k$.
(2) $A_i^{(k)} = L_k^t A_i L_k$ for $i = 1, \ldots, m$.

Then, for $(Y,\alpha) = T_k(X)$, we have:

$$\langle C_k, Y\rangle - z^*\alpha = \alpha[\langle C, X\rangle - z^*].$$

Furthermore, the constraints $\langle A_i, X\rangle = b_i$ for $i = 1, \ldots, m$ and $X \in K$ are equivalent to the conditions:

$$\left\{ \begin{array}{l} \langle A_i^{(k)}, Y\rangle - \alpha b_i = 0, \; i = 1, \ldots, m, \\ \langle I, Y\rangle + \alpha = n+1, \\ Y \in K, \; \alpha \geq 0. \end{array} \right\}$$

It results that solving (SDP) is equivalent to solving the problem:

$$0 = \min_{(Y,\alpha)} \left[\langle C_k, Y\rangle - z^*\alpha \;:\; \begin{array}{l} \langle A_i^{(k)}, Y\rangle - \alpha b_i = 0, i = 1, \ldots, m \\ \langle I, Y\rangle + \alpha = n+1, \\ Y \in K, \; \alpha \geq 0. \end{array} \right] \qquad (E_k^*)$$

Notice that $(I,1)$ is a strictly feasible solution of problem (E_k^*). Because the value z^* is unknown, we consider its approximation

$$z_k = \langle C, X_k\rangle = \langle C_k, I\rangle.$$

Then $z_k > z^*$. Next, we consider the semidefinite program

$$m_k = \min_{(Y,\alpha)} \left[\langle C_k, Y\rangle - z_k\alpha \;:\; \begin{array}{l} \langle A_i^{(k)}, Y\rangle - \alpha b_i = 0, i = 1, \ldots, m \\ \langle I, Y\rangle + \alpha = n+1, \\ Y \in K, \; \alpha \geq 0. \end{array} \right] \qquad (E_k)$$

In a similar way to Karmarkar's method for classical linear programs, we relax problem (E_k) into the convex optimization problem

$$m_k(\beta) = \min_{(Y,\alpha)} \left[\langle C_k, Y\rangle - z_k\alpha \;:\; \begin{array}{l} \langle A_i^{(k)}, Y\rangle - \alpha b_i = 0, i = 1, \ldots, m \\ \langle I, Y\rangle + \alpha = n+1, \\ \|Y - I\|^2 + (\alpha-1)^2 \leq \beta^2 \end{array} \right] \qquad (E_k^r)$$

with $\beta > 0$. Here again, $(I,1)$ is a strictly feasible solution and therefore $m_k(\beta) \leq 0$. Moreover, if $\beta \in (0,1)$, the feasible set of (E_k^r) is contained in the feasible set of (E_k) and then $m_k(\beta) \geq m_k$. Let us turn our interest to the function m_k. It is clear that if $0 < \beta < \beta'$ we have

$$0 \geq m_k(\beta) \geq m_k(\beta').$$

The next proposition shows that the function m_k is actually strictly negative on $(0\,,\,+\infty)$.

Proposition 3.1.

$$m_k(\beta) < 0 \ \ \textit{for all} \ \ \beta > 0.$$

*Proof.*Assume, for contradiction, that $m_k(\beta) = 0$. $(I, 1)$ being a feasible solution of (E_k^r), is optimal too. Apply the first order optimality condition: there exist $\lambda \in \mathbb{R}^m$ and μ such that

$$C_k + \sum_{i=1}^{m} \lambda_i A_i^{(k)} + \mu I = 0 \quad \text{ and } \quad - z_k - \sum_{i=1}^{m} b_i \lambda_i + \mu = 0.$$

Since

$$C_k = L_k^t C L_k \quad \text{ and } \quad A_i^{(k)} = L_k^t A_i L_k$$

the first equation is equivalent to

$$C = - \sum_{i=1}^{m} \lambda_i A_i - \mu X_k^{-1}.$$

It follows that:

$$\begin{aligned} z_k &= \langle C, X_k \rangle = - \textstyle\sum_{i=1}^{m} \lambda_i \langle A_i, X_k \rangle - \mu n, \\ z_k &= - \textstyle\sum_{i=1}^{m} b_i \lambda_i - \mu n. \end{aligned}$$

Hence $\mu = 0$ and therefore

$$z_k = - \sum_{i=1}^{m} b_i \lambda_i \quad \text{and} \quad C + \sum_{i=1}^{m} \lambda_i A_i = 0.$$

Thus, $-\lambda$ is a feasible solution of $(DSDP)$ and therefore

$$z_k = \sum_{i=1}^{m} b_i (-\lambda_i) \leq m_d = z^* < z_k,$$

which is not possible. □

Set $V = Y - I$ and $v = \alpha - 1$, the problem (E_k^r) is equivalent to getting the optimal solution of the convex optimization problem

$$\min_{(V,v)} \left[\langle C_k, V \rangle - z_k v \quad : \quad \begin{array}{l} \langle A_i^{(k)}, V \rangle - b_i v = 0, \ i = 1, \ldots, m, \\ \langle I, V \rangle + v = 0, \\ \|V\|^2 + v^2 \leq \beta^2. \end{array} \right] \qquad (E_k^t)$$

In view of the necessary and sufficient condition for optimality, the problem consists in finding $(V, v, \lambda, \mu, t) \in E \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times [0, +\infty[$ such that:

$$C_k + \sum_{i=1}^{m} \lambda_i A_i^{(k)} + \mu I + tV = 0 \tag{1}$$

$$-z_k - \sum_{i=1}^{m} b_i \lambda_i + \mu + tv = 0 \tag{2}$$

$$\langle A_j^{(k)}, V \rangle - b_j v = 0; \quad j = 1, \ldots, m \tag{3}$$

$$\langle I, V \rangle + v = 0 \tag{4}$$

$$t(\|V\|^2 + v^2 - \beta^2) = 0 \tag{5}$$

$$\|V\|^2 + v^2 \leq \beta^2. \tag{6}$$

Note that one has necessarily $t > 0$, (if not $(V\ ,\ v) = (0\ ,\ 0)$ would be an optimal solution of (E_k^t)). Hence, from (1) and (2), we get

$$V = -t^{-1} V_k \ \text{ and } \ v = -t^{-1} v_k,$$

where

$$V_k = C_k + \sum_{i=1}^{m} \lambda_i A_i^{(k)} + \mu I \ \text{ and } \ v_k = -\sum_{i=1}^{m} b_i \lambda_i + \mu - z_k.$$

By construction, V_k and V are symmetric. Replacing V and v in (3) and (4), we obtain

$$\mu = 0$$

and λ is a solution of the $m \times m$ linear system

$$M\ \lambda = d, \tag{7}$$

where for $i, j = 1, \ldots, m$

$$M_{ij} = \langle A_i^{(k)}, A_j^{(k)} \rangle + b_i b_j,$$
$$d_i = -b_i z_k - \langle C_k, A_i^{(k)} \rangle.$$

By construction, M is symmetric positive semidefinite. It is also positive definite by assumption 1. Hence the system (7) has one solution which can be obtained *via* the Cholesky method. Thus V_k and v_k are easily obtained.

Next, the optimal solution (V, v) of (E_k^t) is given by

$$V = -\beta P_k \ \text{ and } \ v = -\beta p_k,$$

where

$$P_k = \frac{V_k}{\tau}, \quad p_k = \frac{v_k}{\tau} \quad \text{ and } \quad \tau = (\|V_k\|^2 + v_k^2)^{\frac{1}{2}}.$$

Let us return to problem (E_k^r). We see that the optimum is reached for

$$Y(\beta) = I - \beta P_k \quad \text{and} \quad \alpha(\beta) = 1 - \beta p_k. \tag{8}$$

We choose β in such a way that the matrix $Y(\beta)$ stays positive definite and the scalar $\alpha(\beta)$ stays positive. Then, the next iterate X_{k+1} is obtained by the formula:

$$X_{k+1} = T_k^{-1}(Y(\beta)\ ,\ \alpha(\beta)).$$

It is clear that X_{k+1} is a strictly feasible solution for (SDP). Besides, the matrices $V_k, P_k, Y(\beta)$ and X_{k+1} are symmetric.

The next proposition gives an easily checked criteria in order that X_{k+1} stays in the strictly feasible solution set of (SDP).

Proposition 3.2. *Define*

$$\tilde{\beta}_k = \left[\max\left(p_k\ ,\ \overline{\lambda} + \sigma\sqrt{n-1}\right)\right]^{-1}$$

where

$$\overline{\lambda} = \frac{1}{n}\sum_{i=1}^{n}(P_k)_{ii} \quad \textit{and} \quad \sigma = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(P_k)_{ij}^2 - \overline{\lambda}^2.$$

Then $X_{k+1} = T_k^{-1}(Y(\beta)\ ,\ \alpha(\beta))$ *is a strictly feasible solution of* (SDP) *for any* $\beta \in (0, \tilde{\beta}_k)$ *if* $\tilde{\beta}_k > 0$ *and for any* $\beta > 0$ *otherwise.*

Proof. We must prove that the matrix $Y(\beta)$ is positive definite and the scalar $\alpha(\beta)$ is positive.

Let us denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of the matrix P_k, then from [12], we have

$$\overline{\lambda} - \sigma\sqrt{n-1} \le \min_i \lambda_i \le \overline{\lambda} - \frac{\sigma}{\sqrt{n-1}},$$
$$\overline{\lambda} + \frac{\sigma}{\sqrt{n-1}} \le \max_i \lambda_i \le \overline{\lambda} + \sigma\sqrt{n-1}.$$

It follows that $Y(\beta) = I - \beta P_k$ is positive definite when

$$1 - \beta \max_i \lambda_i > 0$$

i.e., when

$$\frac{1}{\beta} > \overline{\lambda} + \sigma\sqrt{n-1}. \tag{a}$$

On the other hand, $\alpha(\beta) = 1 - \beta p_k$ is strictly positive when

$$\frac{1}{\beta} > p_k. \tag{b}$$

Summarizing, X_{k+1} is strictly feasible when $0 < \beta < \tilde{\beta}_k$. □

The next proposition shows that for $\beta \in (0, \tilde{\beta}_k)$, we obtain a reduction of the value of the objective function of (SDP).

Proposition 3.3. *For any $\beta \in (0\ ,\ \tilde{\beta}_k)$, it holds*

$$X_{k+1} = X_k - \frac{\beta}{1-\beta p_k}(L_k P_k L_k^t - p_k X_k),$$

$$m_k(\beta) = -\beta[\langle C_k, P_k\rangle - z_k p_k],$$

and

$$\langle C, X_{k+1}\rangle - \langle C, X_k\rangle = \frac{1}{1-\beta p_k} m_k(\beta) < 0.$$

Proof. Replacing $Y(\beta)$ and $\alpha(\beta)$ in the formula

$$X_{k+1} = T_k^{-1}(Y(\beta)\ ,\ \alpha(\beta)) = \frac{1}{\alpha(\beta)} L_k Y(\beta) L_k^t,$$

we obtain

$$\begin{aligned} X_{k+1} &= X_k - \tfrac{\beta}{1-\beta p_k} L_k(P_k - p_k I)L_k^t, \\ &= X_k - \tfrac{\beta}{1-\beta p_k}(L_k P_k L_k^t - p_k X_k). \end{aligned}$$

On the other hand,

$$\begin{aligned} m_k(\beta) &= \langle C_k, Y(\beta)\rangle - z_k \alpha(\beta), \\ &= \langle C_k, I\rangle - z_k - \beta[\langle C_k, P_k\rangle - z_k p_k], \\ &= \langle C, X_k\rangle - z_k - \beta[\langle C_k, P_k\rangle - z_k p_k], \\ &= -\beta[\langle C_k, P_k\rangle - z_k p_k]. \end{aligned}$$

It follows that

$$\langle C, X_{k+1}\rangle - \langle C, X_k\rangle = \frac{1}{1-\beta p_k} m_k(\beta).$$

We know by Proposition 3.1 that $m_k(\beta) < 0$ for all $\beta > 0$. □

Now, we summarize the algorithm.

Description of the algorithm

a) Initialization:

(1) $k = 0$, X_0 is a strictly feasible solution of the problem.
(2) We choose a parameter $\rho \in (0, 1)$ and a small $\varepsilon > 0$ (for the stopping rule).

b) Step k: At the beginning of the step, X_k is a strictly feasible solution of (SDP).

(1) Set $z_k = \langle C, X_k\rangle$.
(2) Determine L_k such that $X_k = L_k L_k^t$. Next, compute
   (a) $C_k = L_k^t C L_k$,
   (b) $A_i^{(k)} = L_k^t A_i L_k$, $i = 1, \dots, m$.
(3) Compute the matrix M and the vector d as:
   (a) $M_{ij} = \langle A_i^{(k)}, A_j^{(k)}\rangle + b_i b_j$, $i, j = 1, \dots, m$,
   (b) $d_i = -b_i z_k - \langle C_k, A_i^{(k)}\rangle$ $i = 1, \dots, m$.

(4) Solve the linear system $M\lambda = d$.
(5) Compute
  (a) $V_k = (C_k + \sum_{i=1}^m \lambda_i A_i^{(k)})$,
  (b) $v_k = (-\sum_{i=1}^m b_i\lambda_i - z_k)$,
  (c) $\tau = (\|V_k\|^2 + v_k^2)^{\frac{1}{2}}$,
  (d) $\overline{\lambda} = \frac{1}{n}\sum_{i=1}^n (P_k)_{ii}$,
  (e) $\sigma = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n (P_k)_{ij}^2 - \overline{\lambda}^2$,
  (f) $\beta_k = \rho\left[\max\left(p_k\ ,\ \overline{\lambda} + \sigma\sqrt{n-1}\right)\right]^{-1}$.
(6) Compute
  (a) $X_{k+1} = X_k - \frac{\beta_k}{\tau - \beta_k v_k} L_k(V_k - v_k I)L_k^t$.
(7) **Stopping rule**
  (a) If $\beta_k[\langle C_k, V_k\rangle - z_k v_k] < \tau\varepsilon$: **STOP,**
  (b) if not, do $k = k+1$ and go back to **Step k**.

4. Finding an initial feasible solution

The strict feasibility problem of (SDP) consists of finding a $n \times n$ matrix X such that:

$$X \in \text{int (K)}, \quad \langle A_i, X\rangle = b_i \ \text{ for } i = 1, \ldots, m. \qquad (F)$$

In order to solve this problem, we introduce the linear semidefinite program:

$$\min_{(X,\lambda)} \left[\lambda \ : \ \begin{array}{l} \langle A_i, X\rangle + \lambda(b_i - \langle A_i, X_0\rangle) = b_i \ \text{ for } i = 1, \ldots, m, \\ X \in K,\ \lambda \geq 0. \end{array} \right] \qquad (AP)$$

Then X^* is a solution of problem (F) if and only if $(X^*, 0)$ is an optimal solution of problem (AP) and $X^* \in \text{int(K)}$.

Note that (AP) can be reformulated as:

$$\min_{X'} [\ \langle C', X'\rangle \ : \ X' \in K,\ \langle A_i', X'\rangle = b_i \ \text{ for } i = 1, \ldots, m], \qquad (AP)$$

where C' is the $(n+1) \times (n+1)$ symmetric matrix defined by

$$C'[i,j] = \begin{cases} 1 & \text{if } i = j = n+1, \\ 0 & \text{otherwise} \end{cases}$$

and A_i', $i = 1, \cdots, n$ is the $(n+1) \times (n+1)$ symmetric matrix defined by

$$A_i' = \begin{pmatrix} A_i & 0 \\ 0 & b_i - \langle A_i, X_0\rangle \end{pmatrix}.$$

Finally, X' is the $(n+1) \times (n+1)$ matrix such that

$$X' = \begin{pmatrix} X & 0 \\ 0 & \lambda \end{pmatrix}.$$

Choose some $X_0 \in \text{int(K)}$ (for instance the identity matrix). Then, $X' = \begin{pmatrix} X_0 & 0 \\ 0 & 1 \end{pmatrix}$ is a strictly feasible solution of (AP). Apply the algorithm described in Section 3 to (AP).

5. Numerical tests

The algorithm has been tested on some benchmark problems issued from the library of test problems SDPLIB [13]. We have taken $\rho = 0.90$ and the stopping criterion $\epsilon = 10^{-8}$. The first phase (phase 1) corresponds to the search of an initial strictly feasible solution and the second one (phase 2) is the resolution of the problem itself.

Examples	Size (m,n)	Nbr. of iterations Phase 1	Nbr. of iterations Phase 2
control1	(21,15)	10	106
hinf1	(13,14)	6	27
hinf2	(13,16)	7	43
hinf3	(13,16)	6	109
hinf4	(13,16)	7	39
hinf5	(13,16)	7	42
hinf7	(13,16)	7	38
hinf9	(13,16)	5	28
hinf10	(21,18)	7	57
truss1	(6,13)	15	17
truss4	(12,19)	25	21

We have also tested the infeasible problems infd1 and infd2 of SDPLIB. The algorithm concludes to their infeasibility.

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