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## SUBHARMONIC SOLUTIONS OF A NONCONVEX NONCOERCIVE HAMILTONIAN SYSTEM

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**Abstract.** In this paper we study the existence of subharmonic solutions of the Hamiltonian system

$$J\dot{x} + u^*\nabla G(t, u(x)) = e(t)$$

where  $u$  is a linear map,  $G$  is a  $C^1$ -function and  $e$  is a continuous function.

### 1. INTRODUCTION

In this paper we are interested in the existence of periodic solutions, of the noncoercive Hamiltonian system

$$J\dot{x} + u^*\nabla G(t, u(x)) = e(t)$$

where  $u: \mathbb{R}^{2n} \rightarrow \mathbb{R}^m$  ( $1 \leq m \leq 2n$ ) is a linear map not identically null with adjoint  $u^*$ ;  $G: \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $(t, y) \rightarrow G(t, y)$  is a continuous function,  $T$ -periodic in the first variable ( $T > 0$ ), differentiable with respect to the second variable and its derivative  $\nabla G$  is continuous;  $e: \mathbb{R} \rightarrow \mathbb{R}^{2n}$  is a continuous,  $T$ -periodic function with mean value zero, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

is the standard symplectic matrix.

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There have been many papers studying the multiplicity of periodic solutions of the Hamiltonian system  $\dot{x} = JH'(t, x)$ . They have been obtained by using many different techniques, for example Morse theory, Minimax methods, ... However, most of the results proving the existence of subharmonic solutions have made use of a convexity and a coercivity assumptions on  $H$ , see [1, 2, 7, 8].

Our first result is the following:

**Theorem 1.1.** *Assume that  $G$  satisfies:*

$$(G_1) \quad \exists M > 0 : \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^m, \|\nabla G(t, x)\| \leq M;$$

$(G_2)$  either

$$(i) \quad \lim_{|x| \rightarrow +\infty} G(t, x) = +\infty, \text{ uniformly in } t \in [0, T],$$

or

$$(ii) \quad \lim_{|x| \rightarrow +\infty} G(t, x) = -\infty, \text{ uniformly in } t \in [0, T],$$

then the Hamiltonian system

$$J\dot{x} + u^* \nabla G(t, u(x)) = e(t) \quad (\mathcal{H}_e)$$

has at least one  $T$ -periodic solution.

In the case where the forcing term  $e$  is null, we obtain:

**Theorem 1.2.** *Assume that  $rgu > n$ .*

*Under the assumptions  $(G_1, G_2)$ , the Hamiltonian system  $(\mathcal{H}_0)$  possesses for all integer  $k \geq 1$ , a  $kT$ -periodic solution  $x_k$  satisfying*

$$\lim_{k \rightarrow +\infty} \|x_k\|_\infty = +\infty,$$

where  $\|x\|_\infty = \sup\{|x(t)|, t \in \mathbb{R}\}$ .

Concerning the minimality of the period, we have the following result:

**Theorem 1.3.** *Assume that  $rgu > n$ .*

*If the Hamiltonian  $G$  satisfies  $(G_1)$  and the assumption  $(G'_2)$  either*

$$(i') \quad \lim_{|x| \rightarrow +\infty} \langle \nabla G(t, x), x \rangle = +\infty, \text{ uniformly in } t \in [0, T],$$

or

$$(ii') \quad \lim_{|x| \rightarrow +\infty} \langle \nabla G(t, x), x \rangle = -\infty, \text{ uniformly in } t \in [0, T],$$

then for any sufficiently large prime number  $k$ , the system  $(\mathcal{H}_0)$  possesses a  $kT$ -periodic solution with minimal period  $kT$ .

## 2. PRELIMINARIES [5]

Let  $X = W \oplus Z$  be a Banach space and  $X_n = W_n \oplus Z_n$  be a sequence of closed subspaces with  $Z_0 \subset Z_1 \subset \dots \subset Z$ ,  $W_0 \subset W_1 \subset \dots \subset W$ ,  $1 \leq \dim W_n < \infty$ . For every  $\varphi : X \rightarrow \mathbb{R}$  we denote by  $\varphi_n$  the function  $\varphi$  restricted to  $X_n$ . Let us recall that, for any  $A \subset X$ ,  $A_n = A \cap X_n$ .

**Definition 2.1.** Let  $c \in \mathbb{R}$  and  $\varphi \in C^1(X, \mathbb{R})$ . The functional  $\varphi$  satisfies the  $(PS)_c^*$  condition if every sequence  $(x_{n_j}) \subset X$  satisfying:  $\varphi_{n_j}(x_{n_j}) \rightarrow c$ ,  $\varphi'_{n_j}(x_{n_j}) \rightarrow 0$ , possesses a subsequence which converges in  $X$  to a critical point of  $\varphi$ .

**Theorem 2.1** (Generalized Saddle Point theorem). *Let  $\varphi \in C^1(X, \mathbb{R})$ . Assume that there exists  $r > 0$  such that, with  $Y = \{w \in W; |w| = r\}$ :*

- a)  $\sup_Y \varphi \leq \inf_Z \varphi$ ;
- b)  $\varphi$  is bounded from above on  $A = \{x \in W; \|x\| \leq r\}$ ;
- c)  $\varphi$  satisfies the  $(PS)_c^*$  condition, where  $c = \inf_{A \in \mathcal{A}} \sup_{x \in A} \varphi(x)$ ,

with

$$\mathcal{A} = \left\{ A \subset X; A \text{ is closed, } Y \subset A, \text{cat}_{X, Y}^\infty(A) = 1 \right\}.$$

Then  $c$  is a critical value of  $\varphi$  and  $c \geq \inf_Z \varphi$ .

**Remark 2.1.** In a) we may replace  $Z$  by  $q + Z$ ,  $q \in W$ .

## 3. PROOF OF THEOREM 1.1

We will prove here the case where  $G$  satisfies  $(G_2)(i)$ , the case  $(G_2)(ii)$  is the same.

Before giving a variational formulation of  $(\mathcal{H}_e)$ , some preliminary materials on function spaces and norms are needed.

Let  $L^2(S^1, \mathbb{R}^{2n})$  be the space of square integrable functions defined on  $S^1 = \mathbb{R}/T\mathbb{Z}$ , with value in  $\mathbb{R}^{2n}$ . Each function  $x \in L^2(S^1, \mathbb{R}^{2n})$  has a Fourier expansion

$$x(t) = \sum_{m \in \mathbb{Z}} \exp\left(\frac{2\pi}{T} mtJ\right) \hat{x}_m$$

where  $\hat{x}_m \in \mathbb{R}^{2n}$  and  $\sum_{m \in \mathbb{Z}} |\hat{x}_m|^2 < \infty$ .

Set

$$H^{1/2}(S^1, \mathbb{R}^{2n}) = \{x \in L^2(S^1, \mathbb{R}^{2n}); \|x\|_{H^{1/2}} < \infty\},$$

where

$$\|x\|_{H^{1/2}} = \left[ \sum_{m \in \mathbb{Z}} (1 + |m|) |\hat{x}_m|^2 \right]^{1/2}.$$

Consider the subspace

$$X = \left\{ x \in H^{1/2}(S^1, \mathbb{R}^{2n}); \hat{x}_0 \in \ker u^\perp \right\}.$$

It is easy to check that the quadratic form  $Q$  defined on  $X$  by

$$Q(x) = 1/2 \int_0^T \langle J\dot{x}, x \rangle dt$$

satisfies, for a smooth  $x \in X$

$$Q(x) = -\pi \sum_{m \in \mathbb{Z}} m |\hat{x}_m|^2. \quad (1)$$

Set

$$\begin{aligned} X^0 &= (\ker u)^\perp, \\ X^+ &= \left\{ x \in X; x(t) = \sum_{m \leq -1} \exp\left(\frac{2\pi}{T} mtJ\right) \hat{x}_m \quad \text{a.e.} \right\} \\ X^- &= \left\{ x \in X; x(t) = \sum_{m \geq 1} \exp\left(\frac{2\pi}{T} mtJ\right) \hat{x}_m \quad \text{a.e.} \right\}, \end{aligned}$$

then  $X = X^+ \oplus X^- \oplus X^0$ , and  $X^+, X^-, X^0$ , is respectively the subspace of  $X$  on which  $Q$  is positive definite, negative definite, and null.  $X^+, X^-, X^0$  are mutually orthogonal with respect to the associated inner product and mutually orthogonal in  $L^2(S^1, \mathbb{R}^{2n})$ .

These remarks show that if  $x = x^+ + x^- + x^0 \in X$ ,

$$\|x\|^2 = |x^0|^2 + Q(x^+) - Q(x^-) \quad (2)$$

serves as an equivalent norm on  $X$ . Henceforth we use the norm defined in (2) as the norm for  $X$ . For each  $p \in [1, \infty[$ ,  $X$  is compactly embedded in  $L^p(S^1, \mathbb{R}^{2n})$ . In particular there is an  $\alpha_p > 0$  such that

$$\forall x \in X, \|x\|_{L^p} \leq \alpha_p \|x\|. \quad (3)$$

Now, consider the functional

$$f(x) = \int_0^T [1/2 \langle J\dot{x}, x \rangle + G(t, u(x)) - \langle e(t), x \rangle] dt$$

defined on the space  $X$ . The functional  $f$  is continuously differentiable and verifies

$$\forall x, y \in X, f'(x).y = \int_0^T \langle J\dot{x} + u^* \nabla G(t, u(x)) - e(t), y \rangle dt. \quad (4)$$

We claim that the critical points of the functional  $f$  correspond to the  $T$ -periodic solutions of the system  $(\mathcal{H}_e)$ . Indeed, let  $x$  be a critical point of  $f$  on  $X$ , then by (4) there exists a constant  $\xi \in \ker u$  such that

$$J\dot{x} + u^* \nabla G(t, u(x)) - e(t) = \xi \quad \text{a.e.} \quad (5)$$

Integrating (5) we obtain

$$u^* \int_0^T \nabla G(t, u(x)) dt = T\xi,$$

so  $\xi \in (\ker u)^\perp$ , which proves that  $\xi = 0$  and  $x$  is a  $T$ -periodic solution of  $(\mathcal{H}_e)$ . Inversely, it is clear that every solution of  $(\mathcal{H}_e)$  is a critical point of  $f$  on  $X$ .

To find critical points of  $f$  we shall apply the Generalized Saddle Point theorem to the functional  $f$  on  $X$  with these subspaces  $W = X^-$ ,  $Z = X^+ \oplus X^0$  and the sequence of closed subspaces

$$X_n = \left\{ x \in X; x(t) = \sum_{|m| \leq n} \exp\left(\frac{2\pi}{T}mtJ\right) \hat{x}_m \quad \text{a.e.}, \right\}.$$

First, from (1), the assumption  $(G_1)$  and the mean value theorem applied to the function  $G(t, \cdot)$ , we have for  $x = x^0 + x^+ \in Z$ ,

$$f(x) \geq \frac{\pi}{2} \|x^+\|^2 - c \|x^+\| + \int_0^T G(t, u(x^0)) dt, \quad (6)$$

where  $c$  is a constant. Since the function  $u$  is invertible on  $X^0$ , we deduce from (6) and the assumption  $(G_2)$  that  $f(x)$  goes to infinity as  $x$  goes to infinity in  $Z$ .

Secondly as in the lastly, there exist two constants  $a, b \in \mathbb{R}$  such that for every  $x \in W$

$$f(x) \leq -\frac{\pi}{2} \|x\|^2 + a \|x\| + b.$$

Consequently there exists  $r > 0$  such that

$$\sup_Y f \leq \inf_Z f$$

where  $Y = \{x \in W; \|x\| = r\}$ . Furthermore,  $f$  is bounded from above on  $D_r$ , where  $D_r$  is the closed disc in  $W$  centered in zero, with radius  $r$ .

Finally we will show that for all  $c \in \mathbb{R}$ , the functional  $f$  satisfies the Palais Smale condition  $(PS)_c^*$  with respect to  $(X_n)$ . Let  $(x_{n_j})$  be a sequence such that

$$n_j \longrightarrow \infty, x_{n_j} \in X_{n_j}, f(x_{n_j}) \longrightarrow c, f'_{n_j}(x_{n_j}) \longrightarrow 0, \quad (7)$$

we set  $x_{n_j} = x_{n_j}^0 + \tilde{x}_{n_j}$ , where  $x_{n_j}^0$  is the projection of  $x_{n_j}$  onto  $X^0$ . We have from (1) and (4):

$$\begin{aligned} f'_{n_j}(x_{n_j}) \left( x_{n_j}^+ - x_{n_j}^- \right) &= 2\pi \sum_{1 \leq |m| \leq n_j} |m| |\hat{x}_{j,m}|^2 \\ &\quad + \int_0^T \langle u^* \nabla G(t, u(x)) - e(t), x_{n_j}^+ - x_{n_j}^- \rangle dt. \end{aligned}$$

It follows from (7) and the assumption  $(G_1)$  that  $(\tilde{x}_{n_j})$  is bounded in  $X$ . Elsewhere, we have

$$f(x_{n_j}) = 1/2 \int_0^T \langle J\tilde{x}_{n_j}, \tilde{x}_{n_j} \rangle dt + \int_0^T G(t, u(x_{n_j})) dt - \int_0^T \langle e, \tilde{x}_{n_j} \rangle dt,$$

so by (7),  $(\int_0^T G(t, u(x_{n_j})) dt)$  is bounded. By applying the mean value theorem to the function  $G(t, \cdot)$ , there exists  $y_{n_j} \in X$  such that

$$\int_0^T G(t, u(x_{n_j})) dt = \int_0^T G(t, u(x_{n_j}^0)) dt + \int_0^T \langle u^* \nabla G(t, u(y_{n_j})), u(\tilde{x}_{n_j}) \rangle dt$$

and we deduce from the assumption  $(G_1)$  that  $(\int_0^T G(t, u(x_{n_j}^0)) dt)$  is bounded, so by the assumption  $(G_2)(i)$ ,  $(x_{n_j}^0)$  is also bounded in  $X$ . Up to a subsequence, we can assume that  $x_{n_j} \rightarrow x$  and  $x_{n_j}^0 \rightarrow x^0$  in  $X$ . Notice that

$$\begin{aligned} Q(x_{n_j}^+ - x^+) &= \langle f'_{n_j}(x_{n_j}) - f'(x), x_{n_j}^+ - x^+ \rangle \\ &\quad - \int_0^T \langle \nabla G(t, u(x_{n_j})) - \nabla G(t, u(x)), u(x_{n_j}^+ - x^+) \rangle dt. \end{aligned}$$

This implies that  $x_{n_j}^+ \rightarrow x^+$  in  $X$ . Similarly  $x_{n_j}^- \rightarrow x^-$  in  $X$ . It follows then that  $x_{n_j} \rightarrow x$  in  $X$  and  $f'(x) = 0$ .

The function  $f$  satisfies all the assumptions of the Generalized Saddle Point theorem so  $f$  has at least one critical point. The proof of Theorem 1.1 is complete.

#### 4. PROOF OF THEOREM 1.2

As in the proof of Theorem 1.1, we will prove here the case where  $G$  satisfies  $(G_1)$  and  $(G_2)(i)$ , the case  $(G_2)(ii)$  is the same. By making the change of variable  $t \rightarrow \frac{t}{k}$ , the system  $(\mathcal{H}_0)$  transforms to:

$$J\dot{x} + ku^* \nabla G(kt, u(x)) = 0. \quad (\mathcal{H}_k)$$

Hence to find  $kT$ -periodic solutions of  $(\mathcal{H}_0)$ , it suffices to find  $T$ -periodic solutions of  $(\mathcal{H}_k)$ , which are the critical points of the continuously differentiable functional

$$f_k(x) = \frac{1}{2} \int_0^T \langle J\dot{x}, x \rangle dt + k \int_0^T G(kt, u(x)) dt$$

defined on the space  $X$  introduced above. By applying the Theorem 2.1 to the functional  $f_k$  on the space  $X$ , we prove as in Theorem 1.1, that for all integer  $k \geq 1$  the system  $(\mathcal{H}_k)$  possesses a  $T$ -periodic solution  $x_k$  such that

$$f_k(x_k) \geq \inf_{x \in Z} f_k(x). \quad (8)$$

Now we will prove that the sequence  $(x_k)$  obtained above has the following property

$$\lim_{k \rightarrow \infty} \frac{1}{k} f_k(x_k) = \infty. \quad (9)$$

This will be done by the following lemma:

**Lemma 4.1.** *Assume that  $G$  satisfies  $(G_2)$ , then:*

$$\lim_{k \rightarrow \infty} \inf_{x \in Z} \frac{f_k \left( k^{\frac{1}{2}}(\varphi + x) \right)}{k} = \infty \quad (10)$$

where  $\varphi = \frac{1}{\pi^{1/2}} \exp \left( \frac{2\pi}{T} t J \right) e_0$  and  $e_0$  is a particular element in  $X^0$ , ( $J e_0 \notin \ker u$ ).

*Proof of Lemma 4.1.* Assume by contradiction that there exist sequences  $k_j \rightarrow \infty$ ,  $x_j = x_j^+ + x_j^0 \in Z$  and a constant  $c \in \mathbb{R}$  such that

$$\forall j \in \mathbb{N}, f_{k_j} \left( k_j^{\frac{1}{2}}(\varphi + x_j) \right) \leq k_j c. \quad (11)$$

Taking  $x_j = x_j^+ + x_j^0$ ,  $x_j^i \in X^i$ ,  $i = +, 0$ , by an easy calculation, we obtain

$$f_{k_j} \left( k_j^{\frac{1}{2}}(\varphi + x_j) \right) = k_j \left[ \|x_j^+\|^2 - \|e_0\|^2 + \int_0^T G \left( k_j t, u \left( k_j^{\frac{1}{2}}(\varphi + x_j) \right) \right) dt \right]; \quad (12)$$

so by  $(G_2)$  there exists a constant  $c_1 > 0$  such that

$$f_{k_j} \left( k_j^{\frac{1}{2}}(\varphi + x_j) \right) \geq k_j (\|x_j^+\|^2 - c_1)$$

and we conclude, from the inequality (11), that  $(x_j^+)$  is a bounded sequence in  $X$ . Taking a subsequence if necessary, we find  $x^+ \in X^+$  such that

$$x_j^+(t) \rightarrow x^+(t) \quad \text{as } j \rightarrow \infty \quad \text{for a.e. } t \in [0, T]. \quad (13)$$

We claim that  $(x_j^0)$  is also bounded in  $X^0$ . If we suppose otherwise, we easily deduce from (13) and the fact that  $u$  is invertible on  $X^0$  that

$$\left| u \left( k_j^{\frac{1}{2}}(\varphi(t) + x_j^+(t) + x_j^0) \right) \right| \rightarrow \infty \quad \text{as } j \rightarrow \infty \quad \text{for a.e. } t \in [0, T].$$

Consequently by  $(G_2)$  and Fatou's lemma, we obtain

$$\int_0^T G \left( k_j t, u \left( k_j^{\frac{1}{2}}(\varphi(t) + x_j^+(t) + x_j^0) \right) \right) dt \rightarrow \infty \quad \text{as } j \rightarrow \infty, \quad (14)$$

and we deduce from (12) that

$$\frac{f_{k_j}(k_j^{\frac{1}{2}}(\varphi + x_j^+ + x_j^0))}{k_j} \longrightarrow \infty \quad \text{as } j \longrightarrow \infty, \quad (15)$$

which contradicts (11) and proves our claim. Going if necessary to a subsequence, we can assume that there exists  $x^0 \in X^0$  such that

$$\varphi(t) + x_j^+(t) + x_j^0 \longrightarrow x(t) = \varphi(t) + x^+(t) + x^0 \quad \text{as } j \longrightarrow \infty \quad \text{for a.e. } t \in [0, T].$$

By Fourier analysis, we have  $x(t) \notin \ker u$  for almost every  $t \in [0, T]$ . Therefore

$$\left| u \left( k_j^{\frac{1}{2}} (\varphi(t) + x_j^+(t) + x_j^0) \right) \right| \longrightarrow \infty \quad \text{as } j \longrightarrow \infty,$$

and by  $(G_2)$  and Fatou's lemma, we obtain (15), which contradicts (11). Thus (10) must hold. The proof of Lemma 4.1 is complete.  $\square$

Hence Theorem 2.1 (Rem. 2.1) implies that for all  $k \in \mathbb{N}$ , we have

$$f_k(x_k) = b_k \geq \inf_{x \in Z} f_k \left( k^{\frac{1}{2}} (\varphi + x) \right).$$

So we have by Lemma 4.1:

$$\frac{b_k}{k} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty.$$

We claim that  $\|x_k\|_\infty \longrightarrow \infty$  as  $k \longrightarrow \infty$ . Indeed, if we suppose otherwise,  $(x_k)$  possesses a bounded subsequence  $(x_{k_p})$ . Since

$$\frac{f_k(x_k)}{k} = -\frac{1}{2} \int_0^T \langle u^* \nabla G(kt, u(x_k)), x_k \rangle dt + \int_0^T G(kt, u(x_k)) dt$$

the sequence  $\left( \frac{b_{k_p}}{k_p} \right)$  is bounded, contrary to (9). Consequently, we have

$$\lim_{k \longrightarrow \infty} \|x_k\|_\infty = \infty,$$

which implies that  $v_k(t) = x_k(\frac{t}{k})$  is a  $kT$ -periodic solution of the system  $(\mathcal{H}_0)$  and verifies:

$$\lim_{k \longrightarrow \infty} \|v_k\|_\infty = \lim_{k \longrightarrow \infty} \|x_k\|_\infty = \infty.$$

That concludes the proof of Theorem 1.2.



## 5. PROOF OF THEOREM 1.3.

We begin the proof by the following lemma:

**Lemma 5.1.** *The assumptions  $(G_1), (G'_2)$  imply the assumption  $(G_2)$ .*

*Proof of Lemma 5.1:* Assume for example  $(G_1)$  and  $(G'_2)(i')$  hold and let us prove  $(G_2)(i)$ . Let  $R > 0$  be such that

$$|x| \geq R \implies \langle \nabla G(t, x), x \rangle \geq 1, \forall t \in \mathbb{R}.$$

Then, for  $|x| \geq R$ , we have

$$\begin{aligned} G(t, x) &= G(t, 0) + \int_0^{\frac{R}{|x|}} \langle \nabla G(t, sx), x \rangle ds + \int_{\frac{R}{|x|}}^1 \langle \nabla G(t, sx), x \rangle ds \\ &\geq G(t, 0) - MR + \int_{\frac{R}{|x|}}^1 \frac{ds}{s} \\ &\geq \inf_{t \in \mathbb{R}} G(t, 0) - MR + \text{Log} \left( \frac{|x|}{R} \right) \end{aligned}$$

and  $(G_2)(i)$  follows.

Let  $k$  be an integer  $\geq 1$ , we consider the functional  $\varphi_k$

$$\varphi_k(x) = \frac{1}{2} \int_0^{kT} \langle J\dot{x}, x \rangle dt + \int_0^{kT} G(t, u(x)) dt$$

defined on the space

$$X^k = \left\{ x \in H^{\frac{1}{2}}(S_k^1, \mathbb{R}^{2n}); \hat{x}_0 \in (\ker u)^\perp \right\},$$

where  $S_k^1 = \mathbb{R}/kT\mathbb{Z}$ .

It is clear that  $x \in X$  is a critical point of  $f_k$  if and only if  $v(t) = x(\frac{t}{k})$  belongs to  $X^k$  and is a critical point of  $\varphi_k$ . By the Lemma 5.1, the assumptions  $(G_1, G_2)$  are satisfied, so we use the Theorem 1.2.

Let  $(v_k) \subset X^k$  be the sequence of critical points of  $\varphi_k$  associated to the sequence  $(x_k) \subset X$  of critical points of  $f_k$  obtained in the proof of Theorem 1.2, the property (9) is written

$$\lim_{k \rightarrow \infty} \frac{1}{k} \varphi_k(v_k) = \infty. \quad (16)$$

On the other hand, let us denote by  $S_T$ , the set of  $T$ -periodic solutions of  $(\mathcal{H}_0)$  which are in  $X$ . We claim that  $S_T$  is bounded in  $X$ . Indeed, assume by contradiction that there exists a sequence  $(v_n)$  in  $S_T$  such that  $\|v_n\|_\infty \rightarrow \infty$ . Let us write  $\tilde{v}_n = v_n^+ + v_n^-$  and  $\bar{v}_n$  the mean value of  $v_n$ , then  $v_n = \tilde{v}_n + \bar{v}_n$ . Multiplying both sides of the identity

$$J\dot{v}_n + u^* \nabla G(t, u(v_n)) = 0 \quad (17)$$

by  $J\dot{v}_n$  and integrating, we obtain:

$$\|\dot{v}_n\|_{\mathbf{L}^2}^2 + \int_0^T \langle \nabla G(t, u(v_n(t))), u(J\dot{v}_n) \rangle dt = 0.$$

Using assumption  $(G_1)$ , we easily deduce that  $(\dot{v}_n)$  is bounded in  $L^2$ . So  $(\tilde{v}_n)$  is bounded in  $L^\infty([0, T])$  and in  $X$ , and then  $(u(\tilde{v}_n))$  is bounded in  $L^\infty([0, T])$ . The identity

$$\|v_n\|^2 = |\bar{v}_n|^2 + \|\tilde{v}_n\|^2$$

shows then that

$$\lim_{n \rightarrow \infty} |\bar{v}_n| = \infty,$$

and then

$$\lim_{n \rightarrow \infty} |u(\bar{v}_n)| = \infty.$$

Consequently, we obtain

$$\lim_{n \rightarrow \infty} \min_{t \in [0, T]} |u(v_n(t))| = \infty. \quad (18)$$

Multiplying (17) by  $v_n$  and integrating, we get

$$\int_0^T \langle J\dot{v}_n, v_n \rangle dt + \int_0^T \langle \nabla G(t, u(v_n)), u(v_n) \rangle dt = 0. \quad (19)$$

Now, since  $(\tilde{v}_n)$  is bounded in  $X$ , we deduce from (19) that

$$\left( \int_0^T \langle \nabla G(t, u(v_n)), u(v_n) \rangle dt \right)$$

is bounded.

But this is in contradiction with (18) and  $(G'_2)$ . The claim follows immediately. As a consequence,  $\varphi_1(S_T)$  is bounded, and since for any  $v \in S_T$  one has  $\varphi_k(u) = k\varphi_1(u)$ , we have

$$\exists c > 0 : \forall v \in S_T, \forall k \geq 1, \frac{1}{k} |\varphi_k(v)| \leq c. \quad (20)$$

Consequently, (16) and (20) show that, for  $k$  sufficiently large,  $v_k \notin S_T$ , and if  $k$  is chosen to be a prime number, the minimal period of  $v_k$  has to be  $kT$  and the proof is complete.  $\square$

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